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Nonlinear evolution equations with delays satisfying a local Lipschitz condition

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Abstract

In this paper, we establish the maximal regularity for the nonlinear functional differential equations with time delay and establish a variation of constant formula for solutions of the given equations. We make use of the regularity of the linear differential equations that appears on given Gelfand triple spaces.

Keywords: Nonlinear evolution equation, regularity, local Lipschtiz continuity, delay, analytic semigroup

AMS Classification Primary 35K58; Secondary 76B03

1 Introduction

In this paper, we consider the following nonlinear functional differential equation with time delays in a Hilbert space H:

$$\begin{cases} x'(t) + Ax(t) = \int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds) + k(t), \quad 0 < t \le T, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h, 0). \end{cases} \tag{1.1}$$

Here, k is a forcing term, and A_0 is the operator associated with a sesquilinear form defined on $V \times V$ satisfying Gårding's inequality, where V is another Hilbert space such that $V \subset H \subset V^*$. The nonlinear term g, which is a locally Lipschitz continuous operator from $L^2(-h,T;V)$ to $L^2(0,T;H)$, is a semilinear version of the quasilinear one considered in Yong and Pan [13]. Precise assumptions are given in the next section.

It is well known that the future state realistic models in the natural sciences, biology economics and engineering depends not only on the present but also on the past state. Such applications are used to study the stability, controllability and the time optimal control problems of hereditary systems. The regular problems the semilinear functional

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differential equations with unbounded delays has been surveyed in Vrabie [12] and Jeong et al. [8].

As for the regularity results for a class of nonlinear evolution equations with the nonlinear operator A were developed in many references [1-4]. Ahmed and Xiang [1] gave some existence results for the initial value problem in case where the nonlinear term is not monotone, which improved Hirano's result [7].

In this paper, we will establish a variation of constant formula for solutions of the given equation with a general condition of the local Lipschitz continuity of the nonlinear operator , which is reasonable and widely used in case of the nonlinear system. The main research direction is to find conditions on the nonlinear term such that the regularity result of (1.1) is preserved under perturbation. In order to prove the solvability of the initial value problem (1.1) in Section 3, we establish necessary estimates applying the result of Di Blasio et al. [6] to (1.1) considered as an equation in H as well as in V^* in Section 2. The important technique used is a successive approach method using the regularity and a variation of solutions of the corresponding linear equations without nonlinear terms.

2 Preliminaries and Assumptions

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v\in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u, u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u,\ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. From the following inequalities

$$\omega_1 ||u||^2 \le \text{Re}\, a(u, u) + \omega_2 |u|^2 \le C|Au|\,|u| + \omega_2 |u|^2 \le \max\{C, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one which continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^* ([5], Section 1.3.3 of [11],).

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0, T; X)$ is the collection of all strongly measurable square integrable functions from $(0, T)$ into X and $W^{1,2}(0, T; X)$ is the set of all absolutely continuous functions on $[0, T]$ such that their derivative belongs to $L^2(0, T; X)$. $C([0, T]; X)$ will denote the set of all continuously functions from $[0, T]$ into X with the supremum norm. If X and Y are two Banach space, $\mathcal{L}(X, Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X, X)$ is simply written as $\mathcal{L}(X)$. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \leq M_0 ||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \leq M_0 ||x||_{\mathcal{W}_1(T)}. \tag{2.5}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \leq M, \quad ||S(t)||_* \leq M.$$

The following Lemma is from Lemma 3.6.2 of [10].

Lemma 2.2. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^*:*

$$|S(t)x| \leq Mt^{-1/2}||x||_*, \quad ||S(t)x|| \leq Mt^{-1/2}|x|.$$

First of all, consider the following linear system

$$\begin{cases} x'(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{2.6}$$

By virtue of Theorem 3.3 of [6](or Theorem 3.1 of [8], [10]), we have the following result on the corresponding linear equation of (2.6).

Lemma 2.3. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 2.1) and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (2.6) belonging to $\mathcal{W}(T) \subset C([0, T]; V)$ and satisfying

$$||x||_{\mathcal{W}(T)} \leq C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.7}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (2.6) belonging to $\mathcal{W}_1(T) \subset C([0, T]; H)$ and satisfying

$$||x||_{\mathcal{W}_1(T)} \leq C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

where C_1 is a constant depending on T.

Lemma 2.4. *Suppose that $k \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \leq t \leq T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;D(A))} \leq C_1||k||_{L^2(0,T;H)}, \tag{2.9}$$

$$||x||_{L^2(0,T;H)} \leq C_2T||k||_{L^2(0,T;H)}, \tag{2.10}$$

and

$$||x||_{L^2(0,T;V)} \leq C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.11}$$

Proof. The assertion (2.9) is immediately obtained by (2.7). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \leq M \int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\leq M \int_0^T t \int_0^t |k(s)|^2 ds dt \leq M \frac{T^2}{2} \int_0^T |k(s)|^2 ds \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \leq T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2.3), (2.9), and (2.10) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

3 Semilinear differential equations

In this Section, we consider the maximal regularity of the following nonlinear functional differential equation

$$\begin{cases} x'(t) + Ax(t) = \int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds) + k(t), \quad 0 < t \le T, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h,0), \end{cases} \tag{3.1}$$

where A is the operator mentioned in Section 2. We need to impose the following conditions.

Assumption (F). Let $\mathcal{L}$ and $\mathcal{B}$ be the Lebesgue σ-field on $[0,\infty)$ and the Borel σ-field on $[-h,0]$, respectively. Let μ be a Borel measure on $[-h,0]$ and $g : [0,\infty)\times[-h,0]\times V\times V \to H$ be a nonlinear mapping satisfying the following:

(i) For any $x, y \in V$ the mapping $g(\cdot,\cdot,x,y)$ is strongly $\mathcal{L}\times\mathcal{B}$-measurable.

(ii) $g(t,s,x,y)$ is locally Lipschitz continuous in x and y, uniformly in $(t,s) \in [0,\infty)\times[-h,0]$, i.e., there exist positive constants $L_0, L_1(r)$ and L_2 such that

$$|g(t,s,x,y) - g(t,s,\hat{x},\hat{y})| \le L_1(r)|x-\hat{x}| + L_2||y-\hat{y}||,$$

for all $(t,s) \in [0,\infty)\times[-h,0]$, $y,\ \hat{y} \in V$, $|x| \le r$ and $|\hat{x}| \le r$.

(iii) There exists a real number L_0 such that

$$|g(t,s,x,y)| \le L_0(1+|x|+|y|), \quad |g(t,s,0,0)| \le L_0,$$

for any $(t,s) \in [0,\infty)\times[-h,0]$, $x \in H$, and $y \in V$.

Remark 3.1. *The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [13].*

For $x \in L^2(-h,T;V)$, $T > 0$ we set

$$G(t,x) = \int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds). \tag{3.2}$$

Here, as in [13] we consider the Borel measurable corrections of $x(\cdot)$.

Let U be a Banach space and the controller operator B be a bounded linear operator from the Banach space $L^2(0,T;U)$ to $L^2(0,T;H)$.

Lemma 3.1. *Let $x \in L^2(-h,T;V), T > 0$ and $||x||_{C([0,T],H)} \le r$. Then the nonlinear term $G(\cdot,x)$ defined by (3.2) belongs to $L^2(0,T;H)$ and*

$$\|G(\cdot,x)\|_{L^2(0,T;H)} \le \mu([-h,0])\big\{L_0\sqrt{T}+(L_1(r)+L_2)\|x\|_{L^2(0,T;V)}+L_2\|g^1\|_{L^2(-h,0;V)}\big\} \quad (3.3)$$

Moreover, if $x_1, x_2 \in L^2(-h,T;V)$, then

$$\begin{aligned}\|G(\cdot,x_1)-G(\cdot,x_2)\|_{L^2(0,T;H)} &\le \mu([-h,0]) \\ &\times \big\{(L_1(r)+L_2)\|x_1-x_2\|_{L^2(0,T;V)} + L_2\|x_1-x_2\|_{L^2(-h,0;V)}\big\} \quad (3.4)\end{aligned}$$

Proof. From (ii) of Assumption (F), it is easily seen that

$$\begin{aligned}\|G(\cdot,x)\|_{L^2(0,T;H)} &\le \mu([-h,0])\big\{L_0\sqrt{T}+L_1(r)\|x\|_{L^2(0,T,V)}+\|x\|_{L^2(-h,T,V)}\big\} \\ &\le \mu([-h,0])\big\{L_0\sqrt{T}+(L_1(r)+L_2)\|x\|_{L^2(0,T,V)}+L_2\|x\|_{L^2(-h,0;v)}\big\}.\end{aligned}$$

The proof of (3.4) is similar. □

From now on, we establish the following results on the local solvability of (3.1) represented by

$$\begin{cases} x'(t)+Ax(t)=G(t,x)+k(t), & t\in(0,T] \\ x(0)=g^0, x(s)=g^1(s), & s\in[-h,0]. \end{cases}$$

Theorem 3.1. *Let Assumption (F) be satisfied. Assume that $(g^0, g^1) \in H \times L^2(-h,0;V)$, $k \in L^2(0,T;V^*)$. Then, there exists a time $T_0 \in (0,T)$ such that the equation (3.1) admits a solution*

$$x \in L^2(-h,T_0;V) \cap W^{1,2}(0,T_0;V^*) \subset C([0,T_0];H). \quad (3.5)$$

Proof. For a solution of (3.1) in the wider sense, we are going to find a solution of the following integral equation

$$x(t) = S(t)g^0 + \int_0^t S(t-s)\{G(s,x)+k(s)\}ds. \quad (3.6)$$

To prove a local solution, we will use the successive iteration method. First, put

$$x_0(t) = S(t)g^0 + \int_0^t S(t-s)k(s)ds$$

and define $x_{j+1}(t)$ as

$$x_{j+1}(t) = x_0(t) + \int_0^t S(t-s)G(\cdot,x_j)ds. \quad (3.7)$$

By virtue of Lemma 2.3, we have $x_0(\cdot) \in \mathcal{W}_1(t)$, so that

$$||x_0||_{\mathcal{W}_1(t)} \le C_1(|x_0| + ||k||_{L^2(0,t;V^*)}),$$

where C_1 is a constant in Lemma 2.3. Choose $r > C_1 M_0^{-1}(|x_0| + ||k||_{L^2(0,t;V^*)})$, where M_0 is the constant of (2.5). Putting $p(t) = \int_0^t S(t-s)G(\cdot, x_0)ds$, by (2.11) of Lemma 2.4, we have

$$\begin{aligned} &||p||_{L^2(0,t;V)} \le C_2\sqrt{t}||G(\cdot, x_0)||_{L^2(0,t;H)} \\ &\le C_2\sqrt{t}\big\{\mu([-h,0])L_0\sqrt{t} + (L_1(r)+L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\ &= C_2\mu([-h,0])L_0 t + C_2\mu([-h,0])\big[(L_1(r)+L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big]\sqrt{t}. \end{aligned} \tag{3.8}$$

So that, from(3.5) and (3.6),

$$\begin{aligned} &||x_1||_{L^2(0,t;V)} \\ &\le r + C_2\mu([-h,0])t + C_2\mu([-h,0])\{(L_1(r)+L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\}\sqrt{t} \\ &\le 3r \end{aligned}$$

for any

$$\begin{aligned} m = \min\{&r(C_2\mu([-h,0]))^{-1}, \\ &r\{(C_2\mu([-h,0]))\big((L_1(r)+L_2)\|x\|_{L^2(0,T;v)} + \|g^1\|_{L^2(-h,0;V)}\big)\}^{-2}\}, \end{aligned}$$

$0 \le t \le m$. By induction, it can be shown that for all $j = 1, 2, ...$

$$||x_j||_{L^2(0,t;V)} \le 3r, \quad 0 \le t \le m. \tag{3.9}$$

Hence, from the equation

$$x_{j+1}(t) - x_j(t) = \int_0^t S(t-s)\{G(t, x_j) - G(t, x_{j-1})\}ds$$

From (2.11), (3.7) and Assumption (F), we can observe that the inequality

$$\begin{aligned} ||x_{j+1} - x_j||_{L^2(0,t;V)} &\le C_2\sqrt{t}||G(\cdot, x_j) - G(\cdot, x_{j-1})||_{L^2(0,t;H)} \\ &\le \frac{\{C_2\mu([-h,0])(L_1(3r)+L_2)\sqrt{t}\}^j}{j!}||x_1 - x_0||_{L^2(0,t;V)} \end{aligned}$$

holds for any $0 \le t \le m$. Choose $T_0 > 0$ satisfying

$$T_0 < \min\{m\,,\ \{C_2\mu([-h,0])(L_1(3r)+L_2)\}^{-2}\}. \tag{3.10}$$

Then $\{x_j\}$ is strongly convergent to a function x in $L^2(0, T_0; V)$ uniformly on $0 \le t \le T_0$. By letting $j \to \infty$ in (3.7), we obtain (3.6). Next, we prove the uniqueness of the solution. Let $\epsilon > 0$ be given. For $\epsilon \le t \le T_0$, set

$$x^\epsilon(t) = S(t)g^0 + \int_0^{t-\epsilon} S(t-s)\{G(s, x^\epsilon) + k(s)\}ds. \tag{3.11}$$

Then we have $x^\epsilon \in \mathcal{W}_1(T_0)$ and for $x^\epsilon,\ y^\epsilon \in B_r(T_0)$ which is a ball with radius r in $L^2(0,T_0;V)$, since

$$\begin{aligned} x(t)-x^\epsilon(t) &= \int_0^t S(t-s)\{G(s,x)-G(s,x^\epsilon)\}ds \\ &\quad + \int_{t-\epsilon}^t S(t-s)\{G(s,x^\epsilon)+k(s)\}ds, \end{aligned}$$

with aid of Lemma 2.4,

$$\begin{aligned} &||x-x^\epsilon||_{L^2(0,T_0;V)} \le C_2\mu([-h,0])(L_1(r)+L_2)\sqrt{T_0}||x-x^\epsilon||_{L^2(0,T_0;V)} \\ &+ C_2\sqrt{\epsilon}\mu([-h,0])\{(L_0\sqrt{T}_0+(L_1+L_2)||x||_{L^2(0,T_0;V)}+\sqrt{T}_0||k||_{L^2(0,T_0;H)}\}. \end{aligned}$$

we have $x^\epsilon \to x$ as $\epsilon \to 0$ in $L^2(0,T_0;V)$. Suppose y is another solution of (3.1) and y_ϵ is defined as (3.11) with the initial data (g^0,g^1). Let $x^\epsilon,\ y^\epsilon \in B_r$. Then From Lemma 2.2, it follows that

$$\begin{aligned} ||x^\epsilon-y^\epsilon||_{L^2(0,T_0;V)} &\le \big[\int_0^{T_0} ||\int_0^{s-\epsilon} S(s-\tau)\{(G(\cdot,x^\epsilon)-G(\cdot,y^\epsilon))\}d\tau||^2 ds\big]^{1/2} \\ &\le M\big[\int_0^{T_0}\big(\int_0^{s-\epsilon}(s-\tau)^{-1/2}|G(\cdot,x^\epsilon)-G(\cdot,y^\epsilon)|d\tau\big)^2 ds\big]^{1/2} \\ &\le M\mu([-h,0])L_1(r)\big[\int_0^{T_0}\int_0^{s-\epsilon}(s-\tau)^{-1}d\tau\int_0^{s-\epsilon}||x^\epsilon(\tau)-y^\epsilon(\tau)||^2 d\tau ds\big]^{1/2} \\ &\le M\mu([-h,0])L_1(r)\log\frac{T_0}{\epsilon}\int_0^{T_0}||x^\epsilon-y^\epsilon||_{L^2(0,s;V)}ds, \end{aligned}$$

so that by using Gronwall's inequality, independently of ϵ, we get $x^\epsilon = y^\epsilon$ in $L^2(0,T_0;V)$, which proves the uniqueness of solution of (3.1) in $\mathcal{W}_1(T_0)$. □

From now on, we give a norm estimation of the solution of (3.3) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.2. *Under the Assumption (F) for the nonlinear mapping G, there exists a unique solution x of (3.1) such that*

$$x \in \mathcal{W}_1(T) \subset C([0,T];H). \tag{3.12}$$

for any $(g^0,g^1) \in H \times L^2(0,T;V)$, $k \in L^2(0,T;V^*)$. *Moreover, there exists a constant* C_3 *such that*

$$||x||_{\mathcal{W}_1} \le C_3(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{3.13}$$

where C_3 *is a constant depending on* T.

Proof. Let $y \in B_r$ be the solution of the following linear functional differential equation parabolic type;

$$\begin{cases} y'(t)+Ay(t)=k(t), \quad t\in(0,T_1]. \\ y(0)=g^0. \end{cases}$$

Let the constant T_1 satisfy (3.10) and the following inequality:

$$C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)<1. \tag{3.14}$$

Then we have

$$\begin{cases} d(x-y)(t)/dt + A((x-y)(t)) = G(t,x), \quad t\in(0,T_1]. \\ (x-y)(0)=0. \end{cases}$$

Hence, in view of (F) and Lemmas 2.3 and 3.1,

$$\begin{aligned} &||x-y||_{L^2(0,T_1;D(A))\cap W^{1,2}(0,T_1;H)} \le C_1||G(\cdot,x)||_{L^2(0,T_1;H)} \\ &\le C_1\mu([-h,0])\big\{L_0\sqrt{T_1}+(L_1(r)+L_2)\|x\|_{L^2(0,T_1;V)}+L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\ &\le C_1\mu([-h,0])(L_1(r)+L_2)\big(||x-y||_{L^2(0,T_1:V)}+||y||_{L^2(0,T_1;V)}\big) \\ &\quad + C_1\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big). \end{aligned}$$

Thus, by the above inequality and arguing and (2.3),

$$\begin{aligned} ||x-y||_{L^2(0,T_1;V)} &\le C_0||x-y||^{\frac{1}{2}}_{L^2(0,T_1;D(A))}||x-y||^{\frac{1}{2}}_{L^2(0,T_1;H)} \\ &\le C_0||x-y||^{\frac{1}{2}}_{L^2(0,T_1;D(A))}\{\frac{T_1}{\sqrt{2}}||x-y||_{W^{1,2}(0,T_1;H)}\}^{\frac{1}{2}} \\ &\le C_0(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}||x-y||_{L^2(0,T_1;D(A))\cap W^{1,2}(0,T_1;H)} \\ &\le C_0(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\big\{C_1\mu([-h,0])(L_1(r)+L_2)||y||_{L^2(0,T_1;V)} \\ &\qquad + C_1\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big)\big\} \\ &\quad + C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)||x-y||_{L^2(0,T_1:V)}. \end{aligned}$$

Therefore, since

$$\begin{aligned} ||x-y||_{L^2(0,T_1;V)} \le& \frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}||y||_{L^2(0,T_1;V)} \\ &+\frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big)}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}, \end{aligned}$$

we have

$$\begin{aligned} ||x||_{L^2(0,T_1;V)} \le& \frac{1}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}||y||_{L^2(0,T_1;V)} \\ &\frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big)}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}, \end{aligned}$$

and hence, with the aid of (2.8) in Lemma 2.3 and Lemma 3.1, we obtain

$$\begin{aligned} &||x||_{L^2(0,T_1;V)\cap W^{1,2}(0,T_1;V^*)} \\ \le & C_1(|g^0| + ||G(\cdot,x)||_{L^2(0,T_1;V^*)} + ||k||_{L^2(0,T_1:V^*)}) \\ \le & C_1\big[|g^0| + \mu([-h,0])\big\{L_0\sqrt{T_1} + (L_1(r)+L_2)\|x\|_{L^2(0,T_1;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\ & \quad + ||k||_{L^2(0,T_1:V^*)}\big] \\ \le & C_3(|g^0| + ||k||_{L^2(0,T_1:V^*)}). \end{aligned} \tag{3.15}$$

for some constant C_3. Now from (2.5) and (3.15), it follows that

$$|x(T_1)| \le ||x||_{C([0,T_1];H)} \le M_0 C_3(|g^0| + ||k||_{L^2(0,T_1;V^*)}). \tag{3.16}$$

So, we can solve the equation in $[T_1, 2T_1]$ with the initial data $(x(T_1), x_{T_1})$, and obtain an analogous estimate to (3.15). Since the condition (3.14) is independent of initial values, the solution of (3.1) can be extended the internal $[0, nT_1]$ for a natural number n, i.e., for the initial $u(nT_1)$ in the interval $[nT_1, (n+1)T_1]$, as analogous estimate (3.15) holds for the solution in $[0, (n+1)T_1]$. □

By the similar way to Theorems 3.1 and 3.2, we also obtain the following results for (3.1) under Assumption (F) corresponding to 1) of Lemma 2.3.

Corollary 3.1. *Let $(g^0, g^1) \in V \times L^2(-h, 0; D(A))$ and $k \in L^2(0,T;H)$. Then there exists a unique solution x of (3.1) such that*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V).$$

Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;D(A)\cap W^{1,2}(0,T;H)} \le C_3(||g^0|| + ||k||_{L^2(0,T;H)}),$$

where C_3 is a constant depending on T.

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Investigation of α-C-class functions with applications

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Abstract: In this paper, we introduce the new idea of α-C-class function and establish new fixed point results in a complete metric space. It can be stated that the results that have come into being give substantial generalizations and improvements of several well known results in the existing comparable literature.

1 Introduction and preliminaries

In 1973, Geraghty [7] studied a generalization of Banach contraction principle. In 2012, Samet et al. [20] introduced a concept of α-ψ-contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. The notion of an α-admissible mapping has been characterized in many direction. For details, see [2, 4, 8, 9, 10, 11, 12, 14, 16, 17, 18, 21, 22, 23] and references therein.

Now, we give some basic definitions, examples and fundamental results which play an essential role in proving our results.

Definition 1 [20] *Let $S : X \to X$ be a self mapping and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that S is α-admissible if $x, y \in X$ with $\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Sy) \geq 1$.*

Example 2 [15] *Consider $X = [0, \infty)$ and define $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = 2x$ and*

$$\alpha(x, y) = \begin{cases} e^{\frac{y}{x}}, & x \geq y, x \neq 0, \\ 0, & x < y. \end{cases}$$

Then S is α-admissible.

Definition 3 [1] *Let $S, T : X \to X$ be self mappings and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that the pair (S, T) is α-admissible if $x, y \in X$ such that $\alpha(x, y) \geq 1$, then we have $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$.*

Example 4 *Let $X = [0, \infty)$ and define a pair of self mappings $S, T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = 2x$, $Tx = x^2$ and*

$$\alpha(x, y) = \begin{cases} e^{xy}, & x, y \geq 0, \\ 0, & otherwise. \end{cases}$$

Then a pair (S, T) is α-admissible.

Definition 5 [13] *Let $S : X \to X$ be a self mapping and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that S is triangular α-admissible if $x, y \in X$ with $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$.*

Example 6 [13] *Let $X = [0, \infty)$, $Sx = x^2 + e^x$ and*

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1], \\ 0, & otherwise. \end{cases}$$

Hence S is a triangular α-admissible mapping.

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Definition 7 [13] *Let $S : X \to X$ be a self mapping and let $\alpha : X \times X \to \mathbb{R}$ be a function. We say that S is a triangular α-admissible mapping if*

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Sy) \geq 1$, $x, y \in X$;
(T2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$, $x, y, z \in X$.

Example 8 [13] *Let $X = \mathbb{R}$, $Sx = \sqrt[3]{x}$ and $\alpha(x, y) = e^{x-y}$. Then S is a triangular α-admissible mapping. Indeed, if $\alpha(x, y) = e^{x-y} \geq 1$, then $x \geq y$ which implies $Sx \geq Sy$. That is, $\alpha(Sx, Sy) = e^{Sx-Sy} \geq 1$. Also, if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $x - z \geq 0$, $z - y \geq 0$. That is, $x - y \geq 0$ and so $\alpha(x, y) = e^{x-y} \geq 1$.*

Definition 9 [1] *Let $S, T : X \to X$ be self mappings and let $\alpha : X \times X \to \mathbb{R}$ be a function. We say that a pair (S, T) is triangular α-admissible if*

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$, $x, y \in X$;
(T2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$, $x, y, z \in X$.

Example 10 *Let $X = \mathbb{R}$ and define a pair of self mappings $S, T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ by $Sx = \sqrt{x}$, $Tx = x^2$ and $\alpha(x, y) = e^{xy}$ for all $x, y \in X$. Then a pair (S, T) is a triangular α-admissible mapping.*

Definition 11 [19] *Let $S : X \to X$ be a self mapping and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that S is an α-admissible mapping with respect to η if $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Sx, Sy) \geq \eta(Sx, Sy)$.*

Note that if we take $\eta(x, y) = 1$, then this definition reduces to the definition in [20]. Also if we take $\alpha(x, y) = 1$, then we say that S is an η-subadmissible mapping.

Example 12 *Let $X = [0, \infty)$ and $S : X \to X$ be defined by $Sx = \frac{x}{2}$. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by $\alpha(x, y) = 3$ and $\eta(x, y) = 1$ for all $x, y \in X$. Then S is an α-admissible mapping with respect to η.*

Lemma 13 [6] *Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Lemma 14 *Let $S, T : X \to X$ be a pair of triangular α-admissible. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $x_{2i+1} = Sx_{2i}$, and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \cdots$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

We denote by Ω the family of all functions $\beta : [0, +\infty) \to [0, 1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

Theorem 15 [7] *Let (X, d) be a metric space. Let $S : X \to X$ be a self mapping. Suppose that there exists $\beta \in \Omega$ such that, for all $x, y \in X$,*

$$d(Sx, Sy) \leq \beta\left(d(x, y)\right) d(x, y).$$

Then S has a fixed unique point $p \in X$ and $\{S^n x\}$ converges to p for each $x \in X$.

In 2014, Ansari [3] introduced the concept of C-class functions which cover a large class of contractive conditions.

Definition 16 [3] *A continuous function $f : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if for all $s, t \in [0, \infty)$, the following conditions hold:*
(1) *$f(s, t) \leq s$;*
(2) *$f(s, t) = s$ implies that either $s = 0$ or $t = 0$.*

An extra condition on f that $f(0,0)=0$ could be imposed in some cases if required. The letter $\mathcal{C}$ will denote the class of all C-class functions.

Example 17 [3] *The following examples show that the class $\mathcal{C}$ is nonempty:*

1. $f(s,t)=s-t$.
2. $f(s,t)=ms$ for some $m\in(0,1)$.
3. $f(s,t)=\frac{s}{(1+t)^r}$ for some $r\in(0,\infty)$.
4. $f(s,t)=\log(t+a^s)/(1+t)$ for some $a>1$.
5. $f(s,t)=\ln(1+a^s)/2$ for $e>a>1$. Indeed, $f(s,t)=s$ implies that $s=0$.
6. $f(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1$ for $r\in(0,\infty)$.
7. $f(s,t)=s\log_{t+a}a$ for $a>1$.
8. $f(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$.
9. $f(s,t)=s\beta(t)$, where $\beta:[0,\infty)\to[0,1)$ is continuous.
10. $f(s,t)=s-\frac{t}{k+t}$.
11. $f(s,t)=s-\varphi(s)$, where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0$ if and only if $t=0$.
12. $f(s,t)=sh(s,t)$, where $h:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function such that $h(t,s)<1$ for all $t,s>0$.
13. $f(s,t)=s-(\frac{2+t}{1+t})t$.
14. $f(s,t)=\sqrt[n]{\ln(1+s^n)}$.
15. $f(s,t)=\phi(s)$, where $\phi:[0,\infty)\to[0,\infty)$ is a upper semicontinuous function such that $\phi(0)=0$ and $\phi(t)<t$ for $t>0$.
16. $f(s,t)=\frac{s}{(1+s)^r}$, $r\in(0,\infty)$.

Let Φ_u denote the class of functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:

(a) φ is continuous;

(b) $\varphi(t)>0, t>0$ and $\varphi(0)\geq 0$.

Lemma 18 [5] *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1})\to 0$ as $n\to\infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon>0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k)>n(k)>k$ such that $d(x_{m(k)},x_{n(k)})\geq\varepsilon$, $d(x_{m(k)-1},x_{n(k)})<\varepsilon$ and*
(i) $\lim_{k\to\infty}d(x_{m(k)-1},x_{n(k)+1})=\varepsilon$;
(ii) $\lim_{k\to\infty}d(x_{m(k)},x_{n(k)})=\varepsilon$;
(iii) $\lim_{k\to\infty}d(x_{m(k)-1},x_{n(k)})=\varepsilon$.

We can also show that $\lim_{k\to\infty}d(x_{m(k)+1},x_{n(k)+1})=\varepsilon$ and $\lim_{k\to\infty}d(x_{m(k)},x_{n(k)-1})=\varepsilon$.

2 Main results

In this section, we prove some fixed point theorems satisfying α-Geraghty contraction type mappings in a complete metric space.

Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Two self mappings $S,T : X \to X$ are called a pair of generalized α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that, for all $x,y \in X$,

$$\alpha(x,y)d(Sx,Ty) \leq \beta\left(M(x,y)\right)M(x,y),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

If $S = T$, then T is called a generalized α-Geraghty contraction type mapping if there exists $\beta \in \Omega$ such that, for all $x,y \in X$,

$$\alpha(x,y)d(Sx,Ty) \leq \beta\left(N(x,y)\right)N(x,y),$$

where

$$N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Let (X,d) be a metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Two self mappings $S,T : X \to X$ are called a pair of generalized α-C-class function contraction type mappings if there exists $F \in \mathcal{C}$ such that, for all $x,y \in X$,

$$\alpha(x,y)d(Sx,Ty) \leq F(M(x,y), \varphi(M(x,y))), \tag{1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

If $S = T$, then T is called a generalized α-C-class function contraction type mapping if there exists $F \in \mathcal{C}$ such that, for all $x,y \in X$,

$$\alpha(x,y)d(Tx,Ty)) \leq F(N(x,y), \varphi(N(x,y))),$$

where

$$N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Theorem 19 *Let (X,d) be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ be two self mappings. Suppose that the following hold:*
(i) *(S,T) is a pair of generalized α-C-class function contraction type mappings;*
(ii) *(S,T) is triangular α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*
(iv) *S and T are continuous.*
Then (S,T) has a common fixed point.

Proof. Let $x_1 \in X$ be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that

$$x_{2i+1} = Sx_{2i} \;\; and \;\; x_{2i+2} = Tx_{2i+1}, \;\; where \;\; i = 0,1,2,\cdots.$$

By assumption, $\alpha(x_0, x_1) \geq 1$ and a pair (S,T) is α-admissible. By Lemma 14, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \;\; for\ all \;\; n \in \mathbb{N} \cup \{0\}.$$

Then we have

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &= d(Sx_{2i}, Tx_{2i+1}) \leq \alpha(x_{2i}, x_{2i+1}) d(Sx_{2i}, Tx_{2i+1}) \\ &\leq F(M(x_{2i}, x_{2i+1}), \varphi(M(x_{2i}, x_{2i+1}))) \leq M(x_{2i}, x_{2i+1}) \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i}, Sx_{2i}), d(x_{2i+1}, Tx_{2i+1}), \frac{d(x_{2i}, Tx_{2i+1}) + (x_{2i+1}, Sx_{2i})}{2}\right\} \\ &= \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+2})}{2}\right\} \\ &\leq \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+1}) + d(x_{2i+1}, x_{2i+2})}{2}\right\} \\ &= \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\right\}. \end{aligned}$$

Thus

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq F(M(x_{2i}, x_{2i+1}), \varphi(M(x_{2i}, x_{2i+1}))) \\ &\leq F(d(x_{2i}, x_{2i+1}), \varphi(d(x_{2i}, x_{2i+1}))) \leq d(x_{2i}, x_{2i+1}), \end{aligned} \tag{2}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \cup \{0\}$$

for all $n \in \mathbb{N}$. So the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing.

Now, we prove that $d(x_n, x_{n+1}) \to 0$. It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Therefore, there exists some positive number r such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. From (2), by taking limit $n \to \infty$, we have

$$r \leq F(r, \varphi(r)),$$

that is,

$$r = 0 \quad or \quad \varphi(r) = 0.$$

Therefore, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k, we have $m_k > n_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}) \end{aligned} \tag{4}$$

for all $k \in \mathbb{N}$. In the view of (3) and (4), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{5}$$

Again using the triangle inequality, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}).$$

Taking limit as $k \to +\infty$ and using (3) and (5), we obtain

$$\lim_{k\to+\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

By Lemma 14 and $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$, we have

$$\begin{aligned} d(x_{n_{k+1}}, x_{m_{k+2}})) &= d(Sx_{n_k}, Tx_{m_{k+1}})) \leq \alpha(x_{n_k}, x_{m_{k+1}})d(Sx_{n_k}, Tx_{m_{k+1}})) \\ &\leq F(M(x_{n_k}, x_{m_{k+1}}), \varphi(M(x_{n_k}, x_{m_{k+1}}))). \end{aligned}$$

Keeping (3) in mind and letting $k \to +\infty$ in the above inequality, we obtain

$$\epsilon \leq F(\epsilon, \varphi(\epsilon)),$$

that is,

$$\epsilon = 0 \quad or \quad \varphi(\epsilon) = 0.$$

So $\epsilon = 0$, which is a contradiction. Using a similar technique for other cases, it can be easily seen that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $x_n \to p$ implies that $x_{2i+1} \to p$ and $x_{2i+2} \to p$. Since S and T are continuous, we get $Tx_{2i+1} \to Tp$ and $Sx_{2i+2} \to Sp$. Thus $p = Sp$. Similarly, $p = Tp$ and so we have $Sp = Tp = p$. Then (S, T) has a common fixed point. ■

In the following theorem, we drop the continuity.

Theorem 20 *Let (X, d) be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S, T : X \to X$ be two self mappings. Suppose that the following hold:*

(i) *(S, T) is a pair of generalized α-C-class function contraction type mappings;*

(ii) *(S, T) is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.*

Then (S, T) has a common fixed point.

Proof. The proof follows from similar lines of Theorem 19. Define a sequence $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \cdots$, which converges to $p \in X$. By the hypotheses of (iv) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, p) \geq 1$ for all k. Now by using (1), for all k, we have

$$\begin{aligned} d(x_{2n_k+1}, Tp)) &= d(Sx_{2n_k}, Tp)) \leq \alpha(x_{2n_k}, p)d(Sx_{2n_k}, Tp)) \\ &\leq F(M(x_{2n_k}, p), \varphi(M(x_{2n_k}, p))). \end{aligned}$$

On the other hand, we obtain

$$M(x_{2n_k}, p) = \max\left\{d(x_{2n_k}, p), d(x_{2n_k}, Sx_{2n_k}), d(p, Tp), \frac{d(x_{2n_k}, Tp) + d(p, Sx_{2n_k})}{2}\right\}.$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} M(x_{2n_k}, p) = d(p, Tp).$$

Suppose that $d(p,Tp) > 0$. Letting $k \to \infty$ in the above inequality, we have

$$d(p,Tp) \leq F(d(p,Tp), \varphi(d(p,Tp)))$$

and so we obtain that $d(p,Tp) = 0$, which is a contradiction. Thus we find that $d(p,Tp) = 0$ implies $p = Tp$. Similarly, $p = Sp$. Thus $p = Tp = Sp$. ■

If $M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Sy), \frac{d(y,Sx)+d(x,Sy)}{2}\right\}$ and $S = T$ in Theorems 19 and 20, then we have the following corollaries.

Corollary 21 *Let (X,d) be a complete metric space and let S be an α-admissible mapping such that the following hold:*

(i) *S is a generalized α-Geraghty contraction type mapping;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, T_0) \geq 1$;*

(iv) *S is continuous.*

Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 22 *Let (X,d) be a complete metric space and let S be an α-admissible mapping such that the following hold:*

(i) *S is a generalized α-Geraghty contraction type mapping;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.*

Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

If $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Sy)\}$ and $S = T$ in Theorems 19 and 20, then we obtain the following corollaries.

Corollary 23 [6] *Let (X,d) be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping. Suppose that the following hold:*

(i) *S is a generalized α-Geraghty contraction type mapping;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *S is continuous.*

Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 24 [6] *Let (X,d) be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping. Suppose that the following hold:*

(i) *S is a generalized α-Geraghty contraction type mapping;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.*

Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Let (X,d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Two self mappings $S, T : X \to X$ are called a pair of generalized α-η-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that, for all $x, y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow d(Sx,Ty) \leq \beta\left(M(x,y)\right) M(x,y),$$

where

$$M(x,y)=\max\left\{d(x,y),d(x,Sx),d(y,Ty),\frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

Let (X,d) be a metric space and $\alpha,\eta : X\times X\to\mathbb{R}$ be two functions. Two self mappings $S,T : X\to X$ are called a pair of generalized α-η-C-class function contraction type mappings if there exists $F\in C$ such that, for all $x,y\in X$,

$$\alpha(x,y)\geq\eta(x,y)\Rightarrow d(Sx,Ty)\leq F(M(x,y),\varphi(M(x,y)),$$

where

$$M(x,y)=\max\left\{d(x,y),d(x,Sx),d(y,Ty),\frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

Theorem 25 *Let (X,d) be a complete metric space. Let S be an α-admissible mapping with respect to η such that the following hold:*

(i) *(S,T) is a pair of generalized α-η-C-class function contraction type mappings;*

(ii) *(S,T) is triangular α-admissible;*

(iii) *there exists $x_0\in X$ such that $\alpha(x_0,Sx_0)\geq\eta(x_0,Sx_0)$;*

(iv) *S and T are continuous.*

Then (S,T) has a common fixed point.

Proof. Let x_1 in X be such that $x_1=Sx_0$ and $x_2=Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that

$$x_{2i+1}=Sx_{2i}, and\ x_{2i+2}=Tx_{2i+1}, where i=0,1,2,\ldots.$$

By assumption $\alpha(x_0,x_1)\geq\eta(x_0,x_1)$ and a pair (S,T) is α-admissible with respect to η, we have, $\alpha(Sx_0,Tx_1)\geq\eta(Sx_0,Tx_1)$ from which we deduce that $\alpha(x_1,x_2)\geq\eta(x_1,x_2)$ which also implies that $\alpha(Tx_1,Sx_2)\geq\eta(Tx_1,Sx_2)$. Continuing in this way we obtain $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}\cup\{0\}$.

$$\begin{aligned}d(x_{2i+1},x_{2i+2}) &= d(Sx_{2i},Tx_{2i+1})\leq\alpha(x_{2i},x_{2i+1})d(Sx_{2i},Tx_{2i+1})\\ &\leq F(M(x_{2i},x_{2i+1}),\varphi(M(x_{2i},x_{2i+1}))\end{aligned}$$

for all $i\in\mathbb{N}\cup\{0\}$. Now

$$\begin{aligned}M(x_{2i},x_{2i+1}) &= \max\left\{d(x_{2i},x_{2i+1}),d(x_{2i},Sx_{2i}),d(x_{2i+1},Tx_{2i+1}),\frac{d(x_{2i},Tx_{2i+1})+(x_{2i+1},Sx_{2i})}{2}\right\}\\ &= \max\left\{d(x_{2i},x_{2i+1}),d(x_{2i},x_{2i+1}),d(x_{2i+1},x_{2i+2}),\frac{d(x_{2i},x_{2i+2})}{2}\right\}\\ &\leq \max\left\{d(x_{2i},x_{2i+1}),d(x_{2i+1},x_{2i+2}),\frac{d(x_{2i},x_{2i+1})+d(x_{2i+1},x_{2i+2})}{2}\right\}\\ &= \max\left\{d(x_{2i},x_{2i+1}),d(x_{2i+1},x_{2i+2})\right\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}d(x_{2i+1},x_{2i+2}) &\leq F(M(x_{2i},x_{2i+1}),\varphi(M(x_{2i},x_{2i+1}))\\ &\leq F(d(x_{2i},x_{2i+1}),\varphi(d(x_{2i},x_{2i+1}))\leq d(x_{2i},x_{2i+1}).\end{aligned}$$

This implies that

$$d(x_{n+1},x_{n+2})\leq d(x_n,x_{n+1}), foralln\in\mathbb{N}\cup\{0\}.$$

The rest of the proof follows from similar lines of Theorem 19.

Hence p is a common fixed point of S and T. ■

Theorem 26 *Let (X,d) be a complete metric space and let (S,T) be a pair of α-admissible mappings with respect to η such that the following hold:*

(i) *(S,T) is a pair of generalized α-C-class function contraction type mappings;*

(ii) *(S,T) is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all k.*

Then S and T have a common fixed point.

Proof. The proof follows from similar lines of Theorem 20. ■

If $M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Sy), \frac{d(y,Sx)+d(x,Sy)}{2}\right\}$ and $S = T$ in Theorems 25 and 26, then we get the following corollaries.

Corollary 27 *Let (X,d) be a complete metric space and let S be an α-admissible mapping with respect to η such that the following hold:*

(i) *S is a generalized α-Geraghty contraction type mapping;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*

(iv) *S is continuous.*

Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 28 *Let (X,d) be a complete metric space and let S be an α-admissible mapping with respect to η such that the following hold:*

(i) *S is a generalized α-Geraghty contraction type mapping;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all k.*

Then S has a fixed point $p \in X$ and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Example 29 *Let $X = \{a, b, c\}$ with metric*

$$d(x,y) = \begin{cases} 0 \ \ if \ x = y \\ \frac{5}{7} \ \ if \ x, y \in X - \{b\} \\ 1 \ \ if \ x, y \in X - \{c\} \\ \frac{4}{7} \ \ if \ x, y \in X - \{a\}. \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1 & if \ \ x, y \in X \\ 0 & otherwise \end{cases}.$$

Define a mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} a \ \ if \ x \neq b \\ c \ \ if \ x = b \end{cases}$$

and $\beta : [0, +\infty) \to [0, 1)$. Then

$$\alpha(x,y) d(Tx, Ty) \nleq \beta(M(x,y)) M(x,y).$$

Indeed, let $x = b$ and $y = c$. Then

$$\begin{aligned} M(b,c) &= \max\left\{d(b,c), d(b,T(b)), d(c,T(c)), \frac{d(b,T(c))+d(c,T(b))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{1}{2}\right\} = \frac{5}{7}. \end{aligned}$$

[6, Theorem 2.1] *is not valid to get a fixed point of T, since*

$$\alpha(b,c)d\left(T(b), T(c)\right) \nleq \beta(M(b,c))M(b,c).$$

Now, we prove that Theorem 19 can be applied to a common fixed point of S and T.

Now, consider a mapping $S : X \to X$ be such that $Sx = a$ for each $x \in X$, where

$$\begin{aligned} M(b,c) &= \max\left\{d(b,c), d(b,S(b)), d(c,T(c)), \frac{d(b,T(c))+d(c,S(b))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, 1, \frac{5}{7}, \frac{12}{14}\right\} = 1, \end{aligned}$$

$$d(Sb, Tc) = d(a,a) = 0,$$

$$\alpha(x,y)d(Sx,Ty) \leq F(M(x,y)), \varphi\left(M(x,y)\right) \leq M(x,y).$$

Hence the hypothesis of Theorem 19 is satisfied, So S and T have a common fixed point.

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Generalizations of Hua's inequality in Hilbert C^*-modules

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Abstract

We establish a new extended Hua's inequality in the setting of Hilbert C^*-modules. As for its application, we get several generalizations of norm Hua's inequality and more generalized inequalities of the Hua inequality type.

Keywords: Hilbert C^*-module, Hua's inequality, C^*-algebra, norm inequality.

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1 Introduction and Preliminaries

The classical Hua's inequality states that for any $\alpha, \delta > 0$ and real numbers $x_1, x_2, \cdots, x_n$,

$$(\delta - x_1 - \cdots - x_n)^2 + \alpha(x_1^2 + \cdots + x_n^2) \geq \frac{\alpha}{n+\alpha}\delta^2, \tag{1}$$

and the equality holds iff $x_1 = x_2 = \cdots = x_n = \frac{\delta}{n+\alpha}$.

This inequality has been generalized by Wang [14] as follows. If $\alpha, \delta > 0$ and $p \geq 1$, then

$$(\delta - x_1 - \cdots - x_n)^p + \alpha^{p-1}(x_1^p + \cdots + x_n^p) \geq (\frac{\alpha}{n+\alpha})^{p-1}\delta^p \tag{2}$$

for all non-negative numbers $x_1, x_2, \cdots, x_n$ with $x_1 + \cdots + x_n \leq \delta$. A number of researchers discussed the above inequality from different angles [1, 2, 6–15]. In [8], the Hua's inequality for real convex function was given. Dragomir and

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Yang [1] have proved Hua's inequality in the framework of real inner product spaces. Their result was generalized by Pečarić [9]. Drnovšek [2] give an operator version of Hua's inequality for positive conjugate exponents $p, q \in \mathbb{R}$. We also infer to another interesting Radas and Šikić [10] of this type. In particular, Moslehian [6] extended an operator Hua's inequality in Hilbert C^*-modules, which is equivalent to operator convexity of given continuous real function. In recent years, Su, Miao and Li [11] generalize a new Hua's inequality and apply it to proof the boundedness of composition operator. Moslehian and Fujii [7] have shown another type of Hua's operator inequality. There are other interpretation of Hua's inequality [13] and references therein.

In this paper, we establish an extended Hua's inequality in the setting of Hilbert C^*-modules. As for its application, we get several generalizations of norm Hua's inequality and more generalized inequalities of the Hua inequality type. For this purpose, we first set up some notations.

Throughout the paper, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. The notations $B(\mathcal{X}, \mathcal{Y})$ denote the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Let $g : [0, \infty) \to (0, \infty)$ be a function such that $g(t) \geq t + M$ for some $M > 0$.

Recall that an element $a \in \mathcal{A}$ is positive if a is selfadjoint with a positive real spectrum or a is the form of u^*u for some $u \in \mathcal{A}$. We write $a \geq 0$ if a is positive. For more information on the theory of C^*-algebra and Hilbert C^*-module the reader is referred to [5] and [4], respectively.

2 Hua type inequality in Hilbert C^*-modules

Before prove the main results, we need following auxiliary result.

Lemma 1. *[12] Let (G, +) be a semigroup, and let φ and ψ be nonnegative functions on G. Suppose φ is subadditive on G and there is a positive constant λ such that $\varphi(x) \leq \lambda\psi(x)$ for $x \in G$. If f is a nondecreasing convex function on [0, ∞), then*

$$f(\varphi(a)) + \lambda f(\psi(b)) \geq (1 + \lambda) f(\frac{\varphi(a + b)}{1 + \lambda}) \tag{3}$$

holds for any $a, b \in G$. When f is strictly convex, the equality holds in (3) iff

$$\varphi(a + b) = \varphi(a) + \varphi(b), \ \varphi(b) = \lambda\psi(b), \ \varphi(a) = \psi(b).$$

We now state our main result, which is an extended Hua's inequality in the setting of Hilbert C^*-modules.

Theorem 1. *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - x(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|x\|^p \geq (\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}})^{p-1}\|\delta\|^p \tag{4}$$

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for all $x, \delta \in \mathcal{X}$ and all positive $c \in \mathcal{A}$. The equality holds iff

$$\|x(g(c)-c)^{\frac{1}{2}}\| = \|x\|\|g(c)-c\|^{\frac{1}{2}}, \|x\| = \frac{\|\delta\|\|g(c)-c\|^{\frac{q-1}{2}}}{\|g(c)-c\|^{\frac{q}{2}} + \|c\|}.$$

Proof. By the functional calculus, $g(c)-c$ is positive and invertible. Put $G = \mathcal{X}$. Let's define $\varphi : \mathcal{X} \to \mathbb{C}$ by $\varphi(x) = \|x(g(c)-c)^{\frac{1}{2}}\|$ and $\psi : \mathcal{X} \to \mathbb{C}$ by $\psi(x) = \|c\|\|g(c)-c\|^{\frac{1-q}{2}}\|x\|$ for any $x \in \mathcal{X}$. So $\varphi(x) = \|x(g(c)-c)^{\frac{1}{2}}\| \leq \lambda\psi(x)(x \in \mathcal{X})$, where $\lambda = \frac{\|g(c)-c\|^{\frac{q}{2}}}{\|c\|}$. Moreover, putting $f(t) = t^p$ $(t \geq 0)$, clear f is nondecreasing and convex on $[0, \infty)$. Hence, Lemma 1 yields that

$$\|a(f(c)-c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|b\|^p \geq (\frac{\|c\|}{\|c\| + \|(g(c)-c)\|^{\frac{q}{2}}]})^{p-1}\|(a+b)(g(c)-c)^{\frac{1}{2}}\|^p \tag{5}$$

holds for $a, b \in \mathcal{X}$. The equality holds iff

$$\|(a+b)(g(c)-c)^{\frac{1}{2}}\| = \|a(g(c)-c)^{\frac{1}{2}}\| + \|b(g(c)-c)^{\frac{1}{2}}\|, \tag{6}$$

$$\|b(g(c)-c)^{\frac{1}{2}}\| = \|b\|\|g(c)-c\|^{\frac{1}{2}}, \tag{7}$$

$$\|a(g(c)-c)^{\frac{1}{2}}\| = \|c\|\|b\|\|g(c)-c\|^{\frac{1-q}{2}}. \tag{8}$$

By choosing $z \in \mathcal{X}$ such that $z(g(c)-c)^{\frac{1}{2}} = \delta$ and replacing a and b by $z-x$ and x, therefore we can get (4). The equality holds in (4) iff

$$\|\delta\| = \|\delta - x(g(c)-c)^{\frac{1}{2}}\| + \|x(g(c)-c)^{\frac{1}{2}}\|, \tag{9}$$

$$\|x(g(c)-c)^{\frac{1}{2}}\| = \|x\|\|g(c)-c\|^{\frac{1}{2}}, \tag{10}$$

$$\|\delta - x(g(c)-c)^{\frac{1}{2}}\| = \|c\|\|x\|\|g(c)-c\|^{\frac{1-q}{2}}. \tag{11}$$

Observe that an easy computation shown that $\|x\| = \frac{\|\delta\|\|g(c)-c\|^{\frac{q-1}{2}}}{\|g(c)-c\|^{\frac{q}{2}}+\|c\|}$ from above three equations. Consequently, we have

$$\|x(g(c)-c)^{\frac{1}{2}}\| = \|x\|\|g(c)-c\|^{\frac{1}{2}},\ \|x\| = \frac{\|\delta\|\|g(c)-c\|^{\frac{q-1}{2}}}{\|g(c)-c\|^{\frac{q}{2}} + \|c\|}. \tag{12}$$

The simple computation shows that (12) implies (9), (10) and (11). Now this observation completes the proof. □

Example 1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, then $B(\mathcal{H}, \mathcal{K})$ becomes a $B(\mathcal{H})$-module via $\langle T, S\rangle = T^*S$. Replacing x, δ in (4) by T, S and taking p=2 we get*

$$\|(S - T(g(c)-c)^{\frac{1}{2}})^*(S - T(g(c)-c)^{\frac{1}{2}})\| + c\|T^*T\| \geq \frac{c}{g(c)}\|S^*S\|$$

for all $c > 0$ and all $T, S \in B(\mathcal{H}, \mathcal{K})$. The equality holds iff $\|T\| = \frac{\|S\|}{g(c)}$.

3

If $\mathcal{X}$ is a Hilbert space $\mathcal{H}$, which is a Hilbert $\mathbb{C}$-module, then we have the following corollary.

Corollary 1. *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - (g(c) - c)^{\frac{1}{2}} x\|^p + c^{p-1}\|x\|^p \geq \left(\frac{c}{c + (g(c) - c)^{\frac{q}{2}}}\right)^{p-1}\|\delta\|^p \tag{13}$$

for any $c > 0$, $x, \delta \in \mathcal{H}$.

We also have the following extension of Hua's inequality in the framework of Hilbert C^*-module.

Theorem 2. *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - T(x)(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|T\|^p\|x\|^p \geq \left(\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}}\right)^{p-1}\|\delta\|^p \tag{14}$$

for all $x \in \mathcal{X}, \delta \in \mathcal{Y}$, all positive $c \in \mathcal{A}$, and all operators $T \in B(\mathcal{X}, \mathcal{Y})$.

Proof. Substituting $T(x)$ for x in (4) we get

$$\|\delta - T(x)(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|Tx\|^p \geq \left(\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}}\right)^{p-1}\|\delta\|^p$$

utilizing the facts that $\|T(x)\| \leq \|T\|\|x\|$ we obtain

$$\|\delta - T(x)(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|T\|^p\|x\|^p \geq \left(\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}}\right)^{p-1}\|\delta\|^p.$$

□

Recall that the operator $T = u \bigotimes v$ is defined by $T(x) = u\langle v, x\rangle (u, v, x \in \mathcal{X})$ and noting the fact that $\|T\| = \|u\|\|v\|$ we get the following corollary.

Corollary 2. *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - u\langle v, x\rangle(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|u\|^p\|v\|^p\|x\|^p \geq \left(\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}}\right)^{p-1}\|\delta\|^p \tag{15}$$

for all $x, \delta, u, v \in \mathcal{X}$ and all positive $c \in \mathcal{A}$.

When $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, let $A \in B(\mathcal{X}, \mathcal{Y})$, $g(t) = t+1, c = \|A\|^{\frac{p}{1-p}}$, $\delta = y$, the Theorem 2 reduces to Theorem 2 of [2].

Corollary 3. *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and let A be a bounded operator from $\mathcal{X}$ to $\mathcal{Y}$. If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, then*

$$\|y - A(x)\|^p + \|x\|^p \geq \frac{\|y\|^p}{(1 + \|A\|^q)^{p-1}}.$$

4

If we set $p = q$=2 and take $\delta = y(g(c)-c)^{-\frac{1}{2}}$ in Theorem 2 then the following corollary is obtained.

Corollary 4. *Let $p, q > 1$ be conjugate components. Then*

$$\|y(g(c)-c)^{-\frac{1}{2}} - T(x)(g(c)-c)^{\frac{1}{2}}\|^2 + \|c\|\|T\|^2\|x\|^2 \geq \frac{\|c\|\|y(g(c)-c)^{-\frac{1}{2}}\|^2}{\|c\| + \|(g(c)-c)\|}$$

for all $x \in \mathcal{X}, y \in \mathcal{Y}$, all positive $c \in \mathcal{A}$ and all operators $T \in B(\mathcal{X}, \mathcal{Y})$.

Next consider inner spaces $\mathcal{H}$ and $\mathcal{K}$, then $\mathcal{A} = \mathbb{C}$. Let $A \in B(\mathcal{H}, \mathcal{K})$, $g(t) = t+1$ and $c = \frac{\alpha}{\|A\|^2}$, then we deduce the main result of [10] from Corollary 4 as follows.

Corollary 5. *Suppose that $\mathcal{H}$ and $\mathcal{K}$ are inner product spaces, A: $\mathcal{H} \to \mathcal{K}$ is a bounded linear operator and $\alpha > 0$. Then*

$$\|y - Ax\|^2 + \alpha\|x\|^2 \geq \frac{\alpha\|y\|^2}{\|A\|^2 + \alpha} \tag{16}$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

Remark 1. *Applying Corollary 5 for elements of the n-fold inner product space $\mathcal{H}^n$, then inequality 16 can be restated as the following form which is, as noted in [6], a generalization of the main theorem of [1].*

$$\|y - \sum_{i=1}^{n} w_i x_i\|^2 + \alpha \sum_{i=1}^{n}(|w_i|^2\|x_i\|^2) \geq \frac{\alpha\|y\|^2}{\sum_{i=1}^{n}|w_i|^2 + \alpha},$$

where $w_i \in \mathbb{C}(1 \leq i \leq n)$, $A(x_1, \cdots, x_n) = \sum_{i=1}^{n} w_i x_i$ and $\|A\|^2 = \sum_{i=1}^{n}|w_i|^2$. The special case, where $\mathcal{H} = \mathbb{C}$ and $w_i = 1(1 \leq i \leq n)$, give rise to the classical Hua's inequality.

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FOURIER SERIES OF FUNCTIONS RELATED TO HIGHER-ORDER GENOCCHI POLYNOMIALS

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Abstract. In this paper, we consider three types of functions related to higher-order Genocchi functions and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

1. Introduction

The Genocchi polynomials $G_n^{(r)}(x)$ of order r $(r \in \mathbb{Z}_{>0})$ are defined by the generating function

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt} = \sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!}, \quad \text{(see [2-5,8,16,17,20,22])}. \tag{1.1}$$

When $x = 0$, $G_m^{(r)} = G_m^{(r)}(0)$ are called the Genocchi numbers of order r. For $r = 1$, $G_m(x) = G_m^{(1)}(x)$, and $G_m = G_m^{(1)}$ are called the Genocchi polynomials and Genocchi numbers, respectively.

Clearly, $G_m^{(r)}(x) = 0$, for $0 \le m \le r-1$, and $G_r^{(r)}(x) = r!$. Thus we will assume that $m \ge r+1 \ge 2$. Also, as $G_m^{(r)}(x) = \frac{m!}{(m-r)!}E_{m-r}^{(r)}(x)$, $(m \ge r)$, $\deg G_m^{(r)}(x) = m-r$, $(m \ge r)$, and $G_m^{(r)} = \frac{m!}{(m-r)!}E_{m-r}^{(r)}$.

From (1.1), we see that

$$\begin{aligned}&\frac{d}{dx}G_m^{(r)}(x) = mG_{m-1}^{(r)}(x),\ (m \ge 0),\\ &G_m^{(r)}(x+1) = 2mG_{m-1}^{(r-1)}(x) - G_m^{(r)}(x),\ (m \ge 0).\end{aligned} \tag{1.2}$$

In turn, these imply that

$$\begin{aligned}&G_m^{(r)}(1) = 2mG_{m-1}^{(r-1)} - G_m^{(r)},\ (m \ge 0),\\ &\int_0^1 G_m^{(r)}(x)dx = \frac{2}{m+1}\left((m+1)G_m^{(r-1)} - G_{m+1}^{(r)}\right),\ (m \ge 0).\end{aligned} \tag{1.3}$$

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* corresponding author.

We also recall from [14] that, for $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned}&\int_0^1 G_m^{(r)}(x)e^{-2\pi i n x}dx \\ &= -\sum_{k=1}^{m-1} \frac{2(m)_{k-1}}{(2\pi i n)^k}\left((m-k+1)G_{m-k}^{(r-1)} - G_{m-k+1}^{(r)}\right).\end{aligned} \tag{1.4}$$

For any real number x, we let

$$< x >= x - \lfloor x \rfloor \in [0,1), \tag{1.5}$$

denote the fractional part of x.

The Bernoulli polynomials $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}. \tag{1.6}$$

We are going to use the following facts about Bernoulli functions $B_m(< x >)$ later:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \tag{1.7}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.8}$$

Here we will consider the following three types of functions $\alpha_m(< x >), \beta_m(< x >)$, and $\gamma_m(< x >)$ involving higher-order Genocchi polynomials. We will derive their Fourier series expansions and in addition express them in terms of Bernoulli functions.

(1) $\alpha_m(< x >) = \sum_{k=r}^{m} G_k^{(r)}(< x >) < x >^{m-k}$, $(m \geq r+1)$;

(2) $\beta_m(< x >) = \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(< x >) < x >^{m-k}$, $(m \geq r+1)$;

(3) $\gamma_m(< x >) = \sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) < x >^{m-k}$, $(m \geq r+1)$.

The reader may refer to any book for elementary facts about Fourier analysis (for example, see [1,18,23]).

As to $\gamma_m(< x >)$, we note that the polynomial identity (1.9) follows immediately from Theorems 4.1 and 4.2, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$.

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$$\begin{aligned}
&\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(x) x^{m-k} \\
&= \frac{1}{m} \sum_{s=0}^{m-r} \binom{m}{s} \Big\{ 2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) \\
&\times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1} \Big\} B_s(x) \\
&+ \frac{2}{m} (H_{m-1} - H_{r-1}) \sum_{s=m-r+1}^{m-1} \binom{m}{s} (G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) B_s(x),
\end{aligned} \tag{1.9}$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers and $\Lambda_l = \sum_{k=r}^{l-1} \frac{1}{k(l-k)} \Big(2kG_{k-1}^{(r-1)} - G_k^{(r)} \Big)$. The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle)$ we can derive the Faber-Pandharipande-Zagier identity (see [7,12,13]) and the Miki's identity (see [6,9,12,13,19,21]). Recent works on Fourier series expansions for analogous functions can be found in the papers [10,11,15]. From now on, we will assume that $r \geq 2$.

2. The function $\alpha_m(< x >)$

Let $\alpha_m(x) = \sum_{k=r}^{m} G_k^{(r)}(x) x^{m-k}$, $(m \geq r+1)$. Then we will consider the function

$$\alpha_m(< x >) = \sum_{k=r}^{m} G_k^{(r)}(< x >) < x >^{m-k}, \tag{2.1}$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(< x >) e^{-2\pi i n x} dx \\
&= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx.
\end{aligned} \tag{2.3}$$

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Before proceeding further, we need to observe the following.

$$\begin{aligned}\alpha_m'(x) &= \sum_{k=r}^{m}\left(kG_{k-1}^{(r)}(x)x^{m-k} + (m-k)G_k^{(r)}(x)x^{m-k-1}\right)\\ &= \sum_{k=r+1}^{m} kG_{k-1}^{(r)}(x)x^{m-k} + \sum_{k=r}^{m-1}(m-k)G_k^{(r)}(x)x^{m-k-1}\\ &= \sum_{k=r}^{m-1}(k+1)G_k^{(r)}(x)x^{m-1-k} + \sum_{k=r}^{m-1}(m-k)G_k^{(r)}(x)x^{m-1-k}\\ &= (m+1)\alpha_{m-1}(x).\end{aligned} \tag{2.4}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.6}$$

For $m \geq r+1$, we put

$$\begin{aligned}\Delta_m &= \alpha_m(1) - \alpha_m(0)\\ &= \sum_{k=r}^{m}\left(G_k^{(r)}(1) - G_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=r}^{m}\left(2kG_{k-1}^{(r-1)} - G_k^{(r)} - G_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=r}^{m}\left(2kG_{k-1}^{(r-1)} - G_k^{(r)}\right) - G_m^{(r)}.\end{aligned} \tag{2.7}$$

Now, we see that

$$\alpha_m(1) = \alpha_m(0) \Longleftrightarrow \Delta_m = 0, \tag{2.8}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.

Case $1 : n \neq 0$.

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx\\ &= -\frac{1}{2\pi i n}[\alpha_m(x)e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x)e^{-2\pi i n x}dx\\ &= -\frac{1}{2\pi i n}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x)e^{-2\pi i n x}dx\\ &= \frac{m+1}{2\pi i n}A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m,\end{aligned} \tag{2.10}$$

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from which by induction on m, we can easily show

$$\begin{aligned} A_n^{(m)} &= -\sum_{j=1}^{m-r} \frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} \\ &= -\frac{1}{m+2}\sum_{j=1}^{m-r}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}. \end{aligned} \tag{2.11}$$

Case $2: n=0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.12}$$

$\alpha_m(<x>), (m \geq r+1)$ is piecewise C^∞. Moreover, $\alpha_m(<x>)$ is continuous for those integers $m \geq r+1$ with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq r+1$ with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for an integer $m \geq r+1$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(<x>)$ is piecewise C^∞ , and continuous. Thus the Fourier series of $\alpha_m(<x>)$ converges uniformly to $\alpha_m(<x>)$, and

$$\begin{aligned} \alpha_m(<x>) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-r}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-r}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-r}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\ &+ \Delta_m \times \begin{cases} B_1(<x>), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

Now, we can state our first result.

Theorem 2.1. *For each integer* $l \geq r+1$*, we put*

$$\Delta_l = \sum_{k=r}^{l}\left(2kG_{k-1}^{(r-1)} - G_k^{(r)}\right) - G_l^{(r)}.$$

Assume that $\Delta_m = 0$ *, for an integer* $m \geq r+1$*. Then we have the following.*
(a) $\sum_{k=r}^m G_k^{(r)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=r}^{m} G_k^{(r)}(<x>) <x>^{m-k} \\ =&\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-r}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}, \end{aligned} \tag{2.14}$$

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for all $x \in \mathbb{R}$*, where the convergence is uniform.*

$$(b) \sum_{k=r}^{m} G_k^{(r)}(<x>) <x>^{m-k} = \frac{1}{m+2} \sum_{j=0, j \neq 1}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} B_j(<x>), \tag{2.15}$$

for all x *in* $\mathbb{R}$*, where* $B_j(<x>)$ *is the Bernoulli function.*

Assume next that $\Delta_m \neq 0$, for an integer $m \geq r+1$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(<x>)$ is piecewise C^∞ , and discontinuous with jump discontinuities at integers.
The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.16}$$

for $x \in \mathbb{Z}$.

We now state our second result.

Theorem 2.2. *For each integer* $l \geq r+1$*, we put*

$$\Delta_l = \sum_{k=r}^{l} \left(2kG_{k-1}^{(r-1)} - G_k^{(r)}\right) - G_l^{(r)}.$$

Assume that $\Delta_m \neq 0$ *, for an integer* $m \geq r+1$*. Then we have the following.*

$$(a) \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-r} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=r}^{m} G_k^{(r)}(<x>) <x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ G_m^{(r)} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.17}$$

$$(b) \frac{1}{m+2} \sum_{j=0}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} B_j(<x>) = \sum_{k=r}^{m} G_k^{(r)}(<x>) <x>^{m-k}, \text{ for } x \notin \mathbb{Z}; \tag{2.18}$$

$$\frac{1}{m+2} \sum_{j=0, j\neq 1}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} B_j(<x>) = G_m^{(r)} + \frac{1}{2}\Delta_m, \text{ for } x \in \mathbb{Z}. \tag{2.19}$$

3. The function $\beta_m(<x>)$

Let $\beta_m(x) = \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(x) x^{m-k}$, $(m \geq r+1)$. Then we will consider the function

$$\beta_m(<x>) = \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k},$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(<x>) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx.$$

To proceed further, we need to observe the following.

$$\begin{aligned}
\beta_m'(x) &= \sum_{k=r}^{m} \left\{ \frac{k}{k!(m-k)!} G_{k-1}^{(r)}(x) x^{m-k} + \frac{(m-k)}{k!(m-k)!} G_k^{(r)}(x) x^{m-k-1} \right\} \\
&= \sum_{k=r+1}^{m} \frac{1}{(k-1)!(m-k)!} G_{k-1}^{(r)}(x) x^{m-k} + \sum_{k=r}^{m-1} \frac{1}{k!(m-k-1)!} G_k^{(r)}(x) x^{m-k-1} \\
&= \sum_{k=r}^{m-1} \frac{1}{k!(m-1-k)!} G_k^{(r)}(x) x^{m-1-k} + \sum_{k=r}^{m-1} \frac{1}{k!(m-1-k)!} G_k^{(r)}(x) x^{m-1-k} \\
&= 2\beta_{m-1}(x).
\end{aligned} \tag{3.1}$$

From this, we get

$$\left(\frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x),$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \Big(\beta_{m+1}(1) - \beta_{m+1}(0) \Big).$$

For $m \geq r+1$, we let

$$\begin{aligned}
\Omega_m &= \beta_m(1) - \beta_m(0) \\
&= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} \big(G_k^{(r)}(1) - G_k^{(r)} \delta_{m,k} \big) \\
&= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} \left\{ 2k G_{k-1}^{(r-1)} - G_k^{(r)} - G_k^{(r)} \delta_{m,k} \right\} \\
&= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} \Big(2k G_{k-1}^{(r-1)} - G_k^{(r)} \Big) - \frac{1}{m!} G_m^{(r)}.
\end{aligned} \tag{3.2}$$

From this, we now see that

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.3}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \Omega_{m+1}. \tag{3.4}$$

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We are now ready to determine the Fourier coefficients $B_n^{(m)}$.
Case 1:$n \neq 0$

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\Big[\beta_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta'_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\ &= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m, \end{aligned}$$

from which by induction on m we can easily derive

$$B_n^{(m)} = -\sum_{j=1}^{m-r} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}.$$

Case 2: $n = 0$

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(< x >)$, $(m \geq r+1)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq r+1$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq r+1$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for an integer $m \geq r+1$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} &\beta_m(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\Big(-\sum_{j=1}^{m-r}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-r}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\Big(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\Big) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-r}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) \\ &+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned}$$

Now, we are going to state our first result.

Theorem 3.1. *For each positive integer $l \geq r+1$, we set*

$$\Omega_l = \sum_{k=r}^{l}\frac{1}{k!(l-k)!}\Big(2kG_{k-1}^{(r-1)} - G_k^{(r)}\Big) - \frac{1}{l!}G_l^{(r)}.$$

Assume that $\Omega_m = 0$, for an integer $m \geq r+1$. Then we have the following.

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(a) $\sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m-r} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x},\end{aligned} \tag{3.5}$$

for all $x \in \mathbb{R}$*, where the convergence is uniform.*

$$\begin{aligned}(b)\ &\sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-r} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>),\end{aligned} \tag{3.6}$$

for all $x \in \mathbb{R}$*, where* $B_j(<x>)$ *is the Bernoulli function.*

Assume next that $\Omega_m \neq 0$, for an integers $m \geq r+1$. Then, $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \notin \mathbb{Z}$, and convergence to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

Now, we are going to state our second result.

Theorem 3.2. *For each positive integer* $l \geq r+1$ *, we set*

$$\Omega_l = \sum_{k=r}^{l} \frac{1}{k!(l-k)!} \Big(2k G_{k-1}^{(r-1)} - G_k^{(r)} \Big) - \frac{1}{l!} G_l^{(r)}.$$

Assume that $\Omega_m \neq 0$*, for an integer* $m \geq r+1$ *. Then we have the following.*

$$\begin{aligned}(a)\ &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m-r} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{m!} G_m^{(r)} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases} \\ (b)\ &\sum_{j=0}^{m-r} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k}, \quad \text{for } x \notin \mathbb{Z}; \\ &\sum_{j=0, j\neq 1}^{m-r} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \frac{1}{m!} G_m^{(r)} + \frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}.\end{aligned}$$

4. The function $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(x) x^{m-k}$, $(m \geq r+1)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) < x >^{m-k},$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where $C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx$. Before proceeding further, we need to observe the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=r+1}^{m-1} \frac{1}{m-k} G_{k-1}^{(r)}(x) x^{m-k} + \sum_{k=r}^{m-1} \frac{1}{k} G_k^{(r)}(x) x^{m-k-1} \\
&= \sum_{k=r}^{m-2} \frac{1}{m-1-k} G_k^{(r)}(x) x^{m-1-k} + \sum_{k=r}^{m-1} \frac{1}{k} G_k^{(r)}(x) x^{m-1-k} \\
&= \sum_{k=r}^{m-2} \Big(\frac{1}{m-1-k} + \frac{1}{k} \Big) G_k^{(r)}(x) x^{m-1-k} + \frac{1}{m-1} G_{m-1}^{(r)}(x) \\
&= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}^{(r)}(x),
\end{aligned}$$

from which we see that

$$\Big(\frac{1}{m} (\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}^{(r)}(x)) \Big)' = \gamma_m(x). \tag{4.1}$$

This entails that

$$\begin{aligned}
\int_0^1 \gamma_m(x) dx &= \frac{1}{m} \Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} (G_{m+1}^{(r)}(1) - G_{m+1}^{(r)}) \Big) \\
&= \frac{1}{m} \Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{2}{m(m+1)} ((m+1) G_m^{(r-1)} - G_{m+1}^{(r)}) \Big).
\end{aligned}$$

For $m \geq r+1$, we put

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=r}^{m-1} \frac{1}{k(m-k)} \Big(G_k^{(r)}(1) - G_k^{(r)} \delta_{m,k} \Big) \\
&= \sum_{k=r}^{m-1} \frac{1}{k(m-k)} \Big(2k G_{k-1}^{(r-1)} - G_k^{(r)} - G_k^{(r)} \delta_{m,k} \Big) \\
&= \sum_{k=r}^{m-1} \frac{1}{k(m-k)} \Big(2k G_{k-1}^{(r-1)} - G_k^{(r)} \Big).
\end{aligned}$$

Note here that

$$\gamma_m(0)=\gamma_m(1) \ \Leftrightarrow \ \Lambda_m=0,$$

and

$$\int_0^1 \gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right).$$

We are now ready to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n\neq 0$

$$\begin{aligned}
C_n^{(m)}&=\int_0^1 \gamma_m(x)e^{-2\pi inx}dx\\
&=-\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1+\frac{1}{2\pi in}\int_0^1\gamma_m'(x)e^{-2\pi inx}dx\\
&=-\frac{1}{2\pi in}\Big(\gamma_m(1)-\gamma_m(0)\Big)+\frac{1}{2\pi in}\int_0^1\left((m-1)\gamma_{m-1}(x)+\frac{1}{m-1}G_{m-1}^{(r)}(x)\right)e^{-2\pi inx}dx\\
&=\frac{m-1}{2\pi in}C_n^{(m-1)}-\frac{1}{2\pi in}\Lambda_m+\frac{1}{2\pi in(m-1)}\int_0^1 G_{m-1}^{(r)}(x)e^{-2\pi inx}dx\\
&=\frac{m-1}{2\pi in}C_n^{(m-1)}-\frac{1}{2\pi in}\Lambda_m-\frac{1}{2\pi in(m-1)}\Theta_m,
\end{aligned}$$

where

$$\Theta_m=\sum_{k=1}^{m-2}\frac{2(m-1)_{k-1}}{(2\pi in)^k}\left((m-k)G_{m-k-1}^{(r-1)}-G_{m-k}^{(r)}\right).$$

By proceeding induction on m we can show that

$$C_n^{(m)}=-\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1}-\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}.$$

Here we note that

$$\begin{aligned}
&\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}\\
=&\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j-1}\frac{2(m-j)_{k-1}}{(2\pi in)^k}\left((m-j-k+1)G_{m-j-k}^{(r-1)}-G_{m-j-k+1}^{(r)}\right)\\
=&\sum_{j=1}^{m-r}\frac{1}{m-j}\sum_{k=1}^{m-j-1}\frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k}}\left((m-j-k+1)G_{m-j-k}^{(r-1)}-G_{m-j-k+1}^{(r)}\right)\\
=&\sum_{j=1}^{m-r}\frac{1}{m-j}\sum_{s=j+1}^{m-1}\frac{2(m-1)_{s-2}}{(2\pi in)^s}\left((m-s+1)G_{m-s}^{(r-1)}-G_{m-s+1}^{(r)}\right)
\end{aligned} \tag{4.2}$$

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$$\begin{aligned}
=&\sum_{s=1}^{m-r}\frac{2(m-1)_{s-2}}{(2\pi in)^s}\left((m-s+1)G_{m-s}^{(r-1)}-G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{m-s})\\
&+\sum_{s=m-r+1}^{m-1}\frac{2(m-1)_{s-2}}{(2\pi in)^s}\left((m-s+1)G_{m-s}^{(r-1)}-G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{r-1})\\
=&\frac{1}{m}\sum_{s=1}^{m-r}\frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{m-s})\\
&+\frac{1}{m}\sum_{s=m-r+1}^{m-1}\frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{r-1}).
\end{aligned}\tag{4.3}$$

Also, we note that

$$\begin{aligned}
&\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1}\\
=&\frac{1}{m}\sum_{s=1}^{m-r}\frac{(m)_s}{(2\pi in)^s}\Lambda_{m-s+1}.
\end{aligned}\tag{4.4}$$

Putting everything altogether, we have:

$$\begin{aligned}
C_n^{(m)}=&-\frac{1}{m}\sum_{s=1}^{m-r}\frac{(m)_s}{(2\pi in)^s}\Big(2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big)\\
&-\frac{1}{m}\sum_{s=m-r+1}^{m-1}\frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{r-1}).
\end{aligned}\tag{4.5}$$

Case 2: $n=0$

$$C_0^{(m)}=\int_0^1\gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right).$$

$\gamma_m(<x>)$, $(m\geq r+1)$ is piecewise C^∞. Moreover, $\gamma_m(<x>)$ is continuous for those integers $m\geq r+1$ with $\Lambda_m=0$, and discontinuous with jump discontinuities at integers for those integer $m\geq r+1$ with $\Lambda_m\neq 0$.

Assume first that $\Lambda_m=0$, for an integer $m\geq r+1$. Then $\gamma_m(0)=\gamma_m(1)$. Hence $\gamma_m(<x>)$ is piecewise C^∞ and continuous. Thus the Fourier series of $\gamma_m(<x>)$

converges uniformly to $\gamma_m(<x>)$, and

$$
\begin{aligned}
&\gamma_m(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right)\\
&+\sum_{n=-\infty,n\neq 0}^{\infty}\Big\{-\frac{1}{m}\sum_{s=1}^{m-r}\frac{(m)_s}{(2\pi in)^s}\Big(2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big)\Big\}e^{2\pi inx}\\
&+\sum_{n=-\infty,n\neq 0}^{\infty}\Big\{-\frac{1}{m}\sum_{s=m-r+1}^{m-1}\frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)\\
&\times(H_{m-1}-H_{r-1})\Big\}e^{2\pi inx}\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right)\\
&+\frac{1}{m}\sum_{s=1}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big\}\Big(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\Big)\\
&+\frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times\Big(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\Big)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right)\\
&+\frac{1}{m}\sum_{s=2}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big\}B_s(<x>)\\
&+\frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases}\\
&=\frac{1}{m}\sum_{s=0,s\neq 1}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big\}B_s(<x>)\\
&+\frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>)
\end{aligned}
$$

$$+\Lambda_m \times \begin{cases} B_1(<x>), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Now, we get the following theorem.

Theorem 4.1. *For each integer $l \geq r+1$, we let*

$$\Lambda_l = \sum_{k=r}^{l-1} \frac{1}{k(l-k)} \Big(2kG_{k-1}^{(r-1)} - G_k^{(r)} \Big).$$

Assume that $\Lambda_m = 0$, for an integer $m \geq r+1$. Then we have the following.

(a) $\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}((m+1)G_m^{(r-1)} - G_{m+1}^{(r)}) \right) \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{ -\frac{1}{m} \sum_{s=1}^{m-r} \frac{(m)_s}{(2\pi i n)^s} \Big(2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) \\ &\times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1} \Big) \Big\} e^{2\pi i n x} \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{ -\frac{1}{m} \sum_{s=m-r+1}^{m-1} \frac{2(m)_s}{(2\pi i n)^s} \left(G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)} \right) \\ &\times (H_{m-1} - H_{r-1}) \Big\} e^{2\pi i n x}, \end{aligned} \tag{4.6}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned} &\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}((m+1)G_m^{(r-1)} - G_{m+1}^{(r)}) \right) \\ &+ \frac{1}{m} \sum_{s=0, s\neq 1}^{m-r} \binom{m}{s} \Big\{ 2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) \\ &\times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1} \Big\} B_s(<x>) \\ &+ \frac{2}{m}(H_{m-1} - H_{r-1}) \sum_{s=m-r+1}^{m-1} \binom{m}{s} (G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) B_s(<x>), \end{aligned} \tag{4.7}$$

for all $x \in \mathbb{R}$, where $B_s(<x>)$ is the Bernoulli function.

Assume next that $\Lambda_m \neq 0$, for an integer $m \geq r+1$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0)+\gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m,$$

for $x \in \mathbb{Z}$.

Now, we have the following theorem.

Theorem 4.2. *For each integer $l \geq r+1$, we let*

$$\Lambda_l = \sum_{k=r}^{l-1} \frac{1}{k(l-k)}\Big(2kG_{k-1}^{(r-1)} - G_k^{(r)}\Big).$$

Assume that $\Lambda_m \neq 0$, for an integer $m \geq r+1$. Then we have the following.

$$(a)\ \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}((m+1)G_m^{(r-1)} - G_{m+1}^{(r)})\right)$$
$$+ \sum_{n=-\infty, n\neq 0}^{\infty}\Big\{-\frac{1}{m}\sum_{s=1}^{m-r}\frac{(m)_s}{(2\pi i n)^s}\Big(2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)})$$
$$\times (H_{m-1}-H_{m-s}) + \Lambda_{m-s+1}\Big)\Big\}e^{2\pi i n x}$$
$$+ \sum_{n=-\infty, n\neq 0}^{\infty}\Big\{-\frac{1}{m}\sum_{s=m-r+1}^{m-1}\frac{2(m)_s}{(2\pi i n)^s}\left(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)$$
$$\times (H_{m-1}-H_{r-1})\Big\}e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=r}^{m-1}\frac{1}{k(m-k)}G_k^{(r)}(<x>)<x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

$$(b)\ \frac{1}{m}\sum_{s=0}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)})$$
$$\times (H_{m-1}-H_{m-s}) + \Lambda_{m-s+1}\Big\}B_s(<x>)$$
$$+ \frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>)$$
$$= \sum_{k=r}^{m-1}\frac{1}{k(m-k)}G_k^{(r)}(<x>)<x>^{m-k},\ \text{for } x \notin \mathbb{Z};$$

$$\frac{1}{m}\sum_{s=0, s\neq 1}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)})$$
$$\times (H_{m-1}-H_{m-s}) + \Lambda_{m-s+1}\Big\}B_s(<x>)$$
$$+ \frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>)$$
$$= \frac{1}{2}\Lambda_m,\ \text{for } x \in \mathbb{Z}.$$

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Value distribution and uniqueness of certain types of q-difference polynomials *

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Abstract: In this paper, we consider certain types of q-difference polynomials in the complex plane by using the Nevanlinna's theory. Some results about the value distribution and uniqueness are obtained, which are the counterparts of the properties of the general difference polynomials.

Keywords: Value distribution; q-Difference; Share fixed-points.

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1. Introduction and Results

Throughout this paper, we assume $f(z)$, $g(z)$ be non-constant meromorphic (or entire) functions in the complex plane and use the basic notations of the Nevanlinna's theory [1,2,12]. In particular, the order of growth of $f(z)$ is represented by $\sigma(f)$ and the exponent of convergence of the zeros of $f(z)$ is represented by $\lambda(f)$. In addition, $S(r,f)$ represents any quantify which satisfies $S(r,f) = o(T(r,f))(\ r \to \infty)$, possibly outside a set of finite logarithmic measure.

If $f(z) - 1$ and $g(z) - 1$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share 1 CM. If $f(z) - z$ and $g(z) - z$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share z CM, or say that $f(z)$ and $g(z)$ have the same fixed-points[9].

In the past decade, many scholars have focused on complex difference and difference equations and presented many results[3-5] on value distribution theory of meromorphic functions. Meanwhile, q-difference is also becoming an important topic in complex analysis, so the research of it is very meaningful. The aim of this paper is to investigate the value distribution and uniqueness of certain types of q-difference polynomials.

We now introduce some related results. Liu and Laine [3] discussed the problem when a difference polynomial assumes a nonzero small function, and showed the following result.

Theorem A *Let $f(z)$ be a transcendental entire function of finite order, not of period c, where c is a nonzero complex constant, and let $s(z)$ be a nonzero function, small compared*

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to $f(z)$. Then the difference polynomial $f(z)^n + f(z+c) - f(z) - s(z)$ has infinitely many zeros in the complex plane, provided that $n \geq 3$.

Chen [4] investigated the value distribution of a certain difference and obtained the following theorem.

Theorem B *Let $f(z)$ be a transcendental entire function of finite order, and let $a, c \in C \setminus \{0\}$ be constants, with such that $f(z+c) \not\equiv f(z)$. Set $\psi_n(z) = \Delta f(z) - af(z)^n$, where $\Delta f(z) = f(z+c) - f(z)$ and $n \geq 3$ is an integer. Then $\psi_n(z)$ assumes all finite values infinitely often, and for every $b \in C$ one has $\lambda(\psi_n(z) - b) = \sigma(f)$.*

Laine and Yang [5] analyzed the difference $f(z)^n f(z+c)$, and presented the following result.

Theorem C *Let $f(z)$ be a transcendental entire function of finite order and c be non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in C$ infinitely often.*

In this paper, we first prove the analogous results in q-difference type as follows.

Theorem 1 *Let $f(z)$ be a transcendental meromorphic (entire) function of zero order and let $\alpha(z)$ be a non-zero function, small compared to $f(z)$, q is a non-zero complex constant. Then for $n \geq 6 (n \geq 2)$, $f(z)^n f(qz) - \alpha(z)$ has infinitely many zeros in the complex plane.*

Corollary 1 *Let $f(z)$ be a transcendental meromorphic (entire) function of zero order and q is a non-zero complex constant. Then for $n \geq 6 (n \geq 2)$, $f(z)^n f(qz) = 1$ has infinitely many solutions in the complex plane.*

Corollary 2 *Let $f(z)$ be a transcendental meromorphic (entire) function of zero order and q is a non-zero complex constant. Then for $n \geq 6 (n \geq 2)$, the $f(z)^n f(qz)$ has infinitely many fixed-points in the complex plane.*

Theorem 2 *Let $f(z)$ be a transcendental entire function of zero order, and let $\alpha(z)$ be a non-zero function, small compared to $f(z)$. $q \in C \setminus \{0\}$ is a complex constant. Set $\psi_n(z) = f(z)^n + \Delta_q f(z)$, where $\Delta_q f(z) = f(qz) - f(z)$ and $n \geq 2$ is an integer. Then $\psi_n(z) - \alpha(z)$ has infinitely zeros in the complex plane and $\lambda(\psi_n(z) - \alpha(z)) = 0$.*

We now recall the following Theorem D[6].

Theorem D *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic(entire) functions, $n \geq 11 (n \geq 6)$ a positive integer. If $f(z)^n f(z)'$ and $g(z)^n g(z)'$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.*

Naturally, we ask whether there is a corresponding uniqueness theorem in q-difference polynomials. In this paper we give an affirmative answer to this question, and obtain the following results.

Theorem 3 *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (entire) functions of zero order. Suppose that q is a non-zero complex constant and n is an integer $n \geq 8 (n \geq 4)$. If $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share z CM, then $f(z) \equiv tg(z)$ for $t^{n+1} = 1$.*

Theorem 4 *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (entire) functions of zero order. Suppose that q is a non-zero complex constant and n is an integer $n \geq 8 (n \geq 4)$. If $f(z)^n (f(z) - 1) f(qz)$ and $g(z)^n (g(z) - 1) g(qz)$ share z CM, then $f(z) \equiv g(z)$.*

2. Some Lemmas

In this section, we summarize some lemmas, which will be used to prove our main results.
Lemma 2.1[7] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in C \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T(r, f(z))). \tag{2.1}$$

on a set of logarithmic density 1.
Lemma 2.2[8] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in C \setminus \{0\}$. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z)). \tag{2.2}$$

on a set of logarithmic density 1.
Remark 2.1 Equation (2.2) implies that

$$T(r, f(qz)) = T(r, f(z)) + S(r, f). \tag{2.3}$$

Lemma 2.3[8] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in C \setminus \{0\}$. Then*

$$N(r, f(qz)) = (1 + o(1))N(r, f(z)). \tag{2.4}$$

on a set of logarithmic density 1.
Lemma 2.4 *Let $f(z)$ be a transcendental entire function of zero order and q be non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(qz)$ is not a constant.*
Proof Let $F(z) = f(z)^n f(qz)$. If $F(z)$ is a constant c. Then $f(z)^n = \frac{c}{f(qz)}$. From the Lemma 2.2 and an identity due to Valiron-Mohon'ko [10, 11], we get

$$\begin{aligned} nT(r, f(z)) &= T\left(r, f(z)^n\right) \\ &= T\left(r, \frac{c}{f(qz)}\right) \\ &= T(r, f(z)) + S(r, f), \end{aligned}$$

which is a contradiction for $n \geq 2$. Therefore $F(z)$ is not a constant.

3. Proof of Theorem 1

Proof Denote $F(z) = f(z)^n f(qz)$. We claim that $F(z) - \alpha(z)$ is transcendental if $n \geq 2$. Otherwise, we suppose that $F(z) - \alpha(z) = \beta(z)$, where $\beta(z)$ is a rational function. Combining Lemma 2.2 and the identity of Valiron-Mohon'ko, we have

$$\begin{aligned} nT(r, f(z)) &= T(r, f(z)^n) \\ &= T\left(r, \frac{\alpha(z) + \beta(z)}{f(qz)}\right) \\ &\leq T(r, \alpha(z)) + T(r, \beta(z)) + T(r, f(qz)) + S(r, f) \\ &= T(r, f(z)) + S(r, f). \end{aligned}$$

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This contradicts the fact that $n \geq 2$. Hence $F(z) - \alpha(z)$ is transcendental. Then, we consider the following two cases.

Case 1. Suppose that $f(z)$ is a meromorphic function. From Lemma 2.2, Lemma 2.3 and the second main Theorem for three small targets [2], we get

$$\begin{aligned} nT(r, f(z)) &= T(r, f(z)^n) \\ &= T(r, \tfrac{F(z)}{f(qz)}) \\ &\leq T(r, F(z)) + T(r, f(z)) + S(r, f) \\ &\leq \overline{N}(r, F(z)) + \overline{N}\left(r, \tfrac{1}{F(z)}\right) + \overline{N}\left(r, \tfrac{1}{F(z)-\alpha(z)}\right) \\ &+ T(r, f(z)) + S(r, f), \end{aligned} \tag{3.1}$$

$$\begin{aligned} \overline{N}(r, F(z)) &= \overline{N}(r, f(z)^n f(qz)) \\ &\leq \overline{N}(r, f(z)^n) + \overline{N}(r, f(qz)) \\ &= \overline{N}(r, f(z)) + \overline{N}(r, f(qz)) \\ &\leq 2T(r, f(z)) + S(r, f), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \overline{N}\left(r, \tfrac{1}{F(z)}\right) &= \overline{N}\left(r, \tfrac{1}{f(z)^n f(qz))}\right) \\ &\leq \overline{N}\left(r, \tfrac{1}{f(z)}\right) + \overline{N}\left(r, \tfrac{1}{f(qz)}\right) \\ &\leq T\left(r, \tfrac{1}{f(z)}\right) + T\left(r, \tfrac{1}{f(qz)}\right) \\ &= 2T(r, f(z)) + S(r, f). \end{aligned} \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$\overline{N}\left(r, \frac{1}{F(z) - \alpha(z)}\right) \geq (n-5)T(r, f(z)) + S(r, f).$$

The assertion follows by $n \geq 6$.

Case 2. Suppose that $f(z)$ is an entire function. Applying Lemma 2.1 − 2.3 and the second main Theorem for three small targets, we obtain

$$\begin{aligned} (n+1)T(r, f(z)) &= T(r, f(z)^{n+1}) \\ &= m(r, f(z)^{n+1}) \\ &\leq m\left(r, \tfrac{f(z)}{f(qz)}\right) + m(r, F(z)) + S(r, f) \\ &= T(r, F(z)) + S(r, f) \\ &\leq \overline{N}(r, F(z)) + \overline{N}\left(r, \tfrac{1}{F(z)}\right) + \overline{N}\left(r, \tfrac{1}{F(z)-\alpha(z)}\right) + S(r, f), \end{aligned} \tag{3.4}$$

Since $f(z)$ is a zero-order entire function, $F(z) = f(z)^n f(qz)$ is an entire function with zero-order, then

$$\overline{N}(r, F(z)) = 0. \tag{3.5}$$

It follows from (3.3) − (3.5) that

$$\overline{N}\left(r, \frac{1}{F(z) - \alpha(z)}\right) + S(r, f) \geq (n-1)T(r, f(z)).$$

This holds for $n \geq 2$. The proof of Theorem 1 is completed.

4. Proof of Theorem 2

Proof We claim that $\psi_n(z)-\alpha(z)$ is transcendental if $n \geq 2$. On the contrary, we suppose that $\psi_n(z)-\alpha(z)=\beta(z)$, here $\beta(z)$ is a rational function. Then

$$f(z)^n=\alpha(z)+\beta(z)-\Delta_q(z).$$

An application of Lemma 2.1 and the identity due to Valiron-Mohon'ko yields

$$\begin{aligned} T(r,f(z)^n) &= nT(r,f(z))+S(r,f) \\ &= T(r,\alpha(z)+\beta(z)-\Delta_q(z)) \\ &\leq T(r,\alpha(z))+T(r,\beta(z))+T(r,f(qz)-f(z))+S(r,f) \\ &\leq m\left(r,\frac{f(qz)-f(z)}{f(z)}\right)+m(r,f(z))+S(r,f) \\ &= T(r,f(z))+S(r,f). \end{aligned}$$

This contradicts the fact that $n \geq 2$. Hence $\psi_n(z)-\alpha(z)$ is transcendental. Thus we discuss the following two cases.

Case 1. Suppose that $\alpha(z)$ is an entire function. Clearly, $\psi_n(z)-\alpha(z)$ is a transcendental entire function for $n \geq 2$.

Case 2. Suppose that $\alpha(z)$ is a meromorphic function. Set $\alpha(z)=\frac{h(z)}{g(z)}$, where $g(z)$ and $h(z)$ are entire functions with $T(r,g(z))=o(T(r,f(z)))$ and $T(r,h(z))=o(T(r,f(z)))$, respectively. Then

$$\psi_n(z)-\alpha(z)=f(z)^n+f(qz)-f(z)-\frac{h(z)}{g(z)}=\frac{(f(z)^n+f(qz)-f(z))g(z)-h(z)}{g(z)}.$$

If $\psi_n(z)-\alpha(z)$ has finitely many zeros, then $(f(z)^n+f(qz)-f(z))g(z)-h(z)$ must be a polynomial. Denote by $p(z)=(f(z)^n+f(qz)-f(z))g(z)-h(z)$, where $p(z)$ is a polynomial. From Lemma 2.1 , we have

$$\begin{aligned} T(r,f(z)^n) &= nT(r,f(z))+S(r,f) \\ &= T\left(r,\frac{p(z)+h(z)}{g(z)}-f(qz)+f(z)\right) \\ &\leq T(r,p(z))+T(r,g(z))+T(r,\alpha(z))+T(r,f(qz)-f(z)) \\ &\leq m\left(r,\frac{f(qz)-f(z)}{f(z)}\right)+m(r,f(z))+S(r,f) \\ &= T(r,f(z))+S(r,f), \end{aligned}$$

which gives a contradiction since $n \geq 2$. Hence $\psi_n(z)-\alpha(z)$ has infinitely many zeros in the complex plane.

Moreover, by the fact $0 \leq \lambda(\psi_n(z)-\alpha(z)) \leq \sigma(f(z))=0$, it follows that $\lambda(\psi_n(z)-\alpha(z))=0$. We finish the proof of Theorem 2.

5. Proof of Theorem 3

Proof From $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share z CM, we know that $\frac{f(z)^n f(qz)}{z}$ and $\frac{g(z)^n g(qz)}{z}$ share 1 CM.

By the assumption of Theorem 3, there exists an entire function $p(z)$ such that

$$\frac{\frac{f(z)^n f(qz)}{z} - 1}{\frac{g(z)^n g(qz)}{z} - 1} = e^{p(z)}. \tag{5.1}$$

Since the order of $f(z)$ and $g(z)$ is of zero, then $e^{p(z)}$ is a non-zero constant, let it be c. Rewriting the equation (5.1), it follows that

$$c\frac{g(z)^n g(qz)}{z} = \frac{f(z)^n f(qz)}{z} - 1 + c. \tag{5.2}$$

Denote $F(z) = \frac{f(z)^n f(qz)}{z}$ and $G(z) = \frac{g(z)^n g(qz)}{z}$.

First, assume that $c \neq 1$. We take into account the following two cases.

Case 1. Suppose that $f(z)$ and $g(z)$ are meromorphic functions. Combing Lemma 2.2, Lemma 2.3 and equation (5.2), we obtain

$$T(r, F(z)) \leq \overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}\left(r, \frac{1}{F(z) - 1 + c}\right) + S(r, f), \tag{5.3}$$

$$\begin{aligned} \overline{N}(r, F(z)) &= \overline{N}\left(r, \tfrac{f(z)^n f(qz)}{z}\right) \\ &\leq \overline{N}(r, f(z)^n) + \overline{N}(r, f(qz)) + \overline{N}\left(r, \tfrac{1}{z}\right) \\ &= \overline{N}(r, f(z)) + \overline{N}(r, f(qz)) + S(r, f) \\ &< 2T(r, f(z)) + S(r, f), \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \overline{N}\left(r, \tfrac{1}{F(z)}\right) &= \overline{N}\left(r, \tfrac{z}{f(z)^n f(qz)}\right) \\ &\leq \overline{N}\left(r, \tfrac{1}{f(z)^n}\right) + \overline{N}\left(r, \tfrac{1}{f(qz)}\right) + \overline{N}(r, z) \\ &= \overline{N}\left(r, \tfrac{1}{f(z)}\right) + \overline{N}\left(r, \tfrac{1}{f(qz)}\right) \\ &< 2T(r, f(z)) + S(r, f). \end{aligned} \tag{5.5}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{F(z) - 1 + c}\right) = \overline{N}\left(r, \frac{1}{cG(z)}\right) \leq 2T(r, g(z)) + S(r, g). \tag{5.6}$$

By substituting (5.4) – (5.6) into (5.3), it follows that

$$T(r, F(z)) \leq 4T(r, f(z)) + 2T(r, g(z)) + S(r, f) + S(r, g). \tag{5.7}$$

On the other hand, from Lemma 2.2, we have

$$\begin{aligned} T(r, f(z)^n) &= nT(r, f(z)) + S(r, f) \\ &= T\left(r, \frac{zF(z)}{f(qz)}\right) \\ &\leq T(r, z) + T(r, F(z)) + T\left(r, \frac{1}{f(qz)}\right) \\ &= T(r, f(qz)) + T(r, F(z)) + S(r, f) \\ &= T(r, f(z)) + T(r, F(z)) + S(r, f), \end{aligned}$$

which means

$$(n-1)T(r, f(z)) \leq T(r, F(z)) + S(r, f). \tag{5.8}$$

Substituting (5.8) into (5.7), we have

$$(n-5)T(r, f(z)) \leq 2T(r, g(z)) + S(r, f) + S(r, g). \tag{5.9}$$

Similarly, we can get

$$(n-5)T(r, g(z)) \leq 2T(r, f(z)) + S(r, f) + S(r, g). \tag{5.10}$$

Combining the above two inequalities (5.9) and (5.10), we obtain

$$(n-7)(T(r, f(z)) + T(r, g(z))) \leq S(r, f) + S(r, g),$$

which contradicts with the assumption $n \geq 8$.

Case 2. Suppose that $f(z)$ and $g(z)$ are entire functions. From $\overline{N}(r, f(z)) = \overline{N}(r, g(z)) = 0$, then we have

$$\overline{N}(r, F(z)) = \overline{N}\left(r, \frac{f(z)^n f(qz)}{z}\right) \leq \overline{N}(r, f(z)^n) + \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{z}\right) = S(r, f). \tag{5.11}$$

Substituting (5.11), (5.5), (5.6) into (5.3), we obtain

$$T(r, F(z)) \leq 2T(r, f(z)) + 2T(r, g(z)) + S(r, f) + S(r, g). \tag{5.12}$$

On the other hand, by using Lemma 2.1 to obtain

$$\begin{aligned} T(r, f(z)^{n+1}) &= (n+1)T(r, f(z)) + S(r, f) \\ &= m(r, f(z)^{n+1}) \\ &= m\left(r, \frac{f(z)}{f(qz)} zF(z)\right) \\ &\leq m(r, F(z)) + m\left(r, \frac{f(z)}{f(qz)}\right) + S(r, f) \\ &\leq T(r, F(z)) + S(r, f), \end{aligned}$$

which implies

$$(n+1)T(r, f(z)) \leq T(r, F(z)) + S(r, f). \tag{5.13}$$

By substituting (5.13) into (5.12), we get

$$(n-1)T(r,f(z)) \leq 2T(r,g(z)) + S(r,f) + S(r,g). \tag{5.14}$$

Similarly, we can obtain

$$(n-1)T(r,g(z)) \leq 2T(r,f(z)) + S(r,f) + S(r,g). \tag{5.15}$$

Combining (5.14) and (5.15) yields

$$(n-3)(T(r,f(z)) + T(r,g(z))) \leq S(r,f) + S(r,g),$$

this is impossible when $n \geq 4$.

Then, assume that $c = 1$. From (5.2), we can get

$$\frac{f(z)^n f(qz)}{z} = \frac{g(z)^n g(qz)}{z}.$$

Let $h(z) = \frac{f(z)}{g(z)}$, then we have

$$h(z)^n h(qz) = 1. \tag{5.16}$$

From Lemma 2.2, we obtain

$$T(r,h(z)^n) = nT(r,h(z)) + S(r,h) = T\left(r, \frac{1}{h(qz)}\right) = T(r,h(z)) + S(r,h).$$

So $h(z)$ must be constant from $n \geq 4$. Suppose that $h(z) \equiv t$. We conclude that $t^{n+1} = 1$ from (5.16). Thus, Theorem 3 is proved.

6. Proof of Theorem 4

Proof From $f(z)^n(f(z)-1)f(qz)$ and $g(z)^n(g(z)-1)g(qz)$ share z CM, we know that $\frac{f(z)^n(f(z)-1)f(qz)}{z}$ and $\frac{g(z)^n(g(z)-1)g(qz)}{z}$ share 1 CM.
Denote

$$F(z) = \frac{f(z)^n(f(z)-1)f(qz)}{z} \quad \text{and} \quad G(z) = \frac{g(z)^n(g(z)-1)g(qz)}{z}. \tag{6.1}$$

It follows from Lemma 2.1 that

$$\begin{aligned} T(r,f(z)^{n+1}(f(z)-1)) &= (n+2)T(r,f(z)) + S(r,f) \\ &= m(r,f(z)^{n+1}(f(z)-1)) \\ &= m\left(r, \frac{f(z)}{f(qz)} zF(z)\right) \\ &\leq m(r,F(z)) + m\left(r, \frac{f(z)}{f(qz)}\right) + S(r,f) \\ &\leq T(r,F(z)) + S(r,f), \end{aligned}$$

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which implies

$$(n+2)T(r,f(z)) \leq T(r,F(z)) + S(r,f). \tag{6.2}$$

Since $F(z)$ and $G(z)$ share 1 CM, then by the same arguments in the proof of Theorem 3, there exists a non-zero constant c such that

$$F(z) - 1 = c(G(z) - 1). \tag{6.3}$$

Assume that $c \neq 1$. By using Lemma 2.2, Lemma 2.3, (6.1), (6.3) and the second main theorem to $F(z)$, we deduce that

$$T(r,F(z)) \leq \overline{N}(r,F(z)) + \overline{N}\left(r,\frac{1}{F(z)}\right) + \overline{N}\left(r,\frac{1}{F(z)-1+c}\right) + S(r,f), \tag{6.4}$$

$$\begin{aligned} \overline{N}(r,F(z)) &= \overline{N}\left(r,\tfrac{f(z)^n(f(z)-1)f(qz)}{z}\right) \\ &\leq \overline{N}(r,f(z)^n) + \overline{N}(r,f(z)-1) + \overline{N}(r,f(qz)) + \overline{N}\left(r,\tfrac{1}{z}\right) \\ &= S(r,f), \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \overline{N}\left(r,\tfrac{1}{F(z)}\right) &= \overline{N}\left(r,\tfrac{z}{f(z)^n(f(z)-1)f(qz)}\right) \\ &\leq \overline{N}\left(r,\tfrac{1}{f(z)^n}\right) + \overline{N}\left(r,\tfrac{1}{f(qz)}\right) + \overline{N}(r,z) + \overline{N}\left(r,\tfrac{1}{f(z)-1}\right) \\ &\leq 3T(r,f(z)) + S(r,f). \end{aligned} \tag{6.6}$$

Similarly, we can get

$$\overline{N}\left(r,\frac{1}{F(z)-1+c}\right) = \overline{N}\left(r,\frac{1}{cG(z)}\right) \leq 3T(r,g(z)) + S(r,g). \tag{6.7}$$

Substituting (6.5) – (6.7) into (6.4), we have

$$T(r,F(z)) \leq 3T(r,f(z)) + 3T(r,g(z)) + S(r,f) + S(r,g). \tag{6.8}$$

It follows from (6.2) and (6.8) that

$$(n-1)T(r,f(z)) \leq 3T(r,g(z)) + S(r,f) + S(r,g). \tag{6.9}$$

Similarly,

$$(n-1)T(r,g(z)) \leq 3T(r,f(z)) + S(r,f) + S(r,g). \tag{6.10}$$

Combing (6.9) and (6.10) yields

$$(n-4)(T(r,f(z)) + T(r,g(z))) \leq S(r,f) + S(r,g).$$

Clearly, it isn't established for $n \geq 6$.

Assume that $c = 1$, this means

$$\frac{f(z)^n(f(z)-1)f(qz)}{z} = \frac{g(z)^n(g(z)-1)g(qz)}{z}.$$

Denote $h(z) = \frac{f(z)}{g(z)}$, we obtain

$$g(z)(h(z)^{n+1}h(qz) - 1) = h(z)^n h(qz) - 1. \tag{6.11}$$

Assume $h(z)$ is not a constant. By using Lemma 2.4, we know that $h(z)^{n+1}h(qz)$ is also not a constant. If there exists a point z_0 such that $h(z_0)^{n+1}h(qz_0) = 1$. Combing (6.11) and $g(z)$ is an entire function, we obtain $h(z_0)^n h(qz_0) = 1$. Hence $h(z_0) = 1$, then it follows that

$$\begin{aligned} \overline{N}\left(r, \frac{1}{h(z)^{n+1}h(qz) - 1}\right) &= \overline{N}\left(r, \frac{g(z)}{h(z)^n h(qz) - 1}\right) \\ &\leq \overline{N}(r, g(z)) + \overline{N}\left(r, \frac{1}{h(z)^n h(qz) - 1}\right) \\ &\leq \overline{N}\left(r, \frac{1}{h(z) - 1}\right) \\ &\leq T(r, h(z)) + S(r, h), \end{aligned}$$

i.e.,

$$\overline{N}\left(r, \frac{1}{h(z)^{n+1}h(qz) - 1}\right) \leq T(r, h(z)) + S(r, h). \tag{6.12}$$

We now set $H(z) = h(z)^{n+1}h(qz)$. Applying the second main Theorem to $H(z)$, we have

$$T(r, H(z)) \leq \overline{N}(r, H(z)) + \overline{N}\left(r, \frac{1}{H(z)}\right) + \overline{N}\left(r, \frac{1}{H(z) - 1}\right) + S(r, h). \tag{6.13}$$

Combing Lemma 2.2 and Lemma 2.3 yields

$$\overline{N}(r, H(z)) \leq 2T(r, h(z)) + S(r, h) \tag{6.14}$$

and

$$\overline{N}\left(r, \frac{1}{H(z)}\right) \leq 2T(r, h(z)) + S(r, h). \tag{6.15}$$

Substituting (6.12), (6.14), (6.15) into (6.13), we get

$$T(r, H(z)) \leq 5T(r, h(z)) + S(r, h). \tag{6.16}$$

It follows from Lemma 2.2 and (6.16) that

$$\begin{aligned} T(r, h(z)^{n+1}) &= (n+1)T(r, h(z)) + S(r, h) \\ &= T\left(r, \frac{H(z)}{h(qz)}\right) \\ &\leq T(r, H(z)) + T(r, h(z)) + S(r, h) \\ &\leq 6T(r, h(z)) + S(r, h). \end{aligned}$$

Obviously, it is a contradiction with the assumption $n \geq 6$. Thus, $h(z)$ is a constant, let it be t. Then, substituting it into (6.11), we have

$$g(z)(t^{n+2} - 1) = t^{n+1} - 1. \tag{6.17}$$

Since $g(z)$ is a transcendental entire function, from (6.17), we know that $t^{n+2} = 1$ and $t^{n+1} = 1$, which means $t = 1$. Consequently, $f(z) \equiv g(z)$. The proof of Theorem 4 is completed.

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New Exact Penalty Function Methods with ϵ-approximation and Perturbation Convergence for Solving Nonlinear Bilevel Programming Problems

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Abstract. In this paper, in order to solve a class of nonlinear bilevel programming problems, we equivalently transform the nonlinear bilevel programming problems into corresponding single level nonlinear programming problems by using the Karush-Kuhn-Tucker optimality condition. Then, based on penalty function theory, we construct a smooth approximation method for obtaining optimal solutions of classic l_1-exact penalty function optimality problems, which is equivalent to the single level nonlinear programming problems. Furthermore, using ϵ-approximate optimal solution theory, we prove convergence of a simple ϵ-approximate optimal algorithm. Finally, through adding parameters in the constraint set of objective function, we prove some perturbation convergence results for solving the nonlinear bilevel programming problems.

Key Words and Phrases: Nonlinear bilevel programming problem, new exact penalty function method, smooth approximation, ϵ-approximate algorithm, perturbation convergence.

AMS Subject Classification: 49K30, 65K05, 90C30, 90C59.

1 Introduction

Since 1980s, bilevel programming problems had been very widely used in supply chain management, engineering design, network planning and other fields [1]. The theory and algorithms for bilevel programming problems have been deeply explored by many researchers. See, for example, [2, 3] and the reference therein. Recently, there are quite mature theoretical support and algorithm design on how to solve bilevel programming problems. For instance, by using the most famous pole search method, the global optimal solution of the problems can ultimately be obtained (see [4]). Zheng et al. [5] pointed out that a class of exact penalty function methods to solve the weak linear bilevel programming problem is feasible. But the present research to nonlinear bilevel programming problems is mainly focused on some special structure problems, and the proposed methods for solving the problems are mostly applied to aim at some particular examples which are of special properties or structure. In 2010, replacing the lower level problem with its Kuhn-Tucker optimality condition, Pan et al. [6] transformed a class of nonlinear bilevel programming problems into normal nonlinear programming problems with the complementary slackness constraint condition, and introduced and studied a penalty function method to solve the problems. Through appending the duality gap of the lower level problem to the upper level objective with a penalty and obtaining a penalized problem, Lv [7] presented an exact penalty function method for finding solutions of a class of special nonlinear bilevel programs, i.e. the lower level problem is linear programs. Gupta et al. [8] provided a fuzzy goal programming approach to solve a multivariate stratified population problem which was turned out to be a non-linear bilevel programming problem.

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Very recently, based on definition of partial calmness for a single level optimization problem, Lü and Wan [9] constructed an exact penalized problem of a semi-vectorial bilevel programming problem by using the dual theory of linear programming. Based on approximate approach, Hosseini [10] attempted to develop an effective method for solving a nonlinear bilevel programming problem in virtue of transforming the nonlinear bilevel programming problem into a smooth single problem via using the Karush-Kuhn-Tucker conditions and Fischer-Burmeister functions. Hosseini and Kamalabadi [11] proposed a modified genetic algorithm combining particle swarm optimization using a heuristic function and constructed an effective hybrid approach, which is a fast approximate method for solving the non-linear bilevel programming problems. Based on a novel coding scheme, Li [12] developed a genetic algorithm with global convergence to solve a class of nonlinear bilevel programming problems where the follower is a linear fractional program. Moreover, Miao et al. [13] introduced and studied a bilevel genetic algorithms to solve a class of particular mixed integer nonlinear bilevel programming problems, which have been widely appeared in product family problems. Based on exact penalty function method, Di Pillo [14] proposed an efficient derivative-free unconstrained global minimization technique and proved that for every global minimum point, there exists a neighborhood of attraction for the local search under suitable assumptions. By using a simple exact penalty function method, Gao [15] studied an optimal control problem subject to the terminal state equality constraint and continuous inequality constraints on the control and the state. However, a general method to solve nonlinear bilevel programming problems has not yet been dealt with in the literature.

Motivated and inspired by the above works and this work is organized as follows: In Section 2, a class of nonlinear bilevel programming problems are equivalently transformed into corresponding single level nonlinear programming problems by using the Karush-Kuhn-Tucker optimality condition. Further, based on penalty function theory, we construct a smooth approximation method for obtaining optimal solutions of classic l_1-exact penalty function optimality problems. By using ϵ-approximate optimal solution theory, convergence of a simple ϵ-approximate optimal algorithm is proved in Section 3. In Section 4, by adding parameters in the constraint set of objective function, we discuss some perturbation convergence results for solving the nonlinear bilevel programming problems.

2 Smooth approximation method

In this section, by using penalty function theory and Karush-Kuhn-Tucker optimality condition, we shall construct a smooth approximation method for solving a class of nonlinear bilevel programming problems.

In this paper, we consider the following nonlinear bilevel programming problem:

$$\begin{aligned} &\min_{(x,y)\in R^{n+m}} f(x,y) \\ &\min_{(x,y)\in R^{n+m}} F(x,y) \\ &\quad s.t. \quad g_i(x,y) \le 0, \quad i=1,2,\cdots,l, \end{aligned} \tag{2.1}$$

where $f(x,y), F(x,y), g_i(x,y) : R^{n+m} \to R$ are continuously differentiable mappings for $i=1,2,\cdots,l$. By using Karush-Kuhn-Tucker optimality condition (see [16]), the lower level programming problem in (2.1) can be rewritten as follows:

$$\begin{aligned} &\nabla_y F(x,y) + \sum_{i=1}^{l} \lambda_i \nabla_y g_i(x,y) = 0, \\ &\sum_{i=1}^{l} \lambda_i g_i(x,y) = 0, \\ &g_i(x,y) \le 0, \quad i=1,2,\cdots,l \\ &\lambda_i \ge 0, \quad i=1,2,\cdots,l. \end{aligned}$$

Thus the problem (2.1) can be expressed as the following single level nonlinear programming problem:

$$\begin{aligned}
&\min_{(x,y)\in R^{n+m}} f(x,y) \\
&\quad s.t. \quad g_i(x,y) \le 0, \quad i=1,2,\cdots,l, \\
&\qquad \nabla_y F(x,y) + \sum_{i=1}^{l} \lambda_i \nabla_y g_i(x,y) = 0, \\
&\qquad \lambda_i g_i(x,y) = 0, \quad i=1,2,\cdots,l, \\
&\qquad -\lambda_i \le 0 \quad i=1,2,\cdots,l.
\end{aligned} \tag{2.2}$$

Let $z = (x, y, \lambda_1, \lambda_2, ...\lambda_l) \in R^{n+m+l}$. Then we have

$$\begin{aligned}
&h_1(z) := \nabla_y F(x,y) + \sum_{i=1}^{l} \lambda_i \nabla_y g_i(x,y) = 0, \\
&h_{1+i}(z) := \lambda_i g_i(x,y) = 0, \quad i=1,2,\cdots,l \\
&h_{1+l+i}(z) := g_i(x,y) \le 0, \quad i=1,2,\cdots,l \\
&h_{1+2l+i}(z) := -\lambda_i \le 0, \quad i=1,2,\cdots,l.
\end{aligned} \tag{2.3}$$

It follows from (2.3) that the problem (2.2) can be stated as

$$\begin{aligned}
&\min_{z\in R^{n+m+l}} f(z) \\
&\quad s.t. \quad h_i(z) = 0, \quad i=1,2,\cdots,1+l, \\
&\qquad\quad h_j(z) \le 0, \quad j=1,2,\cdots,2l,
\end{aligned} \tag{2.4}$$

where $f(z) : R^{n+m+l} \to R$ is a continuously differentiable mapping. Let $D = \{z | h_j(z) \le 0\}$ be the feasible set of the single level nonlinear programming problem (2.4). According to theory of the penalty function, we give the following l_1-exact penalty function programming problem:

$$\min_{(z,\mu)\in R^{n+m+l}\times R^+} l_1(z,\mu) = f(z) + \mu \sum_{j=1}^{1+3l} [h_j(z)]^+, \tag{2.5}$$

where μ is called a penalty factor and $[h_j(z)]^+ = \max\{0, h_j(z)\}$ for $j = 1, 2, \cdots, 1+3l$.

Now we prove that the problem (2.5) is equivalent to the problem (2.4).

Theorem 2.1 *Suppose that $(z^*, \mu) \in R^{n+m+l} \times R^+$ is optimal solution of the l_1-exact penalty function programming problem (2.5), where $R^+ = (0, +\infty)$ and μ is large enough. Then, z^* must be the optimal solution of the single level nonlinear programming problem (2.4).*

Proof. Let z_1^* be an optimal solution of the problem (2.4), and $(z_2^*, \mu_{z_2^*})$ be an optimal solution of the problem (2.5), where penalty parameter $\mu_{z_2^*} \in R^+$ must exist. Then we get

$$[h_j(z_1^*)]^+ = 0, \tag{2.6}$$

and

$$l_1(z_2^*, \mu_{z_2^*}) \le l_1(z_1^*, \mu_{z_2^*}). \tag{2.7}$$

By (2.7) and (2.5), now we know that

$$f(z_2^*) + \sum_{j=1}^{1+3l} \mu_{z_2^*} [h_j(z_2^*)]^+ \le f(z_1^*) + \sum_{j=1}^{1+3l} \mu_{z_2^*} [h_j(z_1^*)]^+,$$

and so it follows from (2.6) that

$$f(z_2^*) \le f(z_1^*). \tag{2.8}$$

If $z_2^* \in D$, then we have

$$f(z_1^*) \leq f(z_2^*). \tag{2.9}$$

Otherwise, there must exists a $j^{'} \in J$ such that $h_{j'}(z_2^*) > 0$ holds. Thus, we have $\mu h_{j'}(z_2^*) \to +\infty$ with $\mu \to +\infty$. Hence, z_2^* may not be an optimal solution of the single level nonlinear programming problem (2.4). Therefore, $z_2^* \in D$ must be satisfied.

Combining (2.8) and (2.9), we know that the result of Theorem 2.1 is right. It completes the proof. ■

Next, we establish a new smooth function for equivalently approximating the l_1-exact penalty function in (2.5).

Theorem 2.2 *Give the following programming problem:*

$$\min_{(z,\mu,r)\in R^{n+m+l}\times R^+\times R^+} L(z,\mu,r) = f(z) + \sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r, \tag{2.10}$$

where $\mu, r > 0$ are two parameters. Then smooth approximation of optimal solution for the L-exact penalty function programming problem (2.10) *is the optimal solution of the l_1-exact penalty function programming problem* (2.5) *as $r \to 0$.*

Proof. For all $j = 1, 2, \cdots, 3l$, if $h_j(z) \leq 0$, then

$$[h_j(z)]^+ = 0, \tag{2.11}$$

where $[h_j(z)]^+$ is the same as in (2.5). Further, letting $t = \frac{1}{r}$, then we have $t \to +\infty$ as $r \to 0^+$ and

$$\lim_{r\to 0^+} ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r = \lim_{t\to+\infty} \frac{ln\left[1+e^{t\mu h_j(z)}\right]}{t} = 0. \tag{2.12}$$

By (2.11) and (2.12), one can see that the optimal solution of the problem (2.10) is equivalent to the optimal solution of the problem (2.5) as $r \to 0^+$.

If $h_j(z) > 0$, then taking $r = \frac{1}{t}$, and we get

$$\begin{aligned} \lim_{r\to 0^+} ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r &= \lim_{t\to+\infty} \frac{ln\left[1+e^{t\mu h_j(z)}\right]}{t} \\ &= \mu h_j(z)\cdot \lim_{t\to+\infty}\left[1-\frac{1}{1+e^{t\mu h_j(z)}}\right] \\ &= \mu h_j(z) > 0. \end{aligned} \tag{2.13}$$

Thus, it follows from (2.13) that $ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r = \mu\,[h_j]^+$ as $r \to 0^+$, where $[h_j]^+$ is the same as in (2.5), and so $\lim_{r\to 0^+} L(z,\mu,r) = l_1(z,\mu)$.

From the above, it completes the proof. ■

3 ϵ-approximation algorithm

In this section, we shall construct an ϵ-approximation algorithm to solve the nonlinear bilevel programming problem (2.1) via using ϵ-approximate optimal solution theory.

Definition 3.1 *Let $\bar{z}$ be an optimal solution of the nonlinear bilevel programming problem* (2.1)*. Then a point z_0 is called ϵ-approximate optimal solution of the problem* (2.1)*, if for given constant $\epsilon > 0$, the following inequality holds:*

$$f(\bar{z}) - f(z_0) < \epsilon, \tag{3.1}$$

where $f(z)$ is defined as in (2.4) *for any $z \in R^{n+m+l}$.*

Lemma 3.2 *Let* $\varphi(z,\mu) = \lim_{r\to 0^+}\sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r$ *be a nonlinear function for all* $z \in R^{n+m+l}$ *and* $\mu \in R^+$, *and let* $(\bar{z},\bar{\mu})$ *be an optimal solution of the* l_1*-exact penalty function programming problem* (2.5) *with enough large* $\bar{\mu}$. *If there exists* $(z^*,\mu^*) \in R^{n+m+l}\times R^+$ *such that for each* $\epsilon > 0$,

$$\varphi(z^*,\mu^*) < \epsilon, \tag{3.2}$$

then z^* *must be an* ϵ*-approximate optimal solution of the nonlinear bilevel programming problem* (2.1).

Proof. Since $(\bar{z},\bar{\mu})$ is an optimal solution of the problem (2.5), we have

$$l_1(\bar{z},\bar{\mu}) \le l_1(z^*,\mu^*),$$

i.e.,

$$f(\bar{z}) + \bar{\mu}\sum_{j=1}^{1+3l}[h_j(\bar{z})]^+ \le f(z^*) + \mu^*\sum_{j=1}^{1+3l}[h_j(z^*]^+ . \tag{3.3}$$

By Theorem 2.1, we have

$$\bar{\mu}\sum_{j=1}^{1+3l}[h_j(\bar{z})]^+ = 0. \tag{3.4}$$

Thus, it follows from Theorem 2.2 and (3.2) that

$$\mu^*\sum_{j=1}^{1+3l}[h_j(z^*)]^+ = \lim_{r\to 0^+}\sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu^* h_j(z^*)}{r}}\right]^r = \varphi(z^*,\mu^*) < \epsilon. \tag{3.5}$$

Combining (3.4) and (3.5) into (3.3), we get

$$f(\bar{z}) - f(z^*) < \epsilon, \tag{3.6}$$

which implies that the point z^* is an ϵ-approximate optimal solution of the nonlinear bilevel programming problem (2.1). ■

By Lemma 3.2, now we propose the following ϵ-approximation algorithm.

Algorithm 3.3 Step 1. *Give a constant* $\epsilon > 0$, *initial points* $\mu^1 > 0$ *and* $r^1 \in (0, 0.01)$, *a positive integer* $N > 1$, $k := 1$.

Step 2. *Find optimal solution of the following smooth programming problem with the gradient descent method for* (μ^k, r^k), *and denote by* (z^k,μ^k,r^k):

$$\min_{(z,\mu,r)\in R^{n+m+l}\times R^+\times R^+} L(z,\mu^k,r^k) = f(z) + \sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu^k h_j(z)}{r^k}}\right]^{r^k}. \tag{3.7}$$

Step 3. *Let* $\bar{\varphi}(z,\mu,r) = \sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r$. *If the point* (z^k,μ^k,r^k) *satisfies*

$$\bar{\varphi}(z^k,\mu^k,r^k) - \bar{\varphi}(z,\mu^k,r^k) \le \epsilon, \quad \forall\, z \in D,$$

then stop. Otherwise, let $r^{k+1} = \left(r^k\right)^N$ *and* $\mu^{k+1} = N\mu^k$, $k := k+1$, *and go to* Step 2.

Theorem 3.4 *Assume that* $\{(z^k,\mu^k,r^k)\}$ *is a sequence generated by Algorithm* 3.3, *and the feasible region* $D = \{z|h_j(z) \le 0, j = 1, 2, ..., 1+3l\}$ *of the single level nonlinear programming problem* (2.4) *is nonempty. Then the following results hold:*

(i) *If* $z^k \in D$, *then*

$$L(z^k,\mu^k,r^k) \ge L(z^{k+1},\mu^{k+1},r^{k+1}).$$

(ii) *when* $z^k \notin D$, *we have*

$$\lim_{k\to\infty} L(z^k,\mu^k,r^k) \to +\infty.$$

Proof. By Algorithm 3.3, we know that z^k and z^{k+1} are the minimum points of the L-exact penalty function (3.7) with respect to (μ^k, r^k) and (μ^{k+1}, r^{k+1}), respectively. Thus, we have

$$\begin{aligned} L(z^{k+1}, \mu^{k+1}, r^{k+1}) \quad &= f(z^{k+1}) + \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1}h_j(z^{k+1})}{r^{k+1}}}\right]^{r^{k+1}} \\ &\le f(z^k) + \sum_{k=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1}h_j(z^k)}{r^{k+1}}}\right]^{r^{k+1}} \\ &= f(z^k) + r^{k+1}\sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1}h_j(z^k)}{r^{k+1}}}\right]. \end{aligned} \tag{3.8}$$

Let $\bar{\varphi}(z, \mu, r) = \sum_{j=1}^{1+3l} ln[1 + e^{\frac{\mu h_j(z)}{r}}]^r$. For $z \in D$, it follows that $-\frac{\mu h_j(z)}{r} > 0$ and

$$\frac{\partial \bar{\varphi}(z, \mu, r)}{\partial r} = \frac{\left[1 + e^{\frac{\mu h_j(z)}{r}}\right] ln\left[1 + e^{\frac{\mu h_j(z)}{r}}\right] - \frac{\mu h_j(z)}{r} e^{\frac{\mu h_j(z)}{r}}}{1 + e^{\frac{\mu h_j(z)}{r}}} > 0. \tag{3.9}$$

Further, if $z^k \in D$, then it follows from $r^{k+1} < r^k$, (3.9) and $\mu^{k+1} > \mu^k$ that for any $j = 1, 2, \cdots, 1 + 3l$, $h_j(z^k) < 0$, $\mu^{k+1}h_j(z^k) < \mu^k h_j(z^k)$ and

$$\begin{aligned} & f(z^k) + r^{k+1}\sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1}h_j(z^k)}{r^{k+1}}}\right] \\ & \le f(z^k) + r^k \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1}h_j(z^k)}{r^k}}\right] \\ & \le f(z^k) + r^k \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k}h_j(z^k)}{r^k}}\right] \\ & = L(z^k, \mu^k, r^k). \end{aligned} \tag{3.10}$$

Thus, by (3.8) and (3.10), we know that for $z^k \in D$,

$$L(z^{k+1}, \mu^{k+1}, r^{k+1}) \le L(z^k, \mu^k, r^k). \tag{3.11}$$

Moreover, if $z^k \notin D$, then there must exists a positive integer $j_a \in \{1, 2, ..., 1 + 3l\}$ such that $h_{j_a}(z^k) > 0$. It follows from Theorem 2.2 that $r^k \to 0$ and $\mu^k \to +\infty$ as $k \to \infty$, and

$$\begin{aligned} \lim_{k\to\infty} L(z^k, \mu^k, r^k) \quad &= \lim_{k\to\infty}\left\{ f(z^k) + \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}\right]^{r^k}\right\} \\ &= \lim_{k\to\infty} f(z^k) + \lim_{k\to\infty}\sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}\right]^{r^k} \\ &\ge \lim_{k\to\infty} f(z^k) + \lim_{k\to\infty}\left[\mu^k h_{j_a}(z^k)\right] \\ &= +\infty. \end{aligned} \tag{3.12}$$

It completes the proof. ■

From Theorem 3.4, we have the following result.

Theorem 3.5 *Let the feasible region of the single level nonlinear programming problem* (2.4) *denoted by* $D = \{z|h_j(z) \le 0\}$ *be nonempty. Let* $\{(z^k, \mu^k, r^k)\}$ *be a sequence generated by Algorithm* 3.3. *Then there must exists a subsequence of sequence* $\{(z^k, \mu^k, r^k)\}$ *to converge to an optimal solution of the nonlinear bilevel programming problem* (2.1).

Proof. Let $f(z) \geq 0$ always hold. Otherwise, let $f(z) := e^{f(z)} + 1$. Let $\{(z^{k_t}, \mu^{k_t}, r^{k_t})\}$ be a subsequence of the sequence $\{(z^k, \mu^k, r^k)\}$ with $z^{k_t} \in D$. Thus, from Theorem 3.4, it follows that $L(z^{k_t}, \mu^{k_t}, r^{k_t})$ is monotone and bounded. Let z^* be an optimal solution of the nonlinear bilevel programming problem (2.1). Since $r^k > 0$ and $ln[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}] > ln1 = 0$ for every $k \geq 1$ and $j = 1, 2, \cdots, 1 + 3l$, we have $r^k ln[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}] > 0$ and

$$\begin{aligned} L(z^{k_t}, \mu^{k_t}, r^{k_t}) \quad &= f(z^{k_t}) + \sum_{j=1}^{1+3l} r^{k_t} ln\left[1 + e^{\frac{\mu^{k_t} h_j(z^{k_t})}{r^{k_t}}}\right] \\ &> f(z^{k_t}) \geq f(z^*). \end{aligned} \tag{3.13}$$

From (2.12), we have for $r^{k_t} \to 0^+$ as $k_t \to +\infty$ and

$$\lim_{r^{k_t} \to 0^+} \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k_t} h_j(z^{k_t})}{r^{k_t}}}\right]^{r^{k_t}} = 0. \tag{3.14}$$

It follows from (i) of Theorem 3.4 that $L(z^{k_t}, \mu^{k_t}, r^{k_t})$ is monotone decreasing and bounded for all $z^{k_t} \in D$. Combining (3.13) and (3.14), we get

$$\begin{aligned} \lim_{k_t \to \infty} L(z^{k_t}, \mu^{k_t}, r^{k_t}) \quad &= \lim_{k_t \to \infty} f(z^{k_t}) + \lim_{r^{k_t} \to 0^+} \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k_t} h_j(z^{k_t})}{r^{k_t}}}\right]^{r^{k_t}} \\ &= \lim_{k_t \to \infty} f(z^{k_t}) = f(z^*). \end{aligned} \tag{3.15}$$

Thus, from the above, we have that $L(z^{k_t}, \mu^{k_t}, r^{k_t})$ and z^{k_t} converge to $f(z^*)$ and z^* as $k_t \to \infty$, respectively. Combining the equivalence relation between the single level nonlinear programming problem (2.4) and the nonlinear bilevel programming problem (2.1), it completes the proof. ■

4 Perturbation theorem

By adding parameters in the constraint set of objective function, we will discuss some perturbation convergence results for solving the nonlinear bilevel programming problems (2.1) in this section. Let Ω_α be a set defined by

$$\Omega_\alpha = \{z \in R^{n+m+l} | h_j(z) \leq \alpha\}, \tag{4.1}$$

where $\alpha \geq 0$. If $\alpha = 0$, we can obtain that Ω_0 is a feasible set of the single level nonlinear programming problem (2.4). Let $\phi_f(\alpha)$ be perturbation function of the single level nonlinear programming problem (2.4) defined as follows

$$\phi_f(\alpha) = \inf_{z \in \Omega_\alpha} f(z), \forall \alpha > 0, \tag{4.2}$$

where f is the same function as in (2.4). By (4.2), we know that $\phi_f(\alpha)$ is monotone decreasing at $\alpha > 0$, and so $\phi_f(\alpha)$ is a upper semi-continuous function at $\alpha = 0^+$. Denote

$$\phi_f(0) = \inf_{z \in \Omega_0} f(z), \tag{4.3}$$

and

$$\psi_f(0) = \min_{z \in \Omega_0} f(z). \tag{4.4}$$

It is easy to see that the optimization problem (4.4) is equivalent to the single level nonlinear programming problem (2.4).

Theorem 4.1 *If $\phi_f(\alpha)$ defined in (4.2) is a lower semi-continuous function at $\alpha = 0^+$, then (4.3) is equivalent to the nonlinear bilevel programming problem (2.1).*

Proof. From (4.1) and (4.2), it follows that $\phi_f(\alpha)$ is a upper semi-continuous function at $\alpha = 0^+$. If $\phi_f(\alpha)$ is also a lower semi-continuous function at $\alpha = 0^+$, then $\phi_f(\alpha)$ is continuous at $\alpha = 0^+$. Hence, $\phi_f(0) = \psi_f(0)$.

On the other hand, the optimization problem (4.4) is equivalent to the single level nonlinear programming problem (2.4). Combining the equivalence relation between the nonlinear bilevel programming problem (2.1) and the single level nonlinear programming problem (2.4), we know that (4.3) is equivalent to the original programming problem (2.1). It completes the proof. ■

Theorem 4.2 *Let $\{(z^k, \mu^k, r^k)\}$ be a sequence generated by Algorithm 3.3. Assume that feasible set $D = \{z^{R^{n+m+l}} | h_j(z) \leq 0\}$ of the single level nonlinear programming problem (2.4) is nonempty. Then, there must exists a subsequence $\{(z^{k_p}, \mu^{k_p}, r^{k_p})\}$ of the sequence $\{(z^k, \mu^k, r^k)\}$ such that for $z^{k_p} \in D$*

$$\lim_{k_p \to \infty} \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k_p} h_j(z^{k_p})}{r^{k_p}}} \right]^{r^{k_p}} = 0. \tag{4.5}$$

Proof. By (3.9), we know that for $z \in D$,

$$\frac{\partial \bar{\varphi}(z, \mu, r)}{\partial r} > 0. \tag{4.6}$$

Let $\{(z^{k_p}, \mu^{k_p}, r^{k_p})\}$ be a subsequence of the sequence $\{(z^k, \mu^k, r^k)\}$ generated by Algorithm 3.3 with $z^{k_p} \in D$. From (4.6), one can know that for each z^{k_p},

$$\begin{aligned} \bar{\varphi}(z^{k_p}, \mu^k, r^k) \quad &= \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^k h_j(z^{k_p})}{r^k}} \right]^{r^k} \\ &> \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k+1} h_j(z^{k_p})}{r^{k+1}}} \right]^{r^{k+1}} \\ &= \bar{\varphi}(z^{k_p}, \mu^{k+1}, r^{k+1}). \end{aligned} \tag{4.7}$$

Since $\bar{\varphi}(z^{k_p}, \mu^{k+1}, r^{k+1}) > 0$ holds invariably, it follows from (4.7) that

$$\lim_{k \to \infty} \bar{\varphi}(z^{k_p}, \mu^k, r^k) = 0. \tag{4.8}$$

Taking $\mu^k := \mu^{k_p}$ and $r^k := r^{k_p}$, then it follows from (4.8) that

$$\lim_{k_p \to \infty} \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) = 0,$$

and so

$$\lim_{k_p \to \infty} \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k_p}}{r^{k_p}}} \right]^{r^{k_p}} = 0.$$

It completes the proof. ■

Theorem 4.3 *If $\phi_f(\alpha)$ defined in (4.2) is a lower semi-continuous function at $\alpha = 0^+$, and a subsequence $\{z^{k_p}, \mu^{k_p}, r^{k_p}\}$ is the same as in Theorem 4.2, then z^{k_p} converges to an optimal solution of the nonlinear bilevel programming problem (2.1).*

Proof. If there exists a subsequence $\{z^{k_p}, \mu^{k_p}, r^{k_p}\}$ satisfying (4.5), then we know that for each $z \in D$,

$$\lim_{k_p \to \infty} \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) = \lim_{k_p \to \infty} \bar{\varphi}(z, \mu^{k_p}, r^{k_p}) = 0,$$

and so for any positive number ϵ, there exists a positive integer M such that when $k_p \geq M$, we have

$$\bar{\varphi}(z, \mu^{k_p}, r^{k_p}) - \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) \leq \epsilon. \tag{4.9}$$

By (3.7), we know for each $z \in D$

$$L(z^{k_p}, \mu^{k_p}, r^{k_p}) \leq L(z, \mu^{k_p}, r^{k_p}),$$

i.e.,

$$f(z^{k_p}) + \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) \leq f(z) + \bar{\varphi}(z, \mu^{k_p}, r^{k_p}). \quad (4.10)$$

Combining (4.9) into (4.10), we have for $k_p \geq M$,

$$\begin{aligned} f(z^{k_p}) \quad & \leq f(z) + \bar{\varphi}(z, \mu^{k_p}, r^{k_p}) - \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) \\ & \leq f(z) + \epsilon. \end{aligned} \quad (4.11)$$

If $\phi_f(\alpha)$ is a lower semi-continuous function at $\alpha = 0^+$, from Theorem 4.1, it follows that $\inf_{z \in D} f(z) = \phi_f(0)$. Let $f(z) = \phi_f(0)$. By (4.11), now we know that

$$f(z^{k_p}) \leq \phi_f(0) + \epsilon,$$

which implies

$$\phi_f(0) \leq f(z^{k_p}) \leq \phi_f(0) + \epsilon. \quad (4.12)$$

Thus, when $\epsilon \to 0$, it follows from (4.12) that there exists an accumulation $\hat{z}$ for the sequence $\{z^{k_p}\}$ such that $f(\hat{z}) = \phi_f(0)$. Hence, from Theorem 4.1, we know that z^{k_p} converges to an optimal solution of the nonlinear bilevel programming problem (2.1). ■

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APPROXIMATE n-JORDAN $*$-DERIVATIONS ON INDUCED FUZZY C^*-ALGEBRAS

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Abstract. Using the fixed point alternative theorem, we investigate the Hyers-Ulam stability of of n-Jordan $*$-derivations on induced fuzzy C^*-algebras associated with the following functional equation $f(x-y+z)+f(x-z)+f(2x+y)=f(4x)$.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [39] concerning the stability of group homomorphisms. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [34] for linear mappings by considering an unbounded Cauchy difference. Those results have been recently complemented in [7]. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [18], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 12, 13], [20]–[28], [35]–[37]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 1.1 (see [11, 15]). *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$, for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y,y^*) \leq \frac{1}{1-L} d(y,Jy)$ for all $y \in Y$.*

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By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [8, 10, 11, 9, 16, 25, 29, 30, 33, 42]).

In 1984, Katsaras [24] defined a fuzzy norm on a linear space and at the same year Wu and Fang [40] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [5], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [4, 17, 27, 38, 41]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [26]. In 2003, Bag and Samanta [4] modified the definition of Cheng and Mordeson [14] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:
(N_1) $N(x, a) = 0$ for $a \leq 0$;
(N_2) $x = 0$ if and only if $N(x, a) = 1$ for all $a > 0$;
(N_3) $N(ax, b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
(N_4) $N(x + y, a + b) \geq min\{N(x, a), N(y, b)\}$;
(N_5) $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x, a) = 1$;
(N_6) For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.
The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement the norm of x is less than or equal to the real number a'.

Definition 1.2. Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

Definition 1.3. A sequence x_n in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector space X, Y is continuous at point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [2])

Definition 1.4. [32] Let X be a $*$-algebra and (X, N) a fuzzy normed space.

(1) The fuzzy normed space (X, N) is called a fuzzy normed $*$-algebra if

$$N(xy, st) \geq N(x, s) \cdot N(y, t) \quad \text{and} \quad N(x^*, t) = N(x, t).$$

(2) A complete fuzzy normed $*$-algebra is called a *fuzzy Banach $*$-algebra*.

Example 1.5. *Let $(X, \|.\|)$ be a normed $*$-algebra. Let*

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0 \ , x \in X, \\ 0, & a \le 0, x \in X. \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on X and $(X, N(x,t))$ is a fuzzy normed $*$-algebra.

Definition 1.6. Let $(X, \| \cdot \|)$ be a C^*-algebra and N a fuzzy norm on X.

(1) The fuzzy normed $*$-algebra (X, N) is called an induced fuzzy normed $*$-algebra.
(2) The fuzzy Banach $*$-algebra (X, N) is called an induced fuzzy C^*-algebra.

Definition 1.7. Let $(X, \| \cdot \|)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a *fuzzy n-Jordan $*$-derivation* if

$$\begin{aligned} D(x^n) &= D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-2}D(x)x + x^{n-1}D(x), \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x \in X$.

Throughout this paper, assume that (X, N) is an induced fuzzy C^*-algebra.

2. Main results

Lemma 2.1. *Let (Z, N) be a fuzzy normed vector space and $f : X \to Z$ be a mapping such that*

$$N\left(f\left(x-y+z\right)+f\left(x-z\right)+f\left(2x+y\right),t\right) \ge N\left(f\left(4x\right), \frac{t}{2}\right) \tag{2.1}$$

for all $x, y, z \in X$ and all $t > 0$. Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \ge N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(0) = 0$.

Letting $x = y = 0$ in (2.1), we get

$$N(f(z) + f(-z) + f(0), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that $f(-z) + f(z) = 0$ for all $z \in X$. Thus

$$f(-z) = -f(z)$$

for all $z \in X$.

Letting $x = 0$ in (2.1), we get

$$N(f(z-y) + f(-z) + f(y), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that

$$f(y) + f(-z) + f(-y+z) = 0$$

for all $y, z \in X$. Thus

$$f(y+z) = f(y) + f(z)$$

for all $y, z \in X$, as desired. □

Theorem 2.2. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2}\phi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping such that

$$\begin{aligned} &N\left(f\left(\mu(x-y+z)\right) + f\left(\mu(x-z)\right) + f\left(\mu(2x+y)\right) - \mu f\left(4x\right), t\right) \\ &\geq \frac{t}{t+\phi(x,y,z)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &N\left(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w)\right. \\ &\left.+f(v^*) - f(v)^*, t\right) \geq \frac{t}{t+\phi(w,v,0)} \end{aligned} \tag{2.4}$$

for all $x, y, z, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1 := \{c \in \mathbb{C} : |c| = 1\}$. Then the limit $A(x) = N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - A(x), t) \geq \frac{2(1-L)t}{2(1-L)t + L\phi\left(x, 0, x\right)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $\mu = 1, y = 0$, $z = x$ in (2.3), we have

$$N\left(2f\left(x\right) - f(2x), t\right) \geq \frac{t}{t+\phi\left(\frac{x}{2}, 0, \frac{x}{2}\right)} \tag{2.6}$$

and so

$$N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t+\phi\left(\frac{x}{4}, 0, \frac{x}{4}\right)} \geq \frac{t}{t+\frac{L}{4}\phi\left(x, 0, x\right)}$$

for all $x \in X$. Thus

$$N\left(2f\left(\frac{x}{2}\right) - f(x), \frac{L}{4}t\right) \geq \frac{\frac{L}{4}t}{\frac{L}{4}t + \frac{L}{4}\phi\left(x, 0, x\right)} = \frac{t}{t+\phi\left(x, 0, x\right)} \tag{2.7}$$

for all $x \in X$.

Consider the set

$$G := \{g : X \to X\}$$

and introduce the generalized metric on G:

$$d(g, h) := \inf\{a \in \mathbb{R}^+ : N(g(x) - h(x), at) \geq \frac{t}{t+\phi\left(\frac{x}{2}, 0, \frac{x}{2}\right)}\}$$

for all $x \in X$ and all $t > 0$, where $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [**?**, Lemma 2.1]

Now, we consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in G$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\phi\left(x, 0, x\right)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Qg(x) - Qh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \phi\left(\frac{x}{2}, 0, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\phi\left(x, 0, x\right)} \\ &= \frac{t}{t + \phi\left(x, 0, x\right)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus $d(g,h) = \varepsilon$ implies that $d(Qg, Qh) \leq L\varepsilon$. This means that

$$d(Qg, Qh) \leq Ld(g,h)$$

for all $g, h \in G$.

It follows from (2.7) that $d(f, Qf) \leq \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $A : X \to X$ satisfying the following:

(1) A is a fixed point of Q, *i.e.*,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of Q in the set

$$M = \{g \in G : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists an $a \in (0, \infty)$ satisfying

$$N(f(x) - A(x), at) \geq \frac{t}{t + \phi\left(x, 0, x\right)}$$

for all $x \in X$.

(2) $d(Q^k f, A) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Qf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{4(1-L)}.$$

This implies that the inequality (2.5) holds.

Next we show that A is additive. It follows from (2.2) that

$$\begin{aligned} \sum_{k=0}^{\infty} 2^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right) &= \phi(x,y,z) + 2\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) + 2^2\phi\left(\frac{x}{2^2}, \frac{y}{2^2}, \frac{z}{2^2}\right) + \cdots \\ &\leq \phi(x,y,z) + L\phi(x,y,z) + L^2\phi(x,y,z) + \cdots \\ &= \frac{1}{1-L}\phi(x,y,z) < \infty \end{aligned}$$

for all $x, y, z \in X$.

By (2.3),

$$N\left(2^k f\left(\mu\frac{x-y+z}{2^k}\right)+2^k f\left(\mu\frac{x-z}{2^k}\right)+f\left(\mu\frac{2x+y}{2^k}\right)-2^k\mu f\left(\frac{4}{2^k}x\right),2^k t\right) \geq \frac{t}{t+\phi\left(\frac{x}{2^k},\frac{y}{2^k},\frac{z}{2^k}\right)}$$

and so

$$N\left(2^k f\left(\mu\frac{x-y+z}{2^k}\right)+2^k f\left(\mu\frac{x-z}{2^k}\right)+2^k f\left(\mu\frac{2x+y}{2^k}\right)-2^k\mu f\left(\frac{4}{2^k}x\right),t\right) \geq \frac{\frac{t}{2^k}}{\frac{t}{2^k}+\phi\left(\frac{x}{2^k},\frac{y}{2^k},\frac{z}{2^k}\right)}=\frac{t}{t+2^k\phi\left(\frac{x}{2^k}\cdot\frac{y}{2^k},\frac{z}{2^k}\right)}$$

for all $x,y,z\in X$, all $t>0$ and all $\mu\in\mathbb{T}^1$. Since $\lim_{k\to\infty}\frac{t}{t+2^k\phi\left(\frac{x}{2^k},\frac{y}{2^k},\frac{z}{2^k}\right)}=1$ for all $x,y,z\in X$ and all $t>0$,

$$N\left(A\left(\mu(x-y+z)\right)+A\left(\mu(x-z)\right)+A\left(\mu(2x+y)\right)-\mu A\left(4x\right),t\right)=1$$

for all $x,y,z\in X$, all $t>0$ and all $\mu\in\mathbb{T}^1$. So

$$A\left(\mu(x-y+z)\right)+A\left(\mu(x-z)\right)+A\left(\mu(2x+y)\right)=\mu A\left(4x\right) \tag{2.9}$$

for all $x,y,z\in X$, all $t>0$ and all $\mu\in\mathbb{T}^1$. Letting $x=y=z=0$ in (2.9), we have $A(0)=0$. Let $\mu=1$, $x=0$ in (2.9), by the same reasoning as in the proof of Lemma 2.1, one can easily show that A is additive. Letting $y=2x$, $z=0$ in (2.9), we get

$$\mu A(x)=2A\left(\mu\frac{x}{2}\right)=A(\mu x)$$

for all $x\in X$ and $\mu\in\mathbb{T}^1$. The mapping $A:X\to X$ is $\mathbb{C}$-linear by [31, Theorem 2.1].

By (2.4) and letting $v=0$ in (2.4), we get

$$N\left(2^{nk}f\left(\frac{w^n}{2^{nk}}\right)-2^{nk}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-1}-2^{nk}\frac{w}{2^k}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-2}-\cdots \right.$$
$$\left.-2^{nk}\left(\frac{w}{2^k}\right)^{n-2}f\left(\frac{w}{2^k}\right)w-2^{nk}\left(\frac{w}{2^k}\right)^{n-1}f\left(\frac{w}{2^k}\right),2^{nk}t\right)\geq\frac{t}{t+\phi(\frac{w}{2^k},0,0)}$$

for all $w\in X$ and all $t>0$. Thus

$$N\left(2^{nk}f\left(\frac{w^n}{2^{nk}}\right)-2^{nk}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-1}-2^{nk}\frac{w}{2^k}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-2}-\cdots \right.$$
$$\left.-2^{nk}\left(\frac{w}{2^k}\right)^{n-2}f\left(\frac{w}{2^k}\right)w-2^{nk}\left(\frac{w}{2^k}\right)^{n-1}f\left(\frac{w}{2^k}\right),t\right)\geq\frac{\frac{t}{2^{nk}}}{\frac{t}{2^{nk}}+\phi(\frac{w}{2^k},0,0)}$$
$$\geq\frac{t}{t+(2^{n-1}L)^k\phi(w,0,0)}$$

for all $w\in X$ and all $t>0$. Since $\lim_{k\to\infty}\frac{t}{t+(2^{n-1}L)^k\phi(w,0,0)}=1$ for all $w\in X$ and all $t>0$, we get

$$N(D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w),t)=1$$

for all $x\in X$ and all $t>0$. So

$$D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w)=0$$

for all $w \in X$.

Letting $w = 0$ in (2.4), similarly, we get $D(v^*) - D(v)^* = 0$ for all $v \in X$.

Therefore, the mapping $D : X \to X$ is a fuzzy n-Jordan $*$-derivation. □

Corollary 2.3. *Let p be a real number with $p > 1$, $\theta \geq 0$, and X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying*

$$\begin{aligned} &N\left(f\left(\mu(x-y+z)\right) + f\left(\mu(x-z)\right) + f\left(\mu(2x+y)\right) - \mu f\left(4x\right), t\right) \\ &\geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &N\left(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) \right. \\ &\left. + f(v^*) - f(v)^*, t\right) \geq \frac{t}{t + \theta(\|w\|^p + \|v\|^p)} \end{aligned} \tag{2.11}$$

for all $x, y, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Then the limit $A(x) = N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{1-p}$. □

Theorem 2.4. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$3L\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \phi(x, y, z)$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying (2.3) and (2.4). Then the limit $A(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - A(x), t) \geq \frac{2(1 - L)t}{2(1 - L)t + \phi\left(x, 0, x\right)} \tag{2.12}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (G, d) be a generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

It follow from (2.6) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \phi\left(x, 0, x\right)}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Qf) \leq \frac{1}{2}$. Hence

$$d(f, A) \leq \frac{1}{2(1 - L)},$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a positive real number with $p < 1$. Let X be a normed vector space with normed $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying* (2.10) *and* (2.11). *Then $A(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in X$ and defines a fuzzy n-Jordan $*$-derivation $A : X \to X$ such that*

$$N(f(x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{p-1}$. □

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RECURRENCE FORMULAS FOR EULERIAN POLYNOMIALS OF TYPE B AND TYPE D

DAN-DAN SU AND YUAN HE

ABSTRACT. We perform a further investigation for the Eulerian polynomials of type B and type D. By making use of the generating function methods and Padé approximation techniques, we establish some new recurrence formulas for the Eulerian polynomials of type B and type D. Some of these results presented here are the corresponding extensions of some known formulas.

1. Introduction

When computing values of the alternating ζ-function (also called Dirichlet eta function)

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots\quad(\mathrm{Re}(s)>0)\tag{1.1}$$

at negative integers, Leonhard Euler introduced the Eulerian polynomials $A_n(t)$ given by the following generating function

$$\frac{t-1}{t-e^{x(t-1)}}=\sum_{n=0}^{\infty}A_n(t)\frac{x^n}{n!},\tag{1.2}$$

and determined $\eta(-n)=2^{-n-1}A_n(-1)$ for positive integer n. It is interesting to point out that the Eulerian polynomials can be computed by the recurrence relation (see, e.g., [7])

$$A_0(t)=1,\quad A_n(t)=[1+(n-1)t]A_{n-1}(t)+t(1-t)\frac{\partial}{\partial t}(A_{n-1}(t))\quad(n\geq 1),\tag{1.3}$$

and some classical polynomials and numbers can be expressed by the Eulerian polynomials (see [15] for details). The Eulerian polynomials are also called the Eulerian polynomials of type A, and various combinatorial identities for them have been explored by many authors (see, e.g., [8, 10, 12, 13, 14, 15, 16, 20]). Perhaps the best known result is Leonhard Euler's recurrence formula (see, e.g., [7])

$$A_0(t)=1,\quad A_n(t)=\sum_{k=0}^{n-1}\binom{n}{k}A_k(t)(t-1)^{n-1-k}\quad(n\geq 1).\tag{1.4}$$

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We now turn to the Eulerian polynomials of type B and the Eulerian polynomials of type D, which are defined by means of the generating function (see, e.g., [3, 6])

$$\frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}=\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}, \tag{1.5}$$

and

$$\frac{(1-t)e^{x(1-t)}-xt(1-t)e^{2x(1-t)}}{1-te^{2x(1-t)}}=\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}, \tag{1.6}$$

respectively. Like the recurrence relation (1.3) of the Eulerian polynomials of type A, the Eulerian polynomials of type B satisfy the recurrence relation (see, e.g., [3])

$$B_0(t)=1,\quad B_n(t)=[1+(2n-1)t]B_{n-1}(t)+2(t-t^2)\frac{\partial}{\partial t}(B_{n-1}(t))\quad (n\geq 1), \tag{1.7}$$

and the Eulerian polynomials of type D obey the recurrence relation (see, e.g., [6]): $D_0(t)=D_1(t)=1$,

$$\begin{aligned} D_{n+2}(t) &= [n(1+5t)+4t]D_{n+1}(t)+4t(1-t)\frac{\partial}{\partial t}(D_{n+1}(t)) \\ &\quad +[(1-t)^2-n(1+3t)^2-4n(n-1)t(1+2t)]D_n(t) \\ &\quad -[4nt(1-t)(1+3t)+4t(1-t)^2]\frac{\partial}{\partial t}(D_n(t))-4t^2(1-t)^2\frac{\partial^2}{\partial^2 t}(D_n(t)) \\ &\quad +[2n(n-1)t(3+2t+3t^2)-4n(n-1)(n-2)t^2(1+t)]D_{n-1}(t) \\ &\quad +[2nt(1-t)^2(3+t)+8n(n-1)t^2(1-t)(1+t)]\frac{\partial}{\partial t}(D_{n-1}(t)) \\ &\quad +4nt^2(1-t)^2(1+t)\frac{\partial^2}{\partial^2 t}(D_{n-1}(t))\quad (n\geq 1). \end{aligned} \tag{1.8}$$

In the year 2016, Hyatt [11] discovered the corresponding recurrence formulas analogous to (1.4) for the Eulerian polynomials of type B and type D, as follows,

$$\begin{aligned} B_n(t) &= \sum_{k=0}^{n-1}\binom{n}{k}B_k(t)(t-1)^{n-1-k}+t^n\sum_{k=0}^{n-1}\binom{n}{k}B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k} \\ &= P_n(t)+t^nP_n(1/t)\quad (n\geq 1), \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} D_n(t) &= \sum_{k=0}^{n-1}\binom{n}{k}D_k(t)(t-1)^{n-1-k}+t^n\sum_{k=0}^{n-1}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k} \\ &= Q_n(t)+t^nQ_n(1/t)\quad (n\geq 2), \end{aligned} \tag{1.10}$$

say, and interpreted them combinatorially. It is worth mentioning that the polynomials $P_n(t)$, $t^nP_n(1/t)$, $Q_n(t)$, $t^nQ_n(1/t)$ were introduced by Savage and Visontai [21] and used to prove Brenti's [3] conjecture that the Eulerian polynomials of type D have only real roots. See also [11] for a further exploration for $P_n(t)$, $t^nP_n(1/t)$, $Q_n(t)$, $t^nQ_n(1/t)$.

Motivated and inspired by the work of Hyatt [11], in this paper we establish some new recurrence formulas for the Eulerian polynomials of type B and type D by making use of the generating function methods and Padé approximation techniques. Some of these results presented here are the corresponding extensions of Hyatt's recurrence formulas (1.9) and (1.10).

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This paper is organized as follows. In the second section, we recall Padé approximation to general series and their expression in the case of the exponential function. In the third section, we give some recurrence formulas for Eulerian polynomials of type B, and show the recurrence formula (1.9) is obtained as a special case. In the fourth section, we establish some recurrence formulas for Eulerian polynomials of type D, by virtue of which the recurrence formula (1.10) is deduced.

2. Padé approximants

It is well known that Padé approximants provide rational approximations to functions formally defined by a power series expansion, and have played important roles in many fields of mathematics, physics and engineering (see, e.g., [4, 17]). We here recall the definition of Padé approximation to general series and their expression in the case of the exponential function.

Let m, n be non-negative integers and let $\mathcal{P}_k$ be the set of all polynomials of degree $\leq k$. Given a function f with a Taylor expansion

$$f(t) = \sum_{k=0}^{\infty} c_k t^k \tag{2.1}$$

in a neighborhood of the origin, a Padé form of type (m, n) is a pair (P, Q) such that

$$P = \sum_{k=0}^{m} p_k t^k \in \mathcal{P}_m, \quad Q = \sum_{k=0}^{n} q_k t^k \in \mathcal{P}_n \quad (Q \not\equiv 0), \tag{2.2}$$

and

$$Qf - P = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \to 0. \tag{2.3}$$

It is clear that every Padé form of type (m, n) for $f(t)$ always exists and satisfies the same rational function, and the uniquely determined rational function P/Q is called the Padé approximant of type (m, n) for $f(t)$; see for example, [1, 5]. For nonnegative integers m, n, the Padé approximant of type (m, n) for the exponential function e^t is the unique rational function (see, e.g., [9, 18])

$$R_{m,n}(t) = \frac{P_m(t)}{Q_n(t)} \quad (P_m \in \mathcal{P}_m, Q_n \in \mathcal{P}_n, Q_n(0) = 1), \tag{2.4}$$

which obeys the property

$$e^t - R_{m,n}(t) = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \to 0. \tag{2.5}$$

In fact, the explicit formulas for P_m and Q_n can be expressed as follows (see, e.g., [2, 19]):

$$P_m(t) = \sum_{k=0}^{m} \frac{(m+n-k)! \cdot m!}{(m+n)! \cdot (m-k)!} \cdot \frac{t^k}{k!}, \tag{2.6}$$

$$Q_n(t) = \sum_{k=0}^{n} \frac{(m+n-k)! \cdot n!}{(m+n)! \cdot (n-k)!} \cdot \frac{(-t)^k}{k!}, \tag{2.7}$$

and

$$Q_n(t)e^t - P_m(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \int_0^1 x^n (1-x)^m e^{xt} \mathrm{d}x, \tag{2.8}$$

where $P_m(t)$ and $Q_n(t)$ is called the Padé numerator and denominator of type (m, n) for e^t, respectively.

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We shall use the above properties of Padé approximants to the exponential function to establish some new recurrence formulas for the Eulerian polynomials of type B and type D in next sections.

3. Recurrence formulas for Eulerian polynomials of type B

Let m, n be non-negative integers. It is easily seen that if we denote the right hand side of (2.8) by $S_{m,n}(t)$ then we have

$$e^t = \frac{P_m(t) + S_{m,n}(t)}{Q_n(t)}. \tag{3.1}$$

By multiplying the numerator and denominator in the left hand side of (1.5) by $e^{x(t-1)}$ and then respectively substituting $x(t-1)$ and $x(1-t)$ for t in (3.1), we discover

$$\Bigg(\frac{P_m(x(t-1)) + S_{m,n}(x(t-1))}{Q_n(x(t-1))} - t\frac{P_m(x(1-t)) + S_{m,n}(x(1-t))}{Q_n(x(1-t))}\Bigg)\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} = 1-t, \tag{3.2}$$

which means

$$\begin{aligned}&[P_m(x(t-1)) + S_{m,n}(x(t-1))]Q_n(x(1-t))\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}\\ &\quad - t[P_m(x(1-t)) + S_{m,n}(x(1-t))]Q_n(x(t-1))\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}\\ &\quad = (1-t)Q_n(x(t-1))Q_n(x(1-t)).\end{aligned} \tag{3.3}$$

We now apply the exponential series $e^{xt} = \sum_{k=0}^{\infty} x^k t^k/k!$ in the right hand side of (2.8). With the help of the beta function, we obtain

$$\begin{aligned}S_{m,n}(t) &= (-1)^n \frac{t^{m+n+1}}{(m+n)!}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 x^{n+k}(1-x)^m dx\\ &= \sum_{k=0}^{\infty}\frac{(-1)^n \cdot m! \cdot (n+k)!}{(m+n)! \cdot (m+n+k+1)!}\cdot\frac{t^{m+n+k+1}}{k!}.\end{aligned} \tag{3.4}$$

Let $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ be the coefficients of the polynomials $P_m(t), Q_n(t)$ and $S_{m,n}(t)$ given by

$$P_m(t) = \sum_{k=0}^{m} p_{m,n;k}t^k, \quad Q_n(t) = \sum_{k=0}^{n} q_{m,n;k}t^k, \quad S_{m,n}(t) = \sum_{k=0}^{\infty} s_{m,n;k}t^{m+n+k+1}. \tag{3.5}$$

It follows from (2.6), (2.7) and (3.4) that

$$p_{m,n;k} = \frac{m! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (m-k)!}, \quad q_{m,n;k} = \frac{(-1)^k \cdot n! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (n-k)!}, \tag{3.6}$$

and

$$s_{m,n;k} = \frac{(-1)^n \cdot m! \cdot (n+k)!}{k! \cdot (m+n)! \cdot (m+n+k+1)!}. \tag{3.7}$$

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If we apply (3.5) to (3.3) then we get

$$\begin{aligned}
&\Big(\sum_{i=0}^{m} p_{m,n;i}x^i(t-1)^i+\sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(t-1)^{m+n+i+1}\Big)\\
&\quad\times\sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j\sum_{k=0}^{\infty}B_k(t)\frac{x^k}{k!}\\
&-t\Big(\sum_{i=0}^{m} p_{m,n;i}x^i(1-t)^i+\sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(1-t)^{m+n+i+1}\Big)\\
&\quad\times\sum_{j=0}^{n} q_{m,n;j}x^j(t-1)^j\sum_{k=0}^{\infty}B_k(t)\frac{x^k}{k!}\\
&=(1-t)\Big(\sum_{i=0}^{n} q_{m,n;i}x^i(t-1)^i\Big)\Big(\sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j\Big),
\end{aligned}\tag{3.8}$$

which together with the Cauchy product yields

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}x^l\\
&+\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\ge 0}} s_{m,n;i}(t-1)^{m+n+i+1} q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\ge 0}} s_{m,n;i}(1-t)^{m+n+i+1} q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}x^l\\
&=(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l\\ i,j\ge 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j x^l.
\end{aligned}\tag{3.9}$$

Comparing the coefficients of x^l in (3.9) gives that for non-negative integer l with $0\le l\le m+n$,

$$\begin{aligned}
&\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}\\
&-t\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}\\
&=(1-t)\sum_{\substack{i+j=l\\ i,j\ge 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j.
\end{aligned}\tag{3.10}$$

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Observe that

$$\frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}=\frac{(1-\frac{1}{t})e^{xt(1-\frac{1}{t})}}{1-\frac{1}{t}e^{2xt(1-\frac{1}{t})}}. \tag{3.11}$$

It follows from (1.5) and (3.11) that

$$B_n(t)=t^nB_n\left(\frac{1}{t}\right)\quad (n\geq 0). \tag{3.12}$$

By applying (3.12) to (3.10), we get

$$\begin{aligned}&\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{B_k(t)}{k!}\\ &-t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\ &=-(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\geq 0}}(-1)^j q_{m,n;i}q_{m,n;j}.\end{aligned} \tag{3.13}$$

Thus, applying (3.6) to (3.13) gives the following result.

Theorem 3.1. *Let m,n be non-negative integers. Then, for non-negative integer l with $0\leq l\leq m+n$,*

$$\begin{aligned}&\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{B_k(t)}{k!}\\ &\quad -t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\ &\quad\quad =-(t-1)^{l+1}\sum_{i=0}^{l}\binom{n}{i}\binom{n}{l-i}(-1)^i(m+n-i)!\cdot(m+n+i-l)!.\end{aligned} \tag{3.14}$$

We next discuss some special cases of Theorem 3.1. By taking $l=m+n$ in Theorem 3.1, we have

Corollary 3.2. *Let m,n be non-negative integers. Then*

$$\begin{aligned}&\sum_{\substack{i+j+k=m+n\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{B_k(t)}{k!}\\ &-t^{m+n+1}\sum_{\substack{i+j+k=m+n\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\ &\quad =-(t-1)^{m+n+1}\sum_{i=0}^{m+n}\binom{n}{i}\binom{n}{m+n-i}(-1)^i(m+n-i)!\cdot i!.\end{aligned} \tag{3.15}$$

If we take $n=0$ in Theorem 3.1 then we have

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Corollary 3.3. *Let m be non-negative integer. Then, for non-negative integer l with $0 \le l \le m$,*

$$\sum_{\substack{i+k=l\\ i,k\ge 0}} \binom{m}{i}(m-i)!\cdot(t-1)^i\frac{B_k(t)}{k!} - t^{l+1}\sum_{\substack{i+k=l\\ i,k\ge 0}} \binom{m}{i}(m-i)!\cdot\left(\frac{1}{t}-1\right)^i\frac{B_k(\frac{1}{t})}{k!}$$
$$= -(t-1)^{l+1}\binom{0}{l}\cdot(m-l)!. \quad (3.16)$$

In particular, by taking $l=m$ and substituting n for m in Corollary 3.3, we have

Corollary 3.4. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}B_k(t)(t-1)^{n-k} = t^{n+1}\sum_{k=0}^{n}\binom{n}{k}B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-k}. \quad (3.17)$$

The above Corollary 3.4 can be easily used to give Hyatt's recurrence formula (1.9). For example, by multiplying the both sides of (3.17) by $1/(t-1)$, we get that for positive integer n,

$$\sum_{k=0}^{n}\binom{n}{k}B_k(t)(t-1)^{n-1-k} = -t^{n}\sum_{k=0}^{n}\binom{n}{k}B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k}, \quad (3.18)$$

which means

$$\frac{B_n(t)}{t-1} + \sum_{k=0}^{n-1}\binom{n}{k}B_k(t)(t-1)^{n-1-k}$$
$$= -t^{n}\sum_{k=0}^{n-1}\binom{n}{k}B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k} + \frac{t^{n+1}}{t-1}B_n\left(\frac{1}{t}\right). \quad (3.19)$$

Hence, applying (3.12) to (3.19) gives the recurrence formula (1.9) immediately.

We next consider the case l being a positive integer with $l \ge m+n+1$ in (3.9). By comparing the coefficients of x^l in (3.9), we obtain

$$\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}$$
$$+ \sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\ge 0}} s_{m,n;i}(t-1)^{m+n+i+1}q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}$$
$$-t\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}$$
$$-t\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\ge 0}} s_{m,n;i}(1-t)^{m+n+i+1}q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}$$
$$= (1-t)\sum_{\substack{i+j=l\\ i,j\ge 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j, \quad (3.20)$$

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which together (3.12) gives

$$\begin{aligned}
&\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{B_k(t)}{k!}\\
&+(t-1)^{m+n+1}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}}(-1)^j s_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{B_k(t)}{k!}\\
&-t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\
&-t^{l+1}\left(\frac{1}{t}-1\right)^{m+n+1}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}}(-1)^j s_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\
&=-(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\geq 0}}(-1)^j q_{m,n;i}q_{m,n;j}. \qquad (3.21)
\end{aligned}$$

Thus, by taking $l=m+n+1$ and applying (3.6) and (3.7) to (3.21), in view of $B_0(t)=1$, we get the following result.

Theorem 3.5. *Let m,n be non-negative integers with $m\geq n$. Then*

$$\begin{aligned}
&\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{B_k(t)}{k!}\\
&-t^{m+n+2}\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\
&=-[(-1)^m t+(-1)^n](t-1)^{m+n+1}\frac{m!\cdot n!}{m+n+1}. \qquad (3.22)
\end{aligned}$$

If we take $n=0$ and substitute n for m in Theorem 3.5, we have

Corollary 3.6. *Let n be a non-negative integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-k}-t^{n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-k}\\
&=-[(-1)^n t+1]\frac{(t-1)^{n+1}}{n+1}. \qquad (3.23)
\end{aligned}$$

If we multiply the both sides of (3.23) by $1/(t-1)$, we get that for non-negative integer n,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}\\
&=-[(-1)^n t+1]\frac{(t-1)^{n}}{n+1}, \qquad (3.24)
\end{aligned}$$

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which implies

$$\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}$$
$$=\frac{t^{n+2}}{t-1}\cdot\frac{B_{n+1}(\frac{1}{t})}{n+1}-\frac{1}{t-1}\cdot\frac{B_{n+1}(t)}{n+1}-[(-1)^n t+1]\frac{(t-1)^n}{n+1}. \quad (3.25)$$

It follows from (3.12) and (3.25) that

$$\frac{B_{n+1}(t)}{n+1}=\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-1-k}$$
$$+t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}$$
$$+[(-1)^n t+1]\frac{(t-1)^n}{n+1} \quad (n\geq 1), \quad (3.26)$$

which can be regarded as an analogous version to Hyatt's recurrence formula (1.9).

4. Recurrence formulas for Eulerian polynomials of type D

We now multiply the numerator and denominator in the left hand side of (1.6) by $e^{x(t-1)}$, we have

$$\frac{(1-t)-xt(1-t)e^{x(1-t)}}{e^{x(t-1)}-te^{x(1-t)}}=\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}, \quad (4.1)$$

which together with (3.1) gives

$$\left(\frac{P_m(x(t-1))+S_{m,n}(x(t-1))}{Q_n(x(t-1))}\right.$$
$$\left.-t\frac{P_m(x(1-t))+S_{m,n}(x(1-t))}{Q_n(x(1-t))}\right)\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}$$
$$=(1-t)-xt(1-t)\frac{P_m(x(1-t))+S_{m,n}(x(1-t))}{Q_n(x(1-t))}. \quad (4.2)$$

It is obvious that (4.2) can be rewritten as

$$[P_m(x(t-1))+S_{m,n}(x(t-1))]Q_n(x(1-t))\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}$$
$$-t[P_m(x(1-t))+S_{m,n}(x(1-t))]Q_n(x(t-1))\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}$$
$$=(1-t)Q_n(x(t-1))Q_n(x(1-t))$$
$$-xt(1-t)[P_m(x(1-t))+S_{m,n}(x(1-t))]Q_n(x(t-1)). \quad (4.3)$$

If we apply (3.5) to (4.3) then we have

$$\begin{aligned}
&\Bigg(\sum_{i=0}^{m} p_{m,n;i}x^i(t-1)^i+\sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(t-1)^{m+n+i+1}\Bigg)\\
&\quad\times\sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j\sum_{k=0}^{\infty} D_k(t)\frac{x^k}{k!}\\
&-t\Bigg(\sum_{i=0}^{m} p_{m,n;i}x^i(1-t)^i+\sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(1-t)^{m+n+i+1}\Bigg)\\
&\quad\times\sum_{j=0}^{n} q_{m,n;j}x^j(t-1)^j\sum_{k=0}^{\infty} D_k(t)\frac{x^k}{k!}\\
&=(1-t)\Bigg(\sum_{i=0}^{n} q_{m,n;i}x^i(t-1)^i\Bigg)\Bigg(\sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j\Bigg)\\
&\quad-xt(1-t)\Bigg(\sum_{i=0}^{m} p_{m,n;i}x^i(1-t)^i+\sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(1-t)^{m+n+i+1}\Bigg)\\
&\quad\times\sum_{j=0}^{n} q_{m,n;j}x^j(t-1)^j. \qquad (4.4)
\end{aligned}$$

It follows from (4.4) and the Cauchy product that

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{D_k(t)}{k!}x^l\\
&+\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(t-1)^{m+n+i+1} q_{m,n;j}(1-t)^j\frac{D_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{D_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(1-t)^{m+n+i+1} q_{m,n;j}(t-1)^j\frac{D_k(t)}{k!}x^l\\
&=(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l\\ i,j\geq 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j x^l\\
&\quad-t(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l-1\\ i,j\geq 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j x^l\\
&\quad-t(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l-m-n-2\\ i,j,k\geq 0}} s_{m,n;i}(1-t)^{m+n+i+1} q_{m,n;j}(t-1)^j x^l. \qquad (4.5)
\end{aligned}$$

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By comparing the coefficients of x^l in (4.5), we get that for non-negative integer l with $0 \le l \le m+n$,

$$\begin{aligned}&\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\,\frac{D_k(t)}{k!}\\ &-t\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\,\frac{D_k(t)}{k!}\\ &=(1-t)\sum_{\substack{i+j=l\\ i,j\ge 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\\ &\quad -t(1-t)\sum_{\substack{i+j=l-1\\ i,j\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j. \end{aligned} \tag{4.6}$$

Observe that

$$\frac{(1-t)e^{x(1-t)}-xt(1-t)e^{2x(1-t)}}{1-te^{2x(1-t)}}=\frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}-xt\frac{t-1}{t-e^{2x(t-1)}}. \tag{4.7}$$

Applying (1.2), (1.5) and (1.6) to (4.7) gives

$$D_n(t)=B_n(t)-n2^{n-1}tA_{n-1}(t)\quad (n\ge 0). \tag{4.8}$$

It follows from (3.12) and (4.8) that

$$D_n(t)=t^nD_n\left(\frac{1}{t}\right)+n2^{n-1}t^{n-1}A_{n-1}\left(\frac{1}{t}\right)-n2^{n-1}tA_{n-1}(t)\quad (n\ge 0). \tag{4.9}$$

Hence, in light of (4.9), we can rewrite (4.6) as

$$\begin{aligned}&\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}(t-1)^{i+j}\,\frac{D_k(t)}{k!}\\ &-t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{D_k(\frac{1}{t})}{k!}\\ &=t\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}(1-t)^{i+j}\,\frac{k2^{k-1}(t^{k-1}A_{k-1}(\frac{1}{t})-tA_{k-1}(t))}{k!}\\ &\quad -(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\ge 0}} (-1)^j q_{m,n;i}q_{m,n;j}-t(1-t)^l\sum_{\substack{i+j=l-1\\ i,j\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}\\ &=t\sum_{\substack{i+j+k=l-1\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}(1-t)^{i+j}\,\frac{2^k(t^kA_k(\frac{1}{t})-tA_k(t))}{k!}\\ &\quad -(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\ge 0}} (-1)^j q_{m,n;i}q_{m,n;j}\\ &\quad -t(1-t)^l\sum_{\substack{i+j=l-1\\ i,j\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}. \end{aligned} \tag{4.10}$$

Noticing that from (1.2) we have

$$\begin{aligned} 1-t &= \frac{(1-t)e^{x(t-1)}}{e^{x(t-1)}-te^{x(1-t)}} - \frac{t(1-t)e^{x(1-t)}}{e^{x(t-1)}-te^{x(1-t)}} \\ &= \sum_{n=0}^{\infty}\left[(2t)^n A_n\left(\frac{1}{t}\right) - 2^n t A_n(t)\right]\frac{x^n}{n!}, \end{aligned} \tag{4.11}$$

which implies

$$2^0\left(t^0 A_0\left(\frac{1}{t}\right) - tA_0(t)\right) = 1-t, \quad 2^n\left(t^n A_n\left(\frac{1}{t}\right) - tA_n(t)\right) = 0 \quad (n \geq 1). \tag{4.12}$$

So from (4.10) and (4.12), we obtain

$$\begin{aligned} &\sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} (-1)^j p_{m,n;i} q_{m,n;j} (t-1)^{i+j} \frac{D_k(t)}{k!} \\ &-t^{l+1} \sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} (-1)^j p_{m,n;i} q_{m,n;j} \left(\frac{1}{t}-1\right)^{i+j} \frac{D_k(\frac{1}{t})}{k!} \\ &= -(t-1)^{l+1} \sum_{\substack{i+j=l \\ i,j\geq 0}} (-1)^j q_{m,n;i} q_{m,n;j}. \end{aligned} \tag{4.13}$$

Thus, applying (3.6) to (4.13) gives the following result.

Theorem 4.1. *Let m, n be non-negative integers. Then, for non-negative integer l with $0 \leq l \leq m+n$,*

$$\begin{aligned} &\sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j} (m+n-i)! \cdot (m+n-j)! \cdot (t-1)^{i+j} \frac{D_k(t)}{k!} \\ &- t^{l+1} \sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j} (m+n-i)! \cdot (m+n-j)! \cdot \left(\frac{1}{t}-1\right)^{i+j} \frac{D_k(\frac{1}{t})}{k!} \\ &= -(t-1)^{l+1} \sum_{i=0}^{l} \binom{n}{i}\binom{n}{l-i} (-1)^i (m+n-i)! \cdot (m+n+i-l)!. \end{aligned} \tag{4.14}$$

It becomes obvious that taking $l = m+n$ in Theorem 4.1 gives the following result.

Corollary 4.2. *Let m, n be non-negative integers. Then*

$$\begin{aligned} &\sum_{\substack{i+j+k=m+n \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j} (m+n-i)! \cdot (m+n-j)! \cdot (t-1)^{i+j} \frac{D_k(t)}{k!} \\ &- t^{m+n+1} \sum_{\substack{i+j+k=m+n \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j} (m+n-i)! \cdot (m+n-j)! \cdot \left(\frac{1}{t}-1\right)^{i+j} \frac{D_k(\frac{1}{t})}{k!} \\ &= -(t-1)^{m+n+1} \sum_{i=0}^{m+n} \binom{n}{i}\binom{n}{m+n-i} (-1)^i (m+n-i)! \cdot i!. \end{aligned} \tag{4.15}$$

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If we take $n=0$ in Theorem 4.1 then we have

Corollary 4.3. *Let m be non-negative integer. Then, for non-negative integer l with $0\le l\le m$,*

$$\sum_{\substack{i+k=l\\ i,k\ge 0}}\binom{m}{i}(m-i)!\cdot(t-1)^i\frac{D_k(t)}{k!}-t^{l+1}\sum_{\substack{i+k=l\\ i,k\ge 0}}\binom{m}{i}(m-i)!\cdot\left(\frac{1}{t}-1\right)^i\frac{D_k(\frac{1}{t})}{k!}$$
$$=-(t-1)^{l+1}\binom{0}{l}\cdot(m-l)!. \quad (4.16)$$

In particular, by taking $l=m$ and substituting n for m in Corollary 4.3, we have

Corollary 4.4. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}D_k(t)(t-1)^{n-k}=t^{n+1}\sum_{k=0}^{n}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-k}. \quad (4.17)$$

We now use Corollary 4.4 to give Hyatt's recurrence formula (1.10). By multiplying the both sides of (4.17) by $1/(t-1)$, we get that for positive integer n,

$$\sum_{k=0}^{n}\binom{n}{k}D_k(t)(t-1)^{n-1-k}=-t^{n}\sum_{k=0}^{n}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k}, \quad (4.18)$$

which is equivalent to

$$\frac{D_n(t)}{t-1}+\sum_{k=0}^{n-1}\binom{n}{k}D_k(t)(t-1)^{n-1-k}$$
$$=-t^{n}\sum_{k=0}^{n-1}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k}+\frac{t^{n+1}}{t-1}D_n\left(\frac{1}{t}\right). \quad (4.19)$$

Noticing that from (4.9) and (4.12) we have

$$D_n(t)=t^nD_n\left(\frac{1}{t}\right)\quad (n\ge 2). \quad (4.20)$$

Hence, applying (4.20) to (4.19) gives Hyatt's recurrence formula (1.10) immediately.

We next consider the case $l=m+n+1$ in (4.5). By taking $l=m+n+1$ in (4.5), in view of $D_0(t)=1$, we discover

$$\sum_{\substack{i+j+k=m+n+1\\ i,j,k\ge 0}}p_{m,n;i}(t-1)^iq_{m,n;j}(1-t)^j\frac{D_k(t)}{k!}+(t-1)^{m+n+1}s_{m,n;0}q_{m,n;0}$$
$$-t\sum_{\substack{i+j+k=m+n+1\\ i,j,k\ge 0}}p_{m,n;i}(1-t)^iq_{m,n;j}(t-1)^j\frac{D_k(t)}{k!}-t(1-t)^{m+n+1}s_{m,n;0}q_{m,n;0}$$
$$=(1-t)\sum_{\substack{i+j=m+n+1\\ i,j\ge 0}}q_{m,n;i}(t-1)^iq_{m,n;j}(1-t)^j$$
$$-t(1-t)\sum_{\substack{i+j=m+n\\ i,j\ge 0}}p_{m,n;i}(1-t)^iq_{m,n;j}(t-1)^j. \quad (4.21)$$

So if we apply (4.9) to (4.21), in light of (4.12), we get

$$\begin{aligned}
&\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{D_k(t)}{k!}\\
&-t^{m+n+2}\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{D_k(\frac{1}{t})}{k!}\\
&=-(t-1)^{m+n+2}\sum_{\substack{i+j=m+n+1\\ i,j\geq 0}}(-1)^j q_{m,n;i}q_{m,n;j}\\
&\quad+t(1-t)^{m+n+1}s_{m,n;0}q_{m,n;0}-(t-1)^{m+n+1}s_{m,n;0}q_{m,n;0}. \qquad (4.22)
\end{aligned}$$

Thus, applying (3.6) and (3.7) to (4.22) gives the following result.

Theorem 4.5. *Let m,n be non-negative integers with $m\geq n$. Then*

$$\begin{aligned}
&\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{D_k(t)}{k!}\\
&-t^{m+n+2}\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{D_k(\frac{1}{t})}{k!}\\
&=-[(-1)^m t+(-1)^n](t-1)^{m+n+1}\frac{m!\cdot n!}{m+n+1}. \qquad (4.23)
\end{aligned}$$

If we take $n=0$ and substitute n for m in Theorem 4.5, we have

Corollary 4.6. *Let n be a non-negative integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{D_{k+1}(t)}{k+1}(t-1)^{n-k}-t^{n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{D_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-k}\\
&=-[(-1)^n t+1]\frac{(t-1)^{n+1}}{n+1}. \qquad (4.24)
\end{aligned}$$

We now multiply the both sides of (4.24) by $1/(t-1)$ to obtain that for non-negative integer n,

$$\begin{aligned}
&\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}\\
&=\frac{t^{n+2}}{t-1}\cdot\frac{D_{n+1}(\frac{1}{t})}{n+1}-\frac{1}{t-1}\cdot\frac{D_{n+1}(t)}{n+1}-[(-1)^n t+1]\frac{(t-1)^n}{n+1}. \qquad (4.25)
\end{aligned}$$

It follows from (4.20) and (4.25) that

$$\begin{aligned}
\frac{D_{n+1}(t)}{n+1}&=\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}\\
&\quad+[(-1)^n t+1]\frac{(t-1)^n}{n+1}\quad (n\geq 1), \qquad (4.26)
\end{aligned}$$

which is very analogous to Hyatt's recurrence formula (1.10).

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CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH THE GENERALIZED MEIXNER-POLLACZEK POLYNOMIALS

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ABSTRACT. In the present paper, we introduce and investigate two new subclasses of the function class Σ of bi-univalent functions of complex order defined in the open unit disk, which are associated with the one of the orthogonal polynomial namely Generalized Meixner-Pollaczek polynomials, and satisfying subordinate conditions. Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$ were estimated for functions in new subclass. Furthermore, several known consequences are also investigated.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}$ be the class of functions which are subclass of $\mathcal{A}$ and is univalent in $\mathbb{U}$. Some of the essential and well-scrutinized subclasses of the class $\mathcal{S}$ include, for example, the class $\mathcal{S}^*(\alpha)$ of *starlike functions of order* α in $\mathbb{U}$, and the class $\mathcal{K}(\alpha)$ of *convex functions of order* α in $\mathbb{U}$, with $0 \leq \alpha < 1$.

It is prominent that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

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where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $f(z)$ and $f^{-1}(w)$ are univalent in $\mathbb{U}$, and let Σ denote the class of *bi-univalent functions* in $\mathbb{U}$.

The *convolution* or *Hadamard product* of two function $f, h \in \mathcal{A}$ is denoted by $f * h$, and is defined by

$$(f * h)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where f is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

For complex numbers α_i $(i = 1, 2, \ldots, p)$ and β_j $(j = 1, 2, \ldots, q)$ where $\beta_j \neq 0, -1, -2, \ldots;$ $j = 1, 2, \ldots, q$, the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$${}_pF_q(z) = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.3}$$

where $p \leq q + 1$,

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(n)} = \begin{cases} 1 & \text{if } n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & \text{if } n \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

The series given by (1.3) converges absolutely for $|z| < \infty$ if $p < q + 1$ and for z in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ if $p = q + 1$. For relevant values α_i and β_j, the class of hypergeometric functions ${}_pF_q$ is proximately cognate to classes of analytic and univalent functions. It is well-known that hypergeometric and univalent functions play significant roles in a large variety of problems undergone in applied mathematics, probability and statistics, operations research, signal theory, moment problems, and other areas of science (e.g., see Exton [6, 7], Miller and Mocanu [16] and Rönning [23]). In this sequel, we construct a new pathway for studying the connection between classes of hypergeometric and analytic univalent functions and also derive some new bounds for their respected *Fekete-Szegö* coefficients.

2. PRELIMINARIES

For $p = q + 1 = 2$, the series defined by (1.3) gives rise to the Gaussian hypergeometric series ${}_2F_1(a, b; c; z)$. This reduces to the elementary Gaussian geometric series $1 + z + z^2 + \cdots$ if (i)

$a = c$ and $b = 1$ or (ii) $a = 1$ and $b = c$. For $\Re(c) > \Re(b) > 0$, we obtain

$$_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt.$$

As a special case, we observe that

$$_2F_1(1, 1; a; z) = (a-1) \int_0^1 \frac{t^{b-1}(1-t)^{a-2}}{1-tz} dt$$

and

$$_2F_1(a, 1; 1; z) = \frac{1}{(1-z)^a}$$

so that

$$_2F_1(a, 1; 1; z) * {}_2F_1(a, 1; 1; z) = \frac{1}{1-z} = {}_2F_1(1, 1; 1; z).$$

The classical Koebe function is a function holomorphic in $\mathbb{U} = \{z \in \mathbb{C} : |z|< 1\}$ and given by the formula

$$k_2(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left\{\left(\frac{1+z}{1-z}\right)^2 - 1\right\} = z + 2z^2 + 3z^3 + \dots, \quad z \in \mathbb{U}.$$

The important function $k_2(z)$ follows from extremality for the famous Bieberbach conjecture. The Koebe function is univalent and starlike in $\mathbb{U}$ and maps the unit disk $\mathbb{U}$ onto the complex plane minus a slit $\left(-\infty, -\frac{1}{4}\right]$.

Certain generalizations of k_2 were appeared in the literature. Robertson [22] proved that

$$k_{2(1-\alpha)}(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (0 \leq \alpha < 1)$$

is the extremal function for the functions starlike of order α. The function

$$k_\alpha(z) = \frac{1}{2\alpha}\left\{\left(\frac{1+z}{1-z}\right)^\alpha - 1\right\} \quad (\alpha \in \mathbb{R}\backslash\{0\},\ z \in \mathbb{U})$$

was widely studied by Pommerenke [21], who investigated a universal invariant family $\mathcal{U}_\alpha$.
The definition of k_α was extended for a non-zero complex number α by Yamashita [27]. From the classical result of Hille [11], we see that k_α is univalent in $\mathbb{U}$ if and only if $\alpha \neq 0$ is the union A of the closed disks $\{|z+1|\leq 1\}$ and $\{|z-1|\leq 1\}$. Making use of the geometric properties, Yamashita [27] described how k_α tends to be univalent in the whole $\mathbb{U}$ as α tends to each boundary point of A from outside.
On the other hand, The properties of $\log k_c'$, where

$$k_c(z) = \frac{1}{2c}\left\{\left(\frac{1+z}{1-z}\right)^c - 1\right\} \ (c \in \mathbb{C}\backslash\{0\}) \ \text{and} \ \mathrm{k_0(z)} = \frac{1}{2}\log\left(\frac{1+\mathrm{z}}{1-\mathrm{z}}\right) \ (\mathrm{z} \in \mathbb{U}), \quad (2.1)$$

were studied by Campbell and Pflatzgraff [4]. Pommerenke [21] studied the special case of (2.1), that is,

$$k_{i\gamma}(z) = \frac{1}{2i\gamma}\left\{\left(\frac{1+z}{1-z}\right)^{i\gamma} - 1\right\} \quad (\gamma > 0,\ z \in \mathbb{U}),$$

for which

$$k'_{i\gamma}(z) = \frac{1}{(1+z)^{1-i\gamma}(1-z)^{1-i\gamma}}.$$

An obvious and consequential extension of (2.1) was given by the following formula.

$$k_c(\theta,\psi;z) = \frac{1}{(e^{i\psi}-e^{i\theta})\,c}\left\{\left(\frac{1-ze^{i\theta}}{1-ze^{i\psi}}\right)^{c} - 1\right\} \quad (c \in \mathbb{C}\backslash\{0\},\ e^{i\theta} \neq e^{i\psi},\ \theta,\psi \in \mathbb{R},\ z \in \mathbb{U})$$

and for the case when $c = 0$,

$$k_0(\theta,\psi;z) = \frac{1}{(e^{i\psi}-e^{i\theta})}\log\left(\frac{1-ze^{i\theta}}{1-ze^{i\psi}}\right) \quad (e^{i\theta} \neq e^{i\psi},\ \theta,\psi \in \mathbb{R},\ z \in \mathbb{U}).$$

We have

$$k'_c(\theta,\psi;z) = \frac{1}{(1-ze^{i\theta})^{1-c}(1-ze^{i\psi})^{1+c}} \quad (c \in \mathbb{C}).$$

Comparing

$$k'_{i\gamma}(\theta,\psi;z) = \frac{1}{(1-ze^{i\theta})^{1-i\gamma}(1-ze^{i\psi})^{1+i\gamma}}$$

with the generating function for Meixner-Pollaczek polynomial $P_n^{\lambda}(x;\theta)$ [13], we obtain

$$G^{\lambda}(x;\theta,-\theta;z) = \frac{1}{(1-ze^{i\theta})^{\lambda-i\gamma}(1-ze^{-i\theta})^{\lambda+i\gamma}} = \sum_{n=0}^{\infty} P_n^{\lambda}(x;\theta)z^n,$$

where $\lambda > 0$, $\theta \in (0,\pi)$ and $x \in \mathbb{R}$.

Definition 2.1. For $\lambda > 0$, $\theta \in (0,\pi)$ and $x \in \mathbb{R}$

$$\begin{aligned} zG^{\lambda}(x;\theta,-\theta;z) &= \frac{z}{(1-ze^{i\theta})^{\lambda-i\gamma}(1-ze^{-i\theta})^{\lambda+i\gamma}} = \sum_{n=0}^{\infty} P_n^{\lambda}(x;\theta)z^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(2\lambda)_n}{n!} e^{in\theta}{}_2F_1\left(-n,\lambda+ix,2\lambda;1-e^{-2i\theta}\right) z^{n+1} \\ &= \sum_{n=0}^{\infty} F_{n+1} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} F_n z^n, \end{aligned} \tag{2.2}$$

where $F_{n+1} = \frac{(2\lambda)_n}{n!} e^{in\theta}{}_2F_1\left(-n,\lambda+ix,2\lambda;1-e^{-2i\theta}\right)$ and $z \in \mathbb{U}$

To note the significance of the class, we list the following special cases for various values of λ, x and θ:

(1) $L_n^{\alpha}(x) = \lim_{\phi\to 0} P_n^{\frac{\alpha+1}{2}}\left(-\frac{x}{2\phi}, \phi\right)$, called the Laguerre polynomial.
(2) $H_n(x) = \lim_{\lambda\to\infty} n!\,\lambda^{\frac{-n}{2}} P_n^{\lambda}\left(\frac{x\sqrt{\lambda}-\lambda\cos\phi}{\sin\phi}, \phi\right)$, called the Hermite polynomial.
(3) $U_n(x) = \lim_{\lambda\to 0} P_n^{\lambda}\left(\frac{x}{2}, \frac{\phi}{2}\right)$, called the symmetric Meixner-Pollaczek polynomial.
(4) $P_n^0(x) = \lim_{\lambda\to 0} P_n^{\lambda}(x)$, shows that these polynomials are orthogonal polynomials in a strip $-1 \leq \Im(z) \leq 1$.
(5) $W_n(x) = \lim_{\lambda\to 0} P_n^{\frac{3}{4}}\left(\frac{x}{2}, \frac{\pi}{2}\right)$, arises as the the Mellin transform of the odd Hermite orthogonal functions.

For $\lambda > 0$, $\theta \in (0, \pi)$ and $x \in \mathbb{R}$, using the Generalised Meixner-Pollaczek polynomial (2.2), we introduce convolution operator $\mathcal{F}_{x,\theta}^{\lambda} : \mathcal{A} \to \mathcal{A}$, by

$$\mathcal{F}_{x,\theta}^{\lambda} f(z) := \left(zG^{\lambda}(x;\theta,-\theta;z) * f(z)\right) = z + \sum_{n=2}^{\infty} F_n a_n z^n, \tag{2.3}$$

where

$$F_n = \frac{(2\lambda)_{(n-1)}}{(n-1)!} e^{i(n-1)\theta} {}_2F_1\left(-(n-1), \lambda+ix, 2\lambda; 1-e^{-2i\theta}\right) \quad (z \in \mathbb{U}). \tag{2.4}$$

Let $\mho$ be the class of analytic functions w, normalized by $w(0) = 0$, satisfying the condition $|w(z)| < 1$. For analytic functions f and g, we say that f is subordinate to g in $\mathbb{U}$, denoted by $f \prec g$, if there exists a function $w \in \mho$ so that $f(z) = g(w(z))$ in $\mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Recently there has been triggering interest to study bi-univalent function class Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). But the coefficient problem for each of the following *Taylor-MacLaurin coefficients* $|a_n|$ $((n \geq 3)$ is still an open problem (see [2, 1, 3, 14, 17, 26]). Many researchers (see [8, 10, 15, 24]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$.

In [18], the authors defined the classes of functions $\mathcal{P}_m(\beta)$ as follows:

Definition 2.2. [18] Let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions P, normalized with $P(0) = 1$, and satisfying

$$\int_0^{2\pi} \left| \frac{\mathrm{Re}P(z) - \beta}{1 - \beta} \right| d\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{U}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions p analytic in $\mathbb{U}$, normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} d\mu(t),$$

where μ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m, \quad m \geq 2.$$

Remark that $\mathcal{P} := \mathcal{P}_2$ is the well-known class of Carathéodory functions, i.e. the normalized functions with positive real part in the open unit disk $\mathbb{U}$.

Motivated by the earlier work of Deniz [5], Peng *et al.* [20] (see also [19, 25]) and Goswami *et al.* [9], in the present paper, we introduce new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$, involving Generalised Meixner-Pollaczek polynomial operator $\mathcal{F}^{\lambda}_{x,\theta}$, and we find estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions that belong to these new subclasses of functions of the class Σ. Several related classes are also considered, and connection to earlier known results are made.

Definition 2.3. For $0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}^{\lambda,x,\theta}_{\Sigma}(\gamma, \alpha, \beta)$ if the following two conditions are satisfied:

$$1 + \frac{1}{\gamma} \left[\frac{z \left(\mathcal{F}^{\lambda}_{x,\theta} f(z) \right)'}{(1 - \alpha)z + \alpha \mathcal{F}^{\lambda}_{x,\theta} f(z)} - 1 \right] \in \mathcal{P}_m(\beta) \tag{2.5}$$

and

$$1 + \frac{1}{\gamma} \left[\frac{w \left(\mathcal{F}^{\lambda}_{x,\theta} g(w) \right)'}{(1 - \alpha)w + \alpha \mathcal{F}^{\lambda}_{x,\theta} g(w)} - 1 \right] \in \mathcal{P}_m(\beta), \tag{2.6}$$

where $\gamma \in \mathbb{C}^*$, the function g is given by (1.2), and $z, w \in \mathbb{U}$.

Definition 2.4. For $0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{K}^{\lambda,x,\theta}_{\Sigma}(\gamma, \alpha, \beta)$ if the following two conditions are satisfied:

$$1 + \frac{1}{\gamma} \left[\frac{z \left(\mathcal{F}^{\lambda}_{x,\theta} f(z) \right)' + z^2 \left(\mathcal{F}^{\lambda}_{x,\theta} f(z) \right)''}{(1 - \alpha)z + \alpha z \left(\mathcal{F}^{\lambda}_{x,\theta} f(z) \right)'} - 1 \right] \in \mathcal{P}_m(\beta) \tag{2.7}$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'+w^2\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)''}{(1-\alpha)w+\alpha w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}-1\right]\in\mathcal{P}_m(\beta), \tag{2.8}$$

where $\gamma\in\mathbb{C}^*$, the function g is given by (1.2), and $z,w\in\mathbb{U}$.

On specializing the parameters α, one can state the various new subclasses of Σ as illustrated in the following examples. Thus, taking $\alpha=1$ in the above two definitions, we obtain:

Example 2.1. Suppose that $0\le\beta<1$ and $\gamma\in\mathbb{C}^*$.

(i) A function $f\in\Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'}{\mathcal{F}_{x,\theta}^{\lambda}f(z)}-1\right]\in\mathcal{P}_m(\beta)\ \text{ and }\ 1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}{\mathcal{F}_{x,\theta}^{\lambda}g(w)}-1\right]\in\mathcal{P}_m(\beta),$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

(ii) A function $f\in\Sigma$ is said to be in the class $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)''}{\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'}\right]\in\mathcal{P}_m(\beta)\ \text{ and }\ 1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)''}{\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}\right]\in\mathcal{P}_m(\beta)$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

Taking $\alpha=0$ in the previous two definitions, we obtain the next special cases:

Example 2.2. Suppose that $0\le\beta<1$ and $\gamma\in\mathbb{C}^*$.

(i) A function $f\in\Sigma$ is said to be in the class $\mathcal{H}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'-1\right]\in\mathcal{P}_m(\beta)\ \text{ and }\ 1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'-1\right]\in\mathcal{P}_m(\beta)$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

(ii) A function $f\in\Sigma$ is said to be in the class $\mathcal{Q}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'+z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)''-1\right]\in\mathcal{P}_m(\beta),$$

and

$$1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'+w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)''-1\right]\in\mathcal{P}_m(\beta)$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

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In order to derive our main results, we shall need the following lemma.

Lemma 2.1. [9] Let the function $\Phi(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ $(z \in \mathbb{U})$, such that $\Phi \in \mathcal{P}_m(\beta)$. Then,

$$|h_n| \leq m(1-\beta) \quad (n \geq 1).$$

By employing the techniques which used earlier by Deniz [5], in the following section, we find estimates of the coefficients $|a_2|$ and $|a_3|$ for functions of the above-defined subclasses $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma, \alpha, \beta)$ and $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma, \alpha, \beta)$ of the function class Σ.

3. Coefficient Bounds for the Function Class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma, \alpha, \beta)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions f given by (1.1) belonging to the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma, \alpha, \beta)$.

Supposing that the functions $p, q \in \mathcal{P}_m(\beta)$, with

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathbb{U}) \tag{3.1}$$

and

$$q(w) = 1 + \sum_{k=1}^{\infty} q_k w^k \quad (w \in \mathbb{U}), \tag{3.2}$$

from Lemma 2.1, it follows that

$$|p_k| \leq m(1-\beta) \text{ and} \tag{3.3}$$

$$|q_k| \leq m(1-\beta) \text{ (for all } k \geq 1). \tag{3.4}$$

Theorem 3.1. If the function f given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma, \alpha, \beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}}; \frac{m|\gamma|(1-\beta)}{(2-\alpha)F_2}\right\} \tag{3.5}$$

and

$$\begin{aligned} |a_3| \leq \min\Bigg\{ & \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3} + \frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}; \\ & \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{\alpha}{2-\alpha}\right); \\ & \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{|(\alpha^2-2\alpha)F_2^2+2(3-\alpha)F_3|}{(2-\alpha)^2F_2^2}\right)\Bigg\}, \end{aligned} \tag{3.6}$$

where F_2 and F_3 are given by (2.4).

Proof. Since $f \in \mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$, from the definition relations (2.5) and (2.6), it follows that

$$1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'}{(1-\alpha)z+\alpha\mathcal{F}_{x,\theta}^{\lambda}f(z)}-1\right]$$
$$=1+\frac{2-\alpha}{\gamma}F_2a_2z+\left[\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3a_3\right]z^2+\cdots=:p(z) \quad (3.7)$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}{(1-\alpha)w+\alpha\mathcal{F}_{x,\theta}^{\lambda}g(w)}-1\right]$$
$$=1-\frac{2-\alpha}{\gamma}F_2a_2w+\left[\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3(2a_2^2-a_3)\right]w^2+\cdots=:q(w), \quad (3.8)$$

where $p,q\in\mathcal{P}_m(\beta)$, and are of the form (3.1) and (3.2), respectively.

Now, equating the coefficients in (3.7) and (3.8), we get

$$\frac{2-\alpha}{\gamma}F_2a_2 = p_1, \quad (3.9)$$

$$\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3a_3 = p_2, \quad (3.10)$$

$$-\frac{2-\alpha}{\gamma}F_2a_2 = q_1, \quad (3.11)$$

$$\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3(2a_2^2-a_3) = q_2. \quad (3.12)$$

From (3.9) and (3.11), we find that

$$a_2=\frac{\gamma p_1}{(2-\alpha)F_2}=\frac{-\gamma q_1}{(2-\alpha)F_2}, \quad (3.13)$$

which implies

$$|a_2|\leq\frac{|\gamma|m(1-\beta)}{(2-\alpha)F_2}. \quad (3.14)$$

Adding (3.10) and (3.12), by using (3.13) we obtain

$$\left[2(\alpha^2-2\alpha)F_2^2+2(3-\alpha)F_3\right]a_2^2=\gamma(p_2+q_2).$$

Now, by using (3.3) and (3.4), we get

$$|a_2|^2=\frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|},$$

and hence

$$|a_2|\leq\sqrt{\frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}},$$

which gives the bound on $|a_2|$ as asserted in (3.5).

Next, in order to find the upper-bound for $|a_3|$, by subtracting (3.12) from (3.10), we get

$$2(3-\alpha)F_3a_3 = \gamma(p_2-q_2) + 2(3-\alpha)F_3a_2^2. \tag{3.15}$$

It follows from (3.3), (3.4), (3.14) and (3.15), that

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3} + \frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}.$$

From (3.9) and (3.10), we have

$$a_3 = \frac{1}{(3-\alpha)F_3}\left(\gamma p_2 - \frac{\gamma^2(\alpha^2-2\alpha)p_1^2}{(2-\alpha)^2}\right).$$

and hence

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{\alpha}{(2-\alpha)}\right).$$

Further, from (3.9) and (3.12) we deduce that

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{|(\alpha^2-2\alpha)F_2^2+2(3-\alpha)F_3|}{(2-\alpha)^2F_2^2}\right),$$

and thus we obtain the conclusion (3.6) of our theorem. □

For the special cases $\alpha = 1$ and $\alpha = 0$, Theorem 3.1 reduces to the following corollaries, respectively:

Corollary 3.1. If the function f given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|2F_3-F_2^2|}}; \frac{m|\gamma|(1-\beta)}{F_2}\right\}$$

and

$$\begin{aligned}|a_3| \leq \min\Bigg\{&\frac{m|\gamma|(1-\beta)}{|2F_3-F_2^2|}+\frac{m|\gamma|(1-\beta)}{2F_3}; \frac{m|\gamma|(1-\beta)}{2F_3}\left(1+m|\gamma|(1-\beta)\right);\\ &\frac{m|\gamma|(1-\beta)}{2F_3}\left(1+\frac{m|\gamma|(1-\beta)|4F_3-F_2^2|}{F_2^2}\right)\Bigg\},\end{aligned}$$

where F_2 and F_3 are given by (2.4).

Corollary 3.2. If the function f given by (1.1) belongs to the class $\mathcal{G}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{3F_3}}; \frac{m|\gamma|(1-\beta)}{2F_2}\right\}$$

and

$$|a_3| \leq \min\left\{2\frac{m|\gamma|(1-\beta)}{3F_3};\frac{m|\gamma|(1-\beta)}{3F_3};\frac{m|\gamma|(1-\beta)}{3F_3}\left(1+m|\gamma|(1-\beta)\frac{6F_3}{4F_2^2}\right)\right\},$$

where F_2 and F_3 are given by (2.4).

4. Coefficient Bounds for the Function Class $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$

Theorem 4.1. If the function f given by (1.1) belongs to the class $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|}};\frac{m|\gamma|(1-\beta)}{2(2-\alpha)F_2}\right\} \tag{4.1}$$

and

$$\begin{aligned}|a_3| \leq \min&\left\{\frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{\alpha}{2-\alpha}\right);\right.\\ &\frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}+\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|};\\ &\left.\frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}+\frac{m^2|\gamma|^2(1-\beta)^2}{3(3-\alpha)F_3}\left(\frac{\alpha}{2-\alpha}+\frac{3(3-\alpha)F_3}{2(2-\alpha)^2F_2^2}\right)\right\},\end{aligned} \tag{4.2}$$

where F_2 and F_3 are given by (2.4).

Proof. Since $f \in \mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$, from the definition relations (2.7) and (2.8), it follows that

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'+z^2\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)''}{(1-\alpha)z+\alpha z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'}-1\right]\\ &\quad=1+\frac{2(2-\alpha)}{\gamma}F_2a_2z+\left[\frac{4(\alpha^2-2\alpha)}{\gamma}F_2^2a_2^2+\frac{3(3-\alpha)}{\gamma}F_3a_3\right]z^2+\cdots=:p(z)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'+w^2\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)''}{(1-\alpha)w+\alpha w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}-1\right]\\ &=1-\frac{2(2-\alpha)}{\gamma}F_2a_2w+\left[\frac{4(\alpha^2-2\alpha)}{\gamma}F_2^2a_2^2+\frac{3(3-\alpha)}{\gamma}F_3(2a_2^2-a_3)\right]w^2+\cdots=:q(w),\end{aligned} \tag{4.4}$$

where $p,q \in \mathcal{P}_m(\beta)$, and are of the form (3.1) and (3.2), respectively.

Now, equating the coefficients in (4.3) and (4.4), we get

$$\frac{2(2-\alpha)}{\gamma}F_2a_2 = p_1, \tag{4.5}$$

$$\frac{1}{\gamma}\left[4(\alpha^2-2\alpha)F_2^2a_2^2+3(3-\alpha)F_3a_3\right] = p_2, \tag{4.6}$$

$$-\frac{2(2-\alpha)}{\gamma}F_2a_2 = q_1, \tag{4.7}$$

$$\frac{1}{\gamma}\left[4(\alpha^2-2\alpha)F_2^2a_2^2+3(3-\alpha)F_3(2a_2^2-a_3)\right] = q_2. \tag{4.8}$$

From (4.5) and (4.7), we find that

$$a_2=\frac{\gamma p_1}{2(2-\alpha)F_2}=\frac{-\gamma q_1}{2(2-\alpha)F_2}, \tag{4.9}$$

which implies

$$|a_2|\leq \frac{|\gamma|m(1-\beta)}{2(2-\alpha)F_2}. \tag{4.10}$$

Adding (4.6) and (4.8), by using (4.9), we obtain

$$\left[8(\alpha^2-2\alpha)F_2^2+6(3-\alpha)F_3\right]a_2^2=\gamma(p_2+q_2).$$

Now, by using (3.3) and (3.4), we get

$$|a_2|^2=\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|},$$

and hence

$$|a_2|\leq\sqrt{\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|}},$$

which gives the bound on $|a_2|$ as asserted in (4.1).

Next, in order to find the upper-bound for $|a_3|$, by subtracting (4.8) from (4.6), we get

$$6(3-\alpha)F_3a_3=\gamma(p_2-q_2)+6(3-\alpha)F_3a_2^2. \tag{4.11}$$

It follows from (3.3), (3.4), (4.10) and (4.11), that

$$|a_3|\leq\frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}+\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|}.$$

From (4.5) and (4.6), we have

$$a_3=\frac{1}{3(3-\alpha)F_3}\left(\gamma p_2-\frac{\gamma^2(\alpha^2-2\alpha)p_1^2}{(2-\alpha)^2}\right).$$

and hence

$$|a_3|\leq\frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{\alpha}{2-\alpha}\right).$$

Further, from (4.5) and (4.8), we deduce that

$$|a_3|\leq \frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\left(\frac{\alpha}{2-\alpha}+\frac{3(3-\alpha)F_3}{2(2-\alpha)^2F_2^2}\right)\right),$$

and thus we obtain the conclusion (4.2) of our theorem. □

For the special cases $\alpha=1$ and $\alpha=0$, the Theorem 4.1 reduces to the following corollaries, respectively:

Corollary 4.1. If the function f given by (1.1) belongs to the class $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2|\leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|6F_3-4F_2^2|}};\frac{m|\gamma|(1-\beta)}{2F_2}\right\}$$

and

$$\begin{aligned}|a_3| \;\leq\; \min\Bigg\{&\frac{m|\gamma|(1-\beta)}{|6F_3-4F_2^2|}+\frac{m|\gamma|(1-\beta)}{6F_3};\frac{m|\gamma|(1-\beta)}{6F_3}\left(1+m|\gamma|(1-\beta)\right);\\ &\frac{m|\gamma|(1-\beta)}{6F_3}\left(1+m|\gamma|(1-\beta)\left(1+\frac{6F_3}{2F_2^2}\right)\right)\Bigg\},\end{aligned}$$

where F_2 and F_3 are given by (2.4).

Corollary 4.2. If the function f given by (1.1) belongs to the class $\mathcal{Q}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2|\leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{9F_3}};\frac{m|\gamma|(1-\beta)}{4F_2}\right\}$$

and

$$|a_3|\leq \min\left\{2\frac{m|\gamma|(1-\beta)}{9F_3};\frac{m|\gamma|(1-\beta)}{9F_3};\frac{m|\gamma|(1-\beta)}{9F_3}\left(m|\gamma|(1-\beta)\frac{9F_3}{4F_2^2}\right)\right\},$$

where F_2 and F_3 are given by (2.4).

Remark that, various other interesting corollaries and consequences of our main results, which are asserted by Theorem 3.1 and Theorem 4.1 above, can be derived similarly. The details involved may be left as exercises for the interested reader.

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Integral Inequalities of Simpson's Type for Strongly Extended (s,m)-Convex Functions

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Abstract

In the paper, the authors introduce a new concept "strongly extended (s,m)-convex function" and establish some integral inequalities of Simpson's type for strongly extended (s,m)-convex functions.

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1 Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.2 ([13])**.** For $f : [0,b] \to \mathbb{R}$ with $b > 0$ and $m \in (0,1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that f is an m-convex function on $[0,b]$.

Definition 1.3 ([6])**.** Let $s \in (0,1]$ be a real number. A function $f : \mathbb{R} \to \mathbb{R}_0 = [0,\infty)$ is said to be s-convex (in the second sense) if the inequality

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

If $s = 1$, the s-convex function becomes a convex function on $\mathbb{R}_0$.

Definition 1.4 ([15]). For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if

$$f(\lambda x + (1 - \lambda)y) \le \lambda^s f(x) + (1 - \lambda)^s f(y)$$

is valid for all $x, y \in I$ and $\lambda \in (0, 1)$.

Definition 1.5 ([9]). A function $f : [a, b] \to \mathbb{R}$ is said to be strongly convex with modulus $c > 0$ if

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y) - ct(1 - t)(x - y)^2$$

is valid for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Definition 1.6 ([5]). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be strongly s-convex with modulus $c > 0$ and $s \in (0, 1]$ if

$$f(tx + (1 - t)y) \le t^s f(x) + (1 - t)^s f(y) - ct(1 - t)(x - y)^2$$

is valid for all $x, y \in I$ and $t \in [0, 1]$.

The following theorems for some kinds of convex functions were obtained in recent years.

Theorem 1.1 ([2, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b - a)(|f'(a)| + |f'(b)|)}{8}.$$

Theorem 1.2 ([8, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ and $q \ge 1$, then*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b - a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}$$

and

$$\left|f\left(\frac{a + b}{2}\right) - \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b - a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.3 ([1, Theorems 2.2 to 2.3]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L_1([a, b])$.*

1. *If $|f'|$ is s-convex on $[a, b]$ for some $s \in (0, 1]$, then*

$$\begin{aligned}\left|f\left(\frac{a + b}{2}\right) - \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b - a}{4(s + 1)(s + 2)}\left[|f'(a)| + |f'(b)| + 2(s + 1)\left|f'\left(\frac{a + b}{2}\right)\right|\right] \\ &\le \frac{\left(2^{2-s} + 1\right)(b - a)}{4(s + 1)(s + 2)}\left[|f'(a)| + |f'(b)|\right].\end{aligned}$$

2. If $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$ for $p>1$ and some fixed $s\in(0,1]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\Big\{\big[\big(2^{1-s}+s+1\big)|f'(a)|^q+2^{1-s}|f'(b)|^q\big]^{1/q}+\big[2^{1-s}|f'(a)|^q+\big(2^{1-s}+s+1\big)|f'(b)|^q\big]^{1/q}\Big\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.4 ([5, Theorems 3.1])**.** *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a 2-times differentiable function on I° and $a,b\in I^\circ$ with $a<b$ such that $f''\in L_1([a,b])$. If $|f''|^q$ is strongly s-convex on $[a,b]$ for $q\ge1$ and $s\in(0,1]$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\quad\le\frac{6^{1/q}(b-a)^2}{324}\left\{\left[\frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{(s+1)(s+2)(s+3)3^s}|f''(a)|^q+\frac{1}{(s+2)(s+3)3^s}|f''(b)|^q\right.\right.\\&\quad\left.-\frac{c(b-a)^2}{45}\right]^{1/q}+\left[\frac{(s-1)2^{s+2}+s+5}{(s+1)(s+2)(s+3)3^s}\big(|f''(a)|^q+|f''(b)|^q\big)-\frac{11c(b-a)^2}{270}\right]^{1/q}\\&\quad\left.+\left[\frac{1}{(s+2)(s+3)3^s}|f''(a)|^q+\frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{(s+1)(s+2)(s+3)3^s}|f''(b)|^q-\frac{c(b-a)^2}{45}\right]^{1/q}\right\}.\end{aligned}$$

For more information on this topic, please refer to the papers [3, 4, 5, 7, 10, 11, 12, 14, 16, 17] and the closely related references therein.

In this paper, we will introduce a new concept "strongly extended (s,m)-convex function" and establish some integral inequalities of the Hermite-Hadamard type for strongly extended (s,m)-convex functions.

2 Definition and Lemmas

Now we give a definition of strongly extended (s,m)-convex functions.

Definition 2.1. A function $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ is said to be strongly extended (s,m)-convex with modulus $c>0$ and $(s,m)\in[-1,1]\times(0,1]$ if

$$f(tx+m(1-t)y)\le t^sf(x)+m(1-t)^sf(y)-ct(1-t)(x-y)^2$$

is valid for all $x,y\in[0,b^*]$ and $t\in(0,1)$.

Remark 1. If f is strongly extended (s,m)-convex on $[0,b^*]$ and $m=1$, then we say that f is strongly extended s-convex on $[0,b^*]$.

If f is strongly extended s-convex on $[0,b^*]$ and $s\in(0,1]$, then it is strongly s-convex on $[0,b^*]$.

To establish new Hermite-Hadamard type inequalities for strongly extended (s,m)-convex functions, we need the following lemmas.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $n \in \mathbb{N}_+$. If $f^{(n)} \in L_1([a, b])$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - \sum_{k=1}^{n} \frac{(b-a)^{k-1}\big[f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)\big]}{2(k!)} = \frac{(b-a)^n}{2(n!)}\int_0^1 \big[t^n + (t-1)^n\big] f^{(n)}(ta + (1-t)b)\,\mathrm{d}\,t.$$

Proof. By integration by parts, it follows that

$$\begin{aligned}
&\frac{(b-a)^{n+1}}{2(n!)}\int_0^1 t^n f^{(n)}(ta+(1-t)b)\,\mathrm{d}\,t \\
&= -\frac{(b-a)^n}{2(n!)} f^{(n-1)}(a) + \frac{(b-a)^n}{2[(n-1)!]}\int_0^1 t^{n-1} f^{(n-1)}(ta+(1-t)b)\,\mathrm{d}\,t \\
&= -\frac{(b-a)^n}{2(n!)} f^{(n-1)}(a) - \frac{(b-a)^{n-1}}{2[(n-1)!]} f^{(n-2)}(a) + \frac{(b-a)^{n-1}}{2[(n-2)!]}\int_0^1 t^{n-2} f^{(n-2)}(ta+(1-t)b)\,\mathrm{d}\,t \\
&= -\sum_{k=1}^{n-1} \frac{(b-a)^{k+1} f^{(k)}(a)}{2[(k+1)!]} + \frac{(b-a)^2}{2}\int_0^1 t f'(ta+(1-t)b)\,\mathrm{d}\,t \\
&= -\sum_{k=1}^{n} \frac{(b-a)^k f^{(k-1)}(a)}{2(k!)} + \frac{1}{2}\int_a^b f(x)\,\mathrm{d}\,x
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(b-a)^{n+1}}{2(n!)}\int_0^1 (t-1)^n f^{(n)}(ta+(1-t)b)\,\mathrm{d}\,t \\
&= \frac{(b-a)^n}{2(n!)}(-1)^n f^{(n-1)}(b) + \frac{(b-a)^n}{2[(n-1)!]}\int_0^1 (t-1)^{n-1} f^{(n-1)}(ta+(1-t)b)\,\mathrm{d}\,t \\
&= \frac{(b-a)^n}{2(n!)}(-1)^n f^{(n-1)}(b) + \frac{(b-a)^{n-1}}{2[(n-1)!]}(-1)^{n-1} f^{(n-2)}(b) \\
&\quad + \frac{(b-a)^{n-1}}{2[(n-2)!]}\int_0^1 (t-1)^{n-2} f^{(n-2)}(ta+(1-t)b)\,\mathrm{d}\,t \\
&= \sum_{k=1}^{n} \frac{(-1)^k (b-a)^k f^{(k-1)}(b)}{2(k!)} + \frac{1}{2}\int_a^b f(x)\,\mathrm{d}\,x.
\end{aligned}$$

Adding these two equations leads to Lemma 2.1. □

Lemma 2.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $n \in \mathbb{N}_+$. If $f^{(n)} \in L_1([a, b])$, then*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - \sum_{k=1}^{n} \frac{\big[1+(-1)^{k-1}\big](b-a)^{k-1}}{2^{k-1}(k!)} f^{(k-1)}\left(\frac{a+b}{2}\right) \\
&= \frac{(b-a)^n}{n!}\left[\int_0^{1/2} (-t)^n f^{(n)}((1-t)a+tb)\,\mathrm{d}\,t + \int_{1/2}^1 (1-t)^n f^{(n)}(ta+(1-t)b)\,\mathrm{d}\,t\right].
\end{aligned}$$

Proof. This follows from integration by parts immediately. □

3 Some new integral inequalities of Simpson's type

In this section, we establish integral inequalities of Simpson's type for strongly extended (s,m)-convex functions.

Theorem 3.1. *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be n-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended (s,m)-convex on $\left[0,\frac{b}{m}\right]$ for $c\ge0$, $(s,m)\in(-1,1]\times(0,1]$, and $q\ge1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|\le\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}$$
$$\times\left\{\frac{1-nB(n,s+1)}{n+s+1}\left[\left|f^{(n)}(a)\right|^q+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]-\frac{2c}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right\}^{1/q},$$

where $B(\alpha,\beta)$ denotes the well known beta function which can be defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}\,t,\quad \alpha,\beta>0.$$

Proof. Since $|f^{(n)}|^q$ is strongly extended (s,m)-convex on $\left[0,\frac{b}{m}\right]$, from Lemma 2.1 and Hölder's integral inequality, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|\\
&\le\frac{(b-a)^n}{2(n!)}\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t\\
&\le\frac{(b-a)^n}{2(n!)}\left[\int_0^1\left[t^n+(1-t)^n\right]\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(ta+(1-t)b)\right|^q\mathrm{d}\,t\right]^{1/q}\\
&\le\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}\left\{\int_0^1\left[t^n+(1-t)^n\right]\left[t^s\left|f^{(n)}(a)\right|^q\right.\right.\\
&\quad\left.\left.+m(1-t)^s\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right]\mathrm{d}\,t\right\}^{1/q}\\
&=\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}\left\{\frac{1-nB(n,s+1)}{n+s+1}\left[\left|f^{(n)}(a)\right|^q+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]\right.\\
&\quad\left.-\frac{2c}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right\}^{1/q}.
\end{aligned}$$

The proof of Theorem 3.1 is thus completed. □

Corollary 3.1.1. *Under conditions of Theorem 3.1,*

1. *when $q=1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2(n!)}\left\{\frac{1-nB(n,s+1)}{n+s+1}\left[\big|f^{(n)}(a)\big|+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|\right]-\frac{2c}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right\};$$

2. *when $q=1$ and $m=1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2(n!)}\left\{\frac{1-nB(n,s+1)}{n+s+1}\big[\big|f^{(n)}(a)\big|+|f^{(n)}(b)|\big]-\frac{2c}{(n+2)(n+3)}(b-a)^2\right\}.$$

Corollary 3.1.2. *Under conditions of Theorem 3.1,*

1. *when $s=1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2^{1/q}[(n+1)!]}\left[|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{2c(n+1)}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right]^{1/q};$$

2. *when $s=0$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{(n+1)!}\left[|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c(n+1)}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right]^{1/q}.$$

Theorem 3.2. *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be n-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended (s,m)-convex on $\left[0,\frac{b}{m}\right]$ for $c\ge 0$, $(s,m)\in(-1,1]\times(0,1]$, and $q>1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}\left[\frac{\big|f^{(n)}(a)\big|^q+m\big|f^{(n)}\big(\frac{b}{m}\big)\big|^q}{s+1}-\frac{c}{6}\left(\frac{b}{m}-a\right)^2\right]^{1/q}. \quad (3.1)$$

Proof. From Lemma 2.1 and Hölder's integral inequality, it follows that

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\right|$$

$$\leq \frac{(b-a)^n}{2(n!)}\int_0^1 \big[t^n+(1-t)^n\big]\big|f^{(n)}(ta+(1-t)b)\big|\,\mathrm{d}\,t$$
$$\leq \frac{(b-a)^n}{2(n!)}\bigg[\int_0^1 \big[t^n+(1-t)^n\big]^{q/(q-1)}\,\mathrm{d}\,t\bigg]^{1-1/q}\bigg[\int_0^1 \big|f^{(n)}(ta+(1-t)b)\big|^q\,\mathrm{d}\,t\bigg]^{1/q}.$$

Since $t^n+(1-t)^n\leq 1$ for $t\in[0,1]$, we have

$$\int_0^1 \big[t^n+(1-t)^n\big]^{q/(q-1)}\,\mathrm{d}\,t\leq \int_0^1 \big[t^n+(1-t)^n\big]\,\mathrm{d}\,t=\frac{2}{n+1}.$$

Since $|f^{(n)}|^q$ is a strongly extended (s,m)-convex function, it follows that

$$\int_0^1 \big|f^{(n)}(ta+(1-t)b)\big|^q\,\mathrm{d}\,t\leq \int_0^1\bigg[t^s\big|f^{(n)}(a)\big|^q+m(1-t)^s\bigg|f^{(n)}\bigg(\frac{b}{m}\bigg)\bigg|^q-ct(1-t)\bigg(\frac{b}{m}-a\bigg)^2\bigg]\,\mathrm{d}\,t$$
$$=\frac{\big|f^{(n)}(a)\big|^q+m\big|f^{(n)}\big(\frac{b}{m}\big)\big|^q}{s+1}-\frac{c}{6}\bigg(\frac{b}{m}-a\bigg)^2.$$

Substituting the last two inequalities into the first inequality above and rearranging yield the inequality (3.1). The proof of Theorem 3.2 is thus complete. □

Corollary 3.2.1. *Under the assumptions of Theorem 3.2, we have*

1. *if $s=1$, then*

$$\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^n\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\bigg|$$
$$\leq \frac{(b-a)^n}{(n+1)!}\big(n+1\big)^{1/q}\bigg[\big|f^{(n)}(a)\big|^q+m\big|f^{(n)}\big(\frac{b}{m}\big)\big|^q-\frac{c}{3}\bigg(\frac{b}{m}-a\bigg)^2\bigg]^{1/q};$$

2. *if $s=0$, then*

$$\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^n\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\bigg|$$
$$\leq \frac{(b-a)^n}{2(n!)}\bigg(\frac{2}{n+1}\bigg)^{1-1/q}\bigg[\big|f^{(n)}(a)\big|^q+m\big|f^{(n)}\big(\frac{b}{m}\big)\big|^q-\frac{c}{6}\bigg(\frac{b}{m}-a\bigg)^2\bigg]^{1/q}.$$

Theorem 3.3. *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be n-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended $(-1,m)$-convex on $\big[0,\frac{b}{m}\big]$ for $c\geq 0$, $m\in(0,1]$ and $q\geq 1$, then*

$$\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^n\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\bigg|$$
$$\leq \frac{2(b-a)^n}{n!}\bigg[\frac{1}{2^{n+1}(n+1)}\bigg]^{1-1/q}\Bigg[\Bigg(\sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)}+\ln 2\Bigg)\big|f^{(n)}(a)\big|^q$$
$$+m\frac{1}{2^n n}\bigg|f^{(n)}\bigg(\frac{b}{m}\bigg)\bigg|^q-\frac{c(n+4)}{2^{n+3}(n+2)(n+3)}\bigg(\frac{b}{m}-a\bigg)^2\Bigg]^{1/q}.$$

Proof. Using Lemma 2.2 and Hölder's integral inequality and considering that $|f^{(n)}|^q$ is the strongly extended $(-1,m)$-convex function, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{\left[1+(-1)^{k-1}\right](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{(b-a)^n}{n!}\left[\int_0^{1/2}t^n|f^{(n)}((1-t)a+tb)|\,\mathrm{d}\,t+\int_{1/2}^{1}(1-t)^n|f^{(n)}(ta+(1-t)b)|\,\mathrm{d}\,t\right]\mathrm{d}\,t\\
&\le\frac{(b-a)^n}{n!}\left\{\left(\int_0^{1/2}t^n\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^{1/2}t^n\left((1-t)^{-1}\left|f^{(n)}(a)\right|^q\right.\right.\right.\\
&\quad\left.\left.+mt^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right)\mathrm{d}\,t\right]^{1/q}\\
&\quad+\left[\int_{1/2}^{1}(1-t)^n\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_{1/2}^{1}(1-t)^n\left(t^{-1}\left|f^{(n)}(a)\right|^q\right.\right.\\
&\quad\left.\left.\left.+m(1-t)^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right)\mathrm{d}\,t\right]^{1/q}\right\}\\
&=\frac{2(b-a)^n}{n!}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-1/q}\left[\left(\sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)}+\ln 2\right)\left|f^{(n)}(a)\right|^q\right.\\
&\quad\left.+m\frac{1}{2^n n}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-\frac{c(n+4)}{2^{n+3}(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right)\right]^{1/q}.
\end{aligned}$$

The proof of Theorem 3.3 is thus complete. □

Theorem 3.4. *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be n-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended $(-1,m)$-convex on $\left[0,\frac{b}{m}\right]$ for $c\ge 0$, $m\in(0,1]$, and $q>1$, then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|\le\frac{2(b-a)^n}{n!}\\
&\times\left(\frac{q-1}{2^{\frac{(n+1)q-2}{q-1}}[(n+1)q-2]}\right)^{1-1/q}\left[\frac{\ln 4-1}{2}\left|f^{(n)}(a)\right|^q+\frac{m}{2}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-\frac{5c}{192}\left(\frac{b}{m}-a\right)^2\right]^{1/q}.
\end{aligned}$$

Proof. Using Lemma 2.2 and Hölder's integral inequality and considering that $|f^{(n)}|^q$ is strongly extended $(-1,m)$-convex, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{\left[1+(-1)^{k-1}\right](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{(b-a)^n}{n!}\left\{\left(\int_0^{1/2}t^{\frac{nq-1}{q-1}}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^{1/2}t\left((1-t)^{-1}\left|f^{(n)}(a)\right|^q\right.\right.\right.\\
&\quad\left.\left.+mt^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right)\mathrm{d}\,t\right]^{1/q}
\end{aligned}$$

$$+\left[\int_{1/2}^{1}(1-t)^{\frac{nq-1}{q-1}}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_{1/2}^{1}(1-t)\left(t^{-1}\left|f^{(n)}(a)\right|^{q}\right.\right.$$
$$\left.\left.+\,m(1-t)^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{q}-ct(1-t)\left(\frac{b}{m}-a\right)^{2}\right)\mathrm{d}\,t\right]^{1/q}\Bigg\}$$
$$=\frac{2(b-a)^{n}}{n!}\left(\frac{q-1}{2^{\frac{(n+1)q-2}{q-1}}[(n+1)q-2]}\right)^{1-1/q}$$
$$\times\left[\frac{\ln 4-1}{2}\left|f^{(n)}(a)\right|^{q}+\frac{m}{2}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{q}-\frac{5c}{192}\left(\frac{b}{m}-a\right)^{2}\right)\Bigg]^{1/q}.$$

The proof of Theorem 3.4 is thus completed. □

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FIXED POINTS OF MULTIVALUED NONEXPANSIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACE

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ABSTRACT. In this paper, we give a multivalued version of Picard-Mann hybrid iterative process of Khan [4] in Kohlenbach hyperbolic space. This process converges faster than all of Picard, Mann and Ishikawa iterative processes. By using an idea of Shahzad and Zegeye [8] which removes a restriction on the mapping and the method of direct construction of Cauchy sequence as illustrated by Song and Cho [9], we obtain strong and Δ-convergence theorems of this process for a multivalued mapping. Our results improve corresponding results of Shazad and Zegeye [8], Song and Cho [9] and many other in the contemporary literature in terms of faster iteration, more general space and weaker condition on mapping T.

1. Introduction and Preliminaries

Throughout the paper, we denote the set of positive integers by $\mathbb{N}$. Let (E, d) be a metric space and K be a nonempty subset of E. Then K is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x, k) = \inf\{d(x, y) : y \in K\} = d(x, K)$$

We shall denote the closed and bounded subsets, compact subsets and proximinal bounded subsets of K by $CB(K)$, $C(K)$ and $P(K)$, respectively. Let H be a Hausdorff metric induced by the metric d of E, that is

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for every $A, B \in CB(E)$. A multivalued mapping $T : K \longrightarrow P(K)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that for any $x, y \in K$,

$$H(Tx, Ty) \leq kd(x, y),$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in K$. A point $x \in K$ is called a fixed point of T if $x \in Tx$. Denote the set of all fixed points of T by $F(T)$ and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$.

Markin [1] started the study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric (see also [2]). Moreover, Lim [26] proved the existence of fixed points for multivalued nonexpansive mappings under suitable conditions in uniformly convex Banach spaces. Later on, an interesting and rich fixed point theory for such maps was developed which has applications in

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control theory, convex optimization, differential inclusion and economics (see, [3] and references cited therein). Since then different authors have discussed on the existence and convergence of fixed points for this class of maps in convex metric spaces. For example, Shimizu and Takahashi [18] generalized result of Lim [26] given above from uniformly convex Banach spaces to convex metric spaces.

On the other hand, given x_0 in K (a subset of Banach space), we know that Picard, Mann and Ishikawa iteration processes for a single valued map $T : K \to K$ defined as follows:

$$\text{(Picard)} \qquad x_{n+1} = Tx_n,$$

$$\text{(Mann)} \qquad x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T x_n,$$

and

$$\text{(Ishikawa)} \qquad \begin{aligned} x_{n+1} &= (1 - \alpha_n)\, x_n + \alpha_n T y_n \\ y_n &= (1 - \beta_n)\, x_n + \beta_n T x_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$.

Very recently, Khan [4] introduced a new iterative process which can be seen as a hybrid of Picard and Mann iterative processes. He also proved that the new process converges faster than all of Picard, Mann and Ishikawa iterative processes for contractions. Iteration scheme of Khan [4] defined as follows:

$$\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= (1 - \alpha_n)x_n + \alpha_n T x_n \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

It is well know that the theory of multivalued nonexpansive mappings is harder than according to the theory of single valued nonexpansive mappings. Sastry and Babu [5] defined Mann and Ishikawa iterative processes for a multivalued mapping as follows:

Let K be a nonempty convex subset of E and $T : K \to P(K)$ a multivalued mapping with $p \in Tp$.

(i) Mann iterate sequence is defined by $x_1 \in K$,

$$x_{n+1} = (1 - a_n)x_n + a_n y_n,$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$, and $\{a_n\}$ is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum a_n = \infty$.

(ii) Ishikawa iterate sequence is defined by $x_1 \in K$,

$$\begin{cases} y_n = (1 - b_n)x_n + b_n z_n, \\ x_{n+1} = (1 - a_n)x_n + a_n u_n, \end{cases}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that $\|z_n - p\| = d(p, Tx_n)$ and $\|u_n - p\| = d(p, Ty_n)$, and $\{a_n\}, \{b_n\}$ are real sequences with $0 \le a_n, b_n < 1$ satisfying $\lim_{n\to\infty} b_n = 0$ and $\sum a_n b_n = \infty$.

Sastry and Babu [5] proved that these iterates converge to a fixed point q of T under certain conditions. Moreover, they illustrated that fixed point q may be different from p.

The following is a useful Lemma due to Nadler [2].

Lemma 1.1. *Let $A, B \in CB(E)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a, b) \le H(A, B) + \eta$.*

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In 2007, Panyanak [6] proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain and generalized results of Sastry and Babu [5] to uniformly convex Banach spaces. Furthermore, he gave an open question which was answered by Song and Wang [7].

Later, Shahzad and Zegeye [8] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multivalued maps. They also removed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T, namely, $Tp = \{p\}$ for any $p \in F(T)$.

To achieve this, they defined $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$ for a multivalued mapping $T : K \to P(K)$. They also proved strong convergence results using Ishikawa type iteration process.

In this paper, we first define a multivalued version of the faster iteration scheme of Khan (1.1) in Kohlenbach hyperbolic spaces and then use weaker condition $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$ instead of $Tp = \{p\}$ for any $p \in F(T)$ due to Shahzad and Zegeye [8] to approximate fixed points of a multivalued nonexpansive mapping T. Moreover, we use the method of direct construction of Cauchy sequence as indicated by Song and Cho [9] (and opposed to [8]) but used also by many other authors including [10],[11] and [13]. The algorithm we use in this paper read as under.

Let E be a Kohlenbach hyperbolic space and K be a nonempty convex subset of E. Let $T : K \to P(K)$ be a multivalued map and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Choose $x_0 \in K$ and define $\{x_n\}$ as

$$\begin{cases} x_{n+1} = v_n \\ y_n = W(u_n, x_n, \alpha_n) \end{cases}, \tag{1.2}$$

where $u_n \in P_T(x_n)$, $v_n \in P_T(y_n) = P_T(W(u_n, x_n, \alpha_n))$ and $\{\alpha_n\}$ is a real sequence such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. The iterative sequence (1.2) is called the *modifed Picard-Mann hybrid iterative process* for a multivalued nonexpansive mapping in a Kohlenbach hyperbolic space. In this way, we compute fixed points of a multivalued nonexpansive mapping by modifed Picard-Mann hybrid iterative process in a Kohlenbach hyperbolic space. Our results improve corresponding results of Shazad and Zegeye [8], Song and Cho [9] and many other in the contemporary literature in terms of faster iteration, more general space and weaker condition on mapping T.

Different definitions of hyperbolic space can be found in the literature, we refer the readers to [14] for a detailed discussion. We will study under more general setup Kohlenbach hyperbolic spaces which introduced by Kohlenbach [15] as follows:

Definition 1.2. A metric space (E, d) is said to be Kohlenbach hyperbolic space if there exists a map $W : E^2 \times [0, 1] \to E$ satisfying:

W1. $d(u, W(x, y, \alpha)) \leq (1 - \alpha) d(u, x) + \alpha d(u, y)$
W2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
W3. $W(x, y, \alpha) = W(y, x, (1 - \alpha))$
W4. $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha) d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in E$ *and* $\alpha, \beta \in [0, 1]$.

A metric space (E, d) is called a convex metric space introduced by Takahashi [16] if it satisfieses only W1. Every normed space (and Banach space) is a special

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convex metric space, but the converse of this statement is not true, in general (see [11]).

In the sequel, we shall use the term hyperbolic space instead of Kohlenbach hyperbolic space in view of simplicity. The class of hyperbolic spaces includes normed spaces and convex subsets thereof, the Hilbert ball (see [17] for a book treatment) as well as CAT (0)-spaces.

A hyperbolic space (E, d, W) is said to be uniformly convex [18] if for all $u, x, y \in E$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{r} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta) r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε). A subset K of a hyperbolic space E is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

Now, we discourse concept of Δ-convergence which coined by Lim [19] in general metric spaces. To give the definition of Δ-convergence, we first recall the notions of asymptotic radius and asymptotic center. Let $\{x_n\}$ be a bounded sequence in a metric space E. For $x \in E$, define a continuous functional $r(., \{x_n\}) : E \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. Then the asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in E\}$ and the asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of E is defined as follows:

$$A_K(\{x_n\}) = \{x \in E : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [22] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly convex hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in E is said to Δ-converge to $x \in E$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [20]. In this case, we write Δ -$\lim_n x_n = x$.

We want to point out that $\Delta-$convergence coincides with weak convergence in Banach spaces with Opial's property [23].

Kirk and Panyanak [20] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [21] continued to work in this direction and studied the Δ-convergence of Picard, Mann and Ishikawa iterates in $CAT(0)$ spaces (Theorems 3.1, 3.2 and 3.3 respectively in [21]). Khan et al. [24] was studied this concept in hyperbolic spaces and they gave a couple of helpful lemma as follows.

Lemma 1.3. [24] *Let (E, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in E$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty} d(x_n, x) \leq r, \limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 1.4. [24] *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and*

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$r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.

The following useful lemma can be found in [9] gives some properties of P_T in metric (and hence hyperbolic) spaces.

Lemma 1.5. [9] *Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Then the following are equivalent.*
(1) $x \in F(T)$, that is, $x \in Tx$,
(2) $P_T(x) = \{x\}$, that is, $x = y$ for each $y \in P_T(x)$,
(3) $x \in F(P_T)$, that is, $x \in P_T(x)$.
Moreover, $F(T) = F(P_T)$.

2. Main Results

Before giving our main results, we show that the modifed Picard-Mann hybrid iterative process (1.2) can be employed for the approximation of fixed points of a multivalued nonexpansive mapping in hyperbolic spaces.

Define $f : K \to K$ by $f(x) = v$ for some $v \in P_T(y) = P_T(W(u, x_0, \alpha_0))$ and for some $u \in P_T(x)$. Suppose that P_T is nonexpansive multivalued mappings on K. Existence of x_1 is guaranteed if f has a fixed point. For any $m, n \in K$, let $z \in P_T(m)$, $z' \in P_T(n)$ such that $d(z, z') = d(z, Tn)$, and $y \in P_T(W(z, x_0, \alpha_0))$, $y' \in P_T(W(z', x_0, \alpha_0))$ such that $d(y, y') = d(y, T(W(z', x_0, \alpha_0)))$.

On using definition of condition W4 in hyperbolic sapaces, we have

$$\begin{aligned} d(f(m), f(n)) &= d(y, y') \\ &\leq H\left(P_T\left(W(z, x_0, \alpha_0)\right), P_T\left(W(z', x_0, \alpha_0)\right)\right) \\ &\leq d\left(W(z, x_0, \alpha_0), W(z', x_0, \alpha_0)\right) \\ &\leq (1-\alpha_0)d(z, z') \\ &= (1-\alpha_0)d(z, Tn) \\ &\leq (1-\alpha_0)d(z, P_T(n)) \\ &\leq (1-\alpha_0)H(P_T(m), P_T(n)) \\ &\leq (1-\alpha_0)d(m, n). \end{aligned}$$

Since $\alpha_n \in (0, 1)$, f is a contraction. By Banach contraction principle, f has a unique fixed point. Thus the existence of x_1 is established. Similarly, the existence of $x_2, x_3, \ldots$ is established. Thus the modifed Picard-Mann hybrid iterative process (1.2) is well defined.

We start with the following couple of important lemmas.

Lemma 2.1. *Let K be a nonempty closed convex subset of a hyperbolic space E and let $T : K \to P(K)$ be a multivalued map such that P_T is a nonexpansive map and $F \neq \emptyset$. Then for the modifed Picard-Mann hybrid iterative process $\{x_n\}$ in (1.2), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

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Proof. Let $p \in F$. Then $p \in P_T(p) = \{p\}$. Using (1.2) and Lemma 1.5, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(v_n, p) \\ &\leq H(P_T(y_n), P_T(p)) \\ &\leq d(y_n, p) \\ &= d(W(u_n, x_n, \alpha_n), p) \\ &\leq \alpha_n d(p, u_n) + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{2.1}$$

That is,

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists. □

Lemma 2.2. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and let $T: K \to P(K)$ be a multivalued map such that P_T is a nonexpansive map and $F \neq \emptyset$. Let $\{\alpha_n\}$ satisfy $0 < a \leq \alpha_n \leq b < 1$. Then for the modifed Picard-Mann hybrid iterative process $\{x_n\}$ in (1.2), we have $\lim_{n\to\infty}(x_n, P_T(x_n)) = \lim_{n\to\infty}(x_n, P_T(y_n)) = 0$.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$ for some $c \geq 0$. For $c = 0$, the result is trivial. Suppose $c > 0$.

Now $\lim_{n\to\infty} d(x_{n+1}, p) = c$ can be rewritten as $\lim_{n\to\infty} d(v_n, p) = c$.

Since P_T is nonexpansive, we have

$$\begin{aligned} d(u_n, p) &= d(u_n, P_T(p)) \\ &\leq H(P_T(x_n), P_T(p)) \\ &\leq d(x_n, p). \end{aligned}$$

Hence

$$\lim_{n\to\infty} \sup d(u_n, p) \leq c \tag{2.2}$$

Next

$$\begin{aligned} d(v_n, p) &= d(v_n, P_T(p)) \\ &\leq H(P_T(y_n), P_T(p)) \\ &\leq d(y_n, p) \\ &= d(W(u_n, x_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(u_n, p) + \alpha_n d(x_n, p) \\ &\leq (1-\alpha_n) H(P_T(x_n), P_T(p)) + \alpha_n d(x_n, p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

and so

$$\lim_{n\to\infty} \sup d(v_n, p) \leq c.$$

Further

$$\begin{aligned} d\left(W(u_n, x_n, \alpha_n), p\right) &\leq (1-\alpha_n)\, d\left(u_n, p\right) + \alpha_n d\left(x_n, p\right) \\ &\leq (1-\alpha_n)\, d\left(x_n, p\right) + \alpha_n d\left(x_n, p\right) \\ &= d\left(x_n, p\right). \end{aligned}$$

Taking lim sup, we have

$$\lim_{n\to\infty} \sup d\left(W(u_n, x_n, \alpha_n), p\right) \leq c.$$

Now (2.1) can be rewritten as

$$d\left(x_{n+1}, p\right) \leq d\left(W\left(u_n, x_n, \alpha_n\right), p\right)$$

and so

$$c \leq \lim_{n\to\infty} \inf d\left(W\left(u_n, x_n, \alpha_n\right), p\right).$$

Hence

$$\lim_{n\to\infty} d\left(W\left(u_n, x_n, \alpha_n\right), p\right) = c. \tag{2.3}$$

From $\lim_{n\to\infty} d\left(x_n, p\right) = c$, (2.2), (2.3) and Lemma 1.3, it follows

$$\lim_{n\to\infty} d\left(x_n, u_n\right) = 0.$$

Similarly we can show that

$$\lim_{n\to\infty} d\left(x_n, v_n\right) = 0.$$

Since $d\left(x, P_T\left(x\right)\right) = \inf_{z\in P_T(x)} d\left(x, z\right)$, therefore

$$d\left(x_n, P_T\left(x_n\right)\right) \leq d\left(x_n, u_n\right) \to 0.$$

Similarly

$$d\left(x_n, P_T\left(y_n\right)\right) \leq d\left(x_n, v_n\right) \to 0.$$

□

Now we approximate fixed points of the mapping T through Δ-convergence of the modifed Picard-Mann hybrid iterative process defined in (1.2).

Theorem 2.3. *Let K be a nonempty closed and convex subset of a uniformly convex hyperbolic space E with monotone modulus of uniform convexity η. Let T, P_T and $\{\alpha_n\}$ be as in Lemma 2.2. Then the modifed Picard-Mann hybrid iterative process $\{x_n\}$ Δ-converges to a fixed point of T (or P_T).*

Proof. Let $p \in F(T) = F(P_T)$. From the proof of Lemma 2.1, $\lim_{n\to\infty} d\left(x_n, p\right)$ exists and $\{x_n\}$ is bounded. Thus $\{x_n\}$ has a unique asymptotic center. Therefore $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. By Lemma 2.2, we have $\lim_{n\to\infty}\left(x_n, P_T\left(x_n\right)\right) = 0$. We claim that v is a fixed point of P_T.

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To prove this, take $\{z_m\}$ in $P_T(v)$. Then

$$\begin{aligned} r(z_m,\{v_n\}) &= \lim_{n\to\infty}\sup d(z_m,v_n) \\ &\leq \lim_{n\to\infty}\sup\{d(z_m,P_T(v_n))+d(P_T(v_n),v_n)\} \\ &\leq \lim_{n\to\infty}\sup H(P_T(v),P_T(v_n)) \\ &\leq \lim_{n\to\infty}\sup d(v,v_n) \\ &= r(v,\{v_n\}). \end{aligned}$$

This gives $|r(z_m,\{v_n\}-r(v,\{v_n\})|\to 0$ as $m\to\infty$. By Lemma 1.4, we get $\lim_{m\to\infty} z_m = v$. Note that $Tv\in P(K)$ being proximinal is closed, hence $P_T(v)$ is closed. Moreover, $P_T(v)$ is bounded. Consequently $\lim_{m\to\infty} z_m = v \in P_T(v)$. Hence $v\in F(P_T)$ and so $v\in F(T)$. Since $\lim_{n\to\infty} d(x_n,v)$ exists from Lemma 2.1, therefore by the uniqueness of asymptotic center, we have

$$\begin{aligned} \lim_{n\to\infty}\sup d(v_n,v) &< \lim_{n\to\infty}\sup d(v_n,x) \\ &\leq \lim_{n\to\infty}\sup d(x_n,x) \\ &< \lim_{n\to\infty}\sup d(x_n,v) \\ &= \lim_{n\to\infty}\sup d(v_n,v) \end{aligned}$$

a contradiction. Hence $x=u$. Therefore $A(\{v_n\})=\{v\}$ for every subsequence $\{v_n\}$ of $\{x_n\}$. Hence $\{x_n\}$ Δ-converges to a fixed point of T (or P_T). □

We now prove some strong convergence theorems. Our first strong convergence theorem is valid in a complete hyperbolic space. Then we apply this theorem to obtain two results in a complete and uniformly convex hyperbolic space. We also use the method of direct construction of Cauchy sequence as indicated by Song and Cho [9] (and opposed to [8]) but used also by many other authors including [10],[11] and [13].

Theorem 2.4. *Let K be a nonempty closed and convex subset of a complete hyperbolic space E and, T, P_T and $\{\alpha_n\}$ be as in Lemma 2.2. Let $\{x_n\}$ be the modifed Picard-Mann hybrid iterative process as defined in* (1.2), *then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F(T))=0$. By similar methot in Lemma 2.1, we get

$$d(x_{n+1},p)\leq d(x_n,p),$$

which implies

$$d(x_{n+1},F(T))\leq d(x_n,F(T)).$$

This gives that $\lim\limits_{n\to\infty} d(x_n,F(T))$ exists and so by the hypothesis of our theorem, $\liminf\limits_{n\to\infty} d(x_n,F(T))=0$. Therefore we must have $\lim\limits_{n\to\infty} d(x_n,F(T))=0$.

We now we prove that $\{x_n\}$ is a Cauchy sequence in K. Let $m,n\in\mathbb{N}$ and assume $m>n$. Then it follows (along the lines similar to Lemma 2.1) that

$$d(x_m,p)\leq d(x_n,p)$$

for all $p\in F$. Thus we have

$$d(x_m,x_n)\leq d(x_m,p)+d(p,x_n)\leq 2d(x_n,p).$$

Taking inf on the set F, we have $d(x_m, x_n) \leq d(x_n, F(T))$. On letting $m \to \infty$, $n \to \infty$ the inequality $d(x_m, x_n) \leq d(x_n, F(T))$ shows that $\{x_n\}$ is a Cauchy sequence in K and hence converges, say to $q \in K$. Now it is left to show that $q \in F(T)$. Indeed, by $d(x_n, F(P_T)) = \inf_{y \in F(P_T)} d(x_n, y)$. So for each $\varepsilon > 0$, there exists $p_n^{(\varepsilon)} \in F(P_T)$ such that,

$$d\left(x_n, p_n^{(\varepsilon)}\right) < d(x_n, F(P_T)) + \frac{\varepsilon}{2}.$$

This implies $\lim_{n\to\infty} d\left(x_n, p_n^{(\varepsilon)}\right) \leq \frac{\varepsilon}{2}$. From $d\left(p_n^{(\varepsilon)}, q\right) \leq d\left(x_n, p_n^{(\varepsilon)}\right) + d(x_n, q)$ it follows that

$$\lim_{n\to\infty} d\left(p_n^{(\varepsilon)}, q\right) \leq \frac{\varepsilon}{2}.$$

Finally,

$$\begin{aligned} d(P_T(q), q) &\leq d\left(q, p_n^{(\varepsilon)}\right) + d\left(p_n^{(\varepsilon)}, P_T(q)\right) \\ &\leq d\left(q, p_n^{(\varepsilon)}\right) + H\left(P_T\left(p_n^{(\varepsilon)}\right), P_T(q)\right) \\ &\leq 2d\left(p_n^{(\varepsilon)}, q\right) \end{aligned}$$

yields $d(P_T(q), q) < \varepsilon$. Since ε is arbitrary, therefore $d(P_T(q), q) = 0$. Since F is closed, $q \in F$ as required. □

As appropriate our goal, we give the following definitions. The first is the multivalued version of condition (I) of Senter and Dotson [25] and second is semi-compact map.

Definition 2.5. A multivalued nonexpansive mappings $T : K \to CB(K)$ where K a subset of E, are said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F))$ for all $x \in K$.

Definition 2.6. A map $T : K \to P(K)$ is called semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

We now obtain the following theorem by applying the above theorem in a complete and uniformly convex hyperbolic space where $T : K \to P(K)$ satisfies condition (I).

Theorem 2.7. *Let K be a nonempty closed convex subset of a complete and uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and, T, P_T and $\{\alpha_n\}$ be as in Lemma 2.2. Suppose that P_T satisfy condition (I), then the modifed Picard-Mann hybrid iterative process $\{x_n\}$ defined in* (1.2) *converges strongly to $p \in F$.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. We call it c for some $c \geq 0$.

If $c = 0$, there is nothing to prove.

Assume $c > 0$. Now $d(x_{n+1}, p) \leq d(x_n, p)$ gives that

$$\inf_{p \in F(T)} d(x_{n+1}, p) \leq \inf_{p \in F(T)} d(x_n, p),$$

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which means that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$.Hence $\lim_{n\to\infty} d(x_n, F(T))$ exists. By using condition (I) and Lemma 2.2,we get

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

and so

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

By properties f, we obtain that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Finally applying Theorem 2.4, we get the result. □

The proof of follow theorem is also easy and omitted.

Theorem 2.8. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and, T, P_T and $\{\alpha_n\}$ be as in Lemma 2.2. Suppose that P_T is semi-compact, then the modifed Picard-Mann hybrid iterative process $\{x_n\}$ defined in* (1.2) *converges strongly to $p \in F$.*

Remark 2.9. (1) Theorem 2.4 and Theorem 2.7 extends the corresponding results Khan [4] in three ways: (i) from single valued maps to multivalued maps (ii) from bounded domain to unbounded domain (ii) from uniformly convex Banach space to general setup of uniformly convex hyperbolic spaces.

(2) Our theorems sets analogue corresponding results of Khan [4], for multivalued nonexpansive maps on unbounded domain in a uniformly convex hyperbolic space X.

(3) Since Picard-Mann hybrid iterative process converges faster than Mann and Ishikawa iterative processes, our theorems are better than results of Fukhar-ud-din et al. [27].

(4) Since CAT(0)-spaces are uniformly convex hyperbolic spaces with a 'nice' monotone modulus of uniform convexity $\eta(r, \varepsilon) := \frac{\varepsilon^2}{8}$, then our results valid in CAT(0) spaces besides Banach spaces.

(5) Iteration process (1.2) has not been studied in CAT(0) spaces and Banach spaces for multivalued nonexpansive map so far. Due to hyperbolic spaces are more general than CAT(0) spaces as well as Banach spaces, the iteration process (1.2) does not need to be studied for this class of mappings in CAT(0) spaces or Banach spaces.

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Double-framed soft sets with applications in BE -algebras

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Abstract.

The notions of double-framed soft subalgebras/filters in BE-algebras are introduced and related properties are investigated. We consider characterizations of double-framed soft subalgebras/filters, and establish a new double-framed soft subalgebra/filter from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras/filters is a double framed soft subalgebra/filter.

1. Introduction

In 1966, Imai and Iséki [3] and Iséki [4] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, Kim and Kim [10] introduced the notion of a BE-algebra, and investigated several properties. In [2], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras.

Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [11] described the application of soft set theory to a decision making problem. Jun and Park [9] studied applications of soft sets in ideal theory of BCK/BCI-algebras. Jun et al. [8] introduced the notion of intersectional soft sets, and considered its applications to BCK/BCI-algebras. Also, Jun [5] discussed the union soft sets with applications in BCK/BCI-algebras. Jun et al. [6] introduced the notion of double-framed soft sets, and applied it to BCK/BCI-algebras. They discussed double-frame soft algebras and investigated related properties. Jun et al. [7] studied applications of soft sets in BE-algebras. Ahn et al. [1] introduced the notion of int-soft filters of BE-algebras and investigated related properties.

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In this paper, we introduce the notions of double-framed soft subalgebras/filters in BE-algebras and investigated related properties. We consider characterizations of double-framed soft subalgebras/filters, and establish a new double-framed soft subalgebra/filter from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras/filters is a double framed soft subalgebra/filter.

2. Preliminaries

We recall some definitions and results discussed in [10].

An algebra $(X;*,1)$ of type (2, 0) is called a BE-*algebra* if

(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ (*exchange*)

We introduce a relation "$\le$" on a BE-algebra X by $x\le y$ if and only if $x*y=1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x*x=1$ for all $x\in X$, it is clear that $1\in S$.

Definition 2.1. Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1\in F$;
(F2) $x*y\in F$ and $x\in F$ imply $y\in F$ for all $x,y\in X$.

Proposition 2.2. *Let $(X;*,1)$ be a BE-algebra and let F be a filter of X. If $x\le y$ and $x\in F$ for any $y\in X$, then $y\in F$.*

A soft set theory is introduced by Molodtsov [12]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A,B,C,\cdots\subseteq X$.

Definition 2.3. A *soft set* (f,A) of X over U is defined to be the set of ordered pairs

$$(f,A):=\{(x,f(x)):x\in X,\ f(x)\in\mathscr{P}(U)\},$$

where $f:X\to\mathscr{P}(U)$ such that $f(x)=\emptyset$ if $x\notin A$.

3. Double-framed soft subalgebras

In what follows, we take $E=X$, as a set of parameters, which is a BE-algebra unless otherwise specified.

Definition 3.1. A double-framed pair $\langle(\alpha,\beta);X\rangle$ is called a *double-framed soft set* over U, where α and β are mappings from X to $\mathscr{P}(U)$.

Definition 3.2. A double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U is called a *double-framed soft subalgebra* over U if it satisfies :

(3.1) $(\forall x,y\in X)\,(\alpha(x*y)\supseteq\alpha(x)\cap\alpha(y),\ \beta(x*y)\subseteq\beta(x)\cup\beta(y))\,.$

Example 3.3. Let X be the set of parameters where $X:=\{1,a,b,c,d\}$ is a BE-algebra [7] with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	1
d	1	1	1	1	1

Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over U defined as follows:

$$\alpha : X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\tau_3 & \text{if } x=1,\\ \tau_1 & \text{if } x\in\{a,c,d\},\\ \tau_2 & \text{if } x=b,\end{cases}$$

and

$$\beta : X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\gamma_3 & \text{if } x=1,\\ \gamma_1 & \text{if } x\in\{a,c,d\},\\ \gamma_2 & \text{if } x=b\end{cases}$$

where $\tau_1,\tau_2,\tau_3,\gamma_1,\gamma_2$ and γ_3 are subsets of U with $\tau_1\subsetneq\tau_2\subsetneq\tau_3$ and $\gamma_1\supsetneq\gamma_2\supsetneq\gamma_3$ It is routine to verify that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U.

Lemma 3.4. *Every double-framed soft subalgebra $\langle(\alpha,\beta);X\rangle$ over U satisfies the following condition:*

(3.2) $(\forall x\in X)\,(\alpha(x)\subseteq\alpha(1),\ \beta(x)\supseteq\beta(1))\,.$

Proof. Straightforward. □

Proposition 3.5. *For a double-framed soft subalgebra $\langle(\alpha,\beta);X\rangle$ over U, the following are equivalent:*

(i) $(\forall x\in X)\,(\alpha(x)=\alpha(1),\ \beta(x)=\beta(1))\,.$
(ii) $(\forall x,y\in X)\,(\alpha(y)\subseteq\alpha(y*x),\ \beta(y)\supseteq\beta(y*x))\,.$

Proof. Assume that (ii) is valid. Taking $y:=1$ in (ii) and using (BE3), we have $\alpha(1)\subseteq\alpha(1*x)=\alpha(x)$ and $\beta(1)\supseteq\beta(1*x)=\beta(x)$. It follows from Lemma 3.4 that $\alpha(x)=\alpha(1)$ and $\beta(x)=\beta(1)$.

Conversely, suppose that $\alpha(x)=\alpha(1)$ and $\beta(x)=\beta(1)$ for all $x\in X$. Using (3.1), we have

$$\alpha(y)=\alpha(y)\cap\alpha(1)=\alpha(y)\cap\alpha(x)\subseteq\alpha(y*x),$$
$$\beta(y)=\beta(y)\cup\beta(1)=\beta(y)\cup\beta(x)\supseteq\beta(y*x)$$

for all $x,y\in X$. This completes the proof. □

For two double-framed soft sets $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U, the *double-framed soft int-uni set* of $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ is defined to be a double-framed soft set $\langle(\alpha\tilde{\cap}f,\beta\tilde{\cup}g);X\rangle$ where

$$\alpha\tilde{\cap}f : X \to \mathscr{P}(U),\ x \mapsto \alpha(x)\cap f(x),$$
$$\beta\tilde{\cup}g : X \to \mathscr{P}(U),\ x \mapsto \beta(x)\cup g(x).$$

It is denoted by $\langle(\alpha,\beta);X\rangle \sqcap \langle(f,g);X\rangle = \langle(\alpha\tilde{\cap}f,\beta\tilde{\cup}g);X\rangle$.

Theorem 3.6. *The double-framed soft int-uni set of two double-framed soft subalgebras $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U is a double-framed soft subalgebra over U.*

Proof. For any $x,y\in X$, we have

$$\begin{aligned}(\alpha\tilde{\cap}f)(x*y) =&\alpha(x*y)\cap f(x*y) \supseteq (\alpha(x)\cap\alpha(y))\cap(f(x)\cap f(y))\\ =&(\alpha(x)\cap f(x))\cap(\alpha(y)\cap f(y)) = (\alpha\tilde{\cap}f)(x)\cap(\alpha\tilde{\cap}f)(y)\end{aligned}$$

and

$$\begin{aligned}(\beta\tilde{\cup}g)(x*y) =&\beta(x*y)\cup g(x*y) \subseteq (\beta(x)\cup\beta(y))\cup(g(x)\cup g(y))\\ =&(\beta(x)\cup g(x))\cup(\beta(y)\cup g(y)) = (\beta\tilde{\cup}g)(x)\cup(\beta\tilde{\cup}g)(y).\end{aligned}$$

Therefore $\langle(\alpha,\beta);X\rangle \sqcap \langle(f,g);X\rangle$ is a double-framed soft subalgebra over U. □

For two double-framed soft sets $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U, the *double-framed soft uni-int set* of $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ is defined to be a double-framed soft set $\langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$ where

$$\alpha\tilde{\cup}f : X \to \mathscr{P}(U),\ x \mapsto \alpha(x)\cup f(x),$$
$$\beta\tilde{\cap}g : X \to \mathscr{P}(U),\ x \mapsto \beta(x)\cap g(x).$$

It is denoted by $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$.

The following example shows that the double-framed soft uni-int set of two double-framed soft subalgebras $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U may not be a double-framed soft subalgebra over U.

Example 3.7. Let X be the set of parameters where $X := \{1,a,b,c,d\}$ is a BE-algebra [2] with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Let $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ be double-framed soft sets over U defined, respectively, as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_5 & \text{if } x = 1, \\ \tau_2 & \text{if } x = a, \\ \tau_1 & \text{if } x = b, \\ \tau_3 & \text{if } x = c, \end{cases}$$

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_5 & \text{if } x = 1, \\ \gamma_2 & \text{if } x = a, \\ \gamma_1 & \text{if } x = b, \\ \gamma_3 & \text{if } x = c, \end{cases}$$

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_4 & \text{if } x = 1, \\ \tau_2 & \text{if } x = a, \\ \tau_3 & \text{if } x = b, \\ \tau_1 & \text{if } x = c, \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x = 1, \\ \gamma_2 & \text{if } x = a, \\ \gamma_3 & \text{if } x = b, \\ \gamma_1 & \text{if } x = c, \end{cases}$$

where $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\tau_1,\tau_2,\tau_3,\tau_4$ and τ_5 are subsets of U with $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3 \subsetneq \tau_4 \subsetneq \tau_5$ and $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3 \supsetneq \gamma_4 \supsetneq \gamma_5$. It is routine to verify that $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ are double-framed soft subalgebras over U. But $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f, \beta\tilde{\cap}g);X\rangle$ is not a double-framed soft subalgebra over U, since $(\alpha\tilde{\cup}f)(c*b) = (\alpha\tilde{\cup}f)(a) = \alpha(a)\cup f(a) = \tau_2 \not\supseteq \tau_3 = (\alpha\tilde{\cup}f)(c)\cap(\alpha\tilde{\cup}f)(b)$ and/or $(\beta\tilde{\cap}g)(c*b) = (\beta\tilde{\cap}g)(a) = \beta(a)\cap g(a) = \gamma_2 \nsubseteq \gamma_3 = (\beta\tilde{\cap}g)(c)\cup(\beta\tilde{\cap}g)(b)$.

For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U and two subsets γ and δ of U, the *γ-inclusive set* and the *δ-exclusive set* of $\langle(\alpha,\beta);X\rangle$, denoted by $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$, respectively, are defined as follows: $i_X(\alpha;\gamma) := \{x \in X \mid \gamma \subseteq \alpha(x)\}$ and $e_X(\beta;\delta) := \{x \in X \mid \delta \supseteq \beta(x)\}$, respectively. The set $DF_X(\alpha,\beta)_{(\gamma,\delta)} := \{x \in X \mid \gamma \subseteq \alpha(x),\ \delta \supseteq \beta(x)\}$ is called a *double-framed including set* of $\langle(\alpha,\beta);X\rangle$. It is clear that $DF_X(\alpha,\beta)_{(\gamma,\delta)} = i_X(\alpha;\gamma)\cap e_X(\beta;\delta)$.

Theorem 3.8. *For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, the following are equivalent:*

(i) *$\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U.*
(ii) *For every subsets γ and δ of U with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$, the γ-inclusive set and the δ-exclusive set of $\langle(\alpha,\beta);X\rangle$ are subalgebras of X.*

Proof. Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. Let $x,y \in X$ be such that $x,y \in i_X(\alpha;\gamma)$ and $x,y \in e_X(\beta;\delta)$ for every subsets γ and δ of U with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$. It follows from (3.1) that

$$\alpha(x*y) \supseteq \alpha(x)\cap\alpha(y) \supseteq \gamma \text{ and } \beta(x*y) \subseteq \beta(x)\cup\beta(y) \subseteq \delta.$$

Hence $x * y \in i_X(\alpha;\gamma)$ and $x * y \in e_X(\beta;\delta)$, and therefore $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of X.

Conversely, suppose that (ii) is valid. Let $x, y \in X$ be such that $\alpha(x) = \gamma_x$, $\alpha(y) = \gamma_y$, $\beta(x) = \delta_x$ and $\beta(y) = \delta_y$. Taking $\gamma = \gamma_x \cap \gamma_y$ and $\delta = \delta_x \cup \delta_y$ imply that $x, y \in i_X(\alpha;\gamma)$ and $x, y \in e_X(\beta;\delta)$. Hence $x*y \in i_X(\alpha;\gamma)$ and $x*y \in e_X(\beta;\delta)$, which imply that $\alpha(x*y) \supseteq \gamma = \gamma_x \cap \gamma_y = \alpha(x) \cap \alpha(y)$ and $\beta(x*y) \subseteq \delta = \delta_x \cup \delta_y = \beta(x) \cup \beta(y)$. Therefore $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. □

Corollary 3.9. *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U, then the double-framed including set of $\langle(\alpha,\beta);X\rangle$ is a subalgebra of X.*

For any double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, let $\langle(\alpha^*,\beta^*);X\rangle$ be a double-framed soft set over U defined by

$$\alpha^* : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \alpha(x) & \text{if } x \in i_X(\alpha;\gamma), \\ \eta & \text{otherwise,} \end{cases}$$

$$\beta^* : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \beta(x) & \text{if } x \in e_X(\beta;\delta), \\ \rho & \text{otherwise,} \end{cases}$$

where γ, δ, η and ρ are subsets of U with $\eta \subsetneq \alpha(x)$ and $\rho \supsetneq \beta(x)$.

Theorem 3.10. *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U, then so is $\langle(\alpha^*,\beta^*);X\rangle$.*

Proof. Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. Then $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of X for every subsets γ and δ of U with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$, by Theorem 3.8. Let $x, y \in X$. If $x, y \in i_X(\alpha;\gamma)$, then $x * y \in i_X(\alpha;\gamma)$. Thus

$$\alpha^*(x * y) = \alpha(x * y) \supseteq \alpha(x) \cap \alpha(y) = \alpha^*(x) \cap \alpha^*(y).$$

If $x \notin i_X(\alpha;\gamma)$ or $y \notin i_X(\alpha;\gamma)$, then $\alpha^*(x) = \eta$ or $\alpha^*(y) = \eta$. Hence

$$\alpha^*(x * y) \supseteq \eta = \alpha^*(x) \cap \alpha^*(y).$$

Now, if $x, y \in e_X(\beta;\delta)$, then $x * y \in e_X(\beta;\delta)$. Thus

$$\beta^*(x * y) = \beta(x * y) \subseteq \beta(x) \cup \beta(y) = \beta^*(x) \cup \beta^*(y).$$

If $x \notin e_X(\beta;\delta)$ or $y \notin e_X(\beta;\delta)$, then $\beta^*(x) = \rho$ or $\beta^*(y) = \rho$. Hence

$$\beta^*(x * y) \subseteq \rho = \beta^*(x) \cup \beta^*(y).$$

Therefore $\langle(\alpha^*,\beta^*);X\rangle$ is a double-framed soft subalgebra over U. □

Let $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ be double-framed soft sets over U, where X, Y are BE-algebras. The $(\alpha_\wedge,\ \beta_\vee)$-*product* of $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ is defined to be a double-framed soft set $\langle(\alpha_{X\wedge Y},\beta_{X\vee Y});X \times Y\rangle$ over U in which

$$\alpha_{X\wedge Y} : X \times Y \to \mathscr{P}(U),\ (x,y) \mapsto \alpha(x) \cap \alpha(y),$$

$$\beta_{X\vee Y} : X \times Y \to \mathscr{P}(U),\ (x,y) \mapsto \beta(x) \cup \beta(y).$$

Theorem 3.11. *For any BE-algebras X and Y as sets of parameters, let $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ be double-framed soft subalgebras over U. Then the $(\alpha_\wedge,\ \beta_\vee)$-product of $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ is also a double-framed soft subalgebra over U.*

Proof. Note that $(X\times Y,\ \circledast;\ (1,1))$ is a BE-algebra. For any $(x,y),\ (a,b)\in X\times Y$, we have

$$\begin{aligned}
&\alpha_{X\wedge Y}\left((x,y)\circledast(a,b)\right)=\alpha_{X\wedge Y}\left(x*a,y*b\right)\\
&=\alpha(x*a)\cap\alpha(y*b)\supseteq(\alpha(x)\cap\alpha(a))\cap(\alpha(y)\cap\alpha(b))\\
&=(\alpha(x)\cap\alpha(y))\cap(\alpha(a)\cap\alpha(b))\\
&=\alpha_{X\wedge Y}\left(x,y\right)\cap\alpha_{X\wedge Y}\left(a,b\right)
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{X\vee Y}\left((x,y)\circledast(a,b)\right)=\beta_{X\vee Y}\left(x*a,y*b\right)\\
&=\beta(x*a)\cup\beta(y*b)\subseteq(\beta(x)\cup\beta(a))\cup(\beta(y)\cup\beta(b))\\
&=(\beta(x)\cup\beta(y))\cup(\beta(a)\cup\beta(b))\\
&=\beta_{X\vee Y}\left(x,y\right)\cup\beta_{X\vee Y}\left(a,b\right)
\end{aligned}$$

Hence $\langle(\alpha_{X\wedge Y},\beta_{X\vee Y});E\times F\rangle$ is a double-framed soft subalgebra over U. □

4. Double-framed soft filters

Definition 4.1. A double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U is called a *double-framed soft filter* over U if it satisfies :

(4.1) $(\forall x\in X)\left(\alpha(1)\supseteq\alpha(x),\ \beta(1)\subseteq\beta(x)\right)$.
(4.2) $(\forall x,y\in X)\left(\alpha(x*y)\cap\alpha(x)\subseteq\alpha(y),\ \beta(y)\subseteq\beta(x*y)\cup\beta(x)\right)$.

Example 4.2. Let $E=X$ be the set of parameters where $X:=\{1,a,b,c\}$ is a BE-algebra [1] with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over U defined, respectively, as follows:

$$\alpha:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\gamma_2 & \text{if } x\in\{1,c\},\\ \gamma_1 & \text{if } x\in\{a,b\},\end{cases}$$

and

$$\beta:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\tau_2 & \text{if } x\in\{1,c\},\\ \tau_1 & \text{if } x\in\{a,b\},\end{cases}$$

where γ_1,γ_2,τ_1 and τ_2 are subsets of X with $\gamma_1\subsetneq\gamma_2$ and $\tau_2\subsetneq\tau_1$. Then $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter of X over U.

Example 4.3. Let $E = X$ be the set of parameters where $X := \{1, a, b, c\}$ is a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	c
c	1	a	b	1

Let $\langle(\alpha, \beta); X\rangle$ be a double-framed soft set over U defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_2 & \text{if } x \in \{1, a\}, \\ \tau_1 & \text{if } x \in \{b, c\}, \end{cases}$$

and

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \delta_1 & \text{if } x \in \{1, a\}, \\ \delta_2 & \text{if } x \in \{b, c\}, \end{cases}$$

where τ_1, τ_2, δ_1 and δ_2 are subsets of U with $\tau_1 \subsetneq \tau_2$ and $\delta_1 \subsetneq \delta_2$. Then $\langle(\alpha, \beta); X\rangle$ is a double-framed soft subalgebra over U. But $\langle(\alpha, \beta); X\rangle$ is not a double-framed soft filter of X over U, since $\alpha(a * b) \cap \alpha(a) = \tau_2 \nsubseteq \tau_1 = \alpha(b)$ and/or $\beta(b) = \delta_2 \nsubseteq \delta_1 = \beta(a * b) \cup \beta(a)$.

Theorem 4.4. *For a double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U, the following are equivalent:*

(i) *$\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter over U.*

(ii) *For every subsets γ and δ of U with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$, the γ-inclusive set and the δ-exclusive set of $\langle(\alpha, \beta); X\rangle$ are filters of X.*

Proof. Assume that $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter over U. Let $x, y \in X$ be such that $x * y, x \in i_X(\alpha; \gamma)$ and $x * y, x \in e_X(\beta; \delta)$ for every subsets γ and δ of U with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$. It follows from Definition 4.1 that

$$\alpha(1) \supseteq \alpha(x) \supseteq \gamma,\ \delta \supseteq \beta(x) \supseteq \beta(1),$$

$$\alpha(y) \supseteq \alpha(x * y) \cap \alpha(x) \supseteq \gamma \text{ and } \beta(y) \subseteq \beta(x * y) \cup \beta(x) \subseteq \delta.$$

Hence $1, y \in i_X(\alpha; \gamma)$ and $1, y \in e_X(\beta; \delta)$, and therefore $i_X(\alpha; \gamma)$ and $e_X(\beta; \delta)$ are filters of X.

Conversely, suppose that $i_X(\alpha; \gamma)$ and $e_X(\beta; \delta)$ are filters of X for all $\gamma, \delta \in \mathscr{P}(U)$ with $i_X(\alpha; \gamma) \neq \emptyset$ and $e_X(\beta; \delta) \neq \emptyset$. Put $\alpha(x) = \gamma$ for any $x \in X$. Then $x \in i_X(\alpha; \gamma)$. Since $i_X(\alpha; \gamma)$ is a filter of X, we have $1 \in i_X(\alpha; \gamma)$ and so $\alpha(x) = \gamma \subseteq \alpha(1)$. For any $x, y \in X$, let $\alpha(x * y) = \gamma_{x*y}$ and $\alpha(x) = \gamma_x$. Take $\gamma = \gamma_{x*y} \cap \gamma_x$. Then $x * y \in i_X(\alpha; \gamma)$ and $x \in i_X(\alpha; \gamma)$ which imply $y \in i_X(\alpha; \gamma)$. Hence $\alpha(y) \supseteq \gamma = \gamma_{x*y} \cap \gamma_x = \alpha(x * y) \cap \alpha(x)$.

For any $x \in X$, let $\beta(x) = \delta$. Then $x \in e_X(\beta; \delta)$. Since $e_X(\beta; \delta)$ is a filter of X, we have $1 \in e_X(\beta; \delta)$ and so $\beta(x) = \delta \supseteq \beta(1)$. For any $x, y \in X$, let $\beta(x * y) = \delta_{x*y}$ and $\beta(x) = \delta_x$. Take $\delta = \delta_{x*y} \cup \delta_x$. Then $x * y \in e_X(\beta; \delta)$ and $x \in e_X(\beta; \delta)$ which imply $y \in e_X(\beta; \delta)$. Hence $\beta(y) \subseteq \delta = \delta_{x*y} \cup \delta_x = \beta(x * y) \cup \beta(x)$. Therefore $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter over U. □

Proposition 4.5. *Every double-framed soft filter $\langle(\alpha,\beta);X\rangle$ over U satisfies the following condition:*

(i) $(\forall x,y\in X)\,(x\leq y\Rightarrow \alpha(x)\subseteq\alpha(y),\ \beta(x)\supseteq\beta(y))$,
(ii) $(\forall a,x\in X)\,(\alpha(a)\subseteq\alpha((a*x)*x),\beta(a)\supseteq\beta((a*x)*x))$.

Proof. (i) Assume that $x\leq y$ for all $x,y\in X$. Then $x*y=1$. Hence we have $\alpha(x)=\alpha(1)\cap\alpha(x)=\alpha(x*y)\cap\alpha(x)\subseteq\alpha(y)$ and $\beta(x)=\beta(1)\cup\beta(x)=\beta(x*y)\cup\beta(x)\supseteq\beta(y)$.
(ii) Taking $y:=(a*x)*x$ and $x:=a$ in Definition 4.1, we have $\alpha((a*x)*x)\supseteq\alpha(a*((a*x)*x))\cap\alpha(a)=\alpha((a*x)*(a*x))\cap\alpha(a)=\alpha(1)\cap\alpha(a)=\alpha(a)$ and $\beta((a*x)*x)\subseteq\beta(a*((a*x)*x))\cup\beta(a)=\beta((a*x)*(a*x))\cup\beta(a)=\beta(1)\cup\beta(a)=\beta(a)$. □

Theorem 4.6. *Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over U. Then $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over U if and only if it satisfies the following condition:*

(4.3) $(\forall x,y,z\in X)(z\leq x*y\Rightarrow\alpha(y)\supseteq\alpha(x)\cap\alpha(z)$ *and* $\beta(y)\subseteq\beta(x)\cup\beta(z))$.

Proof. Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over U. Let $x,y,z\in X$ be such that $z\leq x*y$. By Proposition 4.5(i), we have $\alpha(y)\supseteq\alpha(x*y)\cap\alpha(x)\supseteq\alpha(z)\cap\alpha(x)$ and $\beta(y)\subseteq\beta(x*y)\cup\beta(x)\subseteq\beta(z)\cup\beta(x)$.

Conversely, suppose that $\langle(\alpha,\beta);X\rangle$ satisfies (4.3). By (BE2), we have $x\leq x*1=1$. Using (4.3), we obtain $\alpha(1)\supseteq\alpha(x)$ and $\beta(1)\subseteq\beta(x)$ for all $x\in X$. By (BE1) and (BE4), we get $x\leq(x*y)*y$ for all $x,y\in X$. It follows from (4.3) that $\alpha(y)\supseteq\alpha(x*y)\cap\alpha(x)$ and $\beta(y)\subseteq\beta(x*y)\cap\beta(x)$. Therefore $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over U. □

For any double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, let $\langle(\alpha^*,\beta^*);X\rangle$ be a double-framed soft set over U defined by

$$\alpha^*:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\alpha(x) & \text{if } x\in i_X(\alpha;\gamma),\\ \emptyset & \text{otherwise,}\end{cases}$$

$$\beta^*:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\beta(x) & \text{if } x\in e_X(\beta;\delta),\\ U & \text{otherwise,}\end{cases}$$

where γ,δ are nonempty subsets of U.

Theorem 4.7. *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over U, then so is $\langle(\alpha^*,\beta^*);X\rangle$.*

Proof. Assume that $\langle(\alpha,\beta);E\rangle$ is a double-framed soft filter over U. Then $i_X(\alpha;\gamma)(\neq\emptyset)$ and $e_X(\beta;\delta)(\neq\emptyset)$ are filters of X for every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$, by Theorem 4.4. Hence $1\in i_X(\alpha;\gamma),1\in e_X(\beta;\delta)$ and so $\alpha^*(1)=\alpha(1)\supseteq\alpha(x)=\alpha^*(x),\beta^*(1)=\beta(1)\subseteq\beta(x)=\beta^*(x)$ for all $x\in X$. Let $x,y\in X$. If $x*y\in i_X(\alpha;\gamma)$ and $x\in i_X(\alpha;\gamma)$, then $y\in i_X(\alpha;\gamma)$. Hence $\alpha^*(y)=\alpha(y)\supseteq\alpha(x*y)\cap\alpha(x)=\alpha^*(x*y)\cap\alpha^*(x)$. If $x*y\notin i_X(\alpha;\gamma)$ or $x\notin i_X(\alpha;\gamma)$, then $\alpha^*(x*y)=\emptyset$ or $\alpha^*(x)=\emptyset$. Therefore

$$\alpha^*(y)\supseteq\emptyset=\alpha^*(x*y)\cap\alpha^*(x).$$

Now, if $x * y, x \in e_X(\beta;\delta)$, then $y \in e_X(\beta;\delta)$. Thus

$$\beta^*(y) = \beta(y) \subseteq \beta(x*y) \cup \beta(x) = \beta^*(x*y) \cup \beta^*(x).$$

If $x * y \notin e_X(\beta;\delta)$ or $x \notin e_X(\beta;\delta)$, then $\beta^*(x*y) = U$ or $\beta^*(x) = U$. Hence

$$\beta^*(y) \subseteq \beta^*(x*y) \cup \beta^*(x).$$

Therefore $\langle(\alpha^*, \beta^*); X\rangle$ is a double-framed soft filter over U. □

Theorem 4.8. *A double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U is a double-framed soft filter over U if and only if it satisfies the following conditions:*

(i) $(\forall x, y \in X)(\alpha(y * x) \supseteq \alpha(x), \beta(y * x) \subseteq \beta(x))$,
(ii) $(\forall x, a, b \in X)(\alpha((a * (b * x)) * x) \supseteq \alpha(a) \cap \alpha(b), \beta((a * (b * x)) * x) \subseteq \beta(a) \cap \beta(b))$.

Proof. Assume that $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter algebra over U. It follows from Definition 4.1 that $\alpha(y * x) \supseteq \alpha(x * (y * x)) \cap \alpha(x) = \alpha(1) \cap \alpha(x) = \alpha(x)$ and $\beta(y * x) \subseteq \beta(x * (y * x)) \cup \beta(x) = \beta(1) \cup \beta(x) = \beta(x)$ for all $x, y \in X$. Using Proposition 4.5(ii), we have $\alpha((a*(b*x))*x) \supseteq \alpha(b*((a*(b*x))*x)) \cap \alpha(b) = \alpha((a*(b*x))*(b*x)) \cap \alpha(b) \supseteq \alpha(a) \cap \alpha(b)$ and $\beta((a*(b*x))*x) \subseteq \beta(b*((a*(b*x))*x)) \cup \beta(b) = \beta((a*(b*x))*(b*x)) \cup \beta(b) \subseteq \beta(a) \cup \beta(b)$ for any $a, b, x \in X$.

Conversely, let $\langle(\alpha, \beta); X\rangle$ be a double-framed soft set over U satisfying conditions (i) and (ii). If $y := x$ in (i), then $\alpha(1) = \alpha(x * x) \supseteq \alpha(x)$ and $\beta(x * x) = \beta(1) \subseteq \beta(x)$ for all $x \in X$. Using (ii), we have $\alpha(y) = \alpha(1 * y) = \alpha(((x * y) * (x * y)) * y) \supseteq \alpha(x * y) \cap \alpha(x)$ and $\beta(y) = \beta(1 * y) = \beta(((x * y) * (x * y)) * y) \subseteq \beta(x * y) \cap \alpha(x)$ for all $x, y \in X$. Hence $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter of X. □

Theorem 4.9. *The double-framed soft int-uni set of two double-framed soft filters $\langle(\alpha, \beta); X\rangle$ and $\langle(f, g); X\rangle$ over U is a double-framed soft filter over U.*

Proof. For any $x, y \in X$, we have $(\alpha\tilde{\cap}f)(1) = \alpha(1) \cap f(1) \supseteq \alpha(x) \cap f(x) = (\alpha\tilde{\cap}f)(x)$, $(\beta\tilde{\cup}g)(1) = \beta(1) \cup g(1) \subseteq \beta(x) \cup g(x) = (\beta\tilde{\cup}g)(x)$ and

$$\begin{aligned}(\alpha\tilde{\cap}f)(y) =& \alpha(y) \cap f(y)\\ \supseteq& (\alpha(x*y) \cap \alpha(x)) \cap (f(x*y) \cap f(x))\\ =& (\alpha(x*y) \cap f(x*y)) \cap (\alpha(x) \cap f(x))\\ =& (\alpha\tilde{\cap}f)(x*y) \cap (\alpha\tilde{\cap}f)(x)\end{aligned}$$

and

$$\begin{aligned}(\beta\tilde{\cup}g)(y) =& \beta(y) \cup g(y)\\ \subseteq& (\beta(x*y) \cup \beta(x)) \cup (g(x*y) \cup g(x))\\ =& (\beta(x*y) \cup g(x*y)) \cup (\beta(x) \cup g(x))\\ =& (\beta\tilde{\cup}g)(x*y) \cup (\beta\tilde{\cup}g)(x).\end{aligned}$$

Therefore $\langle(\alpha, \beta); X\rangle \sqcap \langle(f, g); X\rangle$ is a double-framed soft filter over U. □

The following example shows that the double-framed soft uni-int set of two double-framed soft filter $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U may not be a double-framed soft filter over U.

Example 4.10. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra [2] with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Let $\langle(\alpha,\beta);X\rangle$, $\langle(f,g);X\rangle$ be double-framed soft sets over U defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x \in \{1, c\}, \\ \gamma_1 & \text{if } x \in \{a, b, d, 0\}, \end{cases}$$

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_3 & \text{if } x \in \{1, c\}, \\ \tau_1 & \text{if } x \in \{a, b, d, 0\}, \end{cases}$$

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x \in \{1, a, d\}, \\ \gamma_2 & \text{if } x \in \{c, d, 0\}, \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_4 & \text{if } x \in \{1, a, b\}, \\ \tau_2 & \text{if } x \in \{c, d, 0\}, \end{cases}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \tau_1, \tau_2, \tau_3$ and τ_4 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$ and $\tau_1 \supsetneq \tau_2 \supsetneq \tau_3 \supsetneq \tau_4$. Then $\langle(\alpha,\beta);X\rangle$, $\langle(f,g);X\rangle$ are double-framed soft filters over U. But $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f, \beta\tilde{\cap}g);X\rangle$ is not a double-framed soft filter over U, since

$$\begin{aligned}(\alpha\tilde{\cup}f)(c*d) \cap (\alpha\tilde{\cup}f)(c) =& (\alpha\tilde{\cup}f)(a) \cap (\alpha\tilde{\cup}f)(c) \\ =& (\alpha(a) \cup f(a)) \cap (\alpha(c) \cup f(c)) \\ =& \gamma_4 \cap \gamma_3 = \gamma_3 \nsubseteq \gamma_2 = \gamma_1 \cup \gamma_2 \\ =& \alpha(d) \cup f(d) = (\alpha\tilde{\cup}f)(d)\end{aligned}$$

and/or

$$\begin{aligned}(\beta\tilde{\cap}g)(c*d) \cup (\beta\tilde{\cap}g)(c) =& (\beta\tilde{\cap}g)(a) \cup (\beta\tilde{\cap}g)(c) \\ =& (\beta(a) \cap g(a)) \cup (\beta(c) \cap g(c)) \\ =& (\tau_1 \cap \tau_4) \cup (\tau_3 \cap \tau_2) = \tau_4 \cup \tau_3 = \tau_3 \\ \nsupseteq& \tau_2 = \tau_1 \cap \tau_2 = \beta(d) \cap g(d) = (\beta\tilde{\cap}g)(d).\end{aligned}$$

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HYERS-ULAM STABILITY OF ADDITIVE FUNCTION EQUATIONS IN PARANORMED SPACES

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ABSTRACT. In this paper, we prove the Hyers-Ulam stability of the following additive functional equations

$$\begin{aligned} f\left(\frac{x+y}{2}+z+w\right) &= \frac{1}{2}f(x)+\frac{1}{2}f(y)+f(z)+f(w), \\ f\left(\frac{x+y+z}{3}+w\right) &= \frac{1}{3}f(x)+\frac{1}{3}f(y)+\frac{1}{3}f(z)+f(w) \end{aligned}$$

in paranormed spaces.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [5] and Steinhaus [23] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [6, 9, 11, 12, 18]). This notion was defined in normed spaces by Kolk [10].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [25] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \le P(x) + P(y)$ (triangle inequality)
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_nx_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. See [2, 3, 4, 13, 14, 15, 17, 19, 20, 21, 22] for more information on the stability problems of functional equations.

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Using the direct method, we prove the Hyers-Ulam stability of the following additive functional equations

$$f\left(\frac{x+y}{2}+z+w\right) = \frac{1}{2}f(x)+\frac{1}{2}f(y)+f(z)+f(w), \tag{1.1}$$

$$f\left(\frac{x+y+z}{3}+w\right) = \frac{1}{3}f(x)+\frac{1}{3}f(y)+\frac{1}{3}f(z)+f(w) \tag{1.2}$$

in paranormed spaces.

Throughout this paper, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

2. Hyers-Ulam stability of the functional equation (1.1)

In this section, we prove the Hyers-Ulam stability of the functional equation (1.1) in paranormed spaces.

Note that $P(3x) \le 3P(x)$ for all $x \in Y$.

Theorem 2.1. *Let r, θ be positive real numbers with $r > 1$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y}{2}+z+w\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)-f(z)-f(w)\right) \le \theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r) \tag{2.1}$$

for all $x, y, w, z \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x)-A(x)) \le \frac{4\theta}{3^r-3}\|x\|^r \tag{2.2}$$

for all $x \in Y$.

Proof. Letting $w = z = y = x$ in (2.1), we get

$$P(f(3x)-3f(x)) \le 4\theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x)-3f\left(\frac{x}{3}\right)\right) \le \frac{4}{3^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$P\left(3^l f\left(\frac{x}{3^l}\right)-3^m f\left(\frac{x}{3^m}\right)\right) \le \sum_{j=l}^{m-1} P\left(3^j f\left(\frac{x}{3^j}\right)-3^{j+1} f\left(\frac{x}{3^{j+1}}\right)\right) \le \frac{4}{3^r}\sum_{j=l}^{m-1}\frac{3^j}{3^{rj}}\theta\|x\|^r \tag{2.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in Y$. It follows from (2.3) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $A : Y \to X$ by

$$A(x) := \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$$

for all $x \in Y$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.3), we get (2.2).

It follows from (2.1) that

$$\begin{aligned}
&P\left(A\left(\frac{x+y}{2}+z+w\right)-\frac{1}{2}A(x)-\frac{1}{2}A(y)-A(z)-A(w)\right)\\
&=\lim_{n\to\infty} P\left(3^n\left(f\left(\frac{x+y}{2\cdot 3^n}+\frac{z+w}{3^n}\right)-\frac{1}{2}f\left(\frac{x}{3^n}\right)-\frac{1}{2}f\left(\frac{y}{3^n}\right)-f\left(\frac{z}{3^n}\right)-f\left(\frac{w}{3^n}\right)\right)\right)\\
&\le \lim_{n\to\infty} 3^n P\left(f\left(\frac{x+y}{2\cdot 3^n}+\frac{z+w}{3^n}\right)-\frac{1}{2}f\left(\frac{x}{3^n}\right)-\frac{1}{2}f\left(\frac{y}{3^n}\right)-f\left(\frac{z}{3^n}\right)-f\left(\frac{w}{3^n}\right)\right)\\
&\le \lim_{n\to\infty} \frac{3^n\theta}{3^{nr}}(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r)=0
\end{aligned}$$

for all $x, y, z, w \in Y$. Hence $A\left(\frac{x+y}{2}+z+w\right)=\frac{1}{2}A(x)+\frac{1}{2}A(y)+A(z)+A(w)$ for all $x, y, z, w \in Y$ and so the mapping $A : Y \to X$ is additive.

Now, let $T : Y \to X$ be another additive mapping satisfying (2.2). Then we have

$$\begin{aligned}
P(A(x)-T(x)) &= P\left(3^n\left(A\left(\frac{x}{3^n}\right)-T\left(\frac{x}{3^n}\right)\right)\right)\\
&\le 3^n P\left(A\left(\frac{x}{3^n}\right)-T\left(\frac{x}{3^n}\right)\right)\\
&\le 3^n\left(P\left(A\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right)\right)+P\left(T\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right)\right)\right)\\
&\le \frac{8\cdot 3^n}{(3^r-3)3^{nr}}\theta\|x\|^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $A(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of A. Thus the mapping $A : Y \to X$ is a unique additive mapping satisfying (2.2). □

Theorem 2.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\|f\left(\frac{x+y}{2}+z+w\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)-f(z)-f(w)\right\| \le P(x)^r+P(y)^r+P(z)^r+P(w)^r \tag{2.4}$$

for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{4}{3-3^r}P(x)^r \tag{2.5}$$

for all $x \in X$.

Proof. Letting $w = z = y = x$ in (2.4), we get

$$\|3f(x)-f(3x)\| \le 4P(x)^r$$

and so

$$\left\|f(x)-\frac{1}{3}f(3x)\right\| \le \frac{4}{3}P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{3^l}f(3^l x)-\frac{1}{3^m}f(3^m x)\right\| \le \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}f(3^j x)-\frac{1}{3^{j+1}}f(3^{j+1}x)\right\| \le \frac{4}{3}\sum_{j=l}^{m-1}\frac{3^{rj}}{3^j}P(x)^r \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{\frac{1}{3^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{3^n} f(3^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.5).

It follows from (2.4) that

$$\begin{aligned}
&\left\| A\left(\frac{x+y}{2} + z + w\right) - \frac{1}{2}A(x) - \frac{1}{2}A(y) - A(z) - A(w) \right\| \\
&= \lim_{n\to\infty} \frac{1}{3^n} \left\| f\left(3^n\left(\frac{x+y}{2} + z + w\right)\right) - \frac{1}{2} f\left(3^n x\right) - \frac{1}{2} f\left(3^n y\right) - f\left(3^n z\right) - f\left(3^n w\right) \right\| \\
&\le \lim_{n\to\infty} \frac{3^{nr}}{3^n} (P(x)^r + P(y)^r + P(z)^r + P(w)^r) = 0
\end{aligned}$$

for all $x, y, z, w \in X$. Thus $A\left(\frac{x+y}{2} + z + w\right) = \frac{1}{2}A(x) + \frac{1}{2}A(y) + A(z) + A(w)$ for all $x, y, z, w \in X$ and so the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \frac{1}{3^n} \left\|A\left(3^n x\right) - T\left(3^n x\right)\right\| \\
&\le \frac{1}{3^n} \left(\left\|A\left(3^n x\right) - f\left(3^n x\right)\right\| + \left\|T\left(3^n x\right) - f\left(3^n x\right)\right\|\right) \\
&\le \frac{8 \cdot 3^{nr}}{(3 - 3^r)3^n} P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.5). □

Similarly, one obtains the following.

Theorem 2.3. *Let r, θ be positive real numbers with $r > \frac{1}{4}$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y}{2} + z + w\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) - f(z) - f(w)\right) \le \theta\|x\|^r\|y\|^r\|z\|^r\|w\|^r$$

for all $x, y, z, w \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x) - A(x)) \le \frac{\theta}{81^r - 3}\|x\|^{4r}$$

for all $x \in Y$.

Theorem 2.4. *Let r be a positive real number with $r < \frac{1}{4}$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\| f\left(\frac{x+y}{2} + z + w\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) - f(z) - f(w) \right\| \le P(x)^r P(y)^r P(z)^r P(w)^r$$

for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{3 - 81^r} P(x)^{4r}$$

for all $x \in X$.

3. Hyers-Ulam stability of the functional equation (1.2)

In this section, we prove the Hyers-Ulam stability of the functional equation (1.2) in paranormed spaces.

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 3.1. *Let r, θ be positive real numbers with $r > 1$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y+z}{3}+w\right)-\frac{1}{3}f(x)-\frac{1}{3}f(y)-\frac{1}{3}f(z)-f(w)\right) \le \theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r) \quad (3.1)$$

for all $x, y, w, z \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x)-A(x)) \le \frac{4\theta}{2^r-2}\|x\|^r$$

for all $x \in Y$.

Proof. Letting $w = z = y = x$ in (3.1), we get

$$P(f(2x)-2f(x)) \le 4\theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x)-2f\left(\frac{x}{2}\right)\right) \le \frac{4}{2^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$P\left(2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right) \le \sum_{j=l}^{m-1} P\left(2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right) \le \frac{4}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|^r$$

for all nonnegative integers m and l with $m > l$ and all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\|f\left(\frac{x+y+z}{3}+w\right)-\frac{1}{3}f(x)-\frac{1}{3}f(y)-\frac{1}{3}f(z)-f(w)\right\| \le P(x)^r+P(y)^r+P(z)^r+P(w)^r \quad (3.2)$$

for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{4}{2-2^r}P(x)^r$$

for all $x \in X$.

Proof. Letting $w = z = y = x$ in (3.2), we get

$$\|2f(x)-f(2x)\| \le 4P(x)^r$$

and so

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le 2P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^m}f(2^m x)\right\| \le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x)-\frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \le 2\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}P(x)^r$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Similarly, one obtains the following.

Theorem 3.3. *Let r, θ be positive real numbers with $r > \frac{1}{4}$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y+z}{3}+w\right)-\frac{1}{3}f(x)-\frac{1}{3}f(y)-\frac{1}{3}f(z)-f(w)\right) \leq \theta\|x\|^r\|y\|^r\|z\|^r\|w\|^r$$

for all $x, y, z, w \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x) - A(x)) \leq \frac{\theta}{16^r - 2}\|x\|^{4r}$$

for all $x \in Y$.

Theorem 3.4. *Let r be a positive real number with $r < \frac{1}{4}$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\|f\left(\frac{x+y+z}{3}+w\right)-\frac{1}{3}f(x)-\frac{1}{3}f(y)-\frac{1}{3}f(z)-f(w)\right\| \leq P(x)^r P(y)^r P(z)^r P(w)^r$$

for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2 - 16^r} P(x)^{4r}$$

for all $x \in X$.

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New Uzawa-type method for nonsymmetric saddle point problems

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Abstract

In this paper, based on the Hermitian and skew-Hermitian splitting of the non-Hermitian positive definite (1, 1)-block of the saddle point matrix, a new Uzawa-type iteration method is proposed for solving a class of nonsymmetric saddle point problems. The convergence properties of this iteration method are analyzed. Numerical results verify the effectiveness and robustness of the proposed method.

Keywords: Saddle-point problem, Uzawa-type iteration method, Convergence
2000 MSC: 65F10, 65F50

1. Introduction

Consider the nonsymmetric saddle point problems of the form

$$\mathcal{A}u = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, $B \in \mathbb{C}^{n\times m}$ is a rectangular matrix of full column rank, $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^m$ are given vectors, with $m \leq n$.

The saddle point problem (1) arises in a variety of scientific and engineering applications, such as computational fluid dynamics, constrained optimization, optimal control, weighted least squares problems, electronic networks and computer graphics, and typically result from mixed or hybrid finite element approximation of second-order elliptic problems or the Stokes equations; see [1, 12] and the references therein.

Since matrix blocks A and B are large and sparse, (1) is suitable for being solved by the iterative methods. Most efficient iterative methods have been studied in many literatures, including Uzawa-type methods [10, 11, 14, 16], Hermitian and skew-Hermitian splitting (HSS) iterative method and its variant schemes [3, 5, 6, 7, 9, 17], preconditioned Krylov subspace iterative methods [3, 15] and so on. See [1, 12] and the references therein for a comprehensive survey about iterative methods and preconditioning techniques.

Within these methods, Uzawa method received wide attention and obtained considerable achievements in recent years. The iteration scheme of Uzawa method can be described, for a positive parameter τ, as

$$\begin{cases} x_{k+1} = A^{-1}(f - By_k), \\ y_{k+1} = y_k + \tau(B^* x_{k+1} - g). \end{cases}$$

Note that there is a linear system $Ax = q$ needs to be solved at each step of Uzawa method, we prefer to use iterative method to approximate its solution since matrix A is always large and sparse. When A is Hermitian positive definite, by using classical splitting iteration to approximate x_{k+1} in each step of Uzawa method, a class of Uzawa-type iteration methods for solving the Hermitian saddle-point problems are studied in [21, 22]. When A is no-Hermitian positive definite, we can split A as

$$A = H + S, \text{ with } H = \frac{1}{2}(A + A^*), \ \ S = \frac{1}{2}(A - A^*), \tag{2}$$

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and then approximate x_{k+1} in each step of Uzawa method by using the efficient HSS method [7], then the Uzawa-HSS method for solving nonsingular non-Hermitian saddle point problem is propsed; see [19, 20].

The HSS method received much attentions as it is an efficient and robust method for solving non-Hermitian positive definite systems of linear equations; see for example [2, 4, 7, 8, 9, 13, 18]. There are two linear subsystems with $\alpha I_n + H$ and $\alpha I_n + S$ needs to be solved at each step of the HSS method. Here and in the sequence of the paper I_i denotes the identity matrix with order i. The solution of linear subsystem with $\alpha I_n + H$ can be easily obtained by CG method, however, the solution of linear subsystem with $\alpha I_n + S$ is not easy to obtain. To avoid solving a shift skew-Hermitian linear subsystem with $\alpha I_n + S$, based on the splitting (2), a new iteration method is presented for solving non-Hermitian positive definite system of linear equations [18] recently. The iteration scheme of new method used for solving $Ax = q$ can be written as

$$\begin{cases} Hx_{k+1/2} = -Sx_k + q, \\ (\alpha I_n + H)x_{k+1} = (\alpha I_n - S)x_{k+1/2} + q. \end{cases} \tag{3}$$

Theoretical analysis as well as numerical experiments show that the new method (3) is also an efficient and robust method for solving non-Hermitian positive definite and normal linear system with strong Hermitian parts [18].

In this paper, to avoid solving a shift skew-Hermitian linear subsystem at each step of Uzawa method, we use the iteration (3) to approximate x_{k+1}, then a new Uzawa-type method is established. The convergence properties of this novel method for saddle point problem (1) will be carefully analyzed. In addition, we test the effectiveness and robustness of the proposed method by comparing its iteration number and elapsed CPU time with those of the Uzawa-HSS [19, 20] and the GMRES methods.

2. A Uzawa-type method

The iteration scheme (3) in [18] used for solving non-Hermitian positive definite and normal linear system $Ax = q$ can be written equivalently as

$$x_{k+1} = T(\alpha)x_k + N(\alpha)q,$$

here α is a positive iteration parameter,

$$\begin{cases} T(\alpha) & = (\alpha I_n + H)^{-1}(\alpha I_n - S)H^{-1}(-S) \\ & = (\alpha I_n + H)^{-1}H^{-1}(\alpha I_n - S)(-S) \\ N(\alpha) & = (\alpha I_n + H)^{-1}\left(I_n + (\alpha I_n - S)H^{-1}\right) \\ & = (\alpha I_n + H)^{-1}H^{-1}(\alpha I_n + H - S). \end{cases}$$

In this paper, we assumption that the (1, 1)-block matrix A of (1) is normal, i.e., $AA^* = A^*A$.

Introducing a Hermitian positive definite preconditioning matrix Q for the iteration scheme, and using iteration (3) to approximate x_{k+1}, then we present the following Uzawa-type method for solving the saddle point problem (1):

Method 2.1. (New Uzawa-type method). *Given initial guesses $x_0 \in \mathbb{C}^n$ and $y_0 \in \mathbb{C}^m$, for $k = 0, 1, 2\cdots$, until x_k and y_k convergence*

(i) *compute x_{k+1} from iteration scheme $x_{k+1} = T(\alpha)x_k + N(\alpha)(f - By_k)$;*

(ii) *compute y_{k+1} from iteration scheme $y_{k+1} = y_k + \tau Q^{-1}(B^*x_{k+1} - g)$.*

The Method 2.1 can be equivalently written in matrix-vector form as:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = G(\alpha,\tau)\begin{bmatrix} x_k \\ y_k \end{bmatrix} + M(\alpha,\tau)\begin{bmatrix} f \\ g \end{bmatrix}. \tag{4}$$

where

$$G(\alpha,\tau) = \begin{bmatrix} T(\alpha) & -N(\alpha)B \\ \tau Q^{-1}B^*T(\alpha) & I_m - \tau Q^{-1}B^*N(\alpha)B \end{bmatrix} \tag{5}$$

is the iteration matrix of Method 21 and

$$M(\alpha,\tau) = \begin{bmatrix} N(\alpha) & 0 \\ \tau Q^{-1}B^*N(\alpha) & -\tau Q^{-1} \end{bmatrix}.$$

Notice that Method 2.1 possess the same iteration scheme as the Uzawa-HSS method [20, 19], hence the efficiency and robustness of the Uzawa-HSS method may be followed by Method 2.1. Moreover, Method 2.1 use iteration (3) to approximate x_{k+1}, the solution of the shift skew-Hermitian subsystem is avoided, we may hope that Method 2.1 uses less CPU time and iteration number comparing with the Uzawa-HSS method.

3. Convergence of Method 2.1

In this section, we study the convergence of Method 2.1 used for solving saddle-point problem (1). It is well known that Method 2.1 is convergent if and only if the spectral radius of $G(\alpha,\tau)$ is less than 1, i.e., $\rho(G(\alpha,\tau)) < 1$. Let λ be an eigenvalue of $G(\alpha,\tau)$ and $[u^*,\ v^*]^*$ be the corresponding eigenvector. Then we have

$$\begin{cases} (\alpha I_n - S)(-S)u - (\alpha I_n + H - S)Bv = \lambda H(\alpha I_n + H)u, \\ \lambda B^* u - \frac{\lambda}{\tau} Qv = -\frac{1}{\tau} Qv. \end{cases} \tag{6}$$

To study the convergence of Method 2.1, a lemma is given first.

Lemma 3.1. [11] *Both roots of the complex quadratic equation $\lambda^2 - \phi\lambda + \psi = 0$ have modulus less than one if and only if $|\phi - \overline{\phi}\psi| + |\psi|^2 < 1$, where $\overline{\phi}$ denotes the conjugate complex of ϕ.*

For the convergence of Method 2.1, we have the following results.

Lemma 3.2. *Let A be non-Hermitian positive definite and normal, and B be of full column rank. If λ is an eigenvalue of iteration matrix $G(\alpha,\tau)$, and $[u^*,\ v^*]^*$ is the corresponding eigenvector with $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$, then $\lambda \neq 1$ and $u \neq 0$.*

Proof. If $\lambda = 1$, noticing that τ is a positive parameter, then from (6) we have

$$\begin{cases} Au + Bv = 0, \\ B^* u = 0. \end{cases}$$

It is easy to see that the coefficient matrix $\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$ is nonsingular, hence we have $u = 0$ and $v = 0$, which contradicts the assumption that $[u^*,\ v^*]^*$ is an eigenvector of the iteration matrix $G(\alpha,\tau)$, so $\lambda \neq 1$.

If $u = 0$ then the first equality in (6) reduce to $Bv = 0$. Because B is a matrix of full column rank, we can obtain $v = 0$, which is a contradiction. Hence $u \neq 0$. □

Theorem 3.1. *Let A be non-Hermitian positive definite and normal, B be of full column rank, Q be Hermitian positive definite. Then Method 2.1 used for solving nonsingular saddle-point problem (1) is convergent if and only if parameters α and τ satisfy*

$$\alpha > \max\left\{ \frac{-\omega_1^3 + \sqrt{\mu_n^2(\omega_n^4 + \mu_n^2\omega_n^2 - \mu_1^4)}}{\omega_1^2 - \mu_n^2},\ 0 \right\},\ \ when\ \ \omega_1^2 > \mu_n^2$$

or

$$0 < \alpha < \frac{\omega_1^3 + \sqrt{\mu_1^2(\omega_1^4 + \omega_1^2\mu_1^2 - \mu_n^4)}}{\mu_n^2 - \omega_1^2},\ \ when\ \ \omega_1^2 < \mu_n^2$$

or

$$\alpha > 0,\ \ when\ \ \omega^2 = \mu^2$$

and

$$0 < \tau < \frac{2[(\alpha\omega_1 + \omega_1^2)^2 - \alpha^2\mu_n^2 - \mu_n^4][\omega_1(\alpha + \omega_1)^2 + \omega_1\mu_1^2]}{t_n\omega_n^2[(\alpha + \omega_n)^2 + \mu_n^2]^2 + t_n\mu_n^2[\alpha^2 + \mu_n^2 - \omega_1^2]^2}.$$

where $\omega = \frac{u^*Hu}{u^*u}$, $t = \frac{u^*BQ^{-1}B^*u}{u^*u}$, and $\mathbf{i}\mu = \frac{u^*(-S)u}{u^*u}$, $\mathbf{i}$ is the imaginary unit, μ_1 and μ_n are the minimum and the maximum value of μ, ω_1 and ω_n are the minimum and the maximum value of ω, t_1 and t_n are the minimum and the maximum value of t, respectively.

Proof Due to the result of Lemma 3.2 that $\lambda \neq 1$ and the assumption Q is Hermitian positive definite, solving v from the second equality of (6) and then taking it into the first equality of (6), we have

$$(\alpha I_n - S)(-S)u + \frac{\tau\lambda}{1-\lambda}(\alpha I_n + H - S)BQ^{-1}B^*v = \lambda H(\alpha I_n + H)u. \tag{7}$$

From Lemma 3.2, we known that $u \neq 0$. Multiplying $u^*/(u^*u)$ to the both sides of (7) from left gives

$$\frac{u^*(\alpha I_n - S)(-S)u}{u^*u} + \frac{\lambda\tau}{1-\lambda}\frac{u^*(\alpha I_n + H - S)BQ^{-1}B^*u}{u^*u} = \lambda\frac{u^*H(\alpha I_n + H)u}{u^*u}. \tag{8}$$

Denote

$$\omega = \frac{u^*Hu}{u^*u},\ t = \frac{u^*BQ^{-1}B^*u}{u^*u},\ \mathbf{i}\mu = \frac{u^*(-S)u}{u^*u},$$

where $\mathbf{i}$ is the imaginary unit. It is easy to see that $\omega, t > 0$, and (8) can be rewritten as

$$\lambda^2 - \phi\lambda + \psi = 0, \tag{9}$$

where

$$\phi = \frac{\alpha\omega + \omega^2 - \mu^2 - \alpha\tau t - \omega\tau t + (\alpha\mu - \tau\mu t)\mathbf{i}}{\alpha\omega + \omega^2},\ \ \psi = \frac{\alpha\mu\mathbf{i} - \mu^2}{\alpha\omega + \omega^2}.$$

It follows from Lemma 3.1 that $|\lambda| < 1$ if and only if $|\phi - \overline{\phi}\psi| + |\psi|^2 < 1$. After some careful calculations we have

$$|\phi - \overline{\phi}\psi| + |\psi|^2 = \frac{\zeta_1(\alpha) + \sqrt{\zeta_2(\alpha,\tau)}}{\zeta_3(\alpha)},$$

where

$$\begin{aligned} \zeta_1(\alpha) &= (\alpha\mu)^2 + (\mu^2)^2, \\ \zeta_2(\alpha,\tau) &= [(\alpha\omega + \omega^2)^2 - \mu^4 - \alpha^2\mu^2 - (\alpha\tau t + \omega\tau t)(\alpha\omega + \omega^2) - \mu^2\omega\tau t]^2 \\ &\quad + [\alpha^2\mu\tau t - \omega^2\mu\tau t + \mu^3\tau t]^2, \\ \zeta_3(\alpha) &= (\alpha\omega + \omega^2)^2. \end{aligned}$$

Therefore, $|\phi - \overline{\phi}\psi| + |\psi|^2 < 1$ if and only if

$$\begin{cases} \zeta_3(\alpha) - \zeta_1(\alpha) > 0, \\ \zeta_2(\alpha,\tau) < [\zeta_3(\alpha) - \zeta_1(\alpha)]^2. \end{cases} \tag{10}$$

Solving (10) yields

$$\alpha > \max\left\{\frac{-\omega_1^3 + \sqrt{\mu_n^2(\omega_n^4 + \mu_n^2\omega_n^2 - \mu_1^4)}}{\omega_1^2 - \mu_n^2},\ 0\right\},\ \ \text{when}\ \ \omega_1^2 > \mu_n^2$$

or

$$0 < \alpha < \frac{\omega_1^3 + \sqrt{\mu_1^2(\omega_1^4 + \omega_1^2\mu_1^2 - \mu_n^4)}}{\mu_n^2 - \omega_1^2},\ \ \text{when}\ \ \omega_1^2 < \mu_n^2$$

or

$$\alpha > 0,\ \ \text{when}\ \ \omega^2 = \mu^2$$

and

$$0 < \tau < \frac{2[(\alpha\omega_1 + \omega_1^2)^2 - \alpha^2\mu_n^2 - \mu_n^4][\omega_1(\alpha + \omega_1)^2 + \omega_1\mu_1^2]}{t_n\omega_n^2[(\alpha + \omega_n)^2 + \mu_n^2]^2 + t\mu_n^2[\alpha^2 + \mu_n^2 - \omega_1^2]^2},$$

where μ_1 and μ_n are the minimum and the maximum value of μ, ω_1 and ω_n are the minimum and the maximum value of ω, t_1 and t_n are the minimum and the maximum value of t, respectively.

Noticing that $\alpha, \tau > 0$, the proof is completed. □

4. Numerical results

In this section, we verify the feasibility and efficiency of the Method 2.1 used for solving nonsingular saddle point problems. In the implementation, all the tested methods are started from zero vector and terminated once the current iterate x_k satisfies

$$\mathrm{RES} = \sqrt{\frac{\|f - Ax_k - By_k\|_2^2 + \|g - B^* x_k\|_2^2}{\|f\|_2^2 + \|g\|_2^2}} < 10^{-6}. \tag{11}$$

All codes were run in MATLAB [version 7.11.0.584 (R2010b)] in double precision and all experiments were performed on a personal computer with 3.10 GHz central processing unit [Intel(R) Core(TM) i5-2400] and 4.00G memory.

To test the efficiency of Method 2.1, we compare the numerical results including iteration steps (denoted as IT), elapsed CPU time in seconds (denoted as CPU) and relative residuals (denoted as RES) of Method 2.1 with those of the Uzawa-HSS method and the GMRES method. The parameters α and τ involved in the Uzawa-HSS method and Method 2.1 are chosen to be the experimentally found optimal ones, which result in the least number of iteration steps of iteration methods. In actual computations, we choose right-hand-side vector $[f^*, g^*]^*$ such that the exact solution of (1) is x^* with all elements 1.

Example 4.1. *Let us consider the nonsingular saddle-point problem (1) with coefficient matrix as*

$$A = \begin{bmatrix} I_l \otimes T + T \otimes I_l & 0 \\ 0 & I_l \otimes T + T \otimes I_l \end{bmatrix} \in \mathbb{R}^{2l^2 \times 2l^2}$$

and

$$B = \begin{bmatrix} I_l \otimes F \\ F \otimes I_l \end{bmatrix} \in \mathbb{R}^{2l^2 \times l^2},$$

where

$$T = \frac{1}{h^2}\mathrm{tridiag}(-1, 2, 1) + \frac{1}{2h}\mathrm{tridiag}(-1, 0, 1) \in \mathbb{R}^{l\times l}, \;\; F = \frac{1}{h}\mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{l\times l},$$

$\otimes$ *denotes the Kronecker product symbol and* $h = 1/(l + 1)$ *is the discretization mesh-size, see [10].*

Table 1: Numerical results for Example 4 with $Q = \mathrm{tridiag}(B^* \mathrm{diag}(A)^{-1} B)$

	Method	α	τ	IT	CPU	RES
$l = 16$	Method 2.1	2.33	0.55	75	0.2184	9.7244e-7
	Uzawa–HSS	466.67	0.35	130	0.2184	9.2829e-7
	GMRES	–	–	140	0.2184	9.3640e-7
$l = 32$	Method 2.1	0.33	0.50	126	0.9204	9.6381e-7
	Uzawa–HSS	966.67	0.20	363	2.1060	9.9231e-7
	GMRES	–	–	280	5.7720	9.1950e-7
$l = 64$	Method 2.1	0.33	0.50	191	5.1012	9.7577e-7
	Uzawa–HSS			> 1000		
	GMRES	–	–	579	63.2116	9.9990e-7

In Table 1, we report the numerical results for Example 4, respectively. The experimentally optimal parameters, α and τ of Method 21 and Uzawa-HSS method, the iteration steps, the elapsed CPU time in seconds and the relative residuals, of Method 21, the Uzawa-HSS method and GMRES methods are listed.

From Table 1, we see that all of the three testing methods can converge to the approximate solution of saddle point problem (1). The Uzawa-HSS and GMRES methods needs more iteration steps and CPU time than Method 2.1 to converges. The proposed method, i.e., Method 2.1, is the most efficient one, which use least iteration steps and CPU times than the Uzawa-HSS and GMRES methods to achieve stopping criterion (11).

5. Conclusions

In this work, based on the Hermitian and skew-Hermitian splitting of the non-Hermitian positive (1, 1)-block of the saddle point matrix, we propose a new Uzawa-type iteration method to solve nonsymmetric saddle point problems (1). We demonstrate the convergence properties of the proposed method for saddle point problem (1) when the parameters satisfy some moderate conditions. Numerical results verified the effectiveness of the proposed method.

However, the proposed method involves two iteration parameters α and τ. The choices of the two parameters was not discussed in this work since it is a very difficult and complicated task. Considering that the efficiency of the proposed method largely depends on the choices of the two parameters, how to determine efficient and easy calculated parameters should be a direction for future research.

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FUZZY HYERS-ULAM STABILITY FOR GENERALIZED ADDITIVE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we prove the Hyers-Ulam stability of the following additive functional equation

$$\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\ne i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)$$

in fuzzy Banach spaces, where m is a positive integer greater than 3.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [35] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Th. M. Rassias [28] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. ([28]) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\|\le \epsilon(\|x\|^p+\|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \le p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x)-L(x)\|\le \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

In this paper, we consider the following functional equation

$$\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\ne i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i) \tag{1}$$

and prove the Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces.

First, we introduce the following lemma due to Najati and Ramjbar [20] with $n = 3$ in (1).

Lemma 1.2. *Let X and Y be linear spaces. A mapping $f : X \to Y$ satisfies the equation*

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2[f(x)+f(y)+f(z)] \tag{2}$$

for all $x, y, z \in X$ if and only if f is additive.

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It is noted that the following equation with $z = 0$ in (2)

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x}{2}+y\right) + f\left(x+\frac{y}{2}\right) = 2f(x) + 2f(y)$$

is equivalent to $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

We introduce the following lemma due to J.M. Rassias and Kim [27].

Lemma 1.3. *Let X and Y be linear spaces and let $m \geq 3$ be a fixed positive integer. A mapping $f : X \to Y$ satisfies the functional equation*

$$\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) = \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)$$

for all $x_1, x_2, \cdots, x_m \in X$ if and only if f is an additive mapping.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [5]–[7], [9, 10, 12, 19], [21]–[25], [29]–[31], [32]–[34]).

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [8], [15]–[18], [26]). In particular, Bag and Samanta [1], following Cheng and Mordeson [3], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [13]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

2. Preliminaries

Definition 2.1. ([1, 17, 18]) Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N1)$ $N(x,t) = 0$ for $t \leq 0$;

$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

$(N3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

$(N4)$ $N(x+y, c+t) \geq min\{N(x,s), N(y,t)\}$;

$(N5)$ $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

$(N6)$ for $x \neq 0$, $N(x,.)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space. The properties of fuzzy normed vector space and examples of fuzzy norms are given in [17, 18].

Example 2.2. Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 2.3. ([1, 17, 18]) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t\to\infty} x_n = x$.

Definition 2.4. ([1, 17, 18]) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [2]).

Definition 2.5. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.6. ([4]) *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

3. Fuzzy stability of the functional equation (1): a direct method

In this section, using the direct method, we prove the Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces. Throughout this section, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed space. Moreover, we assume that $N(x, .)$ is a left continuous function on $\mathbb{R}$.

Theorem 3.1. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i), t\right) \geq N'(\varphi(x_1, \cdots, x_m), t) \tag{3}$$

for all $x_1, \cdots, x_m \in X$, $t > 0$ and $\varphi : X^m \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < \frac{1}{m-1}$ such that

$$N'\left(\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right), t\right) \geq N'\left(\varphi(x_1, \cdots, x_m), \frac{t}{|r|}\right) \tag{4}$$

for all $x_1, \cdots, x_m \in X$ and all $t > 0$. Then there is a unique additive mapping $A : X \to Y$ satisfying (1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2|r|\varphi(x, x, \cdots, x)}{m(m-1)(1 - |r|(m-1))}, t\right) \tag{5}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (4) that

$$\begin{aligned} N'\left(\varphi\left(\frac{x_1}{(m-1)^j}, \frac{x_2}{(m-1)^j}, \cdots, \frac{x_m}{(m-1)^j}\right), t\right) &\geq N'\left(r^{j-1}\varphi(x_1, \cdots, x_m), \frac{t}{|r|}\right) \\ &= N'\left(\varphi(x_1, x_2, \cdots, x_m), \frac{t}{|r|^j}\right), \end{aligned} \tag{6}$$

and so

$$N'\left(\varphi\left(\frac{x_1}{(m-1)^j}, \frac{x_2}{(m-1)^j}, \cdots, \frac{x_m}{(m-1)^j}\right), |r|^j t\right) \geq N'\left(\varphi(x_1, x_2, \cdots, x_m), t\right)$$

for all $x_1, \cdots, x_m \in X$ and all $t > 0$.
Substituting $x_1 = x_2 = \cdots = x_m = x$ in (3), we obtain

$$N\left(\frac{m(m-1)}{2} f((m-1)x) - \frac{m(m-1)^2}{2} f(x), t\right) \geq N'(\varphi(x, x, \cdots, x), t), \tag{7}$$

and so

$$N\left(f(x) - (m-1) f\left(\frac{x}{m-1}\right), \frac{2t}{m(m-1)}\right) \geq N'\left(\varphi\left(\frac{x}{m-1}, \frac{x}{m-1}, \cdots, \frac{x}{m-1}\right), t\right) \tag{8}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{(m-1)^j}$ in (8), we have

$$\begin{aligned} &N\left((m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right), \frac{2(m-1)^{j-1} t}{m}\right) \\ &\geq N'\left(\varphi\left(\frac{x}{(m-1)^{j+1}}, \frac{x}{(m-1)^{j+1}}, \cdots, \frac{x}{(m-1)^{j+1}}\right), t\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^{j+1}}\right) \end{aligned} \tag{9}$$

for all $x \in X$, all $t > 0$ and all integer $j \geq 0$. So

$$\begin{aligned} &N\left(f(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \sum_{j=0}^{n-1} \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right) \\ &= N\left(\sum_{j=0}^{n-1}\left[(m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right)\right], \sum_{j=0}^{n-1} \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right) \\ &\geq \min_{0 \leq j \leq n-1}\left\{N\left((m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right), \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right)\right\} \\ &\geq N'(\varphi(x, x, \cdots, x), t) \end{aligned}$$

which implies

$$\begin{aligned} &N\left((m-1)^{n+p} f\left(\frac{x}{(m-1)^{n+p}}\right) - (m-1)^p f\left(\frac{x}{(m-1)^p}\right), \sum_{j=0}^{n-1} \frac{2(m-1)^{j+p} |r|^{j+1} t}{m(m-1)}\right) \\ &\geq N'\left(\varphi\left(\frac{x}{(m-1)^p}, \frac{x}{(m-1)^p}, \cdots, \frac{x}{(m-1)^p}\right), t\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^p}\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. So

$$N\left((m-1)^{n+p} f\left(\frac{x}{(m-1)^{n+p}}\right) - (m-1)^p f\left(\frac{x}{(m-1)^p}\right), \sum_{j=0}^{n-1} \frac{2(m-1)^{j+p}|r|^{j++p+1}t}{m(m-1)}\right)$$
$$\geq N'(\varphi(x, x, \cdots, x), t)$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. Hence one obtains

$$N\left((m-1)^{n+p} f\left(\frac{x}{(m-1)^{n+p}}\right) - (m-1)^p f\left(\frac{x}{(m-1)^p}\right), t\right) \tag{10}$$
$$\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\frac{2(m-1)^{p-1}|r|^{p+1}}{m} \sum_{j=0}^{n-1} (m-1)^j |r|^j}\right)$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. Since the series $\sum_{j=0}^{\infty} (m-1)^j |r|^j$ is a convergent series, we see by taking the limit $p \to \infty$ in the last inequality that a sequence $\left\{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N) and so it converges in Y.

Therefore, a mapping $A : X \to Y$ defined by $A(x) := N - \lim_{n\to\infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right)$ is well defined for all $x \in X$. It means that

$$\lim_{n\to\infty} N\left(A(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), t\right) = 1 \tag{11}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (10) that

$$N\left(f(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), t\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\frac{2|r|}{m(m-1)} \sum_{j=0}^{n-1} (m-1)^j |r|^j}\right)$$

for all $x \in X$ and all $t > 0$. So

$$N(f(x) - A(x), t)$$
$$\geq \min\left\{N\left(f(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), (1-\epsilon)t\right), N\left(A(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \epsilon t\right)\right\}$$
$$\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\frac{2|r|}{m(m-1)} \sum_{j=0}^{n-1} (m-1)^j |r|^j}\right)$$
$$\geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))\epsilon t}{2|r|}\right)$$

for sufficiently large n and for all $x \in X$, $t > 0$ and ϵ with $0 < \epsilon < 1$. Since ϵ is arbitrary and N' is left continuous, we obtain

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))t}{2|r|}\right)$$

for all $x \in X$ and $t > 0$. It follows from (3) that

$$\begin{aligned}
&N\Big((m-1)^n\Big[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Big],t\Big)\\
&\ge N'\left(\varphi\left(\frac{x_1}{(m-1)^n},\frac{x_2}{(m-1)^n},\cdots,\frac{x_m}{(m-1)^n}\right),\frac{t}{(m-1)^n}\right)\\
&\ge N'\left(\varphi(x_1,x_2,\cdots,x_m),\frac{t}{(m-1)^n|r|^n}\right)
\end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} N'\left(\varphi(x_1,x_2,\cdots,x_m),\frac{t}{(m-1)^n|r|^n}\right)=1,$$

$$N\Big((m-1)^n\Big[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Big],t\Big)\to 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Therefore, we obtain, in view of (11),

$$\begin{aligned}
&N\left(\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2}x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m A(x_i),t\right)\\
&\ge \min\Big\{N\Big(\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2}x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m A(x_i)\\
&-(m-1)^n\Big[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Big],\frac{t}{2}\Big),\\
&N\Big((m-1)^n\Big[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Big],\frac{t}{2}\Big)\Big\}\\
&=N\Big((m-1)^n\Big[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Big],\frac{t}{2}\Big)\\
&\ge N'\left(\varphi(x_1,x_2,\cdots,x_m),\frac{t}{2(m-1)^n|r|^n}\right)\to 1 \text{ as } n\to\infty
\end{aligned}$$

which implies

$$\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2}x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^m A(x_i)$$

for all $x_1, x_2, \cdots, x_m \in X$. Thus $A : X \to Y$ is a mapping satisfying (1) and (5).

To prove the uniqueness, assume that there is another mapping $L : X \to Y$ which satisfies the inequality (5). Since $L((m-1)^n x) = (m-1)^n L(x)$ for all $x \in X$, we have

$$\begin{aligned}
&N(A(x) - L(x), t)\\
&= N\left((m-1)^n A\left(\frac{x}{(m-1)^n}\right) - (m-1)^n L\left(\frac{x}{(m-1)^n}\right), t\right)\\
&\geq \min\left\{N\left((m-1)^n A\left(\frac{x}{(m-1)^n}\right) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \frac{t}{2}\right)\right.\\
&\left. N\left((m-1)^n f\left(\frac{x}{(m-1)^n}\right) - (m-1)^n L\left(\frac{x}{(m-1)^n}\right), \frac{t}{2}\right)\right\},\\
&\geq N'\left(\varphi\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right), \frac{m(m-1)(1-|r|(m-1))t}{4|r|(m-1)^n}\right)\\
&\geq N\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))t}{4|r|^{n+1}(m-1)^n}\right) \to 1 \text{ as } n \to \infty \text{ by (N5)}
\end{aligned}$$

for all $t > 0$. Therefore, $A(x) = L(x)$ for all $x \in X$, which completes the proof. □

Corollary 3.2. *Let X be a normed spaces and $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p < 2$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following inequality*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq N'\left(\theta\left(\sum_{j=1}^{m} \|x_j\|^p\right), t\right)$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfying (1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\theta\|x\|^p}{m^3 - 4m^2 + 5m - 2}, t\right)$$

Proof. Let $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\sum_{j=1}^{m} \|x_j\|^p\right)$ and $|r| = \frac{1}{(m-1)^2}$. Applying Theorem 3.1, we get the desired result. □

Theorem 3.3. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3) and $\varphi : X^m \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < m - 1$ such that*

$$N'(\varphi(x_1, \cdots, x_m), |r|t) \geq N'\left(\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right), t\right) \tag{12}$$

for all $x_1, \cdots, x_m \in X$ and all $t > 0$. Then thers is a unique additive mapping $A : X \to Y$ that satisfying (1) and the following inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\varphi(x, x, \cdots, x)}{m(m-1)(m-1-|r|)}, t\right) \tag{13}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (7) that

$$N\left(\frac{f((m-1)x)}{m-1} - f(x), \frac{2t}{m(m-1)^2}\right) \geq N'(\varphi(x, x, \cdots, x), t) \tag{14}$$

for all $x \in X$ and all $t > 0$. Replacing x by $(m-1)^n x$ in (14), we obtain

$$N\left(\frac{f((m-1)^{n+1}x)}{(m-1)^{n+1}} - \frac{f((m-1)^n x)}{(m-1)^n}, \frac{2t}{m(m-1)^{n+2}}\right) \tag{15}$$
$$\geq N'(\varphi((m-1)^n x, (m-1)^n x, \cdots, (m-1)^n x), t) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^n}\right)$$

and so

$$N\left(\frac{f((m-1)^{n+1}x)}{(m-1)^{n+1}} - \frac{f((m-1)^n x)}{(m-1)^n}, \frac{2|r|^n t}{m(m-1)^{n+2}}\right) \geq N'(\varphi(x, x, \cdots, x), t) \tag{16}$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\left(f(x) - \frac{f((m-1)^n x)}{(m-1)^n}, \sum_{j=0}^{n-1} \frac{2|r|^j t}{m(m-1)^{j+2}}\right) \geq N'(\varphi(x, x, \cdots, x), t)$$

for all $x \in X$, all $t > 0$ and any integer $n > 0$. So

$$\begin{aligned} N\left(f(x) - \frac{f((m-1)^n x)}{(m-1)^n}, t\right) &\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\sum_{j=0}^{n-1} \frac{2|r|^j}{m(m-1)^{j+2}}}\right) \\ &\geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(m-1-|r|)t}{2}\right). \end{aligned} \tag{17}$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let X be a normed spaces and $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p = \sum_{j=1}^m p_j < 2$ such that a mapping $f : X \to Y$ satisfies the following inequality*

$$N\left(\sum_{1 \leq i < j \leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i, j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2} \sum_{i=1}^m f(x_i), t\right) \geq N'\left(\theta\left(\prod_{j=1}^m \|x_j\|^{p_j}\right), t\right)$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfying (1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\theta\|x\|^p}{m(m-1)}, t\right)$$

Proof. Let $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\prod_{j=1}^m \|x_j\|^{p_j}\right)$ and $r = m - 2$. Applying Theorem 3.3, we get the desired result. □

4. Fuzzy stability of the functional equation (1): a fixed point method

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces. Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 4.1. *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right) \leq \frac{L\varphi(x_1, x_2, \cdots, x_m)}{m-1}$$

for all $x_1, x_2, \cdots, x_m \in X$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying

$$N\left(\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \ge \frac{t}{t+\varphi(x_1,x_2,\cdots,x_m)} \tag{18}$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Then the limit

$$A(x) := N\text{-}\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \ge \frac{(m(m-1)^2-m(m-1)^2L)t}{(m(m-1)^2-m(m-1)^2L)t+2L\varphi(x,x,\cdots,x)}. \tag{19}$$

Proof. Putting $x_1 = x_2 = \cdots = x_m = x$ in (18), we have

$$N\left(\frac{m(m-1)f((m-1)x)}{2}-\frac{m(m-1)^2f(x)}{2},t\right) \ge \frac{t}{t+\varphi(x,x,\cdots,x)} \tag{20}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\ ;\ g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu\in\mathbb{R}^+ : N(g(x)-h(x),\mu t) \ge \frac{t}{t+\varphi(x,x,\cdots,x)}, \forall x\in X,\ t>0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [16, Lemma 2.1]). Now we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := (m-1)g\left(\frac{x}{m-1}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$N(g(x)-h(x),\epsilon t) \ge \frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x)-Jh(x),L\epsilon t) &= N\left((m-1)g\left(\frac{x}{m-1}\right)-(m-1)h\left(\frac{x}{m-1}\right),L\epsilon t\right) \\ &= N\left(g\left(\frac{x}{m-1}\right)-h\left(\frac{x}{m-1}\right),\frac{L\epsilon t}{m-1}\right) \\ &\ge \frac{\frac{Lt}{m-1}}{\frac{Lt}{m-1}+\varphi\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right)} \\ &\ge \frac{\frac{Lt}{m-1}}{\frac{Lt}{m-1}+\frac{L\varphi(x_1,x_2,\cdots,x_m)}{m-1}} = \frac{t}{t+\varphi(x,x,\cdots,x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \le L\epsilon$. This means that $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in S$. It follows from (20) that

$$N\left(\frac{m(m-1)\left[f((m-1)x)-(m-1)f(x)\right]}{2},t\right) \ge \frac{t}{t+\varphi(x,x,\cdots,x)}.$$

So

$$N\left(f(x)-(m-1)f\left(\frac{x}{m-1}\right),\frac{2t}{m(m-1)}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right)} \tag{21}$$
$$\geq \frac{t}{t+\frac{L\varphi(x,x,\cdots,x)}{m-1}}=\frac{\frac{(m-1)t}{L}}{\frac{(m-1)t}{L}+\varphi(x,x,\cdots,x)}.$$

Therefore,

$$N\left(f(x)-(m-1)f\left(\frac{x}{m-1}\right),\frac{2Lt}{m(m-1)^2}\right)\geq\frac{t}{t+\varphi(x,x,\cdots,x)}. \tag{22}$$

This means that

$$d(f,Jf)\leq\frac{2L}{m(m-1)^2}.$$

By Theorem 2.6, there exists a mapping $A:X\to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{m-1}\right)=\frac{A(x)}{m-1} \tag{23}$$

for all $x\in X$. The mapping A is a unique fixed point of J in the set $\Omega=\{h\in S: d(g,h)<\infty\}$. This implies that A is a unique mapping satisfying (23) such that there exists $\mu\in(0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t)\geq\frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x\in X$ and $t>0$.
(2) $d(J^nf,A)\to 0$ as $n\to\infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)=A(x)$$

for all $x\in X$.
(3) $d(f,A)\leq\frac{d(f,Jf)}{1-L}$ with $f\in\Omega$, which implies the inequality

$$d(f,A)\leq\frac{2L}{m(m-1)^2-m(m-1)^2L}.$$

This implies that the inequality (19) holds. Furthermore, since

$$N\left(\sum_{1\leq i<j\leq m}A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2}x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m A(x_i),t\right)$$
$$=N-\lim_{n\to\infty}\left((m-1)^n\left[\sum_{1\leq i<j\leq m}f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)\right.\right.$$
$$\left.\left.-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\right],t\right)$$
$$\geq\lim_{n\to\infty}\frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n}+\varphi\left(\frac{x_1}{(m-1)^n},\frac{x_2}{(m-1)^n},\cdots,\frac{x_m}{(m-1)^n}\right)}$$
$$\geq\lim_{n\to\infty}\frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n}+\frac{L^n\varphi(x_1,x_2,\cdots,x_m)}{(m-1)^n}}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n} + \frac{L^n\varphi(x_1,x_2,\cdots,x_m)}{(m-1)^n}} = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$, we deduce that

$$N\left(\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i), t\right) = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Thus the mapping $A : X \to Y$ is additive, as desired. □

Corollary 4.2. *Let $\theta \ge 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the following inequality*

$$N\left(\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \ge \frac{t}{t+\theta\left(\sum_{i=1}^m \|x_i\|^p\right)}$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Then the limit

$$A(x) := N\text{-}\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \ge \frac{((m-1)^p-1)t}{((m-1)^p-1)t + 2(m-1)^{-2}\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\sum_{i=1}^m \|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. Then we can choose $L = (m-1)^{-p}$ and we get the desired result. □

Theorem 4.3. *Let $\varphi : X^m \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, x_2, \cdots, x_m) \le (m-1)L\varphi\left(\frac{x}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right)$$

for all $x_1, x_2, \cdots, x_m \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (18). Then

$$A(x) := N\text{-}\lim_{n\to\infty}\frac{f((m-1)^n x)}{(m-1)^n}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \ge \frac{m(m-1)^2(1-L)t}{m(m-1)^2(1-L)t + 2\varphi(x,x,\cdots,x)} \tag{24}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined as in the proof of Theorem 4.1. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{g((m-1)x)}{m-1}$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then

$$N(g(x)-h(x),\epsilon t) \ge \frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(\frac{g((m-1)x)}{m-1} - \frac{h((m-1)x)}{m-1}, L\epsilon t\right) \\ &= N\Big(g((m-1)x) - h((m-1)x), (m-1)L\epsilon t\Big) \\ &\geq \frac{(m-1)Lt}{(m-1)Lt + \varphi((m-1)x, , (m-1)x, \cdots, (m-1)x)} \\ &\geq \frac{(m-1)Lt}{(m-1)Lt + (m-1)L\varphi(x, x, \cdots, x)} = \frac{t}{t + \varphi(x, x, \cdots, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (20) that

$$N\left(\frac{m(m-1)^2}{2}\left[\frac{f((m-1)x)}{m-1} - f(x)\right], t\right) \geq \frac{t}{t + \varphi(x, x, \cdots, x)} \tag{25}$$

for all $x \in X$ and $t > 0$. So

$$N\left(\frac{f((m-1)x)}{m-1} - f(x), \frac{2t}{m(m-1)^2}\right) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}.$$

Therefore,

$$d(f, Jf) \leq \frac{2}{m(m-1)^2}.$$

By Theorem 2.6, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$(m-1)A(x) = A((m-1)x) \tag{26}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that A is a unique mapping satisfying (26) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies $A(x) = N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$ for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{2}{m(m-1)^2(1-L)}.$$

This implies that the inequality (24) holds.

The rest of the proof is similar to that of the proof of Theorem 4.1. □

Corollary 4.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < \frac{1}{m}$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq \frac{t}{t + \theta\left(\prod_{i=1}^{m} \|x_i\|^p\right)}$$

for all $x_1, x_2, \cdots, x_m \in X$ *and all* $t > 0$. *Then*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$$

exists for each $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{m((m-1)^{p+2} - (m-1)^2)t}{m((m-1)^{p+2} - (m-1)^2)t + 2(m-1)^p\theta\|x\|^{mp}}.$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\prod_{i=1}^m \|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. Then we can choose $L = (m-1)^{-p}$ and we get the desired result. □

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n-JORDAN $*$-DERIVATIONS ON INDUCED C^*-ALGEBRAS

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ABSTRACT. Using the fixed point alternative theorem, we investigate the Hyers-Ulam stability of of n-Jordan $*$-derivations on induced fuzzy C^*-algebras associated with the following functional equation $f(my-x)+f(x-mz)+mf(x-y+z)=f(mx)$, where m is a fixed integer greater than 1.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [42] concerning the stability of group homomorphisms. Hyers [16] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [33] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [15], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 9], [17]–[25], [30, 31], [34]–[38], [40, 41]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see [7, 12]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$, for all $n \ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 11, 13, 22, 27, 32]).

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In 1984, Katsaras [21] defined a fuzzy norm on a linear space and at the same year Wu and Fang [43] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [5], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [4, 14, 24, 39, 44]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [23]. In 2003, Bag and Samanta [4] modified the definition of Cheng and Mordeson [10] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:
(N_1) $N(x,a) = 0$ for $a \leq 0$;
(N_2) $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;
(N_3) $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
(N_4) $N(x+y, a+b) \geq min\{N(x,a), N(y,b)\}$;
(N_5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x,a) = 1$;
(N_6) For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement the norm of x is less than or equal to the real number a'.

Definition 1.2. Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

Definition 1.3. A sequence x_n in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector space X, Y is continuous at point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X(see [2]).

Definition 1.4. [29] Let X be a $*$-algebra and (X,N) a fuzzy normed space.

(1) The fuzzy normed space (X,N) is called a fuzzy normed $*$-algebra if

$$N(xy, st) \geq N(x,s)\cdot N(y,t) \quad \text{and} \quad N(x^*,t) = N(x,t).$$

(2) A complete fuzzy normed $*$-algebra is called a *fuzzy Banach $*$-algebra.*

Example 1.5. *Let $(X, \|.\|)$ be a normed $*$-algebras. Let*

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0\ , x \in X, \\ 0, & a \leq 0, x \in X. \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on X and $(X, N(x,t))$ is a fuzzy normed $*$-algebra.

Definition 1.6. Let $(X, \|\cdot\|)$ be a C^*-algebra and N a fuzzy norm on X.

(1) The fuzzy normed $*$-algebra (X, N) is called an induced fuzzy normed $*$-algebra.
(2) The fuzzy Banach $*$-algebra (X, N) is called an induced fuzzy C^*-algebra.

Definition 1.7. Let $(X, \|\cdot\|)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a *fuzzy n-Jordan $*$-derivation* if

$$\begin{aligned} D(x^n) &= D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-2}D(x)x + x^{n-1}D(x), \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x \in X$.

Throughout this paper, assume that (X, N) is an induced fuzzy C^*-algebra and that m is a fixed integer greater than 1.

2. Main results

Lemma 2.1. *Let (Z, N) be a fuzzy normed vector space and $f : X \to Z$ be a mapping such that*

$$N\left(f\left(my - x\right) + f\left(x - mz\right) + mf\left(x - y + z\right), t\right) \ge N\left(f\left(mx\right), \frac{t}{2}\right) \tag{2.1}$$

for all $x, y, z \in X$ and all $t > 0$. Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$N((m+2)f(0), t) = N\left(f(0), \frac{t}{m+2}\right) \ge N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(0) = 0$.

Letting $x = 0$ and $y = z$ in (2.1), we get

$$N(f(my) + f(-my), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that $f(my) + f(-my) = 0$ for all $y \in X$. Thus

$$f(-y) = -f(y)$$

for all $y \in X$.

Letting $x = z = 0$ in (2.1), we get

$$N(f(my) - mf(y), t) = N(f(my) + mf(-y), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. So $f(my) = mf(y)$ for all $y \in X$.

Letting $x = 0$ and replacing z by $-z$ in (2.1), we get

$$N(f(my) + f(mz) + mf(-y - z), t) = N(mf(y) + mf(z) - mf(y + z), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that

$$mf(y) + mf(z) - mf(y + z) = 0$$

for all $y, z \in X$. Thus

$$f(y + z) = f(y) + f(z)$$

for all $y, z \in X$, as desired. □

Theorem 2.2. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}\right) \le \frac{L}{m}\phi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Let $f : X \to X$ be an odd mapping such that

$$N\left(f\left(\mu(my - x)\right) + f\left(\mu(x - mz)\right) + mf\left(\mu(x - y + z)\right) - \mu f\left(mx\right), t\right) \ge \frac{t}{t + \phi(x, y, z)}, \tag{2.3}$$

$$\begin{aligned} &N\big(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) \\ &\quad + f(v^*) - f(v)^*, t\big) \ge \frac{t}{t + \phi(w, v, 0)} \end{aligned} \tag{2.4}$$

for all $x, y, z, w, v \in X$, all $\mu \in \mathbb{T}^1 := \{c \in \mathbb{C} : |c| = 1\}$ and all $t > 0$. Then the limit $D(x) = N - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$ exists for each $x \in X$ and the mapping $D : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - D(x), t) \ge \frac{m(1 - L)t}{m(1 - L)t + L\phi\left(x, 0, 0\right)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Since f is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all X.

Letting $\mu = 1$ and $y = z = 0$ in (2.3), we have

$$N\left(mf\left(x\right) - f(mx), t\right) \ge \frac{t}{t + \phi\left(x, 0, 0\right)} \tag{2.6}$$

and so

$$N\left(mf\left(\frac{x}{m}\right) - f(x), t\right) \ge \frac{t}{t + \phi\left(\frac{x}{m}, 0, 0\right)} = \frac{t}{t + \frac{L}{m}\phi\left(x, 0, 0\right)}$$

for all $x \in X$ and all $t > 0$. Thus

$$N\left(mf\left(\frac{x}{m}\right) - f(x), \frac{L}{m}t\right) \ge \frac{\frac{L}{m}t}{\frac{L}{m}t + \frac{L}{m}\phi\left(x, 0, 0\right)} = \frac{t}{t + \phi\left(x, 0, 0\right)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$G := \{g : X \to X\}$$

and introduce the generalized metric on G:

$$d(g, h) := \inf\{a \in \mathbb{R}^+ : N(g(x) - h(x), at) \ge \frac{t}{t + \phi\left(x, 0, 0\right)}\}$$

for all $x \in X$ and all $t > 0$, where $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [26, Lemma 2.1]

Now, we consider the linear mapping $Q : G \to G$ such that $Qg(x) := mg\left(\frac{x}{m}\right)$ for all $x \in X$.

Let $g, h \in G$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \phi\left(x, 0, 0\right)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(Qg(x) - Qh(x), L\varepsilon t) = N\left(mg\left(\frac{x}{m}\right) - mh\left(\frac{x}{m}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{m}\right) - h\left(\frac{x}{m}\right), \frac{L}{m}\varepsilon t\right)$$
$$\geq \frac{\frac{Lt}{m}}{\frac{Lt}{m} + \phi\left(\frac{x}{m}, 0, 0\right)} \geq \frac{\frac{Lt}{m}}{\frac{Lt}{m} + \frac{L}{m}\phi\left(x, 0, 0\right)} = \frac{t}{t + \phi\left(x, 0, 0\right)}$$

for all $x \in X$ and all $t > 0$. Thus $d(g, h) = \varepsilon$ implies that $d(Qg, Qh) \leq L\varepsilon$. This means that

$$d(Qg, Qh) \leq Ld(g, h)$$

for all $g, h \in G$.

It follows from (2.7) that $d(f, Qf) \leq \frac{L}{m}$.

By Theorem 1.1, there exists a mapping $D : X \to X$ satisfying the following:

(1) D is a fixed point of Q, *i.e.*,

$$D\left(\frac{x}{m}\right) = \frac{1}{m}D(x) \tag{2.8}$$

for all $x \in X$. The mapping D is a unique fixed point of Q in the set

$$M = \{g \in G : d(f, g) < \infty\}.$$

This implies that D is a unique mapping satisfying (2.8) such that there exists an $a \in (0, \infty)$ satisfying

$$N(f(x) - D(x), at) \geq \frac{t}{t + \phi\left(x, 0, 0\right)}$$

for all $x \in X$.

(2) $d(Q^k f, D) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k\to\infty} m^k f\left(\frac{x}{m^k}\right) = D(x)$$

for all $x \in X$;

(3) $d(f, D) \leq \frac{1}{1-L}d(f, Qf)$, which implies the inequality

$$d(f, D) \leq \frac{L}{m(1-L)}.$$

This implies that the inequality (2.5) holds.

Next we show that D is additive. It follows from (2.2) that

$$\sum_{k=0}^{\infty} m^k \phi\left(\frac{x}{m^k}, \frac{y}{m^k}, \frac{z}{m^k}\right) = \phi(x, y, z) + m\phi\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}\right) + m^2\phi\left(\frac{x}{m^2}, \frac{y}{m^2}, \frac{z}{m^2}\right) + \cdots$$
$$\leq \phi(x, y, z) + L\phi(x, y, z) + L^2\phi(x, y, z) + \cdots = \frac{1}{1-L}\phi(x, y, z) < \infty$$

for all $x, y, z \in X$.

By (2.3),

$$N\left(m^k f\left(\mu\frac{my - x}{m^k}\right) + m^k f\left(\mu\frac{x - mz}{m^k}\right) + m^{k+1} f\left(\mu\frac{x - y + z}{m^k}\right) - m^k \mu f\left(\frac{m}{m^k}x\right), m^k t\right)$$
$$\geq \frac{t}{t + \phi\left(\frac{x}{m^k}, \frac{y}{m^k}, \frac{z}{m^k}\right)}$$

and so

$$N\left(m^k f\left(\mu\frac{my-x}{m^k}\right)+m^k f\left(\mu\frac{x-mz}{m^k}\right)+m\cdot m^k f\left(\mu\frac{x-y+z}{m^k}\right)-m^k\mu f\left(\frac{m}{m^k}x\right),t\right)$$
$$\geq \frac{\frac{t}{m^k}}{\frac{t}{m^k}+\phi\left(\frac{x}{m^k},\frac{y}{m^k},\frac{z}{m^k}\right)}=\frac{t}{t+m^k\phi\left(\frac{x}{m^k}.\frac{y}{m^k},\frac{z}{m^k}\right)}$$

for all $x,y,z\in X$, all $\mu\in\mathbb{T}^1$ and all $t>0$. Since $\lim_{k\to\infty}\frac{t}{t+m^k\phi\left(\frac{x}{m^k},\frac{y}{m^k},\frac{z}{m^k}\right)}=1$ for all $x,y,z\in X$ and all $t>0$,

$$N\left(D\left(\mu(my-x)\right)+D\left(\mu(x-mz)\right)+mD\left(\mu(x-y+z)\right)-\mu D\left(mx\right),t\right)=1$$

for all $x,y,z\in X$, all $\mu\in\mathbb{T}^1$ and all $t>0$. So

$$D\left(\mu(my-x)\right)+D\left(\mu(x-mz)+mD\left(\mu(x-y+z)\right)=\mu D\left(mx\right)\right) \tag{2.9}$$

for all $x,y,z\in X$ and all $\mu\in\mathbb{T}^1$. Let $\mu=1$ in (2.9). By the same reasoning as in the proof of Lemma 2.1, one can easily show that D is additive.

Since f is odd, it is easy to show that D is odd. Letting $\mu=1$ and $y=z=0$ in (2.9), we get $mD(x)=D(mx)$ for all $x\in X$. Letting $y=z=0$ in (2.9), we get $mD(\mu x)=\mu D(mx)=m\mu D(x)$ and so

$$D(\mu x)=\mu D(x)$$

for all $x\in X$ and all $\mu\in\mathbb{T}^1$. Thus the mapping $D:X\to X$ is $\mathbb{C}$-linear by [28, Theorem 2.1].

By (2.4) and letting $v=0$ in (2.4), we get

$$N\left(m^{nk}f\left(\frac{w^n}{m^{nk}}\right)-m^{nk}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-1}-m^{nk}\frac{w}{m^k}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-2}-\cdots\right.$$
$$\left.-m^{nk}\left(\frac{w}{m^k}\right)^{n-2}f\left(\frac{w}{m^k}\right)w-m^{nk}\left(\frac{w}{m^k}\right)^{n-1}f\left(\frac{w}{m^k}\right),m^{nk}t\right)\geq\frac{t}{t+\phi(\frac{w}{m^k},0,0)}$$

for all $w\in X$ and all $t>0$. Thus

$$N\left(m^{nk}f\left(\frac{w^n}{m^{nk}}\right)-m^{nk}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-1}-m^{nk}\frac{w}{m^k}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-2}-\cdots\right.$$
$$\left.-m^{nk}\left(\frac{w}{m^k}\right)^{n-2}f\left(\frac{w}{m^k}\right)w-m^{nk}\left(\frac{w}{m^k}\right)^{n-1}f\left(\frac{w}{m^k}\right),t\right)\geq\frac{\frac{t}{m^{nk}}}{\frac{t}{m^{nk}}+\phi(\frac{w}{m^k},0,0)}$$
$$\geq\frac{t}{t+(m^{n-1}L)^k\phi(w,0,0)}$$

for all $w\in X$ and all $t>0$. Since $\lim_{k\to\infty}\frac{t}{t+(m^{n-1}L)^k\phi(w,0,0)}=1$ for all $w\in X$ and all $t>0$, we get

$$N(D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w),t)=1$$

for all $x\in X$ and all $t>0$. So

$$D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w)=0$$

for all $w\in X$.

Similarly, letting $w=0$ in (2.4), we get $D(v^*)-D(v)^*=0$ for all $v\in X$.

Therefore, the mapping $D:X\to X$ is a fuzzy n-Jordan $*$-derivation. □

Corollary 2.3. *Let p be a real number with $p > 1$, $\theta \geq 0$, and X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to X$ be an odd mapping satisfying*

$$\begin{aligned} &N\left(f\left(\mu(my-x)\right)+f\left(\mu(x-mz)\right)+mf\left(\mu(x-y+z)\right)-\mu f\left(mx\right),t\right) \\ &\geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p+\|z\|^p)}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &N\big(f(w^n)-f(w)w^{n-1}-wf(w)w^{n-2}-\cdots-w^{n-2}f(w)w-w^{n-1}f(w) \\ &+f(v^*)-f(v)^*,t\big) \geq \frac{t}{t+\theta(\|w\|^p+\|v\|^p)} \end{aligned} \tag{2.11}$$

for all $x, y, w, v \in X$, all $\mu \in \mathbb{T}^1$ and all $t > 0$. Then the limit $D(x) = N - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$ exists for each $x \in X$ and the mapping $D : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x)-D(x),t) \geq \frac{(m^p-m)t}{(m^p-m)t+\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\phi(x,y,z) = \theta(\|x\|^p+\|y\|^p+\|z\|^p)$$

and $L = m^{1-p}$. □

Theorem 2.4. *Let $\phi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$mL\phi\left(\frac{x}{m},\frac{y}{m},\frac{z}{m}\right) \leq \phi(x,y,z) \tag{2.12}$$

for all $x, y, z \in X$. Let $f : X \to X$ be an odd mapping satisfying (2.3) and (2.4). Then the limit $D(x) = N - \lim_{n\to\infty} \frac{1}{m^n} f(m^n x)$ exists for each $x \in X$ and the mapping $D : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x)-D(x),t) \geq \frac{m(1-L)t}{m(1-L)t+\phi(x,0,0)} \tag{2.13}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (G, d) be generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := \frac{1}{m}g(mx)$$

for all $x \in X$.

It follow from (2.6) that

$$N\left(f(x)-\frac{1}{m}f(mx),\frac{1}{m}t\right) \geq \frac{t}{t+\phi(x,0,0)}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Qf) \leq \frac{1}{m}$. Hence

$$d(f,D) \leq \frac{1}{m(1-L)},$$

which implies that the inequality (2.13) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a positive real number with $p < 1$. Let X be a normed vector space with normed $\|\cdot\|$. Let $f : X \to X$ be an odd mapping satisfying (2.10) and (2.11). Then $D(x) = N - \lim_{n\to\infty} \frac{1}{m^n} f(m^n x)$ exists for each $x \in X$ and defines a fuzzy n-Jordan $*$-derivation $D : X \to X$ such that*

$$N(f(x) - D(x), t) \geq \frac{(m - m^p)t}{(m - m^p)t + \theta\|x\|^p}$$

for every $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = m^{p-1}$. □

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