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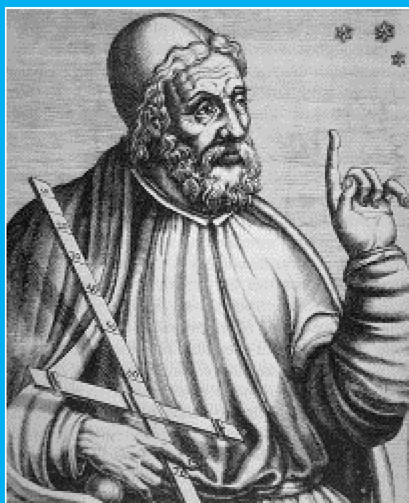

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<u>Contribution to a Book</u>

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NECESSARY AND SUFFICIENT CONDITIONS OF FIRST ORDER NEUTRAL DIFFERENTIAL EQUATIONS

ABHAY KUMAR SETHI[1*] AND JUNG RYE LEE[2*]

ABSTRACT. In this work, we establish the necessary and sufficient conditions for oscillation of a class of functional differential equations of the form

$$((x(t)+p(t)x(t-\sigma))'+q(t)\phi(x(t-\tau))+v(t)\psi(x(t-\eta))=0$$

of a neutral coefficient $p(t)$, by using the Knaster-Tarski fixed point theorem and Banach's fixed point theorem.

1. Introduction

Consider a class of first-order nonlinear neutral differential equations of the form

$$((x(t)+p(t)x(t-\sigma)))'+q(t)\phi(x(t-\tau))+v(t)\psi(x(t-\eta))=0, \tag{1.1}$$

where $r,\ q,\ v,\ \tau,\ \sigma,\eta\in C(\mathbb{R}_+,\mathbb{R}_+),\ p\in C(\mathbb{R}_+,\mathbb{R}),\ \phi\in C(\mathbb{R},\mathbb{R})$ such that $x\phi(x)>0, x\phi(x)>0$ for $x\neq 0$ and $\phi,\ \psi\in C(\mathbb{R},\mathbb{R})$ satisfying the property $x\phi(x)>0,\ \ u\psi(u)>0$ for $x,u\neq 0$.

In this work, our objective is to establish the necessary and sufficient condition results for oscillation of all solutions of (1.1), where

(A_0) $p\in C([0,\infty),\mathbb{R}),\ f\in C(\mathbb{R},\mathbb{R}),\ q,\tau,\sigma,\eta\in C(\mathbb{R}_+,\mathbb{R}_+)$ such that $t-\tau<t,\ t-\sigma<t$ and $t-\eta<t$;

(A_1) $\phi,\psi\in C(\mathbb{R},\mathbb{R})$ are nondecreasing and satisfy $u\phi(u)>0,\ u\psi(u)>0$ for $u,v\neq 0$.

Fatima *et al.* [1] studied the nonlinear neutral differential equation (NDDE) of the form

$$[r(t)(x(t)+p(t)x(t-\tau))]'+q(t)x(t-\sigma)=0, \tag{1.2}$$

where $p\in C[[t_0,\infty)],\mathbb{R}],\ r,q\in C[[t_0,\infty),\mathbb{R}^+],\ \tau,\sigma^+\in\mathbb{R}^+$, and they obtained new sufficient conditions for all solutions of NDDE (1.2) to be oscillatory.

Graef *et al.* [8] studied the first order neutral delay differential equations of the form

$$[x(t)+p(t)x(t-\tau)]'+q(t)f(x(t-\sigma))=0, \tag{1.3}$$

under the conditions

(a) $p\in\mathbb{R},\tau$ *and* σ *are positive constants*;

(b) $q:[t_0,\infty)\to\mathbb{R}$ *is a continuous function with* $q(t)>0$;

(c) $f:\mathbb{R}\to\mathbb{R}$ *is continuous with* $uf(u)>0$ *for* $u\neq 0$, *and there is a positive constant* M *such that* $\frac{f(u)}{u^\alpha}\geq M>0$, *where* α *is a ratio of odd positive integers.* They established internal conditions for all solutions of nonlinear first order neutral delay differential equations.

Grammatikopoulos *et al.* [9] studied first order neutral delay differential equations of the form

$$[x(t)-p(t)x(t-\tau)]'+Q(t)x(t-\delta))=0, \tag{1.4}$$

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where $p, Q, \delta \in C([t_0, \infty)], \mathbb{R}^+)$, and $\lim_{t\to\infty}(t - \delta(t)) = \infty$. They established sufficient conditions for oscillation of all solutions of the the neutral delay differential equations.

The motivation of the present work comes from the above studies. In this work, an attempt is made to establish the necessary and sufficient condition for asymptotic behaviour of solutions of (1.1), under various ranges in the neutral coefficient $p(t)$. Clearly, (1.2), (1.3) and (1.4) are special cases of (1.1). However, there are few results to study the oscillation of (1.1). The purpose of this work is to obtain some sufficient condition results for oscillation of (1.1). This work would be interesting than the works of [15, 19] as long as (1.1) is concerned.

Neutral delay differential equations find numerous applications in electric network. For example, they are frequently used for the study of distributed networks containing lossless transmission lines which arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits (see for example [12]). The problem of obtaining sufficient conditions to ensure the second order differential equations which are special cases of (1.1) is oscillatory has received a great attention. Since the first order equations have the applied applications, there is permanent interest in obtaining new sufficient conditions for oscillation or nonoscillation of solutions of varietal type of the first order equations (see [2–7, 11, 13, 14, 16–18, 20]).

Definition 1.1. *By a solution of (1.1), we mean a continuously differentiable function $x(t)$ which is defined for $t \geq T^* = \min\{(t-\sigma_0), (t-\tau_0), (t-\eta_0)\}$ such that $x(t)$ satisfies (1.1) for all $t \geq t_0$. In the sequel, it will always be assumed that the solution of (1.1) exists on some half line $[t_1, \infty)$, $t_1 \geq t_0$. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.*

2. Oscillation results

This section deals with the oscillation results for necessary and sufficient conditions for oscillation of all solutions of (1.1), Throughout our discussion, we use the following notation

$$z(t) = x(t) + p(t)x(t-\sigma).$$

Lemma 2.1. *[10] Let $p, x, z \in C([0,\infty), \mathbb{R})$ be such that $z(t) = x(t) + p(t)x(t-\sigma)$, $t \geq \tau > 0$, $x(t) > 0$ for $t \geq t_1 > \tau$, $\liminf_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = L$ exists. Let $p(t)$ satisfy one of the following conditions:*

$$i)\ 0 \leq p_1 \leq p(t) \leq p_2 < 1,\ ii)\ 1 < p_3 \leq p(t) \leq p_4 < \infty,\ iii)\ -\infty < -p_5 \leq p(t) \leq 0,$$

where $r_i > 0$, $1 \leq i \leq 5$.
Then $L = 0$.

Theorem 2.2. *Assume that (A_0) and (A_1) hold and $0 \leq a_1 \leq p(t) \leq a_2 < 1$ for $t \in \mathbb{R}_+$. Let ϕ, ψ be Lipschitzian on intervals of the form $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$. Then every solution of (1.1) converges to zero as $t \to \infty$ if and only if*

(A_2) $\int_t^\infty [q(s) + v(s)]ds = \infty$.

Proof. Assume that (A_2) holds. Let $x(t)$ be a solution of (1.1) on $[t_x, \infty]$, $t_x \geq 0$. Let $x(t) > 0$ for $t \geq t_x$. Set

$$z(t) = x(t) + p(t)x\big(t-\sigma\big),\ t \geq t_0. \tag{2.1}$$

Then (1.1) becomes

$$z'(t) = -q(t)\phi(x(t-\tau)) - v(t)\psi(x(t-\eta)) < 0, \tag{2.2}$$

and hence $z(t)$ is a decreasing function for $t \geq t_1 > t_0 + \rho$. Since $z(t) > 0$ for $t \geq t_2$, $\lim_{t\to\infty} z(t)$ exists. Consequently, $z(t) > x(t)$ implies that $x(t)$ is bounded. Our aim is to show that $\lim_{t\to\infty} x(t) = 0$. For this, we need to show that $\liminf_{t\to\infty} x(t) = 0$. If $\liminf_{t\to\infty} x(t) \neq 0$, then there exist $t_3 > t_2$ and $\beta > 0$ such that $x(t-\sigma) \geq \beta > 0$ for $t \geq t_3$. Ultimately,

$$\begin{aligned} \int_{t_3}^{t} [\phi(x(t-\tau)) + v(t)\psi(x(t-\eta))]\, dt &\geq \phi(\beta)[q(t)]dt + \psi(\beta) \int_{t_3}^{t} [v(t)]dt \\ &\to +\infty, \ \ as \ \ t \to \infty, \end{aligned}$$

due to (A_2).

On the other hand, we integrate (2.2) from t_3 to $t(> t_3)$ to obtain

$$\begin{aligned} \int_{t_3}^{t} [q(t)\phi(x(t-\tau)) + v(t)\psi(x(t-\eta))]\, dt &\leq -z(t) + z(t_3) \\ &< \infty, \ \ as \ \ t \to \infty, \end{aligned}$$

which is a contradiction. Therefore, $\liminf_{t\to\infty} x(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$ due to Lemma 2.1. Thus we obtain

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}(x(t) + p(t)x(t-\sigma)) \\ &\geq \limsup_{t\to\infty} x(t), \end{aligned}$$

which implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

Assume that $x(t) < 0$ for $t \geq t_0$. Setting $y(t) = -x(t)$ for $t \geq t_0$ in (1.1), we obtain

$$((y(t) + p(t)y(t-\sigma)))' + q(t)\phi(y(t-\tau)) + v(t)\psi(y(t-\eta)) = 0,$$

and proceeding as above it is easy to prove that $\lim_{t\to\infty} y(t) = 0$.

In order to prove the condition (A_2) is necessary, we suppose that

$$\int_{t}^{\infty} [q(s) + v(s)]\, ds < \infty \tag{2.3}$$

and we need to show that the equation (1.1) admits a nonoscillatory solution which does not tend to zero as $t \to \infty$ when the limit exists. If possible, let there exist $t_1 > 0$ such that

$$\int_{t}^{\infty} [q(s) + v(s)]\, ds < \frac{1-a_1}{10c},$$

where $C = \max\{C_1, \frac{C_2}{L}, \phi(1), \psi(1)\}$, C_1 is the Lipschitz constant of ϕ and C_2 is the Lipschitz constant of ψ on $\left[\frac{2(1-a_1)}{5}, 1\right]$. For $t_2 > t_1$, set $Y = BC([t_2, \infty), \mathbb{R})$, the space of real valued bounded continuous functions on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||Y|| = \sup\{|Y(t)| : t \geq t_2\}.$$

Let's define

$$S = \left\{ u \in Y : \ \frac{2(1-a_1)}{5} \leq u(t) \leq 1, \ t \geq t_2 \right\}.$$

Clearly, S is a closed and convex subspace of Y. Let $T: S \to S$ be defined by

$$Ty(t) = \begin{cases} Ty(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -p(t)y(t-\sigma) + \frac{2+3a_1}{5} + \int_t^\infty [q(s)\phi(y(t-\tau)) + v(s)\psi(y(t-\eta))]\, ds, t \geq t_2+\rho. \end{cases}$$

For every $y \in S$,

$$\begin{aligned} Ty(t) &\leq \frac{2+3a_1}{5} + \phi(1)\int_t^\infty [q(s)]ds + \psi(1)\int_t^\infty [v(s)]ds \\ &< \frac{2+3a_1}{5} + \frac{1-a_1}{10} = \frac{1+a_1}{2} < 1 \end{aligned}$$

and

$$\begin{aligned} Ty(t) &\geq -p(t)y(t-\tau) + \frac{2+3a_1}{5} \\ &\geq -a_1 + \frac{2+3a_1}{5} = \frac{2(1-a_1)}{5} \end{aligned}$$

which imply that $Ty \in S$. Now, for $y_1, y_2 \in S$,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq |p(t)||y_1(t-\tau) - y_2(t-\tau)| \\ &+ C_1\int_t^\infty q(s)|y_1(s-\sigma) - y_2(s-\sigma)|ds + C_2\int_t^\infty v(s)|y_1(s-\eta) - y_2(s-\eta)|ds, \end{aligned}$$

that is,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq a_2||y_1 - y_2|| + C_1||y_1 - y_2||\int_t^\infty [q(s)]ds + C_2||y_1 - y_2||\int_t^\infty [v(s)]ds \\ &< \left(a_1 + \frac{1-a_1}{10}\right)||y_1 - y_2||, \end{aligned}$$

which implies that

$$||Ty_1 - Ty_2|| \leq \mu||y_1 - y_2||,$$

that is, T is a contraction mapping, where $\mu = a_1 + \frac{1-a_1}{10} = \frac{1+9a_1}{10} < 1$. Since S is complete and T is a contraction on S, by the Banach's fixed point theorem, T has a unique fixed point on $\left[\frac{2(1-a_1)}{5}, 1\right]$. Hence $Ty = y$ and

$$y(t) = \begin{cases} y(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -p(t)y(t-\sigma) + \frac{2+3a_1}{5}\left[\int_t^\infty q(s)\phi(y(s-\tau)) + \int_t^\infty v(s)\psi(y(s-\eta)))\right] ds, t \geq t_2+\rho \end{cases}$$

is a nonoscillatory solution of (1.1). Therefore, (A_2) is necessary. This completes the proof of the theorem. □

Theorem 2.3. *Assume that (A_0) and (A_1) hold and $1 < a_3 \leq p(t) \leq a_4 < \infty$ such that $a_3^2 > a_4$ for $t \in \mathbb{R}_+$. Let ϕ, ψ be Lipschitzian on intervals of the form $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$. Then every solution of (1.1) converges to zero as $t \to \infty$ if and only if (A_2) holds.*

Proof. The sufficient part is the same as in the proof of Theorem 2.2.

For the necessary part, we suppose that (2.2) holds. It is possible to find a $t_1 > 0$ such that

$$\int_t^\infty [q(s) + v(s)]\, ds < \frac{a_3 - 1}{2K},$$

where $K = \max\{K_1, \frac{K_2}{L}\}$, K_1, K_2 are Lipschitz constants of ϕ and ψ on $[a,b]$ and $K_2 = \phi(a), \psi(b)$ such that

$$a = \frac{2\lambda(a_3{}^2 - a_4) - a_4(a_3 - 1)}{2a_3{}^2 a_4},$$
$$b = \frac{a_3 - 1 + 2\lambda}{2a_3}, \qquad \lambda > \frac{a_4(a_3 - 1)}{2(a_3{}^2 - a_4)} > 0.$$

Let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of real valued bounded continuous functions on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Define

$$S = \{u \in Y : \; a \leq u(t) \leq b, \; t \geq t_2\}.$$

It is easy to see that S is a closed convex subspace of Y. Let $T : S \to S$ be such that

$$Tx(t) = \begin{cases} Tx(t_2 + \rho), & t \in [t_2, t_2 + \rho] \\ -\frac{x\left(t+\sigma\right)}{p\left(t+\sigma\right)} + \frac{\lambda}{p\left(t+\sigma\right)} + \frac{1}{p\left(t+\sigma\right)} \left[\int_{s+\sigma}^{\infty} q(s)\phi(x(s-\tau))ds + \int_{s+\sigma}^{\infty} v(s)\psi(x(s-\eta)))ds\right], t \geq t_2 + \rho. \end{cases}$$

For every $x \in S$,

$$\begin{aligned} Tx(t) &\leq \frac{\phi(b)}{p\big(t+\sigma)\big)} \left[\int_{s+\sigma}^{\infty} q(s)ds + \frac{\psi(b)}{p\big(t+\sigma)\big)} \int_{s+\sigma}^{\infty} v(s)ds\right] + \frac{\lambda}{p\big(s+\sigma)\big)} \\ &\leq \frac{1}{a_3}\left[\frac{a_3 - 1}{2} + \lambda\right] = b \end{aligned}$$

and

$$\begin{aligned} Tx(t) &\geq -\frac{x\big(t+\tau)\big)}{p\big(t+\tau)\big)} + \frac{\lambda}{p\big(t+\tau)\big)} \\ &> -\frac{b}{a_3} + \frac{\lambda}{a_4} \\ &= -\frac{a_3 - 1 + 2\lambda}{2a_3^2} + \frac{\lambda}{a_4} \\ &= \frac{2\lambda(a_3{}^2 - a_4) - a_4(a_3 - 1)}{2a_3{}^2 a_4} = a, \end{aligned}$$

which imply that $Tx \in S$. For $y_1, y_2 \in S$,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| \leq \frac{1}{|p(t+\sigma)|}&|y_1(t+\sigma) - y_2(t+\sigma)| \\ &+ \frac{K_1}{|p(t+\sigma)|}\left[\int_{s+\sigma}^{\infty} q(s)|y_1(s-\tau) - y_2(s-\tau)|\right] ds \\ &+ \frac{K_2}{|p(t+\sigma)|}\left[\int_{s+\sigma}^{\infty} v(s)|y_1(s-\eta) - y_2(s-\eta)|\right] ds, \end{aligned}$$

that is,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq \frac{1}{p_3}||y_1 - y_2|| + \frac{K_1}{a_3}||y_1 - y_2|| \int_T^{\infty} q(s)ds + \frac{K_2}{a_3}||y_1 - y_2|| \int_T^{\infty} v(s)ds \\ &< \left(\frac{1}{a_3} + \frac{a_3 - 1}{2a_3}\right)||y_1 - y_2||, \end{aligned}$$

which implies that

$$||Ty_1 - Ty_2|| \le \mu ||y_1 - y_2||,$$

that is, T is a contraction, where $\mu = \left(\frac{1}{a_3} + \frac{a_3-1}{2a_3}\right) < 1$. Hence by the Banach's fixed point theorem, T has a unique fixed point which is a nonoscillatory solution of (1.1) on $[a, b]$. Thus the proof of the theorem is complete. □

Theorem 2.4. *Assume that* (A_0) *and* (A_1) *hold and* $-1 < -a_5 \le p(t) \le 0$, $a_5 > 0$ *for* $t \in \mathbb{R}_+$. *Then every solution of (1.1) converges to zero as* $t \to \infty$ *if and only if* (A_2) *holds.*

Proof. Proceeding as in the proof of Theorem 2.2, we obtain (2.2). Hence $r(t)z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. Let $z(t) > 0$ for $t \ge t_2$. Then $\lim_{t\to\infty} z(t)$ exists. Let $z(t) < 0$ for $t \ge t_2$. We claim that $x(t)$ is bounded. If not, there exists $\{\eta_n\}$ such that $\tau(\eta_n) \le \tau_n$ and $\eta_n \to \infty$ as $n \to \infty$, $x(\eta_n) \to \infty$ as $n \to \infty$ and

$$x(\eta_n) = \max\{x(s) : t_2 \le s \le \eta_n\}.$$

Therefore,

$$\begin{aligned} z(\eta_n) &= x(\eta_n) + p(\eta_n)x(\eta_n - \sigma) \\ &\ge (1 - a_5)x(\eta_n) \\ &\to +\infty, \ \ as \ \ n \to \infty, \end{aligned}$$

which is a contradiction to the fact $z(t) > 0$. So our claim holds. Consequently, $z(t) \le x(t)$ implies that $\lim_{t\to\infty} z(t)$ exists. Hence for any $z(t)$, $x(t)$ is bounded. Using the same type of argument as in the proof of Theorem 2.2, it is easy to show that $\liminf_{t\to\infty} x(t) = 0$ and by Lemma 2.1, $\lim_{t\to\infty} z(t) = 0$. Indeed,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\big(x(t) + p(t)x(t-\sigma))\big) \\ &\ge \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-a_5 x(t-\sigma)\big) \\ &= (1 - a_5)\limsup_{t\to\infty} x(t) \end{aligned}$$

which implies that $\limsup_{t\to\infty} x(t) = 0$. The rest of the proof follows from Theorem 2.2.

Next, we suppose that (2.2) holds. Then there exists $t_1 > 0$ such that

$$\int_s^\infty [q(s) + v(s)]\, ds < \frac{1 - a_5}{5\phi(1)\psi(1)}, \ \ t \ge t_1.$$

For $t_2 > t_1$, let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of all real valued bounded continuous functions defined on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \ge t_2\}.$$

Let $K = \{y \in Y : y(t) \ge 0, \ t \ge t_2\}$. Then Y is a partially ordered Banach space (see [8]). For $u, v \in Y$, we define $u \le v$ if and only if $u - v \in K$. Let

$$S = \left\{X \in Y : \ \frac{1 - p_5}{5} \le x(t) \le 1, \ t \ge t_2\right\}.$$

If $x_0(t) = \frac{1-a_5}{5}$, then $x_0 \in S$ and $x_0 =$ g.l.b S. Further, if $\phi \subset S^* \subset S$, then

$$S^* = \left\{x \in Y : \ l_1 \le x(t) \le l_2, \ \frac{1 - a_5}{5} \le l_1, \ l_2 \le 1\right\}.$$

Let $v_0(t) = l_2'$, $t \geq t_3$, where $l_2' = \sup\{l_2 : \frac{1-a_5}{5} \leq l_2 \leq 1\}$. Then $v_0 \in S$ and $v_0 = $ l.u.b S^*. For $t_3 = t_2 + \rho$, define $T : S \to S$ by

$$Tx(t) = \begin{cases} Tx(t_3), & t \in [t_2, t_3] \\ -p(t)x(t-\sigma) + \frac{1-a_5}{5}\left[\int_s^\infty q(\eta)\phi(x(s-\tau))ds + \int_s^\infty v(s)\psi(x(s-\eta))ds\right], & t \geq t_3. \end{cases}$$

For every $x \in S$, $Tx(t) \geq \frac{1-a_5}{5}$ and

$$\begin{aligned} Tx(t) &\leq a_5 + \frac{1-a_5}{5} + \phi(1)\int_s^\infty [q(s)]ds + \psi(1)\int_s^\infty [v(s)]ds \\ &< \frac{2+3a_5}{5} < 1 \end{aligned}$$

which imply that $Tx \in S$. Now, for $x_1, x_2 \in S$, it is easy to verify that $x_1 \leq x_2$ implies that $Tx_1 \leq Tx_2$. Hence by the Knaster-Tarski fixed point theorem ([8, Theorem 1.7.3]), T has a unique fixed point such that $\lim_{t\to\infty} x(t) \neq 0$. This completes the proof of the theorem. □

Theorem 2.5. *Assume that (A_0) and (A_1) hold and $-\infty < -a_6 \leq p(t) \leq -a_7 < -1$, $a_6, a_7 > 0$ for $t \in \mathbb{R}_+$. Let ϕ, ψ be Lipschitzian on intervals of the form $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$. Then every bounded solution of (1.1) converges to zero as $t \to \infty$ if and only if (A_2) holds.*

Proof. The proof of the theorem follows from Theorem 2.2. For the necessary part, we need to mention the following:

$$\int_s^\infty [q(s) + v(s)]\, ds < \frac{a_7 - 1}{2K},$$

where $K = \max\{K_1, K_2\}$, K_1, K_2 are Lipschitz constants of ϕ and ψ on $[a, b]$, $K_2 = \phi(a)\psi(b)$ such that

$$a = \frac{2\lambda a_7 - a_6(a_7 - 1)}{2a_6 a_7}, \quad b = \frac{\lambda}{a_7 - 1}$$

for

$$\lambda > \frac{a_6(a_7 - 1)}{2a_7} > 0,$$

and

$$Tx(t) = \begin{cases} Tx(t_2 + \rho), & t \in [t_2, t_2 + \rho] \\ -\frac{x\left(t+\sigma\right)}{p\left(t+\sigma\right)} - \frac{\lambda}{p\left(t+\sigma\right)} + \frac{1}{p\left(t+\sigma\right)}\left[\int_{s+\sigma}^\infty q(s)\phi(x(s-\tau))ds + \int_{s+\sigma}^\infty v(s)\psi(x(s-\eta))ds\right], \end{cases}$$

where $t \geq t_2 + \rho$. This completes the proof of the theorem. □

Remark 2.6. *In the above theorems, ϕ and ψ could be linear, sublinear or superlinear.*

Remark 2.7. *Lemma 2.1 does not include $p(t) \equiv 1$ for all t (see [8]). The present analysis does not allow the case $p(t) \equiv -1$ for all t. Hence in our discussion, a necessary and sufficient condition is established excluding $p(t) = \pm 1$ for all t. It seems that a different approach is necessary to study the case $p(t) = \pm 1$.*

3. An example

Example 3.1. *Consider*

$$((x(t) + x(t-\pi)))' + e^t\phi(x(t-2\pi)) + e^t\psi(x(t-3\pi)) = 0, t \geq 2\pi,$$

where $\phi(x) = \psi(x) = x^3$. Then all the conditions of Theorem 2.2 are satisfied for (1.1). Hence every solution of (1.1) oscillates. In particular, $x(t) = sint$ is one of such solution of (1.1).

Clearly, all the conditions of Theorem 2.2 are satisfied. Hence, by Theorem 2.2 every solutions of (1.1) oscillates.

4. Conclusion

In this work, we established the necessary and sufficient conditions for oscillation of a class of functional differential equations of the form

$$((x(t)+p(t)x(t-\sigma))'+q(t)\phi(x(t-\tau))+v(t)\psi(x(t-\eta))=0$$

of a neutral coefficient $p(t)$, by using the Knaster-Tarski fixed point theorem and Banach's fixed point theorem.

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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On a class of $k+1$ th-order difference equations with variable coefficients

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Abstract

A Lie point symmetry analysis of a class of higher order difference equations with variable coefficients is considered and new symmetries are found. These symmetries are utilized to investigate the existence of solutions. The results in this paper generalize some results in the literature.

Key words: Difference equation; symmetry; reduction; group invariant solutions; periodicity

1 Introduction

Recently, rational difference equations have become a centre of interest of many authors, see [1–4]. Many methods have been developed to solve difference equations in closed form, that is, when every solution can be written in terms of the initial values and the indexing variable index n only. Among others, is the Lie symmetry approach used for differential equations. This differential equations method for difference equations was studied by P. Hydon and others (see [5–7, 9–11]). In [6], the author introduced an algorithm for obtaining symmetries and conservation laws of second-order difference equations. Now, it is known that these tools can be used to lower the order, via the invariants of the Lie group of transformations, as it was established for differential equations.

In this work, we aim to use the Lie symmetry approach to solve the following difference equations:

$$x_{n+1} = \frac{x_{n-k}}{\beta_n + \gamma_n \prod\limits_{i=0}^{k} x_{n+i}}, \tag{1}$$

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where β_n and γ_n are real sequences. The definitions and notation in this paper follow the ones used by Hydon in [6]. Therefore, we will have to shift the equation k times and study

$$u_{n+k+1} = \frac{u_n}{B_n + A_n \prod_{i=0}^{k} u_{n+i}}, \tag{2}$$

instead.
Our work is a natural generalization of the results by Elabbasy, et. al. [1]. These authors used induction method to give solutions of

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n+i}}, \tag{3}$$

where the parameters α, β and γ are non-negative real numbers and the initial values are positive numbers.

2 Definitions and algorithm

As mentioned earlier, the definitions and notation used in this paper follow those adopted by Hydon in [6].

Definition 2.1 *A parameterized set of point transformations,*

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon), \tag{4}$$

where $x = x_i$, $i = 1, \ldots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. *Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$*

2. *$\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0*

3. *Each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore*

$$\hat{x}_i(x : \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, ..., p. \tag{5}$$

Consider the $k+1$th-order difference equation

$$u_{n+k+1} = \Omega(u_n, u_{n+1}, \ldots, u_{n+k}), \tag{6}$$

for some function Ω. We shall restrict our attention to Lie point symmetries where $\hat{u}_n$ is a function of n and u_n only. In other words, we assume that the Lie point symmetries are of the form

$$\hat{n} = n; \qquad \hat{u}_n = u_n + \epsilon Q(n, u_n) \tag{7}$$

and that the analogous prolonged infinitesimal symmetry generator takes the form

$$X^{[k]} = \sum_{i=0}^{k} Q(n+i, u_{n+i}) \frac{\partial}{\partial u_{n+i}}, \tag{8}$$

where $Q = Q(n, u_n)$ is referred to as the characteristic. We define the symmetry condition as

$$\hat{u}_{n+k+1} = \Omega(n, \hat{u}_n, \hat{u}_{n+1}, ..., \hat{u}_{n+k}) \tag{9}$$

whenever (6) holds. Substituting the Lie point symmetries (7) into the symmetry condition (9) leads to the linearized symmetry condition

$$Q(n+k+1, u_{n+k+1}) - X^{[k]}\Omega = 0, \tag{10}$$

whenever (6) holds.
One can solve for the characteristic $Q(n, u_n)$ using the method of elimination and thereafter lower the order the difference equation (6) via the canonical coordinate [8]

$$S_n = \int \frac{du_n}{Q(n, u_n)}. \tag{11}$$

3 Main results

3.1 Symmetries

Consider the $k+1$ th-order difference equations of the form (2), i.e.,

$$u_{n+k+1} = \Omega = \frac{u_n}{B_n + A_n \prod_{i=0}^{k} u_{n+i}}.$$

We impose the symmetry condition (10) on (2) to get

$$Q(n+k+1, u_{n+k+1}) - \sum_{i=0}^{k} \Omega_{,u_{n+i}} Q(n+i, u_{n+i}) = 0, \tag{12}$$

where $\Omega_{,y}$ denotes the partial derivative of Ω with respect to y.
The characteristic in (12) takes different arguments and one can eliminate the undesirable variable by implicit differentiation. In this optic, we differentiate (12) with respect to u_{n+1} (keeping Ω fixed) and viewing u_{n+2} as a function of $u_n, u_{n+1}, \ldots, u_{n+k}$ and Ω, that is, we act the operator

$$L = \frac{\partial}{\partial u_{n+1}} + \frac{\partial u_{n+2}}{\partial u_{n+1}} \frac{\partial}{\partial u_{n+2}} = \frac{\partial}{\partial u_{n+1}} - \frac{\Omega_{,u_{n+1}}}{\Omega_{,u_{n+2}}} \frac{\partial}{\partial u_{n+2}} \tag{13}$$

on (12). This yields

$$\begin{aligned} &- \Omega_{,u_{n+1}} Q'(n+1, u_{n+1}) + \Omega_{,u_{n+1}} Q'(n+2, u_{n+2}) \\ &- \sum_{i=0}^{k} \left[\Omega_{,u_{n+i}u_{n+1}} - \frac{\Omega_{,u_{n+1}}}{\Omega_{,u_{n+2}}} \Omega_{,u_{n+i}u_{n+2}} \right] Q(n+i, u_{n+i}) = 0 \end{aligned} \tag{14}$$

which simplifies to

$$\begin{aligned} &- u_{n+1}u_{n+2}Q'(n+2, u_{n+2}) + u_{n+1}u_{n+2}Q'(n+1, u_{n+1}) - u_{n+2}Q(n+1, u_{n+1}) \\ &+ u_{n+1}Q(n+2, u_{n+2}) = 0 \end{aligned} \tag{15}$$

after a set of rather long calculations. Note that $'$ stands for the derivative with respect to the continuous variable. The differentiation of (15) with respect to u_{n+1} twice (keeping u_{n+2} fixed) leads to

$$[u_{n+1}Q'(n+1, u_{n+1}) - Q(n+1, u_{n+1})]'' = 0 \tag{16}$$

after simplification. The solution of (16) is given by

$$Q\left(n, u_n\right) = a_n u_n + b_n u_n \ln u_n + c_n \tag{17}$$

for some functions a_n, b_n and c_n of n. These functions are obtained by substituting (17) in (12) and by splitting the resulting equations with respect to product of shifts of u_n, since they are functions of n only. It turns out that $b_n = c_n = 0$ and we are left with the following reduced system:

$$1 \quad : \quad a_{n+k+1} - a_n = 0 \tag{18a}$$

$$u_n \ldots u_{n+k} \quad : \quad a_{n+1} + a_{n+2} + \cdots + a_{n+k} + a_{n+k+1} = 0, \tag{18b}$$

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or equivalently

$$a_n + a_{n+1} + a_{n+2} + \cdots + a_{n+k} = 0. \tag{19}$$

We have found that

$$a_n = \exp\left(\frac{2\pi n s}{k+1} i\right), \quad 1 \leq s \leq k. \tag{20}$$

Thus, the k infinitesimal generators are given by

$$X_s = \exp\left(\frac{2\pi n s}{k+1} i\right) u_n \frac{\partial}{\partial u_n}, \quad 1 \leq s \leq k. \tag{21}$$

3.2 Reduction and exact solutions

Let

$$\theta_s = \exp\left(\frac{2\pi s}{k+1} i\right) \quad \text{and} \quad Q_s(n, u_n) = (\theta_s)^n u_n. \tag{22}$$

To lower the order of (2), we introduce the canonical coordinate defined in (11). We have

$$S_n = \int \frac{du_n}{Q_s(n, u_n)} = \frac{1}{(\theta_s)^n} \ln |u_n|. \tag{23}$$

Thanks to (19), we have proved that

$$X_s \left[(\theta_s)^n S_n + (\theta_s)^{n+1} S_{n+1} + \cdots + (\theta_s)^{n+k} S_{n+k}\right] = 0, \quad 1 \leq s \leq k. \tag{24}$$

So,

$$r_n = (\theta_s)^n S_n + (\theta_s)^{n+1} S_{n+1} + \cdots + (\theta_s)^{n+k} S_{n+k} \tag{25}$$

is an invariant function of $X_s, \; s = 0, 1, 2, \ldots, k$. For convenience, we consider

$$|\tilde{r_n}| = \exp\{-r_n\} = \pm \frac{1}{\prod_{i=0}^{k} u_{n+i}}, \tag{26}$$

instead. We choose $\tilde{r_n} = 1/\prod_{i=0}^{k} u_{n+i}$ and the reader can readily check that $\tilde{r}_n$ satisfies

$$\tilde{r}_{n+1} = B_n\tilde{r}_n + A_n \tag{27}$$

and that

$$\tilde{r}_n = \tilde{r}_0\left(\prod_{k_1=0}^{n-1} B_{k_1}\right) + \sum_{l=0}^{n-1}\left(A_l \prod_{k_2=l+1}^{n-1} B_{k_2}\right). \tag{28}$$

Thanks to (26) and (2), we have that

$$u_{n+k+1} = \frac{\tilde{r}_n}{\tilde{r}_{n+1}} u_n \tag{29}$$

and thus

$$u_{(k+1)n+j} = u_j \prod_{s=0}^{n-1} \frac{\tilde{r}_{(k+1)s+j}}{\tilde{r}_{(k+1)s+j+1}}, \qquad j = 0, 1, \ldots, k. \tag{30}$$

We have

$$\begin{aligned} u_{(k+1)n+j} =& u_j \prod_{s=0}^{n-1} \frac{\tilde{r}_0\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} B_{k_1}\right) + \sum\limits_{l=0}^{(k+1)s+j-1}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} B_{k_2}\right)}{\tilde{r}_0\left(\prod\limits_{k_1=0}^{(k+1)s+j} B_{k_1}\right) + \sum\limits_{l=0}^{(k+1)s+j}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j} B_{k_2}\right)} \\ =& u_j \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} B_{k_1}\right) + \left(\prod\limits_{i=0}^{k} u_i\right)\sum\limits_{l=0}^{(k+1)s+j-1}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} B_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{(k+1)s+j} B_{k_1}\right) + \left(\prod\limits_{i=0}^{k} u_i\right)\sum\limits_{l=0}^{(k+1)s+j}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j} B_{k_2}\right)} \end{aligned} \tag{31}$$

for $j = 0, 1, \ldots, k$. The solution to the sequence $\{x_n\}$ is then given by

$$x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} \beta_{k_1}\right) + \mathcal{P}\sum\limits_{l=0}^{(k+1)s+j-1}\left(\gamma_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} \beta_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{(k+1)s+j} \beta_{k_1}\right) + \mathcal{P}\sum\limits_{l=0}^{(k+1)s+j}\left(\gamma_l \prod\limits_{k_2=l+1}^{(k+1)s+j} \beta_{k_2}\right)}$$

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where $j = 0, 1, 2, \ldots, k$ and $\mathcal{P} = \prod_{i=0}^{k} x_{-i}$. In the subsequent sections, we investigate solutions to special cases of the difference equations.

4 The case when β_n and γ_n are 1-periodic

In this case, we assume that $\beta_0 = \beta_j$ for all $j \geq 1$ and $\gamma_0 = \gamma_j$ for all $j \geq 1$.

4.1 The case when $\beta_0 \neq 1$

The solution becomes

$$x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{\beta_0^{(k+1)s+j} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-\beta_0^{(k+1)s+j}}{1-\beta_0}\gamma_0}{\beta_0^{(k+1)s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-\beta_0^{(k+1)s+j+1}}{1-\beta_0}\gamma_0}, \qquad j = 0, 1, 2, \ldots, k.$$

Set $\beta_0 = \gamma_0 = \frac{1}{a}$ where a is a constant. Then the solution reduces to

$$x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{(a^{-1})^{(k+1)s+j} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-(a^{-1})^{(k+1)s+j}}{1-a^{-1}} a^{-1}}{(a^{-1})^{(k+1)s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-(a^{-1})^{(k+1)s+j+1}}{1-a^{-1}} a^{-1}},$$

which is equivalent to

$$x_{(k+1)n+j-k} = x_{j-k} a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+j-1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+j} a^l}.$$

More explicitly, we have

$$x_{(k+1)n-k} = x_{-k} a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s-1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s} a^l},$$

$$x_{(k+1)n+1-k} = x_{1-k}a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+1} a^l},$$

$$x_{(k+1)n+2-k} = x_{2-k}a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+1} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+2} a^l},$$

$$\vdots$$

$$x_{(k+1)n-1} = x_{-1}a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k-2} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k-1} a^l}$$

and

$$x_{(k+1)n} = x_0a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k-1} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k} a^l}.$$

This solution has appeared in [1].

4.1.1 The special case $\beta = -1$ and k is odd

The solution simplifies to

$$x_{(k+1)n-k} = x_{-k}(-1+(x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

$$x_{(k+1)n+1-k} = x_{1-k}(-1+(x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{n},$$

$$x_{(k+1)n+2-k} = x_{2-k}(-1+(x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

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$$\vdots$$

$$x_{(k+1)n-1} = x_{-1}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\ldots x_{-1}x_0)\gamma_0)^{-n},$$

$$x_{(k+1)n} = x_{j-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\ldots x_{-1}x_0)\gamma_0)^{n},$$

as long as the denominator does not vanish.
However, the solution above can be written in compact form as

$$x_{(k+1)n+j-k} = x_{j-k}\left(-1 + (x_{-k}x_{-k+1}x_{-k+2}\ldots x_{-1}x_0)\gamma_0\right)^{(-1)^{j+1}n}$$

for $j = 0, 1, \ldots, k$.
This solution has appeared in [1] (See Theorem 9).

Remark 4.1 *Note that if* $\gamma_0 \prod_{i=0}^{k} x_{-i} = 2$*, the solution is periodic with period* $k+1$*.*

4.1.2 The special case $\beta = -1$ and k is even

In this case, we have

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\prod_{s=0}^{n-1}\frac{(-1)^{s+j} + \left(\prod_{i=0}^{k} x_{-i}\right)^{\frac{1-(-1)^{s+j}}{2}}\gamma_0}{(-1)^{s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right)^{\frac{1-(-1)^{s+j+1}}{2}}\gamma_0} \\ &= x_{j-k}\prod_{\substack{s\geq 0,\\ s-j \text{ is even}}}^{n-1}\frac{1}{-1 + \left(\prod_{i=0}^{k} x_{-i}\right)\gamma_0}\prod_{\substack{s\geq 0,\\ s-j \text{ is odd}}}^{n-1}\left(-1 + \left(\prod_{i=0}^{k} x_{-i}\right)\gamma_0\right). \end{aligned}$$

If j is even and n is odd,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1 + \gamma_0\prod_{i=0}^{k} x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor-1}\left(-1 + \gamma_0\prod_{i=0}^{k} x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor} \\ &= x_{j-k}\left(-1 + \gamma_0\prod_{i=0}^{k} x_{-i}\right)^{-1}. \end{aligned}$$

If j is odd and n is odd,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor+1} \\ &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right). \end{aligned}$$

If j is even and n is even,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor-1}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor+1} \\ &= x_{j-k}. \end{aligned}$$

If j is odd and n is even,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor-1}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor+1} \\ &= x_{j-k}. \end{aligned}$$

In summary, and more compactly, the solution is

$$x_{(k+1)n+j-k} = \begin{cases} x_{j-k}\left(-1+\gamma_0\prod\limits_{i=0}^{k}x_{-i}\right)^{(-1)^{j+1}}, & \text{if } n \text{ is odd} \\ x_{j-k}, & \text{if } n \text{ is even.} \end{cases}$$

This solution has appeared in [1] (See Theorem 8).

4.1.3 The case when $\beta_0 = 1$

The solution is given by

$$x_{(k+1)n+j-k} = x_{j-k}\prod_{s=0}^{n-1}\frac{1+\left(\prod\limits_{i=0}^{k}x_{-i}\right)((k+1)s+j)\gamma_0}{1+\left(\prod\limits_{i=0}^{k}x_{-i}\right)((k+1)s+j+1)\gamma_0}, \qquad j=0,1,2,\dots,k.$$

5 Conclusion

We have utilized symmetry analysis to find point symmetries for certain $(k+1)$ th-order difference equations. We performed the group reduction of the equations using one of these symmetries and solutions were given in a unified manner. Our results generalise those in [1] in the sense that (a) α, β and γ need not necessarily be non-negative integers and (b) the constants can be replaced with sequences (variable constants).

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SUPERSTABILITY OF THE PEXIDER TYPE SINE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we find solutions and investigate the superstability bounded by function for the sine functional equation (S) from the approximate inequality of the Pexider type functional equation:

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)k(y).$$

Furthermore, the results are extended to Banach algebras. As a consequence, we obtain the superstability for the exponential functional equations, the hyperbolic functional equations, and the jointed Pexider Lobacevski equation.

Keywords: stability, superstability, sine functional equation, cosine functional equation.

MSC 2020: 39B82, 39B52.

1. Introduction

In 1979, Baker *et al.* [4] announced the *superstability* as the new concept as follows: If f satisfies $|f(x+y) - f(x)f(y)| \le \varepsilon$ for some fixed $\varepsilon > 0$, then either f is bounded or f satisfies the exponential functional equation

$$f(x+y) = f(x)f(y). \tag{E}$$

D'Alembert, in 1769, introduced the cosine (d'Alembert) functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{C}$$

whose superstability was proved on Abelian group by Baker [3] in 1980.

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y) + f(x-y) = 2f(x)g(y), \tag{W}$$

$$f(x+y) + f(x-y) = 2g(x)f(y), \tag{K}$$

in which (W) is called the Wilson equation, and (K) was raised by Kim [9].

The superstability of the cosine (C), Wilson (W) and Kim (K) was founded in Badora [1], Ger [2], Kannappan and Kim [9], Kim [13, 15, 16, 20], and in [5, 7, 22].

In 1983, Cholewa [6] investigated the superstability of the sine functional equation

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = f(x)f(y) \tag{S}$$

under the condition bounded by constant (Hyers sense). His result was improved to the condition bounded by a function (Găvruta's sense in [8]) in Badora and Ger [2].

Their results were also improved by Kim [11, 12, 14], which are the superstability of the generalized sine functional equations:

$$f\big(\frac{x+y}{2}\big)^2 - f\big(\frac{x-y}{2}\big)^2 = f(x)g(y) \tag{S_{fg}}$$

$$f\big(\frac{x+y}{2}\big)^2 - f\big(\frac{x-y}{2}\big)^2 = g(x)f(y) \tag{S_{gf}}$$

$$f\big(\frac{x+y}{2}\big)^2 - f\big(\frac{x-y}{2}\big)^2 = g(x)g(y) \tag{S_{gg}}$$

$$f\big(\frac{x+y}{2}\big)^2 - f\big(\frac{x-y}{2}\big)^2 = g(x)h(y) \tag{S_{gh}}$$

under the condition bounded by a constant or a function.

The aim of this paper is to find solutions and to investigate the superstability bounded by the function (Găvruta sense in [8]) for the sine functional equation (S) from an approximate inequality of the Pexider type functional equation:

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)k(y), \tag{S_{fghk}}$$

which is represented by the exponential equations, hyperbolic cosine(sine) equations, and the jointed Pexider Lobacevski equation (PL).

As corollaries, we obtain the superstability bounded by a constant or the function for the sine functional equation (S) from an approximate inequality of the sine type functional equations (S_{fg}), (S_{gf}), (S_{gg}), (S_{gh}), and the Pexider type functional equations:

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)h(y) \tag{S_{fghh}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)f(y) \tag{S_{fghf}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)g(y) \tag{S_{fghg}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = f(x)h(y) \tag{S_{fgfh}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = f(x)g(y) \tag{S_{fgfg}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = f(x)f(y) \tag{S_{fgff}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)h(y) \tag{S_{fggh}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)g(y) \tag{S_{fggg}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)f(y). \qquad (S_{fggf})$$

Furthermore, the obtained results are extended to Banach algebras.

In this paper, let $(G,+)$ be an uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, and G the field of real numbers. f, g, h, k are nonzero functions and ε is a nonnegative real constant, $\varphi : G \to \mathbb{R}$ be a mapping.

2. Creation of the equations and its solution.

The purpose of this chapter is to show the creation and the solution for the frequently risen function equations dued by the trigonometric function.

Let us recall the trigonometric formula, except for (C), (W) (K).

$\sin(x+y) + \cos(x-y) = [\sin(x) + \cos(x)][\sin(y) + \cos(y)]$ *implies*

$$f(x+y) + g(x-y) = [f(x) + g(x)][f(y) + g(y)] = h(x)h(y). \qquad (fghh)$$

$\cos(x+y) + \sin(x-y) = [\cos(x) + \sin(x)][\cos(y) - \sin(y)]$ *implies*

$$f(x+y) + g(x-y) = [f(x) + g(x)][f(y) - g(y)] = h(x)k(y). \qquad (fghk)$$

$\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$ *implies*

$$f(x+y) - f(x-y) = 2g(x)f(y). \qquad (T_{gf})$$

$\cos(x+y) - \cos(x-y) = -2\sin(x)\sin(y)$ *implies*

$$f(x+y) - f(x-y) = -2g(x)g(y) = 2g(x)h(y). \qquad (T_{gh})$$

$\cos(x+y) - \sin(x-y) = [\cos(x) - \sin(x)][\cos(y) + \sin(y)]$ *implies*

$$f(x+y) - g(x-y) = [f(x) - g(x)][f(y) + g(y)] = h(x)k(y). \qquad (T_{fghk})$$

$\sin(x+y) - \cos(x-y) = [\sin(x) - \cos(x)][\cos(y) - \sin(y)]$ *implies*

$$f(x+y) - g(x-y) = [f(x) - g(x)][g(y) - f(y)] = h(x)k(y). \qquad (T_{fghk})$$

$$f(x+y) - f(x-y) = 2f(x)f(y). \qquad (T)$$

Like the cosine and the sine, the above functional equations are also derived simultaneously by the hyperbolic cosine (sine), exponential equation, and Jensen equation, as can be seen in the following relations:

$$\cosh(x+y) \pm \cosh(x-y) = 2\cosh(x)\cosh(y)\big(= -2\sinh(x)\sinh(y)\big)$$
$$\sinh(x+y) \pm \sinh(x-y) = 2\sinh(x)\cosh(y)\big(= 2\cosh(x)\sinh(y)\big)$$
$$a^{x+y} \pm a^{x-y} = 2a^x \tfrac{a^y \pm a^{-y}}{2} \approx 2e^x \tfrac{e^y \pm e^{-y}}{2} = 2e^x\cosh(y)\big(= 2e^x\sinh(y)\big)$$
$$(n(x+y)+c) \pm (n(x-y)+c) = 2(nx+c)\big(= 2n(y)\big)$$
$$: \text{Jensen equation, for } f(x) = nx+c, g(y) = 1,$$

where the subtraction corresponds into parentheses ( ).

Since the trigonometric and hyperbolic functions are expressed by an exponential function as following: $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, respectively, all of the above functional equations naturally have exponential and hyperbolic functions as solution.

Now, let's bring the quadratic functional equation generated by a product or a square of the above equations, which is the target of this paper.

It is well known that the sine functional equation (S) is derived as follows:

$$\begin{aligned} f\left(\tfrac{x+y}{2}\right)^2 - f\left(\tfrac{x-y}{2}\right)^2 &= \sin\left(\tfrac{x+y}{2}\right)^2 - \sin\left(\tfrac{x-y}{2}\right)^2 \\ &= \sin(x)\sin(y) = f(x)f(y). \end{aligned}$$

Eq. (S) has simultaneously an exponential solution as follows :

$$\left(\frac{1}{2i}\left(e^{i\frac{x+y}{2}} - e^{-i\frac{x+y}{2}}\right)\right)^2 - \left(\frac{1}{2i}\left(e^{i\frac{x-y}{2}} - e^{-i\frac{x-y}{2}}\right)\right)^2 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{iy} - e^{-iy}}{2i}\right).$$

Also, simultaneously, (S) is satisfied for the hyperbolic sine function as follows:

$$\begin{aligned} & f\left(\tfrac{x+y}{2}\right)^2 - f\left(\tfrac{x-y}{2}\right)^2 = \sinh\left(\tfrac{x+y}{2}\right)^2 - \sinh\left(\tfrac{x-y}{2}\right)^2 \\ &= \left(\tfrac{1}{2}\left(e^{\frac{x+y}{2}} - e^{-\frac{x+y}{2}}\right)\right)^2 - \left(\tfrac{1}{2}\left(e^{\frac{x-y}{2}} - e^{-\frac{x-y}{2}}\right)\right)^2 = \left(\tfrac{1}{2}\left(e^x - e^{-x}\right)\right)\left(\tfrac{1}{2}\left(e^y - e^{-y}\right)\right) \\ &= \sinh(x)\sinh(y) = f(x)f(y), \end{aligned}$$

which is added solutions as the hyperbolic sine, exponential function.

Also, the other examples of the Pexider type quadratic functional equations

$$f(x)f(y) = \begin{cases} (i)\ f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \\ (ii)\ g\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 \\ (iii)\ g\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \\ (iv)\ -\left(f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2\right) \end{cases}$$

have solutions to the hyperbolic sine(cosine) as follows:

$$\sinh(x)\sinh(y) = \begin{cases} (i)\ \sinh^2\left(\frac{x+y}{2}\right) - \sinh^2\left(\frac{x-y}{2}\right) \\ (ii)\ \cosh^2\left(\frac{x+y}{2}\right) - \cosh^2\left(\frac{x-y}{2}\right) \\ (iii)\ \cosh^2\left(\frac{x+y}{2}\right) - \sinh^2\left(\frac{x-y}{2}\right) \\ (iv)\ -\left(\sinh^2\left(\frac{x+y}{2}\right) - \cosh^2\left(\frac{x-y}{2}\right)\right). \end{cases}$$

Next, the Lobacevski equation

$$f\left(\frac{x+y}{2}\right)^2 = f(x)f(y) \tag{L}$$

is considered to the exponential equation (E) by $f\left(\frac{x+y}{2}\right)^2 = \left(e^{\frac{x+y}{2}}\right)^2 = e^{x+y} = e^x e^y = f(x)f(y)$.

The Lobacevski equation (L) was generalized by Kim [17, 18] Kim and Park [21] to the Pexider type Lobacevski equations

$$f\left(\frac{x+y}{2}\right)^2 = g(x)h(y), \qquad f\Big(\frac{x+y}{n}\Big)^m = g(x)h(y). \tag{PL}$$

Hence (S_{fghk}) and all (S) type equations are also represented as joint of (L) and (PL) as follows:

$$\begin{aligned} f\left(\frac{x+y}{m}\right)^m - g\left(\frac{x-y}{n}\right)^n &= \left(\sqrt[m]{p}\left(a^{\frac{x+y}{m}}\right)\right)^m - \left(\sqrt[n]{q}\left(a^{-\frac{x-y}{n}}\right)\right)^n \\ &= p\left(a^{x+y}\right) - q\left(a^{-x+y}\right) = \left(pa^x - qa^{-x}\right)a^y \\ &= h(x)k(y), \end{aligned}$$

where $f(x) = \sqrt[m]{p}(a^x)$, $g(x) = \sqrt[n]{q}(a^{-x})$, $h(x) = pa^x - qa^{-x}$, $k(x) = a^x$.

As a result, the target function equation (S_{fghk}) has solutions as the trigonometric, exponential, hyperbolic function, jointed Pexider Lobacevski equation.

In the following, we show examples of solution applied to the trigonometric function for (S_{gg}), (S_{gh}), (S_{fghk}), (S_{fghh}). Of course, it is also natural to have the its solutions as exponential function, hyperbolic sine(cosine) function, Pexider Lobacevski equation. Their description will be skip.

Solution 1. *The functions $f, g, h : G \longrightarrow \mathbb{C}$ satisfy (S_{gh}) if and only if f, g, h are solutions with $f(x) = \cos x$, $g(x) = \sin x$, and $h(x) = -\sin x$.*

In particular, if the functions $f, g : G \longrightarrow \mathbb{C}$ satisfy the functional equation (S_{gg}) if and only if f, g are solutions with $f(x) = \cos x$ and $g(x) = i \sin x$.

Proof. $f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = \cos\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = -\sin x \sin y = g(x)h(y)$. In particular case, it is established that $f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = \cos\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = -\sin x \sin y = i^2 \sin x \sin y = g(x)g(y)$. □

Solution 2. *The functions $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy (S_{fghk}) if and only if f, g, h, k are solutions with $f(x) = \sin(x)$, $g(x) = \cos(x)$, $h(x) = (\sin^2 - \cos^2)(x)$, $k(x) = (\cos^2 - \sin^2)(x)$.*

Proof. $f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = \sin\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = (\sin^2 x - \cos^2 x)(\cos^2 y - \sin^2 y) = h(x)k(y)$. □

Solution 3. *(i) The functions $f, g, h : G \longrightarrow \mathbb{C}$ satisfy (S_{fghh}) if and only if f, g, h are solutions with $f(x) = \cos(2x)$, $g(x) = \sin(2x)$, $h(x) = (\cos^2 - \sin^2)(x)$.*

(ii) The functions $f, g, h : G \longrightarrow \mathbb{C}$ satisfy (S_{fghh}) if and only if f, g, h are solutions with $f(x) = \sin(x)$, $g(x) = \cos(x)$, $h(x) = i(\sin^2 - \cos^2)(x)$.

Proof. (i) $f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = \cos\left(2\frac{x+y}{2}\right)^2 - \sin\left(2\frac{x-y}{2}\right)^2 = \cos\left(x+y\right)^2 - \sin\left(x-y\right)^2 = (\cos^2 x - \sin^2 x)(\cos^2 y - \sin^2 y) = h(x)h(y)$.

(ii) $f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = \sin\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = (\sin^2 x - \cos^2 x)(\cos^2 y - \sin^2 y) = i^2(\sin^2 x - \cos^2 x)(\sin^2 y - \cos^2 y) = h(x)h(y)$. □

Remark 1. *(i) It is trivial that all 1∼3 have solutions as an exponential functions, hyperbolic sine (cosine) functions, and Lovachevsky equations.*

(ii) The investigation of solutions associated with the generative method for (S_{fghk}) can be further extended to that for the Pexider type function equation:

$$f\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x-y}{2}\right)^2 = h(x)k(y). \tag{P_{fghk}}$$

3. Superstability of (S) from the approximate inequality of (S_{fghk})

We investigate the superstability of the sine functional equation (S) from the approximate inequality of the Pexider type functional equation (S_{fghk}) related to (S). As a corollary, we obtain the superstability of the sine functional equation (S).

Theorem 1. *Assume that $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right| \le \varphi(y) \quad \forall x, y \in G, \tag{3.1}$$

which satisfies one of the cases $k(0) = 0$, $f(x)^2 = g(x)^2$.

Then either h is bounded or k satisfies (S). In addition, if h satisfies (C)*, k and h satisfy (T_{gf}):= $k(x+y) - k(x-y) = 2h(x)k(y)$.*

Proof. The inequality (3.1) may equivalently be written as

$$|f(x+y)^2 - g(x-y)^2 - h(2x)k(2y)| \le \varphi(2y), \quad \forall\, x, y \in G. \tag{3.2}$$

Let h be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \ne |h(2x_n)| \to \infty, \quad \text{as} \quad n \to \infty. \tag{3.3}$$

Taking $x = x_n$ in (3.2), we obtain

$$\left| \frac{f(x_n+y)^2 - g(x_n-y)^2}{h(2x_n)} - k(2y) \right| \le \frac{\varphi(2y)}{|h(2x_n)|},$$

and so by (3.3), we have

$$k(2y) = \lim_{n\to\infty} \frac{f(x_n+y)^2 - g(x_n-y)^2}{h(2x_n)}. \tag{3.4}$$

Using (3.1) we have

$$\begin{aligned} 2\varphi(y) \ge & \left| h(2x_n+x)k(y) - f\left(\frac{2x_n+x+y}{2}\right)^2 + g\left(\frac{2x_n+x-y}{2}\right)^2 \right| \\ & + \left| h(2x_n-x)k(y) - f\left(\frac{2x_n-x+y}{2}\right)^2 + g\left(\frac{2x_n-x-y}{2}\right)^2 \right| \\ \ge & \,|\, (h(2x_n+x) + h(2x_n-x))\, k(y) \\ & - \left(f\left(x_n + \frac{x+y}{2}\right)^2 - g\left(x_n - \frac{x+y}{2}\right)^2 \right) \\ & - \left(f\left(x_n + \frac{-x+y}{2}\right)^2 - g\left(x_n - \frac{-x+y}{2}\right)^2 \right) \Bigg| \end{aligned} \tag{3.5}$$

for all $x, y \in G$ and all $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}\frac{2\varphi(y)}{|h(2x_n)|} \geq \Bigg| & \frac{h(2x_n+x)+h(2x_n-x)}{h(2x_n)}k(y) \\ & - \frac{f\left(x_n+\frac{x+y}{2}\right)^2 - g\left(x_n-\frac{x+y}{2}\right)^2}{h(2x_n)} \\ & - \frac{f\left(x_n+\frac{-x+y}{2}\right)^2 - g\left(x_n-\frac{-x+y}{2}\right)^2}{h(2x_n)}\Bigg| \end{aligned} \tag{3.6}$$

for all $x, y \in G$ and all $n \in \mathbb{N}$.

Taking the limit as $n \longrightarrow \infty$ with the use of (3.4) and (3.6), we conclude that, for every $x \in G$, there exists the limit function

$$L_1(x) := \lim_{n\to\infty} \frac{h(2x_n+x)+h(2x_n-x)}{h(2x_n)},$$

where the obtained function $L_1 : G \to \mathbb{C}$ satisfies the equation as even

$$k(x+y)+k(-x+y) = L_1(x)k(y) \quad \forall x, y \in G. \tag{3.7}$$

First, let us consider the case $k(0) = 0$. Then it forces by (3.7) that k is odd. So (3.7) is

$$k(x+y)-k(x-y) = L_1(x)k(y) \quad \forall x, y \in G. \tag{3.8}$$

By means of (3.8) and the oddness of k, we have the following

$$\begin{aligned} k(x+y)^2 - k(x-y)^2 &= [k(x+y)+k(x-y)]L_1(x)k(y) \\ &= \big[k(2x+y)+k(2x-y)\big]k(y) \\ &= \big[k(y+2x)-k(y-2x)\big]k(y) \\ &= L_1(y)k(2x)k(y). \end{aligned} \tag{3.9}$$

Putting $x = y$ in (3.8), we conclude that

$$k(2y) = L_1(y)k(y) \text{ for all } x, y \in G. \tag{3.10}$$

The equation (3.9), in return, leads with (3.10) to the equation

$$k(x+y)^2 - k(x-y)^2 = k(2x)k(2y), \tag{3.11}$$

which, by 2-divisibility of G, states nothing else but (S).

In addition, if h satisfies (C), L_1 forces $2h$, so (3.8) forces that k and h satisfy (T_{gf}).

For the other case $f(x)^2 = g(x)^2$, it is enough to show that $k(0) = 0$. Suppose that this is not the case. Then, we may assume that $k(0) = c$: constant.

Putting $y = 0$ in (3.1), from the above assumption, we obtain the inequality

$$|h(x)| \leq \frac{\varphi(0)}{c} \quad \forall\, x \in G.$$

This inequality means that h is globally bounded, which is a contradiction by unboundedness assumption. Thus the claimed $k(0) = 0$ holds, so the proof is completed. □

Theorem 2. *Suppose that $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right| \leq \varphi(x) \quad \forall x, y \in G, \tag{3.12}$$

which satisfies one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$.

Then either k is bounded or h satisfies (S). In addition, if k satisfies (C)*, h and k satisfy the Wilson equation $(W) := h(x+y) + h(x-y) = 2h(x)k(y)$.*

Proof. Let k be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that $k(2y_n)| \to \infty$ as $n \to \infty$. An obvious slight change in the proof steps applied in the start of Theorem 1 gives us

$$h(2x) = \lim_{n\to\infty} \frac{f(x+y_n)^2 - g(x-y_n)^2}{k(2y_n)}. \tag{3.13}$$

Replacing y by $y + 2y_n$ and $-y + 2y_n$ in (3.12), the same procedure of (3.5) and (3.6) allows, with an applying of (3.13), one to state the existence of a limit function

$$L_2(y) := \lim_{n\to\infty} \frac{k(y+2y_n) + k(-y+2y_n)}{k(2y_n)},$$

where $L_2 : G \longrightarrow \mathbb{C}$ satisfies the equation

$$h(x+y) + h(x-y) = h(x)L_2(y) \quad \forall x, y \in G. \tag{3.14}$$

For the case $h(0) = 0$, it forces by (3.14) that h is odd.

Putting $y = x$ in (3.14), we get

$$h(2x) = h(x)L_2(x) \quad \forall x, \in G. \tag{3.15}$$

From (3.14), the oddness of h and (3.15), we obtain the equation

$$\begin{aligned} h(x+y)^2 - h(x-y)^2 &= h(x)L_2(y)[h(x+y) - h(x-y)] \\ &= h(x)[h(x+2y) - h(x-2y)] \\ &= h(x)[h(2y+x) + h(2y-x)] \\ &= h(x)h(2y)L_2(x) \\ &= h(2x)h(2y), \end{aligned}$$

which, by 2-divisibility of G, states (S).

In addition, if k satisfies (C), L_2 forces $2k$, so (3.14) forces that h and k satisfy (W).

The other case $f(x)^2 = g(-x)^2$ also is established $h(0) = 0$ for the same reason as that of Theorem 1, so the proof is completed. □

From Theorems 1 and 2, we obtain the following result as a corollary.

Theorem 3. *Suppose that $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right| \leq \min\{\varphi(x), \varphi(y)\} \tag{3.16}$$

for all $x, y \in G$. Then

(i) either h under the cases $k(0)=0$ or $f(x)^2=g(x)^2$ is bounded or k satisfies (S). In addition, if h satisfies (C), *k and h satisfy (T_{gf}):= $k(x+y)-k(x-y)=2h(x)k(y)$;*

(ii) either k under the cases $h(0)=0$ or $f(x)^2=g(-x)^2$ is bounded, or h satisfies (S). In addition, if k satisfies (C), *h and k satisfy the Wilson equation (W):=$h(x+y)+h(x-y)=2h(x)k(y)$.*

As a corollary, we obtain the stability of the sine functional equation (S) from Theorems 1, 2, 3.

Corollary 1. *Assume that $f: G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2-f(x)f(y)\right| \leq \begin{cases}(i)\ \varphi(y),\\ (ii)\ \varphi(x),\\ (iii)\ \min\{\varphi(x),\varphi(y)\}.\end{cases}$$

Then, either f is bounded or f satisfies (S).

Proof. Assumption $f(0)=0$ in Theorems is simply eliminated (see [2, Theorem 5]). □

4. Application of the equations (S_{fghh}), (S_{fghf}), (S_{fgfh}), (S_{fgfg})

Replacing according to the location by f, g, or h for the functions k, h in Theorems 1, 2, and 3, as corollaries, we obtain the stability of the sine functional equation (S) from the approximate inequalities of (S_{fghh}), (S_{fghf}), (S_{fgfh}), (S_{fgfg}). Other cases are skipped. All proofs follow from that of Theorems 1, 2, 3.

4.1. Stability of the equation (S_{fghh}).

Corollary 2. *Suppose that $f,g,h: G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-h(x)h(y)\right| \leq \begin{cases}(i)\ \varphi(y)\\ (ii)\ \varphi(x)\\ (iii)\ \min\{\varphi(x),\varphi(y)\}\end{cases} \qquad \forall x,y \in G.$$

Then, either h is bounded or h satisfies (S) under one of the cases $h(0)=0$, $f(x)^2=g(x)^2$, $f(x)^2=g(-x)^2$, respectively.

4.2. Stability of the equation (S_{fghf}).

Corollary 3. *Assume that $f,g,h: G \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-h(x)f(y)\right| \leq \varphi(y), \quad \forall x,y \in G$$

which satisfies one of the cases $f(0)=0$, $f(x)^2=g(x)^2$.

Then, either h is bounded or f satisfy (S). In addition, if h satisfies (C), *then f and h satisfy (T_{gf}):= $f(x+y)-f(x-y)=2h(x)f(y)$.*

Corollary 4. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)f(y) \right| \le \varphi(x), \quad \forall x, y \in G$$

which satisfies one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$.

Then, either f is bounded or h satisfies (S). In addition, if f satisfies (C)*, h and f satisfy the Wilson equation (W):=$h(x+y) + h(x-y) = 2h(x)f(y)$.*

The following result follows from Corollaries 3 and 4.

Corollary 5. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)f(y) \right| \le \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in G$. Then

(i) either h is bounded under one of the cases $f(0) = 0$, $f(x)^2 = g(x)^2$ or f satisfy (S). In addition, if h satisfies (C)*, f and h satisfy (T_{gf}):= $f(x+y) - f(x-y) = 2h(x)f(y)$;*

(ii) either f is bounded under one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$ or h satisfies (S). In addition, if f satisfies (C)*, h and f satisfy the Wilson equation (W):=$h(x+y) + h(x-y) = 2h(x)f(y)$.*

4.3. **Stability of the equation** (S_{fgfh})**.**

Corollary 6. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right| \le \varphi(y),$$

which satisfies one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$.

Then, either f is bounded or h satisfies (S). In addition, if f satisfies (C)*, h and f satisfy (T_{gf}):= $h(x+y) - h(x-y) = 2f(x)h(y)$.*

Corollary 7. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right| \le \varphi(x),$$

which satisfies one of the cases $f(0) = 0$, $f(x)^2 = g(-x)^2$.

Then, either h is bounded or f satisfies (S). In addition, if h satisfies (C)*, f and h satisfy the Wilson equation (W):= $f(x+y) + f(x-y) = 2f(x)h(y)$.*

Corollary 8. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right| \le \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in G$. Then

(i) either f is bounded under one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$ or h satisfies (S). In addition, if f satisfies (C)*, h and f satisfy (T_{gf}):= $h(x+y) - h(x-y) = 2f(x)h(y)$;*

(ii) either h is bounded under one of the cases $f(0)=0$, $f(x)^2=g(-x)^2$ or f satisfies (S). In addition, if h satisfies (C), *f and h satisfy the Wilson equation* (W):$=f(x+y)+f(x-y)=2f(x)h(y)$.

4.4. **Stability of the equation (S_{fgfg}).**

Corollary 9. *Suppose that $f,g:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-f(x)g(y)\right|\le\varphi(y),$$

which satisfies one of the cases $g(0)=0$, $f(x)^2=g(x)^2$.

Then, either f is bounded or g satisfies (S). In addition, if f satisfies (C), *g and f satisfy (T_{gf}):$=g(x+y)-g(x-y)=2f(x)g(y)$.*

Corollary 10. *Suppose that $f,g:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-f(x)g(y)\right|\le\varphi(x),$$

which satisfies one of the cases $f(0)=0$, $f(x)^2=g(-x)^2$.

Then, either g is bounded or f satisfies (S). In addition, if g satisfies (C), *then f and g satisfy the Wilson equation (W).*

Corollary 11. *Suppose that $f,g:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-f(x)g(y)\right|\le\min\{\varphi(x),\varphi(y)\}$$

for all $x,y\in G$. Then

(i) either f is bounded under one of the cases $g(0)=0$, $f(x)^2=g(x)^2$ or g satisfies (S). In addition, if f satisfies (C), *g and f satisfy (T_{gf}):$=g(x+y)-g(x-y)=2f(x)g(y)$;*

(ii) either g is bounded under one of the cases $f(0)=0$, $f(x)^2=g(-x)^2$ or f satisfies (S). In addition, if g satisfies (C), *then f and g satisfy the Wilson equation (W).*

Remark 2. *As corollaries, we obtain more stability results for the following reduced equations of (S_{fghk}).*

(i) The stability for the functional equations (S_{fghg}), (S_{fggh}), (S_{fggf}), (S_{fgff}), (S_{fggg}), and (S_{gh}), (S_{gf}), (S_{fg}), (S_{gg}) is skipped by same reason as the cases (S_{fghh}), (S_{fghf}), (S_{fgfh}), (S_{fgfg}). In particular, the stability for the equations (S_{gh}), (S_{gf}), (S_{fg}), (S_{gg}) is found in papers (see [11, 14, 19]*).*

(ii) Applying $\varphi(x)=\varphi(y)=\varepsilon$ in all results containing (i), then it imply the stability results.

5. Extension of the stability results to Banach algebras

All the results in Sections 3 and 4 can be also extended to Banach algebras. The following theorem is an extension dued by Theorem 1, Theorem 2, and Theorem 3.

Theorem 4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h, k : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition $x^* \circ h$ under the cases $k(0) = 0$ or $f(x)^2 = g(x)^2$ is bounded or k satisfies (S), In addition, if h satisfies* (C), *k and h satisfy (T_{gf}):= $k(x+y) - k(x-y) = 2h(x)k(y)$;*

(ii) *either the superposition $x^* \circ k$ under the cases $h(0) = 0$ or $f(x)^2 = g(-x)^2$ is bounded or h satisfies (S). In addition, if k satisfies* (C), *h and k satisfy the Wilson equation (W):= $h(x+y) + h(x-y) = 2h(x)k(y)$;*

(iii) (i) *and* (ii) *hold.*

Proof. Assume that (i) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned} \varphi(y) &\geq \left\| h(x)k(y) - f\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x-y}{2}\right)^2 \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(h(x)k(y) - f\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x-y}{2}\right)^2 \right) \right| \\ &\geq \left| x^*(h(x)) \cdot x^*(k(y)) - x^*\left(f\left(\frac{x+y}{2}\right) \right) + x^*\left(g\left(\frac{x-y}{2}\right)^2 \right) \right|, \end{aligned}$$

which states that the superpositions $x^* \circ h$ and $x^* \circ k$ yield a solution of stability inequality (3.1) of Theorem 1. Since, by assumption, the superposition $x^* \circ h$ is unbounded, an appeal to Theorem 1 forces that the function $x^* \circ k$ solves the sine equation (S). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference $\mathcal{D}S : G \longrightarrow E$ defined by

$$\mathcal{D}S(x,y) := k\left(\frac{x+y}{2}\right)^2 - k\left(\frac{x-y}{2}\right)^2 - k(x)k(y)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}S(x,y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathcal{D}S(x,y) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed. The cases(ii), (iii) also are the same. $\square$

Corollary 12. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)h(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ h$ is bounded or h satisfies (S) under one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$, $f(x)^2 = g(-x)^2$, respectively.

Corollary 13. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)f(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition $x^* \circ h$ under one of the cases $f(0) = 0$, $f(x)^2 = g(x)^2$ is bounded or f satisfies (S), In addition, if h satisfies* (C), *f and h satisfy (T_{gf});*

(ii) *either the superposition $x^* \circ f$ under one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$ is bounded or h satisfies (S). In addition, if f satisfies* (C), *h and f satisfy the Wilson equation (W);*

(iii) (i) *and* (ii) *hold.*

Corollary 14. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i)*) either the superposition $x^* \circ f$ under one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$ is bounded or h satisfy (S);*

In addition, if f satisfies (C), *h and f satisfy (T_{gf}):= $h(x+y) - h(x-y) = 2f(x)h(y)$.*

(ii) *either the superposition $x^* \circ h$ under the cases $f(0) = 0$ or $f(x)^2 = f(-x)^2$ is bounded or f satisfies (S). In addition, if h satisfies* (C), *f and h satisfy the Wilson equation (W):= $f(x+y) + f(x-y) = f(x)h(y)$;*

(iii) (i) *and* (ii) *hold.*

Corollary 15. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f : G \longrightarrow E$ satisfies the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)f(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ f$ is bounded or f satisfies (S).

Remark 3. *All items of Remark 2 also hold to same results for all functional equations on Banach algebras.*

6. Conclusion

We investigated the superstability bounded by function for the sine functional equation (S) from the approximate inequality of the Pexider type functional equation (S_{fghk}), and we studied a creative process for the sine, cosine(d'Alembert), Wilson, Kim's, (S) type functional equations, which are a frequently arisen function equations related for the sine functional equation (S) and the Pexider type functional equation(S_{fghk}).

As a result, all (S) types functional equations related with (S) and (S_{fghk}) can be represented by the trigonometric, exponential, hyperbolic function, jointed Pexider Lobacevski equation. Furthermore, we showed the application of our results to a myriad of equations and the results were extended to Banach algebra.

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TERNARY HOM-DERIVATION-HOMOMORPHISM

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Abstract. In this paper, we introduce and solve the following additive-additive (s,t)-functional inequality

$$\begin{aligned}
&\|g(x+y+z)-g(x)-g(y)-g(z)\| \\
&+\|h(x+y+z)+h(x-2y+z)+h(x+y-2z)-3h(x)\| \\
&\leq \left\|s\left(3g\left(\frac{x+y+z}{3}\right)-g(x)-g(y)-g(z)\right)\right\| \\
&+\left\|t\left(3h\left(\frac{x+y+z}{3}\right)+h(x-2y+z)+h(x+y-2z)-3h(x)\right)\right\|,
\end{aligned} \tag{0.1}$$

where s and t are fixed nonzero complex numbers with $|s|<1$ and $|t|<1$. Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of ternary hom-derivations and ternary homomorphisms in C^*-ternary algebras, associated to the additive-additive (s,t)-functional inequality (0.1) and the following functional inequality

$$\begin{aligned}
&\|g([x,y,z])-[g(x),h(y),h(z)]-[h(x),g(y),h(z)]-[h(x),h(y),g(z)]\| \\
&+\|h([x,y,z])-[h(x),h(y),h(z)]\| \leq \varphi(x,y,z).
\end{aligned} \tag{0.2}$$

1. Introduction and preliminaries

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\mapsto[x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$, and satisfies $\|[x,y,z]\|\leq\|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\|=\|x\|^3$ (see [33]).

Let A be a C^*-ternary algebra. A $\mathbb{C}$-linear mapping $g:A\to A$ is a ternary derivation if $g:A\to A$ satisfies

$$g([x,y,z])=[g(x),y,z]+[x,g(y),z]+[x,y,g(z)]$$

for all $x,y,z\in A$, and a $\mathbb{C}$-linear mapping $h:A\to A$ is a ternary homomorphism if $h:A\to A$ satisfies

$$h([x,y,z])=[h(x),h(y),h(z)]$$

for all $x,y,z\in A$ (see [1, 18]). For a ternary derivation $g:A\to A$ and a ternary homomorphism $h:A\to A$,

$$g\circ h([x,y,z])=[g\circ h(x),h(y),h(z)]+[h(x),g\circ h(y),h(z)]+[h(x),h(y),g\circ h(z)]$$

for all $x,y,z\in A$. The $\mathbb{C}$-linear mapping $g\circ h:A\to A$ is called a ternary hom-derivation, which is defined as follows:

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Definition 1.1. Let A be a C^*-ternary algebra and $H : A \to A$ be ternary homomorphism. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *ternary hom-derivation* in A if $D : A \to A$ satisfies

$$D([x, y, z]) = [D(x), H(y), H(z)] + [H(x), D(y), H(z)] + [H(x), H(y), D(z)]$$

for all $x, y, z \in A$.

The stability problem of functional equations originated from a question of Ulam [31] concerning the stability of group homomorphisms. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [26] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. Park [21, 22, 24] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [8, 9, 10, 11, 12, 14, 19, 27, 28, 29, 30, 32]).

We recall a fundamental result in fixed point theory.

Theorem 1.2. [3, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 7, 23, 25]).

In this paper, we solve the additive-additive (s, t)-functional inequality (0.1). Furthermore, we investigate ternary hom-derivations and ternary homomorphisms in C^*-ternary algebras associated to the additive-additive (s, t)-functional inequality (0.1) and the functional inequality (0.2) by using the direct method and by the fixed point method.

Throughout this paper, assume that A is a C^*-ternary algebra and that s and t are fixed nonzero complex numbers with $|s| < 1$ and $|t| < 1$.

2. Stability of additive-additive (s,t)-functional inequality (0.1): a direct method

In this section, we solve and investigate the additive-additive (s,t)-functional inequality (0.1) in C^*-ternary algebras.

Lemma 2.1. *If mappings $g, h : A \to A$ satisfy $g(0) = h(0) = 0$ and*

$$\begin{aligned}
&\|g(x+y+z) - g(x) - g(y) - g(z)\| \\
&+\|h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x)\| \\
&\leq \left\| s\left(3g\left(\frac{x+y+z}{3}\right) - g(x) - g(y) - g(z)\right)\right\| \\
&+ \left\| t\left(3h\left(\frac{x+y+z}{3}\right) + h(x-2y+z) + h(x+y-2z) - 3h(x)\right)\right\|
\end{aligned} \tag{2.1}$$

for all $x, y, z \in A$, then the mappings $g, h : A \to A$ are additive.

Proof. Letting $x = y = z$ in (2.1), we get

$$\|g(3x) - 3g(x)\| + \|h(3x) - 3h(x)\| \leq 0$$

for all $x \in A$. So $g(3x) = 3g(x)$ and $h(3x) = 3h(x)$ for all $x \in A$. It follows from (2.1) that

$$\begin{aligned}
&\|g(x+y+z) - g(x) - g(y) - g(z)\| \\
&+\|h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x)\| \\
&\leq \|s(g(x+y+z) - g(x) - g(y) - g(z))\| \\
&+ \|t(h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x))\|
\end{aligned}$$

for all $x, y, z \in A$. Thus

$$g(x+y+z) = g(x) + g(y) + g(z),$$

$$h(x+y+z) + h(x-2y+z) + h(x+y-2z) = 3h(x)$$

for all $x, y, z \in A$, since $|s| < 1$ and $|t| < 1$. So the mappings $g, h : A \to A$ are additive. $\square$

Lemma 2.2. [20, Theorem 2.1] *Let $f : A \to A$ be an additive mapping such that*

$$f(\lambda a) = \lambda f(a)$$

for all $\lambda \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and all $a \in A$. Then the mapping $f : A \to A$ is $\mathbb{C}$-linear.

Using the direct method, we prove the Hyers-Ulam stability of pairs of ternary hom-derivations and ternary homomorphisms in C^*-ternary algebras associated to the additive-additive (s,t)-functional inequality (2.1).

Theorem 2.3. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 27^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) < \infty \tag{2.2}$$

for all $x, y, z \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and

$$\begin{aligned}
&\|g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z))\| \\
&+\|h(\lambda(x+y+z)) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\| \\
&\le \left\| s\left(3g\left(\lambda\frac{x+y+z}{3}\right) - \lambda(g(x)+g(y)+g(z))\right)\right\| \\
&+ \left\| t\left(3h\left(\lambda\frac{x+y+z}{3}\right) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\right)\right\| + \varphi(x,y,z)
\end{aligned} \tag{2.3}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$. If the mappings $g, h : A \to A$ satisfy

$$\begin{aligned}
&\|g([x,y,z]) - [g(x),h(y),h(z)] - [h(x),g(y),h(z)] - [h(x),h(y),g(z)]\| \\
&+\|h([x,y,z]) - [h(x),h(y),h(z)]\| \le \varphi(x,y,z)
\end{aligned} \tag{2.4}$$

for all $x, y, z \in A$, then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \sum_{j=1}^{\infty} 3^{j-1}\varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) \tag{2.5}$$

for all $x \in A$.

Proof. Letting $\lambda = 1$ and $y = z = x$ in (2.3), we get

$$\|g(3x) - 3g(x)\| + \|h(3x) - 3h(x)\| \le \varphi(x,x,x) \tag{2.6}$$

and so

$$\left\|g(x) - 3g\left(\frac{x}{3}\right)\right\| + \left\|h(x) - 3h\left(\frac{x}{3}\right)\right\| \le \varphi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right)$$

for all $x \in A$. Thus

$$\begin{aligned}
&\left\|3^l g\left(\frac{x}{3^l}\right) - 3^m g\left(\frac{x}{3^m}\right)\right\| + \left\|3^l h\left(\frac{x}{3^l}\right) - 3^m h\left(\frac{x}{3^m}\right)\right\| \\
&\quad\le \sum_{j=l}^{m-1}\left\|3^j g\left(\frac{x}{3^j}\right) - 3^{j+1} g\left(\frac{x}{3^{j+1}}\right)\right\| + \sum_{j=l}^{m-1}\left\|3^j h\left(\frac{x}{3^j}\right) - 3^{j+1} h\left(\frac{x}{3^{j+1}}\right)\right\| \\
&\quad\le \sum_{j=l+1}^{m} 3^{j-1}\varphi\left(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}\right)
\end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.7) that the sequences $\{3^k g(\frac{x}{3^k})\}$ and $\{3^k h(\frac{x}{3^k})\}$ are Cauchy for all $x \in A$. Since Y is a Banach space, the sequences $\{3^k g(\frac{x}{3^k})\}$ and $\{3^k h(\frac{x}{3^k})\}$ converge. So one can define the mappings $D, H : A \to A$ by

$$D(x) := \lim_{k\to\infty} 3^k g\left(\frac{x}{3^k}\right), \qquad \& \qquad H(x) := \lim_{k\to\infty} 3^k h\left(\frac{x}{3^k}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (2.5).

It folllows from (2.3) that

$$\begin{aligned}
&\|D\left(\lambda(x+y+z)\right)-\lambda(D(x)+D(y)+D(z))\| \\
&+\|H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\| \\
&=\lim_{n\to\infty}3^n\left\|g\left(\lambda\frac{x+y+z}{3^n}\right)-\lambda\left(g\left(\frac{x}{3^n}\right)+g\left(\frac{y}{3^n}\right)+g\left(\frac{z}{3^n}\right)\right)\right\| \\
&+\lim_{n\to\infty}3^n\left\|h\left(\lambda\frac{x+y+z}{3^n}\right)+h\left(\lambda\frac{x-2y+z}{3^n}\right)+h\left(\lambda\frac{x+y-2z}{3^n}\right)-3\lambda h\left(\frac{x}{3^n}\right)\right\| \\
&\le\lim_{n\to\infty}3^n\left\|s\left(3g\left(\lambda\frac{x+y+z}{3^{n+1}}\right)-\lambda\left(g\left(\frac{x}{3^n}\right)+g\left(\frac{y}{3^n}\right)+g\left(\frac{z}{3^n}\right)\right)\right)\right\| \\
&+\lim_{n\to\infty}3^n\left\|t\left(3h\left(\lambda\frac{x+y+z}{3^{n+1}}\right)+h\left(\lambda\frac{x-2y+z}{3^n}\right)+h\left(\lambda\frac{x+y-2z}{3^n}\right)-3\lambda h\left(\frac{x}{3^n}\right)\right)\right\| \\
&\quad+\lim_{n\to\infty}3^n\varphi\left(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right) \\
&=\left\|s\left(3D\left(\lambda\frac{x+y+z}{3}\right)-\lambda(D(x)+D(y)+D(z))\right)\right\| \\
&+\left\|t\left(3H\left(\lambda\frac{x+y+z}{3}\right)+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\right)\right\|
\end{aligned}$$

for all $\lambda\in\mathbb{T}^1$ and all $x,y,z\in A$. So

$$\begin{aligned}
&\|D\left(\lambda(x+y+z)\right)-\lambda(D(x)+D(y)+D(z))\| \\
&+\|H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\| \\
&\le\left\|s\left(3D\left(\lambda\frac{x+y+z}{3}\right)-\lambda(D(x)+D(y)+D(z))\right)\right\| \qquad (2.8)\\
&+\left\|t\left(3H\left(\lambda\frac{x+y+z}{3}\right)+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\right)\right\|
\end{aligned}$$

for all $\lambda\in\mathbb{T}^1$ and all $x,y,z\in A$.

Let $\lambda=1$ in (2.8). By Lemma 2.1, the mappings $D,H:A\to A$ are additive.

It follows from (2.8) and the additivity of D and H that

$$\begin{aligned}
&\|D\left(\lambda(x+y+z)\right)-\lambda(D(x)+D(y)+D(z))\| \\
&+\|H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\| \\
&\le\|s\left(D\left(\lambda(x+y+z)\right)-\lambda(D(x)+D(y)+D(z))\right)\| \\
&+\|t\left(H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\right)\|
\end{aligned}$$

for all $\lambda\in\mathbb{T}^1$ and all $x,y,z\in A$. Since $|s|<1$ and $|t|<1$,

$$\begin{aligned}
D\left(\lambda(x+y+z)\right)-\lambda(D(x)+D(y)+D(z)) &= 0, \\
H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x) &= 0
\end{aligned}$$

and so $D(\lambda x)=\lambda D(x)$ and $H(\lambda x)=\lambda H(x)$ for all $\lambda\in\mathbb{T}^1$ and all $x,y,z\in A$. Thus by Lemma 2.2, the additive mappings $D,H:A\to A$ are $\mathbb{C}$-linear.

It follows from (2.4) and the additivity of D, H that

$$\begin{aligned}
&\|D([x,y,z]) - [D(x),H(y),H(z)] - [H(x),D(y),H(z)] - [H(x),H(y),D(z)]\| \\
&+\|H([x,y,z]) - [H(x),H(y),H(z)]\| \\
&= 27^n \left\| g\left(\frac{[x,y,z]}{27^n}\right) - \left[g\left(\frac{x}{3^n}\right), h\left(\frac{y}{3^n}\right), h\left(\frac{z}{3^n}\right)\right]\right. \\
&\qquad \left. - \left[h\left(\frac{x}{3^n}\right), g\left(\frac{y}{3^n}\right), h\left(\frac{z}{3^n}\right)\right] - \left[h\left(\frac{x}{3^n}\right), h\left(\frac{y}{3^n}\right), g\left(\frac{z}{3^n}\right)\right]\right\| \\
&+27^n \left\| h\left(\frac{[x,y,z]}{27^n}\right) - \left[h\left(\frac{x}{3^n}\right), h\left(\frac{y}{3^n}\right), h\left(\frac{z}{3^n}\right)\right]\right\| \le 27^n \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right),
\end{aligned}$$

which tends to zero as $n \to \infty$, by (2.2). So

$$\begin{aligned}
D([x,y,z]) - [D(x),H(y),H(z)] - [H(x),D(y),H(z)] - [H(x),H(y),D(z)] &= 0, \\
H([x,y,z]) - [H(x),H(y),H(z)] &= 0
\end{aligned}$$

for all $x, y, z \in A$. Hence the mapping $D : A \to A$ is a ternary hom-derivation and the mapping $H : A \to A$ is a ternary homomorphism. $\square$

Corollary 2.4. *Let $r > 3$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and*

$$\begin{aligned}
&\|g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z))\| \\
&+\|h(\lambda(x+y+z)) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\| \qquad (2.9)\\
&\le \left\| s\left(3g\left(\lambda\frac{x+y+z}{3}\right) - \lambda(g(x)+g(y)+g(z))\right)\right\| \\
&+ \left\| t\left(3h\left(\lambda\frac{x+y+z}{3}\right) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\right)\right\| \\
&\qquad +\theta(\|x\|^r + \|y\|^r)
\end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$. If the mappings $g, h : A \to A$ satisfy

$$\begin{aligned}
&\|g([x,y,z]) - [g(x),h(y),h(z)] - [h(x),g(y),h(z)] - [h(x),h(y),g(z)]\| \qquad (2.10)\\
&+\|h([x,y,z]) - [h(x),h(y),h(z)]\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned}$$

for all $x, y, z \in A$, then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \frac{3\theta}{3^r - 3}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. $\square$

Theorem 2.5. *Let $\varphi : A^3 \to [0,\infty)$ be a function and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3), (2.4) and*

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{3^j}\varphi(3^j x, 3^j y, 3^j z) < \infty \qquad (2.11)$$

for all $x, y, z \in A$. Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{1}{3}\Phi(x, x, x) \tag{2.12}$$

for all $x \in A$.

Proof. It follows from (2.6) that

$$\left\|g(x) - \frac{1}{3}g(3x)\right\| + \left\|h(x) - \frac{1}{3}h(3x)\right\| \leq \frac{1}{3}\varphi(x, x, x) \tag{2.13}$$

for all $x \in A$. Thus

$$\begin{aligned} &\left\|\frac{1}{3^l}g\left(\frac{x}{3^l}\right) - \frac{1}{3^m}g\left(3^m x\right)\right\| + \left\|\frac{1}{3^l}h\left(3^l x\right) - \frac{1}{3^m}h\left(3^m x\right)\right\| \\ &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}g\left(3^j x\right) - \frac{1}{3^{j+1}}g\left(3^{j+1}x\right)\right\| + \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}h\left(3^j x\right) - \frac{1}{3^{j+1}}h\left(3^{j+1}x\right)\right\| \\ &\leq \frac{1}{3}\sum_{j=l}^{m-1}\frac{1}{3^j}\varphi\left(3^j x, 3^j x, 3^j x\right) \end{aligned} \tag{2.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.14) that the sequences $\{\frac{1}{3^k}g(3^k x)\}$ and $\{\frac{1}{3^k}h(3^k x)\}$ are Cauchy for all $x \in A$. Since Y is a Banach space, the sequences $\{\frac{1}{3^k}g(3^k x)\}$ and $\{\frac{1}{3^k}h(3^k x)\}$ converge. So one can define the mappings $D, H : A \to A$ by

$$D(x) := \lim_{k\to\infty}\frac{1}{3^k}g\left(3^k x\right),$$
$$H(x) := \lim_{k\to\infty}\frac{1}{3^k}h\left(3^k x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.14), we get (2.12).

By the same reasoning as in the proof of Theorem 2.3, one can show that the mappings $D, H : A \to A$ are $\mathbb{C}$-linear.

It follows from (2.4) and the additivity of D and H that

$$\begin{aligned} &\|D([x, y, z]) - [D(x), H(y), H(z)] - [H(x), D(y), H(z)] - [H(x), H(y), D(z)]\| \\ &+\|H([x, y, z]) - [H(x), H(y), H(z)]\| \\ &= \frac{1}{27^n}\|g\left(27^n[x, y, z]\right) - [g\left(3^n x\right), h(3^n y), h(3^n z)] \\ &\qquad -[h\left(3^n x\right), g(3^n y), h(3^n z)] - [h\left(3^n x\right), h(3^n y), g(3^n z)]\| \\ &+\frac{1}{27^n}\|h\left(27^n[x, y, z]\right) - [h\left(3^n x\right), h\left(3^n y\right), h(3^n z)]\| \\ &\leq \frac{1}{27^n}\varphi\left(3^n x, 3^n y, 3^n z\right) \leq \frac{1}{3^n}\varphi\left(3^n x, 3^n y, 3^n z\right), \end{aligned}$$

which tends to zero as $n \to \infty$, by (2.11). So

$$\begin{aligned} D([x, y, z]) - [D(x), H(y), H(z)] - [H(x), D(y), H(z)] - [H(x), H(y), D(z)] &= 0, \\ H([x, y, z]) - [H(x), H(y), H(z)] &= 0 \end{aligned}$$

for all $x, y, z \in A$. Hence the mapping $D : A \to A$ is a ternary hom-derivation and the mapping $H : A \to A$ is a ternary homomorphism. $\square$

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.9) and (2.10). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{3\theta}{3 - 3^r}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.5 by $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. □

3. Stability of additive-additive (s, t)-functional inequality (0.1): a fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of pairs of hom-derivations and homomorphisms in C^*-ternary algebras associated to the additive-additive (s, t)-functional inequality (0.1).

Theorem 3.1. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \leq \frac{L}{27}\varphi(x, y, z) \leq \frac{L}{3}\varphi(x, y, z) \tag{3.1}$$

for all $x, y, z \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3) and (2.4). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{L}{3(1 - L)}\varphi(x, x, x) \tag{3.2}$$

for all $x \in A$.

Proof. It follows from (3.1) that

$$\sum_{j=1}^{\infty} 27^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) \leq \sum_{j=1}^{\infty} 27^j \frac{L^j}{27^j}\varphi(x, y) = \frac{L}{1 - L}\varphi(x, y, z) < \infty$$

for all $x, y, z \in A$. By Theorem 2.3, there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ satisfying (2.5).

Letting $\lambda = 1$ and $y = z = x$ in (2.3), we get

$$\|g(3x) - 3g(x)\| + \|h(3x) - 3h(x)\| \leq \varphi(x, x, x) \tag{3.3}$$

for all $x \in A$.

Consider the set

$$S := \{(g, h) : (A, A) \to (A, A), \quad g(0) = h(0) = 0\}$$

and introduce the generalized metric on S:

$$d((g, h), (g_1, h_1)) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - g_1(x)\| + \|h(x) - h_1(x)\| \leq \mu\varphi(x, x, x), \ \forall x \in A\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [17]).

Now we consider the linear mapping $J : S \to S$ such that

$$J(g,h)(x) := \left(3g\left(\frac{x}{3}\right), 3h\left(\frac{x}{3}\right)\right)$$

for all $x \in A$.

Let $(g,h),(g_1,h_1) \in S$ be given such that $d((g,h),(g_1,h_1)) = \varepsilon$. Then

$$\|g(x) - g_1(x)\| + \|h(x) - h_1(x)\| \le \varepsilon\varphi(x,x,x)$$

for all $x \in A$. Since

$$\begin{aligned} &\left\|3g\left(\frac{x}{3}\right) - 3g_1\left(\frac{x}{3}\right)\right\| + \left\|3h\left(\frac{x}{3}\right) - 3h_1\left(\frac{x}{3}\right)\right\| \\ &\quad \le 3\varepsilon\varphi\left(\frac{x}{3},\frac{x}{3},\frac{x}{3}\right) \le 3\varepsilon\frac{L}{3}\varphi(x,x,x) = L\varepsilon\varphi(x,x,x) \end{aligned}$$

for all $x \in A$, t $d(J(g,h),J(g_1,h_1)) \le L\varepsilon$. This means that

$$d(J(g,h),J(g_1,h_1)) \le Ld((g,h),(g_1,h_1))$$

for all $(g,h),(g_1,h_1) \in S$.

It follows from (3.3) that

$$\left\|g(x) - 3g\left(\frac{x}{3}\right)\right\| + \left\|h(x) - 3h\left(\frac{x}{3}\right)\right\| \le \varphi\left(\frac{x}{3},\frac{x}{3},\frac{x}{3}\right) \le \frac{L}{3}\varphi(x,x,x)$$

for all $x \in A$. So $d((g,h),(Jg,Jh)) \le \frac{L}{3}$.

By Theorem 1.2, there exist mappings $D, H : A \to A$ satisfying the following:

(1) (D,H) is a fixed point of J, i.e.,

$$D(x) = 3D\left(\frac{x}{3}\right), \qquad H(x) = 3H\left(\frac{x}{3}\right) \tag{3.4}$$

for all $x \in A$. The mapping (D,H) is a unique fixed point of J. This implies that (D,H) is a unique mapping satisfying (3.4) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \quad \le \quad \mu\varphi(x,x,x)$$

for all $x \in A$;

(2) $d(J^l(g,h),(D,H)) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 3^l g\left(\frac{x}{3^l}\right) = D(x), \quad \lim_{l\to\infty} 3^l h\left(\frac{x}{3^l}\right) = H(x)$$

for all $x \in A$;

(3) $d((g,h),(D,H)) \le \frac{1}{1-L}d((g,h),J(g,h))$, which implies

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \frac{L}{3(1-L)}\varphi(x,x,x)$$

for all $x \in A$. Thus we get the inequality (3.2).

The rest of the proof is the same as in the proof of Theorem 2.3. □

Corollary 3.2. *Let $r > 3$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.9) and (2.10). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{3\theta}{3^r - 3}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking $L = 3^{1-r}$ and $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. □

Theorem 3.3. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq 27L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{3.5}$$

for all $x, y, z \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3) and (2.4). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{1}{3(1 - L)}\varphi(x, x, x) \tag{3.6}$$

for all $x \in A$.

Proof. It follows from (3.5) that

$$\sum_{j=1}^{\infty} \frac{1}{27^j}\varphi\left(3^j x, 3^j y, 3^j z\right) \leq \sum_{j=1}^{\infty} \frac{1}{27^j}(27L)^j \varphi(x, y, z) = \frac{L}{1 - L}\varphi(x, y, z) < \infty$$

for all $x, y, z \in A$. By Theorem 2.5, there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ satisfying (2.12).

Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the linear mapping $J : S \to S$ such that

$$J(g, h)(x) := \left(\frac{1}{3}g(3x), \frac{1}{3}h(3x)\right)$$

for all $x \in A$.

It follows from (3.3) that

$$\left\|g(x) - \frac{1}{3}g(3x)\right\| + \left\|h(x) - \frac{1}{3}h(3x)\right\| \leq \frac{1}{3}\varphi(x, x, x)$$

for all $x \in A$. Thus we get the inequality (3.6).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.9) and (2.10). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{3\theta}{3 - 3^r}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking $L = 3^{r-1}$ and $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. □

4. Conclusions

We have introduced the additive-additive (s, t)-functional inequality (0.1), and using the direct method and the fixed point method, we have proved the Hyers-Ulam stability of ternary hom-derivations and ternary homomorphisms in C^*-ternary algebras, associated to the additive-additive (s, t)-functional inequality (0.1) and the functional inequality (0.2).

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MULTIFARIOUS FUNCTIONAL EQUATIONS IN CONNECTION WITH THREE GEOMETRICAL MEANS

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Abstract. In this article, we introduce a new generalized multifarious radical reciprocal functional equation by generalizing the equation employed by Narasimman *et al.* in [5] and combining three classical Pythagorean means arithmetic, geometric and harmonic. Also, we illustrate the geometrical interpretation. Mainly, we find its general solution and stabilities related to Ulam problem in modular spaces by using fixed point approach.

1. Introduction and preliminaries

In the development of broad field functional equations, we come acrossing various types like additive, quadratic, cubic and so on. In recent research many researchers modeled functional

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equations from physical phenomena. In particular, by geometrical construction authors introduced remarkable reciprocal type functional equations.

In 2010, Ravi and Senthil Kumar [6] introduced functional equation of reciprocal type

$$s(z+w) = \frac{s(z)s(w)}{s(z)+s(w)}. \tag{1.1}$$

with solution $s(z) = \frac{c}{z}$.

In 2014, Bodaghi and Kim [1] introduced the quadratic reciprocal functional equation, which was generalized by Song andSong citeAM.

In 2015, Narasimman, Ravi and Pinelas [5] introduced the radical reciprocal quadratic functional equation

$$s\left(\sqrt[2]{z^2+w^2}\right) = \frac{s(z)s(w)}{s(z)+s(w)}, \ z,w \in (0,\infty), \tag{1.2}$$

which is satisfied by $s(z) = \frac{c}{z^2}$. Also, they provided the solution and stability of (1.2) with geometrical interpretation and application.

For the necessary introduction on stability related to Ulam problem and the notion of modular spaces one can refer to [7, 8, 9, 10, 12].

2. Main results

Definition 2.1. A reciprocal functional equation is a functional equation with solution of the form $\frac{1}{s(z)}$. When $s(z) = z, z^2, z^3 \dots$ we have various type of reciprocal functional equations like reciprocal additive, reciprocal quadratic, reciprocal cubic and so on.

Definition 2.2. Pythagorean means [3] The three classical Pythagorean means are the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM), which are defined by

$$AM(a_1,a_2,...,a_n) = \frac{1}{n}(a_1+...+a_n),$$
$$GM(a_1,a_2,...,a_n) = \sqrt[n]{a_1+...+a_n},$$
$$HM(a_1,a_2,...,a_n) = \frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}.$$

Definition 2.3. A functional equations which are arisen from the relations between three Pythagorean means (arithmetic, geometric and harmonic) are known as Pythagorean mean functional equations.

Definition 2.4. A reciprocal Pythagorean mean functional equation which shall possess the nature of any type of functional equation like additive, quadratic, cubic and so on is said to be a multifarious reciprocal Pythagorean mean functional equation.

In this paper, using Pythagorean means, we introduce the new generalized $2-$dimensional and $3-$dimensional multifarious radical reciprocal functional equations.

The following $2-$dimensional and $3-$dimensional multifarious radical reciprocal functional equations are obtained by generalizing (1.1) and (1.2)

$$s\left(\sqrt[m]{z^m+w^m}\right)=\frac{s(z)s(w)}{s(z)+s(w)}, \tag{2.3}$$

$$s\left(\sqrt[m]{z_1^m+z_2^m+z_3^m}\right)=\frac{s(z_1)s(z_2)s(z_3)}{s(z_1)s(z_2)+s(z_2)s(z_3)+s(z_1)s(z_3)}, \tag{2.4}$$

which are satisfied by $s(z)=\frac{c}{z^m}$, for all $z,w,z_1,z_2,z_3\in(0,\infty), m\in\mathbb{N}$. Observe that if $m=1$ and $m=2$ in (2.3), we have (1.1) and (1.2), respectively. Further, if $m=3,4,\cdots$ in (2.3), then we have various type of functional equations. Hence the functional equation (2.3) is known as two dimensional multifarious radical reciprocal functional equation. By similar argument, (2.4) is known as three dimensional multifarious radical reciprocal functional equation.

2.1. **Geometrical construction and geometrical interpretation of multifarious radical reciprocal functional equations.** Geometric construction of three Pythagorean means of two variables can be constructed geometrically as showed in Figure 1. Geometric construction of geometric mean of three variables are not possible but the other Pythagorean means can be constructed for any number of variables, one can refer [4, 11].

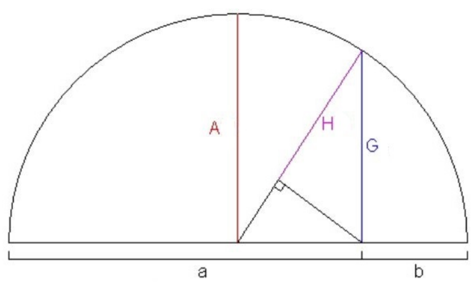

FIGURE 1. Pythagorean means of a and b. A is the arithmetic mean, H is the harmonic mean and G is the geometric mean.

The relations between three Pythagorean means of $p-$objects $z_1,z_2,\cdots,z_p$ are represented by the following equation

$$H\left(z_1,z_2,\cdots,z_p\right)=\frac{G\left(z_1,z_2,\cdots,z_p\right)^p}{A\left(\frac{1}{z_1}\prod_{i=1}^p z_i,\frac{1}{z_2}\prod_{i=1}^p z_i,\cdots,\frac{1}{z_p}\prod_{i=1}^p z_i\right)}. \tag{2.5}$$

Consider two spheres S_1 and S_2 of radii r_1 and r_2 with $r_1>r_2$, which are located along the $x-$axis centered at $C_1(0,0,0)$ and $C_2(d,0,0)$, respectively.

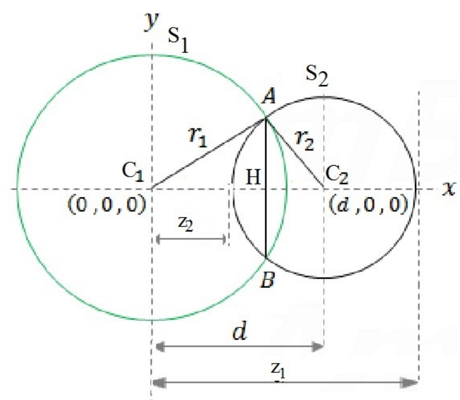

FIGURE 2. Intersecting two spheres S_1 and S_2.

We can show that the length of C_2C_1 is $\frac{z_1+z_2}{2}$ which is the arithmetic mean of z_1 and z_2. We can find the length AC_1, using Pythagoras' theorem, is the geometric mean $\sqrt{z_1 z_2}$ of z_1 and z_2. Also, we can obtain the length HC_1 is $\frac{2z_1z_2}{z_1+z_2}$, which is the harmonic mean of z_1 and z_2, since C_2AC_1 and AHC_1 are similar.

From Figure 2, we have the equality $HC_1 = \frac{AC_1^2}{C_2C_1}$, that is

$$H(z_1, z_2) = \frac{G(z_1, z_2)^2}{A\left(\frac{1}{z_1}\prod_{i=1}^{2} z_i, \frac{1}{z_2}\prod_{i=1}^{2} z_i\right)}, \tag{2.6}$$

which is the particular case of (2.5) by assuming $p = 2$ and which implies

$$\frac{1}{\frac{1}{z_1}+\frac{1}{z_2}} = \frac{z_1z_2}{z_1+z_2}. \tag{2.7}$$

Assuming $z_1 = \frac{1}{z}$ and $z_2 = \frac{1}{w}$ in (2.7), we obtain

$$\frac{1}{z+w} = \frac{\frac{1}{z}\frac{1}{w}}{\frac{1}{z}+\frac{1}{w}}. \tag{2.8}$$

In that case, (1.1) is valid by (2.8), which is satisfied by $s(z) = \frac{c}{z}$. Assuming $z_1 = \frac{1}{z^2}$ and $z_2 = \frac{1}{w^2}$ in (2.7) leads

$$\frac{1}{z^2+w^2} = \frac{\frac{1}{z^2}\frac{1}{w^2}}{\frac{1}{z^2}+\frac{1}{w^2}}. \tag{2.9}$$

In that case (1.2) is valid by (2.9), which is satisfied by $s(z) = \frac{c}{z^2}$. In general, assuming $z_1 = \frac{1}{z^m}$ and $z_2 = \frac{1}{w^m}$ in (2.7), we have

$$\frac{1}{z^m+w^m} = \frac{\frac{1}{z^m}\frac{1}{w^m}}{\frac{1}{z^m}+\frac{1}{w^m}}. \tag{2.10}$$

In that case, (2.3) is valid by (2.10), which is satisfied by $s(z) = \frac{c}{z^m}$.

In Figure 2, AB is the diameter of common circle. The common circle is the solution of the system

$$\begin{aligned} z_1^2+z_2^2+z_3^2 &= r_1^2, \\ (z_1-d)^2+z_2^2+z_3^2 &= r_2^2, \end{aligned} \tag{2.11}$$

which implies

$$\begin{aligned} \frac{1}{z_1^2+z_2^2+z_3^2} &= \frac{1}{r_1^2}, \\ \frac{1}{(z_1-d)^2+z_2^2+z_3^2} &= \frac{1}{r_2^2}. \end{aligned} \tag{2.12}$$

The system (2.12) can be expressed by radical reciprocal quadratic functional equations of the form

$$\begin{aligned} s\left(r_1^2\right) &= \frac{s(z_1)s(z_2)s(z_3)}{s(z_1)s(z_2)+s(z_2)s(z_3)+s(z_1)s(z_3)}, \\ s\left(r_2^2\right) &= \frac{s(z_1-d)s(z_2)s(z_3)}{s(z_1-d)s(z_2)+s(z_2)s(z_3)+s(z_1-d)s(z_3)}, \end{aligned} \tag{2.13}$$

for $z_1, z_2, z_3, r_1, r_2 \in (0,\infty)$, which is satisfied by $s(z_1)=\frac{c}{z_1^2}$ and the denominators are not equal to zero. Also, observe that the equation (2.13) is the particular case of (2.4) for $m=2$. By assuming $p=3$ in (2.5), we obtain

$$H\left(z_1,z_2,z_3\right) = \frac{G\left(z_1,z_2,z_3\right)^3}{A\left(\frac{1}{z_1}\prod_{i=1}^3 z_i, \frac{1}{z_2}\prod_{i=1}^3 z_i, \frac{1}{z_3}\prod_{i=1}^3 z_i\right)}, \tag{2.14}$$

which gives

$$\frac{1}{\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}} = \frac{z_1z_2z_3}{z_2z_3+z_1z_3+z_1z_2}. \tag{2.15}$$

Assuming $z_1=\frac{1}{z_1^m}$, $z_2=\frac{1}{z_2^m}$ and $z_3=\frac{1}{z_3^m}$ in (2.15), we have

$$\frac{1}{z_1^m+z_2^m+z_3^m} = \frac{\frac{1}{z_1^m}\frac{1}{z_2^m}\frac{1}{z_3^m}}{\frac{1}{z_1^m}+\frac{1}{z_2^m}+\frac{1}{z_3^m}}. \tag{2.16}$$

In that case (2.4) is valid by (2.16), which is satisfied by $s(z_1)=\frac{c}{z_1^m}$.

3. General solution of the multifarious radical reciprocal functional equations

The following theorems give the solution of (2.3) and (2.4) through motivated by the work of Ger [?].

Theorem 3.1. *A general solution of (2.3) is* $s(z)=\frac{c}{z^m}$; $z\in(0,\infty)$ *with* $\frac{s(z)}{\frac{1}{z^m}}$ *a quotient at zero.*

Proof. Assuming $z, w = z$ in (2.3), we have

$$s(\sqrt[m]{2}z) = \frac{1}{2}s(z) \tag{3.17}$$

for all $z \in (0, \infty)$. Assuming

$$g(z) = \frac{s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.18}$$

for all $z \in (0, \infty)$, we have

$$\overset{lim}{z \to 0^+} \frac{g(z)}{\frac{1}{z^{\frac{m}{2}}}} =: c \in \mathbb{R}$$

for all $z \in (0, \infty)$. Dividing (3.17) by $\frac{1}{z^{\frac{m}{2}}}$, we obtain

$$\frac{s(\sqrt[m]{2}z)}{\frac{\sqrt{2}}{\sqrt{2}z^{\frac{m}{2}}}} = \frac{\frac{1}{2}s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.19}$$

for all $z \in (0, \infty)$. Using (3.18) in (3.19), we have

$$g(\sqrt[m]{2}z) = \frac{1}{\sqrt{2}}g(z), \tag{3.20}$$

for all $z \in (0, \infty)$. Replacing z by $\frac{z}{\sqrt[m]{2}}$ in (3.20), we get

$$\sqrt{2}g(z) = g\left(\frac{z}{\sqrt[m]{2}}\right). \tag{3.21}$$

Again, replacing z by $\frac{z}{\sqrt[m]{2}}$ in (3.21), we have

$$(\sqrt{2})^2 g(z) = g\left(\frac{z}{(\sqrt[m]{2})^2}\right), \tag{3.22}$$

for all $z \in (0, \infty)$. Continuing the same process k times, we obtain

$$(\sqrt{2})^k g(z) = g\left(\frac{z}{(\sqrt[m]{2})^k}\right), \tag{3.23}$$

for all $z \in (0, \infty)$.

Now,

$$\frac{g(z)}{\frac{1}{z^{\frac{m}{2}}}} = \frac{(\sqrt{2})^k g(z)}{(\sqrt{2})^k \frac{1}{z^{\frac{m}{2}}}} = \frac{g\left(\frac{1}{(\sqrt[m]{2})^k}z\right)}{\frac{(\sqrt{2})^k}{z^{\frac{m}{2}}}} \to c \quad as \quad k \to \infty,$$

for all $z \in (0, \infty)$. Eq. (3.18) implies that

$$s(z) = \frac{1}{z^{\frac{m}{2}}}g(z) = \frac{1}{z^{\frac{m}{2}}}\frac{1}{z^{\frac{m}{2}}}c = \frac{c}{z^m}$$

for all $z \in (0, \infty)$. This completes the proof. □

Theorem 3.2. *A general solution of (2.4) is* $s(z) = \frac{c}{z^m}$; $z \in (0, \infty)$ *with* $\frac{s(z)}{\frac{1}{z^m}}$ *a quotient at zero.*

Proof. Assuming $z_1, z_2, z_3 = z$ in (2.4), we have

$$s(\sqrt[m]{3}z) = \frac{1}{3}s(z), \tag{3.24}$$

and assuming

$$h(z) = \frac{s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.25}$$

we obtain

$$\lim_{z\to 0^+} \frac{h(z)}{\frac{1}{z^{\frac{m}{2}}}} =: c \in \mathbb{R}.$$

Dividing (3.24) by $\frac{1}{z^{\frac{m}{2}}}$, we get

$$\frac{s(\sqrt[m]{3}z)}{\frac{\sqrt{3}}{\sqrt{3}z^{\frac{m}{2}}}} = \frac{\frac{1}{3}s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.26}$$

and substituting (3.25) in (3.26), we obtain

$$h(\sqrt[m]{3}z) = \frac{1}{\sqrt{3}}h(z), \tag{3.27}$$

and replacing z by $\frac{z}{\sqrt[m]{3}}$ in (3.27), we have

$$\sqrt{3}h(z) = h\left(\frac{z}{\sqrt[m]{3}}\right). \tag{3.28}$$

Again, replacing z by $\frac{z}{\sqrt[m]{3}}$ in (3.28), we get

$$(\sqrt{3})^2 h(z) = h\left(\frac{z}{(\sqrt[m]{3})^2}\right), \tag{3.29}$$

for all $z \in (0, \infty)$. Continuing the same process k times, we have

$$(\sqrt{3})^k h(z) = h\left(\frac{z}{(\sqrt[m]{3})^k}\right), \tag{3.30}$$

for all $z \in (0, \infty)$. Now,

$$\frac{h(z)}{\frac{1}{z^{\frac{m}{2}}}} = \frac{(\sqrt{3})^k h(z)}{(\sqrt{3})^k \frac{1}{z^{\frac{m}{2}}}} = \frac{h\left(\frac{1}{(\sqrt[m]{3})^k}z\right)}{\frac{(\sqrt{3})^k}{z^{\frac{m}{2}}}} \to c \quad as \quad k \to \infty,$$

for all $z \in (0, \infty)$. Eqs. (3.25) and (3.30) imply that

$$s(z) = \frac{1}{z^{\frac{m}{2}}}h(z) = \frac{1}{z^{\frac{m}{2}}}\frac{1}{z^{\frac{m}{2}}}c = \frac{c}{z^m}$$

for all $z \in (0, \infty)$. This completes the proof. □

In the following theorem, we obtain general solution of (2.3) and (2.4) by derivative method.

Theorem 3.3. *Let $s : (0, \infty) \to \mathbb{R}$ be a continuously differentiable function with nowhere vanishing derivatives s'. Then s yields a solution to the functional equation (2.3) if and only if there exists a nonzero real constant c such that $s(z) = \frac{c}{z^m}$, $z \in (0, \infty)$.*

Proof. Differentiating (2.3) with respect to z on both side, we get

$$s'(\sqrt[m]{z^m + w^m})\frac{(z)^{m-1}}{(\sqrt[m]{z^m + w^m})^{m-1}} = \frac{\Big(s'(z)s(w)\Big)\Big[s(z) + s(w)\Big] - \Big(s(z)s(w)\Big)\Big[s'(z)\Big]}{\Big(s(z) + s(w)\Big)^2}. \tag{3.31}$$

Assuming $z, w = z$ in (3.31), we obtain

$$s'(\sqrt[m]{2}\ z) = \frac{1}{2\sqrt[m]{2}}s'(z), \tag{3.32}$$

and setting $z = \sqrt[m]{2}z$ and $w = z$ in (3.31) and making use of (3.17) and (3.32), we get

$$s'(\sqrt[m]{3}\ z) = \frac{1}{(3)\sqrt[m]{3}}s'(z) \tag{3.33}$$

for all $z \in (0, \infty)$. By making use of (3.32) and (3.33), we have

$$s'\left((\sqrt[m]{2})^k(\sqrt[m]{3})^l\ z\right) = \frac{1}{2^k(\sqrt[m]{2})^k}\frac{1}{(3)^l(\sqrt[m]{3})^l}s'(z)$$

for all integers k, l. We derive its linearity by assuming $\lambda = (\sqrt[m]{2})^k(\sqrt[m]{3})^l$ and $z = 1$,

$$s'(\lambda) = s'(1)\frac{1}{(\lambda)^{m+1}}$$

for $\lambda \in (0, \infty)$. Therefore, there exist real numbers $c \neq 0, d$ such that $s(z) = \frac{c}{z^m} + d$ for $z \in (0, \infty)$. Note that we have $d = 0$ because of the equality $s(\sqrt[m]{2}z) = \frac{1}{2}s(z)$ valid for all positive z. This completes the proof. $\square$

Theorem 3.4. *Let $s : (0, \infty) \to \mathbb{R}$ be a continuously differentiable function with nowhere vanishing derivatives s'. Then s yields a solution to the functional equation (2.4) if and only if there exists a nonzero real constant c such that $s(z) = \frac{c}{z^m}$, $z \in (0, \infty)$.*

Proof. Differentiating (2.4) with respect to z_1 on both side, we obtain

$$\begin{aligned} &s'(\sqrt[m]{z_1^m + z_2^m})\frac{(z_1)^{m-1}}{(\sqrt[m]{z_1^m + z_2^m})^{m-1}} + s'(\sqrt[m]{z_1^m + z_{p+1}^m})\frac{(z_1)^{m-1}}{(\sqrt[m]{z_1^m + z_{p+1}^m})^{m-1}} \\ &= \frac{s'(z_1)\left(s(z_2)\right)^2}{\left(s(z_1) + s(z_2)\right)^2} + \frac{s'(z_1)\left(s(z_{p+1})\right)^2}{\left(s(z_1) + s(z_{p+1})\right)^2}, \end{aligned} \tag{3.34}$$

and (3.24) implies

$$s'(\sqrt[m]{2}z) = \frac{1}{2\sqrt[m]{2}}s'(z). \tag{3.35}$$

Assuming $z_1 = z$ and $z_2 = z_{p+1} = \sqrt[m]{2}z$ in (3.34) and making use of (3.24) and (3.35), we get

$$s'(\sqrt[m]{3}z) = \frac{1}{3\sqrt[m]{3}}s'(z), \tag{3.36}$$

and from (3.35) and (3.36), we get

$$s'\left((\sqrt[m]{2})^k(\sqrt[m]{3})^l z\right) = \frac{1}{2^k(\sqrt[m]{2})^k}\frac{1}{3^l(\sqrt[m]{3})^l}s'(z),$$

for all integers k, l. We derive its linearity by assuming $\lambda = (\sqrt[m]{2})^k(\sqrt[m]{3})^l$ and $z = 1$,

$$s'(\lambda) = s'(1)\frac{1}{(\lambda)^{m+1}}$$

for $\lambda \in (0, \infty)$. Therefore, there exist real numbers $c \neq 0, d$ such that $s(z) = \frac{c}{z^m} + d$ for $z \in (0, \infty)$. Note that we have to have $d = 0$ because of the equality $s(\sqrt[m]{2}z) = \frac{1}{2}s(z)$ exists. This completes the proof. □

4. Generalized Hyers-Ulam stability of two dimensional multifarious functional equation

This section deals the generalized Hyers-Ulam stability of two dimensional multifarious functional equation (2.3) in modular spaces by making use of fixed point approach.

Theorem 4.1. *Consider a mapping $\eta : M^2 \to [0, +\infty)$ with*

$$\lim_{k\to\infty} \frac{1}{\left(\frac{1}{2}\right)^k}\eta\left((2)^{\frac{k}{m}}z, (2)^{\frac{k}{m}}w\right) = 0, \tag{4.37}$$

and

$$\begin{aligned}\eta\left((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}w\right)& \\ \leq \frac{1}{2}\psi\eta\{z, w\}, \forall z, w \in M,&\end{aligned} \tag{4.38}$$

for $\psi < 1$. Assume that $s : M \to Z_\xi$ fulfills

$$\xi\left(M_1 s(z, w)\right) \leq \eta(z, w), \tag{4.39}$$

for all $z, w \in M$. In that case, there is a unique reciprocal mapping $R : M \to Z_\xi$ such that

$$\xi(R(z) - s(z)) \leq \frac{1}{\frac{1}{2}(1-\psi)}\eta(z, z),\ \forall z \in M. \tag{4.40}$$

Proof. Assume $N = \xi'$ and define ξ' on N as,

$$\xi'(q) =: \inf\{(2)^{\frac{1}{m}} > 0 : \xi(h(j)) \leq (2)^{\frac{1}{m}}\eta(z,w), \forall z \in M\}.$$

One can easily prove that ξ' is a convex modular with Fatou property on N and $N_{\xi'}$ is $\xi-$complete, see [2]. Consider the function $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma q(z) = \frac{1}{2}q(2^{\frac{1}{m}}z), \tag{4.41}$$

for all $z \in M$ and $q \in N_{\xi'}$. Let $q, r \in N_{\xi'}$ and $(2)^{\frac{1}{m}} \in [0,1]$ with $\xi'(q-r) < (2)^{\frac{1}{m}}$. By definition of ξ', we get

$$\xi(q(z) - r(z)) \leq (2)^{\frac{1}{m}}\eta(z,w), \forall z, w \in M. \tag{4.42}$$

By making use of (4.38) and (4.42), we get

$$\begin{aligned}\xi\left(\frac{q((2)^{\frac{1}{m}}z)}{\frac{1}{2}} - \frac{r((2)^{\frac{1}{m}}z)}{\frac{1}{2}}\right) &\leq \frac{1}{\frac{1}{2}}\xi\left(q((2)^{\frac{1}{m}}z) - r((2)^{\frac{1}{m}}z)\right)\\ &\leq \frac{1}{\frac{1}{2}}(2)^{\frac{1}{m}}\eta\left((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}w\right) \leq (2)^{\frac{1}{m}}\psi\eta\left(z,w\right),\end{aligned}$$

for all $z, w \in M$. In that case, σ is a $\xi'-$contraction and (4.39) implies

$$\xi\left(\frac{s((2)^{\frac{1}{m}}z)}{\frac{1}{2}} - s(z)\right) \leq \frac{1}{\frac{1}{2}}\eta(z,z), \forall z \in M, \tag{4.43}$$

and replacing z by $(2)^{\frac{1}{m}}z$ in (4.43), we get

$$\xi\left(\frac{s((2)^{\frac{2}{m}}z)}{\frac{1}{2}} - s((2)^{\frac{1}{m}}z)\right) \leq \frac{\eta((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}z)}{\frac{1}{2}},\ \forall z \in M. \tag{4.44}$$

By making use of (4.43) and (4.44), we get

$$\xi\left(\frac{s((2)^{\frac{2}{m}}z)}{\frac{1}{2^2}} - s(z)\right) \leq \frac{1}{\frac{1}{2^2}}\eta((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}z) + \frac{1}{\frac{1}{p}}\eta(z,z), \tag{4.45}$$

for all $z \in M$ and by generalization, we get

$$\begin{aligned}\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}} - s(z)\right) &\leq \sum_{i=1}^{k}\frac{1}{\frac{1}{2^i}}\eta(((2)^{\frac{1}{m}})^{i-1}z, ((2)^{\frac{1}{m}})^{i-1}z)\\ &\leq \frac{1}{\psi\frac{1}{2}}\eta(z,z)\sum_{i=1}^{k}\psi^i\\ &\leq \frac{1}{\frac{1}{2}(1-\psi)}\eta(z,z),\ \forall z \in M. \end{aligned}\tag{4.46}$$

We obtain from (4.46),

$$\begin{aligned}
&\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}\right) &(4.47)\\
&\leq \frac{1}{2}\xi\left(2\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-2s(z)\right)+\frac{1}{2}\xi\left(2\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}-2s(z)\right)\\
&\leq \frac{\kappa}{2}\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-s(z)\right)+\frac{\kappa}{2}\xi\left(\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}-s(z)\right)\\
&\leq \frac{\kappa}{\frac{1}{2}(1-\psi)}\eta(z,z),\ \forall z\in M
\end{aligned}$$

where $k,u\in\mathfrak{N}$. Thus

$$\xi'(\sigma^k s-\sigma^u s)\leq \frac{\kappa}{\frac{1}{2}(1-\psi)},$$

and hence the boundedness of an orbit of σ at s is given. $\{\tau^k s\}$ is $\xi'-$converges to $R\in N_{\xi'}$ by Theorem 1.5 in [2]. By $\xi'-$contractivity of σ, we get

$$\xi'(\sigma^k s-\sigma R)\leq \psi\xi'(\sigma^{k-1}s-R).$$

Letting $k\to\infty$ and by Fatou property of ξ', we get

$$\begin{aligned}
\xi'(\sigma R-R)&\leq \lim_{2\to\infty}\inf\xi'(\sigma R-\sigma^k s)\\
&\leq \psi\lim_{k\to\infty}\inf\xi'(R-\sigma^{k-1}s)=0.
\end{aligned}$$

Hence R is a fixed point of σ. In (4.39), replacing (z,W) by $\left((2)^{\frac{k}{m}}z,(2)^{\frac{k}{m}}w\right)$, we get

$$\xi\left(\frac{1}{\frac{1}{2^k}}M_1s((2)^{\frac{k}{m}}z,(2)^{\frac{k}{m}}w)\right)\leq\frac{1}{\frac{1}{2^k}}\eta((2)^{\frac{k}{m}}z,(2)^{\frac{k}{m}}w). \tag{4.48}$$

By Theorems 3.1, 3.3 and letting $k\to\infty$, we obtain that R is a reciprocal mapping and using (4.46), we obtain (4.40). For the uniqueness of R, consider another multifarious type reciprocal mapping $T:M\to Z_\xi$ satisfying (4.40). Then T is a fixed point of σ such that

$$\xi'(R-T)=\xi'(\sigma R-\sigma T)\leq\psi\xi'(R-T). \tag{4.49}$$

From (4.49), we get $R=T$. This completes the proof. □

The proofs of the following corollaries 4.2 and 4.4 follow from the fact that, each normed space implies a modular space with modular $\xi(z)=\|z\|$.

Corollary 4.2. *Assume η is a function from M^2 to $[0,+\infty)$ for*

$$\lim_{k\to\infty}\frac{1}{\frac{1}{2^k}}\eta\{(2^{\frac{k}{m}})z,(2^{\frac{k}{m}})w\}=0, \tag{4.50}$$

and

$$\eta\{(2^{\frac{1}{m}})z,(2^{\frac{1}{m}})w\}\leq\frac{1}{2}\psi\eta\{z,w\},\ \psi<1. \tag{4.51}$$

Assume that $s:M\to Z$ satisfies the condition, for a Banach space Z,

$$\|M_1s(z,w)\|\leq\eta(z,w), \tag{4.52}$$

for all $z,w\in M$. Then there is a unique reciprocal mapping $R:M\to Z$ such that

$$\|R(z)-s(z)\|\leq\frac{\eta(z,z)}{\frac{1}{2}(1-\psi)}, \tag{4.53}$$

for all $z\in M$.

Theorem 4.3. *Assume η is a function from M^2 to $[0,+\infty)$ with*

$$\lim_{k\to\infty}\frac{1}{\kappa^k}\eta\left(\frac{z}{(2)^{\frac{k}{m}}},\frac{w}{(2)^{\frac{k}{m}}}\right)=0, \tag{4.54}$$

and

$$\eta\left(\frac{z}{(2)^{\frac{1}{m}}},\frac{w}{(2)^{\frac{1}{m}}}\right)\leq\frac{\psi}{\frac{1}{2}}\rho\{z,w\}, \tag{4.55}$$

for all $z,w\in M,\psi<1$. Assume that $s:M\to Z_\xi$ fulfills

$$\xi\left(M_1s(z,w)\right)\leq\eta(z,w). \tag{4.56}$$

Then there is a unique reciprocal mapping $R:M\to Z_\xi$ such that

$$\xi(R(z)-s(z))\leq\frac{p\psi}{1-\psi}\eta(z,z),\ \forall z\in M. \tag{4.57}$$

Proof. Replacing z by $\frac{z}{(2)^{\frac{1}{m}}}$ in (4.41) of Theorem 4.1 and using a similar method to that of Theorem 4.1, we complete the proof. $\square$

Corollary 4.4. *Assume η is a function from M^2 to $[0,+\infty)$ with*

$$\lim_{k\to\infty}\frac{1}{\sigma^k}\eta\left(\frac{z}{(2)^{\frac{k}{m}}},\frac{w}{(2)^{\frac{k}{m}}}\right)=0, \tag{4.58}$$

and

$$\eta\left(\frac{z}{(2)^{\frac{1}{m}}},\frac{w}{(2)^{\frac{1}{m}}}\right)\leq\frac{\psi}{\frac{1}{2}}\eta\{z,w\},\psi<1. \tag{4.59}$$

Assume that $s : M \to Z$ fulfills

$$\|M_1 s(z,w)\| \leq \eta(z,w), \tag{4.60}$$

for all $z, w \in M$. Then there is a unique reciprocal mapping $R : M \to Z$ such that

$$\|R(z) - s(z)\| \leq \frac{p\psi}{1-\psi}\eta(z,z), \tag{4.61}$$

for all $z \in M$.

Using Corollaries 4.2 and 4.4, we obtain the following corollaries.

Corollary 4.5. *Assume η is a function from M^2 to $[0,+\infty)$, Z is a Banach space and $\epsilon \geq 0$ is a real number such that*

$$\lim_{k\to\infty} \frac{1}{\frac{1}{2^k}} \eta\{(2)^{\frac{k}{m}} z, (2)^{\frac{k}{m}} w\} = 0, \tag{4.62}$$

and

$$\eta\{(2)^{\frac{1}{m}} z, (2)^{\frac{1}{m}} w\} \leq \frac{1}{2}\psi\eta\{z,w\}, \psi < 1. \tag{4.63}$$

Assume that $s : M \to Z$ fulfills

$$\|M_1 s(z,w)\| \leq \epsilon, \tag{4.64}$$

for all $z, w \in M$. Then there is a unique reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((2)^{\frac{k}{m}} z\right)}{\frac{1}{2^k}}$, such that

$$\|R(z) - s(z)\| \leq 2\epsilon, \tag{4.65}$$

for all $z \in M$.

Proof. Assume that $\eta(z,w) = \epsilon$ for all $z, w \in Z$. The Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq 2\epsilon,$$

for all $z \in Z$ and making use of Corollary 4.4, we get

$$\|R(z) - s(z)\| \leq 2\epsilon,$$

for all $z \in Z$. $\square$

Corollary 4.6. *Assume that $s : M \to X$ fulfills the following, for a linear space M and a Banach space Z , respectively,*

$$\|M_1 s(z,w)\| \leq \epsilon \left(\|z\|^u + \|w\|^u\right), \tag{4.66}$$

for all $z, w \in M$ with $0 \leq u < -m$ or $u > -m$ for some $\epsilon \geq 0$. Then there is a reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((2)^{\frac{k}{m}} z\right)}{\frac{1}{2^k}}$, such that

$$\|R(z) - s(z)\| \leq \frac{4\epsilon}{\left|1 - 2^{\frac{m+u}{m}}\right|} \|z\|^u, \ \ \forall z \in M. \tag{4.67}$$

Proof. If we choose $\eta(z,w) = \epsilon\big(\|z\|^u + \|w\|^u\big)$, then Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq \frac{4\epsilon}{1 - 2^{\frac{m+u}{m}}} \|z\|^u,$$

for all $z \in Z$ and $u < -m$. Using Corollary-4.4, we obtain

$$\|R(z) - s(z)\| \leq \frac{4\epsilon}{2^{\frac{m+u}{m}} - 1} \|z\|^u,$$

for all $z \in Z$ and $u > -m$. □

The following is an example to elucidate (2.3), which is not stable for $u = -m$ in Corollary 4.6.

Example 4.7. Define $\phi : \mathbb{R} \to \mathbb{R}$ with $a > 0$ as

$$\phi(z) = \begin{cases} \frac{a}{z^m}, & \text{if } z \in (1, \infty) \\ a\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k} z)}{2^{mk}}$. Then s fulfills

$$\|M_1 s(z,w)\| \leq \frac{a 2^{2m}(3)}{2(2^m - 1)} \times \left(\left|\frac{1}{z^m}\right| + \left|\frac{1}{w^m}\right|\right) \tag{4.68}$$

for all $z_1, w \in \mathbb{R}$. In that case there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ as

$$|s(z) - R(z)| \leq \beta \left|\frac{1}{z^m}\right|, \ \beta > 0, \forall z \in \mathbb{R}. \tag{4.69}$$

Corollary 4.8. *Let $s : Z_1 \to Z_2$ be a mapping. Assume that there exists $\epsilon \geq 0$ such that*

$$\|M_1 s(z,w)\| \leq \epsilon \left(\|z\|^{\frac{u}{2}} \|w\|^{\frac{u}{2}}\right)$$

for all $z, w \in Z_1$. Then there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ satisfying (2.3) and

$$\|r(z) - s(z)\| \leq \begin{cases} \frac{2\epsilon}{1-2^{\frac{m+u}{m}}}\|z\|^u & \text{for } u < -m \\ \frac{2\epsilon}{2^{\frac{m+u}{m}}-1}\|z\|^u & \text{for } u > -m \end{cases}$$

for all $z \in Z_1$.

Proof. Replace $\eta(z, w)$ by $\epsilon\left(\|z\|^{\frac{u}{2}}\|w\|^{\frac{u}{2}}\right)$. Then Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{1 - 2^{\frac{m+u}{m}}}\|z\|^2,$$

for $u < -m$ and for all $z \in Z_1$ and making use of Corollary-4.4, we get

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{2^{\frac{m+u}{m}} - 1}\|z\|^2, \tag{4.70}$$

for $u > -m$ and for all $z \in Z_1$. □

Corollary 4.9. *Let $\epsilon > 0$ and $\alpha < -\frac{m}{2}$ or $\alpha > -\frac{m}{2}$ be real numbers, and $s : Z_1 \to Z_2$ be a mapping satisfying the functional inequality*

$$\|M_1 s(z, w)\| \leq \epsilon\left\{\|z\|^{2\alpha} + \|w\|^{2\alpha} + (\|z\|^{\alpha}\|w\|^{\alpha})\right\}.$$

Then e there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ fulfilling (2.3) and

$$\|R(z) - s(z)\| \leq \begin{cases} \frac{6\epsilon}{1-2^{\frac{2\alpha+m}{m}}}\|z\|^{2\alpha} & \text{for } \alpha < -\frac{m}{2} \\ \frac{6\epsilon}{2^{\frac{2\alpha+m}{m}}-1}\|z\|^{2\alpha} & \text{for } \alpha > -\frac{m}{2} \end{cases}$$

for all $z \in Z_1$.

Proof. Set $\epsilon\Big\{\|z\|^{2\alpha} + \|w\|^{2\alpha} + (\|z\|^{\alpha}\|w\|^{\alpha})\Big\}$ instead of $\eta(z, w)$. Then Corollary 4.4 implies

$$\|R(z) - s(z)\| \leq \frac{6\epsilon}{1 - 2^{\frac{2\alpha+m}{m}}}\|z\|^{2\alpha},$$

for $\alpha < -\frac{m}{2}$ and for all $z \in Z_1$ and making use of Corollary-4.4, we get

$$\|R(z) - s(z)\| \leq \frac{6\epsilon}{2^{\frac{2\alpha+m}{m}} - 1}\|z\|^{2\alpha},$$

for $\alpha > -\frac{m}{2}$ and for all $z \in Z$. □

The following is an example to elucidate (2.3), which is not stable for $\alpha = -\frac{m}{2}$ in Corollary 4.9.

Example 4.10. Define $\phi : \mathbb{R} \to \mathbb{R}$ with a constant $l > 0$ as

$$\phi(z) = \begin{cases} \frac{l}{z^m}, & \text{if } z \in (1, \infty) \\ l\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k}z)}{2^{mk}}$. Then s fulfills

$$\|M_1 s(z,w)\| \leq \frac{a2^{2m}(3)}{2(2^m - 1)} \times \left(\left|\frac{1}{z^m}\right| + \left|\frac{1}{w^m}\right| + \left|\frac{1}{z^m}\right| \left|\frac{1}{w^m}\right| \right) \tag{4.71}$$

for all $z, w \in \mathbb{R}$. In that case, there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ as

$$|s(z) - R(z)| \leq \beta \left|\frac{1}{z^m}\right|,\ \beta > 0, \forall z \in \mathbb{R}. \tag{4.72}$$

5. Generalized Hyers-Ulam stability of three dimensional multifarious functional equation

This section deals the Hyers-Ulam stability of the three dimensional multifarious functional equation (2.4) in modular spaces by making use of fixed point approach.

Theorem 5.1. *Consider a mapping $\eta : M^2 \to [0, +\infty)$ with*

$$\lim_{k\to\infty} \frac{1}{\left(\frac{1}{3}\right)^k} \eta\left((3)^{\frac{k}{m}} z_1, (3)^{\frac{k}{m}} z_2, (3)^{\frac{k}{m}} z_3\right) = 0, \tag{5.73}$$

and

$$\eta\left((3)^{\frac{1}{m}} z_1, (3)^{\frac{1}{m}} z_2, (3)^{\frac{1}{m}} z_3\right) \leq \frac{1}{3}\psi\eta\{z_1, z_2, z_3\}, \forall z_1, z_2, z_3 \in M, \tag{5.74}$$

for $\psi < 1$. Assume that $s : M \to Z_\xi$ fulfills

$$\xi\left(M_1 s(z_1, z_2, z_3)\right) \leq \eta(z_1, z_2, z_3), \tag{5.75}$$

for all $z_1, z_2, z_3 \in M$. Then there is a unique reciprocal mapping $R : M \to Z_\xi$ such that

$$\xi(R(z) - s(z)) \leq \frac{1}{\frac{1}{3}(1-\psi)} \eta(z, z, z),\ \forall z \in M. \tag{5.76}$$

Proof. Assume $N = \xi'$ and define ξ' on N by

$$\xi'(q) =: \inf\{(3)^{\frac{1}{m}} > 0 : \xi(h(j)) \leq (3)^{\frac{1}{m}} \eta(z_1, z_2, z_3), \forall z \in M\}.$$

One can easily prove that ξ' is a convex modular with Fatou property on N and $N_{\xi'}$ is $\xi-$complete, see [2]. Consider the mapping $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma q(z) = \frac{1}{3} q(3^{\frac{1}{m}} z), \tag{5.77}$$

for all $z \in M$ and $q \in N_{\xi'}$. Let $q, r \in N_{\xi'}$ and $(3)^{\frac{1}{m}} \in [0,1]$ with $\xi'(q-r) < (3)^{\frac{1}{m}}$. By definition of ξ', we get

$$\xi(q(z) - r(z)) \leq (3)^{\frac{1}{m}} \eta(z_1, z_2, z_3), \forall z_1, z_2, z_3 \in M. \tag{5.78}$$

By making use of (5.74) and (5.78), we have

$$\begin{aligned}
\xi\left(\frac{q((3)^{\frac{1}{m}}z)}{\frac{1}{3}} - \frac{r((3)^{\frac{1}{m}}z)}{\frac{1}{3}}\right) &\leq \frac{1}{\frac{1}{3}} \xi\left(q((3)^{\frac{1}{m}}z) - r((3)^{\frac{1}{m}}z)\right) \\
&\leq \frac{1}{\frac{1}{3}}(3)^{\frac{1}{m}} \eta\left((3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z_2, (3)^{\frac{1}{m}}z_3\right) \leq (3)^{\frac{1}{m}} \psi \eta\left(z_1, z_2, z_3\right),
\end{aligned}$$

for all $z_1, z_2, z_3 \in M$. Then σ is a $\xi'-$contraction and (5.75) implies

$$\xi\left(\frac{s((3)^{\frac{1}{m}}z)}{\frac{1}{3}} - s(z)\right) \leq \frac{1}{\frac{1}{3}} \eta(z, z, z), \forall z \in M, \tag{5.79}$$

and replacing z by $(3)^{\frac{1}{m}}z$ in (5.79), we get

$$\xi\left(\frac{s((3)^{\frac{2}{m}}z)}{\frac{1}{3}} - s((3)^{\frac{1}{m}}z)\right) \leq \frac{\eta((3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z, \ldots, (3)^{\frac{1}{m}}z)}{\frac{1}{3}}, \ \forall z \in M \tag{5.80}$$

and by making use of (5.79) and (5.80), we get

$$\xi\left(\frac{s((3)^{\frac{2}{m}}z)}{\frac{1}{9}} - s(z)\right) \leq \frac{1}{\frac{1}{9}} \eta((3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z) + \frac{1}{\frac{1}{3}} \eta(z, z, z),$$

for all $z \in M$ and by generalization, we get

$$\begin{aligned}
\xi\left(\frac{s((3)^{\frac{k}{m}}z)}{\frac{1}{3^k}} - s(z)\right) &\leq \sum_{i=1}^{k} \frac{1}{\frac{1}{3^i}} \eta(((3)^{\frac{1}{m}})^{i-1}z, ((3)^{\frac{1}{m}})^{i-1}z, ((3)^{\frac{1}{m}})^{i-1}z) \\
&\leq \frac{1}{\psi \frac{1}{3}} \eta(z, z, z) \sum_{i=1}^{k} \psi^i \\
&\leq \frac{1}{\frac{1}{3}(1-\psi)} \eta(z, z, z), \ \forall z \in M.
\end{aligned} \tag{5.81}$$

We obtain from (5.81),

$$
\begin{aligned}
&\xi\left(\frac{s((3)^{\frac{k}{m}}3)}{\frac{1}{3^k}}-\frac{s((3)^{\frac{u}{m}}z)}{\frac{1}{3^u}}\right)\\
&\leq \frac{1}{2}\xi\left(2\frac{s((3)^{\frac{k}{m}}z)}{\frac{1}{3^k}}-2s(z)\right)+\frac{1}{2}\xi\left(2\frac{s((3)^{\frac{u}{m}}z)}{\frac{1}{3^u}}-2s(z)\right)\\
&\leq \frac{\kappa}{2}\xi\left(\frac{s((3)^{\frac{k}{m}}z)}{\frac{1}{3^k}}-s(z)\right)+\frac{\kappa}{2}\xi\left(\frac{s((3)^{\frac{u}{m}}z)}{\frac{1}{3^u}}-s(z)\right)\\
&\leq \frac{\kappa}{\frac{1}{3}(1-\psi)}\eta(z,z,z),\ \forall z\in M
\end{aligned}
$$

where $k,u\in\mathfrak{N}$. Thus

$$\xi'(\sigma^k s-\sigma^u s)\leq \frac{\kappa}{\frac{1}{3}(1-\psi)},$$

and hence the boundedness of an orbit of σ at s is given. So $\{\tau^k s\}$ is $\xi'-$convergent to $R\in N_{\xi'}$ by Theorem 1.5 in [2]. By $\xi'-$contractivity of σ, we get

$$\xi'(\sigma^k s-\sigma R)\leq \psi\xi'(\sigma^{k-1}s-R).$$

Taking $k\to\infty$ and by Fatou property of ξ', we get

$$\xi'(\sigma R-R)\leq \lim_{k\to\infty}\inf\xi'(\sigma R-\sigma^k s)\leq\psi\lim_{k\to\infty}\inf\xi'(R-\sigma^{k-1}s)=0.$$

Hence R is a fixed point of σ. In (5.75), replacing (z_1,z_2,z_3) by $\left((3)^{\frac{k}{m}}z_1,(3)^{\frac{k}{m}}z_2,(3)^{\frac{k}{m}}z_3\right)$, we get

$$\xi\left(\frac{1}{\frac{1}{3^k}}M_1 s((3)^{\frac{k}{m}}z_1,(3)^{\frac{k}{m}}z_2,(3)^{\frac{k}{m}}z_3)\right)\leq\frac{1}{\frac{1}{3^k}}\eta((3)^{\frac{k}{m}}z_1,(3)^{\frac{k}{m}}z_2,(3)^{\frac{k}{m}}z_3).$$

By Theorems 3.1, 3.3 and taking $k\to\infty$, we obtain that R is a reciprocal mapping and using (5.81), we have (5.76). For the uniqueness of R, consider another multi-type reciprocal mapping $T:M\to Z_\xi$ satisfying (5.76). Then T is a fixed point of σ such that

$$\xi'(R-T)=\xi'(\sigma R-\sigma T)\leq\psi\xi'(R-T). \tag{5.82}$$

From (5.82), we get $R=T$. This completes the proof. □

The proofs of Corollaries 5.2 and 5.4 follows from the fact that every normed space is a modular space of modular $\xi(z)=\|z\|$.

Corollary 5.2. *Assume η is a function from M^2 to $[0,+\infty)$ such that*

$$\lim_{k\to\infty}\frac{1}{\frac{1}{3^k}}\eta\{(3^{\frac{k}{m}})z_1,(3^{\frac{k}{m}})z_2,(3^{\frac{k}{m}})z_3\}=0,$$

and

$$\eta\{(3^{\frac{1}{m}})z_1, (3^{\frac{1}{m}})z_2, (3^{\frac{1}{m}})z_3\} \leq \frac{1}{3}\psi\eta\{z_1, z_2, z_3\},\ \psi < 1.$$

Assume that $s : M \to Z$ *satisfies the following, for a Banach space* Z*,*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \eta(z_1, z_2, z_3),$$

for all $z_1, z_2, z_3 \in M$*. Then there is a unique reciprocal mapping* $R : M \to Z$ *such that*

$$\|R(z) - s(z)\| \leq \frac{\eta(z, z, z)}{\frac{1}{3}(1-\psi)},$$

for all $z \in M$*.*

Theorem 5.3. *Assume* η *is a function from* M^2 *to* $[0, +\infty)$ *with*

$$\lim_{k\to\infty} \frac{1}{\kappa^k}\eta\left(\frac{z_1}{(3)^{\frac{k}{m}}}, \frac{z_2}{(3)^{\frac{k}{m}}}, \frac{z_3}{(3)^{\frac{k}{m}}}\right) = 0,$$

and

$$\eta\left(\frac{z_1}{(3)^{\frac{1}{m}}}, \frac{z_2}{(3)^{\frac{1}{m}}}, \frac{z_3}{(3)^{\frac{1}{m}}}\right) \leq \frac{\psi}{\frac{1}{3}}\rho\{z_1, z_2, z_3\},$$

for all $z_1, z_2, z_3 \in M, \psi < 1$*. Assume that* $s : M \to Z_\xi$ *fulfills*

$$\xi\left(M_1 s(z_1, z_2, z_3)\right) \leq \eta(z_1, z_2, z_3).$$

Then there is a unique reciprocal mapping $R : M \to Z_\xi$ *such that*

$$\xi(R(z) - s(z)) \leq \frac{p\psi}{1-\psi}\eta(z, z, z),\ \forall z \in M.$$

Proof. Replacing z by $\frac{z}{(3)^{\frac{1}{m}}}$ in (5.77) of Theorem 5.1 and by a similar method to that of Theorem 5.1, we complete the proof. □

Corollary 5.4. *Assume* η *is a function from* M^2 *to* $[0, +\infty)$ *with*

$$\lim_{k\to\infty} \frac{1}{\sigma^k}\eta\left(\frac{z_1}{(3)^{\frac{k}{m}}}, \frac{z_2}{(3)^{\frac{k}{m}}}, \frac{z_3}{(3)^{\frac{k}{m}}}\right) = 0,$$

and

$$\eta\left(\frac{z_1}{(3)^{\frac{1}{m}}}, \frac{z_2}{(3)^{\frac{1}{m}}}, \frac{z_3}{(3)^{\frac{1}{m}}}\right) \leq \frac{\psi}{\frac{1}{3}}\eta\{z_1, z_2, z_3\}, \psi < 1.$$

Assume that $s : M \to Z$ *fulfills*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \eta(z_1, z_2, z_3),$$

for all $z_1, z_2, z_3 \in M$. Then there is a unique reciprocal mapping $R : M \to Z$ such that

$$\|R(z) - s(z)\| \leq \frac{p\psi}{1-\psi}\eta(z, z, z),$$

for all $z \in M$.

Using Corollaries 5.2 and 5.4, we obtain the following corollaries.

Corollary 5.5. *Assume η is a function from M^2 to $[0, +\infty)$, Z is a Banach space and $\epsilon \geq 0$ is a real number such that*

$$\lim_{k\to\infty} \frac{1}{\frac{1}{3^k}}\eta\{(3)^{\frac{k}{m}}z_1, (3)^{\frac{k}{m}}z_2, (3)^{\frac{k}{m}}z_3\} = 0,$$

and

$$\eta\{(3)^{\frac{1}{m}}z_1, (3)^{\frac{1}{m}}z_2, (3)^{\frac{1}{m}}z_3\} \leq \frac{1}{3}\psi\eta\{z_1, z_2, z_3\}, \psi < 1.$$

Assume that $s : M \to Z$ fulfills

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon,$$

for all $z_1, z_2, z_3 \in M$. Then there is a unique reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((3)^{\frac{k}{m}}z\right)}{\frac{1}{3^k}}$, such that

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{2},$$

for all $z \in M$.

Proof. Assume that $\eta(z_1, z_2, z_3) = \epsilon$ for all $z_1, z_2, z_3 \in Z$. Then Corollary 5.2 implies

$$\|R(z) - s(z)\| \leq \frac{p\epsilon}{2},$$

for all $z \in Z$ and $p \neq 0, \pm 1$ and making use of Corollary 5.4, we get

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{2},$$

for all $z \in Z$. □

Corollary 5.6. *If $s : M \to X$ fulfills the following inequality, for a linear space M and a Banach space Z, respectively,*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon\left(\|z_1\|^u + \|z_2\|^u + \|x_3\|^u\right),$$

for all $z_1, z_2, z_3 \in M$ with $0 \leq u < -m$ or $u > -m$ for some $\epsilon \geq 0$. Then there is a reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((3)^{\frac{k}{m}} z\right)}{\frac{1}{3^k}}$, such that

$$\|R(z) - s(z)\| \leq \frac{9\epsilon}{\left|1 - 3^{\frac{m+u}{m}}\right|} \|z\|^u, \ \forall z \in M.$$

Proof. If we choose $\eta(z_1, z_2, z_3) = \epsilon\big(\|z_1\|^u + \|z_2\|^u + \|z_3\|^u\big)$, then Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq \frac{9\epsilon}{1 - 3^{\frac{m+u}{m}}} \|z\|^u,$$

for all $z \in Z$ and $u < -m$. Using Corollary 4.4, we obtain

$$\|R(z) - s(z)\| \leq \frac{9\epsilon}{3^{\frac{m+u}{m}} - 1} \|z\|^u,$$

for all $z \in Z$ and $u > -m$. □

The following is an example to elucidate (2.4), which is not stable for $u = -m$ in Corollary 5.6.

Example 5.7. Define $\phi : \mathbb{R} \to \mathbb{R}$ with $a > 0$ as

$$\phi(z) = \begin{cases} \frac{a}{z^m}, & \text{if } z \in (1, \infty) \\ a\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(3^{-k} z)}{3^{mk}}$. Then s fulfills

$$\|M_1 s(z_1, z_2, z_3)\| \leq \frac{a 3^{2m}(4)}{3(3^m - 1)} \times \left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \left|\frac{1}{z_3^m}\right|\right)$$

for all $z_1, z_2, z_3 \in \mathbb{R}$. In that case, there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ such that

$$|s(z) - R(z)| \leq \beta \left|\frac{1}{z^m}\right|, \beta > 0, \forall z \in \mathbb{R}.$$

Corollary 5.8. *Assume $s : Z_1 \to Z_2$ is a mapping. Assume that there exists $\epsilon \geq 0$ such that*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon \left(\|z_1\|^{\frac{u}{3}} \|z_2\|^{\frac{u}{3}} \|z_3\|^{\frac{u}{p}}\right)$$

for all $z_1, z_2, z_3 \in Z_1$. Then there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ fulfilling (2.4) and

$$\|r(z) - s(z)\| \leq \begin{cases} \frac{3\epsilon}{1 - 3^{\frac{m+u}{m}}} \|z\|^u & \textit{for } u < -m \\ \frac{3\epsilon}{3^{\frac{m+u}{m}} - 1} \|z\|^u & \textit{for } u > -m \end{cases}$$

for all $z \in Z_1$.

Proof. Replace $\eta(z_1, z_2, z_3)$ by $\epsilon \left(\|z_1\|^{\frac{u}{3}}\|z_2\|^{\frac{u}{3}}\|z_3\|^{\frac{u}{3}}\right)$. Then Corollary 5.2 implies

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{1 - 3^{\frac{m+u}{m}}}\|z\|^3,$$

for $u < -m$ and for all $z \in Z_1$ and making use of Corollary 5.4, we get

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{3^{\frac{m+u}{m}} - 1}\|z\|^3,$$

for $u > -m$ and for all $z \in Z_1$. □

Corollary 5.9. *Let $\epsilon > 0$ and $\alpha < -\frac{m}{3}$ or $\alpha > -\frac{m}{3}$ be real numbers, and $s : Z_1 \to Z_2$ be a mapping satisfying the functional inequality*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon \left\{\|z_1\|^{3\alpha} + \|z_2\|^{3\alpha} + \|z_3\|^{3\alpha} + (\|z_1\|^{\alpha}\|z_2\|^{\alpha}\|z_3\|^{\alpha})\right\}.$$

Then there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ fulfilling (2.4) and

$$\|R(z) - s(z)\| \leq \begin{cases} \frac{12\epsilon}{1-3^{\frac{3\alpha+m}{m}}}\|z\|^{3\alpha} & \text{for } \alpha < -\frac{m}{3} \\ \frac{12\epsilon}{3^{\frac{3\alpha+m}{m}}-1}\|z\|^{3\alpha} & \text{for } \alpha > -\frac{m}{3} \end{cases}$$

for all $z \in Z_1$.

Proof. Replace $\eta(z_1, z_2, z_3)$ by $\epsilon\Big\{\|z_1\|^{3\alpha} + \|z_2\|^{3\alpha} + \|z_3\|^{3\alpha} + (\|z_1\|^{\alpha}\|z_2\|^{\alpha}\|z_3\|^{\alpha})\Big\}$. Then Corollary 5.4 implies

$$\|R(z) - s(z)\| \leq \frac{12\epsilon}{1 - 3^{\frac{3\alpha+m}{m}}}\|z\|^{3\alpha},$$

for $\alpha < -\frac{m}{3}$ and for all $z \in Z_1$ and making use of Corollary 5.4, we get

$$\|R(z) - s(z)\| \leq \frac{12\epsilon}{3^{\frac{3\alpha+m}{m}} - 1}\|z\|^{3\alpha},$$

for $\alpha > -\frac{m}{3}$ and for all $z \in Z$. □

The following is an example to elucidate (2.4), which is not stable for $\alpha = -\frac{m}{3}$ in Corollary 5.9.

Example 5.10. Define $\phi : \mathbb{R} \to \mathbb{R}$ with a constant $l > 0$ as

$$\phi(z) = \begin{cases} \frac{l}{z^m}, & \text{if } z \in (1, \infty) \\ l\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(3^{-k}z)}{3^{mk}}$. Then s fulfills

$$\|M_1 s(z_1, z_2, z_3)\| \leq \frac{a3^{2m}(4)}{3(3^m - 1)} \times \left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \left|\frac{1}{z_3^m}\right| + \left|\frac{1}{z_1^m}\right|\left|\frac{1}{z_2^m}\right|\left|\frac{1}{z_3^m}\right|\right)$$

for all $z_1, z_2, z_3 \in \mathbb{R}$. In that case, there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ such that

$$|s(z) - R(z)| \leq \beta \left| \frac{1}{z^m} \right|, \ \beta > 0, \forall z \in \mathbb{R}.$$

6. Conclusion

In this work, we introduced the new generalized multifarious type radical reciprocal functional equations combining three classical Pythagorean means arithmetic, geometric and harmonic. Importantly, we obtained their general solution and stabilities related to Ulam problem with suitable counter examples in modular spaces by using fixed point approach. Furthermore, we illustrated their geometrical interpretation.

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WEIGHTED DIFFERENTIATION SUPERPOSITION OPERATOR FROM H^∞ TO nth WEIGHTED-TYPE SPACE

CHENG-SHI HUANG AND ZHI-JIE JIANG*

Abstract. Let $H(\mathbb{D})$ be the set of all analytic functions on the open unit disk $\mathbb{D}$ of $\mathbb{C}$, $u \in H(\mathbb{D})$ and ϕ an entire function on $\mathbb{C}$. In this paper, we characterize the boundedness and compactness of the weighted differentiation superposition operator $D_u^m S_\phi$ from H^∞ to the nth weighted-type space.

1. Introduction

Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$.

First, we present some of the most interesting linear operators studied on some subspaces of $H(\mathbb{D})$. Let $z \in \mathbb{D}$, then the multiplication operator with symbol $u \in H(\mathbb{D})$ is defined by $M_u(f)(z) = u(z)f(z)$, and composition operator with symbol $\varphi \in S(\mathbb{D})$ is defined by $C_\varphi(f)(z) = f(\varphi(z))$.

Let $m \in \mathbb{N}_0$ and $f \in H(\mathbb{D})$, then the mth differentiation operator is defined by

$$D^m f(z) = f^{(m)}(z), \quad z \in \mathbb{D}, \tag{1}$$

where $f^{(0)} = f$. If $m = 1$, then it is the standard differentiation operator D. In recent years, there has been a lot of interest in the study of products of differential operator and others. For example, products DC_φ and $C_\varphi D$, which are the most basic product-type operators involving the differentiation operator, have been studied, for example, in [1–9]. Many other results have evolved from them, for example, the following six operators were studied in [10]

$$DM_uC_\varphi,\ DC_\varphi M_u,\ C_\varphi DM_u,\ C_\varphi M_uD,\ M_uC_\varphi D, M_uDC_\varphi. \tag{2}$$

An operator, namely including all the operators in (2), was introduced and investigated in [11, 12]. In some studies, for example, Wang et al. in [13] generalized operators in (2) and studied the following operators

$$D^nM_uC_\varphi,\ D^nC_\varphi M_u,\ C_\varphi D^nM_u,\ C_\varphi M_uD^n,\ M_uC_\varphi D^n,\ M_uD^nC_\varphi. \tag{3}$$

Some other product-type operators on subspaces of $H(\mathbb{D})$ can be found (see, e.g., [14–17] and the related references therein).

Next, we introduce the superposition operator (see, for example, [18] or [19]). Let ϕ be a complex-valued function on $\mathbb{C}$. Then the superposition operator S_ϕ on $H(\mathbb{D})$ is defined as

$$S_\phi f = \phi(f(z)), \quad z \in \mathbb{D}.$$

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Assume that X and Y are two metric spaces of analytic functions on $\mathbb{D}$ and S_ϕ maps X into Y. Note that if X contain the linear functions, then ϕ must be an entire function. Recently, the boundedness and compactness of S_ϕ have been characterized on or between some analytic function spaces (see, for example, [19–26]).

The following weighted differentiation superposition operator, which is introduced in [27], is a class of nonlinear operators. Let $m \in \mathbb{N}_0$, $u \in H(\mathbb{D})$ and ϕ be an entire function on $\mathbb{C}$. The weighted differentiation superposition operator denoted as $D_u^m S_\phi$ on some subspaces of $H(\mathbb{D})$ is defined by

$$(D_u^m S_\phi f)(z) = u(z)\phi^{(m)}(f(z)), \quad z \in \mathbb{D}.$$

Our goal of this paper is to improve results of Kamal and Eissa in [27]. Here, we rethink the boundedness and compactness of this operator from H^∞ space to nth weighted-type space, which can be regarded as a continuation of our work (see, for example, [19]).

Now, we introduce the important Bell polynomial (see, for example, [13, 15]). Let $n, k \in \mathbb{N}_0$. Then the Bell polynomial is defined as

$$B_{n,k} := B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{\prod_{i=1}^{n-k-1} j_i!} \prod_{i=1}^{n-k-1} \left(\frac{x_i}{i!}\right)^{j_i}, \tag{4}$$

where the sum is taken over all non-negative integer sequences $j_1, j_2, \dots, j_{n-k+1}$ satisfying $\sum_{i=1}^{n-k+1} j_i = k$ and $\sum_{i=1}^{n-k+1} ij_i = n$. In particular, if $k = 0$, we have $B_{0,0} = 1$ and $B_{n,0} = 0$ for $n \in \mathbb{N}$.

Next, we collect some needed spaces as follows (see [7]). The symbol H^∞ denotes the space of all bounded analytic functions f on $\mathbb{D}$ such that

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < +\infty.$$

Let μ be a weight function (i.e. a positive continuous function on $\mathbb{D}$) and $n \in \mathbb{N}_0$. Then the nth weighted-type space $\mathcal{W}_\mu^{(n)}(\mathbb{D}) := \mathcal{W}_\mu^{(n)}$ consists of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(n)}}(f) := \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < +\infty.$$

If $n = 0$, it is the weighted-type space H_μ^∞ (see, for example, [28–30]). If $n = 1$, the Bloch-type space $\mathcal{B}_\mu$, and if $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu$. If $\mu(z) = 1 - |z|^2$, we correspondingly get the classical weighted-type space, Bloch space and Zygmund space. Some information on these classical function spaces and some operators on them can be found, for example, in [31–37].

Let $n \in \mathbb{N}$, then the quantity $b_{\mathcal{W}_\mu^{(n)}}(f)$ is a seminorm on $\mathcal{W}_\mu^{(n)}$ and a norm on $\mathcal{W}_\mu^{(n)}/\mathbb{P}_{n-1}$, where $\mathbb{P}_{n-1}$ is the class of all polynomials whose degrees are less than or equal to $n-1$. A natural norm on $\mathcal{W}_\mu^{(n)}$ can be introduced as follows

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathcal{W}_\mu^{(n)}}(f).$$

The set $\mathcal{W}_\mu^{(n)}$ with this norm becomes a Banach space. The little nth weighted-type space $\mathcal{W}_{\mu,0}^{(n)}$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \mu(z)|f^{(n)}(z)| = 0.$$

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It is easy to see that $\mathcal{W}_{\mu,0}^{(n)}$ is a closed subspace of $\mathcal{W}_{\mu}^{(n)}$ and the set of all polynomials is dense in $\mathcal{W}_{\mu,0}^{(n)}$. If $n=1$ and $\mu(z)=1-|z|^2$, then it is the little Bloch space $\mathcal{B}_0$.

Finally, we will introduce the boundedness and compactness of a operator T. Let X and Y be two Banach spaces, and $T:X\to Y$ be a operator. If there is a positive constant K such that

$$\|Tf\|_Y \le K\|f\|_X$$

for all $f\in X$, we say that T is bounded. The operator $T:X\to Y$ is compact if it maps bounded sets into relatively compact sets.

As usual, some positive constants are denoted by C, and they may differ from one occurrence to another. The notation $a\lesssim b$ (resp. $a\gtrsim b$) means that there is a positive constant C such that $a\le Cb$ (resp. $a\ge Cb$). When $a\lesssim b$ and $b\gtrsim a$, we write $a\asymp b$.

2. Preliminary results

In this section, we need several auxiliary results for proving the main results. First, we have the following useful result which can be found in [38].

Lemma 2.1. *Let $f\in H^\infty$. Then for every $n\in\mathbb{N}$, there exists a constant $C>0$ independent of f such that*

$$\sup_{z\in\mathbb{D}}(1-|z|)^n|f^{(n)}(z)|\le C\|f\|_\infty.$$

The following lemma is introduced in [31].

Lemma 2.2. *Let $f\in\mathcal{B}$. Then for every $n\in\mathbb{N}$*

$$\|f\|_{\mathcal{B}}\asymp\sum_{j=0}^{n-1}|f^{(j)}(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^n|f^{(n)}(z)|.$$

The following lemma shows that any bounded analytic function on $\mathbb{D}$ is in Bloch space (see Proposition 5.1.2 in [39]).

Lemma 2.3. *$H^\infty\subset\mathcal{B}$. Moreover, $\|f\|_{\mathcal{B}}\le\|f\|_\infty$ for all $f\in H^\infty$.*

The following gives an important test function (see [40]).

Lemma 2.4. *For fixed $t\ge 0$ and $w\in\mathbb{C}$, the following function is in H^∞*

$$g_{w,t}(z)=\left(\frac{1-|w|^2}{(1-\langle z,w\rangle)}\right)^{t+1}.$$

Moreover,

$$\sup_{w\in\mathbb{C}}\|g_{w,t}\|_\infty\lesssim 1.$$

We construct some suitable linear combinations of the functions in Lemma 2.4, which will be used in the proofs of the main results.

Lemma 2.5. *Let $w\in\mathbb{C}$. Then there are constants $c_0,c_1,\dots,c_n$ such that the function*

$$h_w(z)=\sum_{k=0}^{n}c_k g_{w,k}(z)$$

satisfies

$$h_w^{(s)}(w) = \frac{\overline{w}^s}{(1-|w|^2)^s}, \quad 0 \le s \le n \quad \text{and} \quad h_w^{(l)}(w) = 0, \tag{5}$$

where $l \in \{0, 1, \dots, n\} \backslash \{s\}$. *Moreover,*

$$\sup_{w \in \mathbb{C}} \|h_w\|_\infty < +\infty.$$

Proof. For the simplicity sake, we write $d_k = k+1$. By a direct calculation, it is easy to see that the system (5) is equivalent to the following system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_0 & d_1 & \cdots & d_n \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{s-1} d_k & \prod_{k=0}^{s-1} d_{k+1} & \cdots & \prod_{k=0}^{s-1} d_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{n-1} d_k & \prod_{k=0}^{n-1} d_{k+1} & \cdots & \prod_{k=0}^{n-1} d_{k+n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_s \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}. \tag{6}$$

Since $d_k > 0$, $k = \overline{0, n}$, by Lemma 5 in [41], the determinant of system (6) is $D_{n+1}(d_0) = \prod_{j=1}^n j!$, which is different from zero. Therefore, there exist constants $c_0, c_1, \dots, c_n$ such that the system (5) holds. Furthermore, we obtain $\sup_{w \in \mathbb{C}} \|h_w\|_\infty < +\infty$. □

Remark 2.1. In Lemma 2.5, it is clear that, if $s = 0$, then there are constants $c_0, c_1, \dots, c_n$ such that the function $h_w(z)$ satisfies $h_w^{(0)}(w) = h_w(w) = 1$ and $h_w^{(l)}(w) = 0$ for $l = \overline{1, n}$.

We also have the following characterization of compactness which can be proved similar to that in [42] (Proposition 3.11), and so we omit the proof.

Lemma 2.6. *Let* $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $u \in H(\mathbb{D})$ *and* ϕ *be an entire function. Then the bounded operator* $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ *is compact if and only if for each bounded sequence* $\{f_k\}(k \in \mathbb{N}) \subset H^\infty$ *such that* $f_k \to 0$ *uniformly on any compact subsets of* $\mathbb{D}$ *as* $k \to \infty$, *it follows that*

$$\lim_{k \to \infty} \|D_u^m S_\phi f_k\|_{\mathcal{W}_\mu^{(n)}} = 0.$$

Finally, we need the following result proved in [34]. So, the details are omitted.

Lemma 2.7. *A closed set* K *in* $\mathcal{W}_{\mu,0}^{(n)}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z| \to 1} \sup_{f \in K} \mu(z) |f^{(n)}(z)| = 0.$$

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3. Main results and proofs

Now, we begin to characterize the boundedness and compactness of the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ (or $\mathcal{W}_{\mu,0}^{(n)}$).

Theorem 3.1. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $u \in H(\mathbb{D})$ and ϕ an entire function with $\phi^{(m)}(1) \neq 0$ and $\phi^{(m+1)}(0) \neq 0$. Then the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is bounded if and only if*

$$M_i := \sup_{z\in\mathbb{D}} \frac{\mu(z)|u^{(n-i)}(z)|}{(1-|z|^2)^i} < +\infty \tag{7}$$

for $i = \overline{0,n}$.

Moreover, if the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is bounded, then the following asymptotic relationship holds

$$\|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} \asymp \sum_{i=0}^n M_i. \tag{8}$$

Proof. Assume that condition (7) holds. Then for each $z \in \mathbb{D}$ and $f \in H^\infty$, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \mu(z)\big|(D_u^m S_\phi f)^{(n)}(z)\big| &= \sup_{z\in\mathbb{D}} \mu(z)\Big|\sum_{j=0}^n \Big(\sum_{i=j}^n C_n^i u^{(n-i)}(z) B_{i,j}(f(z))\Big)\phi^{(m+j)}(f(z))\Big| \\ &\le \sup_{z\in\mathbb{D}} \mu(z)\sum_{j=0}^n \Big(\sum_{i=j}^n C_n^i \big|u^{(n-i)}(z)\big|\big|B_{i,j}(f(z))\big|\Big)\big|\phi^{(m+j)}(f(z))\big|,\end{aligned}$$

where

$$B_{i,j}(f(z)) := B_{i,j}\left(f'(z), f''(z), \ldots, f^{(i-j+1)}(z)\right), \quad 0 \le j \le i \le n.$$

Applying formula (4) and Lemma 2.1, we obtain

$$\begin{aligned}\big|B_{i,j}(f(z))\big| &= \Big|B_{i,j}\left(f'(z), f''(z), \ldots, f^{(i-j+1)}(z)\right)\Big| \\ &\le B_{i,j}\left(\frac{\|f\|_\infty}{1-|z|^2}, \frac{\|f\|_\infty}{(1-|z|^2)^2}, \ldots, \frac{\|f\|_\infty}{(1-|z|^2)^{i-j+1}}\right).\end{aligned} \tag{9}$$

For the convenience, we write

$$\widehat{B}_{i,j}(f,z) = B_{i,j}\left(\frac{\|f\|_\infty}{1-|z|^2}, \frac{\|f\|_\infty}{(1-|z|^2)^2}, \ldots, \frac{\|f\|_\infty}{(1-|z|^2)^{i-j+1}}\right). \tag{10}$$

From (9) and (10), we get

$$\sup_{z\in\mathbb{D}} \mu(z)\big|(D_u^m S_\phi f)^{(n)}(z)\big| \le \sup_{z\in\mathbb{D}} \mu(z)\sum_{j=0}^n \Big(\sum_{i=j}^n C_n^i \big|u^{(n-i)}(z)\big|\widehat{B}_{i,j}(f,z)\Big)\big|\phi^{(m+j)}(f(z))\big|. \tag{11}$$

For $i > j$, we have $\widehat{B}_{i,j}(f,z) = 0$. Let $f \in H^\infty$ and $\|f\|_\infty \le M$. Then, we obtain

$$\widehat{B}_{i,j}(f,z) \lesssim \frac{1}{(1-|z|^2)^i}, \quad 0 \le j \le i, \tag{12}$$

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where $i = \overline{0,n}$. From (11) and (12), we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}\mu(z)\big|(D_u^m S_\phi f)^{(n)}(z)\big| \le& \sup_{z\in\mathbb{D}}\mu(z)\sum_{j=0}^{n}\Big(\sum_{i=j}^{n}C_n^i\big|u^{(n-i)}(z)\big|\widehat{B}_{i,j}(f,z)\Big)\big|\phi^{(m+j)}(f(z))\big|\\ \le& C\sup_{z\in\mathbb{D}}\mu(z)\Bigg(|u^{(n)}(z)||\phi^{(m)}(f(z))|\\ &+\sum_{i=1}^{n}\frac{|u^{(n-i)}(z)|}{(1-|z|^2)^i}\Big(\sum_{j=1}^{i}\big|\phi^{(m+j)}(f(z))\big|\Big)\Bigg). \quad (13)\end{aligned}$$

Since $f\in H^\infty$ and $\|f\|_\infty \le M$ and ϕ is an entire function, we obtain

$$\big|\phi^{(m+j)}(f(z))\big| \le \max_{|w|=M}\big|\phi^{(m+j)}(w)\big| = L_j < +\infty \quad (14)$$

for each $z\in\mathbb{D}$ and $j=\overline{0,n}$. From (13) and (14), we have

$$\sup_{z\in\mathbb{D}}\mu(z)\big|(D_u^m S_\phi f)^{(n)}(z)\big| \le C\sup_{z\in\mathbb{D}}\Bigg(\mu(z)|u^{(n)}(z)|+\sum_{i=1}^{n}\frac{\mu(z)|u^{(n-i)}(z)|}{(1-|z|^2)^i}\Bigg). \quad (15)$$

On the other hand, we also have that for every $l=\overline{0,n-1}$

$$\big|(D_u^m S_\phi f)^{(l)}(0)\big| \le \Big|\sum_{j=0}^{l}\Big(\sum_{i=j}^{l}C_l^i u^{(l-i)}(0)B_{i,j}(f(0))\Big)\phi^{(m+j)}(f(0))\Big| < +\infty. \quad (16)$$

From Lemma 2.2, (7), (15) and (16), we see that the operator $D_u^m S_\phi : \mathcal{B}\to\mathcal{W}_\mu^{(n)}$ is bounded. By Lemma 2.3 (or (7) and (15)), it is obvious that the operator $D_u^m S_\phi : H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded. Moreover, it follows that

$$\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \le C\sum_{i=0}^{n}M_i. \quad (17)$$

Now assume that the operator $D_u^m S_\phi : H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded, then there is a positive constant C independent of f such that

$$\|D_u^m S_\phi f\|_{\mathcal{W}_\mu^{(n)}} \le C\|f\|_\infty \quad (18)$$

for each $f\in H^\infty$. By Remark 2.1, there is a function $h_w\in H^\infty$ such that

$$h_w(w)=1 \quad \text{and} \quad h_w^{(l)}(w)=0 \quad (19)$$

for $l=\overline{1,n}$. Let $L_0=\|h_w\|_\infty$. Then, from (18) and (19), we obtain

$$\begin{aligned}L_0\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \ge& \|D_u^m S_\phi h_w\|_{\mathcal{W}_\mu^{(n)}}\\ =& \sup_{z\in\mathbb{D}}\mu(z)\Big|\sum_{j=0}^{n}\Big(\sum_{i=j}^{n}C_n^i u^{(n-i)}(z)B_{i,j}(h_w(z))\Big)\phi^{(m+j)}(h_w(z))\Big|\\ \ge& \mu(w)\big|u^{(n)}(w)\big|\big|B_{0,0}(h_w(w))\big|\big|\phi^{(m)}(1)\big|\\ =& \mu(w)\big|u^{(n)}(w)\big|\big|\phi^{(m)}(1)\big|. \quad (20)\end{aligned}$$

Since $|\phi^{(m)}(1)|\neq 0$, we have

$$L_0\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \ge \|D_u^m S_\phi h_w\|_{\mathcal{W}_\mu^{(n)}} \ge C\mu(z)|u^{(n)}(z)|, \quad (21)$$

for each $z\in\mathbb{D}$, which implies that $M_0 < +\infty$.

By Lemma 2.4, there is a function $\tilde{h}_w \in H^\infty$ such that

$$\tilde{h}_w^{(n)}(w) = \frac{\overline{w}^n}{(1-|w|^2)^n} \quad \text{and} \quad \tilde{h}_w^{(l)}(w) = 0 \tag{22}$$

for $l = \overline{0, n-1}$. Let $L_n = \|\tilde{h}_w\|_\infty$. Then, from (18) and (22), we have

$$\begin{aligned}
L_n\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \geq & \|D_u^m S_\phi \tilde{h}_w\|_{\mathcal{W}_\mu^{(n)}} \\
= & \sup_{z\in\mathbb{D}} \mu(z)\Big|\sum_{j=0}^{n}\Big(\sum_{i=j}^{n} C_n^i u^{(n-i)}(z)B_{i,j}(\tilde{h}_w(z))\Big)\phi^{(m+j)}(\tilde{h}_w(z))\Big| \\
\geq & \mu(w)\big|u(w)B_{n,1}(\tilde{h}_w(w))\phi^{(m+1)}(0) + u^{(n)}(w)B_{0,0}(\tilde{h}_w(w))\phi^{(m)}(0)\big| \\
= & \mu(w)\Big|\frac{u(w)\overline{w}^n}{(1-|w|^2)^n}\phi^{(m+1)}(0) + u^{(n)}(w)\phi^{(m)}(0)\Big| \\
\geq & \mu(w)\Big|\frac{u(w)\overline{w}^n}{(1-|w|^2)^n}\phi^{(m+1)}(0)\Big| - \mu(w)\big|u^{(n)}(w)\phi^{(m)}(0)\big|,
\end{aligned} \tag{23}$$

where

$$B_{i,j}(\tilde{h}_w(z)) := B_{i,j}\left(\tilde{h}_w'(z), \tilde{h}_w''(z), \ldots, \tilde{h}_w^{(i-j+1)}(z)\right).$$

From (21) and (23), we have

$$\begin{aligned}
\mu(w)\Big|\frac{u(w)\overline{w}^n}{(1-|w|^2)^n}\phi^{(m+1)}(0)\Big| &\leq L_n\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} + \mu(w)\big|u^{(n)}(w)\phi^{(m)}(0)\big| \\
&\leq (L_n + CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}.
\end{aligned}$$

Since $|\phi^{(m+1)}(0)| \neq 0$, we have

$$(L_n + CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \geq \|D_u^m S_\phi \tilde{h}_w\|_{\mathcal{W}_\mu^{(n)}} \geq C\frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n}. \tag{24}$$

From (24), we have

$$(L_n + CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \geq C\sup_{|z|>1/2}\frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n} \geq \frac{C}{2^n}\sup_{|z|>1/2}\frac{\mu(z)|u(z)|}{(1-|z|^2)^n}. \tag{25}$$

One the other hand, we have

$$\sup_{|z|\leq 1/2}\frac{\mu(z)|u(z)|}{(1-|z|^2)^n} \leq \left(\frac{4}{3}\right)^n \sup_{|z|\leq 1/2}\mu(z)|u(z)|. \tag{26}$$

From (25) and (26), we get that $M_n < +\infty$.

By Lemma 2.4, there is a function $\hat{h}_w \in H^\infty$ such that

$$\hat{h}_w^{(n-1)}(w) = \frac{\overline{w}^{n-1}}{(1-|w|^2)^{n-1}} \quad \text{and} \quad \hat{h}_w^{(l)}(w) = 0, \tag{27}$$

where $l \in \{0, 1, \ldots, n\} \setminus \{n-1\}$. Let $L_{n-1} = \|\hat{h}_w\|_\infty$. From (18) and (27), we have

$$
\begin{aligned}
L_{n-1}\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \geq& \|D_u^m S_\phi \hat{h}_w\|_{\mathcal{W}_\mu^{(n)}} \\
=& \sup_{z\in\mathbb{D}} \mu(z)\Big|\sum_{j=0}^{n}\Big(\sum_{i=j}^{n} C_n^i u^{(n-i)}(z) B_{i,j}(\hat{h}_w(z))\Big)\phi^{(m+j)}(\hat{h}_w(z))\Big| \\
\geq& \mu(w)\Big|C_n^{n-1}u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0) \\
&+\sum_{j=1}^{n} u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)+u^{(n)}(w)\phi^{(m)}(0)\Big| \\
\geq& \mu(w)\Big|C_n^{n-1}u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0) \\
&+\sum_{j=1}^{n} u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big|-\mu(w)\big|u^{(n)}(w)\phi^{(m)}(0)\big|,
\end{aligned}
\tag{28}
$$

where $B_{i,j}(\hat{h}_w(z)) := B_{i,j}\Big(\hat{h}_w'(z), \hat{h}_w''(z), \ldots, \hat{h}_w^{(i-j+1)}(z)\Big)$. From (21) and (28), by using the triangle inequality, we obtain

$$
\begin{aligned}
&(L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \\
\geq& \mu(w)\Big|C_n^{n-1}u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0)+\sum_{j=1}^{n} u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big| \\
\geq& \mu(w)\Big|u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0)\Big|-\mu(w)\Big|\sum_{j=1}^{n} u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big|.
\end{aligned}
\tag{29}
$$

From (29), we have

$$
\begin{aligned}
&\mu(w)\big|u'(w)B_{n-1,1}(\hat{h}_w)\phi^{(m+1)}(0)\big| \\
\leq& (L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+\mu(w)\Big|\sum_{j=1}^{n} u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big| \\
\leq& (L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+\frac{\mu(w)|u(w)||w|^n}{(1-|w|^2)^n}\Big(\sum_{j=1}^{n}\big|\phi^{(m+j)}(0)\big|\Big).
\end{aligned}
\tag{30}
$$

Since $|\phi^{(m+1)}(0)| \neq 0$, by using (24) and (30), we obtain

$$
\begin{aligned}
C\frac{\mu(z)|u'(z)||z|^{n-1}}{(1-|z|^2)^{n-1}}\big| \leq& (L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+C\frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n} \\
\leq& (L_n+L_{n-1}+2CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}.
\end{aligned}
\tag{31}
$$

From (31), we have

$$
\begin{aligned}
(L_n+L_{n-1}+2CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \geq& C\sup_{|z|>1/2}\frac{\mu(z)|u'(z)||z|^{n-1}}{(1-|z|^2)^{n-1}} \\
\geq& \frac{C}{2^{n-1}}\sup_{|z|>1/2}\frac{\mu(z)|u'(z)|}{(1-|z|^2)^{n-1}}.
\end{aligned}
\tag{32}
$$

One the other hand, we have

$$\sup_{|z|\le 1/2}\frac{\mu(z)|u'(z)|}{(1-|z|^2)^{n-1}}\le\left(\frac{4}{3}\right)^{n-1}\sup_{|z|\le 1/2}\mu(z)|u'(z)|. \tag{33}$$

From (32) and (33), we get that $M_{n-1}<+\infty$.

Now, assume that (7) holds for $k\le i\le n$, where $1\le k\le n-2$. Let $L_{k-1}=\|h_w\|_\infty$. By using the function in Lemma 2.4, we have

$$\begin{aligned}
L_{k-1}\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\ge&\|D_u^m S_\phi h_w\|_{\mathcal{W}_\mu^{(n)}}\\
=&\sup_{z\in\mathbb{D}}\mu(z)\Big|\sum_{j=0}^{n}\Big(\sum_{i=j}^{n}C_n^i u^{(n-i)}(z)B_{i,j}(h_w(z))\Big)\phi^{(m+j)}(h_w(z))\Big|\\
\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\\
&+\sum_{i=k}^{n}\sum_{j=1}^{i}C_n^i u^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)+u^{(n)}(w)\phi^{(m)}(0)\Big|\\
\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\\
&+\sum_{i=k}^{n}\sum_{j=1}^{i}C_n^i u^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|-\mu(w)\big|u^{(n)}(w)\phi^{(m)}(0)\big|
\end{aligned} \tag{34}$$

for each $w\in\mathbb{D}$. From (21) and (34), we have

$$\begin{aligned}
(L_{k-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\\
&+\sum_{i=k}^{n}\sum_{j=1}^{i}C_n^i u^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|\\
\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\Big|\\
&-\mu(w)\Big|\sum_{i=k}^{n}\sum_{j=1}^{i}C_n^i u^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\Big|\\
\le&(L_{k-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\\
&+\mu(w)\Big|\sum_{i=k}^{n}\sum_{j=1}^{i}C_n^i u^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|\\
\le&(L_{k-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\\
&+C\sum_{i=k}^{n}\sum_{j=1}^{i}\mu(w)\Big|u^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|\\
\le&(L_{k-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}
\end{aligned}$$

$$+C\sum_{i=k}^{n}\frac{\mu(w)|u^{(n-i)}(w)||w|^i}{(1-|w|^2)^i}\Big(\sum_{j=1}^{i}\big|\phi^{(m+j)}(0)\big|\Big) \tag{35}$$

Since $|\phi^{(m+1)}(0)|\neq 0$, from (35) and the assumption (7), we have

$$\begin{aligned}
&C\frac{\mu(z)|u^{(n-(k-1))}(z)||z|^{k-1}}{(1-|z|^2)^{k-1}}\\
\leq&(L_{k-1}+CL_0)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+C\sum_{i=k}^{n}\frac{\mu(w)|u^{(n-i)}(w)||w|^i}{(1-|w|^2)^i}\\
\leq&\Big(\sum_{t=k-1}^{n}L_t+(n-k+2)CL_0\Big)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}.
\end{aligned} \tag{36}$$

From (36), we have

$$\begin{aligned}
\Big(\sum_{t=k-1}^{n}L_t+(n-k+2)CL_0\Big)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}&\geq C\frac{\mu(z)|u^{(n-(k-1))}(z)||z|^{k-1}}{(1-|z|^2)^{k-1}}\\
&\geq\frac{C}{2^{k-1}}\frac{\mu(z)|u^{(n-(k-1))}(z)|}{(1-|z|^2)^{k-1}}
\end{aligned} \tag{37}$$

One the other hand, we have

$$\sup_{|z|\leq 1/2}\frac{\mu(z)|u^{(n-(k-1))}(z)|}{(1-|z|^2)^{k-1}}\leq\Big(\frac{4}{3}\Big)^{n-(k-1)}\sup_{|z|\leq 1/2}\mu(z)|u^{(n-(k-1))}(z)|. \tag{38}$$

From (37) and (38), we get that $M_{k-1}<+\infty$. Hence, from the mathematical induction it follows that (7) holds for every $i=\overline{0,n}$. Moreover, we also obtain

$$\sum_{i=0}^{n}M_i\leq C\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}. \tag{39}$$

From (17) and (39), then the asymptotic relation (8) follows, as desired. □

Theorem 3.2. *Let $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, $u\in H(\mathbb{D})$ and ϕ an entire function with $\phi^{(m)}(1)\neq 0$ and $\phi^{(m+1)}(0)\neq 0$. Then the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_{\mu,0}^{(n)}$ is bounded if and only if the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded and for each $i\in\{0,1,\dots,n\}$*

$$\lim_{|z|\to 1}\mu(z)|u^{(n-i)}(z)|=0. \tag{40}$$

Proof. Assume that $D_u^mS_\phi:H^\infty\to\mathcal{W}_{\mu,0}^{(n)}$ is bounded. Then for each $f\in H^\infty$, we have

$$\lim_{|z|\to 1}\mu(z)|(D_u^mS_\phi f)^{(n)}(z)|=0. \tag{41}$$

Clearly, the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded. Hence, from (24), we obtain

$$\frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n}\leq C\mu(z)|(D_u^mS_\phi\tilde{h}_w)^{(n)}(z)|. \tag{42}$$

From (42), we obtain

$$\mu(z)|u(z)||z|^n\leq C\mu(z)|(D_u^mS_\phi\tilde{h}_w)^{(n)}(z)| \tag{43}$$

By taking $|z|\to 1$ in (43) and using (41), it follows that (40) holds for $i=n$. Hence, by the proof of Theorem 3.1, we get that (40) holds for each $i=\overline{0,n}$.

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Conversely, assume that $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is bounded and condition (40) holds. Let $\hat{p} \in H^\infty$ and $\|\hat{p}\|_\infty \leq M$. Then, we have

$$\left|\phi^{(m+j)}(\hat{p}(z))\right| < +\infty.$$

For every polynomial $\hat{p}$, we have

$$\begin{aligned}\mu(z)\left|(D_u^m S_\phi \hat{p})^{(n)}(z)\right| &= \sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{j=0}^{n}\Big(\sum_{i=j}^{n} C_n^i u^{(n-i)}(z) B_{i,j}(\hat{p}(z))\Big)\phi^{(m+j)}(\hat{p}(z))\right| \\ &\leq \sup_{z\in\mathbb{D}} \mu(z)\sum_{j=0}^{n}\Big(\sum_{i=j}^{n} C_n^i \left|u^{(n-i)}(z)\right|\left|B_{i,j}(\hat{p}(z))\right|\Big)\left|\phi^{(m+j)}(\hat{p}(z))\right| \to 0\end{aligned}$$

as $|z| \to 1$. From this, we have that for every polynomial $\hat{p}$, $D_u^m S_\phi \hat{p} \in \mathcal{W}_{\mu,0}^{(n)}$. Since the set of all polynomials is dense in H^∞, we have that for each $f \in H^\infty$ there exist a sequence of polynomial $\{\hat{p}_k\}$ such that

$$\lim_{k\to\infty} \|f - \hat{p}_k\|_\infty = 0. \tag{44}$$

From (44) and using the boundedness of $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$, we obtain

$$\|D_u^m S_\phi f - D_u^m S_\phi \hat{p}_k\|_{\mathcal{W}_\mu^{(n)}} \leq \|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} \|f - \hat{p}_k\|_\infty \to 0 \tag{45}$$

as $k \to \infty$. Hence, $D_u^m S_\phi(H^\infty) \subseteq \mathcal{W}_{\mu,0}^{(n)}$ and the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_{\mu,0}^{(n)}$ is bounded. The proof is finished. □

Theorem 3.3. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $u \in H(\mathbb{D})$ and ϕ an entire function with $\phi^{(m)}(1) \neq 0$ and $\phi^{(m+1)}(0) \neq 0$. Then the following statements are equivalent:*

(a) *The operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is compact.*

(b) *The operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_{\mu,0}^{(n)}$ is compact.*

(c) *For each $i \in \{0, 1, \ldots, n\}$, it follows that*

$$\lim_{|z|\to 1} \frac{\mu(z)|u^{(n-i)}(z)|}{(1-|z|^2)^i} = 0. \tag{46}$$

Proof. $(c) \Rightarrow (b)$. From (13) and using (46), we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_\infty \leq 1} \mu(z)|(D_u^m S_\phi f)^n(z)| = 0.$$

Obviously, the set is bounded. Hence, by Lemma 2.6 the compactness of the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_{\mu,0}^{(n)}$ follows.

$(b) \Rightarrow (a)$ is obvious.

$(a) \Rightarrow (c)$. Suppose that $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is compact. Then it is clear that the operator is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|z_k| \to 1$ as $k \to \infty$. If such a sequence does not exist, then condition (46) is vacuously satisfied. Let $\tilde{h}_k = \tilde{h}_{z_k}$, where $\tilde{h}_w$ is defined in the proof of the Theorem 3.1 (or Lemma 2.4). Since $\lim_{k\to\infty} \tilde{h}_{z_k} = 0$, we have $\tilde{h}_k \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $k \to \infty$. Hence, by Lemma 2.5 we have

$$\lim_{k\to\infty} \|D_u^m S_\phi \tilde{h}_k\|_{\mathcal{W}_\mu^{(n)}} = 0. \tag{47}$$

On the other hand, from (25), for sufficiently large k it follows that

$$\|D_u^m S_\phi \tilde{h}_k\|_{\mathcal{W}_\mu^{(n)}} \geq \frac{\mu(z_k)|u(z_k)|}{(1-|z_k|^2)^n}, \tag{48}$$

which along with (47) and letting $k \to \infty$ in inequality (48) and since $\{z_k\}$ is an arbitrary sequence such that $|z_k| \to 1$ as $k \to \infty$, implies that (46) holds for $i = n$. By the proof of the Theorem 3.1, we get that equality (46) holds for each $i \in \{0, 1, ..., n\}$. This completes the proof. □

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Inertial hybrid and shrinking projection methods for sums of three monotone operators

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Abstract.

In this paper, we introduce two iterative algorithms for finding the solution of the sum of two monotone operators by using hybrid projection methods and shrinking projection methods. Under some suitable conditions, we prove strong convergence theorems of such sequences to the solution of the sum of an inverse-strongly monotone and a maximal monotone operator. Finally, we present a numerical result of our algorithm which defined by the hybrid method.

Keywords: Hybrid projection methods, Shrinking projection methods, Monotone operators and Resolvent.

AMS Classification: 47J25, 47H05, 65K10, 65K15, 90C25.

1 Introduction

In this work, we consider the problem is finding a zero point of the sum of three monotone operators that is,

$$\text{find } z \in H \text{ such that } 0 \in (A+B+C)z, \tag{1.1}$$

where A is a multi-valued maximal monotone operator and B, C are two single monotone operators. In 2017, Davis and Yin [5] shown that the problem (1.1) can be related to a convex optimization problem, that is,

$$\text{minimize}_{x\in H} F(x) + G(x) + M(x),$$

where $A = \partial R, B = \partial S$ and $C = \nabla P$ with ∂R and ∂S denote the subdiferentials of R and S, respectively. The convex optimization problem involves several specific problems that have emerged in material sciences, medical and image processing and signal and image processing (see more in [6, 7]). Moreover, the monotone inclusion problems (1.1) includes some special cases. For example, when $B = 0$, problem (1.1) becomes find $x \in H$, such that

$$0 \in Ax + Cx. \tag{1.2}$$

If $C = 0$, problem (1.1) reduces to find $x \in H$, such that

$$0 \in Ax + Bx. \tag{1.3}$$

If $B = 0$ and $C = 0$, problem (1.1) reduces to the simple monotone inclusion find $x \in H$ such that

$$0 \in Ax. \tag{1.4}$$

So, we have the problem (1.1) is very important. Many researcher study and develop algorithm methods to solve the solution. Davis and Yin [5] introduced the fixed-point equation for solving monotone inclusions with three operators. In 2018, Cevher et al. [8] extended the three-operator splitting algorithm [5] from the determinist setting to the stochastic setting for solving the problem (1.1). Similarly, Yurtsever et al. [9] introduced a stochastic three-composite minimization algorithm to solve the convex minimization of the sum of three convex functions. In addition, Yu et al. [10] introduced an outer reflected forward-backward splitting algorithm to solve this problem as

$$x_{n+1} = J_r^A(x_n - \lambda Bx_n - \lambda Cx_n) - r(Bx_n - Bx_{n-1}). \tag{1.5}$$

The sequence $\{x_n\}$ converges weakly to solution of the problem (1.1).

Motivated and inspired by all above contributions, in this work, we will introduce two iterative algorithms for finding the solution of the sum of three monotone operators by using hybrid projection method and shrinking projection method. Under some suitable conditions, we prove strong convergence theorems of such sequences to the solution of the sum of three monotone operators. Finally, we will present a numerical result of our algorithm which defined by the hybrid method and applied to image inpainting.

2 Preliminaries

Let H be a real Hilbert space and $\mathcal{C}$ be a nonempty closed convex subset of H. Denote that $\to$ and $\rightharpoonup$ are a weak and strong convergence, respectively. I denotes the identity operator on H. For a given sequence, let $\omega_w(x_n) := \{x : \exists x_{n_k} \rightharpoonup x\}$ denote the weak ω-limit set of $\{x_n\}$.

Lemma 2.1. *Let $x \in H$ and $z \in \mathcal{C}$. Then we have*

(i) $z = P_{\mathcal{C}}(x)$ *if* $\langle x - z, z - y\rangle \geq 0$, *for all* $y \in \mathcal{C}$.

(ii) $\|P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y)\| \leq \|x - y\|$, *for all* $x, y \in H$

(iii) $\|x - P_{\mathcal{C}}(x)\|^2 \leq \|x - y\|^2 - \|y - P_{\mathcal{C}}(x)\|^2$ *for all* $y \in \mathcal{C}$.

Definition 2.2. *[1] Let $T : H \to H$ be a single-valued operator. Then*

(i) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in H.$$

(ii) T is said to be firmly nonexpansive if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \quad \text{for all } x, y \in H.$$

It is obvious that a firmly nonexpansive operator is nonexpansive.

(iii) T is said to be L-Lipschitz continuous, for some $L > 0$, if

$$\|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in H.$$

If $L = 1$, then T is nonexpansive.

(iv) T is said to be c-cocoercive (or c-inverse strongly monotone), if

$$\langle x-y, Tx-Ty\rangle \geq c\|Tx-Ty\|, \quad \text{for all } x, y \in H,$$

where $c > 0$.

(v) T is said to be monotone if

$$\langle Tx-Ty, x-y\rangle \geq 0, \quad \text{for all } x, y \in H.$$

Remark 2.3. *If C is c-cocoercive, then C is $1/c$-Lipschitz continuous and monotone. By using the L-Lipschitz continuity of B, we obtain that $B+C$ is $(L+1/c)$-Lipschitz continuous. Moreover, since C is c-cocoercive, we have C is monotone.*

Definition 2.4. *Let $A : H \to 2^H$ be a set-valued operator and the domain of A be $D(A) = \{x \in H : Ax \neq \emptyset\}$. The graph of A is denoted by $Graph(A) = \{(x,u) \in H \times H : u \in Ax\}$. Then the operator A is monotone if $\langle x_1 - x_2, z_1 - z_2\rangle \geq 0$ whenever $z_1 \in Ax_1$ and $z_2 \in Ax_2$.*

A monotone operator A is maximal if for any $(x,z) \in H \times H$ such that

$$\langle x-y, z-w\rangle \geq 0$$

for all $(y,w) \in Graph(A)$ implies $z \in Ax$.

Let A be a maximal monotone operator and $r > 0$. Then we can define the resolvent $J_r : R(I+rA) \to D(A)$ by

$$J_r^A = (I+rA)^{-1}$$

where $D(A)$ is the domain of A. We know that J_r^A is nonexpensive and we can study the other properties in references [12, 11, 13].

Lemma 2.5. *[4] Let $A : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a Lipschitz continuous and monotone mapping. Then $A+B$ is maximally monotone.*

Lemma 2.6. [2] *Let $\mathcal{C}$ be a closed convex subset of a real Hilbert space H, $x \in H$ and $z = P_{\mathcal{C}}x$. If $\{x_n\}$ is a sequence in $\mathcal{C}$ such that $\omega_w(x_n) \subset \mathcal{C}$ and*

$$\|x_n - x\| \leq \|x - z\|,$$

for all $n \geq 1$, then the sequence $\{x_n\}$ converges strongly to a point z.

Lemma 2.7. *[3] Let $\mathcal{C}$ be a closed convex subset a real Hilbert space H, and $x, y, z \in H$. Then, for given $a \in \mathbb{R}$, the set*

$$U = \{v \in \mathcal{C} : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

3 Hybrid Projection Methods

In this section, we introduce a intertial hybrid projection method and prove a strong convergence theorem.

(A1) $A : H \to 2^H$ is maximal monotone.

(A2) $B : H \to H$ is monotone and L-Lipchitz continuous, for some $L > 0$.

(A3) $C : H \to H$ is c-cocoercive.

(A4) $\Omega := (A+B+C)^{-1}(0) \neq \emptyset$.

The method is of the following form.

Algorithm 3.1 : Inertial hybrid projection algorithm (IHP Algorithm) Initialization : Choose $x_0, x_1 \in H, \alpha_n \in [0,1)$.
Iterative step : Compute x_{n+1} via

$$\begin{cases} w_n = x_n + \alpha_n(x_n + x_{n-1}), \\ y_n = J^A_{r_n}(w_n - r_nBw_n - r_nCw_n), \\ z_n = y_n - r_n(By_n - Bw_n), \\ C_n = \{z \in H : \|z_n - z\|^2 \leq \|w_n - z\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{3.1}$$

where

$$0 < r_n < \min\{c, \frac{1}{2L}\} \quad \text{and} \quad \lim_{n\to\infty} r_n = 0.$$

Lemma 3.1. *Let $\{z_n\}$ be a sequence generated by IHP Algorithm. If conditions $(A1)-(A4)$ hold, we have*

$$\|z_n - u\|^2 \leq \|w_n - u\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2, \quad \textit{for all } u \in \Omega. \tag{3.2}$$

Proof. Let $a_n = r_n^2\|By_n - Bw_n\|^2 - 2r_n\langle y_n - u, By_n - Bw_n\rangle$. Thus

$$\begin{aligned} \|z_n - u\|^2 &= \|y_n - r_n(By_n - Bw_n) - u\|^2 \\ &= \|y_n - u\|^2 - 2r_n\langle y_n - u, By_n - Bw_n\rangle + r_n^2\|By_n - Bw_n\|^2 \\ &= \|w_n - u\|^2 + \|y_n - w_n\|^2 + 2\langle w_n - u, y_n - w_n\rangle + a_n \\ &= \|w_n - u\|^2 + \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - u\rangle + a_n \\ &= \|w_n - u\|^2 - \|y_n - w_n\|^2 - 2\langle y_n - u, w_n - y_n + r_n(By_n - Bw_n)\rangle \\ &\quad + r_n^2\|By_n - Bw_n\|^2. \end{aligned} \tag{3.3}$$

Since B is L-Lipchitz continuous, we have

$$\|Bw_n - By_n\| \leq L\|w_n - y_n\|. \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\|z_n - u\|^2 \leq \|w_n - u\|^2 - (1 - L^2r_n^2)\|w_n - y_n\|^2 - 2\langle y_n - u, w_n - y_n + r_n(By_n - Bw_n)\rangle. \tag{3.5}$$

Since $y_n = J^A_{r_n}(w_n - r_nBw_n - r_nCw_n)$, we have $(I - r_nB - r_nC)w_n \in (I + r_nA)y_n$. So, we obtain

$$\frac{1}{r_n}(w_n - r_nBw_n - r_nCw_n - y_n) \in Ay_n. \tag{3.6}$$

Since $0 \in (A+B+C)u$, we have

$$-Bu - Cu \in Au. \tag{3.7}$$

Since the operator A is maximal monotone, one gets

$$\frac{1}{r_n}\langle w_n - r_nBw_n - r_nCw_n - y_n + r_nBu + r_nCu, y_n - u\rangle \geq 0.$$

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This implies that

$$\langle w_n - r_nBw_n - r_nCw_n - y_n + r_nBu + r_nCu, y_n - u\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle w_n - y_n + r_n(By_n - Bw_n), y_n - u\rangle &\geq \langle r_nBy_n - r_nBu - r_nCu + r_nCw_n, y_n - u\rangle \\ &= \langle r_nBy_n - r_nBu, y_n - u\rangle + \langle r_nCw_n - r_nCu, y_n - u\rangle \\ &\geq \langle r_nCw_n - r_nCu, y_n - u\rangle \end{aligned} \tag{3.8}$$

and since C is c-cococercive, we have

$$\begin{aligned}2r_n\langle Cw_n - Cu, y_n - u\rangle &= 2r_n\langle Cw_n - Cu, y_n - w_n\rangle + 2r_n\langle Cw_n - Cu, w_n - u\rangle \\ &\geq -2r_n\|Cw_n - Cu\|\|y_n - w_n\| + 2cr_n\|Cw_n - Cu\|^2 \\ &\geq -2cr_n\|Cw_n - Cu\|^2 - \frac{r_n}{2c}\|y_n - w_n\|^2 + 2cr_n\|Cw_n - Cu\|^2 \\ &= -\frac{r_n}{2c}\|y_n - w_n\|^2. \end{aligned} \tag{3.9}$$

Combining the equation (3.8) and (3.9), we obtain

$$-2\langle w_n - y_n + r_n(By_n - Bw_n), y_n - u\rangle \leq \frac{r_n}{2c}\|y_n - w_n\|^2. \tag{3.10}$$

Combining the equation (3.5) and (3.10), we obtain

$$\|z_n - u\|^2 \leq \|w_n - u\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2, \quad \text{for all } u \in \Omega.$$

This completed the proof.

Lemma 3.2. *Let the operators A, B and C satisfies conditions $(A1)-(A4)$. The three sequences $\{x_n\}, \{w_n\}$ and $\{y_n\}$ generated by IHP Algorithm. Assume that $\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|w_n - y_n\| = 0$. If a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to some $x^* \in H$, then $x^* \in \Omega$ where $\Omega := (A+B+C)^{-1}(0)$.*

Proof. Suppose that $(u, v) \in Graph(A+B+C)$. Thus $v - Bu - Cu \in Au$. Since $y_{n_k} = J^A_{r_{n_k}}(w_{n_k} - r_{n_k}Bw_{n_k} - r_{n_k}Cw_{n_k})$, we have $(I - r_n(B+C)) \in (I + r_{n_k}A)y_{n_k}$. This implies that

$$\frac{1}{r_{n_k}}(w_{n_k} - y_{n_k} - r_{n_k}(B+C)w_{n_k}) \in Ay_{n_k}.$$

By using the maximal monotonicity of A, we get

$$\langle u - y_{n_k}, v - Bu - Cu - \frac{1}{r_{n_k}}(w_{n_k} - y_{n_k} - r_{n_k}(B+C)w_{n_k})\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle u - y_{n_k}, v\rangle &\geq \langle u - y_{n_k}, (B+C)u + \frac{1}{r_{n_k}}(w_{n_k} - y_{n_k} - r_{n_k}(B+C)w_{n_k})\rangle \\ &= \langle u - y_{n_k}, (B+C)u - (B+C)w_{n_k}\rangle + \frac{1}{r_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle \\ &= \langle u - y_{n_k}, (B+C)u - (B+C)y_{n_k}\rangle + \langle u - y_{n_k}, (B+C)y_{n_k} - (B+C)w_{n_k}\rangle \\ &\quad + \frac{1}{r_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle \\ &\geq \langle u - y_{n_k}, (B+C)y_{n_k} - (B+C)w_{n_k}\rangle + \frac{1}{r_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle. \end{aligned}$$

Since $\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|w_n - y_n\| = 0$ and $B + C$ is Lipschitz continuous, we have $\lim_{n\to\infty}\|(B + C)y_{n_k} - (B + C)w_{n_k}\| = 0$. From $0 < r_n < \min\{c, \frac{1}{2L}\}$, one get

$$\lim_{n\to\infty}\langle u - y_{n_k}, v\rangle = \langle u - x^*, v\rangle \geq 0.$$

Since $A+B+C$ is maximal monotone, we have $0 \in (A+B+C)x^*$. We can conclude that $x^* \in \Omega$. This completed the proof.

Theorem 3.3. *Let the operators A, B and C satisfy conditions $(A1) - (A4)$. Then, the sequence $\{x_n\}$ generated by IHP Algorithm converges strongly to $x^* = P_\Omega(x_0)$.*

Proof. It is obvious that C_n and Q_n are closed convex for every $n \in \mathbb{N}$. First, we will prove that $\Omega \subset C_n$, for all $n \in \mathbb{N}$. By using Lemma 3.1, we obtain $\Omega \subset C_n$, for all $n \in \mathbb{N}$. Next, we prove that $\Omega \subset Q_n$ for all $n \in \mathbb{N}$ by the mathematical induction. By the definition of Q_n in IHP Algorithm, we have $Q_1 = H$. For $n = 1$, we note that $\Omega \subset H = Q_1$. Suppose that $\Omega \subset Q_k$ for some $k \in \mathbb{N}$. Since $C_k \cap Q_k$ is closed and convex, we can define

$$x_{k+1} = P_{C_k\cap Q_k}(x_0).$$

This implies that

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0 \quad \text{for all } z \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we have $\Omega \subset Q_{k+1}$. It follows that $\Omega \subset Q_n$, for all $n \in \mathbb{N}$. So, $\{x_n\}$ is well defined. Next, we show that $\{x_n\}$ is a bounded sequence and $\lim_{n\to\infty}\|w_n - y_n\|^2 = 0$. Since $\Omega \subset C_n \cap Q_n$, for all $n \in \mathbb{N}$, and $x_{n+1} = P_{C_n\cap Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \leq \|x^* - x_0\|.$$

This mean that $\{x_n\}$ is bounde, so $\{w_n\}$ is also bounded. From the definition of Q_n, we obtain $x_n = P_{Q_n}(x_0)$. Since $x_{n+1} \in Q_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \text{ for all } n \in \mathbb{N}.$$

This implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Therefore,

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}$$

It follows that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $x_{n+1} \in C_n \cap Q_n \subset C_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2.$$

Since $0 \leq r_n < \min\{c, \frac{1}{2L}\}$, we have $\|z_n - x_{n+1}\| \leq \|w_n - x_{n+1}\|$. Moreover, by the definition of $\{w_n\}$, we get

$$\|w_n - x_n\| = \|x_n + \alpha_n(x_n - x_{n+1}) - x_n\| = |\alpha_n|\|x_n - x_{n+1}\|$$

This implies that $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$. Therefore,

$$(1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2 \leq \|w_n - x_{n+1}\|^2 - \|z_n - x_{n+1}\|^2.$$

Since $\lim_{n\to\infty} r_n = 0$, we have $\lim_{n\to\infty}(1 - \frac{r_n}{2c} - L^2r_n^2) = 1$. It follows that $\lim_{n\to\infty}\|w_n - y_n\| = 0$. Finally, we show that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x_0)$. Let $x^* = P_\Omega(x_0)$. Therefore,

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \leq \|x_0 - x^*\|.$$

By Lemma 3.2, we have every sequential weakcluster point of the sequence $\{x_n\}$ belong to Ω. That is $\omega_w(x_n) \subset \Omega$. Hence by Lemma 2.6, we can conclude that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(x_0)$. This completes the proof. □

3 The Inertial Shrinking projection methods

In this section, we introduce a intertial shrinking projection method and prove a strong convergence theorem.

Algorithm 3.2 : Inertial shrinking projection algorithm (ISP Algorithm) Initialization : Choose $x_0, x_1 \in H, \alpha_n \in [0,1)$. Let $C_1 = H$

Iterative step : Compute x_{n+1} via

$$\begin{cases} w_n = x_n + \alpha_n(x_n + x_{n-1}), \\ y_n = J^A_{r_n}(w_n - r_nBw_n - r_nCw_n), \\ z_n = y_n - r_n(By_n - Bw_n), \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 \leq \|w_n - z\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases} \tag{3.11}$$

where

$$0 < r_n < \min\{c, \frac{1}{2L}\} \text{ and } \lim_{n\to\infty} r_n = 0.$$

Theorem 3.4. *Let the operators A, B and C satisfy conditions $(A1)-(A4)$. Then, the sequence $\{x_n\}$ generated by ISP Algorithm converges strongly to $x^* = P_\Omega(x_0)$.*

Proof. By Lemma 3.1, we obtain

$$\|z_n - u\|^2 \leq \|w_n - u\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2, \text{ for all } u \in \Omega.$$

It follows from $x_n = P_{C_n}(x_0)$ and $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$ that

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|.$$

On the other hand, since $x^* \in \Omega \in C_n$ and $x_n = P_{C_n}(x_0)$, we have $\|x_n - x_0\| \leq \|x^* - x_0\|$. Thus $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Similarly proof of Theorem 3.3, we can proof that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|w_n - y_n\| = 0$. By Lemma 2.6 and Lemma 3.2, we can conclude that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x_0)$. This completes the proof. $\square$

4 Numerical results

In this section, we firstly present by following the ideas of He et al. [14] and Dong et al. [15]. For $C = H$, we can write the algorithm 3.1 as in the following

$$\begin{cases} x_0, z_0 \in H, \\ y_n = \alpha_n z_n + (1 - \alpha_n)x_n, \\ z_{n+1} = J^A_{r_n}(y_n - r_n(B + C)y_n), \\ u_n = \alpha_n z_n + (1 - \alpha_n)x_n - z_{n+1}, \\ v_n = (\alpha_n\|z_n\|^2 + (1 - \alpha_n)\|x_n\|^2 - \|z_{n+1}\|^2)/2, \\ C_n = \{z \in C : \langle u_n, z\rangle \leq v_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = p_n, \quad \text{if } p_n \in Q_n, \\ x_{n+1} = q_n, \quad \text{if } p_n \notin Q_n, \end{cases} \tag{4.1}$$

where

$$\begin{aligned}
p_n &= x_0 - \frac{\langle u_n, x_0\rangle - v_n}{\|u_n\|^2} u_n, \\
q_n &= \left(1 - \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, w_n - p_n\rangle}\right) p_n + \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, w_n - p_n\rangle} w_n, \\
w_n &= x_n - \frac{\langle u_n, x_n\rangle - v_n}{\|u_n\|^2}.
\end{aligned}$$

Next, we will applies the above to image inpainting. We consider the degradation model that represents an actual image restoration problems or through the least useful mathematical abstractions thereof.

$$y = Hx + w$$

where y, H, x and w are the degraded image, degradation operator, or blurring operator; original image; and noise operator, respectively.

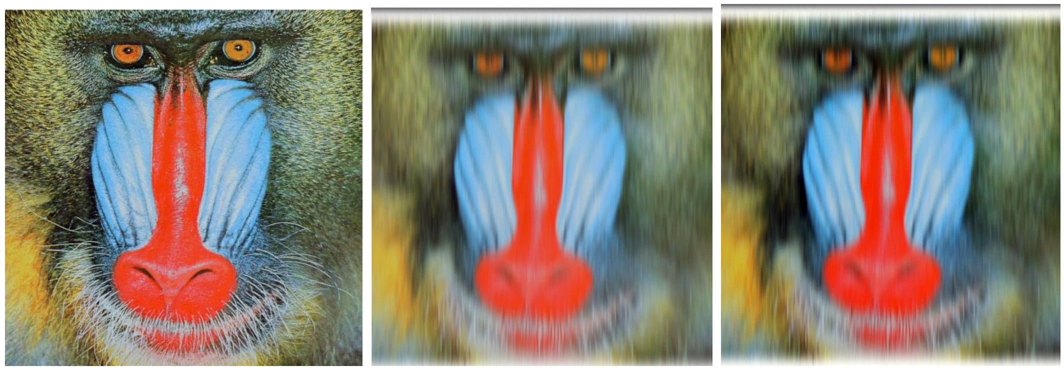
The regularized least-squares problem can be solve to obtain the reconstructed image is the following

$$\min\{\frac{1}{2}\|H(x) - y\|_2^2 + \mu\varphi(y)\} \tag{4.2}$$

where $\mu > 0$ is the regularization parameter and $\varphi(.)$ is the regularization functional. A well-known regularization function used to remove noise in the restoration problem is the l_1 norm, which is called Tikhonov regularization [**?**]. The problem (4.2) can be written in the form of the following problem as:

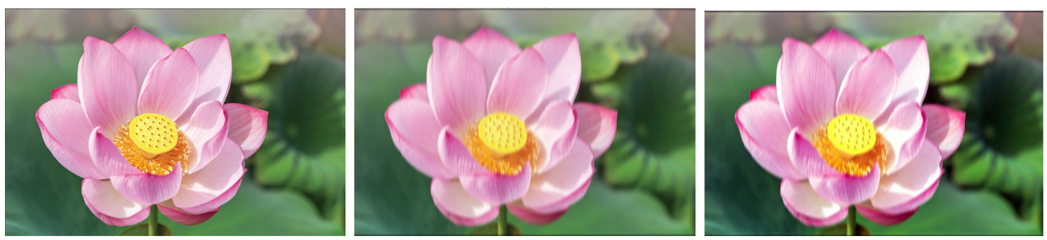
$$\min_{x\in\mathbb{R}^k}\{\frac{1}{2}\|H(x) - y\|_2^2 + \mu\|x\|_1\} \tag{4.3}$$

Note that problem (4.3) is a spacial case of the problem (1.1) by setting $A = \partial f(.), B = 0$, and $C = \nabla L(.)$ where $f(x) = \|x\|_1$ and $L(x) = \frac{1}{2}\|Hx - y\|_2^2$ This setting we have that $C(x) = \nabla L(x) = H'(Hx-y)$, where H' is a transpose of H. We begin the problem by choosing images and degrade them by random noise and different types of blurring. The random noise in this study is provided by Gaussian white noise of zero mean and 0.0001 variance. We solve the problem in (4.3) by using the above algorithm. We set $c = 70n^2, L = 0.001$ and $r_n = \frac{1}{100n+1}$. All the experiments were implemented in Matlab R2015 running on a Desktop with Intel(R) Core(TM) i5-7200u CPU 2.50 GHz, and 4 GB RAM. We obtain the following results.



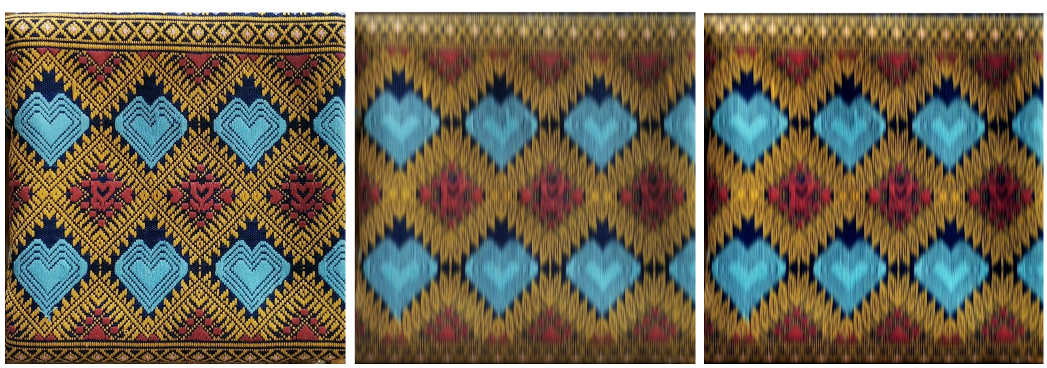
(a) Mandril **(b)** Gaussian blur **(c)** Our algorithm

Figure 1: Pictures of animals



(a) Lotus **(b)** Gaussian blur **(c)** Our algorithm

Figure 2: Pictures of lotus



(a) Fabric **(b)** Gaussian blur **(c)** Our algorithm

Figure 3: Pictures of Thai fabric

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Abstract Cauchy Problems in Two Variables and Tensor Product of Banach Spaces

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Abstract

In this paper we study linear abstract Cauchy problem in two variables. Theory of two-parameter semigroups of linear operators and tensor product of Banach spaces is needed to study the solution of such equation.

Keywords: Two-parameter semigroup, Abstract Cauchy problem, Operator valued function, Banach space.

MSC 2010: 26A33

1 Introduction

Ordinary and partial differential equations in which the unknown function and its derivatives take values in some abstract space such as Hilbert space or Banach space are called abstract differential equations. One of the most powerful tools for solving linear abstract differential equations is the method of semigroups of linear operators on Banach spaces. The basics of this method was originated, independently, by both E. Hille in (1948) [1] and K. Yosida in (1948) [2]. The power of the semigroup approach became clear through contribution by W. Feller in (1952, 1954) [3]. One of the classical vector valued differential equations that can be handled via semigroups of operators is the so called the abstract Cauchy problem which has the form:

$$\frac{du}{dt} = Au(t), \qquad t \geq 0,$$
$$u(0) = x,$$

where $A : D(A) \subseteq X \to X$ a linear operator of an appropriate type, $x \in X$ is given and $u : [0, \infty) \to X$ is the unknown function. For both linear and nonlinear abstract Cauchy problems, there are many applications in engineering and applied sciences. For any abstract Cauchy problem, one can associate a family of bounded linear operators that is known as a semigroup of operators.

Let X be a Banach space, and $L(X, X)$ be the space of all bounded linear operators on X. A one-parameter semigroup is a family of linear operators, namely, $\{T(t)\}_{t\geq 0} \subseteq L(X, X)$ such that

$$(i)\ T(0) = I, \text{ the identity operator of } X,$$
$$(ii)\ T(s + t) = T(s)T(t) \text{ fore very } t, s \geq 0.$$

If, in addition, for each fixed $x \in X$, $T(t)x \to x$ as $t \to 0+$, then the semigroup is called c_0-semigroup or strongly continuous semigroup.

The fact that every non-zero continuous real or complex function that satisfies the fact $g(s+t) = g(s)g(t)$ for every $t, s \geq 0$ has the form $g(t) = e^{ax}$, and that g is determined by the number $a = g'(0)$, reveals the association of an operator A to $\{T(t)\}_{t\geq 0}$ such that $Ax := \lim\limits_{t\to 0+} \frac{T(0+t)x - T(0)x}{t}$; $x \in D(A)$ and is called the infinitesimal generator of $\{T(t)\}_{t\geq 0}$.

In 2004, Khalil etal presented the definition of the infinitesimal generator for two parameter semigroups [4]. Recently, in 2019, M. Akkouchi et al.[5] carried out a theoretical framework for two-parameter semigroups of bounded linear operators on a Banach space. For more related works, we refer the reader to [6, 7, 8, 9].

The object of this paper is study the abstract Cauchy problem in two variables by considering two characteristics, namely, the concept of two-parameter semigroup of linear operators and the theory of tensor product of Banach spaces.

2 Preliminaries

Definition 1 *Let X be a Banach space, and $L(X,X)$ be the space of all bounded linear operators on X. A map defined by $T:[0,\infty)\times[0,\infty)\to L(X,X)$ is called a two-parameter semigroup or semigroup in two variables if*

(i) $T(0,0)=I$, *the identity operator of X,*

(ii) $T((s_1,t_1)+(s_2,t_2))=T(s_1,t_1)T(s_2,t_2)$ *fore very $t,s\geq 0$.*

Remark 1 *From the above definition, it follows that*

$$\begin{aligned} T(s,t) &= T((s,0)+(0,t)) \\ &= T(s,0)T(0,t). \end{aligned}$$

This implies that a semigroup in two variables is the product of two semigroups in one variable.

Definition 2 *The linear operator $L(1,1)$ defined by*

$$L(1,1)x = A_1x + A_2x,$$

where

$$\begin{aligned} A_1x &:= \lim_{s\to 0+}\tfrac{(T(s,0)-I)x}{s}, \\ A_2x &:= \lim_{t\to 0+}\tfrac{(T(0,t)-I)x}{t}. \end{aligned}$$

is the infinitesimal generator of the two-parameter semigroup $\{T(s,t)\}_{t,s\geq 0}$, A_1 and, A_2 are the generators of $T(s,0)$ and $T(0,t)$, respectively. We write L for $L(1,1)x$.

Theorem 1 *Let $\{T(s,t)\}_{t,s\geq 0}$ be a two-parameter semigroup and L be its infinitesimal generator. Then*

$$DT(s,t)\begin{pmatrix}1\\1\end{pmatrix}x = (A_1+A_2)\,T(s,t)x.$$

3 Main Results

3.1 Abstract Cauchy Problems in Two Variables

In this section, we provide the solution of the non-homogeneous abstract Cauchy problems in two variables.

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Consider the abstract Cauchy problem in two variables:

$$\frac{\partial u(s,t)}{\partial s}+\frac{\partial u(s,t)}{\partial t}=Lu(s,t)+f(s)+g(t), \tag{1}$$

where $L:X\to X$, closed linear operator with $Dom(L)\subseteq Rang(u)$. Let us assume the initial condition $u(0,0)=x_\circ$.

Procedure

(1) Consider the semigroup of operators

$$T(s,t)x=e^{(s+t)L}x,\ \ \forall s,t>0 \text{ and } x\in X. \tag{2}$$

To make life easy, let us assume L to be of exponential order, in the sense:

(i) L is densely defined,
(ii) $\sum_{i=m}^{n}\frac{(s+t)^n}{n!}\left\|L^n x\right\|<\infty,\ \forall x\in Dom(L)$,
(iii) If $x\in Dom(L)$, then $C\left(x,L\right)=\left\{x,\ Lx,\ L^2x,\cdots\right\}\subseteq Dom(L)$.

(2) Let

$$\begin{aligned} u(s,t) &= T(s,0)x_\circ+T(0,t)x_\circ \\ &\quad+\int_0^s T(s-\theta,0)f(\theta)\,d\theta \\ &\quad+\int_0^t T(0,t-w)g(w)\,dw. \end{aligned} \tag{3}$$

The claim is such $u(s,t)$ is a solution of (1).

Indeed, using the form of $T(s-\theta,0)x=e^{sL}e^{-\theta L}x$ for any x in the domain of L, and similarly for$T(0,t-w)$ we get

$$\begin{aligned} \frac{\partial u}{\partial s} &= LT(s,0)x_\circ+L\int_0^s T(s-\theta,0)f(\theta)\,d\theta+f(s) \\ \frac{\partial u}{\partial t} &= LT(0,t)x_\circ+L\int_0^t T(0,t-w)g(w)\,dw+g(t). \end{aligned} \tag{4}$$

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Hence,

$$\begin{aligned}\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} &= L\left[T(s,0)+T(0,t)\right]x_{\circ} \\ &\quad +L\int_0^s T(s-\theta,0)f(\theta)\,d\theta+f(s) \\ &\quad +L\int_0^t T(0,t-w)g(w)\,dw+g(t) \\ &= L\left[u(s,t)\right]+f(s)+g(t). \end{aligned} \tag{5}$$

Thus, such $u(s,t)$ given in (ref A-3) satisfies equation (1).

3.2 Tensor Product Abstract Cauchy Problem in Two Variables

Definition 3 *Let X and Y be any two Banach spaces and X^* is the dual space of X. For $x \in X$ and $y \in Y$, the operator $T : X^* \to Y$, defined by*

$$T(x^*) = x^*(x)y = \langle x,\, x^*\rangle\, y,$$

is a bounded one rank linear operator. We write $x \otimes y$ for such T . Such operators are called atoms.

Atoms are used in theory of best approximation in Banach spaces [10] and they are considered among the fundamental ingredients in the theory of tensor product of Banach spaces. For more related work on tensor product of Banach spaces, we refer reader to [10, 11, 12, 17] .

Definition 4 *Let X and Y be Banach spaces and $A : Dom(A) \subseteq X \to Y$ be linear. The operator A is called of exponential order if:*

(i) If $x \in Dom(A)$ then $\{x,\, Ax,\, A^2x,\, \cdots\} \subseteq Dom(A)$,
(ii) $\sum_{n=1}^{\infty} \frac{t^n}{n!} A^n x < \infty$, $\forall x \in Dom(A)$.

Clearly every bounded linear operator is of exponential order.
We will write $e^{tA}x$ for $\sum_{n=1}^{\infty} \frac{t^n}{n!} \|A^n x\|$.
Now, consider the abstract Cauchy problem.

$$\begin{aligned} u'(s)\otimes v'(t) &= Au(s)\otimes Bv(t)+f(s)\otimes Bv(t) \\ &\quad +Au(s)\otimes g(t)+f(s)\otimes g(t) \end{aligned} \tag{6}$$

5

where X and Y are Banach spaces, $u : [0,\infty) \to X$, $\upsilon : [0,\infty) \to Y$, $A : Dom(A) \subseteq X \to X$, $Rang(u) \subseteq Dom(A)$, $B : Dom(B) \subseteq Y \to Y$, $Rang(\upsilon) \subseteq Dom(B)$, A, B are closed operators of exponential order, and both $f : [0,\infty) \to X$, $g : [0,\infty) \to Y$ are given. Moreover, let us assume $u(0) = x_\circ$ and $\upsilon(0) = y_\circ$.

Procedure

Let

$$\begin{aligned} u(s) &= e^{sA} x_\circ + \int_0^s e^{(s-\theta)A} f(\theta)\, d\theta, && (7) \\ \upsilon(s) &= e^{tB} y_\circ + \int_0^t e^{(t-\omega)B} g(\omega)\, d\omega. && (8) \end{aligned}$$

Then, using the same technique as in section 1 for differentiating the integral we get:

$$\begin{aligned} u'(s) &= Ae^{sA} x_\circ + A\int_0^s e^{(s-\theta)A} f(\theta)\, d\theta + f(s) \\ &= Au(s) + f(s), && (9) \\ \upsilon'(s) &= Be^{tB} y_\circ + B\int_0^t e^{(t-\omega)B} g(\omega)\, d\omega + g(t) \\ &= B\upsilon(s) + g(t). && (10) \end{aligned}$$

Compiling both (9) and (10) in tensor product form yields (6).

Remark 2 *Both $e^{sA} x_\circ$ and $e^{tB} y_\circ$ have no meaning unless A and B are, respectively, of exponential order. Thus, one can summarize the above result as follows:*

If A and B are of exponential order, then (6) has a unique solution where $u(0) = x_\circ$ and $\upsilon(0) = y_\circ$.

4 Conclusions

This paper has successfully introduced analytical methods for handling non-homogeneous abstract Cauchy problem in two variables. The first method is based on the new concept of two-parameter semigroup of linear operators and its infinitesimal generator. While the second method utilizes the theory of tensor product of Banach spaces coupled with the tensor product properties to formulate a solution to a general tensor version of nonhomogeneous abstract Cauchy problem. In both cases the obtained results seem to be very interesting and promising in the sense that they could be extended for further classes of abstract Cauchy problems.

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APPROXIMATE EULER–LAGRANGE QUADRATIC MAPPINGS IN FUZZY BANACH SPACES *

ICK-SOON CHANG, HARK-MAHN KIM †, AND JOHN M. RASSIAS

ABSTRACT. For any rational numbers k, l with $kl(l-1) \neq 0$, we prove the generalized Hyers–Ulam stability of the Euler-Lagrange quadratic functional equation

$$f(kx+ly)+f(kx-ly)+2(l-1)[k^2f(x)-lf(y)] = l[f(kx+y)+f(kx-y)]$$

using both the direct method and fixed point method in fuzzy Banach spaces.

1. INTRODUCTION.

Some mathematicians have established fuzzy spaces with fuzzy norms on linear spaces from various points of view [2, 12, 18, 34]. Xiao and Zhu [34], Cheng and Mordeson [6], and Bag and Samanta [2, 3] gave the idea of fuzzy norms over linear spaces in such a manner that the corresponding fuzzy metric may be of Kramosil and Michalek type [17] and investigated some properties of fuzzy linear operators on fuzzy normed spaces.

Now, we introduce the definition of fuzzy normed spaces given in [2, 21, 22].

Definition 1.1 [2, 21, 22]**.** Let X be a real linear space. A function $N : X \times \mathbf{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbf{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ for $c \neq 0$;
(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N_5) $N(x, \cdot)$ is a non-decreasing function on $\mathbf{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbf{R}$.

The pair (X, N) is called a fuzzy normed (linear) space. The properties of fuzzy normed linear spaces and examples of fuzzy norms are given in [21, 23].

Definition 1.2 [2, 21, 22]**.** Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent or to converge to x if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$, and we denote it by N-$\lim_{n\to\infty} x_n = x$.

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Definition 1.3 [2, 21, 22]**.** Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space. They say that a mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at $x_0 \in X$ if for each sequence $\{x_n\}$ converging to each $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [3, 21]).

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [14] gave the first affirmative partial answer to the question of Ulam for additive mappings on Banach spaces. Hyers's theorem has been generalized by Aoki [1], Th.M. Rassias [28] and Găvruta [13] by considering an unbounded Cauchy difference. The classical functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

associated with the parallelogram equality $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ in inner product spaces, is called a quadratic functional equation, and every solution of the quadratic functional equation is said to be a quadratic mapping. First of all, the Hyers-Ulam stability problem for the quadratic functional equation has been established by Skof [32], Cholewa [7] and Czerwik [9]. In particular, Isac and Th.M Rassias [15] have provided a new application of fixed point theorems to prove the stability theory of functional equations. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 8, 31, 23, 27, 26]).

We recall the fixed point theorem from [19], which is needed in Section 3.

Theorem 1.4 [4, 19]**.** Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with the Lipschitz constant $L < 1$. Then, for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

On the other hand, J.M. Rassias investigated the Hyers–Ulam stability for the relative Euler–Lagrange functional equation

$$f(ax + by) + f(bx - ay) = (a^2 + b^2)[f(x) + f(y)]$$

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in [29, 30]. The stability problems of several quadratic functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [5, 24, 10, 11]). In the paper [16], the authors have proved the generalized Hyers–Ulam stability of the Euler–Lagrange quadratic functional equation

$$\begin{aligned} f(kx+ly)+f(kx-ly) \\ = kl[f(x+y)+f(x-y)]+2(k-l)[kf(x)-lf(y)] \end{aligned} \tag{1.1}$$

in fuzzy Banach spaces, where k, l are nonzero rational numbers with $k \neq l$.

Motivated to research stability results of the Euler–Lagrange functional equation, we investigate the generalized Hyers–Ulam stability of the following modified Euler–Lagrange functional equation

$$\begin{aligned} f(kx+ly)+f(kx-ly)+2(l-1)[k^2f(x)-lf(y)] \\ = l[f(kx+y)+f(kx-y)] \end{aligned} \tag{1.2}$$

using both the fixed point method and the direct method in fuzzy Banach spaces in the paper, where k, l are nonzero rational numbers with $kl(l-1) \neq 0$. Throughout the paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed space.

2. General solution of (1.2).

The following lemma can be found in the paper [16].

Lemma 2.1. [16] A mapping $f : X \to Y$ between linear spaces satisfies the functional equation

$$f(rx+y)+f(rx-y) = r[f(x+y)+f(x-y)]+2(r-1)[rf(x)-f(y)]$$

for any fixed rational numbers r with $r \neq 0, 1$ if and only if f is quadratic.

Now, we present the general solution of the functional equation (1.2).

Theorem 2.2. A mapping $f : X \to Y$ between vector spaces satisfies the functional equation (1.2) if and only if $f - f(0)$ is quadratic, where $f(0) = 0$ whenever $k^2 \neq l+1$.

Proof. First of all, replacing $(x, y) := (0, 0)$ in the functional equation (1.2), we find $f(0) = 0$ whenever $k^2 \neq l+1$. Substituting $(x, y) := (x, 0)$ in (1.2), we get $f(kx) = k^2 f(x)$ for all $x \in X$. Putting $(x, y) := (0, x)$ in (1.2), one has

$$f(lx)+f(-lx) = (2l^2-l)f(x)+lf(-x) \tag{2.1}$$

for all $x \in X$. Replacing x by $-x$ in (2.1), one gets

$$f(-lx)+f(lx) = (2l^2-l)f(-x)+lf(x) \tag{2.2}$$

for all $x \in X$. Subtracting equation (2.1) from (2.2), we find $f(-x) = f(x)$ and so $f(lx) = l^2 f(x)$ for all $x \in X$. Thus the equation (1.2) can be rewritten as

$$f(x+\frac{ly}{k})+f(x-\frac{ly}{k}) = l[f(x+\frac{y}{k})+f(x-\frac{y}{k})]-2(l-1)[f(x)-lf(\frac{y}{k})],$$

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which yields by switching (x,y) with (y,kx)

$$f(lx+y)+f(lx-y)=l[f(x+y)+f(x-y)]+2(l-1)[lf(x)-f(y)]$$

for all $x,y\in X$. Therefore, it follows from Lemma 2.1 that f is quadratic.

Conversely, if a mapping f is quadratic, then it is obvious that f satisfies the equation (1.2).

3. Stability of equation (1.2) by fixed point method.

For notational convenience, we define the difference operator $D_{kl}f : X^2\to Y$ of the equation (1.2) for a given mapping $f : X\to Y$ as

$$\begin{aligned} D_{kl}f(x,y) &:= f(kx+ly)+f(kx-ly)+2(l-1)[k^2f(x)-lf(y)] \\ &\quad -l[f(kx+y)+f(kx-y)] \end{aligned}$$

for all $x,y\in X$. Now, we are going to consider a stability problem concerning the stability of equation (1.2) by using the fixed point theorem for contraction mappings on generalized complete metric spaces.

Theorem 3.1. Assume that a mapping $f : X\to Y$ with $f(0)=0$ satisfies the functional inequality

$$N(D_{kl}f(x,y),t_1+t_2)\ge\min\{N'(\varphi(x),t_1^q),N'(\varphi(y),t_2^q)\} \tag{3.1}$$

for all $x,y\in X$ and all $t_i>0$ $(i=1,2)$, and for some $q>0$, and assume in addition that there exists a constant $s\in\mathbf{R}$ with $|s|\neq 1$, $0<|s|^{\frac{1}{q}}<k^2$ such that a constrained function $\varphi : X\to Z$ satisfies the inequality

$$N'(\varphi(kx),t)\ge N'(s\varphi(x),t), \tag{3.2}$$

for all $x\in X$ and all $t>0$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X\to Y$ satisfying the equation $D_{kl}Q(x,y)=0$ and the approximate functional inequality

$$\begin{aligned} N(f(x)-Q(x),t) &\ge \min\Big\{N'\Big(\frac{\varphi(x)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q},t^q\Big), \\ &\qquad N'\Big(\frac{\varphi(0)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q},t^q\Big)\Big\} \end{aligned} \tag{3.3}$$

near f for all $x\in X$ and all $t>0$.

Proof. We consider the set of functions

$$\Omega := \{g : X\to Y | g(0)=0\}$$

and define a generalized metric on Ω as follows:

$$\begin{aligned} d_\Omega(g,h) := \inf\Big\{K\in[0,\infty] : N(g(x)-h(x),Kt) &\ge \min\{N'(\varphi(x),t^q),N'(\varphi(0),t^q)\}, \\ &\forall x\in X,\forall t>0\Big\}. \end{aligned}$$

Then one can easily see that (Ω,d_Ω) is a complete generalized metric space [20].

Now, we define an operator $J : \Omega \to \Omega$ as

$$Jg(x) = \frac{g(kx)}{k^2}$$

for all $g \in \Omega$, $x \in X$.

We first prove that J is strictly contractive on Ω. For any $g, h \in \Omega$, let $\varepsilon \in [0, \infty)$ be any constant with $d_\Omega(g, h) \leq \varepsilon$. Then it follows from the use of (3.2) and the definition of $d_\Omega(g, h) \leq \varepsilon$ that

$$\begin{aligned} &N(g(x) - h(x), \varepsilon t) \geq \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\}, \\ &\quad\Rightarrow N\Big(\frac{g(kx)}{k^2} - \frac{h(kx)}{k^2}, \frac{|s|^{\frac{1}{q}}}{k^2}\varepsilon t\Big) \geq \min\{N'(\varphi(kx), |s|t^q), N'(\varphi(0), |s|t^q)\}, \\ &\quad\Rightarrow N\Big(Jg(x) - Jh(x), \frac{|s|^{\frac{1}{q}}}{k^2}\varepsilon t\Big) \geq \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\}, \\ &\quad\Rightarrow d_\Omega(Jg, Jh) \leq \frac{|s|^{\frac{1}{q}}}{k^2}\varepsilon, \qquad \forall x \in X, t > 0. \end{aligned}$$

Since ε is an arbitrary constant with $d_\Omega(g, h) \leq \varepsilon$, we see that for any $g, h \in \Omega$,

$$d_\Omega(Jg, Jh) \leq \frac{|s|^{\frac{1}{q}}}{k^2} d_\Omega(g, h),$$

which implies J is strictly contractive with the constant $\frac{|s|^{\frac{1}{q}}}{k^2} < 1$ on Ω.

We now want to show that $d_\Omega(f, Jf) < \infty$. If we put $y := 0$, $t_i := t$ $(i = 1, 2)$ in (3.1), then we arrive at

$$N\Big(f(x) - \frac{f(kx)}{k^2}, \frac{t}{|l-1|k^2}\Big) \geq \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\},$$

which yields $d_\Omega(f, Jf) \leq \frac{1}{|l-1|k^2} < \infty$, and so

$$d_\Omega(J^n f, J^{n+1} f) \leq d_\Omega(f, Jf) \leq \frac{1}{|l-1|k^2}$$

for all $n \in \mathbf{N}$. Now, applying the fixed point theorem of the alternative for contractions on generalized complete metric spaces due to Margolis and Diaz [19], we obtain the following approximate functional inequalities (a), (b) and (c):

(a) There is a mapping $Q : X \to Y$ with $Q(0) = 0$ such that

$$d_\Omega(f, Q) \leq \frac{1}{1 - \frac{|s|^{\frac{1}{q}}}{k^2}} d_\Omega(f, Jf) \leq \frac{1}{|l-1|(k^2 - |s|^{\frac{1}{q}})},$$

and thus Q is a fixed point of the operator J, that is, $\frac{1}{k^2}Q(kx) = JQ(x) = Q(x)$ for all $x \in X$. Thus we arrive at

$$\begin{aligned}
&N\Big(f(x)-Q(x), \frac{t}{|l-1|(k^2-|s|^{\frac{1}{q}})}\Big) \geq \min\{N'(\varphi(x),t^q), N'(\varphi(0),t^q)\},\\
&N(f(x)-Q(x),t) \geq \min\Big\{N'\Big(\varphi(x), |l-1|^q(k^2-|s|^{\frac{1}{q}})^q\, t^q\Big),\\
&\qquad\qquad N'\Big(\varphi(0), |l-1|^q(k^2-|s|^{\frac{1}{q}})^q\, t^q\Big)\Big\}
\end{aligned}$$

for all $t>0$ and all $x \in X$, which implies the approximation (3.3).

(b) Since $d_\Omega(J^n f, Q) \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}
&N\Big(\frac{f(k^n x)}{k^{2n}} - Q(x), t\Big)\\
&= N(f(k^n x) - Q(k^n x), k^{2n}t)\\
&\geq \min\Big\{N'\Big(\frac{\varphi(k^n x)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, k^{2nq}t^q\Big), N'\Big(\frac{\varphi(0)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, k^{2nq}t^q\Big)\Big\}\\
&\geq \min\Big\{N'\Big(\frac{\varphi(x)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, \Big(\frac{k^{2q}}{|s|}\Big)^n t^q\Big), N'\Big(\frac{\varphi(0)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, \Big(\frac{k^{2q}}{|s|}\Big)^n t^q\Big)\Big\}\\
&\to 1 \quad \text{as} \quad n \to \infty \quad \Big(\frac{k^{2q}}{|s|} > 1\Big)
\end{aligned}$$

for all $t > 0$ and all $x \in X$, that is, the mapping $Q : X \to Y$ given by

$$N\text{-}\lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}} = Q(x) \tag{3.4}$$

is well defined for all $x \in X$. In addition, it follows from the conditions (3.1), (3.2) and (N_4) that

$$\begin{aligned}
N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, t\Big) &\geq \min\Big\{N'\Big(\varphi(k^n x), \frac{k^{2nq}t^q}{2^q}\Big), N'\Big(\varphi(k^n y), \frac{k^{2nq}t^q}{2^q}\Big)\Big\}\\
&\geq \min\Big\{N'\Big(|s|^n\varphi(x), \frac{k^{2nq}t^q}{2^q}\Big), N'\Big(|s|^n\varphi(y), \frac{k^{2nq}t^q}{2^q}\Big)\Big\}\\
&\geq \min\Big\{N'\Big(\varphi(x), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{2^q}\Big), N'\Big(\varphi(y), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{2^q}\Big)\Big\}\\
&\to 1 \quad \text{as} \quad n \to \infty, \quad t > 0, \qquad (3.5)
\end{aligned}$$

for all $x \in X$. Therefore we obtain, by use of (N_4), (3.4) and (3.5),

$$\begin{aligned} N(D_{kl}Q(x,y),t) &\geq \min\Big\{N\Big(D_{kl}Q(x,y) - \frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big), \\ &\qquad\qquad N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big)\Big\} \\ &= N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big), \qquad \text{(for sufficiently large } n) \\ &\geq \min\Big\{N'\Big(\varphi(x), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{4^q}\Big), N'\Big(\varphi(y), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{4^q}\Big)\Big\} \\ &\to 1 \quad \text{as} \quad n \to \infty, \quad t > 0, \end{aligned}$$

which implies $D_{kl}Q(x,y) = 0$ by (N_2), and so the mapping Q is quadratic satisfying equation (1.2).

(c) The mapping Q is a unique fixed point of the operator J in the set $\Delta = \{g \in \Omega | d_\Omega(f,g) < \infty\}$. Thus, if we assume that there exists another Euler-Lagrange type quadratic mapping $Q' : X \to Y$ satisfying inequality (3.3), then

$$Q'(x) = \frac{Q'(kx)}{k^2} = JQ'(x), \quad d_\Omega(f, Q') \leq \frac{1}{|l-1|(k^2 - |s|^{\frac{1}{q}})} < \infty,$$

and so Q' is a fixed point of the operator J and $Q' \in \Delta = \{g \in \Omega | d_\Omega(f,g) < \infty\}$. By the uniqueness of the fixed point of J in Δ, we find that $Q = Q'$, which proves the uniqueness of Q satisfying inequality (3.3). This ends the proof of the theorem.

We observe that if $0 < |s| < 1$ in Theorem 3.1, then

$$\min\big\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\big\} = N'(\varphi(x), t^q)$$

for all $x \in X$ and all $t > 0$ since $N'(\varphi(0), t^q) \geq N'\Big(\varphi(0), \frac{t^q}{|s|^n}\Big) \to 1$ as $n \to \infty$ by the condition (3.2).

Theorem 3.2 Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl}f(x,y), t_1 + t_2) \geq \min\{N'(\varphi(x), t_1^q), N'(\varphi(y), t_2^q)\}$$

for all $x, y \in X$ and all $t_i > 0$ $(i = 1, 2)$ and for some $q > 0$, and furthermore assume that there exists a constant $s \in \mathbf{R}$ with $|s| \neq 1$, $|s|^{\frac{1}{q}} > k^2$ such that a constrained function $\varphi : X \to Z$ satisfies

$$N'\Big(\varphi\Big(\frac{x}{k}\Big), t\Big) \geq N'\Big(\frac{1}{s}\varphi(x), t\Big)$$

for all $x \in X$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x,y) = 0$ and the approximate functional inequality

$$N(f(x)-Q(x),t) \geq \min\Big\{N'\Big(\frac{\varphi(x)}{|l-1|^q(|s|^{\frac{1}{q}}-k^2)^q},t^q\Big), N'\Big(\frac{\varphi(0)}{|l-1|^q(|s|^{\frac{1}{q}}-k^2)^q},t^q\Big)\Big\}, \tag{3.6}$$

for all $t>0$ and all $x \in X$.

Proof. Finally, applying the same argument as in the proof of Theorem 3.1, we can find a mapping $Q : X \to Y$ defined by

$$N\text{-}\lim_{n\to\infty} k^{2n} f(\frac{x}{k^n}) = Q(x)$$

satisfying the equation $D_{kl}Q(x,y)=0$ and the approximate functional inequality (3.6) near f.

4. Stability of equation (1.2) by direct method.

In the following, we are going to investigate alternatively generalized Hyers–Ulam stability of the Euler–Lagrange functional equation (1.2) via the direct method in fuzzy Banach spaces.

Theorem 4.1. Assume that a mapping $f : X \to Y$ with $f(0)=0$ satisfies the inequality

$$N(D_{kl}f(x,y),t) \geq N'(\varphi(x,y),t) \tag{4.1}$$

and assume in addition that there exists a constant $s \in \mathbf{R}$ subject to $0<|s|<k^2$ such that a constrained function $\varphi : X^2 \to Z$ satisfies the functional inequality

$$N'(\varphi(kx,ky),t) \geq N'(s\varphi(x,y),t) \tag{4.2}$$

for all $x \in X$ and all $t>0$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x,y)=0$ and the approximate inequality

$$N(f(x)-Q(x),t) \geq N'\Big(\frac{\varphi(x,0)}{2|l-1|(k^2-|s|)},t\Big), \quad t>0, \tag{4.3}$$

for all $x \in X$.

Proof. It follows from the assumption (4.2) that

$$\begin{aligned} N'(\varphi(k^n x,k^n y),t) &\geq N'(s^n\varphi(x,y),t) \\ &= N'\Big(\varphi(x,y),\frac{t}{|s|^n}\Big), \quad t>0, \end{aligned}$$

which yields

$$N'(\varphi(k^n x,k^n y),|s|^n t) \geq N'(\varphi(x,y),t), \quad t>0, \tag{4.4}$$

for all $x, y \in X$. Putting $(x, y) := (x, 0)$ in (4.1), we have

$$N(2(l-1)f(kx) - 2(l-1)k^2 f(x), t) \geq N'(\varphi(x,0), t),$$

$$\text{or,} \quad N\Big(f(x) - \frac{f(kx)}{k^2}, \frac{t}{2|l-1|k^2}\Big) \geq N'(\varphi(x,0), t) \tag{4.5}$$

for all $x \in X$. Therefore it follows from (4.4), (4.5) that

$$N\Big(\frac{f(k^n x)}{k^{2n}} - \frac{f(k^{n+1}x)}{k^{2(n+1)}}, \frac{|s|^n t}{2|l-1|k^{2(n+1)}}\Big) \geq N'(\varphi(k^n x, 0), |s|^n t) \geq N'(\varphi(x,0), t)$$

for all $x \in X$ and any integer $n \geq 0$. Thus, we deduce the functional inequality

$$\begin{aligned} N\Big(f(x) - \frac{f(k^n x)}{k^{2n}}, \sum_{i=0}^{n-1} \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \qquad (4.6)\\ = N\Big(\sum_{i=0}^{n-1}\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(i+1)}}\Big), \sum_{i=0}^{n-1}\frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \\ \geq \min_{0\leq i\leq n-1}\Big\{N\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(i+1)}}, \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big)\Big\} \\ \geq N'(\varphi(x,0), t), \quad t > 0, \end{aligned}$$

which implies

$$\begin{aligned} &N\Big(\frac{f(k^m x)}{k^{2m}} - \frac{f(k^{m+p}x)}{k^{2(m+p)}}, \sum_{i=m}^{m+p-1}\frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \\ &= N\Big(\sum_{i=m}^{m+p-1}\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(i+1)}}\Big), \sum_{i=m}^{m+p-1}\frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \\ &\geq \min_{m\leq i\leq m+p-1}\Big\{N\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(n+1)}}, \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big)\Big\} \\ &\geq \quad N'(\varphi(x,0), t), \quad t > 0, \end{aligned}$$

for all $x \in X$ and any integers $p > 0, m \geq 0$. Therefore, one concludes

$$N\Big(\frac{f(k^m x)}{k^{2m}} - \frac{f(k^{m+p}x)}{k^{2(m+p)}}, t\Big) \geq N'\Big(\varphi(x,0), \frac{t}{\sum_{i=m}^{m+p-1}\frac{|s|^i}{2|l-1|k^{2(i+1)}}}\Big) \tag{4.7}$$

for all $x \in X$ and any integers $p > 0, m \geq 0, t > 0$. Since $\sum_{i=m}^{m+p-1}\frac{|s|^i}{k^{2i}}$ is a convergent series, we know that the sequence $\{\frac{f(k^n x)}{k^{2n}}\}$ is Cauchy in the fuzzy Banach space (Y, N), and so it converges in Y. Therefore a mapping $Q : X \to Y$ defined by

$$Q(x) := N\text{-}\lim_{n\to\infty}\frac{f(k^n x)}{k^{2n}} \Leftrightarrow \lim_{n\to\infty} N\Big(\frac{f(k^n x)}{k^{2n}} - Q(x), t\Big) = 1, \ \forall t > 0,$$

is well defined for all $x \in X$. In addition, we see from (4.6) that

$$N\Big(f(x) - \frac{f(k^n x)}{k^{2n}}, t\Big) \geq N'\Big(\varphi(x,0), \frac{t}{\sum_{i=0}^{n-1}\frac{|s|^i}{2|l-1|k^{2(i+1)}}}\Big), \tag{4.8}$$

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and thus for any ε with $0 < \varepsilon < 1$ the following inequality

$$\begin{aligned} N(f(x) - Q(x), t) &\geq \min\Big\{N\Big(f(x) - \frac{f(k^n x)}{k^{2n}}, (1-\varepsilon)t\Big), \\ &\qquad\qquad N\Big(\frac{f(k^n x)}{k^{2n}} - Q(x), \varepsilon t\Big)\Big\} \\ &\geq N'\Big(\varphi(x,0), \frac{(1-\varepsilon)t}{\sum_{i=0}^{n-1} \frac{|s|^i}{2|l-1|k^{2(i+1)}}}\Big) \\ &\geq N'(\varphi(x,0), 2|l-1|(1-\varepsilon)(k^2 - |s|)t), \end{aligned} \tag{4.9}$$

holds good for sufficiently large n, and for all $x \in X$ and all $t > 0$. Since ε is arbitrary and N' is a left continuous function, we obtain

$$N(f(x) - Q(x), t) \geq N'(\varphi(x,0), 2|l-1|(k^2 - |s|)t), \quad t > 0,$$

for all $x \in X$, which yields the approximation (4.3).

On the other hand, it is clear from (4.1) and (N_5) that the relation

$$\begin{aligned} N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, t\Big) &\geq N'(\varphi(k^n x, k^n y), k^{2n} t) \\ &\geq N'\Big(\varphi(x,y), \frac{k^{2n}}{|s|^n} t\Big) \\ &\to 1 \quad \text{as} \quad n \to \infty \end{aligned}$$

holds for all $x, y \in X$ and all $t > 0$. Therefore, we figure out by definition of $\lim_{n\to\infty} N(\frac{f(k^n x)}{k^{2n}} - Q(x), t) = 1$ for all $(t > 0)$ that

$$\begin{aligned} N(D_{kl}Q(x,y), t) &\geq \min\Big\{N\Big(D_{kl}Q(x,y) - \frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big), \\ &\qquad\qquad N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big)\Big\} \\ &= N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big) \qquad \text{(for sufficiently large } n\text{)} \\ &\geq N'\Big(\varphi(x,y), \frac{k^{2n}}{2|s|^n} t\Big), \quad t > 0 \\ &\to 1 \quad \text{as} \quad n \to \infty, \end{aligned}$$

which implies $D_{kl}Q(x,y) = 0$ by (N_2). Thus we find that Q is a quadratic mapping satisfying equation (1.2) and inequality (4.3) near the approximate quadratic mapping $f : X \to Y$.

To prove the uniqueness, we now assume that there is another quadratic mapping $Q' : X \to Y$ which satisfies the approximate inequality (4.3). Then it follows from the equality $Q'(k^n x) = k^{2n} Q'(x)$, $Q(k^n x) = k^{2n} Q(x)$ and (4.3) that

$$\begin{aligned} N(Q(x) - Q'(x), t) &= N\Big(\frac{Q(k^n x)}{k^{2n}} - \frac{Q'(k^n x)}{k^{2n}}, t\Big) \\ &\geq \min\Big\{N\Big(\frac{Q(k^n x)}{k^{2n}} - \frac{f(k^n x)}{k^{2n}}, \frac{t}{2}\Big), N\Big(\frac{f(k^n x)}{k^{2n}} - \frac{Q'(k^n x)}{k^{2n}}, \frac{t}{2}\Big)\Big\} \\ &\geq N'(\varphi(k^n x, 0), (k^2 - |s|)k^{2n} t) \\ &\geq N'\Big(\varphi(x, 0), \frac{(k^2 - |s|)k^{2n} t}{|s|^n}\Big), \quad t > 0, \end{aligned}$$

for all $n \in \mathbf{N}$, which tends to 1 as $n \to \infty$ by (N_5). Therefore one obtains $Q(x) = Q'(x)$ for all $x \in X$, completing the proof of uniqueness. This completes the proof of the theorem.

We remark that if $k = 1$ in Theorem 4.1, then

$$N'(\varphi(x, y), t) \geq N'\big(\varphi(x, y), \frac{t}{|s|^n}\big) \to 1$$

as $n \to \infty$, and so $\varphi(x, y) = 0$ for all $x, y \in X$. Hence, $D_{kl} f(x, y) = 0$ for all $x, y \in X$ and thus f is itself an Euler–Lagrange quadratic mapping.

Theorem 4.2. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl} f(x, y), t) \geq N'(\varphi(x, y), t) \tag{4.10}$$

and assume in addition that there exists a constant $s \in \mathbf{R}$ subject to $|s| > k^2$ such that a constrained function $\varphi : X^2 \to Z$ satisfies the inequality

$$N'\Big(\varphi\Big(\frac{x}{k}, \frac{y}{k}\Big), t\Big) \geq N'\Big(\frac{1}{s}\varphi(x, y), t\Big), \quad t > 0, \tag{4.11}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl} Q(x, y) = 0$ and the approximate inequality

$$N(f(x) - Q(x), t) \geq N'\Big(\frac{\varphi(x, 0)}{2|l - 1|(|s| - k^2)}, t\Big), \quad t > 0, \tag{4.12}$$

for all $x \in X$.

Proof. It follows from (4.5) and (4.11) that

$$N\Big(f(x) - k^2 f\Big(\frac{x}{k}\Big), \frac{t}{2|l - 1||s|}\Big) \geq N'(\varphi(x, 0), t), \quad t > 0,$$

for all $x \in X$. Therefore it follows that

$$N\Big(f(x) - k^{2n} f\Big(\frac{x}{k^n}\Big), \sum_{i=0}^{n-1} \frac{k^{2i}}{2|l - 1||s|^{i+1}} t\Big) \geq N'(\varphi(x, 0), t), \quad t > 0,$$

for all $x \in X$ and any integer $n > 0$. Thus we see from the last inequality that

$$\begin{aligned} N\Big(f(x) - k^{2n} f\Big(\frac{x}{k^n}\Big), t\Big) &\geq N'\Big(\varphi(x,0), \frac{t}{\sum_{i=0}^{n-1} \frac{k^{2i}}{2|l-1||s|^{i+1}}}\Big) \\ &\geq N'(\varphi(x,0), 2|l-1|(|s|-k^2)t), \quad t > 0. \end{aligned}$$

The remaining assertions go through the corresponding part of Theorem 4.1 by the similar way.

We also observe that if $k = 1$ in Theorem 4.2, then

$$N'(\varphi(x,y), t) \geq N'(\varphi(x,y), |s|^n t) \to 1$$

as $n \to \infty$, and so $\varphi(x,y) = 0$ for all $x, y \in X$. Hence, $D_{kl} f = 0$ and thus f is itself an Euler–Lagrange quadratic mapping.

Corollary 4.3. Let X be a normed space and $(\mathbf{R}, N')$ be a fuzzy normed space. Assume that there exist real numbers $\theta_1 \geq 0$, $\theta_2 \geq 0$ and that p is a real number such that either $0 < p < 2$ or $p > 2$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl} f(x,y), t) \geq N'(\theta_1 \|x\|^p + \theta_2 \|y\|^p, t)$$

for all $x, y \in X$ and all $t > 0$, then we can find a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl} Q(x,y) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq \begin{cases} N'\big(\frac{\theta_1 \|x\|^p}{2|l-1|(k^2 - |k|^p)}, t\big), & \text{if } 0 < p < 2,\ |k| > 1,\ (p > 2,\ |k| < 1) \\ N'\big(\frac{\theta_1 \|x\|^p}{2|l-1|(|k|^p - k^2)}, t\big), & \text{if } p > 2,\ |k| > 1,\ (0 < p < 2,\ |k| < 1) \end{cases}$$

for all $x \in X$ and all $t > 0$.

Proof. Taking $\varphi(x,y) = \theta_1 \|x\|^p + \theta_2 \|y\|^p$ and applying Theorem 4.1 and Theorem 4.2, we obtain the desired approximations, respectively.

Corollary 4.4. Assume that for $k \neq 1$, there exists a real number $\theta \geq 0$ such that the mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl} f(x,y), t) \geq N'(\theta, t)$$

for all $x, y \in X$ and all $t > 0$. Then we can find a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl} Q(x,y) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq N'\Big(\frac{\theta}{2|l-1||k^2-1|}, t\Big)$$

for all $x \in X$ and all $t > 0$.

We remark that if $\theta = 0$, then $N(D_{kl} f(x,y), t) \geq N'(0, t) = 1$, and so $D_{kl} f(x,y) = 0$. Thus we get $f = Q$ is itself an Euler–Lagrange quadratic mapping.

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On linear fuzzy real numbers

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Abstract. In this paper we introduce the notion of liner fuzzy real numbers, and show that the set of all positive (or negative) symmetric linear fuzzy real numbers forms a semiring. Moreover, we discuss a complex transform of linear fuzzy real numbers.

1. Introduction

A. Kaufmann and M. M. Gupta [2] introduced the notion of a trapezoidal fuzzy number, and D. Dubois and H. Prade [1] generalized the notion of trapezoidal fuzzy numbers. X. F. Zhang and G. W. Meng [12] introduced the notion of an isoceles triangular fuzzy number, and discussed the simplification of addition and subtraction operations of fuzzy numbers. A. Kumar et al. [2] studied an RM approach for ranking of generalized trapezoidal fuzzy numbers, and showed that the ranking function satisfies all the reasonable properties of fuzzy quantities proposed by X. Wang and E. E. Kerre [11]. Neggers and Kim researched fuzzy posets [7] and created Linear Fuzzy Real numbers [8]. Linear Fuzzy Real numbers were used by Monk [4]. In [9], Rogers et al. focused on linear fuzzy programming problems. Rogers [10] focused on solving and manipulating Fuzzy Nonlinear problems in the Linear Fuzzy Real number system using the Gradient Descent. In texts on fuzzy logic, fuzzy subsets of the real numbers may also be referred to as *fuzzy real numbers*, thus obviating the needs to talk about fuzzy subsets of the real numbers. Actually, equating these concepts may be a disadvantage since in some way we expect numbers, whether fuzzy or not, to behave differently from (sub)sets, whether fuzzy or not. It is with this in mind that we seek to introduce among several models of systems of fuzzy real numbers, the system of *linear fuzzy real numbers* discussed below. For general concepts for fuzzy set theory we refer to [5, 6].

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2. Linear fuzzy real numbers

A mapping $\mu : \mathbf{R} \to [0,1]$ is called a *linear fuzzy real number* [8, 4] if there is a triple of real numbers a, b, c $(a \le b \le c)$ such that

(1) $\mu(b) = 1$,
(2) $\mu(x) = 0$ if $x < a$ or $x > c$,
(3) $\mu(x) = \frac{x-a}{b-a}$ if $a \le x < b$,
(4) $\mu(x) = \frac{c-x}{c-b}$ if $b < x \le c$.

We denote such a linear fuzzy real number by a triple $\mu =< a, b, c >$ or $\mu = \mu(a, b, c)$ where $a \le b \le c$. Notice that the integral $\int_{-\infty}^{\infty} \mu(t)dt = \frac{b-a}{2} + \frac{c-b}{2} = \frac{c-a}{2}$, i.e., if the *goodness* of the fuzzy subset $\mu : (-\infty, \infty) \to [0,1]$ is defined by

$$G(\mu) := \frac{e^{\gamma(\mu)} - 1}{e^{\gamma(\mu)} + 1}, \quad \gamma(\mu) = [\int_{-\infty}^{\infty} \mu(t)dt]^{-1},$$

then in the case of a linear fuzzy real number $\mu =< a, b, c >$, it follows that

$$G(\mu) = \frac{e^{\frac{2}{c-a}} - 1}{e^{\frac{2}{c-a}} + 1}$$

In particular, if we let $c - a \to 0^+$, then $G(\mu) \to 1$, so that for any $\mu =< a, a, a >$, we set $G(\mu) = 1$. On the other hand, if $c - a \to \infty$, then $G(\mu) \to 0^+$, and $0 \le G(\mu) \le 1$. If $\mu =< a, a, a >$, then $\mu(t) = 0$ if $t \ne a$, while $\mu(a) = 1$, i.e., $\mu = \delta_a$, the *characteristic function* of the real number a. For any two linear fuzzy real numbers $\mu_i =< a_i, b_i, c_i >$ $(i = 1, 2)$, we define $\mu_1 + \mu_2 :=< a_1 + a_2, b_1 + b_2, c_1 + c_2 >$.

Theorem 2.1. *If $\mu_i =< a_i, b_i, c_i >$ $(i = 1, 2)$ be linear fuzzy real numbers, then $G(\mu_1 + \mu_2) \le \min\{G(\mu_1), G(\mu_2)\}$.*

Proof. If we let $F(x) := \frac{e^{\frac{2}{x}}-1}{e^{\frac{2}{x}}+1}$ then

$$\begin{aligned} F'(x) &= \frac{(e^{\frac{2}{x}} + 1)e^{\frac{2}{x}}(-\frac{2}{x^2}) - (e^{\frac{2}{x}} - 1)e^{\frac{2}{x}}(-\frac{2}{x^2})}{(e^{\frac{2}{x}} + 1)^2} \\ &= \frac{-4e^{\frac{2}{x}}}{x^2(e^{\frac{2}{x}} + 1)^2} \\ &< 0. \end{aligned}$$

Hence, if $x_1 \geq x_2$, then $F(x_1) \leq F(x_2)$. Since $(c_1+c_2)-(a_1+a_2) \geq c_i - a_i$, $(i=1,2)$,

$$\begin{aligned} G(\mu_1+\mu_2) &= F((c_1+c_2)-(a_1+a_2)) \\ &\leq F(c_i-a_i) \\ &= G(\mu_i), \end{aligned}$$

proving the theorem. □

Corollary 2.2. *If $G(\mu_1+\mu_2) = G(\mu_1)$, then μ_2 is the characteristic function of a real number.*

Proof. If $G(\mu_1+\mu_2) = G(\mu_1)$, then $(c_1+c_2)-(a_1+a_2) = c_1-a_1$ and hence $c_2-a_2=0$, which means that $\mu_2 =< b_2, b_2, b_2 >= \delta_{b_2}$ for some $b_2 \in \mathbf{R}$. □

Proposition 2.3. *Let $\mu_i =< a_i, b_i, c_i >$, $(i=1,2,3)$, be linear fuzzy real numbers and $\delta_0 =< 0,0,0 >$. Then*
(i) $\mu_i + \delta_0 = \mu_i$,
(ii) $\mu_1+\mu_2 = \mu_2+\mu_1$,
(iii) $(\mu_1+\mu_2)+\mu_3 = \mu_1+(\mu_2+\mu_3)$,
(iv) *If $\mu_1+\mu_2=\delta_0$, then $\mu_1 = \delta_{b_1}, \mu_2 = \delta_{b_2}$ and $b_2=-b_1$.*

Proof. (iv). If we let $\mu_1+\mu_2=\delta_0$, then $G(\mu_1+\mu_2) = G(\delta_0) = 1 \leq \min\{G(\mu_1), G(\mu_2)\}$ by Theorem 2.1. Hence $G(\mu_1) = G(\mu_2) = 1$, i.e., $\mu_1 = \delta_{b_1}$ and $\mu_2 = \delta_{b_2}$ for some $b_1, b_2 \in \mathbf{R}$ and obviously $b_2 = -b_1$. □

By Proposition 2.3, we obtain the following theorem.

Theorem 2.4. *If $\mathcal{L}$ is the collection of all linear fuzzy real numbers with the operation "+", then $(\mathcal{L}, +)$ is a commutative semigroup with a neutral element δ_0.*

Remark. In view of Theorem 2.1, there exists a function $G : \mathcal{L} \to [0,1]$ such that $G(\mu_1+\mu_2) \leq \min\{G(\mu_1), G(\mu_2)\}$, denoting "the goodness" of the linear fuzzy real number.

Let $\mu_i :=< a_i, b_i, c_i >$, $(i=1,2)$, be linear fuzzy real numbers. We define a new linear fuzzy real number $\mu_1 \ominus \mu_2 :=< a_1-a_2, b_1-b_2, c_1-c_2 >$ when $a_1-a_2 \leq b_1-b_2 \leq c_1-c_2$, which is called the *subtraction* of μ_1 by μ_2.

Proposition 2.5. *If $\mu_1 \ominus \mu_2 = \delta_b$ for some real number b, then $G(\mu_1) = G(\mu_2)$.*

Proof. If $\mu_1 \ominus \mu_2 = \delta_b$, then $a_1-a_2 = b_1-b_2 = c_1-c_2 = b$, i.e., μ_1 and μ_2 have the same "shape", and $G(\mu_1) = G(\mu_2)$. □

Proposition 2.6. *If $\mu_1 \ominus \mu_2$ is defined, then $G(\mu_1) \leq \min\{G(\mu_1 \ominus \mu_2), G(\mu_2)\}$.*

Proof. If $\mu_1 \ominus \mu_2$ is defined, then $\mu_1 = (\mu_1 \ominus \mu_2) + \mu_2$. By applying Theorem 2.1, we obtain the result. □

Remark. The fact $G(\mu_1) \le G(\mu_2)$ is not sufficient to determine that $\mu_1 \ominus \mu_2$ is defined. For example, consider $\mu_1 =< 2, 3, 7 >$ and $\mu_2 =< 1, 3, 4 >$. Then $G(\mu_1) = \frac{e^{2/5}-1}{e^{2/5}+1} \le \frac{e^{2/3}-1}{e^{2/3}+1} = G(\mu_2)$, but $\mu_1 \ominus \mu_2$ is not defined, since $2-1 > 3-3, 3-3 < 7-4$.

3. Multiplcations of linear fuzzy real numbers

Let $\mathcal{A}$ be the set of all linear fuzzy real numbers $\mu =< a, b, c >$ with $a \ne c$. Given $\mu_i =< a_i, b_i, c_i > \in \mathcal{A}$ $(i = 1, 2)$, we construct $\mu_1 \odot \mu_2$ as follows:

$$\begin{aligned} a &= \inf\{t_1t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ c &= \sup\{t_1t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ b &= \frac{a+c}{2}\{\frac{b_1-a_1}{c_1-a_1} + \frac{b_2-a_2}{c_2-a_2}\}. \end{aligned}$$

For example, $< -3, -2, -1 > \odot < -5, -1, 4 >=< -12, b, 15 >$ where $b = \frac{3}{2}\{\frac{1}{2} + \frac{4}{9}\} = \frac{17}{12}$, and $< -3, -2, 2 > \odot < -5, -1, 4 >=< -12, \frac{29}{30}, 15 >$.

Remark. The associative law for the product $\odot$ fails for linear fuzzy real numbers. Consider $\mu_1 =< -10, -9, -1 >, \mu_2 =< -8, 1, 2 >$ and $\mu_3 =< 1, 4, 5 >$. Then $(\mu_1 \odot \mu_2) \odot \mu_3 =< -20, \frac{91}{3}, 80 > \odot < 1, 4, 5 >=< -100, 94, 400 >$, but $\mu_1 \odot (\mu_2 \odot \mu_3) =< -10, -9, -1 > \odot < -40, -\frac{94}{5}, 10 >=< -100, \frac{1204}{15}, 400 >$. Hence $(\mu_1 \odot \mu_2) \odot \mu_3 \ne \mu_1 \odot (\mu_2 \odot \mu_3)$.

Consider a linear fuzzy real number $\mu =< a, \frac{a+c}{2}, c >$ with $a \ne c$. We call such a fuzzy subset μ a *symmetric linear fuzzy real number*. Let $\mathcal{B}$ be the set of all symmetric linear fuzzy real numbers $\mu =< a, \frac{a+c}{2}, c >$ with $a \ne c$. It is easy to show that if $\sigma_1, \sigma_2 \in \mathcal{B}$, then $\sigma_1 + \sigma_2 \in \mathcal{B}$, and $\sigma_1 \odot \sigma_2 \in \mathcal{B}$. Furthermore, it is easy to show that $(\sigma_1 \odot \sigma_2) \odot \sigma_3 = \sigma_1 \odot (\sigma_2 \odot \sigma_3)$ and $\sigma_1 \odot \sigma_2 = \sigma_2 \odot \sigma_1$. We summarize:

Theorem 3.1. *Let $\mathcal{B}$ be the set of all symmetric linear fuzzy real numbers $\mu =< a, \frac{a+c}{2}, c >$ with $a \ne c$. Then $(\mathcal{B}, \odot)$ is a commutative semigroup. Moreover, $\mu \odot \delta_0 = \delta_0$ for all $\mu \in \mathcal{B}$.*

Remark. Given the symmetric linear fuzzy real numbers $\mu_1 =< -2, -1.5, -1 >, \mu_2 =< -3, -0.5, 2 >, \mu_3 =< -4, -1, 2 > \in \mathcal{B}$, we have $(\mu_1 + \mu_2) \odot \mu_3 =< -10, 5, 20 >$, but $\mu_1 \odot \mu_2 + \mu_2 \odot \mu_3 =< -12, 4, 20 >$. Hence the distributive law fails.

A linear fuzzy real number $\mu =< a, b, c >$ is said to be *positive*(*negative*, resp.) if $a > 0$ ($c < 0$, resp.).

Proposition 3.2. *Let $\mu_i \in \mathcal{B}$ $(i = 1, 2, 3)$. If μ_3 is positive (or negative), then $(\mu_1 + \mu_2) \odot \mu_3 = \mu_1 \odot \mu_3 + \mu_2 \odot \mu_3$. If μ_1 is positive (or negative), then $\mu_1 \odot (\mu_2 + \mu_3) = \mu_1 \odot \mu_2 + \mu_1 \odot \mu_3$.*

Proof. Straightforward. □

Given $\mu_i =< a_i, b_i, c_i >\in \mathcal{A}$ $(i = 1, 2)$, we consider a "*weighted product*" $\mu_1 \otimes \mu_2 :=< a, b, c >$ where

$$\begin{aligned} a &= \inf\{t_1t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ c &= \sup\{t_1t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ b &= (a+c)\Big[\frac{\frac{b_1-a_1}{(c_1-a_1)^2} + \frac{b_2-a_2}{(c_2-a_2)^2}}{\frac{1}{c_1-a_1} + \frac{1}{c_2-a_2}}\Big]. \end{aligned}$$

Thus, if $c_1 - a_1 = c_2 - a_2$, then we obtain for b the formula:

$$b = (a+c)\frac{(b_1 - a_1) + (b_2 - a_2)}{(c_1 - a_1) + (c_2 - a_2)}$$

If $\frac{b_1-a_1}{c_1-a_1} = \frac{b_2-a_2}{c_2-a_2} = \frac{1}{2}$, then

$$\begin{aligned} b &= (a+c)\,\frac{\frac{1}{2}(\frac{1}{c_1-a_1}) + \frac{1}{2}(\frac{1}{c_2-a_2})}{\frac{1}{c_1-a_1} + \frac{1}{c_2-a_2}} \\ &= \frac{a+c}{2} \end{aligned}$$

so that for $\mu_1, \mu_2 \in \mathcal{B}$, we obtain $\mu_1 \otimes \mu_2 = \mu_1 \odot \mu_2$. We summarize:

Proposition 3.3. *If $\mu_1, \mu_2 \in \mathcal{B}$, then $\mu_1 \otimes \mu_2 = \mu_1 \odot \mu_2$.*

Remark. The weighted product $\sigma_1 \otimes \sigma_2$ is not associative in general. For example, let $\sigma_1 :=< -10, -9, -1 >, \sigma_2 :=< -8, 1, 2 >$ and $\sigma_3 :=< 1, 4, 5 >$. Then $\sigma_1 \otimes \sigma_2 =< -20, \frac{1{,}658}{57}, 80 >$ and $\sigma_2 \otimes \sigma_3 =< -40, -\frac{333}{14}, 10 >$ and so $(\sigma_1 \otimes \sigma_2) \otimes \sigma_3 =< -100, \frac{109{,}673}{494}, 400 >$ and $\sigma_1 \otimes (\sigma_2 \otimes \sigma_3) =< -100, \frac{106{,}774}{2{,}478}, 400 >$.

Actually, in general case, products $\mu_1 \odot \delta_{b_2}$ and $\mu_1 \otimes \delta_{b_2}$ are troublesome to define in a simple way since $a_2 = b_2 = c_2$ produce singularities.

Theorem 3.4. *Let $\mathcal{C}$ be the set of all positive (or negative) symmetric linear fuzzy real numbers. Then $(\mathcal{C}, +, \odot)$ is a semiring.*

Proof. It follows from Theorems 2.4 and 3.1, and Proposition 3.2. □

4. Complex transforms of linear fuzzy real numbers

Given the linear fuzzy real number $\mu = \mu(a, b, c)$ we may associate with it the complex transform $T(\mu)$, where

$$T(\mu)(s) = \int_{-\infty}^{\infty} s^2 e^{st} \mu(a, b, c)(t)dt \tag{5}$$

By integration by parts we obtain

$$\begin{aligned} T(\mu)(s) &= \int_a^c s^2 e^{st} \mu(a, b, c)(t)dt \\ &= \int_a^b s^2 e^{st} \frac{t-a}{b-a} dt + \int_b^c s^2 e^{st} \frac{c-t}{c-b} dt \\ &= \frac{1}{c-b}(e^{sc} - e^{sb}) - \frac{1}{b-a}(e^{sb} - e^{sa}) \end{aligned}$$

We summarize:

Proposition 4.1. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number, then its associated complex transformation $T(\mu)$ is*

$$T(\mu)(s) = \frac{1}{c-b}(e^{sc} - e^{sb}) - \frac{1}{b-a}(e^{sb} - e^{sa}) \tag{6}$$

Example 4.2. If $\mu_0 = \mu(-1, 0, 1)$, then $T(\mu_0)(s) = (e^s - 1) - (1 - e^{-s}) = (e^s + e^{-s}) - 2 = 2\cosh s - 2$ or "inversely" $\cosh s = \frac{T(\mu_0)(s)+2}{2}$ indicating that we may consider the functions $T(\mu)(s)$ to be "pseudo-hyperbolic" in nature.

Proposition 4.3. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda \neq 0$, then*

$$T(\mu)(\lambda s) = \frac{1}{c-b}(e^{\lambda sc} - e^{\lambda sb}) - \frac{1}{b-a}(e^{\lambda sb} - e^{\lambda sa}) \tag{7}$$

Proof. If $\lambda \neq 0$, then

$$\begin{aligned} T(\mu)(\lambda s) &= \int_{-\infty}^{\infty} (\lambda s)^2 e^{\lambda st} \mu(a, b, c)(t)dt \\ &= \lambda^2 \int_a^b s^2 e^{\lambda st} \frac{t-a}{b-a} dt + \lambda^2 \int_b^c s^2 e^{\lambda st} \frac{c-t}{c-b} dt \end{aligned}$$

By integration by parts, we obtain $\int_a^b s^2 e^{\lambda st} \frac{t-a}{b-a} dt = \frac{s}{\lambda} e^{\lambda sb} - \frac{1}{b-a}\frac{1}{\lambda^2}(e^{\lambda sb} - e^{\lambda sa})$ and $\int_b^c s^2 e^{\lambda st} \frac{c-t}{c-b} dt = -\frac{s}{\lambda} e^{\lambda sb} + \frac{1}{c-b}\frac{1}{\lambda^2}(e^{\lambda sc} - e^{\lambda sb})$, which proves the proposition. $\square$

Given a linear fuzzy real number $\mu = \mu(a, b, c)$ and $\lambda \in \mathbf{R}$, we define a new linear fuzzy real number $\lambda\mu$ as follows:

$$\lambda\mu(a, b, c) := \begin{cases} \mu(\lambda a, \lambda b, \lambda c) & \text{if } \lambda \geq 0 \\ \mu(\lambda c, \lambda b, \lambda a) & \text{otherwise} \end{cases}$$

Proposition 4.4. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda > 0$, then*

$$T(\mu)(\lambda s) = \lambda T(\lambda\mu)(s)$$

Proof. If $\mu = \mu(a, b, c)$ and $\lambda > 0$, then $\lambda\mu = \mu(\lambda a, \lambda b, \lambda c)$ and hence

$$\begin{aligned} T(\lambda\mu)(s) &= \int_{\lambda a}^{\lambda c} s^2 e^{st} \mu(\lambda a, \lambda b, \lambda c)(t) dt \\ &= \int_{\lambda a}^{\lambda b} s^2 e^{st} \frac{t - \lambda a}{\lambda b - \lambda a} dt + \int_{\lambda b}^{\lambda c} s^2 e^{st} \frac{\lambda c - t}{\lambda c - \lambda b} dt \\ &= \frac{1}{\lambda(c - b)}(e^{s\lambda c} - e^{s\lambda b}) - \frac{1}{\lambda(b - a)}(e^{s\lambda b} - e^{s\lambda a}) \end{aligned}$$

By multiplying λ to both sides and by applying Proposition 4.3, we proves the proposition. $\square$

Proposition 4.5. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda < 0$, then*

$$T(\mu)(\lambda s) = (-\lambda)T(\lambda\mu)(s)$$

Proof. If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda < 0$, then $\lambda\mu = \mu(-|\lambda|c, -|\lambda|b, -|\lambda|a)$. By applying Proposition 4.1, we obtain

$$T(\lambda\mu)(s) = \frac{1}{(-|\lambda|a) - (-|\lambda|b)}(e^{s(-|\lambda|a)} - e^{s(-|\lambda|b)}) - \frac{1}{(-|\lambda|b) - (-|\lambda|c)}(e^{s(-|\lambda|b)} - e^{s(-|\lambda|c)}).$$

Using Proposition 4.3 we obtain

$$\begin{aligned} \lambda T(\lambda\mu)(s) &= (-|\lambda|)T(\lambda\mu)(s) \\ &= \frac{1}{a - b}(e^{\lambda sa} - e^{\lambda sb}) - \frac{1}{b - c}(e^{\lambda sb} - e^{\lambda sc}) \\ &= -T(\mu)(\lambda s) \end{aligned}$$

$\square$

Combining Propositions 4.4 and 4.5 we obtain:

Theorem 4.6. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda \in \mathbf{R}$, then*

$$T(\mu)(\lambda s) = |\lambda|T(\lambda\mu)(s) \tag{8}$$

Proof. For non-zero real number λ, it was proved by Propositions 4.4 and 4.5. If $\lambda = 0$, then $T(\mu)(0s) = 0$, and so (8) holds trivially. □

Given a fuzzy real number $\mu = \mu(a, b, c)$ and $\lambda = -1$, we have $T(\mu)(-s) = T(\mu)((-1)s) = |-1|T((-1)\mu)(s) = T(\mu(-c, -b, -a))(s)$, i.e.,

$$T(\mu(a, b, c))(-s) = T(\mu(-c, -b, -a))(s) \tag{9}$$

If we let $s := -s$ in (9), then we have

$$T(\mu(a, b, c))(s) = T(\mu(-c, -b, -a))(-s) \tag{10}$$

For $\lambda = 2$, we obtain from (8) a "doubling formula".

$$T(\mu)(2s) = 2T(2\mu)(s) \tag{11}$$

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Octagonal Fuzzy DEMATEL Approach to Study the Risk Factors of Stomach Cancer

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Abstract

Fuzzy number (FN) plays a vital role in decision making problems as it is used to represent the uncertain terms. Researchers in the field of decision-making analysis have used triangular and trapezoidal FN to solve the problem in the uncertain environment. FN have also been extended recently such as pentagonal, hexagonal, and heptagonal and so on. This paper aims to generalize the Hexadecagonal fuzzy number (GHFN) which contains set of 16-tuples. Membership functions and alpha cuts of linear and nonlinear GHFN with symmetry and asymmetry have also been derived.

1 Introduction

Fuzzy sets have been presented by (Zadeh, 1965) to handle imprecision informa tion, all things considered, issues [1]. In 2003 (Coxe & Reiter, 2003), utilized fuzzy automata on a hexagonal foundation utilizing straightforward number juggling mixes of neighboring fuzzy qualities [2]. In 2013, (Rajarajeswari & Sudha, 2013) involved stretch math in another activity for expansion, deduction and duplication of Hexagonal Fuzzy number based on alpha cut sets of fuzzy numbers [3]. (Rajarajeswari & Sudha, 2014) summed up hexagonal fuzzy numbers by rank, mode, uniqueness and spreads to improve the independent direction, estimate and chance investigation [4]. In the extended time of 2015, (Dhurai & Karpagam, 2016) utilized span math to present another enrollment capability and fulfilled the activity of expansion, deduction and duplication of hexagonal fuzzy number based on alpha cut sets of fuzzy numbers [5].Hexagonal, heptagonal, nonagon, decagonal fuzzy numbers have additionally been acquainted with tackle the dubiousness [6, 7, 13].Sankar and Manimohan embraced pentagonal fuzzy number, determined direct and non-straight pentagonal fuzzy number and add- ressed fuzzy conditions utilizing pentagonal fuzzy number [14].Karthik et. al., proposed straight and nonlinear enrollment capabilities for the summed up heptagonal fuzzy number and presented Haar positioning technique for hexagonal fuzzy number [9]. Malini and Kennedy tackled fuzzy transportation issue by utilizing octagonal fuzzy numbers [10]. Felix et.al., proposed the nonagonal fuzzy number and its math activities and determined alpha cuts for nonagonal fuzzy number [7]. Venkatesh and Britto presented a positioning technique utilizing decagonal fuzzy number for diet control [15]. Nagadevi and Rosario tackled transportation issue, in which decagonal fuzzy numbers are utilized to address transportation expenses to find least transportation cost [11]. Naveena and Rajkumar presented turn around request pentadecagonal, nonagonal and decagonal fuzzy numbers and their math activities [12]. Necessary preliminaries are cited therein [8–10].

2 Hexadecagonal Fuzzy Number(HFN) and It's Variation

Hexadecagonal Fuzzy Number: A HFN $\tilde{A} =$
$(\Omega_1, \Omega_2, \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7,\ \Omega_8,\ \Omega_9,\ \Omega_{10},\ \Omega_{11},\ \Omega_{12},\ \Omega_{13},\ \Omega_{14},\ \Omega_{15},\ \Omega_{16})$
where
$\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8, \Omega_9, \Omega_{10}, \Omega_{11}, \Omega_{12}, \Omega_{13}, \Omega_{14}, \Omega_{15}, \Omega_{16} \in R$ must hold the consequent conditions

•$\mu_{\tilde{A}}(\theta)$ is a continuous function(breifly, $cts.fn$) in $[0, 1]$.

•$\mu_{\tilde{A}}(\theta)$ is strictly increasing and $cts.fn$ on $[\Omega_1, \Omega_2], [\Omega_2, \Omega_3], [\Omega_3,\ \Omega_4]$ and $[\Omega_4,\ \Omega_5], [\Omega_5,\ \Omega_6], [\Omega_6,\ \Omega_7]$ and $[\Omega_7,\ \Omega_8]$.

•$\mu_{\tilde{A}}(\theta)$ is strictly decreasing and $cts.fn$ on $[\Omega_9,\ \Omega_{10}], [\Omega_{10},\ \Omega_{11}], [\Omega_{11},\ \Omega_{12}]$, $[\Omega_{12},\ \Omega_{13}], [\Omega_{13},\ \Omega_{14}], [\Omega_{14},\ \Omega_{15}]$ and $[\Omega_{15},\ \Omega_{16}]$.

3.1.1 Equality of two HFN'S: Two HFN'S $\tilde{A} = (\Omega_1, \Omega_2, \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7,\ \Omega_8,\ \Omega_9,\ \Omega_{10},\ \Omega_{11},\ \Omega_{12},\ \Omega_{13},\ \Omega_{14},\ \Omega_{15},\ \Omega_{16})$ and $\tilde{B} = (\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4,\ \varphi_5,\ \varphi_6,\ \varphi_7,\ \varphi_8,\ \varphi_9,\ \varphi_{10},\ \varphi_{11},\ \varphi_{12},\ \varphi_{13},\ \varphi_{14},\ \varphi_{15},\ \varphi_{16})$ are equal iff $\Omega_1 = \varphi_1, \Omega_2 = \varphi_2, \Omega_3 = \varphi_3,\ \Omega_4 = \varphi_4,\ \Omega_5 = \varphi_5,\ \Omega_6 = \varphi_6,\ \Omega_8 = \varphi_8,\ \Omega_9 = \varphi_9,\ \Omega_{10} = \varphi_{10},\ \Omega_{11} =$

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$\varphi_{11},\ \Omega_{12}=\varphi_{12},\ \Omega_{13}=\varphi_{13},\ \Omega_{14}=\varphi_{14},\ \Omega_{15}=\varphi_{15},\ \Omega_{16}=\varphi_{16}$

Linear Hexadecagonal Symmetry:

$$\mu_x=\begin{cases} p\left(\frac{x-\Omega_1}{\Omega_2-\Omega_1}\right), \Omega_1\le x\le\Omega_2\\ p+(q-p)\left(\frac{x-\Omega_2}{\Omega_3-\Omega_2}\right), \Omega_2\le x\le\Omega_3\\ q+(r-q)\left(\frac{x-\Omega_3}{\Omega_4-\Omega_3}\right), \Omega_3\le x\le\Omega_4\\ r+(s-r)\left(\frac{x-\Omega_4}{\Omega_5-\Omega_4}\right), \Omega_4\le x\le\Omega_5\\ s+(t-s)\left(\frac{x-\Omega_5}{\Omega_6-\Omega_5}\right), \Omega_5\le x\le\Omega_6\\ t+(u-t)\left(\frac{x-\Omega_6}{\Omega_7-\Omega_6}\right), \Omega_6\le x\le\Omega_7\\ u+(1-u)\left(\frac{x-\Omega_7}{\Omega_8-\Omega_7}\right), \Omega_7\le x\le\Omega_8\\ 1, \Omega_8\le x\le\Omega_9\\ u-(u-1)\left(\frac{\Omega_{10}-x}{\Omega_{10}-\Omega_9}\right), \Omega_9\le x\le\Omega_{10}\\ t-(t-u)\left(\frac{\Omega_{11}-x}{\Omega_{11}-\Omega_{10}}\right), \Omega_{10}\le x\le\Omega_{11}\\ s-(s-t)\left(\frac{\Omega_{12}-x}{\Omega_{12}-\Omega_{11}}\right), \Omega_{11}\le x\le\Omega_{12}\\ r-(r-s)\left(\frac{\Omega_{13}-x}{\Omega_{13}-\Omega_{12}}\right), \Omega_{12}\le x\le\Omega_{13}\\ q-(q-r)\left(\frac{\Omega_{14}-x}{\Omega_{14}-\Omega_{13}}\right), \Omega_{13}\le x\le\Omega_{14}\\ p-(p-q)\left(\frac{\Omega_{15}-x}{\Omega_{15}-\Omega_{14}}\right), \Omega_{14}\le x\le\Omega_{15}\\ p\left(\frac{\Omega_{16}-x}{\Omega_{16}-\Omega_{15}}\right), \Omega_{15}\le x\le\Omega_{16}\\ 0, x\le\Omega_1 \quad\text{and}\quad x\ge\Omega_{16} \end{cases}$$

$$A_\alpha=\begin{cases} A_{1L}(\alpha)=\Omega_1+(\frac{\alpha}{p})(\Omega_2-\Omega_1) for\alpha\in[0,p]\\ A_{2L}(\alpha)=\Omega_2+(\frac{\alpha-p}{q-p})(\Omega_3-\Omega_2) for\alpha\in[p,q]\\ A_{3L}(\alpha)=\Omega_3+(\frac{\alpha-q}{r-q})(\Omega_4-\Omega_3) for\alpha\in[q,r]\\ A_{4L}(\alpha)=\Omega_4+(\frac{\alpha-r}{s-r})(\Omega_5-\Omega_4) for\alpha\in[r,s]\\ A_{5L}(\alpha)=\Omega_5+(\frac{\alpha-s}{t-s})(\Omega_6-\Omega_5) for\alpha\in[s,t]\\ A_{6L}(\alpha)=\Omega_6+(\frac{\alpha-t}{u-t})(\Omega_7-\Omega_6) for\alpha\in[t,u]\\ A_{7L}(\alpha)=\Omega_7+(\frac{\alpha-u}{1-u})(\Omega_8-\Omega_7) for\alpha\in[u,1]\\ A_{7R}(\alpha)=\Omega_{10}+(\frac{\alpha-u}{u-1})(\Omega_{10}-\Omega_9) for\alpha\in[u,1]\\ A_{6R}(\alpha)=\Omega_{11}+(\frac{\alpha-t}{t-u})(\Omega_{11}-\Omega_{10}) for\alpha\in[t,u]\\ A_{5R}(\alpha)=\Omega_{12}+(\frac{\alpha-s}{s-t})(\Omega_{12}-\Omega_{11}) for\alpha\in[s,t]\\ A_{4R}(\alpha)=\Omega_{13}+(\frac{\alpha-r}{r-s})(\Omega_{13}-\Omega_{12}) for\alpha\in[r,s]\\ A_{3R}(\alpha)=\Omega_{14}+(\frac{\alpha-q}{q-r})(\Omega_{14}-\Omega_{13}) for\alpha\in[q,r]\\ A_{2R}(\alpha)=\Omega_{15}+(\frac{\alpha-p}{p-q})(\Omega_{15}-\Omega_{14}) for\alpha\in[p,q]\\ A_{1R}(\alpha)=\Omega_{16}-(\frac{\alpha}{p})(\Omega_{16}-\Omega_{15}) for\alpha\in[0,p] \end{cases}$$

Linear Haxadecagonal Asymmetry

$$\mu_x = \begin{cases} p\left(\frac{x-\Omega_1}{\Omega_2-\Omega_1}\right), \Omega_1 \leq x \leq \Omega_2 \\ p+(q-p)\left(\frac{x-\Omega_2}{\Omega_3-\Omega_2}\right), \Omega_2 \leq x \leq \Omega_3 \\ q+(r-q)\left(\frac{x-\Omega_3}{\Omega_4-\Omega_3}\right), \Omega_3 \leq x \leq \Omega_4 \\ r+(s-r)\left(\frac{x-\Omega_4}{\Omega_5-\Omega_4}\right), \Omega_4 \leq x \leq \Omega_5 \\ s+(t-s)\left(\frac{x-\Omega_5}{\Omega_6-\Omega_5}\right), \Omega_5 \leq x \leq \Omega_6 \\ t+(u-t)\left(\frac{x-\Omega_6}{\Omega_7-\Omega_6}\right), \Omega_6 \leq x \leq \Omega_7 \\ u+(1-u)\left(\frac{x-\Omega_7}{\Omega_8-\Omega_7}\right), \Omega_7 \leq x \leq \Omega_8 \\ 1, \Omega_8 \leq x \leq \Omega_9 \\ e-(e-1)\left(\frac{\Omega_{10}-x}{\Omega_{10}-\Omega_9}\right), \Omega_9 \leq x \leq \Omega_{10} \\ f-(f-e)\left(\frac{\Omega_{11}-x}{\Omega_{11}-\Omega_{10}}\right), \Omega_{10} \leq x \leq \Omega_{11} \\ g-(g-f)\left(\frac{\Omega_{12}-x}{\Omega_{12}-\Omega_{11}}\right), \Omega_{11} \leq x \leq \Omega_{12} \\ h-(h-g)\left(\frac{\Omega_{13}-x}{\Omega_{13}-\Omega_{12}}\right), \Omega_{12} \leq x \leq \Omega_{13} \\ i-(i-h)\left(\frac{\Omega_{14}-x}{\Omega_{14}-\Omega_{13}}\right), \Omega_{13} \leq x \leq \Omega_{14} \\ j-(j-i)\left(\frac{\Omega_{15}-x}{\Omega_{15}-\Omega_{14}}\right), \Omega_{14} \leq x \leq \Omega_{15} \\ j\left(\frac{\Omega_{16}-x}{\Omega_{16}-\Omega_{15}}\right), \Omega_{15} \leq x \leq \Omega_{16} \\ 0, x \leq \Omega_1 and x \geq \Omega_{16} \end{cases}$$

$$A_\alpha = \begin{cases} A_{1L}(\alpha) = \Omega_1 + (\frac{\alpha}{p})(\Omega_2 - \Omega_1) for \alpha \in [0,p] \\ A_{2L}(\alpha) = \Omega_2 + (\frac{\alpha-p}{q-p})(\Omega_3 - \Omega_2) for \alpha \in [p,q] \\ A_{3L}(\alpha) = \Omega_3 + (\frac{\alpha-q}{r-q})(\Omega_4 - \Omega_3) for \alpha \in [q,r] \\ A_{4L}(\alpha) = \Omega_4 + (\frac{\alpha-r}{s-r})(\Omega_5 - \Omega_4) for \alpha \in [r,s] \\ A_{5L}(\alpha) = \Omega_5 + (\frac{\alpha-s}{t-s})(\Omega_6 - \Omega_5) for \alpha \in [s,t] \\ A_{6L}(\alpha) = \Omega_6 + (\frac{\alpha-t}{u-t})(\Omega_7 - \Omega_6) for \alpha \in [t,u] \\ A_{7L}(\alpha) = \Omega_7 + (\frac{\alpha-u}{1-u})(\Omega_8 - \Omega_7) for \alpha \in [u,1] \\ A_{7R}(\alpha) = \Omega_{10} + (\frac{\alpha-e}{e-1})(\Omega_{10} - \Omega_9) for \alpha \in [e,1] \\ A_{6R}(\alpha) = \Omega_{11} + (\frac{\alpha-f}{f-e})(\Omega_{11} - \Omega_{10}) for \alpha \in [f,e] \\ A_{5R}(\alpha) = \Omega_{12} + (\frac{\alpha-g}{g-f})(\Omega_{12} - \Omega_{11}) for \alpha \in [g,f] \\ A_{4R}(\alpha) = \Omega_{13} + (\frac{\alpha-h}{h-g})(\Omega_{13} - \Omega_{12}) for \alpha \in [h,g] \\ A_{3R}(\alpha) = \Omega_{14} + (\frac{\alpha-i}{i-h})(\Omega_{14} - \Omega_{13}) for \alpha \in [i,h] \\ A_{2R}(\alpha) = \Omega_{15} + (\frac{\alpha-j}{j-i})(\Omega_{15} - \Omega_{14}) for \alpha \in [j,i] \\ A_{1R}(\alpha) = \Omega_{16} - (\frac{\alpha}{j})(\Omega_{16} - \Omega_{15}) for \alpha \in [0,j] \end{cases}$$

Nonlinear Haxadecagonal Symmetry:

$$\mu_x = \begin{cases} p\left(\frac{x-\Omega_1}{\Omega_2-\Omega_1}\right)^{S_1}, \Omega_1 \le x \le \Omega_2 \\ p+(q-p)\left(\frac{x-\Omega_2}{\Omega_3-\Omega_2}\right)^{S_2}, \Omega_2 \le x \le \Omega_3 \\ q+(r-q)\left(\frac{x-\Omega_3}{\Omega_4-\Omega_3}\right)^{S_3}, \Omega_3 \le x \le \Omega_4 \\ r+(s-r)\left(\frac{x-\Omega_4}{\Omega_5-\Omega_4}\right)^{S_4}, \Omega_4 \le x \le \Omega_5 \\ s+(t-s)\left(\frac{x-\Omega_5}{\Omega_6-\Omega_5}\right)^{S_5}, \Omega_5 \le x \le \Omega_6 \\ t+(u-t)\left(\frac{x-\Omega_6}{\Omega_7-\Omega_6}\right)^{S_6}, \Omega_6 \le x \le \Omega_7 \\ u+(1-u)\left(\frac{x-\Omega_7}{\Omega_8-\Omega_7}\right)^{S_7}, \Omega_7 \le x \le \Omega_8 \\ 1, \Omega_8 \le x \le \Omega_9 \\ u-(u-1)\left(\frac{\Omega_{10}-x}{\Omega_{10}-\Omega_9}\right)^{P_1}, \Omega_9 \le x \le \Omega_{10} \\ t-(t-u)\left(\frac{\Omega_{11}-x}{\Omega_{11}-\Omega_{10}}\right)^{P_2}, \Omega_{10} \le x \le \Omega_{11} \\ s-(s-t)\left(\frac{\Omega_{12}-x}{\Omega_{12}-\Omega_{11}}\right)^{P_3}, \Omega_{11} \le x \le \Omega_{12} \\ r-(r-s)\left(\frac{\Omega_{13}-x}{\Omega_{13}-\Omega_{12}}\right)^{P_4}, \Omega_{12} \le x \le \Omega_{13} \\ q-(q-r)\left(\frac{\Omega_{14}-x}{\Omega_{14}-\Omega_{13}}\right)^{P_5}, \Omega_{13} \le x \le \Omega_{14} \\ p-(p-q)\left(\frac{\Omega_{15}-x}{\Omega_{15}-\Omega_{14}}\right)^{P_6}, \Omega_{14} \le x \le \Omega_{15} \\ p\left(\frac{\Omega_{16}-x}{\Omega_{16}-\Omega_{15}}\right)^{P_7}, \Omega_{15} \le x \le \Omega_{16} \\ 0, x \le \Omega_1 and x \ge \Omega_{16} \end{cases}$$

$$A_\alpha = \begin{cases} A_{1L}(\alpha) = \Omega_1 + (\frac{\alpha}{p})(\Omega_2-\Omega_1) for \alpha \in [0,p] \\ A_{2L}(\alpha) = \Omega_2 + (\frac{\alpha-p}{q-p})(\Omega_3-\Omega_2) for \alpha \in [p,q] \\ A_{3L}(\alpha) = \Omega_3 + (\frac{\alpha-q}{r-q})(\Omega_4-\Omega_3) for \alpha \in [q,r] \\ A_{4L}(\alpha) = \Omega_4 + (\frac{\alpha-r}{s-r})(\Omega_5-\Omega_4) for \alpha \in [r,s] \\ A_{5L}(\alpha) = \Omega_5 + (\frac{\alpha-s}{t-s})(\Omega_6-\Omega_5) for \alpha \in [s,t] \\ A_{6L}(\alpha) = \Omega_6 + (\frac{\alpha-t}{u-t})(\Omega_7-\Omega_6) for \alpha \in [t,u] \\ A_{7L}(\alpha) = \Omega_7 + (\frac{\alpha-u}{1-u})(\Omega_8-\Omega_7) for \alpha \in [u,1] \\ A_{7R}(\alpha) = \Omega_{10} + (\frac{\alpha-u}{u-1})(\Omega_{10}-\Omega_9) for \alpha \in [u,1] \\ A_{6R}(\alpha) = \Omega_{11} + (\frac{\alpha-t}{t-u})(\Omega_{11}-\Omega_{10}) for \alpha \in [t,u] \\ A_{5R}(\alpha) = \Omega_{12} + (\frac{\alpha-s}{s-t})(\Omega_{12}-\Omega_{11}) for \alpha \in [s,t] \\ A_{4R}(\alpha) = \Omega_{13} + (\frac{\alpha-r}{r-s})(\Omega_{13}-\Omega_{12}) for \alpha \in [r,s] \\ A_{3R}(\alpha) = \Omega_{14} + (\frac{\alpha-q}{q-r})(\Omega_{14}-\Omega_{13}) for \alpha \in [q,r] \\ A_{2R}(\alpha) = \Omega_{15} + (\frac{\alpha-p}{p-q})(\Omega_{15}-\Omega_{14}) for \alpha \in [p,q] \\ A_{1R}(\alpha) = \Omega_{16} - (\frac{\alpha}{p})(\Omega_{16}-\Omega_{15}) for \alpha \in [0,p] \end{cases}$$

Nonlinear Haxadecagonal Asymmetry:

$$\mu_x = \begin{cases} p\left(\frac{x-\Omega_1}{\Omega_2-\Omega_1}\right)^{S_1}, \Omega_1 \leq x \leq \Omega_2 \\ p+(q-p)\left(\frac{x-\Omega_2}{\Omega_3-\Omega_2}\right)^{S_2}, \Omega_2 \leq x \leq \Omega_3 \\ q+(r-q)\left(\frac{x-\Omega_3}{\Omega_4-\Omega_3}\right)^{S_3}, \Omega_3 \leq x \leq \Omega_4 \\ r+(s-r)\left(\frac{x-\Omega_4}{\Omega_5-\Omega_4}\right)^{S_4}, \Omega_4 \leq x \leq \Omega_5 \\ s+(t-s)\left(\frac{x-\Omega_5}{\Omega_6-\Omega_5}\right)^{S_5}, \Omega_5 \leq x \leq \Omega_6 \\ t+(u-t)\left(\frac{x-\Omega_6}{\Omega_7-\Omega_6}\right)^{S_6}, \Omega_6 \leq x \leq \Omega_7 \\ u+(1-u)\left(\frac{x-\Omega_7}{\Omega_8-\Omega_7}\right)^{S_7}, \Omega_7 \leq x \leq \Omega_8 \\ 1, \Omega_8 \leq x \leq \Omega_9 \\ e-(e-1)\left(\frac{\Omega_{10}-x}{\Omega_{10}-\Omega_9}\right)^{P_1}, \Omega_9 \leq x \leq \Omega_{10} \\ f-(f-e)\left(\frac{\Omega_{11}-x}{\Omega_{11}-\Omega_{10}}\right)^{P_2}, \Omega_{10} \leq x \leq \Omega_{11} \\ g-(g-f)\left(\frac{\Omega_{12}-x}{\Omega_{12}-\Omega_{11}}\right)^{P_3}, \Omega_{11} \leq x \leq \Omega_{12} \\ h-(h-g)\left(\frac{\Omega_{13}-x}{\Omega_{13}-\Omega_{12}}\right)^{P_4}, \Omega_{12} \leq x \leq \Omega_{13} \\ i-(i-h)\left(\frac{\Omega_{14}-x}{\Omega_{14}-\Omega_{13}}\right)^{P_5}, \Omega_{13} \leq x \leq \Omega_{14} \\ j-(j-i)\left(\frac{\Omega_{15}-x}{\Omega_{15}-\Omega_{14}}\right)^{P_6}, \Omega_{14} \leq x \leq \Omega_{15} \\ j\left(\frac{\Omega_{16}-x}{\Omega_{16}-\Omega_{15}}\right)^{P_7}, \Omega_{15} \leq x \leq \Omega_{16} \\ 0, x \leq \Omega_1 and x \geq \Omega_{16} \end{cases}$$

$$A_\alpha = \begin{cases} A_{1L}(\alpha) = \Omega_1 + (\frac{\alpha}{p})(\Omega_2 - \Omega_1) for \alpha \in [0,p] \\ A_{2L}(\alpha) = \Omega_2 + (\frac{\alpha-p}{q-p})(\Omega_3 - \Omega_2) for \alpha \in [p,q] \\ A_{3L}(\alpha) = \Omega_3 + (\frac{\alpha-q}{r-q})(\Omega_4 - \Omega_3) for \alpha \in [q,r] \\ A_{4L}(\alpha) = \Omega_4 + (\frac{\alpha-r}{s-r})(\Omega_5 - \Omega_4) for \alpha \in [r,s] \\ A_{5L}(\alpha) = \Omega_5 + (\frac{\alpha-s}{t-s})(\Omega_6 - \Omega_5) for \alpha \in [s,t] \\ A_{6L}(\alpha) = \Omega_6 + (\frac{\alpha-t}{u-t})(\Omega_7 - \Omega_6) for \alpha \in [t,u] \\ A_{7L}(\alpha) = \Omega_7 + (\frac{\alpha-u}{1-u})(\Omega_8 - \Omega_7) for \alpha \in [u,1] \\ A_{7R}(\alpha) = \Omega_{10} + (\frac{\alpha-e}{e-1})(\Omega_{10} - \Omega_9) for \alpha \in [e,1] \\ A_{6R}(\alpha) = \Omega_{11} + (\frac{\alpha-f}{f-e})(\Omega_{11} - \Omega_{10}) for \alpha \in [f,e] \\ A_{5R}(\alpha) = \Omega_{12} + (\frac{\alpha-g}{g-f})(\Omega_{12} - \Omega_{11}) for \alpha \in [g,f] \\ A_{4R}(\alpha) = \Omega_{13} + (\frac{\alpha-h}{h-g})(\Omega_{13} - \Omega_{12}) for \alpha \in [h,g] \\ A_{3R}(\alpha) = \Omega_{14} + (\frac{\alpha-i}{i-h})(\Omega_{14} - \Omega_{13}) for \alpha \in [i,h] \\ A_{2R}(\alpha) = \Omega_{15} + (\frac{\alpha-j}{j-i})(\Omega_{15} - \Omega_{14}) for \alpha \in [j,i] \\ A_{1R}(\alpha) = \Omega_{16} - (\frac{\alpha}{j})(\Omega_{16} - \Omega_{15}) for \alpha \in [0,j] \end{cases}$$

1. Arithmetic operations on linear HFN with symmetry Let $\tilde{A}_{LS} = (\Omega_1, \Omega_2, \Omega_3, \Omega_4, a_5, a_6, a_7, a_8, a_9, \Omega_{10}, \Omega_{11}, \Omega_{12}, \Omega_{13}, \Omega_{14}, \Omega_{15}, \Omega_{16}; m_1, n_1)$ and

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$\tilde{B}_{LS} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8, \varphi_9, \varphi_{10}, \varphi_{11}, \varphi_{12}, \varphi_{13}, \varphi_{14}, \varphi_{15}, \varphi_{16}; m_2, n_2)$ be two linear heptagonal FN's with symmetry, then

(i) The summation of two HFN's is defined as $\tilde{C}_{LS} = \tilde{A}_{LS} + \tilde{B}_{LS} = (\Omega_1 + \varphi_1, \Omega_2 + \varphi_2, \Omega_3 + \varphi_3, \Omega_4 + \varphi_4, \Omega_5 + \varphi_5, \Omega_6 + \varphi_6, \Omega_7 + \varphi_7, \Omega_8 + \varphi_8, \Omega_9 + \varphi_9, \Omega_{10} + \varphi_{10}, \Omega_{11} + \varphi_{11}, \Omega_{12} + \varphi_{12}, \Omega_{13} + \varphi_{13}, \Omega_{14} + \varphi_{14}, \Omega_{15} + \varphi_{15}, \Omega_{16} + \varphi_{16}; m, n)$ Where $m = \min\{m_1, m_2\}$ and $n = \min\{n_1, n_2\}$.

Theorem 2.1.. Let $\tilde{H}_1 = (a^1_{lu}, a^2_{lu}, a^3_{lu}, a^4_{lu}, a^5_{lu}, a^6_{lu}, a^7_{lu}, a^8_{lu} a^9_{lu}, a^{10}_{lu}, a^{11}_{lu}, a^{12}_{lu}, a^{13}_{lu}, a^{14}_{lu}, a^{15}_{lu}, a^{16}_{lu})$ and $\tilde{H}_2 = (a^1_{\alpha\beta}, a^2_{\alpha\beta}, a^3_{\alpha\beta}, a^4_{\alpha\beta}, a^5_{\alpha\beta}, a^6_{\alpha\beta}, a^7_{\alpha\beta}, a^8_{\alpha\beta}, a^9_{\alpha\beta}, a^{10}_{\alpha\beta}, a^{11}_{\alpha\beta}, a^{12}_{\alpha\beta}, a^{13}_{\alpha\beta}, a^{14}_{\alpha\beta}, a^{15}_{\alpha\beta}, a^{16}_{\alpha\beta})$ be two HFN's; then the arithmetic operation of $\tilde{H}_1$and$\tilde{H}_2$, denoted as $\tilde{H}_1 \oplus \tilde{H}_2, \tilde{H}_1 \Theta \tilde{H}_2$ and $\tilde{H}_1 \otimes \tilde{H}_2$ an yield another HFN,

$$1. \tilde{H}_1 \oplus \tilde{H}_2 = \begin{pmatrix} a^1_{lu} + a^1_{\alpha\beta}, a^2_{lu} + a^2_{\alpha\beta}, a^3_{lu} + a^3_{\alpha\beta}, a^4_{lu} + a^4_{\alpha\beta}, a^5_{lu} + a^5_{\alpha\beta}, a^6_{lu} + a^6_{\alpha\beta}, a^7_{lu} + a^7_{\alpha\beta}, a^8_{lu} + a^8_{\alpha\beta} \\ a^9_{lu} + a^9_{\alpha\beta}, a^{10}_{lu} + a^{10}_{\alpha\beta}, a^{11}_{lu} + a^{11}_{\alpha\beta}, a^{12}_{lu} + a^{12}_{\alpha\beta}, a^{13}_{lu} + a^{13}_{\alpha\beta}, a^{14}_{lu} + a^{14}_{\alpha\beta}, a^{15}_{lu} + a^{15}_{\alpha\beta}, a^{16}_{lu} + a^{16}_{\alpha\beta} \end{pmatrix}.$$

$$2. \tilde{H}_1 \Theta \tilde{H}_2 = \begin{pmatrix} a^1_{lu} - a^{16}_{\alpha\beta}, a^2_{lu} - a^{15}_{\alpha\beta}, a^3_{lu} - a^{14}_{\alpha\beta}, a^4_{lu} - a^{13}_{\alpha\beta}, a^5_{lu} - a^{12}_{\alpha\beta}, a^6_{lu} - a^{11}_{\alpha\beta}, a^7_{lu} - a^{10}_{\alpha\beta}, a^8_{lu} - a^9_{\alpha\beta} \\ a^9_{lu} - a^8_{\alpha\beta}, a^{10}_{lu} - a^7_{\alpha\beta}, a^{11}_{lu} - a^6_{\alpha\beta}, a^{12}_{lu} - a^5_{\alpha\beta}, a^{13}_{lu} - a^4_{\alpha\beta}, a^{14}_{lu} - a^3_{\alpha\beta}, a^{15}_{lu} - a^2_{\alpha\beta}, a^{16}_{lu} - a^1_{\alpha\beta} \end{pmatrix}.$$

$$3. \tilde{H}_1 \otimes \tilde{H}_2 = \begin{pmatrix} a^1_{lu} a^1_{\alpha\beta}, a^2_{lu} a^2_{\alpha\beta}, a^3_{lu} a^3_{\alpha\beta}, a^4_{lu} a^4_{\alpha\beta}, a^5_{lu} a^5_{\alpha\beta}, a^6_{lu} a^6_{\alpha\beta}, a^7_{lu} a^7_{\alpha\beta}, a^8_{lu} a^8_{\alpha\beta} \\ a^9_{lu} a^9_{\alpha\beta}, a^{10}_{lu} a^{10}_{\alpha\beta}, a^{11}_{lu} a^{11}_{\alpha\beta}, a^{12}_{lu} a^{12}_{\alpha\beta}, a^{13}_{lu} a^{13}_{\alpha\beta}, a^{14}_{lu} a^{14}_{\alpha\beta}, a^{15}_{lu} a^{15}_{\alpha\beta}, a^{16}_{lu} a^{16}_{\alpha\beta} \end{pmatrix}.$$

$$\lambda \otimes \tilde{H}_1 = (\lambda a^1_{lu}, \lambda a^2_{lu}, \lambda a^3_{lu}, \lambda a^4_{lu}, \lambda a^5_{lu}, \lambda a^6_{lu}, \lambda a^7_{lu}, \lambda a^8_{lu},$$

$$\lambda a^9_{lu}, \lambda a^{10}_{lu}, \lambda a^{11}_{lu}, \lambda a^{12}_{lu}, \lambda a^{13}_{lu}, \lambda a^{14}_{lu}, \lambda a^{15}_{lu}, \lambda a^{16}_{lu}) .$$

Haar Ranking method for Haxadecagonal Fuzzy Number:

Let $\tilde{A} = (\Omega_1, \Omega_2, \Omega_3, \Omega_4, a_5, a_6, a_7, a_8, a_9, \Omega_{10}, \Omega_{11}, \Omega_{12}, \Omega_{13}, \Omega_{14}, \Omega_{15}, \Omega_{16})$ be the HFN. Using HRM(Haar ranking method), the HFN is rewritten as $\tilde{A} = (\Omega_1, \Omega_2, \Omega_3, \Omega_4, a_5, a_6, a_7, a_8, a_9, \Omega_{10}, \Omega_{11}, \Omega_{12}, \Omega_{13}, \Omega_{14}, \Omega_{15}, \Omega_{16})$. The average and elaborate coefficients namely the scaling and wavelet coefficients of HFN can be calculated as follows.

Step-l: Group the HFN in pairs.

$$[\Omega_1, \Omega_2], [\Omega_3, \Omega_4], [a_5, a_6], [a_7, a_8], [a_9, \Omega_{10}], [\Omega_{11}, \Omega_{12}], [\Omega_{13}, \Omega_{14}], [\Omega_{15}, \Omega_{16}]$$

Step-2: Replace the first 4 elements of approximation coefficient with the detailed coefficient.

$\alpha_1 = (\frac{\Omega_1 + \Omega_2}{2}), \alpha_2 = (\frac{\Omega_3 + \Omega_4}{2}), \alpha_3 = (\frac{a_5 + a_6}{2}), \alpha_4 = (\frac{a_7 + a_8}{2})$,

$\alpha_5 = (\frac{a_9 + \Omega_{10}}{2}), \alpha_6 = (\frac{\Omega_{11} + \Omega_{12}}{2}), \alpha_7 = (\frac{\Omega_{13} + \Omega_{14}}{2}), \alpha_8 = (\frac{\Omega_{15} + \Omega_{16}}{2})$

$\beta_1 = (\frac{\Omega_1 - \Omega_2}{2}), \beta_2 = (\frac{\Omega_3 - \Omega_4}{2}), \beta_3 = (\frac{a_5 - a_6}{2}), \beta_4 = (\frac{a_7 - a_8}{2})$

$\beta_5 = (\frac{a_9 - \Omega_{10}}{2}), \beta_6 = (\frac{\Omega_{11} - \Omega_{12}}{2}), \beta_7 = (\frac{\Omega_{13} - \Omega_{14}}{2}), \beta_8 = (\frac{\Omega_{15} - \Omega_{16}}{2})$

The $\tilde{A}_1$ changed into $\tilde{A}_1 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8)$

Step-3: Group the pair of approximation coefficients of $\tilde{A}_1$. Then, find the new approximation coefficients and the detailed coefficients for the pair of approximation coefficient of $\tilde{A}_1$

$$[\alpha_1, \alpha_2], [\alpha_3, \alpha_4], [\alpha_5, \alpha_6], [\alpha_7, \alpha_8]$$

$$\gamma_1 = (\frac{\alpha_1 + \alpha_2}{2}), \gamma_2 = (\frac{\alpha_3 + \alpha_4}{2}), \gamma_3 = (\frac{\alpha_5 + \alpha_6}{2}), \gamma_4 = (\frac{\alpha_7 + \alpha_8}{2})$$

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$\eta_1 = (\frac{\alpha_1 - \alpha_2}{2}), \eta_2 = (\frac{\alpha_3 - \alpha_4}{2}), \eta_3 = (\frac{\alpha_5 - \alpha_6}{2}), \eta_4 = (\frac{\alpha_7 - \alpha_8}{2})$

The $\tilde{A_1}$ changed into $\tilde{A_2} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \eta_1, \eta_2, \eta_3, \eta_4, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8)$

Step-4: Determine the pair of approximation coefficient in $\tilde{A_2}$. Then, find the new approximation and detailed coefficients for the pair of approximation coefficient of $\tilde{A_2}$.

$$[\gamma_1, \gamma_2, \gamma_3, \gamma_4]$$

$\delta_1 = (\frac{\gamma_1 + \gamma_2}{2}), \delta_2 = (\frac{\gamma_3 + \gamma_4}{2}), \mathcal{E}_1 = (\frac{\gamma_1 - \gamma_2}{2}), \mathcal{E}_2 = (\frac{\gamma_3 - \gamma_4}{2})$

The $\tilde{A_2}$ changed into $\tilde{A_3} = (\delta_1, \delta_2, \epsilon_1, \epsilon_2, \eta_1, \eta_2, \eta_3, \eta_4, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8)$

Step-4: Determine the pair of approximation coefficient in $\tilde{A_3}$. Then, find the new approximation and detailed coefficients for the pair of approximation coefficient of $\tilde{A_3}$.

$$[\delta_1, \delta_2]$$

$\mu_8 = (\frac{\delta_1 + \delta_2}{2}), \mu_2 = (\frac{\delta_1 - \delta_2}{2})$

The $\tilde{A_3}$ changed into $H(\tilde{A}) = (\mu_1, \mu_2, \epsilon_1, \epsilon_2, \eta_1, \eta_2, \eta_3, \eta_4, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8)$

Step-5: Determine the Ranking μ.

•$\tilde{A} \prec \tilde{B}$, if the first element of the ordered tuple of $H(\tilde{A})$ is less than the first element of the ordered tuple of $H(\tilde{B})$.

•$\tilde{A} \succ \tilde{B}$, if the first element of the ordered tuple of $H(\tilde{A})$ is greater than the first element of the ordered tuple of $H(\tilde{B})$.

•$\tilde{A} \approx \tilde{B}$ if and only if all the elements of $H(\tilde{A})$ and $H(\tilde{B})$ are term wise equal.

3 Fuzzy Assignment Problem(FAP)

FAP in general defines as follows

$$\min \text{ (or)}\max X = \sum_j \sum_{i=1}^m \tilde{P_{ij}} y_{ij}$$

Subject to

$$\sum_{i=1}^{m} y_{ij} = 1 for i = 1, 2, \ldots, m.$$

$$\sum_{j=1}^{m} y_{ij} = 1 for j = 1, 2, \ldots, m.$$

$y_{ij} = 1$, if the i^{th} job is assigned to j^{th} person
0, if the i^{th} job is not assigned to j^{th} person

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Example 4.1: A FAP with 4 machinesM_1, M_2, M_3, M_4 and 4 jobs J, J_2, J_3, J_4 is premeditated. The cost matrix C_{ij^*} is whose values are depicted by HFN. The problem is to find the minimum assignment cost. Here, the Hungarian method.

After taking the averages of fuzzy cost matrix, the following is obtained

$$A = \begin{bmatrix} 5 & 8.2 & 9.4 & 7.2 \\ 8.3 & 7.1 & 15.1 & 8.3 \\ 10.5 & 9.4 & 10.5 & 10.6 \\ 13.8 & 8.3 & 12.5 & 7.5 \end{bmatrix}$$

Row wise subtraction,

$$A = \begin{bmatrix} 0 & 3.2 & 4.4 & 2.2 \\ 1.2 & 0 & 8 & 1.2 \\ 1.1 & 0 & 1.1 & 1.2 \\ 6.3 & 0.8 & 5 & 0 \end{bmatrix}$$

Column wise subtraction,

$$A = \begin{bmatrix} 0 & 3.2 & 3.3 & 2.2 \\ 1.2 & 0 & 6.9 & 1.2 \\ 1.1 & 0 & 0 & 1.2 \\ 6.3 & 0.8 & 3.9 & 0 \end{bmatrix}$$

Number of rows=Number of squares.

Therefore, the optimal cost is $= 5 + 7.1 + 10.5 + 7.5 = 30.1$.

4 Conclusion

In this present study, the GHFN's have been derived under fuzzy environment which may help to handle uncertainties in the decision-making problems.

These kinds of FN's are helpful when decision maker needs to represent a parameter at 16 different points. The following important outcomes have been attained in this research,

• The membership curve of a generalized linear and nonlinear Haxadecagonal fuzzy number with symmetry and asymmetry has been derived.

• Alpha cuts for all kinds of Haxadecagonal fuzzy number have also been derived.

Generalized Haxadecagonal fuzzy numbers can be used to extend Multi Criteria Decision Making (MCDM) models such as DEMATEL, TOPSIS, VIKOR, and others. These numbers are helpful in transportation problems such fuzzy assignment and transportation.

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A general composite iterative algorithm for monotone mappings and pseudocontractive mappings in Hilbert spaces

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Abstract

In this paper, we introduce a general composite iterative algorithm for finding a common element of the set of solutions of variational inequality problem for a hemicontinuous monotone mapping and the set of fixed points of a hemicontinuous pseudocontractive mapping in a Hilbert space. Under suitable control conditions, we establish strong convergence of the sequence generated by the proposed iterative algorithm to a common element of two sets, which is the unique solution of a certain variational inequality related to a boundedly Lipschitzian and strongly monotone mapping. As a consequence, we obtain the unique minimum-norm common point of two sets.

MSC: 47H06, 47H09, 47H10, 47J20, 49J40, 47J25, 47J05.

Key words: Iterative algorithm, Hemicontinuous monotone mapping, Hemicontiunous pseudocontractive mapping, Boundedly Lipschitzian, η-Strongly monotone mapping, Variational inequality, Fixe points.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be self-mapping on C. We denote by $Fix(S)$ the set of fixed points of S.

Let A be a nonlinear mapping of C into H. The variational inequality problem (shortly, VIP) is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We denote the set of solutions of the VIP (1.1) by $VI(C, A)$. The variational inequality problem has been extensively studied in the literature; see [4,14,15,24] and the references therein.

A fixed point problem (shortly, FPP) is to find a fixed point z of a nonlinear mapping $T : C \to C$ with property:

$$z \in C, \quad Tz = z. \tag{1.2}$$

Fixed point theory is one of the most powerful and important tools of modern mathematics and may be considered a core subject of nonlinear analysis.

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall that a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be *k-strictly pseudocontractive* ([3]) if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings as a subclass, and the class of k-strictly pseudocontractive mappings falls into the one between the class of nonexpansive mappings and the class of pseudocontractive mappings. Moreover, this inclusion is strict due to Example 5.7.1 and Example 5.7.2 in [1].

Recently, in order to study the VIP (1.1) coupled with the FPP (1.2), many authors have introduced some iterative algorithms for finding a common element of the set of the solutions of the VIP (1.1) for an inverse-strongly monotone mapping A and the set of fixed points of a nonexpansive mapping T; see [6,8,9,12,19] and the references therein. Also, some iterative algorithms for finding a common element of the set of the solutions of the VIP (1.1) for a continuous monotone mapping A more general than an inverse-strongly monotone mapping and the set of fixed points of a continuous pseudocontractive mapping T more general than a nonexpansive mapping were considered by many authors: see [20,22,26] and the references therein.

In 2001, Yamada [24] introduced the hybrid steepest descent method for the nonexpansive mapping to solve a variational inequality related to a Lipschitzian and strongly monotone mapping. Since then, in 2009, He and Xu [11] invented a hybrid iterative algorithm for the nonexpansive mapping to obtain the unique solution to the VIP (1.1) related to a boundedly Lipschitzian and strongly monotone mapping. As the result, He and Xu [11] were able to relax the global Lipschitz condition on the mapping to the weaker bounded Lipschitz condition, and improved the Yamada's result [24]. In 2010, He and Liang [10] considered the hybrid steepest descent algorithm for the strict pseudocontractive mapping more general than the nonexpansive mapping to solve a variational inequality related to a boundedly Lipschitzian and strongly monotone mapping, and extended the corresponding results in He and Xu [11].

On the other hand, by using ideas of Yamada [24], Tien [21] and Ceng *et al.* [5] provided general iterative algorithms for finding a fixed point of the nonexpansive mapping, which solves a certain variational inequality related to a Lipschitzian and strongly monotone mapping. Jung [13] gave a general iterative algorithm for finding a fixed point of the k-strictly pseudocontractive mapping.

In this paper, inspired and motivated by the above mentioned results, we introduce a general composite iterative algorithm for finding a common point of the set of solutions of the VIP (1.1) for a hemicontinuous monotone mapping A and the set of fixed points of a hemicontinuous pseudocontractive mapping T. We establish strong convergence of the sequence generated by the proposed iterative algorithm to a common point of the above two sets, which solves a certain variational inequality related to a boundedly Lipschitzian and strongly monotone mapping. As a direct consequence, we find the unique solution of the minimum-norm problem: find $x^* \in Fix(T) \cap VI(C, A)$ such that

$$\|x^*\| = \min\{\|x\| : x \in Fix(T) \cap VI(C, A)\}.$$

Our results extend and unify the corresponding results of Ceng *et al.* [5], Chen *et al.* [6], Iiduka and Takahashi [8], Jung [12], Su et al. [16], Tian [21], Wangkeeree and Nammanee [22], Zegeye [25], Zegeye and Shahzad [26], and some recent results in the literature.

2. Preliminaries and Lemmas

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. We denote by $S(x : R)$ the closed ball with center $x \in H$ and radius $R > 0$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. $P_C(x)$ is characterized by the property:

$$u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall x \in H,\ y \in C. \tag{2.1}$$

We recall that a mapping A of H into H is called

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(i) *monotone* if $\langle x-y, Ax-Ay\rangle \geq 0, \quad \forall x,\ y \in H$;

(ii) *α-inverse-strongly monotone* ([9,14]) if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x,\ y \in H;$$

(iii) *strongly monotone* if there exists a positive real number η such that

$$\langle x-y, Ax-Ay\rangle \geq \eta\|x-y\|^2, \quad \forall x,\ y \in H;$$

(iv) *Lipschitzian continuous* if there exists $L > 0$ such that

$$\|Ax-Ay\| \leq L\|x-y\|, \quad \forall x,\ y \in H;$$

(v) *hemicontinuous* ([1,17]) if, for all $x, y \in H$, the mapping $g : [0,1] \to H$ defined by $g(t) = A(tx+(1-t)y)$ is continuous, where H has a weak topology;

(vi) *boundedly Lipschitzian* on C, if for each nonempty bounded subset S on C, there exists a positive constant $k_S > 0$ depending only on the set S such that $\|Ax-Ay\| \leq k_S\|x-y\|, \quad \forall x,\ y \in S$.

We note that (i) if A is a monotone mapping, then $T = I-A$ is a pseudocontractive mapping, and (ii) the class of the Lipschitzian mappings is a proper subclass of the class of the boundedly Lipschitzian mappings. It is easy to see that if $T : C \to H$ is continuous on C, then T is hemicontinuous on C and bounded on any line segment of C, but the converse is not true (see Example 1.10.14 in [1]).

The following lemmas can be easily proven, and therefore, we omit the proofs (see [10,24]).

Lemma 2.1. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$, and let $F : H \to H$ be a boundedly Lipschitzian and η-strongly monotone mapping with constant $\eta > 0$. Take $x_0 \in H$ arbitrarily and set $\widehat{C} = S(x_0, R)$ for some $R > 0$. Denote by $\widehat{\kappa}$ the Lipschitz constant of F on $\widehat{C}$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu F-\gamma V)x-(\mu F-\gamma V)y, x-y\rangle \geq (\mu\eta-\gamma l)\|x-y\|^2, \ \ \forall x,\ y \in \widehat{C}.$$

That is, $\mu F-\gamma V$ is strongly monotone on $\widehat{C}$ with constant $\mu\eta-\gamma l$.

Lemma 2.2. *Let H be a real Hilbert space H. Let $F : H \to H$ be a boundedly Lipschitzian and η-strongly monotone mapping with constant $\eta > 0$. Take $x_0 \in H$ arbitrarily and set $\widehat{C} = S(x_0, R)$ for some $R > 0$. Denote by $\widehat{\kappa}$ the Lipschitz constant of F on $\widehat{C}$ Let $0 < \mu < \frac{2\eta}{\widehat{\kappa}^2}$ and $0 < t < \rho \leq 1$. Then $G := \rho I - t\mu F$ restricted to $\widehat{C}$ is a contractive mapping with constant $\rho - t\tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\widehat{\kappa}^2)}$.*

By a similar arguments in [2], we obtain the following lemma for the hemicontinuous monotone mapping, which extends Lemma 2.3 of Zegeye [25].

Lemma 2.3. *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a hemicontinuous monotone mapping. Suppose that for each $x, y \in C$, there exists $\tau_{xy} > 0$ such that $A(tx + (1-t)y) < \tau_{xy}$ for all $t \in [0,1]$; that is, A is bounded on any line segment on C. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Az \rangle + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Proof. Since $A : C \to H$ is a hemicontinuous mapping, for $x, y \in C$, the mapping $g : [0,1] \to H$ defined by $g(t) = A(tx + (1-t)y)$ is continuous, where H has a weak topology, and so A is bounded on any line segment on C. Thus, by taking $f(z, y) = \langle y - z, A(z) \rangle$ as a bifunction $f : C \times C \to \mathbb{R}$ in [2], the result follows from a similar argument in [2].

Moreover, by a similar argument in [7,18] together with Lemma 2.3, we have the following lemma, which improves Lemma 2.4 of Zegeye [25].

Lemma 2.4. *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a hemicontinuous monotone mapping. Suppose that for each $x, y \in C$, there exists $\tau_{xy} > 0$ such that $A(tx + (1-t)y) < \tau_{xy}$ for all $t \in [0,1]$; that is, A is bounded on any line segment on C. For $\lambda > 0$ and $x \in H$, define $A_\lambda : H \to C$ by*

$$A_\lambda x = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{\lambda}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) A_λ *is single-valued;*
(ii) A_λ *is firmly nonexpansive, that is,*

$$\|A_\lambda x - A_\lambda y\|^2 \leq \langle x - y, A_\lambda x - A_\lambda y \rangle, \quad \forall x,\ y \in H;$$

(iii) $Fix(A_\lambda) = VI(C, A)$*;*
(iv) $VI(C, A)$ *is a closed convex subset of* C

Proof. Let $f(z, y) = \langle y - z, Az \rangle$ as a bifunction $f : C \times C \to \mathbb{R}$ in [7]. Then the result follows from similar arguments in [2] and [7].

Applying Lemma 2.3 and lemma 2.4, we get the following lemmas for the hemicontinuous pseudocontractive mapping, which generalize Lemma 3.1 and Lemma 3.2 of Zegeye [25], respectively.

Lemma 2.5. Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a hemicontinuous pseudocontractive mapping. Suppose that T is bounded on any line segment on C. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that

$$\langle y - z, Tz \rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

Proof. Let $A := I - T$, where I is the identity mapping on C. Then, T is a hemicontinuous pseudocontractive mapping and T is bounded on any line segment of C, A is clearly hemicontinuous monotone mapping and bounded on any line segment of C. Thus, by Lemma 2.3, there exists $z \in C$ such that $\langle y - z, Az\rangle + (1/r)\langle y - z, z - x\rangle \geq 0$ for all $y \in C$. But this is equivalent to $\langle y - z, Tz\rangle - (1/r)\langle y - z, (1 + r)z - x\rangle \leq 0$ for all $y \in C$. Hence the result holds.

Lemma 2.6. *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a hemicontinuous pseudocontractive mapping. Suppose that T is bounded on any line segment on C. For $r > 0$ and $x \in H$, define $T_r : H \to C$ by*

$$T_r x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$;*
(iv) *$Fix(T)$ is a closed convex subset of C*

Proof. We note that $\langle y - z, Tz\rangle - (1/r)\langle y - z, (1 + r)z - x\rangle \leq 0$, for all $y \in C$, is equivalent to $\langle y - z, Az\rangle + (1/r)\langle y - z, z - x\rangle \geq 0$, for all $y \in C$, where $A := I - T$ is a hemicontinuous monotone mapping and I is the identity mapping on C. Moreover, as T is a self-mapping, we get that $VI(C, A) = Fix(T)$. Thus, by Lemma 2.4, the conclusions of (i)–(iv) hold.

We also need the following lemmas for the proof of our main results.

Lemma 2.7. *In a real Hilbert space H, there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.8. ([23]) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-\lambda_n)s_n + \beta_n + \gamma_n, \quad \forall n \ge 1,$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1-\lambda_n) = 0$;
(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$;
(iii) $\gamma_n \ge 0$ $(n \ge 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$;
- C is a nonempty closed convex subset of H;
- $A : C \to H$ is a hemicontinuous monotone mapping with $VI(C, A) \neq \emptyset$ and is bounded on any line segment of C;
- $T : C \to C$ is a hemicontinuous pseudocontractive mapping with $Fix(T) \neq \emptyset$ and is bounded on any line segment of C;
- $A_{\lambda_n} : H \to C$ is a mapping defined by

$$A_{\lambda_n} x = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{\lambda_n} \langle y - z, z - x \rangle \ge 0, \quad \forall y \in C \right\},$$

  where $\{\lambda_n\} \subset (0, \infty)$;
- $T_{r_n} : H \to C$ is a mapping defined by

$$T_{r_n} x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n} \langle y - z, (1+r)z - x \rangle \le 0, \quad \forall y \in C \right\},$$

  where $\{r_n\} \subset (0, \infty)$;
- $F : H \to H$ is a boundedly Lipschitzian and η-strongly monotone mapping with constant $\eta > 0$;
- $V : H \to H$ is an l-Lipschitzian mapping with constant $l > 0$;
- $\Omega := VI(C, A) \cap Fix(T) \neq \emptyset$

By Lemma 2.4 and Lemma 2.6, we note that A_{λ_n} and T_{r_n} are firmly nonexpansive and so nonexpansive, and $VI(C, A) = Fix(A_{\lambda_n})$ and $Fix(T_{r_n}) = Fix(T)$.

Now, we present a new composite iterative algorithm for hemicontinuous monotone mappings and hemicontinuous pseudocontractive mappings and establish strong convergence of this algorithm.

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Theorem 3.1. *Let $x_0 \in \Omega$ be chosen arbitrarily. Set $\widehat{C} = S(x_0, \frac{\gamma\|Vx_0\|+\mu\|Fx_0\|}{\tau-\gamma l}) \cap C$ and denote by $\widehat{\kappa}$ the Lipschitz constant of F on $\widehat{C}$, where the constants μ, γ and τ are such that $0 < \mu < \frac{2\eta}{\widehat{\kappa}^2}$, $0 \leq \gamma l < \tau$ and, $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\widehat{\kappa}^2)}$, respectively. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} A_{\lambda_n} x_n, \\ x_{n+1} = (1-\beta_n) y_n + \beta_n T_{r_n} A_{\lambda_n} y_n, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1)$ and $\{\lambda_n\}$, $\{r_n\} \subset (0,\infty)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the conditions:

(C1) $\alpha_n \to 0$ $(n \to \infty)$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C4) $\beta_n \subset [0,a)$ *for all* $n \geq 0$ *and for some* $a \in (0,1)$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(C5) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
(C6) $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $q \in \Omega$, which is a solution of the following variational inequality

$$\langle (\gamma V - \mu F)q, q - p \rangle \geq 0, \quad \forall p \in \Omega. \tag{3.2}$$

Proof. Note that from the condition (C1), without loss of generality, we assume that $2\alpha_n(\tau - \gamma l) < 1$ and $\alpha_n < 1 - \beta_n - \alpha_n$ for $n \geq 1$. For $K = P_\Omega$, it follows that $K(I + \gamma V - \mu F)$ is a contractive mapping of $\widehat{C}$ into Ω. In fact, from Lemma 2.2, we have, for any $x, y \in \widehat{C}$,

$$\begin{aligned} \|K(I+\gamma V - \mu F)x - &(I + \gamma V - \mu F)y\| \\ &\leq \|(I + \gamma V - \mu F)x - (I + \gamma V - \mu F)y\| \\ &\leq \gamma\|Vx - Vy\| + \|(I - \mu F)x - (I - \mu F)y\| \\ &\leq \gamma l\|x - y\| + (1-\tau)\|x - y\| \\ &= (1 - (\tau - \gamma l))\|x - y\|. \end{aligned}$$

This is, $K(I + \gamma V - \mu F)$ is a contractive mapping with constant $(1 - (\tau - \gamma l))$. Since $\widehat{C}$ is complete, there exists a unique element $q \in \widehat{C}$ such that $q = P_\Omega(I + \gamma V - \mu F)q$. Equivalently, by (2.1), q is the unique solution of the variational inequality:

$$\langle (\gamma V - \mu F)q, q - p \rangle \geq 0, \quad \forall p \in \Omega.$$

In fact, noting that $0 \leq \gamma l < \tau$ and $\mu\eta \geq \tau \iff \widehat{\kappa} \geq \eta$, it follows from Lemma 2.1 that

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y \rangle \geq (\mu\eta - \gamma l)\|x - y\|^2.$$

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That is, $\mu F - \gamma V$ is strongly monotone on $\widehat{C}$ for $0 \leq \gamma l < \tau \leq \mu\eta$. Hence the variational inequality (3.2) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the variational inequality (3.2):

From now, put $z_n = A_{\lambda_n} x_n$, $u_n = T_{r_n} z_n$, $w_n = A_{\lambda_n} y_n$, and $v_n = T_{r_n} w_n$ for every $n \geq 0$.

Now, we divide the proof into several steps.

Step 1. We show that $x_n \in \widehat{C}$ for all $n \geq 0$ by induction, and hence $\{x_n\}$ is bounded. It is obvious that $x_0 \in \widehat{C}$. First of all, from Lemma 2.4 (iii) and Lemma 2.6 (iii), we observe that $VI(C, A) = Fix(A_{\lambda_n})$ and $Fix(T) = Fix(T_{r_n})$. Then, it follows that

$$\|z_n - x_0\| = \|A_{\lambda_n} x_n - x_0\| \leq \|x_n - x_0\|,$$

and

$$\|w_n - x_0\| = \|A_{\lambda_n} y_n - x_0\| \leq \|y_n - x_0\|.$$

Now, suppose that we have proved $x_n \in \widehat{C}$, that is,

$$\|x_n - x_0\| \leq \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}.$$

Using lemma 2.2, Lemma 2.4 (ii), and Lemma 2.6 (ii), we derive that

$$\begin{aligned}
\|y_n - x_0\| &= \|\alpha_n(\gamma V x_n - \mu F x_0) + (I - \alpha_n \mu F) T_{r_n} A_{\lambda_n} x_n - (I - \alpha_n \mu F) x_0\| \\
&\leq \|(I - \alpha_n \mu F) T_{r_n} z_n - (I - \alpha_n \mu F) x_0\| + \|\alpha_n(\gamma V x_n - \mu F x_0)\| \\
&\leq (1 - \tau\alpha_n)\|z_n - x_0\| + \alpha_n \gamma \|V x_n - V x_0\| + \alpha_n \|\gamma V x_0 - \mu F x_0\| \\
&\leq (1 - \tau\alpha_n)\|x_n - x_0\| + \alpha_n \gamma l \|x_n - x_0\| + \alpha_n \|\gamma V x_0 - \mu F x_0\| \\
&\leq (1 - (\tau - \gamma l)\alpha_n)\|x_n - x_0\| + (\tau - \gamma l)\alpha_n \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l} \\
&\leq \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}.
\end{aligned}$$

This implies $y_n \in \widehat{C}$ and

$$\begin{aligned}
\|x_{n+1} - x_0\| &= \|(1 - \beta_n)(y_n - x_0) + \beta_n(T_{r_n} A_{\lambda_n} y_n - x_0)\| \\
&\leq \|(1 - \beta_n)\|y_n - x_0\| + \beta_n \|T_{r_n} w_n - x_0\| \\
&\leq (1 - \beta_n)\|y_n - x_0\| + \beta_n \|w_n - x_0\| \\
&\leq (1 - \beta_n)\|y_n - x_0\| + \beta_n \|y_n - x_0\| \\
&= \|y_n - p\| \\
&\leq \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}.
\end{aligned}$$

It prove that $x_{n+1} \in \widehat{C}$. Therefore, $x_n \in \widehat{C}$ for all $n \geq 0$. Thus, $\{x_n\}$ is bounded.

It is not difficult to verify that that the sequences $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{Vx_n\}$, $\{Fx_n\}$, $\{Fy_n\}$, $\{Fu_n\}$, are bounded. Moreover, since $\|u_n - x_0\| = \|T_{r_n} z_n - x_0\| \leq \|x_n - x_0\|$ and $\|v_n - x_0\| = \|T_{r_n} w_n - x_0\| \leq \|y_n - x_0\|$, $\{u_n\}$ and $\{v_n\}$ are also bounded. And, by the condition (C1), we have

$$\begin{aligned} \|y_n - u_n\| &= \|y_n - T_{r_n} z_n\| \\ &= \alpha_n \|\gamma V x_n - \mu F T_{r_n} z_n\| \\ &\leq \alpha_n (\gamma \|V x_n\| + \mu \|F u_n\|) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.3}$$

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$. Indeed, since $z_n = A_{\lambda_n} x_n$ and $z_{n-1} = A_{\lambda_{n-1}} x_{n-1}$, we have

$$\langle y - z_n, A z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \tag{3.4}$$

and

$$\langle y - z_{n-1}, A z_{n-1} \rangle + \frac{1}{\lambda_{n-1}} \langle y - z_{n-1}, z_{n-1} - x_{n-1} \rangle \geq 0, \quad \forall y \in C, \tag{3.5}$$

Putting $y := z_{n-1}$ in (3.4) and $y := z_n$ in (3.5), we get

$$\langle z_{n-1} - z_n, A z_n \rangle + \frac{1}{\lambda_n} \langle z_{n-1} - z_n, z_n - x_n \rangle \geq 0, \tag{3.6}$$

and

$$\langle z_n - z_{n-1}, A z_{n-1} \rangle + \frac{1}{\lambda_{n-1}} \langle z_n - z_{n-1}, z_{n-1} - x_{n-1} \rangle \geq 0. \tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$\langle z_n - z_{n-1}, A z_{n-1} - A z_n \rangle + \left\langle z_n - z_{n-1}, \frac{z_{n-1} - x_{n-1}}{\lambda_{n-1}} - \frac{z_n - x_n}{\lambda_n} \right\rangle \geq 0,$$

which implies

$$-\langle z_n - z_{n-1}, A z_n - A z_{n-1} \rangle + \left\langle z_n - z_{n-1}, \frac{z_{n-1} - x_{n-1}}{\lambda_{n-1}} - \frac{z_n - x_n}{\lambda_n} \right\rangle \geq 0. \tag{3.8}$$

Since A is monotone, from (3.8) we get

$$\left\langle z_n - z_{n-1}, \frac{z_{n-1} - x_{n-1}}{\lambda_{n-1}} - \frac{z_n - x_n}{\lambda_n} \right\rangle \geq 0,$$

and hence

$$\left\langle z_n - z_{n-1}, z_{n-1} - z_n + z_n - x_{n-1} - \frac{\lambda_{n-1}}{\lambda_n} (z_n - x_n) \right\rangle \geq 0.$$

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Without loss of generality, let us assume that there exists a real number λ such that $\lambda_n > \lambda > 0$ for all $n \geq 0$. Then we have

$$\begin{aligned} \|z_n - z_{n-1}\|^2 &\leq \left\langle z_n - z_{n-1}, x_n - x_{n-1} + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right)(z_n - x_n)\right\rangle \\ &\leq \|z_n - z_{n-1}\| \left\{ \|x_n - x_{n-1}\| + \left|1 - \frac{\lambda_{n-1}}{\lambda_n}\right| \|z_n - x_n\| \right\}, \end{aligned} \tag{3.9}$$

and hence from (3.9) we obtain

$$\begin{aligned} \|z_n - z_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{\lambda_n}|\lambda_n - \lambda_{n-1}|\|z_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_1, \end{aligned} \tag{3.10}$$

where $L_1 = \sup\{\|z_n - x_n\| : n \geq 0\} < \infty$. Using the same method, we also get

$$\|w_n - w_{n-1}\| \leq \|y_n - y_{n-1}\| + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2, \tag{3.11}$$

where $L_2 = \sup\{\|w_n - y_n\| : n \geq 0\} < \infty$.

Moreover, since $u_{n-1} = T_{r_{n-1}} z_{n-1}$ and $u_n = T_{r_n} z_n$, we have

$$\langle y - u_{n-1}, Tu_{n-1}\rangle - \frac{1}{r_{n-1}}\langle y - u_{n-1}, (1 + r_{n-1})u_{n-1} - z_{n-1}\rangle \leq 0, \quad \forall y \in C, \tag{3.12}$$

and

$$\langle y - u_n, Tu_n\rangle - \frac{1}{r_n}\langle y - u_n, (1 + r_n)u_n - z_n\rangle \leq 0, \quad \forall y \in C, \tag{3.13}$$

Putting $y := u_n$ in (3.12) and $y := u_{n-1}$ in (3.13), we get

$$\langle u_n - u_{n-1}, Tu_{n-1}\rangle - \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, (1 + r_{n-1})u_{n-1} - z_{n-1}\rangle \leq 0, \tag{3.14}$$

and

$$\langle u_{n-1} - u_n, Tu_n\rangle - \frac{1}{r_n}\langle u_{n-1} - u_n, (1 + r_n)u_n - z_n\rangle \leq 0. \tag{3.15}$$

Adding (3.14) and (3.15), we obtain

$$\begin{aligned} &\langle u_n - u_{n-1}, Tu_{n-1} - T_{u_n}\rangle \\ &\quad - \left\langle u_n - u_{n-1}, \frac{(1 + r_{n-1})u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{(1 + r_n)u_n - z_n}{r_n}\right\rangle \leq 0, \end{aligned}$$

which implies that

$$\begin{aligned} &\langle u_n - u_{n-1}, (u_n - Tu_n) - (u_{n-1} - Tu_{n-1})\rangle \\ &\qquad\qquad - \left\langle u_n - u_{n-1}, \frac{u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{u_n - z_n}{r_n}\right\rangle \leq 0. \end{aligned}$$

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Now, since T is pseudocontractive, we obtain

$$\left\langle u_n - u_{n-1}, \frac{u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{u_n - z_n}{r_n} \right\rangle \geq 0,$$

and hence

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - z_{n-1} - \frac{r_{n-1}}{r_n}(u_n - z_n)\rangle \geq 0.$$

Also, we can assume that $r_n > r > 0$ for all n and for some $r > 0$. Thus, using the method in (3.9) and (3.10), we deduce

$$\|u_n - u_{n-1}\| \leq \|z_n - z_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_3, \tag{3.17}$$

where $L_3 = \sup\{\|u_n - z_n\| : n \geq 0\}$. Also, using the same method, we have

$$\|v_n - v_{n-1}\| \leq \|w_n - w_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_4, \tag{3.18}$$

where $L_4 = \sup\{\|v_n - w_n\| : n \geq 0\}$.

Now, simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} A_{\lambda_n} x_n - \alpha_{n-1} \gamma V x_{n-1} \\ &\quad - (I - \alpha_{n-1}\mu F) T_{r_{n-1}} A_{\lambda_{n-1}} x_{n-1} \\ &= \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} z_n - \alpha_{n-1}\gamma V x_{n-1} \\ &\quad - (I - \alpha_{n-1}\mu F) T_{r_{n-1}} z_{n-1} \\ &= (\alpha_n - \alpha_{n-1})(\gamma V x_{n-1} - \mu F u_{n-1}) + \alpha_n \gamma (V x_n - V x_{n-1}) \\ &\quad + (I - \alpha_n \mu F) u_n - (I - \alpha_n \mu F) u_{n-1}. \end{aligned}$$

By (3.17) and Lemma 2.2, we obtain

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq |\alpha_n - \alpha_{n-1}|(\gamma \|V x_{n-1}\| + \mu \|F u_{n-1}\|) \\ &\quad + \alpha_n \gamma l \|x_n - x_{n-1}\| + (1 - \tau\alpha_n)\|u_n - u_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|(\gamma \|V x_{n-1}\| + \mu \|F u_{n-1}\|) + \alpha_n \gamma l \|x_n - x_{n-1}\| \\ &\quad + (1 - \tau\alpha_n)\|z_n - z_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_3. \end{aligned} \tag{3.19}$$

Also, observe that

$$\begin{aligned} x_{n+1} - x_n &= (1 - \beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(T_{r_{n-1}} w_{n-1} - y_{n-1}) \\ &\quad + \beta_n (T_{r_n} w_n - T_{r_{n-1}} w_{n-1}) \\ &= (1 - \beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(v_{r_{n-1}} - y_{n-1}) \\ &\quad + \beta_n (v_n - v_{n-1}). \end{aligned} \tag{3.20}$$

By (3.10), (3.11), (3.18), (3.19), and (3.20), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le (1-\beta_n)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \beta_n\|v_n - v_{n-1}\| \\
&\le (1-\beta_n)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \beta_n\|w_n - w_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_4 \\
&\le (1-\beta_n)\|y_n - y_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2 + \frac{1}{r}|r_n - r_{n-1}|L_4 \\
&= \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2 + \frac{1}{r}|r_n - r_{n-1}|L_4 \qquad (3.21)\\
&\le \gamma l\alpha_n\|x_n - x_{n-1}| + (1-\tau\alpha_n)\|z_n - z_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\gamma\|x_{n-1}\| + \mu\|Fu_{n-1}\|) + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2 + \frac{1}{r}|r_n - r_{n-1}|(L_3 + L_4) \\
&\le (1-(\tau-\gamma l)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|Fu_{n-1}\|) \\
&\quad + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|(L_1 + L_2) + \frac{1}{r}|r_n - r_{n-1}|(L_3 + L_4) \\
&\le (1-(\tau-\gamma l)\alpha_n)\|x_n - x_{n-1}\| + M_1|\alpha_n - \alpha_{n-1}| + M_2|\beta_n - \beta_{n-1}| \\
&\quad + M_3|\lambda_n - \lambda_{n-1}| + M_4|r_n - r_{n-1}|,
\end{aligned}$$

where $M_1 = \sup\{\gamma\|Vx_n\| + \mu\|Fu_n\| : n \ge 0\}$, $M_2 = \sup\{\|v_n\| + \|y_n\| : n \ge 0\}$, $M_3 = \frac{1}{\lambda}(L_1 + L_2)$ and $M_4 = \frac{1}{r}(L_3 + L_4)$. From the conditions (C1) – (C6), it is easy to see that

$$\lim_{n\to\infty}(\tau - \gamma l)\alpha_n = 0, \quad \sum_{n=1}^{\infty}(\tau - \gamma l)\alpha_n = \infty,$$

and

$$\sum_{n=2}^{\infty}(M_1|\alpha_n - \alpha_{n-1}| + M_2|\beta_n - \beta_{n-1}| + M_3|\lambda_n - \lambda_{n-1}| + M_4|r_n - r_{n-1}|) < \infty.$$

Applying Lemma 2.8 to (3.21), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Moreover, by (3.10) and (3.19), we also have

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0 \text{ and } \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0.$$

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Step 3. We show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Indeed,

$$\begin{aligned}\|x_{n+1} - y_n\| &= \beta_n\|v_n - y_n\| \\ &\le \beta_n(\|v_n - u_n\| + \|u_n - y_n\|) \\ &\le a(\|w_n - z_n\| + \|u_n - y_n\|) \\ &\le a(\|y_n - x_n\| + \|u_n - y_n\|) \\ &\le a(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|u_n - y_n\|)\end{aligned}$$

which implies that

$$\|x_{n+1} - y_n\| \le \frac{a}{1-a}(\|x_{n+1} - x_n\| + \|u_n - y_n\|).$$

Obviously, by (3.3) and Step 2, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. This implies that that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \ \text{ as } n \to \infty. \tag{3.22}$$

By (3.2) and (3.22), we also have

$$\|x_n - u_n\| \le \|x_n - y_n\| + \|y_n - u_n\| \to 0 \ \text{ as } n \to \infty.$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|y_n - z_n\| = 0$. To this end, let $p \in \Omega$. Since $Fix(T) = Fix(T_{r_n})$ by Lemma 2.6 (iii), from Lemma 2.2, we have

$$\begin{aligned}&\|y_n - p\|^2 \\ &= \|\alpha_n(\gamma V x_n - \mu F p) + (I - \alpha_n\mu F)T_{r_n}A_{\lambda_n}x_n - (I - \alpha_n\mu F)p\|^2 \\ &\le (\alpha_n\|\gamma V x_n - \mu F p\| + \|(I - \alpha_n\mu F)T_{r_n}z_n - (I - \alpha_n\mu F)T_{r_n}p\|)^2 \\ &\le \alpha_n\|\gamma V x_n - \mu F p\|^2 + (1 - \tau\alpha_n)\|z_n - p\|^2 \\ &\quad + 2\alpha_n(1 - \tau\alpha_n)\|\gamma V x_n - \mu F p\|\|z_n - p\|.\end{aligned} \tag{3.23}$$

Moreover, since $VI(C, A) = Fix(A_{\lambda_n})$ by Lemma 2.4 (iii), from Lemma 2.4 (ii), we obtain

$$\begin{aligned}\|z_n - p\|^2 &= \|A_{\lambda_n}x_n - p\|^2 \\ &\le \langle A_{\lambda_n}x_n - A_{\lambda_n}p, x_n - p\rangle^2 \\ &= \langle z_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|z_n - p\|^2 + \|x_n - p\|^2 - \|x_n - z_n\|^2),\end{aligned}$$

and hence

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2. \tag{3.24}$$

Therefore, from (3.23) and (3.24), we deduce

$$\begin{aligned}\|y_n - p\|^2 \le \alpha_n\|\gamma V x_n - \mu F p\|^2 + (1-\tau\alpha_n)(\|x_n - p\|^2 - \|x_n - z_n\|^2) \\ + 2\alpha_n(1-\tau\alpha_n)\|\gamma V x_n - \mu F p\|\|z_n - p\|,\end{aligned}$$

and hence

$$\begin{aligned}&(1-\tau\alpha_n)\|x_n - z_n\|^2 \\ \le&\alpha_n\|\gamma V x_n - \mu F p\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\ &+ 2\alpha_n\|\gamma V x_n - \mu F p\|\|z_n - p\| \\ \le& \alpha_n\|\gamma V x_n - \mu F p\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\ &+ 2\alpha_n\|\gamma V x_n - \mu F p\|\|z_n - p\|.\end{aligned}$$

Since $\alpha_n \to 0$ by condition (C1) and $\|x_n - y_n\| \to 0$ by (3.22), we get $\|x_n - z_n\| \to 0$. Also, from (3.22), it follows that

$$\|y_n - z_n\| \le \|y_n - x_n\| + \|x_n - z_n\| \to 0 \ (n \to \infty). \tag{3.25}$$

Step 5. We show that $\lim_{n\to\infty}\|u_n - z_n\| = \|T_{r_n}z_n - z_n\| = 0$. Indeed, from (3.3) and (3.25), we get

$$\|u_n - z_n\| = \|T_{r_n}z_n - z_n\| \le \|u_n - y_n\| + \|y_n - z_n\| \to 0 \quad \text{as } n \to \infty.$$

Step 6. We show that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F))q, y_n - q\rangle \le 0,$$

where q is the unique solution of the variational inequality (3.2). First of all, from (3.3) and Step 4, without of loss generality, we may assume that u_n, z_n in $\widehat{C}$ for all $n \ge 0$.

First we prove that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)q, u_n - q\rangle \le 0.$$

To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)q, u_n - q\rangle = \lim_{i\to\infty}\langle(\gamma V - \mu F)q, u_{n_i} - q\rangle.$$

Since $\{u_{n_i}\}$ is bounded, we can choose a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ and $z \in H$ such that $u_{n_{i_j}} \rightharpoonup z$. Without loss of generality, we may assume that $u_{n_i} \rightharpoonup z$. Since $\widehat{C}$ is closed and convex, it is weakly closed and hence $z \in \widehat{C}$. Since $u_n - z_n \to 0$ as $n \to \infty$ by Step 5, we have $z_{n_i} \rightharpoonup z$.

Now, we show that $z \in \Omega$. First we prove that $z \in Fix(T)$. In fact, from definition z_{n_i}, we have

$$\langle y - u_{n_i}, Tu_{n_i}\rangle - \frac{1}{r_{n_i}}\langle y - u_{n_i}, (1 + r_{n_i})u_{n_i} - z_{n_i}\rangle \leq 0, \quad \forall y \in C. \tag{3.26}$$

Put $z_t = tv + (1 - t)z$ for all $t \in (0, 1]$ and $v \in C$. Then $z_t \in C$ and from (3.26) and pseudocontractivity of T, it follows that

$$\begin{aligned}
\langle u_{n_i} - z_t, Tz_t\rangle &\geq \langle u_{n_i} - z_t, Tz_t\rangle + \langle z_t - u_{n_i}, Tu_{n_i}\rangle \\
&\quad - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, (1 + r_{n_i})u_{n_i} - z_{n_i}\rangle \\
&= -\langle z_t - u_{n_i}, Tz_t - Tu_{n_i}\rangle - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - z_{n_i}\rangle \\
&\quad - \langle z_t - u_{n_i}, u_{n_i}\rangle \\
&\geq -\|z_t - u_{n_i}\|^2 - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - z_{n_i}\rangle \\
&\quad - \langle z_t - u_{n_i}, u_{n_i}\rangle \\
&= -\langle z_t - u_{n_i}, z_t\rangle - \langle z_t - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{r_{n_i}}\rangle.
\end{aligned} \tag{3.27}$$

Since $u_n - z_n \to 0$ as $n \to \infty$ by Step 5 and $\liminf_{n\to\infty} r_n > 0$ by condition (C6), we have $\frac{u_{n_i} - z_{n_i}}{r_{n_i}} \to 0$ as $i \to \infty$. Therefore, as $i \to \infty$ in (3.27), it follows that

$$\langle z - z_t, Tz_t\rangle \geq \langle z - z_t, z_t\rangle,$$

and hence

$$-\langle v - z, Tz_t\rangle \geq -\langle v - z, z_t\rangle, \quad \forall v \in C.$$

Letting $t \to 0$ and using the fact that T is hemicontinuous, we have

$$-\langle v - z, Tz\rangle \geq -\langle v - z, z\rangle, \quad \forall v \in C.$$

Now, let $v = Tz$. Then we obtain that $z = Tz$ and so $z \in Fix(T)$.

Next, let us show that $z \in VI(C, A)$. From the definition of z_n, we get that

$$\langle y - z_{n_i}, Az_{n_i}\rangle + \langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

Set $v_t = tv + (1 - t)z$ for all $t \in (0, 1]$ and $v \in C$. Then, it follows that $v_t \in C$. From (3.28), we have

$$\begin{aligned}
\langle v_t - z_{n_i}, Av_t\rangle &\geq \langle v_t - z_{n_i}, Av_t\rangle - \langle v_t - z_{n_i}, Az_{n_i}\rangle - \langle v_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\
&= \langle v_t - z_{n_i}, Av_t - Az_{n_i}\rangle - \langle v_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle.
\end{aligned}$$

From the fact that $\|z_n - x_n\| \to 0$ in Step 4 and $\liminf_{n\to\infty} \lambda_n > 0$ by condition (C5), it follows that $\frac{z_{n_i}-x_{n_i}}{\lambda_{n_i}} \to 0$ as $i \to \infty$.. Since A is monotone, we also have $\langle v_t - z_{n_i}, Av_t - Az_{n_i}\rangle \geq 0$. Thus, it follows that

$$0 \leq \lim_{i\to\infty} \langle v_t - z_{n_i}, Av_t\rangle = \langle v_t - z, Av_t\rangle,$$

and hence

$$\langle v - z, Av_t\rangle \geq 0, \quad \forall v \in C.$$

It $t \to 0$, the hemicontinuity A yields that

$$\langle v - z, Az\rangle \geq 0, \quad \forall v \in C.$$

This implies that $z \in VI(C, A)$. Therefore, $z \in \Omega$.

Now, since q is the unique solution of the variational inequality (3.2), from Step 5, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \langle(\gamma V - \mu F)q, u_n - q\rangle \\
&= \lim_{i\to\infty} \langle(\gamma V - \mu F)q, u_{n_i} - z_{n_i}\rangle + \lim_{i\to\infty} \langle(\gamma V - \mu F)q, z_{n_i} - q\rangle \\
&\leq \lim_{i\to\infty} \|(\gamma V - \mu F)q\|\|u_{n_i} - z_{n_i}\| + \lim_{i\to\infty} \langle(\gamma V - \mu F)q, z_{n_i} - q\rangle \\
&= \langle(\gamma V - \mu F)q, z - q\rangle \leq 0.
\end{aligned} \tag{3.29}$$

By (3.3) and (3.29) , we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle(\gamma V - \mu F)q, y_n - q\rangle \\
&\leq \limsup_{n\to\infty} \langle(\gamma V - \mu F)q, y_n - u_n\rangle + \limsup_{n\to\infty} \langle(\gamma V - \mu F)q, u_n - q\rangle \\
&\leq \limsup_{n\to\infty} \|(\gamma V - \mu F)q\|\|y_n - u_n\| + \limsup_{n\to\infty} \langle(\gamma V - \mu F)q, u_n - q\rangle \leq 0.
\end{aligned}$$

Step 7. We show that $\lim_{n\to\infty} \|x_n - q\| = 0$, where q is the unique solution of the variational inequality (3.2). Indeed, from (3.1), Lemma 2.2, and lemma 2.7, we derive

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|y_n - q\|^2 \\
&= \|\alpha_n(\gamma V x_n - \mu F q) + (I - \alpha_n \mu F)T_{r_n} A_{\lambda_n} x_n - (I - \alpha_n \mu F)q\|^2 \\
&\leq \|(I - \alpha_n \mu F)T_{r_n} z_n - (I - \alpha_n \mu F)q\|^2 + 2\alpha_n \langle \gamma V x_n - \mu F q, y_n - q\rangle \\
&\leq (1 - \tau\alpha_n)^2 \|z_n - q\|^2 + 2\alpha_n \gamma \langle V x_n - V q, y_n - q\rangle \\
&\quad + 2\alpha_n \langle \gamma V q - \mu F q, y_n - q\rangle) \\
&\leq (1 - \tau\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \gamma l \|x_n - q\|\|y_n - q\| \\
&\quad + 2\alpha_n \langle(\gamma V - \mu F)q, y_n - q\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\tau\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n\gamma l\|x_n-q\|(\|y_n-x_n\|+\|x_n-q\|)\\
&\quad + 2\alpha_n\langle(\gamma V-\mu F)q, y_n-q\rangle\\
&= (1-2(\tau-\gamma l)\alpha_n)\|x_n-q\|^2\\
&\quad + \alpha_n^2\tau^2\|x_n-q\|^2 + 2\alpha_n\gamma l\|x_n-q\|\|y_n-x_n\|\\
&\quad + 2\alpha_n\langle(\gamma V-\mu F)q, y_n-q\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq (1-2(\tau-\gamma l)\alpha_n)\|x_n-q\|^2 + \alpha_n^2\tau^2M_5^2 + 2\alpha_n\gamma l\|y_n-x_n\|M_5\\
&\quad + 2\alpha_n\langle(\gamma V-\mu F)q, y_n-q\rangle\\
&= (1-\overline{\alpha_n})\|x_n-q\|^2 + \overline{\beta_n},
\end{aligned}$$

where $M_5 = \sup\{\|x_n-q\| : n \geq 1\}$, $\overline{\alpha_n} = 2(\tau-\gamma l)\alpha_n$ and

$$\overline{\beta_n} = \alpha_n[\alpha_n\tau^2M_5^2 + 2\gamma l\|y_n-x_n\|M_5 + 2\langle(\gamma V-\overline{F})q, y_n-q\rangle].$$

From the conditions (C1) and (C2), $\|y_n-x_n\| \to 0$ in Step 3, and Step 6, it is easily seen that $\overline{\alpha_n} \to 0$, $\sum_{n=1}^{\infty}\overline{\alpha_n} = \infty$, and $\limsup_{n\to\infty}\frac{\overline{\beta_n}}{\overline{\alpha_n}} \leq 0$. Hence, by Lemma 2.8, we conclude $x_n \to q$ as $n \to \infty$. This completes the proof. □

By taking $F \equiv I$, $V \equiv 0$, $\mu = 1$, $\tau = 1$, and $l = 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.1. *Let H, C, A, T, T_{r_n} and A_{λ_n} be as in Theorem 3.1. Let $x_0 \in \Omega := Fix(T) \cap VI(C,A)$ be chosen arbitrarily and let $\widehat{C} = S(x_0, \|x_0\|) \cap C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = (1-\alpha_n)T_{r_n}A_{\lambda_n}x_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_nT_{r_n}A_{\lambda_n}y_n, \quad \forall n \geq 0, \end{cases} \tag{3.30}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1)$ and $\{\lambda_n\}$, $\{r_n\} \subset (0,\infty)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the conditions (C1) – (C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves the following minimum-norm problem: find $x^* \in \Omega$ such that*

$$\|x^*\| = \min_{x\in\Omega}\|x\|. \tag{3.31}$$

Proof. Take $F \equiv I$, $V \equiv 0$, $\mu = 1$, $\tau = 1$, and $l = 0$ in Theorem 3.1. Then the variational inequality (3.2) is reduced to the inequality

$$\langle q, q-p\rangle \leq 0, \quad \forall p \in \Omega.$$

This obviously implies that

$$\|q\|^2 \leq \langle q, p\rangle \leq \|q\|\|p\|, \quad \forall p \in \Omega.$$

It turns out that $\|q\| \leq \|p\|$ for all $p \in \Omega$. Therefore q is the minimum-norm point of Ω. $\square$

Taking $\beta_n = 0$ for $n \geq 0$ in Theorem 3.1 and Corollary 3.1, respectively, we derive the following results.

Corollary 3.2. *Let H, C, $\widehat{C}$, A, T, T_{r_n}, A_{λ_n}, F, V, γ, τ, $\widehat{\kappa}$, η, l and μ be as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by $x_0 \in \Omega$ and*

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} A_{\lambda_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\}, \{r_n\} \subset (0,\infty)$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the conditions (C1), (C2), (C3), (C5) *and* (C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in \Omega$, which is the unique solution of the variational inequality* (3.2).

Corollary 3.3. *Let H, C, A, T, T_{r_n} and A_{λ_n} be as in Theorem 3.1. Let $x_0 \in \Omega$ be chosen arbitrarily and let $\widehat{C} = S(x_0, \|x_0\|) \cap C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = (1 - \alpha_n) T_{r_n} A_{r_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $[0,1)$. Let $\{\alpha_n\}$ and $\{\lambda_n\}$, $\{r_n\} \subset (0,\infty)$ satisfy the conditions (C1), (C2), (C3), (C5) *and* C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves the following minimum-norm problem* (3.31).

As direct consequences of Theorem 3.1 along with $\beta_n = 0$ for $n \geq 0$, we also have the following results. First, if, in Theorem 3.1, we take that $A \equiv I$, the identity mapping on C, then we obtain the following corollary.

Corollary 3.4. *Let H, C, $\widehat{C}$, A, T, T_{r_n}, F, V, γ, τ, $\widehat{\kappa}$, η, l and μ be as in Theorem 3.1. Let $x_0 \in Fix(T)$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset [0,1)$ and $\{r_n\} \subset (0,\infty)$. Let $\{\alpha_n\}$ and $\{r_n\}$ satisfy the conditions (C1), (C2), (C3) *and* (C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in Fix(T)$, which is the unique solution of the variational inequality*

$$\langle (\gamma V - \mu F)q, q - p \rangle \geq 0, \quad \forall p \in Fix(T).$$

Next, if, in Theorem 3.1, $T \equiv I$ is the identity mapping on C along with $\beta_n = 0$ for $n \geq 0$, then we have the following corollary.

Corollary 3.5. *Let H, C, $\widehat{C}$, A, A_{λ_n}, F, V, γ, τ, $\widehat{\kappa}$, η, l and μ be as in Theorem 3.1. Let $x_0 \in VI(C,A)$ be chosen arbitrarily, and let $\widehat{C} = S(x_0, \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}) \cap C$.*

Let $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) A_{\lambda_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$*. Let* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy the conditions* (C1)*,* (C2)*,* (C3) *and* (C5) *in Theorem 3.1. Then* $\{x_n\}$ *converges strongly to* $q \in VI(C,A)$*, which is the unique solution of the variational inequality*

$$\langle (\gamma V - \mu F)q, q - p \rangle \geq 0, \quad \forall p \in VI(C,A).$$

Remark 3.1.

1) Our results extend and unify most of the results that have been established for these important classes of nonlinear mappings. In particular, Theorem 3.1 and Corollary 3.2 improve Theorem 3.1 of Jung [12] and Theorem 3.1 of Wangkeeree and Nammanee [22] and Theorem 3.1 of Zegeye and Shahzad [26], respectively, in the sense that our convergence is for more general classes of nonlinear mappings such as hemicintinuos monotone mappings, hemicontinuous pseudocontractive mappings, boundedly Lipschitzian and strongly monotone mappings, and Lipschizian mappings.
2) It is worth pointing out that the variable parameters λ_n and r_n in our iterative algorithms are used in comparison with the corresponding iterative algorithms in [22,25,26].
3) Corollary 3.2 also includes Proposition 3.1 of Chen et al. [6], Theorem 3.1 of Iiduka and Takahashi [8] and Corollary 3.2 of Su et al. [16] in the convergence sense for more general classes of nonlinear mappings mentioned in 1).
4) Corollary 3.1 and Corollary 3.3 are new results for finding the minimum-norm point of $Fix(T) \cap VI(C,A)$.
5) Corollary 3.4 and Corollary 3.5 also improve the corresponding results of Chen et al. [5], Tian [21], Wangkeeree and Nammanee [22] and Zegeye and Shahzad [26] in the sense that our results are for more general classes of nonlinear mappings.
6) As in Corollary 3.1, if we take $F \equiv I$, $V \equiv 0$, $\mu = 1$, $\tau = 1$, and $l = 0$ in Corollary 3.4 and Corollary 3.5, then we can find the minimum-norm point of $Fix(T)$ and $VI(C,A)$, respectively.

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Commutative ideals of BCK-algebras based on makgeolli structures

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Abstract The purpose of this paper is to study by applying the makgeolli structure to commutative ideal in BCK-algebras. The notion of commutative makgeolli ideal is introduced, and their properties are investigated. The relationship between makgeolli ideal and commutative makgeolli ideal is discussed. Example to show that a makgeolli ideal may not be a commutative makgeolli ideal is provided, and then the conditions under which a makgeolli ideal can be a commutative makgeolli ideal are explored. A new commutative makgeolli ideal is established using the given commutative makgeolli ideal, and characterizations of a commutative makgeolli ideal are displayed. Finally, the extension property for a commutative makgeolli ideal is established.

Keywords: BCK-soft universe, makgeolli structure, makgeolli ideal, commutative makgeolli ideal.

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1 Introduction

Many of the problems that need to be solved in the real world often include inherently inaccurate, uncertain, and ambiguous elements. The fuzzy set by Zadeh [26, 27, 28] is useful tool as a means of effectively controlling uncertainty, which is an attribute of information. Uncertainty is limited in handling using traditional mathematical tools, but can be handled using a wide range of theories such as probability theory, (intuitionistic) fuzzy set theory, theory of interval mathematics, vague set theory, rough set theory, and soft set theory etc. Molodtsov [21] introduced the concept of a soft set as a new tool for dealing with uncertainties beyond the difficulties that plagued general theoretical approaches, and he suggested several directions for the application of the soft set. Globally, interest in soft set theory and its application has been growing rapidly in recent years. Following this trend, research in the field of algebraic structure is also showing the use of soft sets. For example, groups, rings, fields and modules etc. (see [1, 3, 4, 5, 12]), and BCK/BCI-algebras etc. (see [9, 10, 11, 13, 14, 15, 16, 17, 22, 24]). In 2019, Ahn et al. [2] introduced the notion of makgeolli structures as a hybrid structure based on fuzzy set and soft set theory, and applied it to BCK/BCI-algebras. Kologani et al. [18] applied the makgeolli structure to hoops, and Song et al. [25] studied positive implicative makgeolli ideals of BCK-algebras.

In this paper, we apply the makgeolli structure to the commutative ideal of BCK-algebras. We introduce the notion of commutative makgeolli ideal, and investigate their properties. We discuss the relationship between makgeolli ideal and commutative makgeolli ideal. We provide example to show that any makgeolli ideal may not be a commutative makgeolli ideal, and then we explore the conditions under which makgeolli ideal can be commutative makgeolli ideal. We make a new commutative makgeolli ideal using the given commutative makgeolli ideal. We explore the characterization of commutative makgeolli ideal and establish the extension property for commutative makgeolli ideal.

2 Preliminaries

2.1 Preliminaries on BCK-algebras

BCI/BCK-algebra is an important type of logical algebra introduced by K. Iséki (see [7] and [8]), and it has been extensively investigated by several researchers. See the books [6, 20] for further information regarding BCI-algebras and BCK-algebras. In this section, we recall the definitions and basic results required in this paper.

Let L be a set with a special element "0" and a binary operation "$*$". If it satisfies the following conditions:

(I1) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\ (((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0)$,

(I2) $(\forall \mathfrak{a}, \mathfrak{b} \in L)\ ((\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0)$,

(I3) $(\forall \mathfrak{a} \in L)\ (\mathfrak{a} * \mathfrak{a} = 0)$,

(I4) $(\forall \mathfrak{a}, \mathfrak{b} \in L)\ (\mathfrak{a} * \mathfrak{b} = 0,\ \mathfrak{b} * \mathfrak{a} = 0\ \Rightarrow\ \mathfrak{a} = \mathfrak{b})$,

(K) $(\forall \mathfrak{a} \in L)\ (0 * \mathfrak{a} = 0)$,

then it is called a *BCK-algebra*, and it is denoted by $(L, *, 0)$.

The order relation " $\leq$ " in a BCK-algebra $(L, *, 0)$ is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \leq \mathfrak{b}\ \Leftrightarrow\ \mathfrak{a} * \mathfrak{b} = 0). \tag{2.1}$$

Every BCK/BCI-algebra $(L, *, 0)$ satisfies the following conditions (see [19, 20]):

$$(\forall \mathfrak{a} \in L)\,(\mathfrak{a} * 0 = \mathfrak{a})\,, \tag{2.2}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\,(\mathfrak{a} \leq \mathfrak{b}\ \Rightarrow\ \mathfrak{a} * \mathfrak{c} \leq \mathfrak{b} * \mathfrak{c},\ \mathfrak{c} * \mathfrak{b} \leq \mathfrak{c} * \mathfrak{a})\,, \tag{2.3}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = (\mathfrak{a} * \mathfrak{c}) * \mathfrak{b})\,. \tag{2.4}$$

Every BCI-algebra $(L, *, 0)$ satisfies (see [6]):

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)\,(\mathfrak{a} * (\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) = \mathfrak{a} * \mathfrak{b})\,, \tag{2.5}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)\,(0 * (\mathfrak{a} * \mathfrak{b}) = (0 * \mathfrak{a}) * (0 * \mathfrak{b}))\,. \tag{2.6}$$

A BCK-algebra $(L, *, 0)$ is said to be *commutative* (see [20]) if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} * (\mathfrak{a} * \mathfrak{b}) = \mathfrak{b} * (\mathfrak{b} * \mathfrak{a})). \tag{2.7}$$

A subset $\mathcal{R}$ of a BCK/BCI-algebra $(L, *, 0)$ is called

- a *subalgebra* of $(L, *, 0)$ (see [6, 20]) if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b} \in \mathcal{R})(\mathfrak{a} * \mathfrak{b} \in \mathcal{R}), \tag{2.8}$$

- an *ideal* of $(L, *, 0)$ (see [6, 20]) if it satisfies:

$$0 \in \mathcal{R}, \tag{2.9}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} * \mathfrak{b} \in \mathcal{R},\ \mathfrak{b} \in \mathcal{R}\ \Rightarrow\ \mathfrak{a} \in \mathcal{R}). \tag{2.10}$$

A subset $\mathcal{R}$ of a BCK-algebra $(L, *, 0)$ is called a *commutative ideal* of $(L, *, 0)$ (see [20]) if it satisfies (2.9) and

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} \in \mathcal{R},\ \mathfrak{c} \in \mathcal{R}\ \Rightarrow\ \mathfrak{a} * (\mathfrak{b} * (\mathfrak{b} * \mathfrak{a})) \in \mathcal{R}). \tag{2.11}$$

Lemma 2.1 ([20])**.** *A nonempty subset $\mathcal{R}$ of a BCK-algebra $(L, *, 0)$ is a commutative ideal of $(L, *, 0)$ if and only if $\mathcal{R}$ is an ideal of $(L, *, 0)$ that satisfies:*

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} * \mathfrak{b} \in \mathcal{R}\ \Rightarrow\ \mathfrak{a} * (\mathfrak{b} * (\mathfrak{b} * \mathfrak{a})) \in \mathcal{R}). \tag{2.12}$$

2.2 Preliminaries on makgeolli structures

Let L be a universal set and $\mathbb{E}$ a set of parameters. We say that the pair $(L, \mathbb{E})$ is a *soft universe*.

Definition 2.2 ([2]). Let $(L, \mathbb{E})$ be a soft universe and let $\mathcal{R}$ and $\mathcal{S}$ be subsets of $\mathbb{E}$. A *makgeolli structure* over $(L, \mathbb{E})$ (related to $\mathcal{R}$ and $\mathcal{S}$) is a structure of the form:

$$\mathbb{M}_{(\mathcal{R},\mathcal{S},L)} := \{\langle(\mathfrak{a},\mathfrak{b},z); f_{\mathcal{R}}(\mathfrak{a}), g_{\mathcal{S}}(\mathfrak{b}), \xi(z)\rangle \mid (\mathfrak{a},\mathfrak{b},z) \in \mathcal{R}\times\mathcal{S}\times L\} \tag{2.13}$$

where $f_{\mathcal{R}} := (f, \mathcal{R})$ and $g_{\mathcal{S}} := (g, \mathcal{S})$ are soft sets over L and ξ is a fuzzy set in L.

A fuzzy set ξ in a set L of the form

$$\xi(\mathfrak{b}) := \begin{cases} t \in (0,1] & \text{if } \mathfrak{b} = \mathfrak{a}, \\ 0 & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be a *fuzzy point* with support $\mathfrak{a}$ and value t and is denoted by $\langle\mathfrak{a}_t\rangle$.

For a fuzzy set ξ in a set L, we say that a fuzzy point $\langle\mathfrak{a}_t\rangle$ is

(i) *contained* in ξ, denoted by $\langle\mathfrak{a}_t\rangle \in \xi$, (see [23]) if $\xi(\mathfrak{a}) \geq t$.

(ii) *quasi-coincident* with ξ, denoted by $\langle\mathfrak{a}_t\rangle \, q \, \xi$, (see [23]) if $\xi(\mathfrak{a}) + t > 1$.

For the sake of simplicity, the makgeolli structure in (2.13) will be denoted by $\mathbb{M}_{(\mathcal{R},\mathcal{S},L)} := (f_{\mathcal{R}}, g_{\mathcal{S}}, \xi)$. The makgeolli structure $\mathbb{M}_{(\mathcal{R},\mathcal{R},L)} := (f_{\mathcal{R}}, g_{\mathcal{R}}, \xi)$ over $(L, \mathbb{E})$ related to a subset $\mathcal{R}$ of $\mathbb{E}$ is simply denoted by $\mathbb{M}_{(\mathcal{R},L)} := (f_{\mathcal{R}}, g_{\mathcal{R}}, \xi)$. If $\mathcal{R} = \mathcal{S} = \mathbb{E}$, we use the notation $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ as the makgeolli structure over $(L, \mathbb{E})$.

We say that a soft universe $(L, \mathbb{E})$ is a *BCK/BCI-soft universe* if L and $\mathbb{E}$ are BCK/BCI-algebras with binary operations "$*$" and "$\oslash$", respectively.

Definition 2.3 ([2]). Let $(L, \mathbb{E})$ be a BCK/BCI-soft universe. A makgeolli structure $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is called a *makgeolli ideal* of $(L, \mathbb{E})$ if it satisfies:

$$\begin{cases} (\forall \mathfrak{a} \in \mathbb{E}) \, (f_{\mathbb{E}}(0) \supseteq f_{\mathbb{E}}(\mathfrak{a}), \;\; g_{\mathbb{E}}(0) \subseteq g_{\mathbb{E}}(\mathfrak{a})) \,. \\ (\forall z \in L) \, (\langle 0/\xi(z)\rangle \in \xi) \,. \end{cases} \tag{2.14}$$

$$\begin{cases} (\forall \mathfrak{a},\mathfrak{b} \in \mathbb{E}) \begin{pmatrix} f_{\mathbb{E}}(\mathfrak{a}) \supseteq f_{\mathbb{E}}(\mathfrak{a} \oslash \mathfrak{b}) \cap f_{\mathbb{E}}(\mathfrak{b}) \\ g_{\mathbb{E}}(\mathfrak{a}) \subseteq g_{\mathbb{E}}(\mathfrak{a} \oslash \mathfrak{b}) \cup g_{\mathbb{E}}(\mathfrak{b}) \end{pmatrix}. \\ (\forall x, y \in L)(\forall t, r \in (0,1]) \begin{pmatrix} \langle (x * y)/t\rangle \in \xi, \; \langle y/r\rangle \in \xi \\ \Rightarrow \; \langle x/\min\{t,r\}\rangle \in \xi \end{pmatrix}. \end{cases} \tag{2.15}$$

Lemma 2.4 ([2]). *Let $(L, \mathbb{E})$ be a BCK/BCI-soft universe. Every makgeolli ideal $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ of $(L, \mathbb{E})$ satisfies the following assertions.*

(i) $$\begin{cases} (\forall \mathfrak{a},\mathfrak{b} \in \mathbb{E}) \left(\mathfrak{a} \leq \mathfrak{b} \;\Rightarrow \begin{cases} f_{\mathbb{E}}(\mathfrak{a}) \supseteq f_{\mathbb{E}}(\mathfrak{b}) \\ g_{\mathbb{E}}(\mathfrak{a}) \subseteq g_{\mathbb{E}}(\mathfrak{b}) \end{cases} \right). \\ (\forall x, y \in L) \, (x \leq y \;\Rightarrow\; \xi(x) \geq \xi(y)) \,. \end{cases}$$

$$\text{(ii)} \left\{ \begin{array}{l} (\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{E}) \left(\mathfrak{a} \oslash \mathfrak{b} \leq \mathfrak{c} \Rightarrow \left\{ \begin{array}{l} f_{\mathbb{E}}(\mathfrak{a}) \supseteq f_{\mathbb{E}}(\mathfrak{b}) \cap f_{\mathbb{E}}(\mathfrak{c}) \\ g_{\mathbb{E}}(\mathfrak{a}) \subseteq g_{\mathbb{E}}(\mathfrak{b}) \cup g_{\mathbb{E}}(\mathfrak{c}) \end{array} \right. \right). \\ (\forall x, y, z \in L) \, (x * y \leq z \Rightarrow \xi(x) \geq \min\{\xi(y), \xi(z)\}). \end{array} \right.$$

Let $(L, \mathbb{E})$ be a BCK/BCI-soft universe. Given a makgeolli structure $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ over $(L, \mathbb{E})$, consider the following sets:

$$\begin{aligned} f_{\mathbb{E}}(\mathbb{E}; \alpha) &:= \{\mathfrak{a} \in \mathbb{E} \mid f_{\mathbb{E}}(\mathfrak{a}) \supseteq \alpha\}, \\ g_{\mathbb{E}}(\mathbb{E}; \delta) &:= \{\mathfrak{b} \in \mathbb{E} \mid g_{\mathbb{E}}(\mathfrak{b}) \subseteq \delta\}, \\ \xi(L; t) &:= \{z \in L \mid \xi(z) \geq t\} \end{aligned}$$

where α and δ are subsets of L and $t \in [0, 1]$.

Lemma 2.5 ([2]). *A makgeolli structure $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ over a BCK/BCI-soft universe $(L, \mathbb{E})$ is a makgeolli ideal of $(L, \mathbb{E})$ if and only if the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$, and the nonempty set $\xi(L; t)$ is an ideal of $(L, *, 0)$ for all subsets α and δ of L and $t \in [0, 1]$.*

3 Commutative makgeolli ideals

In what follows, let $(Y, \mathbb{E})$ be a BCK-soft universe unless otherwise specified.

Definition 3.1. A makgeolli structure $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is called a *commutative makgeolli ideal* of $(Y, \mathbb{E})$ if it satisfies (2.14) and

$$(\forall \check{x}, \check{y}, \check{z} \in \mathbb{E}) \left(\begin{array}{l} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \\ g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \end{array} \right), \tag{3.1}$$

$$(\forall x, y, z \in Y)(\forall t, r \in (0, 1]) \left(\begin{array}{l} \langle ((x * y) * z)/t \rangle \in \xi, \ \langle z/r \rangle \in \xi \\ \quad \Rightarrow \ \langle (x * (y * (y * x)))/\min\{t, r\} \rangle \in \xi \end{array} \right). \tag{3.2}$$

Example 3.2. Consider a BCK-soft universe $(Y, \mathbb{E})$ where $Y := \{0, 1, 2, 3, 4\}$ and $\mathbb{E} := \{0, 1, 2, 3\}$ have binary operations "$*$" and "$\oslash$", respectively, given by Table 1.

Table 1: Cayley tables for the binary operations "$*$" and "$\oslash$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$\oslash$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ be a makgeolli structure over $(Y,\mathbb{E})$ defined as follows:

$$f_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} Y & \text{if } x = 0, \\ \{3,4\} & \text{if } x = 1, \\ \{1,3,4\} & \text{if } x = 2, \\ \{1,2,3,4\} & \text{if } x = 3, \end{cases}$$

$$g_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} \{4\} & \text{if } x = 0, \\ \{0,1,4\} & \text{if } x = 1, \\ \{1,4\} & \text{if } x = 2, \\ \{0,1,3,4\} & \text{if } x = 3, \end{cases}$$

and

$$\xi : Y \to [0,1],\ y \mapsto \begin{cases} 0.79 & \text{if } y = 0, \\ 0.62 & \text{if } y = 1, \\ 0.62 & \text{if } y = 2, \\ 0.45 & \text{if } y = 3, \\ 0.67 & \text{if } y = 4. \end{cases}$$

It is routine to verify that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$.

We discuss the relationship between the commutative makgeolli ideal and the makgeolli ideal.

Theorem 3.3. *Every commutative makgeolli ideal is a makgeolli ideal.*

Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ be a commutative makgeolli ideal of $(Y,\mathbb{E})$. If we put $\breve{y} = 0 = y$ in (3.1) and (3.2) and use (K) and (2.2), then we get (2.15). Hence $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a makgeolli ideal of $(Y,\mathbb{E})$. $\square$

The following example informs the existence of the makgeolli ideal, not the commutative makgeolli ideal.

Example 3.4. Consider a BCK-soft universe $(Y,\mathbb{E})$ in which $Y = \{0,1,2,3,4\} = \mathbb{E}$ with binary operations "$*$" and "$\oslash$", respectively, given by Table 2.

Table 2: Cayley tables for the binary operations "$*$" and "$\oslash$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

$\oslash$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli structure on $(Y, \mathbb{E})$ defined as follows:

$$f_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} Y & \text{if } x = 0, \\ \{1,2,4\} & \text{if } x = 1, \\ \{0,1,3,4\} & \text{if } x = 2, \\ \{1,4\} & \text{if } x = 3, \\ \{0,2\} & \text{if } x = 4, \end{cases}$$

$$g_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} \{4\} & \text{if } x = 0, \\ \{0,2,4\} & \text{if } x = 1, \\ \{1,4\} & \text{if } x = 2, \\ \{0,2,4\} & \text{if } x = 3, \\ \{0,1,2,4\} & \text{if } x = 4, \end{cases}$$

and

$$\xi : Y \to [0,1],\ y \mapsto \begin{cases} 0.73 & \text{if } y = 0, \\ 0.63 & \text{if } y = 1, \\ 0.54 & \text{if } y = 2, \\ 0.42 & \text{if } y = 3, \\ 0.42 & \text{if } y = 4. \end{cases}$$

It is routine to verify that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a makgeolli ideal of $(Y, \mathbb{E})$. But it is not a commutative makgeolli ideal of $(Y, \mathbb{E})$ since

$$f_{\mathbb{E}}(2 \oslash (4 \oslash (4 \oslash 2))) = f_{\mathbb{E}}(2) = \{0,1,3,4\} \not\supseteq \{1,2,4\} = f_{\mathbb{E}}((2 \oslash 4) \oslash 1) \cap f_{\mathbb{E}}(1)$$

and/or $\langle((2 * 3) * 0)/0.71\rangle \in \xi$ and $\langle 0/0.65\rangle \in \xi$, but

$$\langle(2 * (3 * (3 * 2)))/\min\{0.71, 0.65\} = \langle 2/0.65\rangle \,\overline{\in}\, \xi.$$

We explore the conditions for the makgeolli ideal to be the commutative makgeolli ideal.

Theorem 3.5. *In a commutative BCK-algebra, every makgeolli ideal is a commutative makgeolli ideal.*

Proof. Let $(Y, \mathbb{E})$ be a BCK-soft universe in which $(Y, *, 0)$ and $(\mathbb{E}, \oslash, 0)$ are commutative BCK-algebras, and let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli ideal of $(Y, \mathbb{E})$. Using (I1), (I3), (2.1), (2.4) and the commutativity of Y and $\mathbb{E}$, we have

$$(\forall \check{x}, \check{y}, \check{z} \in \mathbb{E})((\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash ((\check{x} \oslash \check{y}) \oslash \check{z}) \leq \check{z}),$$
$$(\forall x, y, z \in \mathbb{E})((x * (y * (y * x))) * ((x * y) * z) \leq z).$$

It follows from Lemma 2.4(ii) that

$$f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}),$$
$$g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}),$$

and

$$\xi(x * (y * (y * x))) \geq \min\{\xi((x * y) * z), \xi(z)\}. \tag{3.3}$$

Let $x, y, z \in Y$ and $t, r \in (0, 1]$ be such that $\langle((x * y) * z)/t\rangle \in \xi$ and $\langle z/r\rangle \in \xi$. Then $\xi((x * y) * z) \geq t$ and $\xi(z) \geq r$, and so

$$\xi(x * (y * (y * x))) \geq \min\{\xi((x * y) * z), \xi(z)\} \geq \min\{t, r\}$$

by (3.3). Hence $\langle(x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi$. Therefore $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. □

Corollary 3.6. *If a BCK-soft universe $(Y, \mathbb{E})$ satisfies any one of the following conditions:*

$$\begin{cases} (\forall \breve{x}, \breve{y} \in \mathbb{E})\,(\breve{x} \oslash (\breve{x} \oslash \breve{y}) \leq \breve{y} \oslash (\breve{y} \oslash \breve{x}))\,, \\ (\forall x, y \in Y)\,(x * (x * y) \leq y * (y * x))\,, \end{cases} \tag{3.4}$$

$$\begin{cases} (\forall \breve{x}, \breve{y} \in \mathbb{E})\,(\breve{x} \leq \breve{y} \;\Rightarrow\; \breve{x} = \breve{y} \oslash (\breve{y} \oslash \breve{x}))\,, \\ (\forall x, y \in Y)\,(x \leq y \;\Rightarrow\; x = y * (y * x))\,, \end{cases} \tag{3.5}$$

$$\begin{cases} (\forall \breve{x}, \breve{y}, \breve{z} \in \mathbb{E})\,(\breve{x} \leq \breve{z},\, \breve{z} \oslash \breve{y} \leq \breve{z} \oslash \breve{x} \;\Rightarrow\; \breve{x} \leq \breve{y})\,, \\ (\forall x, y, z \in Y)\,(x \leq z,\, z * y \leq z * x \;\Rightarrow\; x \leq y)\,, \end{cases} \tag{3.6}$$

then every makgeolli ideal is a commutative makgeolli ideal.

Proof. Straightforward. □

Theorem 3.7. *Let $(Y, \mathbb{E})$ be a BCK-soft universe in which $(Y, *, 0)$ and $(\mathbb{E}, \oslash, 0)$ are lower semilattices with respect to the order relation "$\leq$". Then every makgeolli ideal is a commutative makgeolli ideal.*

Proof. Assume that $(Y, *, 0)$ and $(\mathbb{E}, \oslash, 0)$ are lower semilattices with respect to the order relation "$\leq$" in the BCK-soft universe $(Y, \mathbb{E})$. Let $\breve{x}, \breve{y} \in \mathbb{E}$ and $x, y \in Y$. Then $\breve{x} \oslash (\breve{x} \oslash \breve{y})$ is a common lower bound of $\breve{x}$ and $\breve{y}$; and $x * (x * y)$ is a common lower bound of x and y. Also, $\breve{y} \oslash (\breve{y} \oslash \breve{x})$ is the greatest lower bound of $\breve{x}$ and $\breve{y}$; and $y * (y * x)$ is the greatest lower bound of x and y. Hence $\breve{x} \oslash (\breve{x} \oslash \breve{y}) \leq \breve{y} \oslash (\breve{y} \oslash \breve{x})$ and $x * (x * y) \leq y * (y * x)$. Therefore every makgeolli ideal is a commutative makgeolli ideal by Corollary 3.6. □

Theorem 3.8. *If a makgeolli ideal $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ of $(Y, \mathbb{E})$ satisfies:*

$$(\forall \breve{x}, \breve{y}, \breve{z} \in \mathbb{E}) \begin{pmatrix} f_{\mathbb{E}}((\breve{x} \oslash \breve{z}) \oslash (\breve{y} \oslash (\breve{y} \oslash \breve{x}))) \supseteq f_{\mathbb{E}}(((\breve{x} \oslash \breve{y}) \oslash \breve{z}) \\ g_{\mathbb{E}}((\breve{x} \oslash \breve{z}) \oslash (\breve{y} \oslash (\breve{y} \oslash \breve{x}))) \subseteq g_{\mathbb{E}}(((\breve{x} \oslash \breve{y}) \oslash \breve{z}) \end{pmatrix}, \tag{3.7}$$

$$(\forall x, y, z \in Y)\,(\xi((x * z) * (y * (y * x))) \geq \xi(((x * y) * z))\,, \tag{3.8}$$

then it is a commutative makgeolli ideal of $(Y, \mathbb{E})$.

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Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli ideal of $(Y, \mathbb{E})$ that satisfies the conditions (3.7) and (3.8). Using (2.4), (2.15) and (3.7), we have

$$\begin{aligned} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &\supseteq f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \\ &= f_{\mathbb{E}}((\check{x} \oslash \check{z}) \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \cap f_{\mathbb{E}}(\check{z}) \\ &\supseteq f_{\mathbb{E}}(((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \end{aligned}$$

and

$$\begin{aligned} g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &\subseteq g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \\ &= g_{\mathbb{E}}((\check{x} \oslash \check{z}) \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \cup f_{\mathbb{E}}(\check{z}) \\ &\subseteq g_{\mathbb{E}}(((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}). \end{aligned}$$

Let $x, y, z \in Y$ and $t, r \in (0, 1]$ be such that $\langle((x * y) * z)/t\rangle \in \xi$ and $\langle z/r\rangle \in \xi$. Then

$$\xi((x * (y * (y * x))) * z) = \xi((x * z) * (y * (y * x))) \geq \xi(((x * y) * z) \geq t$$

by (2.4) and (3.8), that is, $\langle((x * (y * (y * x))) * z)/t\rangle \in \xi$. It follows from (2.15) that $\langle(x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi$. Therefore $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. □

Theorem 3.9. *A makgeolli structure $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ over $(Y, \mathbb{E})$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$ if and only if it is a makgeolli ideal of $(Y, \mathbb{E})$ that satisfies:*

$$(\forall \check{x}, \check{y} \in \mathbb{E}) \left(\begin{array}{c} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}(\check{x} \oslash \check{y}) \\ g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}(\check{x} \oslash \check{y}) \end{array} \right), \tag{3.9}$$

$$(\forall x, y \in Y)\, (\xi(x * (y * (y * x))) \geq \xi(x * y))\,. \tag{3.10}$$

Proof. Assume that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. Then it is a makgeolli ideal of $(Y, \mathbb{E})$ (see Theorem 3.3). If we put $\check{z} = 0$ in (3.1) and use (2.2) and (2.14), then

$$\begin{aligned} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash 0) \cap f_{\mathbb{E}}(0) = f_{\mathbb{E}}(\check{x} \oslash \check{y}), \\ g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash 0) \cup g_{\mathbb{E}}(0) = g_{\mathbb{E}}(\check{x} \oslash \check{y}). \end{aligned}$$

Let $t := \xi(x * y)$ for all $x, y \in Y$. Then $t := \xi((x * y) * 0)$, i.e., $\langle((x * y) * 0)/t\rangle \in \xi$. Since $\langle 0/t\rangle \in \xi$ by (2.14), it follows from (3.2) that $\langle(x * (y * (y * x)))/t\rangle \in \xi$. Hence $\xi(x * (y * (y * x))) \geq t = \xi(x * y)$. Therefore (3.9) and (3.10) are valid.

Conversely, let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli ideal of $(Y, \mathbb{E})$ that satisfies (3.9) and (3.10). For every $\check{x}, \check{y}, \check{x} \in \mathbb{E}$, we have

$$\begin{aligned} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}), \\ g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \end{aligned}$$

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by (3.9) and (2.15). Let $x, y, z \in Y$ and $t, r \in (0, 1]$ be such that $\langle z/r\rangle \in \xi$ and $\langle((x * y) * z)/t\rangle \in \xi$. Then $\langle(x * y)/\min\{t, r\}\rangle \in \xi$ by (2.15). It follows from (3.10) that

$$\xi((x * (y * (y * x))) \geq \xi(x * y) \geq \min\{t, r\},$$

i.e., $\langle((x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi$. Consequently, $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. □

Theorem 3.10. *A makgeolli structure $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ over $(Y, \mathbb{E})$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$ if and only if the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ for all subsets α and δ of Y, and the nonempty set $\xi(Y; t)$ is a commutative ideal of $(Y, *, 0)$ for all $t \in [0, 1]$.*

Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a commutative makgeolli ideal of $(Y, \mathbb{E})$. Then it is a makgeolli ideal of $(Y, \mathbb{E})$ (see Theorem 3.3). Hence the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$, and the nonempty set $\xi(Y; t)$ is an ideal of $(Y, *, 0)$ for all subsets α and δ of Y and $t \in [0, 1]$ by Lemma 2.5. Let $\check{x} \oslash \check{y} \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ for all $\check{x}, \check{y} \in \mathbb{E}$ and subsets α and δ of Y. Then $f_{\mathbb{E}}(\check{x} \oslash \check{y}) \supseteq \alpha$ and $g_{\mathbb{E}}(\check{x} \oslash \check{y}) \subseteq \delta$. It follows from (3.9) that

$$f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}(\check{x} \oslash \check{y}) \supseteq \alpha$$

and $g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}(\check{x} \oslash \check{y}) \subseteq \delta$. Hence $\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$, and therefore $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ by Lemma 2.1. Let $x, y \in Y$ and $t \in [0, 1]$ be such that $x * y \in \xi(Y; t)$. Then $\xi(x * y) \geq t$, and so $\xi(x * (y * (y * x))) \geq \xi(x * y) \geq t$ by (3.10), that is, $x * (y * (y * x)) \in \xi(Y; t)$. Thus $\xi(Y; t)$ is a commutative iddeal of $(Y, *, 0)$ by Lemma 2.1.

Conversely, suppose that the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ for all subsets α and δ of Y, and the nonempty set $\xi(Y; t)$ is a commutative ideal of $(Y, *, 0)$ for all $t \in [0, 1]$. Then $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$, and $\xi(Y; t)$ is an ideal of $(Y, *, 0)$. Thus $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a makgeolli ideal of $(Y, \mathbb{E})$ by Lemma 2.5. Let $\check{x}, \check{y} \in \mathbb{E}$ be such that $f_{\mathbb{E}}(\check{x} \oslash \check{y}) = \alpha$ and $g_{\mathbb{E}}(\check{x} \oslash \check{y}) = \delta$. Then $\check{x} \oslash \check{y} \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$, and so $\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ by Lemma 2.1. Hence $f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq \alpha = f_{\mathbb{E}}(x \oslash y)$ and $g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq \delta = f_{\mathbb{E}}(x \oslash y)$. Let $x, y \in Y$ be such that $\xi(x * y) = t$. Then $x * y) \in \xi(Y; t)$, which implies from Lemma 2.1 that $x * (y * (y * x)) \in \xi(Y; t)$. Thus $\xi(x * (y * (y * x))) \geq t = \xi(x * y)$. Therefore $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$ by Theorem 3.9. □

Corollary 3.11. *If $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$, then $f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ and $\xi(Y; t)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ and $(Y, *, 0)$, respectively, for all subsets α and δ of Y and $t \in [0, 1]$.*

Proof. Straightforward. □

The converse of Corollary 3.11 is not true in general as seen in the following example.

Example 3.12. Consider a BCK-soft universe $(Y, \mathbb{E})$ where $Y = \mathbb{E} := \{0, 1, 2, 3, 4\}$ has binary operation "$*(= \oslash)$" given by Table 3.

Table 3: Cayley tables for the binary operations "$*(= \oslash)$"

$*(= \oslash)$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli structure over $(Y, \mathbb{E})$ defined as follows:

$$f_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} Y & \text{if } x = 0, \\ \{3, 4\} & \text{if } x = 1, \\ \{1, 3, 4\} & \text{if } x = 2, \\ \{1, 2, 3, 4\} & \text{if } x = 3, \\ \{4\} & \text{if } x = 4, \end{cases}$$

$$g_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} \{3\} & \text{if } x = 0, \\ \{0, 3\} & \text{if } x = 1, \\ \{0, 2, 3\} & \text{if } x = 2, \\ \{0, 2, 3, 4\} & \text{if } x = 3, \\ Y & \text{if } x = 4, \end{cases}$$

and

$$\xi : Y \to [0, 1],\ y \mapsto \begin{cases} 0.82 & \text{if } y = 0, \\ 0.54 & \text{if } y = 1, \\ 0.75 & \text{if } y = 2, \\ 0.65 & \text{if } y = 3, \\ 0.42 & \text{if } y = 4. \end{cases}$$

It is routine to verify that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a makgeolli ideal of $(Y, \mathbb{E})$ and the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ and $\xi(Y; t)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ and $(Y, *, 0)$, respectively, for all subsets α and δ of Y and $t \in [0, 1]$. We have $f_{\mathbb{E}}(2 \oslash (4 \oslash (4 \oslash 2))) = f_{\mathbb{E}}(2) = \{1, 3, 4\} \not\supseteq Y = f_{\mathbb{E}}(0) = f_{\mathbb{E}}(2 \oslash 4)$ and/or $\xi(1 * (4 * (4 * 1))) = \xi(1) = 0.54 \ngeq 0.82 = \xi(0) = \xi(1 * 4)$. Hence $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is not a commutative makgeolli ideal of $(Y, \mathbb{E})$ by Theorem 3.9.

We make a new commutative makgeolli ideal using the given commutative makgeolli ideal.

Theorem 3.13. *Given a makgeolli structure $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ over $(Y, \mathbb{E})$, let $\mathbb{M}^*_{(Y,\mathbb{E})} := (f^*_{\mathbb{E}},\, g^*_{\mathbb{E}},\, \xi^*)$ be a new makgeolli structure over $(Y, \mathbb{E})$ which is*

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defined by

$$f^*_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ \check{x} \mapsto \begin{cases} f_{\mathbb{E}}(\check{x}) & \text{if } \check{x} \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)), \\ \beta & \text{otherwise,} \end{cases}$$

$$g^*_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ \check{x} \mapsto \begin{cases} g_{\mathbb{E}}(\check{x}) & \text{if } \check{x} \in g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w)), \\ \gamma & \text{otherwise,} \end{cases}$$

$$\xi^* : Y \to [0,1],\ x \mapsto \begin{cases} \xi(x) & \text{if } x \in \xi(Y; \xi(u)), \\ k & \text{otherwise,} \end{cases}$$

where $w \in \mathbb{E}$*,* $u \in Y$*,* $k \in [0,1]$ *and* $\beta, \gamma \in \mathcal{P}(Y)$ *with* $\beta \subsetneq f_{\mathbb{E}}(\check{x})$*,* $\gamma \supsetneq g_{\mathbb{E}}(\check{x})$ *and* $\xi(x) > k$*. If* $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\ g_{\mathbb{E}},\ \xi)$ *is a commutative makgeolli ideal of* $(Y, \mathbb{E})$*, then* $\mathbb{M}^*_{(Y,\mathbb{E})} := (f^*_{\mathbb{E}},\ g^*_{\mathbb{E}},\ \xi^*)$ *is a commutative makgeolli ideal of* $(Y, \mathbb{E})$*.*

Proof. Assume that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\ g_{\mathbb{E}},\ \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. Then the sets $f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w))$ and $g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ for all $w \in \mathbb{E}$, and $\xi(Y; \xi(u))$ is a commutative ideal of $(Y, *, 0)$ for all $u \in Y$. Hence $0 \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w)) \cap \xi(Y; \xi(u))$, and so $f^*_{\mathbb{E}}(0) = f_{\mathbb{E}}(0) \supseteq f_{\mathbb{E}}(\check{x}) \supset f^*_{\mathbb{E}}(\check{x})$ and $g^*_{\mathbb{E}}(0) = g_{\mathbb{E}}(0) \subseteq g_{\mathbb{E}}(\check{x}) \subseteq g^*_{\mathbb{E}}(\check{x})$ for all $\check{x} \in \mathbb{E}$. Also, we get $\xi^*(0) = \xi(0) \geq \xi(x) \geq \xi^*(x)$, i.e., $\langle 0/\xi^*(\check{x})\rangle \in \xi^*$ for all $x \in Y$. Let $\check{x}, \check{y}, \check{z} \in \mathbb{E}$. If $(\check{x} \oslash \check{y}) \oslash \check{z} \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$ and $z \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$, then $\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$. Thus

$$\begin{aligned} f^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &= f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \\ &\supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \\ &= f^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f^*_{\mathbb{E}}(\check{z}) \end{aligned}$$

and

$$\begin{aligned} g^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &= g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \\ &\subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \\ &= g^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g^*_{\mathbb{E}}(\check{z}). \end{aligned}$$

If $(\check{x} \oslash \check{y}) \oslash \check{z} \notin f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w))$ or $z \notin f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w))$, then $f^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) = \beta$ or $f^*_{\mathbb{E}}(\check{z}) = \beta$. Hence $f^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq \beta = f^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f^*_{\mathbb{E}}(z)$. If $(\check{x} \oslash \check{y}) \oslash \check{z} \notin g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$ or $z \notin g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$, then $g^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) = \gamma$ or $g^*_{\mathbb{E}}(\check{z}) = \gamma$. Hence $g^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq \gamma = g^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g^*_{\mathbb{E}}(z)$. Let $x, y, z \in Y$ and $t, r \in (0, 1]$ be such that $\langle((x * y) * z)/t\rangle \in \xi^*$ and $\langle z/r\rangle \in \xi^*$. If $(x * y) * z \in \xi(Y; \xi(u))$ and $z \in \xi(Y; \xi(u))$, then $x * (y * (y * x)) \in \xi(Y; \xi(u))$ and thus

$$\begin{aligned} \xi^*(x * (y * (y * x))) &= \xi(x * (y * (y * x))) \\ &\geq \min\{\xi((x * y) * z), \xi(z)\} \\ &= \min\{\xi^*((x * y) * z), \xi^*(z)\} \\ &\geq \min\{t, r\}, \end{aligned}$$

that is, $\langle (x*(y*(y*x)))/\min\{t,r\}\rangle \in \xi^*$. If $(x*y)*z \notin \xi(Y;\xi(u))$ or $z \notin \xi(Y;\xi(u))$, then $\xi^*((x*y)*z) = k$ or $\xi^*(z) = k$. Thus

$$\xi^*(x*(y*(y*x))) \geq k = \min\{\xi^*((x*y)*z), \xi^*(z)\} \geq \min\{t,r\},$$

and so $\langle (x*(y*(y*x)))/\min\{t,r\}\rangle \in \xi^*$. Therefore $\mathbb{M}^*_{(Y,\mathbb{E})} := (f^*_{\mathbb{E}},\, g^*_{\mathbb{E}},\, \xi^*)$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$. □

Note that a makgeolli ideal might not be a commutative makgeolli ideal (see Example 3.4). But we can consider the extension property for a commutative makgeolli ideal.

Theorem 3.14. *Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ and $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$ be makgeolli ideals of $(Y,\mathbb{E})$ such that $\mathbb{M}_{(Y,\mathbb{E})} \Subset \tilde{\mathbb{M}}_{(Y,\mathbb{E})}$, that is,*

(i) $f_{\mathbb{E}}(0) = \tilde{f}_{\mathbb{E}}(0)$, $g_{\mathbb{E}}(0) = \tilde{g}_{\mathbb{E}}(0)$, $\xi(0) = \tilde{\xi}(0)$,

(ii) $(\forall \check{x} \in \mathbb{E}, \forall x \in Y)$ $(\tilde{f}_{\mathbb{E}}(\check{x}) \supseteq f_{\mathbb{E}}(\check{x}),\, \tilde{g}_{\mathbb{E}}(\check{x}) \subseteq g_{\mathbb{E}}(\check{x}),\, \tilde{\xi}(x) \geq \xi(x))$.

If $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$, then so is $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$.

Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ and $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$ be makgeolli ideals of $(Y,\mathbb{E})$ such that $\mathbb{M}_{(Y,\mathbb{E})} \Subset \tilde{\mathbb{M}}_{(Y,\mathbb{E})}$. Then $f_{\mathbb{E}}(\mathbb{E};\alpha) \subseteq \tilde{f}_{\mathbb{E}}(\mathbb{E};\alpha)$, $g_{\mathbb{E}}(\mathbb{E};\delta) \supseteq \tilde{g}_{\mathbb{E}}(\mathbb{E};\delta)$ and $\xi(Y;t) \subseteq \tilde{\xi}(Y;t)$ for all subsets α and δ of Y and $t \in (0,1]$. Assume that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$. Then the nonempty sets $f_{\mathbb{E}}(\mathbb{E};\alpha)$ and $g_{\mathbb{E}}(\mathbb{E};\delta)$ are commutative ideals of $(\mathbb{E},\oslash,0)$ for all subsets α and δ of Y, and the nonempty set $\xi(Y;t)$ is a commutative ideal of $(Y,*,0)$ for all $t \in (0,1]$ by Theorem 3.10. Since $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$ is a makgeolli ideal of $(Y,\mathbb{E})$, we know from Lemma 2.5 that the nonempty sets $\tilde{f}_{\mathbb{E}}(\mathbb{E};\alpha)$ and $\tilde{g}_{\mathbb{E}}(\mathbb{E};\delta)$ are ideals of $(\mathbb{E},\oslash,0)$ for all subsets α and δ of Y, and the nonempty set $\tilde{\xi}(Y;t)$ is an ideal of $(Y,*,0)$ for all $t \in (0,1]$. Let $x,y \in Y$ and $t \in (0,1]$ be such that $x*y \in \tilde{\xi}(Y;t)$. Using (I3) and (2.4), we have $(x*(x*y))*y = (x*y)*(x*y) = 0 \in \xi(Y;t)$. Since $\xi(Y;t)$ is a commutative ideal of $(Y,*,0)$, using (2.4) and Lemma 2.1 leads to

$$\begin{aligned}
&(x*(y*(y*(x*(x*y))))) * (x*y) \\
&= (x*(x*y))*(y*(y*(x*(x*y)))) \\
&\in \xi(Y;t) \subseteq \tilde{\xi}(Y;t),
\end{aligned}$$

and so $x*(y*(y*(x*(x*y)))) \in \tilde{\xi}(Y;t)$ bacause $\tilde{\xi}(Y;t)$ is an ideal of $(Y,*,0)$.

Note that

$$\begin{aligned}
&(x * (y * (y * x))) * (x * (y * (y * (x * (x * y)))))\\
&\overset{(I1)}{\leq} (y * (y * (x * (x * y)))) * (y * (y * x))\\
&\overset{(I1)}{\leq} (y * x) * (y * (x * (x * y)))\\
&\overset{(I1)}{\leq} (x * (x * y)) * x\\
&\overset{(2.4)}{=} (x * x) * (x * y) \overset{(I3)\&(K)}{=} 0 \in \tilde{\xi}(Y; t).
\end{aligned}$$

Hence $x * (y * (y * x)) \in \tilde{\xi}(Y; t)$, and therefore $\tilde{\xi}(Y; t)$ is a commutative ideal of $(Y, *, 0)$. Let $\check{x}, \check{y} \in \mathbb{E}$ be such that $\check{x} \oslash \check{y} \in \tilde{f}_{\mathbb{E}}(\mathbb{E}; \alpha) \cap \tilde{g_{\mathbb{E}}}(\mathbb{E}; \delta)$. Then

$$(\check{x} \oslash (\check{x} \oslash \check{y})) \oslash \check{y} = (\check{x} \oslash \check{y}) \oslash (\check{x} \oslash \check{y}) = 0 \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$$

by (I3) and (2.4), and so

$$\begin{aligned}
&(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash (\check{x} \oslash (\check{x} \oslash \check{y}))))) \oslash (\check{x} \oslash \check{y})\\
&= (\check{x} \oslash (\check{x} \oslash \check{y})) \oslash (\check{y} \oslash (\check{y} \oslash (\check{x} \oslash (\check{x} \oslash \check{y}))))\\
&\in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta) \subseteq \tilde{f}_{\mathbb{E}}(\mathbb{E}; \alpha) \cap \tilde{g_{\mathbb{E}}}(\mathbb{E}; \delta)
\end{aligned}$$

since $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$. Using (I1), (I3), (K) and (2.4), we have

$$(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash (\check{x} \oslash (\check{y} \oslash (\check{y} \oslash (\check{x} \oslash (\check{x} \oslash \check{y}))))) \leq 0.$$

Since $\tilde{f}_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $\tilde{g_{\mathbb{E}}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$, it follows that

$$\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in \tilde{f}_{\mathbb{E}}(\mathbb{E}; \alpha) \cap \tilde{g_{\mathbb{E}}}(\mathbb{E}; \delta).$$

Hence $\tilde{f}_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $\tilde{g_{\mathbb{E}}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ by Lemma 2.1. Consequently, $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}}, \tilde{g_{\mathbb{E}}}, \tilde{\xi})$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. □

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An Extension of Soft Operations on Generalized Soft Subsets

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Abstract

Existing Literature, Problem and Limitation: To address problems of fuzzy data in various fields, Molodtsov presented soft set theory, a broad mathematical technique for ambiguity. This theory has been used in a variety of pure and practical mathematical fields. It is evident in this theory that soft subsets and soft equal relations significantly contributed to soft topology, lattices, soft groups, etc. Existing research is limited in that various features, such as associative, distributive, etc., are not confirmed by some current soft subsets for soft product operations. **Purpose:** While studying soft subsets, we observe that several algebraic properties have not yet been investigated on various generalized soft subsets to enhance algebraic structures in soft set theory. So, this article investigates some of these algebraic properties on different generalized soft subsets on different soft operations. **Contribution:** This study demonstrates a few counterexamples that some algebraic properties are unsatisfied by generalized soft subsets. Based on this approach, we present some crucial theorems and results that show these significant features on all soft subsets by employing additional conditions. A universal complement property in soft set theory in relation to soft complements (negation complement (^{c}) and relative complement (^{r})) is propounded. **Limitation:** The sole restriction of these results is that two generalised soft subsets (soft J-subset and soft L-subset) do not satisfy the union and intersection condition of classical mathematics as described in section 4.

Keywords: Soft sets, Soft M-subset, Soft L-subset, Soft F-subset, Soft J-subset, Soft Complements etc.

1. Introduction

1.1. Problem Statement:

Most existing techniques for formal reasoning, computing, and modelling have a clear, deterministic, and precise nature. But there are other challenging issues in fields, including economics, engineering, environment etc., that can sometimes involve fuzzy data. Because these challenges contain a variety of forms of uncertainty, we cannot effectively employ classical approaches to solve them. Several theories can be viewed as a framework in mathematics to cope with ambiguities, including the theory of interval mathematics, vague sets, fuzzy sets, and a few others. Molodtsov [1] noted each of these theories includes deficiencies that are inherent to them. To address these issues mathematically, Molodtsov developed the idea of soft set theory. Soft sets could be viewed as a particular type of neighbourhood systems and context-dependent fuzzy sets. The problem of constructing the membership function as well as other related complications are essentially nonexistent in this theory. As a result, it is extremely useful and has potential applications in various areas of mathematics, as shown in [1]. In recent years, many authors [1, 6, 8, 10, 11, 19, 20] worked on operations of soft sets and studied algebraic structures in soft set theory. But we observe that very few works has been done on join($\tilde{\vee}$) and meet($\tilde{\wedge}$) operations of soft sets. Therefore, for extending algebraic structures in soft set theory, we studied these operators on generalized soft subsets, and gives some algebraic properties in this research article.

1.2. Previous Research and Limitations:

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Maji et. al. [11] provided the first detailed explanation of the concept of soft subsets. A complete theoretical examination on soft sets was also written by them, and asserted a few results regarding soft distributive laws with respect to soft products ($\tilde{\wedge}$ and $\tilde{\vee}$) operations of soft sets, but they did not present any supporting data (see 2.6 in [11]). Moreover, according to Ali et al. [6], the results in [10] were inaccurate (see 2.8 in [10]). Therefore, the ideas of generalized soft subsets and soft equal relations were also put forward by Jun and Yang [20]. In an effort to respond to Maji's results (Proposition 2.6 in [11]) and suggested generalized soft distribution laws, they also tried to apply generalized soft equal relations. They started by defining soft J-equal and soft L-equal relations. It is crucial to note that Jun and Yang in [20] and Liu et al. in [19] did not reach the same conclusion on the applicability of distributive laws to all types of soft equal relations. Additionally, Feng and Li [3] thoroughly examined soft product operations, conducted a systematic investigation of five different kinds of soft subsets and developed the free soft algebraic quotient structures linked to soft product operations. But no one study these soft product operations on different types of generalized soft subsets. Therefore, we tried to attempt and explore some operations on various types of generalized soft subsets.

1.3. Motivation:

Yadav and Singh [12] first studied El-algebra in soft sets and introduced the concept of soft El-algebra as well as a number of noteworthy algebraic features. But while studying ES structure [13, 18] on soft sets, we observed that the given structure does not make a lattice structure in the sense of soft M-subsets. Therefore, we studied other generalized soft subsets [19, 20] and found that ES structure makes a lattice structure with respect to soft J-subsets. Furthermore, for defining order reversing involution operator on ES structure, we needed complement operation of soft sets. So, we studied two types of complements [6, 11] in soft sets and observed that no one worked on generalized soft sets on these complements. Therefore, in the present article, we derive some algebraic properties of generalized soft subsets on these complement operations.

1.4. Contribution of the study:

Soft Set Theory is a mathematical framework that deals with ambiguities and vagueness in real-world scenarios. In this research article, the researchers focused on the algebraic properties of generalized soft subsets. They found that some of these algebraic properties were not satisfied by any of generalized soft subsets, and demonstrated these findings with a few counterexamples.

To overcome this issue, the researchers proposed additional conditions on the elements of parameter set, that would satisfy these significant algebraic features on all soft subsets. They presented some crucial theorems and results supported by real life example that showed how these conditions could be used to achieve these algebraic features. Moreover, the researchers studied the universal complement property on all generalized soft subsets in soft set theory. This property relates to two types of soft complements, which include negation soft complement (^{c}) and relative soft complement (^{r}). The researchers proposed a new approach to achieve this property, which can be used to define soft complements more generally. Overall, this study contributes to the advances of soft set theory by addressing some of the algebraic properties of generalized soft subsets and proposing new conditions to satisfy these properties.

1.5. Paper Organization:

This work is divided into six components as: Section 1. provides research problem, previous research, motivation, background etc. Section 2. summarise the fundamental definitions of soft sets and their operations like soft unions ($\tilde{\cup}$), soft intersections ($\tilde{\cap}$), soft products ($\tilde{\vee}$ and $\tilde{\wedge}$) etc. with some basic results. Section 3. gives a brief introduction to four types of soft subsets with an important proposition about their interrelations. Section 4. is devoted to provide some important outcomes on various soft subsets in terms of soft product, soft union and soft intersection operations. In Section 5. we first give a general property of complement on classical sets in mathematics, and then study this property on all generalized soft subsets in the sense of soft negation and soft relative complements [6, 11]. Section 6. provides the conclusion and future work of present study.

1.6. Background:

As we know that Molodtsov [1] presented the idea of soft sets as a unique mathematical technique to dealing with ambiguities. The implementation of this theory to a decision-making issue involving rough sets is described by Maji et. al. [10] by utilizing soft sets in the form of a binary information table, and defined first time parameter reduction on soft sets. Further, in [11], they gave a few definitions and operations of soft sets like soft subset ($\tilde{\subseteq}$), soft superset ($\tilde{\supseteq}$), null soft set ($\tilde{\Phi}$), absolute soft sets ($\tilde{A}$), soft complement, soft union ($\tilde{\cup}$), soft intersection ($\tilde{\cap}$), "AND" and "OR" ($\tilde{\wedge}$ and $\tilde{\vee}$) operations. Further, Feng et. al. [4] provided the definition of soft subset in a different way. But, Cagman and Enginoglu [8] built a uni-int decision-making approach by redefining soft sets operations to improve new results. Ali et. al. [6] also gave some new operations on soft sets like extended and restricted intersection, difference and union, relative and negation complements, and proved De-Morgan's law on these operations. Moreover, in [6, 8], they proved that soft products ($\tilde{\vee}$-product and $\tilde{\wedge}$-product) does not hold commutative and associative properties in the sense of soft M-equal relation. For studying these algebraic properties, Jun and Yang [20] gave the definition of generalized soft subset (Soft J-subset in [19]), and proved distributive property called it generalized distributive law, with respect to soft J-equality and generalized interval-valued fuzzy soft equality. After that Liu et. al. [19] combined fuzzy, rough and soft sets to provide four types of generalized soft subsets with some basic properties. They found that the distributive property given in [20] does holds with respect to soft J-equal, and provided a new generalized soft distributive law of soft L-equal. Moreover, they amended the associative property of Maji [11] with respect to soft L-equal and said that this property can be satisfied only by soft L-equality instead of other existing equality.

In literature, some authors [2, 7, 9] have explored above properties to topological spaces in soft set theory and presented different kinds of soft topological spaces. A full and exhaustive overview of the researches done in soft set theory and the advancements of topological spaces in soft sets was provided by Yadav and Singh [14]. According to Bentley [5], topological spaces can be derived from nearness spaces. Furthermore, Singh with others [15, 16, 17] studied the concepts of soft d-proximity, soft binary heminearness spaces, and nearness of finite order S_n-merotopy respectively.

2. Preliminaries:

Some fundamental definitions of soft sets and associated operators are provided in this section. Throughout the whole article, U and E are non-empty finite universal sets of objects and all possible parameters/attributes respectively. The touple (U, E) or U_E is referred to as a soft universe.

Definition 2.01([1]): Let P(U) be the power set of U and A $\subseteq$ E. Then a couple (F, A) is said to be *soft set* over U, if F is a representation defined as:

$$\mathrm{F} : \mathrm{A} \longrightarrow \mathrm{P(U)}.$$

Here, we writes a soft set (F, A) by F_A, where $\mathrm{F}_A = \{(\alpha, \mathrm{F}(\alpha)) \mid \alpha \in \mathrm{A}, \mathrm{F}(\alpha) \in \mathrm{P(U)}\}$. Set $\mathrm{F}(\alpha)$ can be selected at random. Soft set is not a classical set. Then, a significant quantity of information was provided in [1].

Definition 2.02([11]): 1. A *null soft set* $\tilde{\Phi}$, is a soft set F_A on U, if $\forall\ \alpha \in \mathrm{A}$, $\mathrm{F}(\alpha) = \phi$ (null set). 2. An *absolute soft set* $\tilde{A}$, is a soft set F_A on U, if $\forall\ \alpha \in \mathrm{A}$, $\mathrm{F}(\alpha) = \mathrm{U}$.

Definition 2.03([11]): Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are two soft sets on U. Then *Intersection* of $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ over U is defined as: $\mathrm{F}^1_{A_1}\ \tilde{\cap}\ \mathrm{F}^2_{A_2} = \mathrm{F}^3_{A_3}$, where $\mathrm{A}_3 = \mathrm{A}_1 \cap \mathrm{A}_2$, and $\mathrm{F}^3(\alpha) = \mathrm{F}^2(\alpha)$ or $\mathrm{F}^1(\alpha)$, (as both have similar approximation), $\forall\ \alpha \in \mathrm{A}_3$.

Definition 2.04([6]): The *extended intersection* of $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ over U is written as $\mathrm{F}^1_{A_1}\ \tilde{\sqcap}_{\mathscr{E}}\ \mathrm{F}^2_{A_2}$ and defined as: $\mathrm{F}^1_{A_1}\ \tilde{\sqcap}_{\mathscr{E}}\ \mathrm{F}^2_{A_2} = \mathrm{F}^3_{A_3}$, where $\mathrm{A}_3 = \mathrm{A}_1 \cup \mathrm{A}_2$, and $\forall\ \alpha \in \mathrm{A}_3$

$$F^3(\alpha) = \begin{cases} F^1(\alpha) & ,\alpha \in A_1 - A_2 \\ F^2(\alpha) & ,\alpha \in A_2 - A_1 \\ F^1(\alpha) \cap F^2(\alpha) & ,\alpha \in A_1 \cap A_2. \end{cases}$$

Definition 2.05([6]): Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets on U such as $A_1 \cap A_2 \neq \phi$. Then, the *restricted intersection* of $F^1_{A_1}$ and $F^2_{A_2}$ is written as $F^1_{A_1} \tilde{\Cap} F^2_{A_2}$, described as $F^1_{A_1} \tilde{\Cap} F^2_{A_2} = F^3_{A_3}$, where $A_3 = A_1 \cap A_2$, and $\forall\ \alpha \in A_3$, $F^3(\alpha) = F^1(\alpha) \cap F^2(\alpha)$.

Result 2.06: By the definitions 2.03, 2.04 and 2.05 we can conclude that *intersection and extended intersection* of two non-null soft sets is a non-null soft set, either their approximations are similar or not at the same attribute. However, their *restricted intersection* may be a null soft set, as shown by below example.

Example 2.07: Consider U = $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ be the set of five canditates for an interview in a company, and E = $\{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$ be the set of five types of jobs, where ϱ_1 stands for Network Administrator (NA), ϱ_2 stands for User Experience Designer (UED), ϱ_3 stands for System Analyst (SA), ϱ_4 stands for Database Administrator (DA) and ϱ_5 stands for Development Operations Engineer (DOE). let $F^1_{A_1}$ and $F^2_{A_2}$ are two members in selection board, which provides the names of capable canditates for respective jobs in sets A_1 and A_2 respectively. Here, we consider $F^1_{A_1}$ and $F^2_{A_2}$ are two soft sets over universe set U, defined as:

$F^1_{A_1} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_2, \{\mu_4, \mu_5\})\}$,
$F^2_{A_2} = \{(\varrho_1, \{\mu_3\}), (\varrho_2, \{\mu_1, \mu_2\}), (\varrho_3, \{\mu_4, \mu_5\})\}$.

By definition 2.05, $F^1_{A_1} \tilde{\Cap} F^2_{A_2} = F^3_{A_3}$, where $A_3 = A_1 \cap A_2$. Now, $A_3 = \{\varrho_1, \varrho_2\}$ and hence $F^3_{A_3} = \{(\varrho_1, \phi), (\varrho_2, \phi)\}$. Here, $F^1_{A_1} \tilde{\Cap} F^2_{A_2}$ provides the most suitable canditates for common jobs in respect to the opinions of $F^1_{A_1}$ and $F^2_{A_2}$.

Definition 2.08([11]): Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets on U, then the *soft union* is provided as a soft set F_A, that satisfies the following criteria:
(i) $A = A_1 \cup A_2$,
(ii) $\forall\ \alpha \in A$,

$$F(\alpha) = \begin{cases} F^1(\alpha) & ,\alpha \in A_1 - A_2 \\ F^2(\alpha) & ,\alpha \in A_2 - A_1 \\ F^1(\alpha) \cup F^2(\alpha) & ,\alpha \in A_1 \cap A_2. \end{cases}$$

Definition 2.09([11]): Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets described on U, where $A_1, A_2 \subseteq E$. Then "*AND*" can be defined as: $F^1_{A_1} \tilde{\wedge} F^2_{A_2} = F_{A_1 \times A_2}$, where $\forall\ (\alpha, \beta) \in A_1 \times A_2$, $F(\alpha, \beta) = F^1(\alpha) \cap F^2(\beta)$.

Definition 2.10([11]): Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets described on U, where $A_1, A_2 \subseteq E$. Then "*OR*" can be defined as: $F^1_{A_1} \tilde{\vee} F^2_{A_2} = F_{A_1 \times A_2}$, where $\forall\ (\alpha, \beta) \in A_1 \times A_2$, $F(\alpha, \beta) = F^1(\alpha) \cup F^2(\beta)$.

Result 2.11: Let $F^1_{A_1} \neq \tilde{\Phi}$, $F^2_{A_2} \neq \tilde{\Phi}$. Then, "AND" of $F^1_{A_1}$ and $F^2_{A_2}$ can be a null soft set i.e. $F^1_{A_1} \tilde{\wedge} F^2_{A_2} = \tilde{\Phi}$ (see example 2.12).

Example 2.12: Let U and E are universal sets as given in example 2.07, $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets defined as:

$F^1_{A_1} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_2, \{\mu_5\})\}$,
$F^2_{A_2} = \{(\varrho_1, \{\mu_3\}), (\varrho_3, \{\mu_5\})\}$.

Now, $A_3 = A_1 \times A_2 = \{(\varrho_1, \varrho_1), (\varrho_2, \varrho_1), (\varrho_1, \varrho_3), (\varrho_2, \varrho_3)\}$. Thus, $F^1_{A_1} \tilde{\wedge} F^2_{A_2} = F^3_{A_3} = \{((\varrho_1, \varrho_1), \phi), ((\varrho_2, \varrho_1), \phi), ((\varrho_1, \varrho_3), \phi), ((\varrho_2, \varrho_3), \phi)\}$, where $F^1_{A_1} \tilde{\wedge} F^2_{A_2}$ provides most suitable candidates for one or two perticular jobs at a time with respect to the opinion of $F^1_{A_1}$ and $F^2_{A_2}$.

3. Generalized Soft Subsets:

Maji et. al. [11] and Feng et. al. [4] gave two kinds of soft subsets and soft equal relations. Liu et. al. [19] called it soft M-subset, soft F-subset and soft M-equal relation, soft F-equal relation. A generalization of soft subsets was examined by Jun and Yang [20]. Furthermore, Liu et. al. [19] gave the notions of soft J-subset and soft L-subset and demonstrated that soft M-equal and soft F-equal relations are correlate with one another, while others are distinct in general. Here, we provide an overview of four different soft subsets as:

Definition 3.01([11]): Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are soft sets defined on U. Then $\mathrm{F}^1_{A_1}$ is a *soft subset* (renamed it a soft M-subset in [19]) of $\mathrm{F}^2_{A_2}$ if:

(i) $\mathrm{A}_1 \subseteq \mathrm{A}_2$,

(ii) For each $\mathrm{a}_1 \in \mathrm{A}_1$, $\mathrm{F}^1(\mathrm{a}_1)$ and $\mathrm{F}^2(\mathrm{a}_1)$ are approximations that are similar.

Soft subset is represented by $\mathrm{F}^1_{A_1} \tilde{\subseteq} \mathrm{F}^2_{A_2}$ or $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2}$. Also $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft M-equal or soft equal, written as $\mathrm{F}^1_{A_1} =_M \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_M \mathrm{F}^1_{A_1}$.

Definition 3.02([4]): Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are soft sets on U. Then $\mathrm{F}^1_{A_1}$ is *soft subset* (renamed it a soft F-subset in [19]) of $\mathrm{F}^2_{A_2}$, written as $\mathrm{F}^1_{A_1} \tilde{\subseteq}_F \mathrm{F}^2_{A_2}$, iff $\mathrm{A}_1 \subseteq \mathrm{A}_2$ and $\mathrm{F}^1(\mathrm{a}_1) \subseteq \mathrm{F}^2(\mathrm{a}_1)\ \forall\ \mathrm{a}_1 \in \mathrm{A}_1$. Also, $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft F-equal, denoted as $\mathrm{F}^1_{A_1} =_F \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_F \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_F \mathrm{F}^1_{A_1}$.

Definition 3.03([19]): Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are non-empty soft sets. Then $\mathrm{F}^1_{A_1}$ is called *soft J-subset* of $\mathrm{F}^2_{A_2}$ or $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$ if and only if for any $\mathrm{a}_1 \in \mathrm{A}_1$, $\exists$ $\mathrm{a}_2 \in \mathrm{A}_2$ such that $\mathrm{F}^1(\mathrm{a}_1) \subseteq \mathrm{F}^2(\mathrm{a}_2)$ (see example 3.04). Also, $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft J-equal, denoted as $\mathrm{F}^1_{A_1} =_J \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_J \mathrm{F}^1_{A_1}$.

Example 3.04: Consider U and E are universal sets of candidates and jobs respectively, as given in example 2.07. Let,

$\mathrm{F}^1_{A_1} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_2, \{\mu_3\}), (\varrho_3, \{\mu_2, \mu_3\})\}$,

$\mathrm{F}^2_{A_2} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_3, \{\mu_2, \mu_3\}), (\varrho_4, \{\mu_2\})\}$.

Since $\mathrm{A}_1 \neq \mathrm{A}_2$, so $\mathrm{F}^1_{A_1} \neq_M \mathrm{F}^2_{A_2}$. But we can see that, $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_J \mathrm{F}^1_{A_1}$. Hence, $\mathrm{F}^1_{A_1} \tilde{=}_J \mathrm{F}^2_{A_2}$.

Here, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2}$, then from example 2.07 it indicates that both members ($\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$) of selection board selected same candidates for every job in A_1. Similarly, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$, then it indicates that for every job in A_1, the members selected by $\mathrm{F}^1_{A_1}$ are also selected by $\mathrm{F}^2_{A_2}$ for the same or different job in A_2.

Definition 3.05([19]): Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are non-empty soft sets on U. Then, $\mathrm{F}^1_{A_1}$ is *soft L-subset* of $\mathrm{F}^2_{A_2}$ or $\mathrm{F}^1_{A_1} \tilde{\subseteq}_L \mathrm{F}^2_{A_2}$ if and only if for any $\mathrm{a}_1 \in \mathrm{A}_1$, $\exists$ $\mathrm{a}_2 \in \mathrm{A}_2$ such that $\mathrm{F}^1(\mathrm{a}_1) = \mathrm{F}^2(\mathrm{a}_2)$. Soft sets $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft L-equal, denoted as $\mathrm{F}^1_{A_1} =_L \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_L \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_L \mathrm{F}^1_{A_1}$.

Proposition 3.06([19]): Let $\mathrm{F}^1_{A_1} \neq \tilde{\Phi}$ and $\mathrm{F}^2_{A_2} \neq \tilde{\Phi}$. Then,

(1) $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_F \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$,

(2) $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_L \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$,

(3) $\mathrm{F}^1_{A_1} =_M \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} =_L \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} =_J \mathrm{F}^2_{A_2}$.

But generally, the converse of the aforementioned arguments does not exist (see examples 2.6, 2.9, 3.3 in [19]).

4. Generalized Soft Subsets on Soft Operations:

This section presents some characterizations of above given different types of soft subsets with respect to two properties given below. We can see that all soft subsets satisfy only property 4.01 (1); property 4.01 (2) could be satisfied by soft F-subset and soft M-subset instead of soft J-subset and soft L-subset.

Property 4.01: Let P, Q, X and Y are four crisp subsets of the universe set U such as P $\subseteq$ X and Q $\subseteq$ Y. Then we have,
(1). P $\vee$ Q $\subseteq$ X $\vee$ Y, and P $\wedge$ Q $\subseteq$ X $\wedge$ Y,
(2). P $\cap$ Q $\subseteq$ X $\cap$ Y, and P $\cup$ Q $\subseteq$ X $\cup$ Y.

Proposition 4.02: Let $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets defined on U. Then,
(1). If $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.
(2). If $F^1_{A_1} \tilde{\subseteq}_M F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_M F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.
(3). If $F^1_{A_1} \tilde{\subseteq}_J F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_J F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\wedge} F^4_{A_4}$,
(4). If $F^1_{A_1} \tilde{\subseteq}_L F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_L F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_L F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_L F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.

Proof: We simply demonstrate the correctness of (1) and (3); same method can be used to obtain (2) and (4).

(1). Let $F^1_{A_1} \tilde{\vee} F^3_{A_3} = F_{A_1 \times A_3}$ and $F^2_{A_2} \tilde{\vee} F^4_{A_4} = G_{A_2 \times A_4}$, where $F(\alpha, \gamma) = F^1(\alpha) \cup F^3(\gamma)$, $\forall$ $(\alpha, \gamma) \in A_1 \times A_3$ and $G(\beta, \delta) = F^2(\beta) \cup F^4(\delta)$, $\forall$ $(\beta, \delta) \in A_2 \times A_4$. Since, $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$. So, we have $A_1 \subseteq A_2$, $A_3 \subseteq A_4$, $F^1(\alpha) \subseteq F^2(\alpha)$ $\forall$ $\alpha \in A_1$ and $F^3(\gamma) \subseteq F^4(\gamma)$ $\forall$ $\gamma \in A_3$. It implies that, $A_1 \times A_3 \subseteq A_2 \times A_4$ and $F^1(\alpha) \cup F^3(\gamma) \subseteq F^2(\alpha) \cup F^4(\gamma)$, $\forall$ $(\alpha, \gamma) \in A_1 \times A_3$. Hence, $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\vee} F^4_{A_4}$. Similarly, we can prove that $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.

(3). Since, $F^1_{A_1} \tilde{\subseteq}_J F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_J F^4_{A_4}$, so for every $\alpha \in A_1$, $\exists$ $\beta \in A_2$ such that $F^1(\alpha) \subseteq F^2(\beta)$. Similarly, for every $\gamma \in A_3$, $\exists$ $\delta \in A_4$ such that $F^3(\gamma) \subseteq F^4(\delta)$. Now, let $F^1_{A_1} \tilde{\vee} F^3_{A_3} = F_{A_1 \times A_3}$ and $F^2_{A_2} \tilde{\vee} F^4_{A_4} = G_{A_2 \times A_4}$. Then $F(\alpha, \gamma) = F^1(\alpha) \cup F^3(\gamma)$, $\forall$ $(\alpha, \gamma) \in A_1 \times A_3$ and $G(\beta, \delta) = F^2(\beta) \cup F^4(\delta)$, $\forall$ $(\beta, \delta) \in A_2 \times A_4$. But $F^1(\alpha) \cup F^3(\gamma) \subseteq F^2(\beta) \cup F^4(\delta)$. This implies that, for any $(\alpha, \gamma) \in A_1 \times A_3$, $\exists$ $(\beta, \delta) \in A_2 \times A_4$ such that $F^1(\alpha) \cup F^3(\gamma) \subseteq F^2(\beta) \cup F^4(\delta)$. Hence from definition 3.03, $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\vee} F^4_{A_4}$. Similarly, we can prove that $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.

Proposition 4.03: Let $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets over U. Then,
(1). $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$, implies $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cup} F^4_{A_4}$ and $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cap} F^4_{A_4}$,
(2). If $F^1_{A_1} \tilde{\subseteq}_M F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_M F^4_{A_4}$, implies $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\cup} F^4_{A_4}$ and $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

Proof: We just varify the validity of (1); subsequent work could be use identical methods to establish (2). To this end, Let $F^1_{A_1} \tilde{\cup} F^3_{A_3} = J_{\dot{D}}$, where $\dot{D} = A_1 \cup A_3$, and $F^2_{A_2} \tilde{\cup} F^4_{A_4} = J^{'}_{D'}$, where $D^{'} = A_2 \cup A_4$. By definition 2.08, we have

$$J(\dot{d}) = \begin{cases} F^1(\dot{d}) & , \dot{d} \in A_1 - A_3 \\ F^3(\dot{d}) & , \dot{d} \in A_3 - A_1 \\ F^1(\dot{d}) \cup F^3(\dot{d}) & , \dot{d} \in A_1 \cap A_3, \end{cases} \quad , \forall\ \dot{d} \in \dot{D},$$

$$J^{'}(d^{'}) = \begin{cases} F^2(d^{'}) & , d^{'} \in A_2 - A_4 \\ F^4(d^{'}) & , d^{'} \in A_4 - A_2 \\ F^2(d^{'}) \cup F^4(d^{'}) & , d^{'} \in A_2 \cap A_4. \end{cases} \quad , \forall\ \mathrm{d}^{'} \in \mathrm{D}^{'}$$

To prove $J_{\dot{D}} \tilde{\subseteq}_F J^{'}_{D'}$, we show that $\dot{D} \subseteq D^{'}$ and $J(\alpha) \subseteq J^{'}(\alpha)$, $\forall$ $\alpha \in \dot{D}$. Since $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$, so $A_1 \subseteq A_2$, $F^1(\alpha) \subseteq F^2(\alpha)$, $\forall$ $\alpha \in A_1$, and $A_3 \subseteq A_4$, $F^3(\alpha) \subseteq F^4(\alpha)$, $\forall$ $\alpha \in A_3$. It implies that $\dot{D} = A_1 \cup A_3 \subseteq A_2 \cup A_4 = D^{'}$ and $\dot{D} \cap D^{'} = \dot{D}$. Now,

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Case 1. If $\alpha \in A_1$ - A_3, then $J(\alpha) = F^1(\alpha) \subseteq F^2(\alpha) = J^{'}(\alpha)$.
Case 2. If $\alpha \in A_3$ - A_1 then $J(\alpha) = F^3(\alpha) \subseteq F^4(\alpha) = J^{'}(\alpha)$.
Case 3. If $\alpha \in A_1 \cap A_3$, then $J(\alpha) = F^1(\alpha) \cup F^3(\alpha) \subseteq F^2(\alpha) \cup F^4(\alpha) = J^{'}(\alpha)$.
Hence, $\forall\, \alpha \in \dot{D}$, $J(\alpha) \subseteq J^{'}(\alpha)$ and thus we finally conclude that $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cup} F^4_{A_4}$. Similarly, we can prove that $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

The following examples provides an explanation to how the property 4.01 (2) need not be satisfied by soft J-subsets and soft L-subsets, as we discussed above.

Example 4.04: Let U = {1, 2, 3, 4, 5} and E = {a, b, c, d, e} are the universe set and the essential set of parameters. Consider, $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets defined over U as:

$F^1_{A_1}$ = {(a, {2, 3}), (b, {1, 3, 4})},
$F^2_{A_2}$ = {(c, {2, 3}), (d, {1, 3, 4, 5})},
$F^3_{A_3}$ = {(a, {3, 5})},
$F^4_{A_4}$ = {(c, {4}), (e, {1, 3, 5})}.

Then, it is clear that $F^1_{A_1} \tilde{\subseteq}_J F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_J F^4_{A_4}$, since $F^1(a) \subseteq F^2(c)$, $F^1(b) \subseteq F^2(d)$ and $F^3(a) \subseteq F^4(e)$ respectively. Now, let us write $F^1_{A_1} \tilde{\cup} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cup} F^4_{A_4} = J^{'}_{D'}$, where D = {a, b} and $D^{'}$ = {c, d, e} such that J_D = {(a, {2, 3, 5}), (b, {1, 3, 4})} and $J^{'}_{D'}$ = {(c, {2, 3, 4}), (d, {1, 3, 4, 5}), (e, {1, 3, 5})}. Then, clearly we can see that $J(b) \subseteq J^{'}(d)$ but for $a \in D$ there does not exists any $\alpha^{'} \in D^{'}$ such as $J(a) \subseteq J^{'}(\alpha^{'})$ which gives $J_D \tilde{\nsubseteq}_J J^{'}_{D'}$ or $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\nsubseteq}_J F^2_{A_2} \tilde{\cup} F^4_{A_4}$. Also by definition 2.05, let us write $F^1_{A_1} \tilde{\cap} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cap} F^4_{A_4} = J^{'}_{D'}$, where D = {a} and $D^{'}$ = {c} such that J_D = {(a, {3})} and $J^{'}_{D'}$ = {(c, ϕ)}. Therefore, we have $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\nsubseteq}_J F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

Example 4.05: Let U and E are universal sets as given in example 4.04. Consider, $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets defined over U as:

$F^1_{A_1}$ = {(a, {2}), (b, {1, 3, 4})},
$F^2_{A_2}$ = {(c, {2}), (d, {1, 3, 4}), (e, {5})},
$F^3_{A_3}$ = {(a, {3, 5})},
$F^4_{A_4}$ = {(c, {2, 4}), (e, {3, 5})}.

Then, clearly $F^1_{A_1} \tilde{\subseteq}_L F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_L F^4_{A_4}$, since $F^1(a) = F^2(c)$, $F^1(b) = F^2(d)$ and $F^3(a) = F^4(e)$ respectively. Now, let us write $F^1_{A_1} \tilde{\cup} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cup} F^4_{A_4} = J^{'}_{D'}$, where D = {a, b} and $D^{'}$ = {c, d, e} such that J_D = {(a, {2, 3, 5}), (b, {1, 3, 4})} and $J^{'}_{D'}$ = {(c, {2, 4}), (d, {1, 3, 4}), (e, {3, 5})}. Therefore, we have $J(b) = J^{'}(d)$ but for $a \in R \nexists\, \alpha^{'} \in D^{'}$ such as $J(a) = J^{'}(\alpha^{'})$. Thus, $J_D \tilde{\nsubseteq}_L J^{'}_{D'}$ or $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\nsubseteq}_L F^2_{A_2} \tilde{\cup} F^4_{A_4}$. Also by definition 2.05, let us write $F^1_{A_1} \tilde{\cap} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cap} F^4_{A_4} = J^{'}_{D'}$, where D = {a} and $D^{'}$ = {c, e} such that J_D = {(a, ϕ)} and $J^{'}_{D'}$ = {(c, {2}), (e, {5})}. Therefore, we have $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\nsubseteq}_L F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

5. Complement Property and Generalized Soft Subsets:

In this section, first we define a universal complement property on classical subsets and soft complements (negation ^{c} and relative ^{r} complement) on soft sets. In soft set theory, it is obvious that none of the soft subsets presented in section 4 satisfy the complement property 5.01. However, by applying a restrictions on their parameter sets, we show the validity of the specified complement property on all soft subsets.

Definition 5.01(Universal Complement Property): Let the universe set be X. If $U \subseteq X$ and $V \subseteq X$ such as $U \subseteq V$, then $V^{'} \subseteq U^{'}$, where "$'$" is called complement operator defined as $U^{'}$ = X - U.

Definition 5.02([11]): The soft set $(F_A)^c$ is the complement of soft set F_A, described as $(F_A)^c = F^c_{\rceil A}$, where F^c is a mapping as: $F^c : \rceil A \longrightarrow P(U)$, such that $\forall\, \alpha \in \rceil A$, $F^c(\alpha) = U - F(\neg\alpha)$. Here, $\rceil A$

is read as "NOT set of a set A"; $\rceil$A = $\{\neg\alpha_1, \neg\alpha_2, \ldots, \neg\alpha_n\}$, where $\neg\alpha_i$ = not α_i, $\forall$ i. (It should be observed that the operators $\rceil$ and $\neg$ are distinct). This type of soft complement is called *"negation complement (neg-complement or pseudo-complement [6])"*.

Definition 5.03([6]): The soft set $(\mathrm{F}_A)^r$ is the complement of soft set F_A, defined as $(\mathrm{F}_A)^r = \mathrm{F}^r_A$, where F^r is a mapping: $\mathrm{F}^r : \mathrm{A} \longrightarrow \mathrm{P(U)}$, such as $\forall\ \alpha \in$ A, $\mathrm{F}^r(\alpha)$ = U - F(α). This type of soft complement is called *"Relative Complement"*.

Clearly, $((\mathrm{F}_A)^c)^c = \mathrm{F}_A$ and $((\mathrm{F}_A)^r)^r = \mathrm{F}_A$. However, it is noted that the parameter set in relative complement $(\mathrm{F}_A)^r$ is still the original set A of parameters, instead of $\rceil$A in negation complement $(\mathrm{F}_A)^c$. The following theorem provides an important result that, if $\mathrm{F}_{A_1} \tilde{\subseteq} \mathrm{G}_{A_2}$, then $(\mathrm{F}_{A_1})^c \tilde{\subseteq} (\mathrm{G}_{A_2})^c$ and $(\mathrm{F}_{A_1})^r \tilde{\subseteq} (\mathrm{G}_{A_2})^r$ with respect to soft M-subset and soft L-subset.

Theorem 5.04: Let F_{A_1} and G_{A_2} are soft sets defined on U. Then,
(1). $\mathrm{F}_{A_1} \tilde{\subseteq}_M \mathrm{G}_{A_2} \iff \mathrm{F}^c_{\rceil A_1} \tilde{\subseteq}_M \mathrm{G}^c_{\rceil A_2}$,
(2). $\mathrm{F}_{A_1} \tilde{\subseteq}_M \mathrm{G}_{A_2} \iff \mathrm{F}^r_{A_1} \tilde{\subseteq}_M \mathrm{G}^r_{A_2}$,
(3). $\mathrm{F}_{A_1} \tilde{\subseteq}_L \mathrm{G}_{A_2} \iff \mathrm{F}^c_{\rceil A_1} \tilde{\subseteq}_L \mathrm{G}^c_{\rceil A_2}$,
(4). $\mathrm{F}_{A_1} \tilde{\subseteq}_L \mathrm{G}_{A_2} \iff \mathrm{F}^r_{A_1} \tilde{\subseteq}_L \mathrm{G}^r_{A_2}$.

Proof: We only varify the correctness of (1) and (3); using similar techniques we can give the proof of (2) and (4).

(1). Let $\mathrm{F}_{A_1} \tilde{\subseteq}_M \mathrm{G}_{A_2}$. Then we have $\mathrm{A}_1 \subseteq \mathrm{A}_2$ and for all $\alpha \in \mathrm{A}_1$, F(α) = G(α). Now by defintion 5.02, we write $(\mathrm{F}_{A_1})^c = \mathrm{F}^c_{\rceil A_1}$, where $\mathrm{F}^c(\neg\alpha)$ = U - F(α), $\forall\ \neg\alpha \in \rceil\mathrm{A}_1$ or $\forall\ \alpha \in \mathrm{A}_1$. Since F($\alpha$) = G($\alpha$), and F($\alpha$), G($\alpha$) are crisp subsets of universe set U, so we can find that U - F(α) = U - G(α), for all $\alpha \in \mathrm{A}_1$. Also, we know that $\mathrm{A}_1 \subseteq \mathrm{A}_2$ iff $\rceil\mathrm{A}_1 \subseteq \rceil\mathrm{A}_2$. It implies that, for any $\neg\alpha \in \rceil\mathrm{A}_1$, $\mathrm{F}^c(\neg\alpha) = \mathrm{G}^c(\neg\alpha)$. This shows that $\mathrm{F}^c_{\rceil A_1} \tilde{\subseteq}_M \mathrm{G}^c_{\rceil A_2}$.

Conversely, let us take $\mathrm{F}^c_{\rceil A_1} \tilde{\subseteq}_M \mathrm{G}^c_{\rceil A_2}$. It implies that $\rceil\mathrm{A}_1 \subseteq \rceil\mathrm{A}_2 \implies \mathrm{A}_1 \subseteq \mathrm{A}_2$, and $\forall\ \neg\alpha \in \rceil\mathrm{A}_1$, $\mathrm{F}^c(\neg\alpha) = \mathrm{G}^c(\neg\alpha)$. Thus, U - F($\alpha$) = U - G($\alpha$) which gives F($\alpha$) = G($\alpha$). So we have $\forall\ \alpha \in \mathrm{A}_1$, F($\alpha$) = G($\alpha$). Hence $\mathrm{F}_{A_1} \tilde{\subseteq}_M \mathrm{G}_{A_2}$.

(3). Let $(\mathrm{F}_{A_1})^c = \mathrm{F}^c_{\rceil A_1}$, where $\mathrm{F}^c(\neg\alpha)$ = U - F(α), $\forall\ \neg\alpha \in \rceil\mathrm{A}_1$ or $\forall\ \alpha \in \mathrm{A}_1$. Since $\mathrm{F}_{A_1} \tilde{\subseteq}_L \mathrm{G}_{A_2}$. Therefore, for any $\alpha \in \mathrm{A}_1$, $\exists\ \beta \in \mathrm{A}_2$ such that F(α) = G(β). Consequently U - F(α) = U - G(β). We also know that for any $\alpha \in \mathrm{A}_1$, $\neg\alpha \in \rceil\mathrm{A}_1$. So, we can find that for any $\neg\alpha \in \rceil\mathrm{A}_1\ \exists\ \neg\beta \in \rceil\mathrm{A}_2$ such that $\mathrm{F}^c(\neg\alpha)$ = U - F(α) = U - G(β) = $\mathrm{G}^c(\neg\beta)$. Hence, we have $\mathrm{F}^c_{\rceil A_1} \tilde{\subseteq}_L \mathrm{G}^c_{\rceil A_2}$.

Conversely, let $\mathrm{F}^c_{\rceil A_1} \tilde{\subseteq}_L \mathrm{G}^c_{\rceil A_2}$. Then for any $\neg\alpha \in \rceil\mathrm{A}_1$, $\exists\ \neg\beta \in \rceil\mathrm{A}_2$ such that $\mathrm{F}^c(\neg\alpha) = \mathrm{G}^c(\neg\beta)$. Thus, U - F($\alpha$) = U - G($\beta$) which implies F($\alpha$) = G($\beta$). Also, $\neg\alpha \in \rceil\mathrm{A}_1$ if and only if $\alpha \in \mathrm{A}_1$. Thus, for any $\alpha \in \mathrm{A}_1\ \exists\ \beta \in \mathrm{A}_2$ such as F(α) = G(β). Therefore, $\mathrm{F}_{A_1} \tilde{\subseteq}_L \mathrm{G}_{A_2}$.

Remark 5.05: From the theorem 5.04, we conclude that the property 5.01 is not satisfied by soft L-subset. Whenever soft M-subset satisfies 5.01, the attribute sets should be equal. That is, if A_1 = A_2 and $\mathrm{F}_{A_1} \tilde{\subseteq}_M \mathrm{G}_{A_2}$, then $\mathrm{G}^r_{A_2} \tilde{\subseteq}_M \mathrm{F}^r_{A_1}$ and $\mathrm{G}^c_{\rceil A_2} \tilde{\subseteq}_M \mathrm{F}^c_{\rceil A_1}$ (see theorem 5.09). We give the following example only for soft L-subset. One can also see it for soft M-subsets, when $\mathrm{A}_1 \subset \mathrm{A}_2$.

Example 5.06: Let U and E are universal sets as given in example 4.04, $\mathrm{F}_{A_1} \neq \tilde{\Phi}$ and $\mathrm{G}_{A_2} \neq \tilde{\Phi}$ are defined as: F_{A_1} = {(a, {2, 3}), (b, {1, 4, 5})}, G_{A_2} = {(c, {2, 3}), (d, {3, 4}), (e, {1, 4, 5})}. So, we can see that for a $\in \mathrm{A}_1$, $\exists$ c $\in \mathrm{A}_2$ such as F(a) = G(c), and for b $\in \mathrm{A}_1$, $\exists$ e $\in \mathrm{A}_2$ such that F(b) = G(e). Therefore, $\mathrm{F}_{A_1} \tilde{\subseteq}_L \mathrm{G}_{A_2}$. Now by defintions 5.02 and 5.03, we have $\mathrm{F}^c_{\rceil A_1}$ = {($\neg$a, {1, 4, 5}), ($\neg$b, {2, 3})}, $\mathrm{G}^c_{\rceil A_2}$ = {($\neg$c, {1, 4, 5}), ($\neg$d, {1, 2, 5}), ($\neg$e, {2, 3})}, $\mathrm{F}^r_{A_1}$ = {(a, {1, 4, 5}), (b, {2, 3})} and $\mathrm{G}^r_{A_2}$ = {(c, {1, 4, 5}), (d, {1, 2, 5}), (e, {2, 3})}. So we can see that $\mathrm{A}_2 \nsubseteq \mathrm{A}_1$ and for $\neg$e $\in \rceil\mathrm{A}_2$ there does not exists any element $\neg$e$'$ $\in \rceil\mathrm{A}_1$ such that $\mathrm{G}^c(\neg\mathrm{e}) = \mathrm{F}^c(\neg\mathrm{e}')$. It implies that $\mathrm{G}^c_{\rceil A_2} \tilde{\nsubseteq}_L \mathrm{F}^c_{\rceil A_1}$. Similarly, we can find that $\mathrm{G}^r_{A_2} \tilde{\nsubseteq}_L \mathrm{F}^r_{A_1}$.

Remark 5.07: The above properties given in the theorem 5.04 and definition 5.01 are not satisfied by soft F-subset and soft J-subset. Here, we give an example only for soft J-subset. Similarly, one can give an example for soft F-subset.

Example 5.08: Let U and E are universal sets as given in example 4.04, F_{A_1} and G_{A_2} are soft sets over U, defined as: F_{A_1} = {(a, {2, 3}), (b, {1, 3, 4})}, G_{A_2} = {(c, {1, 2, 3}), (d, {1, 3, 4, 5}), (e, {2, 4})}. Clearly, we can see that $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Now, we have $F^r_{A_1}$ = {(a, {1, 4, 5}), (b, {2, 5})} and $G^r_{A_2}$ = {(c, {4, 5}), (d, {2}), (e, {1, 3, 5})}. It implies that neither $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$ nor $F^r_{A_1} \tilde{\subseteq}_J G^r_{A_2}$. Similarly, we can find that neither $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$ nor $F^c_{\rceil A_1} \tilde{\subseteq}_J G^c_{\rceil A_2}$.

The following theorems proves the validity of the stated complement property 5.01 on all generalized soft subsets by taking an onto mapping between sets of parameters/attributes.

Theorem 5.09: Let F_{A_1} and G_{A_2} are soft sets on U and $A_1 = A_2$. Then,
(1). $F_{A_1} \tilde{\subseteq}_M G_{A_2} \iff G^r_{A_2} \tilde{\subseteq}_M F^r_{A_1}$,
(2). $F_{A_1} \tilde{\subseteq}_M G_{A_2} \iff G^c_{\rceil A_2} \tilde{\subseteq}_M F^c_{\rceil A_1}$,
(3). $F_{A_1} \tilde{\subseteq}_F G_{A_2} \iff G^r_{A_2} \tilde{\subseteq}_F F^r_{A_1}$,
(4). $F_{A_1} \tilde{\subseteq}_F G_{A_2} \iff G^c_{\rceil A_2} \tilde{\subseteq}_F F^c_{\rceil A_1}$.

Proof: We give only the validity of argument (4); the proof of other statements (1), (2) and (3) can be obtained by the same method. Let us take $F_{A_1} \tilde{\subseteq}_F G_{A_2}$. Then $A_1 \subseteq A_2$ and for all $\alpha \in A_1$, $F(\alpha) \subseteq G(\alpha)$ which gives U - G(α) U - F(α). That is, $G^c(\neg\alpha) \subseteq F^c(\neg\alpha)$. Since, $A_1 = A_2$ iff $\rceil A_1 = \rceil A_2$, so we have $\forall\ \neg\alpha \in \rceil A_1$, $G^c(\neg\alpha) \subseteq F^c(\neg\alpha)$. Hence $G^c_{\rceil A_2} \tilde{\subseteq}_F F^c_{\rceil A_1}$.

Conversely, let $G^c_{\rceil A_2} \tilde{\subseteq}_F F^c_{\rceil A_1}$. Then $\rceil A_2 \subseteq \rceil A_1$ and $\forall\ \neg\beta \in \rceil A_2$, $G^c(\neg\beta) \subseteq F^c(\neg\beta)$. Therefore, U - G($\beta$) $\subseteq$ U - F(β) which implies $F(\beta) \subseteq G(\beta)$. Since, $A_1 = A_2$ iff $\rceil A_1 = \rceil A_2$, hence $\forall\ \alpha \in A_1$, $F(\alpha) \subseteq G(\alpha)$.

Theorem 5.10: Let F_{A_1} and G_{A_2} are soft sets on U. If there exists a surjective or onto mapping f : $A_1 \longrightarrow A_2$ as; for any $\alpha \in A_1$, f(α) = β where $\beta \in A_2$, such that $F(\alpha) \subseteq G(f(\alpha))$, then $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Hence, $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$ and $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$.

Proof: Since f is a mapping as, for any $\alpha \in A_1$, f(α) = β, such as $F(\alpha) \subseteq G(f(\alpha))$. So for any $\alpha \in A_1$, $\exists\ \beta = f(\alpha) \in A_1$ such as $F(\alpha) \subseteq G(\beta)$. So we have $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Now, for any $\alpha \in A_1$, $\exists\ \beta \in A_2$ such as $F(\alpha) \subseteq G(\beta)$. That is, U - G($\beta$) $\subseteq$ U - F(α). Consequently $G^c(\neg\beta) \subseteq F^c(\neg\alpha)$. We also know that f is onto mapping, so for every $\beta \in A_2$, $\exists\ \alpha \in A_1$ such as β = f(α). Thus, $\neg\beta$ = f($\neg\alpha$). Hence, for any $\neg\beta \in \rceil A_2$, $\exists\ \neg\alpha \in \rceil A_1$ such as $G^c(\neg\beta) \subseteq F^c(\neg\alpha)$. Therefore, we have $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$. Similarly, we can prove $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$.

Theorem 5.11: Let F_{A_1} and G_{A_2} are soft sets over U. If there exists a surjective or onto mapping f : $A_1 \longrightarrow A_2$ as; for any $\alpha \in A_1$, f(α) = β where $\beta \in A_2$, such as $F(\alpha) = G(f(\alpha))$, then $F_{A_1} \tilde{\subseteq}_L G_{A_2}$. Hence, $G^c_{\rceil A_2} \tilde{\subseteq}_L F^c_{\rceil A_1}$ and $G^r_{A_2} \tilde{\subseteq}_L F^r_{A_1}$.

Proof: Due to similar proof to that of the Theorem 5.10, the proof is excluded.

The following real world example first makes two soft sets as soft J-subset using a mapping between two different parameter sets and then show above given complement property (theorem 5.10) on them. Here, noted that these parameter sets may share common elements.

Example 5.12: Let U = {μ_1, μ_2, μ_3, μ_4, μ_5} be the universal set of five canditates for an interview in a company, and E = {ϱ_1, ϱ_2, ϱ_3, ϱ_4, ϱ_5, $\tilde{\varrho}_1$, $\tilde{\varrho}_2$, $\tilde{\varrho}_3$, $\tilde{\varrho}_4$} be the universal set of all attributes/parameters defined on U, where all ϱ_i's (i $\in$ {1, 2, 3, 4, 5}) represents the name of jobs as given in example 2.07, and all $\tilde{\varrho}_j$'s (j $\in$ {1, 2, 3, 4}) represents the required qualifications for respective jobs such as; $\tilde{\varrho}_1$ indicates bachelor in information technology (Bach. I.T.), $\tilde{\varrho}_2$ indicates bachelor in computer science (Bach. C.S.), $\tilde{\varrho}_3$ indicates bachelor in information technology with course work in design and web developer (Bach. I.T. + C.W. in Dgn and Web devl.), and $\tilde{\varrho}_4$ indicates bachelor in computer science with course work in business administration (Bach. C.S. + C.W. in B.A.). Below table provides the connection between jobs and required qualifications for corresponding jobs.

Table 1: Data table for jobs and required qualifications

Attributes representation of jobs	**Name of jobs**	**Required qualifications of corresponding jobs**	**Representation of required qualifications in attribute form**
ϱ_1	Network Administrator (NA)	Bach. I.T. or Bach. C.S.	$\tilde{\varrho_1}$ or $\tilde{\varrho_2}$
ϱ_2	User Experience Designer (UED)	Bach. I.T. + C.W. in Dgn and Web devl.	$\tilde{\varrho_3}$
ϱ_3	System Analyst (SA)	Bach. C.S. or related fields + C.W. in B.A. or Management or Finance	$\tilde{\varrho_4}$
ϱ_4	Database Administrator (DA)	Bach. C.S.	$\tilde{\varrho_2}$
ϱ_5	Development Operations Engineer (DOE)	Bach. I.T. or Bach. C.S.	$\tilde{\varrho_1}$ or $\tilde{\varrho_2}$

Now, consider two soft sets F_{A_1} and G_{A_2}, where F_{A_1} provides the canditates for corresponding jobs in $A_1 = \{\varrho_1, \varrho_2, \varrho_3\}$ and G_{A_2} provides the canditates according to the qualifications in $A_2 = \{\tilde{\varrho_1}, \tilde{\varrho_2}\}$, defined as:

$F_{A_1} = \{(\varrho_1, \{\mu_1, \mu_3\}), (\varrho_2, \{\mu_2\}), (\varrho_3, \{\mu_3, \mu_5\})\}$,
$G_{A_2} = \{(\tilde{\varrho_1}, \{\mu_2, \mu_4\}), (\tilde{\varrho_2}, \{\mu_1, \mu_3, \mu_5\})\}$.

From given table 1 and soft sets F_{A_1} and G_{A_2}, we can see that there exist an onto mapping f defined as:
$f : A_1 \longrightarrow A_2$, where $f(\varrho_1) = \tilde{\varrho_2}$, $f(\varrho_2) = \tilde{\varrho_1}$, and $f(\varrho_3) = \tilde{\varrho_2}$, such as for every $\varrho_i \in A_1$ $\exists$ $\tilde{\varrho_j} \in A_2$, $F(\varrho_i) \tilde{\subseteq} G(f(\varrho_i))$. So, clearly we have $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Now,
$F^c_{\rceil A_1} = \{(\neg\varrho_1, \{\mu_2, \mu_4, \mu_5\}), (\neg\varrho_2, \{\mu_1, \mu_3, \mu_4, \mu_5\}), (\neg\varrho_3, \{\mu_1, \mu_2, \mu_4\})\}$,
$G^c_{\rceil A_2} = \{(\neg\tilde{\varrho_1}, \{\mu_1, \mu_3, \mu_5\}), (\neg\tilde{\varrho_2}, \{\mu_2, \mu_4\})\}$.
Therefore, $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$. Similarly, we can see that $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$.

By utilizing above results (5.01 - 5.11) of complements on all generalized soft subsets, we provide here some more results on soft product operators. Proofs will be similar to above results. So, we exclude their proofs.

Theorem 5.13:
(1). If $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\vee} F^4_{A_4}$, then $F^{1c}_{\rceil A_1} \tilde{\wedge} F^{3c}_{\rceil A_3} \tilde{\subseteq}_M F^{2c}_{\rceil A_2} \tilde{\wedge} F^{4c}_{\rceil A_4}$. Similarly, if $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\wedge} F^4_{A_4}$, then $F^{1c}_{\rceil A_1} \tilde{\vee} F^{3c}_{\rceil A_3} \tilde{\subseteq}_M F^{2c}_{\rceil A_2} \tilde{\vee} F^{4c}_{\rceil A_4}$. Also, it holds with respect to soft L-subset.
(2). Let $A_1 \times A_3 = A_2 \times A_4$. Then, $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\vee} F^4_{A_4}$, implies that $F^{2c}_{\rceil A_2} \tilde{\wedge} F^{4c}_{\rceil A_4} \tilde{\subseteq}_M F^{1c}_{\rceil A_1} \tilde{\wedge} F^{3c}_{\rceil A_3}$. Similarly, $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\wedge} F^4_{A_4}$, implies that $F^{2c}_{\rceil A_2} \tilde{\vee} F^{4c}_{\rceil A_4} \tilde{\subseteq}_M F^{1c}_{\rceil A_1} \tilde{\vee} F^{3c}_{\rceil A_3}$. Also, it holds with respect to soft F-subset.
Both points are also true for relative complement (^{r}).

Theorem 5.14: Let $F^1_{A_1} \tilde{\vee} F^3_{A_3} = F_{A_1\times A_3}$, $F^2_{A_2} \tilde{\vee} F^4_{A_4} = G_{A_2\times A_4}$ and f be an onto mapping defined as: f : $A_1\times A_3 \longrightarrow A_2\times A_4$; for any $(\alpha_1, \alpha_3) \in A_1\times A_3$, $f(\alpha_1, \alpha_3) = (\alpha_2, \alpha_4)$ where $(\alpha_2, \alpha_4) \in A_2$, A_4, such as:
(1). If $F(\alpha_1, \alpha_3) \subseteq G(f(\alpha_1, \alpha_3))$, then $F_{A_1\times A_3} \tilde{\subseteq}_J G_{A_2\times A_4}$. Hence, $G^c_{\rceil A_1\times\rceil A_3} \tilde{\subseteq}_J G^c_{\rceil A_2\times\rceil A_4}$ and $G^r_{A_1\times A_3} \tilde{\subseteq}_J G^r_{A_2\times A_4}$.
(2). If $F(\alpha_1, \alpha_3) = G(f(\alpha_1, \alpha_3))$, then $F_{A_1\times A_3} \tilde{\subseteq}_L G_{A_2\times A_4}$. Hence, $G^c_{\rceil A_1\times\rceil A_3} \tilde{\subseteq}_L G^c_{\rceil A_2\times\rceil A_4}$ and $G^r_{A_1\times A_3} \tilde{\subseteq}_L G^r_{A_2\times A_4}$.

6. CONCLUSION AND FUTURE WORK

Due to the non-availability of complement property on generalized soft subsets in soft set theory, generalized soft subsets can not be used to study various algebraic structures. This research provides a platform in this area. It presents crucial results on soft operations using various generalized soft subsets. It is also shown here that the classical property of intersection and union (Property 4.01(2)) only holds with respect to soft M-subset and soft F-subset but not for soft J-subset and soft L-subset. Further, we provide the complement property 5.01 for given soft subsets, and prove that the property is not satisfied by any generalized soft subsets for which the relevant counterexamples are given. But, this problem is solved in the given study by an onto mapping between the sets of parameters on all generalized soft subsets.

In classical mathematics, subsets, operators and complements are very important concepts when studying algebraic structures such as topology, lattices, and Boolean algebra. In soft set theory, these concepts are also crucial for studying these structures. To achieve this, some researchers have provided soft topological spaces in various forms using soft union, soft intersection, soft M-subsets, soft F-subsets, and soft complement operators.

In addition, some researchers have focused on soft product operations to enhance algebraic properties, and it has been found that these soft product operations can be utilized in soft lattice structures. As such, it is suggested that future research can expand on these findings by investigating various soft subsets in other algebraic properties to make lattice structures on various generalized soft subsets. By studying these structures in soft set theory, researchers can gain a better understanding of how uncertainty and vagueness can affect the algebraic properties of subsets, operators, and complements. The findings from this research can also have practical applications in various fields such as decision-making, data analysis, and artificial intelligence.

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Trigonometric and Hyperbolic Polya type inequalities

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Abstract

Here based on trigonometric and hyperbolic type Taylor formulae we derive Polya type inequalities in a number of cases.

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Keywords and phrases: trigonometric and hyperbolic Taylor formulae, Polya inequalities.

1 Main Results

We present a collection of Polya's type inequalities.

Theorem 1 *Let $f \in C^2([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We set*

$$M_1 := \max\left\{ \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \|f'' + f\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}, \tag{1}$$

$$M_2 := \max\left\{ \|f'' + f\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \|f'' + f\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)} \right\}, \tag{2}$$

and

$$M_3 := \max\left\{ \|f'' + f\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}, \|f'' + f\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)} \right\}. \tag{3}$$

Then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx \le \min \left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1, \\ \frac{(b-a)^2}{4} M_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} M_3 \end{array} \right\}. \tag{4}$$

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Proof. Here $f \in C^2([a,b], K)$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1$. By Corollary 3.4 of [1], we have

$$f(x) = \int_a^x (f''(t) + f(t)) \sin(x-t)\, dt, \tag{5}$$

and

$$f(x) = \int_x^b (f''(t) + f(t)) \sin(x-t)\, dt. \tag{6}$$

By using $|\sin x| \le |x|$, $\forall\, x \in \mathbb{R}$, we obtain

$$|f(x)| \le \int_a^x |f''(t) + f(t)| \, |\sin(x-t)|\, dt \le$$

$$\int_a^x |f''(t) + f(t)| (x-t)\, dt \le \left(\int_a^x (x-t)\, dt \right) \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]} = \tag{7}$$

$$\frac{(x-a)^2}{2} \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \ \ \forall\, x \in \left[a, \frac{a+b}{2}\right].$$

Also it holds

$$|f(x)| \le (x-a) \left(\int_a^x |f''(t) + f(t)|\, dt \right) \le \tag{8}$$

$$(x-a) \|f'' + f\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \ \ \forall\, x \in \left[a, \frac{a+b}{2}\right].$$

Furthermore, by Hölder's inequality, we have $\left(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1\right)$

$$|f(x)| \le \left(\int_a^x |f''(t) + f(t)|^q\, dt \right)^{\frac{1}{q}} \left(\int_a^x (x-t)^p\, dt \right)^{\frac{1}{p}} \le \tag{9}$$

$$\|f'' + f\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)} \frac{(x-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}}, \ \ \forall\, x \in \left[a, \frac{a+b}{2}\right].$$

We have found that

$$|f(x)| \le \left\{ \begin{array}{l} \frac{(x-a)^2}{2} \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \\ (x-a) \|f'' + f\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \\ \frac{(x-a)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \|f'' + f\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}, \quad p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \end{array} \right\}, \tag{10}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$.

Similarly acting, we get that

$$|f(x)| = \left| \int_x^b (f''(t) + f(t)) \sin(x-t)\, dt \right| \le$$

2

$$\int_x^b \left|f''(t)+f(t)\right| \left|\sin(t-x)\right| dt \leq \int_x^b \left|f''(t)+f(t)\right| (t-x)\, dt, \tag{11}$$

and

$$|f(x)| \leq \left\{ \begin{array}{l} \frac{(b-x)^2}{2} \left\|f''+f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}, \\ (b-x) \left\|f''+f\right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}, \\ \frac{(b-x)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \left\|f''+f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}, \quad p,q>1: \frac{1}{p}+\frac{1}{q}=1 \end{array} \right\}, \tag{12}$$

$\forall\, x \in \left[\frac{a+b}{2}, b\right]$.

Consequently, we obtain

$$\int_a^{\frac{a+b}{2}} |f(x)|\, dx \leq \left\{ \begin{array}{l} \frac{(b-a)^3}{16} \left\|f''+f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \\ \frac{(b-a)^2}{8} \left\|f''+f\right\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} \left\|f''+f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \ p,q>1: \frac{1}{p}+\frac{1}{q}=1. \end{array} \right\} \tag{13}$$

Similarly, we derive that

$$\int_{\frac{a+b}{2}}^b |f(x)|\, dx \leq \left\{ \begin{array}{l} \frac{(b-a)^3}{16} \left\|f''+f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}, \\ \frac{(b-a)^2}{8} \left\|f''+f\right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} \left\|f''+f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}, \quad p,q>1: \frac{1}{p}+\frac{1}{q}=1. \end{array} \right\} \tag{14}$$

We have that

$$\int_a^b |f(x)|\, dx = \int_a^{\frac{a+b}{2}} |f(x)|\, dx + \int_{\frac{a+b}{2}}^b |f(x)|\, dx \leq$$

$$\left\{ \begin{array}{l} \frac{(b-a)^3}{16} \left[\left\|f''+f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\|f''+f\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right], \\ \frac{(b-a)^2}{8} \left[\left\|f''+f\right\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)} + \left\|f''+f\right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)} \right], \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} \left[\left\|f''+f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, + \left\|f''+f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)} \right]. \end{array} \right. \leq \tag{15}$$

$$\left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1, \\ \frac{(b-a)^2}{4} M_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} M_3 \end{array} \right\}. \tag{16}$$

The claim is proved. ■

We continue with

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Theorem 2 *All as in Theorem 1. Denote*

$$M_1^* := \max \left\{ \|f'' - f\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \|f'' - f\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}, \tag{17}$$

$$M_2^* := \max \left\{ \|f'' - f\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \|f'' - f\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)} \right\}, \tag{18}$$

and

$$M_3^* := \max \left\{ \|f'' - f\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \|f'' - f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)} \right\}. \tag{19}$$

Then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx \le \min \cosh(b-a) \times \left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1^*, \\ \frac{(b-a)^2}{4} M_2^*, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} M_3^* \end{array} \right\}. \tag{20}$$

Proof. As similar to Theorem 1 it is omitted. It based on Corollary 3.5 of [1]. Also we use that $|\sinh x| \le \cosh(b-a)|x|$, $\forall\, x \in [-(b-a), b-a]$, by the mean value theorem. ■

It follows

Theorem 3 *Let $f \in C^4([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0,1,2,3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We set*

$$A_1 := \max \left\{ \left\| f^{(4)} - f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| f^{(4)} - f \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}, \tag{21}$$

$$A_2 := \max \left\{ \left\| f^{(4)} - f \right\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \left\| f^{(4)} - f \right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)} \right\}, \tag{22}$$

and

$$A_3 := \max \left\{ \left\| f^{(4)} - f \right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \left\| f^{(4)} - f \right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)} \right\}. \tag{23}$$

Then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx \le$$

$$\min \frac{(\cosh(b-a)+1)}{4} \times \left\{ \begin{array}{l} \frac{(b-a)^3}{4} A_1, \\ \frac{(b-a)^2}{2} A_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} A_3 \end{array} \right\}. \tag{24}$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.6 of [1]. ■

We continue with

4

Theorem 4 *Let $f \in C^4([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0,1,2,3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let also $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. We set*

$$B_1 := \max\left\{ \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \right.$$

$$\left. \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}, \tag{25}$$

$$B_2 := \max\left\{ \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \right.$$

$$\left. \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)} \right\}, \tag{26}$$

and

$$B_3 := \max\left\{ \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}, \right.$$

$$\left. \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)} \right\}. \tag{27}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq$$

$$\min \frac{1}{\left|\beta^2 - \alpha^2\right|} \times \begin{cases} \frac{(b-a)^3}{4} B_1, \\ \frac{(b-a)^2}{2} B_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} B_3 \end{cases}. \tag{28}$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.7 of [1]. ■

We finish with

Theorem 5 *All as in Theorem 4, $|\alpha|, |\beta| < 1$.However here, instead of B_1, B_2, B_3, we set*

$$D_1 := \max\left\{ \left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \right.$$

$$\left. \left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}, \tag{29}$$

$$D_2 := \max\left\{ \left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \right.$$

$$\left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2 \beta^2 f \right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)} \Bigg\}, \tag{30}$$

and

$$D_3 := \max \Bigg\{ \left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2 \beta^2 f \right\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)},$$

$$\left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2 \beta^2 f \right\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)} \Bigg\}. \tag{31}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq$$

$$\min \frac{\cosh(b-a)}{\left|\beta^2 - \alpha^2\right|} \times \begin{cases} \frac{(b-a)^3}{4} D_1, \\ \frac{(b-a)^2}{2} D_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}} (p+1)^{\frac{1}{p}} \left(2+\frac{1}{p}\right)} D_3 \end{cases}. \tag{32}$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.9 of [1]. ■

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Deductive systems and filters of Sheffer stroke Hilbert algebras based on the bipolar-valued fuzzy set environment

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Abstract The notion of bipolar-valued fuzzy set is used to treat the filter and deductive system in Sheffer stroke Hilbert algebras. The concepts of bipolar-valued fuzzy filter and bipolar-valued fuzzy deductive system are introduced and related properties are investigated. Conditions under which the bipolar-valued fuzzy set can be a bipolar-valued fuzzy filter are explored. Characterizations of the bipolar-valued fuzzy filter are examined. A bipolar-valued fuzzy filter is built using a filter. To consider the nomality of bipolar-valued fuzzy filter, the notion of normal bipolar-valued fuzzy filter is introduced and related properties are investigated. The method of normalizing the bipolar-valued fuzzy filter is addressed, and we will see what the normal bipolar-valued fuzzy filter looks like.

Keywords: Sheffer stroke Hilbert algebra, filter, (bipolar-valued fuzzy) deductive system, (normal) bipolar-valued fuzzy filter.

2020 Mathematics Subject Classification. 03B05, 03G25, 06F35, 08A72.

1 Introduction

The shaper stroke, denoted by the symbol "|", is a logical operation for two inputs that produces false results only when both inputs are true, as shown in Table 1.

Table 1: The truth table for the Sheffer stroke "|"

P	Q	$P\vert Q$
F	F	T
F	T	T
T	F	T
T	T	F

The Sheffer stroke has been applied to several algebraic structures, for example, Boolean algebra, MV-algebra, BL-algebra, BCK-algebra, and ortholattices, etc., and it is also being dealt with in the fuzzy environment (see [3, 5, 7, 11, 12, 13, 14, 15]). In 2021, Oner et al. [12] applied the Sheffer stroke to Hilbert algebras. They introduced Sheffer stroke Hilbert algebra and investigated several properties. In [11], Oner et al. introduced the notion of deductive system and filter of Sheffer stroke Hilbert algebras, and dealt with their fuzzification. The bipolar-valued fuzzy set, which is introduced by Lee [9, 10] is a type of fuzzy set where the degree of membership to a set is represented by a value that can take on both positive and negative values, as opposed to traditional fuzzy sets where the degree of membership is represented by a value between 0 and 1. The value 0 in the bipolar-valued fuzzy set represents a lack of information about membership or a neutral position. Also, the negative values represent the degree of non-membership, while the positive values represent the degree of membership to the set. The bipolar-valued fuzzy set is useful for methods such as modeling complex and uncertain situations beyond traditional fuzzy sets. Therefore, the bipolar-valued fuzzy set has been applied in various fields, such as pattern recognition, decision making, and control systems etc. The bipolar-valued fuzzy set has also been widely applied in algebraic structures (see [1, 2, 4, 6, 8])

In this paper, we introduce the notion of the bipolar-valued fuzzy deductive system and the bipolar-valued fuzzy filter in Sheffer stroke Hilbert algebras, and investigate several properties. We first show that the bipolar-valued fuzzy deductive system and the bipolar-valued fuzzy filter are equivalent each other. We explore the conditions under which a bipolar-valued fuzzy set can be a bipolar-valued fuzzy filter. We establish characterization of the bipolar-valued fuzzy filter. Using the filter of Sheffer stroke Hilbert algebra, we make a bipolar-valued fuzzy filter. We discuss the nomality of bipolar-valued fuzzy filter, and we deal with how to normalize the bipolar-valued fuzzy filter. We look into what the normal bipolar-valued fuzzy filter looks like.

2 Preliminaries

Definition 2.1 ([16]). Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation "|" is said to be *Sheffer stroke* or *Sheffer operation* if it satisfies:

(s1) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ (\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a})$,

(s2) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ ((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a})$,

(s3) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ (\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c})$,

(s4) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ ((\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a})$.

Definition 2.2 ([12]). A *Sheffer stroke Hilbert algebra* is a groupoid $\mathcal{L} := (L, |)$ with a Sheffer stroke "|" that satisfies:

(sH1) $(\mathfrak{a}|((A)|(A)))|(((B)|((C)|(C)))|((B)|((C)|(C)))) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a})$,
where $A := \mathfrak{b}|(\mathfrak{c}|\mathfrak{c})$, $B := \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})$ and $C := \mathfrak{a}|(\mathfrak{c}|\mathfrak{c})$,

(sH2) $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{b}|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) \ \Rightarrow\ \mathfrak{a} = \mathfrak{b}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L$.

Let $\mathcal{L} := (L, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation "$\leq_L$" on L is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \leq_L \mathfrak{b} \ \Leftrightarrow\ \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = 1). \tag{2.1}$$

We observe that the relation "$\leq_L$" is a partial order in a Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ (see [12]).

Proposition 2.3 ([12]). *Every Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ satisfies:*

$$(\forall \mathfrak{a} \in L)(\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) = 1), \tag{2.2}$$
$$(\forall \mathfrak{a} \in L)(\mathfrak{a}|(1|1) = 1), \tag{2.3}$$
$$(\forall \mathfrak{a} \in L)(1|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}), \tag{2.4}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \leq_L \mathfrak{b}|(\mathfrak{a}|\mathfrak{a})), \tag{2.5}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = (\mathfrak{b}|(\mathfrak{a}|\mathfrak{a}))|(\mathfrak{a}|\mathfrak{a})), \tag{2.6}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)\,(((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))\,, \tag{2.7}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\,(\mathfrak{a}|((\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))) = \mathfrak{b}|((\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))))\,, \tag{2.8}$$

Definition 2.4 ([11]). Let $(L, |)$ be a Sheffer stroke Hilbert algebra. A subset F of L is called

- a *deductive system* of $(L, |)$ if it satisfies:

$$1 \in F, \tag{2.9}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \in F,\ \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in F \ \Rightarrow\ \mathfrak{b} \in F), \tag{2.10}$$

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- a *filter* of $(L, |)$ if it satisfies (2.9) and

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{b} \in F \;\Rightarrow\; \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in F), \tag{2.11}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)(\mathfrak{b}, \mathfrak{c} \in F \;\Rightarrow\; (\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c}) \in F). \tag{2.12}$$

Definition 2.5 ([11])**.** Let $(L, |)$ be a Sheffer stroke Hilbert algebra. A fuzzy set f in L is called a *fuzzy filter* of $(L, |)$ if it satisfies:

$$(\forall \mathfrak{a} \in L)(f(1) \geq f(\mathfrak{a})), \tag{2.13}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(f(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \geq f(\mathfrak{b})), \tag{2.14}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)(f((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \geq \min\{f(\mathfrak{b}), f(\mathfrak{c})\}). \tag{2.15}$$

Denote by $FS(L)$ the collection of all fuzzy sets in L. Define a relation " $\subseteq$ " on $FS(L)$ by

$$(\forall f, g \in FS(L))(f \subseteq g \;\Leftrightarrow\; (\forall \mathfrak{a} \in L)(f(\mathfrak{a}) \leq g(\mathfrak{a}))).$$

Consider two maps f^- and f^+ on L (; a universe of discourse) as follows:

$$f^- : L \to [-1, 0] \text{ and } f^+ : L \to [0, 1],$$

respectively. A structure

$$\mathfrak{f} := \{(\mathfrak{a}; f^-(\mathfrak{a}), f^+(\mathfrak{a})) \mid \mathfrak{a} \in L\}$$

is called a *bipolar-valued fuzzy set* on L (see [9]), and is will be denoted by simply $\mathfrak{f} := (L;\, f^-,\, f^+)$.

For a BVF-set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L and $(s, t) \in [-1, 0] \times [0, 1]$, we define

$$L(f^-; s) := \{\mathfrak{a} \in L \mid f^-(\mathfrak{a}) \leq s\},$$
$$U(f^+; t) := \{\mathfrak{a} \in L \mid f^+(\mathfrak{a}) \geq t\}$$

which are called the *negative s-cut* and the *positive t-cut* of $\mathfrak{f} := (L;\, f^-,\, f^+)$, respectively.

3 Bipolar-valued fuzzy deductive systems and filters

In what follows, let $\mathcal{L} := (L, |)$ denote the Sheffer stroke Hilbert algebra unless otherwise specified.

Definition 3.1. A bipolar-valued fuzzy set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L is called

- a *bipolar-valued fuzzy deductive system* of $\mathcal{L} := (L, |)$ if it satisfies:

$$(\forall x \in L)(f^-(1) \leq f^-(x),\;\; f^+(1) \geq f^+(x)), \tag{3.1}$$
$$(\forall x, y \in L)\begin{pmatrix} f^-(y) \leq \max\{f^-(x), f^-(x|(y|y))\} \\ f^+(y) \geq \min\{f^+(x), f^+(x|(y|y))\} \end{pmatrix}. \tag{3.2}$$

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- a *bipolar-valued fuzzy filter* of $\mathcal{L} := (L, |)$ if it satisfies (3.1) and

$$(\forall x, y \in L)(f^-(x|(y|y)) \leq f^-(y),\ f^+(x|(y|y)) \geq f^+(y)), \tag{3.3}$$

$$(\forall x, y, z \in L) \left(\begin{array}{l} f^-((x|(y|z))|(y|z)) \leq \max\{f^-(y), f^-(z)\} \\ f^+((x|(y|z))|(y|z)) \geq \min\{f^+(y), f^+(z)\} \end{array} \right). \tag{3.4}$$

Example 3.2. Consider a set $L = \{0, 1, 2, 3, 4, 5, 6, 7\}$. The Hasse diagram and the Sheffer stroke "|" on L are given by Figure 1 and Table 2, respectively.

Figure 1: Hasse Diagram

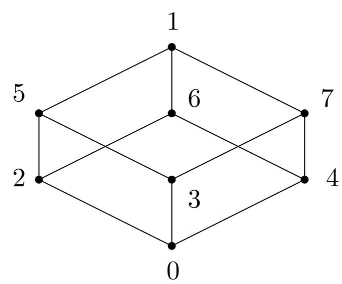

Table 2: Cayley table for the Sheffer stroke "|"

|	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	0

Then $\mathcal{L} := (L, |)$ is a Sheffer stroke Hilbert algebra (see [12]). Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ and $\mathfrak{g} := (L;\ g^-,\ g^+)$ be BVF-sets in L given by Table 3.
It is routine to verify that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy deductive system of $\mathcal{L} := (L, |)$, and $\mathfrak{g} := (L;\ g^-,\ g^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$.

Theorem 3.3. *Given a bipolar-valued fuzzy set* $\mathfrak{f} := (L;\ f^-,\ f^+)$ *in* L*, the following are equivalent to each other.*

(i) $\mathfrak{f} := (L;\ f^-,\ f^+)$ *is a bipolar-valued fuzzy deductive system of* $\mathcal{L} := (L, |)$.

Table 3: Tabular representation of $\mathfrak{f}$ and $\mathfrak{g}$

L	$f^-(x)$	$f^+(x)$	$g^-(x)$	$g^+(x)$
0	−0.42	0.49	−0.48	0.33
2	−0.56	0.68	−0.62	0.33
3	−0.42	0.49	−0.48	0.46
4	−0.42	0.49	−0.48	0.33
5	−0.64	0.79	−0.75	0.46
6	−0.56	0.68	−0.62	0.33
7	−0.42	0.49	−0.48	0.61
1	−0.72	0.83	−0.79	0.67

(ii) $\mathfrak{f} := (L;\, f^-,\, f^+)$ *is a bipolar-valued fuzzy filter of* $\mathcal{L} := (L, |)$.

Proof. Assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy deductive system of $\mathcal{L} := (L, |)$ and let $x, y, z \in L$. Note that $y|((x|(y|y))|(x|(y|y))) = 1$ by (2.1) and (2.5). The use of (3.1) and (3.2) leads to

$$\begin{aligned} f^-(x|(y|y)) &\leq \max\{f^-(y), f^-(y|((x|(y|y))|(x|(y|y))))\} \\ &= \max\{f^-(y), f^-(1)\} = f^-(y) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} f^+(x|(y|y)) &\geq \min\{f^+(y), f^+(y|((x|(y|y))|(x|(y|y))))\} \\ &= \min\{f^+(y), f^+(1)\} = f^+(y). \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} y|(((y|z)|z)|((y|z)|z)) &\overset{(s2)}{=} y|(((y|z)|((z|z)|(z|z)))|((y|z)|((z|z)|(z|z)))) \\ &\overset{(2.8)}{=} (y|z)|((y|((z|z)|(z|z)))|(y|((z|z)|(z|z)))) \\ &\overset{(s2)}{=} (y|z)|((y|z)|(y|z)) \\ &\overset{(2.2)}{=} 1. \end{aligned}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} f^-((y|z)|z) &\leq \max\{f^-(y), f^-(y|(((y|z)|z)|((y|z)|z)))\} \\ &= \max\{f^-(y), f^-(1)\} = f^-(y) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} f^+((y|z)|z) &\geq \min\{f^+(y), f^+(y|(((y|z)|z)|((y|z)|z)))\} \\ &= \min\{f^+(y), f^+(1)\} = f^+(y). \end{aligned} \tag{3.8}$$

Since $z|(((y|z)|(y|z))|((y|z)|(y|z))) \overset{(s2)}{=} z|(y|z) \overset{(s1)}{=} (y|z)|z$, we obtain

$$\begin{aligned} g^-((y|z)|(y|z)) &\le \max\{g^-(z), g^-(z|(((y|z)|(y|z))|((y|z)|(y|z))))\} \\ &= \max\{g^-(z), g^-((y|z)|z)\} \\ &\le \max\{g^-(z), g^-(y)\} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} f^+((y|z)|(y|z)) &\ge \min\{f^+(z), f^+(z|(((y|z)|(y|z))|((y|z)|(y|z))))\} \\ &= \min\{f^+(z), f^+((y|z)|z)\} \\ &\ge \min\{f^+(z), f^+(y)\}. \end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned} & f^-((x|(y|z))|(y|z)) \\ & \overset{(s2)}{=} f^-((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z)))) \\ & \overset{(3.5)}{\le} f^-((y|z)|(y|z)) \\ & \overset{(3.9)}{\le} \max\{f^-(z), f^-(y)\} \end{aligned}$$

and

$$\begin{aligned} & f^+((x|(y|z))|(y|z)) \\ & \overset{(s2)}{=} f^+((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z)))) \\ & \overset{(3.6)}{\ge} f^+((y|z)|(y|z)) \\ & \overset{(3.10)}{\ge} \min\{f^+(z), f^+(y)\}. \end{aligned}$$

Therefore $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$.

Conversely, assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ and let $x, y, z \in L$. If we replace y, z, and x with x, $x|(y|y)$, and y, respectively, in (3.4), then

$$\begin{aligned} f^-(y) &= f^-(((x|x)|(1|1))|(y|y)) \\ &= f^-(((x|x)|((y|(y|y))|(y|(y|y))))|(y|y)) \\ &= f^-(((((x|x)|y)|((x|x)|y))|(y|y))|(y|y)) \\ &= f^-((y|((x|x)|y))|((x|x)|y)) \\ &= f^-(((((x|x)|y)|y)|y)|(((x|x)|y)|y)) \\ &= f^-((y|(x|(x|(y|y))))|(x|(x|(y|y)))) \\ &\le \max\{f^-(x), f^-(x|(y|y))\} \end{aligned}$$

and

$$\begin{aligned} f^+(y) &= f^-(((x|x)|(1|1))|(y|y)) \\ &= f^+(((x|x)|((y|(y|y))|(y|(y|y))))|(y|y)) \\ &= f^+(((((x|x)|y)|((x|x)|y))|(y|y))|(y|y)) \\ &= f^+((y|((x|x)|y))|((x|x)|y)) \\ &= f^+(((((x|x)|y)|y)|y)|(((x|x)|y)|y)) \\ &= f^+((y|(x|(x|(y|y))))|(x|(x|(y|y)))) \\ &\geq \min\{f^+(x), f^+(x|(y|y))\} \end{aligned}$$

by (s1), (s2), (s3), (2.2), (2.3), (2.4) (2.6) and (2.7). Consequently, $\mathfrak{f} := (L; f^-, f^+)$ is a bipolar-valued fuzzy deductive system of $\mathcal{L} := (L, |)$. □

By Theorem 3.3, it can be seen that all the results for the bipolar-valued fuzzy filter covered below can be handled in the same way using the bipolar-valued fuzzy deductive system.

Proposition 3.4. *Every bipolar-valued fuzzy filter* $\mathfrak{f} := (L; f^-, f^+)$ *of* $\mathcal{L} := (L, |)$ *satisfies:*

$$(\forall x, y \in L)\left(\begin{array}{l} f^-((x|(y|y))|(y|y)) \leq f^-(x) \\ f^+((x|(y|y))|(y|y)) \geq f^+(x)\} \end{array}\right). \tag{3.11}$$

$$(\forall x, y \in L)\left(x \leq_L y \ \Rightarrow \begin{cases} f^-(x) \geq f^-(y) \\ f^+(x) \leq f^+(y) \end{cases} \right). \tag{3.12}$$

Proof. Let $\mathfrak{f} := (L; f^-, f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then

$$\begin{aligned} f^-((x|(y|y))|(y|y)) = f^-((y|(x|x))|(x|x)) \leq \max\{f^-(x), f^-(x)\} = f^-(x), \\ f^+((x|(y|y))|(y|y)) = f^+((y|(x|x))|(x|x)) \geq \min\{f^+(x), f^+(x)\} = f^+(x) \end{aligned}$$

for all $x, y \in L$ by (2.6) and (3.4). Therefore, (3.11) is valid. Let $x, y \in L$ be such that $x \leq_L y$. Then $x|(y|y) = 1$, and so

$$f^-(y) = f^-(1|(y|y)) = f^-((x|(y|y))|(y|y)) \leq f^-(x)$$

and

$$f^+(y) = f^+(1|(y|y)) = f^+((x|(y|y))|(y|y)) \geq f^+(x)$$

by (2.4) and (3.11). □

We consider a bipolar-valued fuzzy set $\mathfrak{f} := (L; f^-, f^+)$ in L satisfying the condition (3.12) and question whether it becomes a bipolar-valued fuzzy filter. But the example below shows that the answer to that is negative.

Figure 2: Hasse Diagram

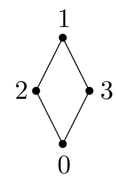

Table 4: Cayley table for the Sheffer stroke "|"

$\vert$	1	2	3	0
1	0	3	2	1
2	3	3	1	1
3	2	1	2	1
0	1	1	1	1

Table 5: Tabular representation of $\mathfrak{f} := (L;\, f^-,\, f^+)$

L	$f^-(x)$	$f^+(x)$
0	-0.12	0.09
2	-0.37	0.16
3	-0.54	0.28
1	-0.81	0.62

Example 3.5. Consider a set $L = \{0, 1, 2, 3\}$. The Hasse diagram and the Sheffer stroke "|" on L are given by Figure 2 and Table 4, respectively. Then $\mathcal{L} := (L, |)$ is a Sheffer stroke Hilbert algebra (see [12]). Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a BVF-set in L given by Table 5. Then $\mathfrak{f} := (L;\, f^-,\, f^+)$ satisfies the condition (3.12). But it is not a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ since

$$f^-((0|(3|2))|(3|2)) = f^-(0) = -0.12 \not\leq -0.37 = \max\{f^-(3), f^-(2)\}$$

and/or $f^+((0|(3|2))|(3|2)) = f^+(0) = 0.09 \not\geq 0.16 = \min\{f^+(3), f^+(2)\}$.

We explore the conditions under which a bipolar-valued fuzzy set can be a bipolar-valued fuzzy filter.

Theorem 3.6. *A bipolar-valued fuzzy set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if it satisfies the condition (3.12) and*

$$(\forall x, y \in L)\left(\begin{array}{l} f^-((x|y)|(x|y)) \leq \max\{f^-(x), f^-(y)\} \\ f^+((x|y)|(x|y)) \geq \min\{f^+(x), f^+(y)\} \end{array}\right). \tag{3.13}$$

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Proof. Let $\mathfrak{f} := (L; f^-, f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then the condition (3.12) is valid by Proposition 3.4. Using (s1), (s2), (2.3), (2.4) and (3.4), we have $f^-((x|y)|(x|y)) = f^-(((1|1)|(x|y))|(x|y)) \leq \max\{f^-(x), f^-(y)\}$ and $f^+((x|y)|(x|y)) = f^+(((1|1)|(x|y))|(x|y)) \geq \min\{f^+(x), f^+(y)\}$ for all $x, y \in L$.

Conversely, assume that $\mathfrak{f} := (L; f^-, f^+)$ satisfies (3.12) and (3.13). Since $x \leq_L 1$ and $y \leq_L x|(y|y)$ for all $x, y \in L$, we have $f^-(1) \leq f^-(x)$, $f^+(1) \geq f^+(x)$, $f^-(x|(y|y)) \leq f^-(y)$, and $f^+(x|(y|y)) \geq f^+(y)$ by (3.12). Using (2.5), (s2), (3.12) and (3.13), we have

$$f^-((x|(y|z))|(y|z)) \leq f^-((y|z)|(y|z)) \leq \max\{f^-(y), f^-(z)\}$$

and $f^+((x|(y|z))|(y|z)) \geq f^+((y|z)|(y|z)) \geq \min\{f^+(y), f^+(z)\}$ for all $x, y \in L$. Therefore $\mathfrak{f} := (L; f^-, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. □

Theorem 3.7. *A bipolar-valued fuzzy set $\mathfrak{f} := (L; f^-, f^+)$ in L is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if its negative s-cut and positive t-cut are filters of $\mathcal{L} := (L, |)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

Proof. Assume that $\mathfrak{f} := (L; f^-, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ and $L(f^-; s) \neq \emptyset \neq U(f^+; t)$ for all $(s, t) \in [-1, 0] \times [0, 1]$. It is clear that $1 \in L(f^-; s) \cap U(f^+; t)$. Let $y, \mathfrak{b} \in L$ be such that $(y, \mathfrak{b}) \in L(f^-; s) \times U(f^+; t)$. Then $f^-(y) \leq s$ and $f^+(\mathfrak{b}) \geq t$. It follows from (3.3) that $f^-(x|(y|y)) \leq f^-(y) \leq s$ and $f^+(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \geq f^+(\mathfrak{b}) \geq t$ for all $x, \mathfrak{a} \in L$. Hence $(x|(y|y), \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \in L(f^-; s) \times U(f^+; t)$. Let $y, \mathfrak{b}, z, \mathfrak{c} \in L$ be such that $(y, \mathfrak{b}) \in L(f^-; s) \times U(f^+; t)$ and $(z, \mathfrak{c}) \in L(f^-; s) \times U(f^+; t)$. Then $f^-(y) \leq s$, $f^-(z) \leq s$, $f^+(\mathfrak{b}) \geq t$, and $f^+(\mathfrak{c}) \geq t$. Using (3.4), we get $f^-((x|(y|z))|(y|z)) \leq \max\{f^-(y), f^-(z)\} \leq s$ and $f^+((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \geq \min\{f^+(\mathfrak{b}), f^+(\mathfrak{c})\} \geq t$, and so

$$((x|(y|z))|(y|z), (\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \in L(f^-; s) \times U(f^+; t).$$

Therefore $L(f^-; s)$ and $U(f^+; t)$ are filters of $\mathcal{L} := (L, |)$.

Conversely, let $\mathfrak{f} := (L; f^-, f^+)$ be a bipolar-valued fuzzy set in L for which its negative s-cut and positive t-cut are filters of $\mathcal{L} := (L, |)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$. If $f^-(1) > f^-(\mathfrak{a})$ or $f^+(1) < f^+(x)$ for some $x, \mathfrak{a} \in L$, then $\mathfrak{a} \in L(f^-; f^-(\mathfrak{a}))$ and $x \in U(f^+; f^+(x))$, but $1 \notin L(f^-; f^-(\mathfrak{a})) \cap U(f^+; f^+(x))$. This is a contradiction, and thus $f^-(1) \leq f^-(x)$ and $f^+(1) \geq f^+(x)$ for all $x \in L$. If $f^-(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) > f^-(\mathfrak{b})$ for some $\mathfrak{a}, \mathfrak{b} \in L$, then $\mathfrak{b} \in L(f^-; f^-(\mathfrak{b}))$ but $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \notin L(f^-; f^-(\mathfrak{b}))$ which is a contradiction. Hence $f^-(x|(y|y)) \leq f^-(y)$ for all $x, y \in L$. If $f^+(x|(y|y)) < f^+(y)$ for some $x, y \in L$, then $y \in U(f^+; f^+(y))$ but $x|(y|y) \notin U(f^+; f^+(y))$, a contadiction. Thus $f^+(x|(y|y)) \geq f^+(y)$ for all $x, y \in L$. Suppose that

$$f^-((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))(\mathfrak{b}|\mathfrak{c})) > \max\{f^-(\mathfrak{b}), f^-(\mathfrak{c})\}$$

or $f^+((x|(y|z))(y|z)) < \min\{f^+(y), f^+(z)\}$ for some $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, x, y, $z \in L$. Then $\mathfrak{b}, \mathfrak{c} \in L(f^-; s)$ or $y, z \in U(f^+; t)$ where $s := \max\{f^-(\mathfrak{b}), f^-(\mathfrak{c})\}$ and $t :=$

$\min\{f^+(y), f^+(z)\}$. But $(\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))(\mathfrak{b}|\mathfrak{c}) \notin L(f^-; s)$ or $(x|(y|z))(y|z) \notin U(f^+; t)$, a contradiction. Therefore $f^-((x|(y|z))(y|z)) \leq \max\{f^-(y), f^-(z)\}$ and

$$f^+((x|(y|z))(y|z)) \geq \min\{f^+(y), f^+(z)\}$$

for all $x, y, z \in L$. Consequently, $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. □

Theorem 3.8. *A bipolar-valued fuzzy set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if the fuzzy sets f_c^- and f^+ are fuzzy filters of $\mathcal{L} := (L, |)$, where $f_c^- : L \to [0, 1],\ x \mapsto 1 - f^-(x)$.*

Proof. Assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. It is clear that f^+ is a fuzzy filter of $\mathcal{L} := (L, |)$. For every $x, y, z \in L$, we have $f_c^-(1) = 1 - f^-(1) \geq 1 - f^-(x) = f_c^-(x)$,

$$f_c^-(x|(y|y)) = 1 - f^-(x|(y|y)) \geq 1 - f^-(y) = f_c^-(y),$$

and

$$\begin{aligned} f_c^-((x|(y|z))|(y|z)) &= 1 - f^-((x|(y|z))|(y|z)) \\ &\geq 1 - \max\{f^-(y), f^-(z)\} \\ &= \min\{1 - f^-(y), 1 - f^-(z)\} \\ &= \min\{f_c^-(y), f_(^- z)\}. \end{aligned}$$

Hence f_c^- is a fuzzy filter of $\mathcal{L} := (L, |)$.

Conversely, let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a bipolar-valued fuzzy set in L for which f_c^- and f^+ are fuzzy filters of $\mathcal{L} := (L, |)$. Then $1 - f^-(1) = f_c^-(1) \geq f_c^-(x) = 1 - f^-(x)$,

$$1 - f^-(x|(y|y)) = f_c^-(x|(y|y)) \geq f_c^-(y) = 1 - f^-(y)$$

and

$$\begin{aligned} &1 - f^-((x|(y|z))|(y|z)) = f_c^-((x|(y|z))|(y|z)) \\ &\geq \min\{f_c^-(y), f_c^-(z)\} \\ &= \min\{1 - f^-(y), 1 - f^-(z)\} \\ &= 1 - \max\{f^-(y), f^-(z)\} \end{aligned}$$

for all $x, y, z \in L$. Hence $f^-(1) \leq f^-(x)$, $f^-(x|(y|y)) \leq f^-(y)$ and

$$f^-((x|(y|z))|(y|z)) \leq \max\{f^-(y), f^-(z)\}$$

for all $x, y, z \in L$. Therefore, $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. □

Theorem 3.9. *Given a nonempty subset F of L, let $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ be a bipolar-valued fuzzy set in L defined as follows:*

$$f_F^- : L \to [-1,0],\ \mathfrak{a} \mapsto \begin{cases} s^- & \text{if } \mathfrak{a} \in F, \\ t^- & \text{otherwise,} \end{cases}$$

and

$$f_F^+ : L \to [0,1],\ x \mapsto \begin{cases} s^+ & \text{if } x \in F, \\ t^+ & \text{otherwise,} \end{cases}$$

where $s^- < t^-$ in $[-1,0]$ and $s^+ > t^+$ in $[0,1]$. Then $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if F is a filter of $\mathcal{L} := (L, |)$. Moreover, we have $F = L_{\mathfrak{f}_F} := \{x \in L \mid f_F^-(x) = f_F^-(1),\, f_F^+(x) = f_F^+(1)\}$.

Proof. Assume that $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then $f_F^-(1) = s^-$ and $f_F^+(1) = s^+$, and so $1 \in F$. Let $x, y \in L$ be such that $y \in F$. Then $f_F^-(y) = s^-$ and $f_F^+(y) = s^+$. It follows from (3.3) that $s^- = f_F^-(y) \geq f_F^-(x|(y|y))$ and $s^+ = f_F^+(y) \leq f_F^+(x|(y|y))$. Hence $f_F^-(x|(y|y)) = s^-$ and $f_F^+(x|(y|y)) = s^+$, from which $x|(y|y) \in F$ is derived. Let $x, y, z \in L$ be such that $y, z \in F$. Using (3.4), we have:

$$\begin{aligned} f_F^-((x|(y|z))|(y|z)) &\leq \max\{f_F^-(y), f_F^-(z)\} = s^-, \\ f_F^+((x|(y|z))|(y|z)) &\geq \min\{f_F^+(y), f_F^+(z)\} = s^+, \end{aligned}$$

and so $f_F^-((x|(y|z))|(y|z)) = s^-$ and $f_F^+((x|(y|z))|(y|z)) = s^+$. This shows that $(x|(y|z))|(y|z) \in F$. Therefore F is a filter of $\mathcal{L} := (L, |)$.

Conversely, let F be a filter of $\mathcal{L} := (L, |)$. Since $1 \in F$, we get $f_F^-(1) = s^- \leq f_F^-(\mathfrak{a})$ and $f_F^+(1) = s^+ \geq f_F^+(x)$ for all $(\mathfrak{a}, x) \in L \times L$. Let $x, y \in L$. If $y \in F$, then $x|(y|y) \in F$, and thus $f_F^-(x|(y|y)) = s^- = f_F^-(y)$ and $f_F^+(x|(y|y)) = s^+ = f_F^+(y)$. If $y \notin F$, then $f_F^-(y) = t^- > f_F^-(x|(y|y))$ and $f_F^+(y) = t^+ < f_F^+(x|(y|y))$. For every $x, y, z \in L$, if $y, z \in F$ then $(x|(y|z))|(y|z) \in F$ which implies that $f_F^-((x|(y|z))|(y|z)) = s^- = \max\{f_F^-(y), f_F^-(z)\}$ and $f_F^+((x|(y|z))|(y|z)) = s^+ = \min\{f_F^+(y), f_F^+(z)\}$. If $y \notin F$ or $z \notin F$, then

$$\begin{aligned} f_F^-((x|(y|z))|(y|z)) &\leq t^- = \max\{f_F^-(y), f_F^-(z)\}, \\ f_F^+((x|(y|z))|(y|z)) &\geq t^+ = \min\{f_F^+(y), f_F^+(z)\}. \end{aligned}$$

Therefore, $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Since F is a filter of $\mathcal{L} := (L, |)$, we get

$$\begin{aligned} L_{\mathfrak{f}_F} &= \{x \in L \mid f_F^-(x) = f_F^-(1),\, f_F^+(x) = f_F^+(1)\} \\ &= \{x \in L \mid f_F^-(x) = s^-,\, f_F^+(x) = s^+\} \\ &= \{x \in L \mid x \in F\} = F. \end{aligned}$$

This completes the proof. □

4 Normality of bipolar-valued fuzzy filters

Definition 4.1. A bipolar-valued fuzzy filter $\mathfrak{f} := (L;\, f^-,\, f^+)$ of $\mathcal{L} := (L, |)$ is said to be *normal* if there exists $(\mathfrak{a}, x) \in L \times L$ such that $f^-(\mathfrak{a}) = -1$ and $f^+(x) = 1$.

Example 4.2. Consider the Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ in Example 3.2. Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a BVF-set in L given by Table 6.

Table 6: Tabular representation of $\mathfrak{f} := (L;\, f^-,\, f^+)$

L	$f^-(x)$	$f^+(x)$
0	-0.42	0.36
2	-0.42	0.36
3	-0.42	0.76
4	-0.57	0.36
5	-0.42	1.00
6	-1.00	0.36
7	-0.57	0.76
1	-1.00	1.00

Then $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$.

Theorem 4.3. *A bipolar-valued fuzzy filter $\mathfrak{f} := (L;\, f^-,\, f^+)$ of $\mathcal{L} := (L, |)$ is normal if and only if $f^-(1) = -1$ and $f^+(1) = 1$.*

Proof. Suppose that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then $f^-(\mathfrak{a}) = -1$ and $f^+(x) = 1$ for some $(\mathfrak{a}, x) \in L \times L$. It follows from (3.1) that $f^-(1) \leq f^-(\mathfrak{a}) = -1$ and $f^+(1) \geq f^+(x) = 1$. Hence $f^-(1) = -1$ and $f^+(1) = 1$. The sufficiency is clear. □

Given two bipolar-valued fuzzy sets $\mathfrak{f} := (L;\, f^-,\, f^+)$ and $\mathfrak{g} := (L;\, g^-,\, g^+)$ in L, the inclusion " $\Subset$" between them is defined as follows:

$$\mathfrak{f} \Subset \mathfrak{g} \;\Leftrightarrow\; (\forall x \in L)(f^-(x) \geq g^-(x),\, f^+(x) \leq g^+(x)).$$

In this case we say that $\mathfrak{g} := (L;\, g^-,\, g^+)$ is larger than $\mathfrak{f} := (L;\, f^-,\, f^+)$.

Theorem 4.4. *Given a bipolar-valued fuzzy set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L, let $\mathfrak{f}_* := (L;\, f_*^-,\, f_*^+)$ be a bipolar-valued fuzzy set in L defined by $f_*^-(\mathfrak{a}) = f^-(\mathfrak{a}) - 1 - f^-(1)$ and $f_*^+(x) = f^+(x) + 1 - f^+(1)$ for all $(\mathfrak{a}, x) \in L \times L$. Then $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if $\mathfrak{f}_* := (L;\, f_*^-,\, f_*^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Moreover, $\mathfrak{f}_* := (L;\, f_*^-,\, f_*^+)$ is normal which is larger than $\mathfrak{f} := (L;\, f^-,\, f^+)$.*

Proof. Assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ and let $x, y \in L$ be such that $x \leq_L y$. Then

$$f_*^-(x) = f^-(x) - 1 - f^-(1) \geq f^-(y) - 1 - f^-(1) = f_*^-(y)$$

and

$$f_*^+(x) = f^+(x) + 1 - f^+(1) \leq f^+(y) + 1 - f^+(1) = f_*^+(y).$$

For every $x, y \in L$, we have:

$$\begin{aligned} f_*^-((x|y)|(x|y)) &= f^-((x|y)|(x|y)) - 1 - f^-(1) \\ &\leq \max\{f^-(x), f^-(y)\} - 1 - f^-(1) \\ &= \max\{f^-(x) - 1 - f^-(1), f^-(y) - 1 - f^-(1)\} \\ &= \max\{f_*^-(x), f_*^-(y)\} \end{aligned}$$

and

$$\begin{aligned} f_*^+((x|y)|(x|y)) &= f^+((x|y)|(x|y)) + 1 - f^+(1) \\ &\geq \min\{f^+(x), f^+(y)\} + 1 - f^+(1) \\ &= \min\{f^+(x) + 1 - f^+(1), f^+(y) + 1 - f^+(1)\} \\ &= \min\{f_*^+(x), f_*^+(y)\}. \end{aligned}$$

Hence $\mathfrak{f}_* := (L; f_*^-, f_*^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 3.6. Suppose that $\mathfrak{f}_* := (L; f_*^-, f_*^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Since $f^-(1) - 1 - f^-(1) = f_*^-(1) \leq f_*^-(\mathfrak{a}) = f^-(\mathfrak{a}) - 1 - f^-(1)$ and $f^+(1) + 1 - f^+(1) = f_*^+(1) \geq f_*^+(x) = f^+(x) + 1 - f^+(1)$ for all $(\mathfrak{a}, x) \in L \times L$, we have $f^-(1) \leq f^-(x)$ and $f^+(1) \geq f^+(x)$ for all $x \in L$. Since

$$f^-(\mathfrak{b}) - 1 - f^-(1) = f_*^-(\mathfrak{b}) \geq f_*^-(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) = f^-(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) - 1 - f^-(1)$$

and $f^+(y) + 1 - f^+(1) = f_*^+(y) \leq f_*^+(x|(y|y)) = f^+(x|(y|y)) + 1 - f^+(1)$ for all $(\mathfrak{a}, x), (\mathfrak{b}, y) \in L \times L$, it follows that $f^-(y) \geq f^-(x|(y|y))$ and $f^+(y) \leq f^+(x|(y|y))$ for all $x, y \in L$. Since

$$\begin{aligned} & f^-((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) - 1 - f^-(1) = f_*^-((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \\ & \leq \max\{f_*^-(\mathfrak{b}), f_*^-(\mathfrak{c})\} \\ & = \max\{f^-(\mathfrak{b}) - 1 - f^-(1), f^-(\mathfrak{c}) - 1 - f^-(1)\} \\ & = \max\{f^-(\mathfrak{b}), f^-(\mathfrak{c})\} - 1 - f^-(1) \end{aligned}$$

and

$$\begin{aligned} & f^+((x|(y|z))|(y|z)) + 1 - f^+(1) = f_*^+((x|(y|z))|(y|z)) \\ & \geq \min\{f_*^+(y), f_*^+(z)\} \\ & = \min\{f^+(y) + 1 - f^+(1), f^+(z) + 1 - f^+(1)\} \\ & = \min\{f^+(y), f^+(z)\} + 1 - f^+(1) \end{aligned}$$

for all $(\mathfrak{a}, x), (\mathfrak{b}, y), (\mathfrak{c}, z) \in L \times L$, we have

$$f^-((x|(y|z))|(y|z)) \leq \max\{f^-(y), f^-(z)\}$$

and $f^+((x|(y|z))|(y|z)) \geq \min\{f^+(y), f^+(z)\}$ for all $x, y, z \in L$. Therefore, $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Since $f_*^-(1) = f^-(1) - 1 - f^-(1) = -1$ and $f_*^+(1) = f^+(1) + 1 - f^+(1) = 1$, we know that $\mathfrak{f}_* := (L;\ f_*^-,\ f_*^+)$ is normal. Also, we have $f_*^-(x) = f^-(x) - 1 - f^-(1) \leq f^-(x)$ and $f_*^+(x) = f^+(x) + 1 - f^+(1) \geq f^+(x)$ for all $x \in L$. This shows that $\mathfrak{f}_* := (L;\ f_*^-,\ f_*^+)$ is larger than $\mathfrak{f} := (L;\ f^-,\ f^+)$. □

Theorem 4.5. *Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then it is normal if and only if $\mathfrak{f}_* = \mathfrak{f}$, that is, $f^-(x) = f_*^-(x)$ and $f^+(x) = f_*^+(x)$ for all $x \in L$.*

Proof. Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then $\mathfrak{f}_* := (L;\ f_*^-,\ f_*^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 4.4. Hence it is clear that if $\mathfrak{f}_* = \mathfrak{f}$, then $\mathfrak{f} := (L;\ f^-,\ f^+)$ is normal.

Conversely, if $\mathfrak{f} := (L;\ f^-,\ f^+)$ is normal, then $f_*^-(x) = f^-(x) - 1 - f^-(1) = f^-(x)$ and $f_*^+(x) = f^+(x) + 1 - f^+(1) = f^+(x)$ for all $x \in L$. Hence $\mathfrak{f}_* = \mathfrak{f}$. □

Proposition 4.6. *Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ and $\mathfrak{g} := (L;\ g^-,\ g^+)$ be bipolar-valued fuzzy filters of $\mathcal{L} := (L, |)$ with $\mathfrak{f} \Subset \mathfrak{g}$. If $f^-(1) = g^-(1)$ and $f^+(1) = g^+(1)$, then $L_{\mathfrak{f}_F} \subseteq L_{\mathfrak{g}_F}$.*

Proof. Straightforward. □

The example below shows that there are bipolar-valued fuzzy filters $\mathfrak{f} := (L;\ f^-,\ f^+)$ and $\mathfrak{g} := (L;\ g^-,\ g^+)$ of $\mathcal{L} := (L, |)$ that satisfy $L_{\mathfrak{f}_F} \subseteq L_{\mathfrak{g}_F}$ and $\mathfrak{f} \not\Subset \mathfrak{g}$.

Example 4.7. Consider the Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ in Example 3.5. Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ and $\mathfrak{g} := (L;\ g^-,\ g^+)$ be bipolar-valued fuzzy sets in L defined by the Table 7.

Table 7: Tabular representation of $\mathfrak{f}$ and $\mathfrak{g}$

L	$f^-(x)$	$f^+(x)$	$g^-(x)$	$g^+(x)$
0	−0.42	0.43	−0.36	0.33
2	−1.00	1.00	−1.00	1.00
3	−0.42	0.43	−0.36	0.33
1	−1.00	1.00	−1.00	1.00

Then $L_{\mathfrak{f}_F} = \{1, 2\} = L_{\mathfrak{g}_F}$ but $\mathfrak{f} \not\Subset \mathfrak{g}$ since $f^-(3) = -0.42 < -0.36 = g^-(3)$ and/or $f^+(0) = 0.43 > 0.33 = g^+(0)$.

Theorem 4.8. *Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then it is normal if and only if there is a bipolar-valued fuzzy filter $\mathfrak{g} := (L;\ g^-,\ g^+)$ of $\mathcal{L} := (L, |)$ such that $\mathfrak{g}_* \Subset \mathfrak{f}$.*

Proof. The necessity is straightforward because if $\mathfrak{f} := (L;\ f^-,\ f^+)$ is normal, then $\mathfrak{f}_* = \mathfrak{f}$.

Conversely, assume that there is a bipolar-valued fuzzy filter $\mathfrak{g} := (L;\ g^-,\ g^+)$ of $\mathcal{L} := (L, |)$ such that $\mathfrak{g}_* \Subset \mathfrak{f}$. Then $-1 = g_*^-(1) \geq f^-(1)$ and $1 = g_*^+(1) \leq f^+(1)$. Thus $f^-(1) = -1$ and $f^+(1) = 1$, and so $\mathfrak{f} := (L;\ f^-,\ f^+)$ is normal. $\square$

Theorem 4.9. *Given a bipolar-valued fuzzy set $\mathfrak{f} := (L;\ f^-,\ f^+)$ in L, consider an increasing mapping $\ell := (\ell^-, \ell^+) : [-1, f^-(1)] \times [0, f^+(1)] \to [-1, 0] \times [0, 1]$. If $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$, then the bipolar-valued fuzzy set $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ in L defined by $f_\ell^-(\mathfrak{a}) = \ell^-(f^-(\mathfrak{a}))$ and $f_\ell^+(x) = \ell^+(f^+(x))$ for all $(\mathfrak{a}, x) \in L \times L$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Moreover, if $f_\ell^-(1) = -1$ and $f_\ell^+(1) = 1$, then $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ is normal, and*

$$(\forall (s,t) \in [-1, f^-(1)] \times [0, f^+(1)])(\ell^-(s) \leq s,\ \ell^+(t) \geq t \ \Rightarrow\ \mathfrak{f} \Subset \mathfrak{f}_\ell).$$

Proof. Assume that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Let $x, y \in L$ be such that $x \leq_L y$. Then $f_\ell^-(x) = \ell^-(f^-(x)) \geq \ell^-(f^-(y)) = f_\ell^-(y)$ and $f_\ell^+(x) = \ell^+(f^+(x)) \leq \ell^+(f^+(y)) = f_\ell^+(y)$. For every $x, y, z \in L$, we have

$$\begin{aligned} f_\ell^-((x|y)|(x|y)) &= \ell^-(f^-((x|y)|(x|y))) \\ &\leq \ell^-(\max\{f^-(x), f^-(y)\}) \\ &= \max\{\ell^-(f^-(x)), \ell^-(f^-(y))\} \\ &= \max\{f_\ell^-(x), f_\ell^-(y)\} \end{aligned}$$

and

$$\begin{aligned} f_\ell^+((x|y)|(x|y)) &= \ell^+(f^+((x|y)|(x|y))) \\ &\geq \ell^+(\min\{f^+(x), f^+(y)\}) \\ &= \min\{\ell^+(f^+(x)), \ell^+(f^+(y))\} \\ &= \min\{f_\ell^+(x), f_\ell^+(y)\}. \end{aligned}$$

Therefore, $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 3.6. If $f_\ell^-(1) = -1$ and $f_\ell^+(1) = 1$, then $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ is normal by Theorem 4.3. Let $(s,t) \in [-1, f^-(1)] \times [0, f^+(1)]$ be such that $\ell^-(s) \leq s$ and $\ell^+(t) \geq t$. Then $f_\ell^-(x) = \ell^-(f^-(x)) \leq f^-(x)$ and $f_\ell^+(x) = \ell^+(f^+(x)) \geq f^+(x)$ for all $x \in L$. Hence $\mathfrak{f} \Subset \mathfrak{f}_\ell$. $\square$

Theorem 4.10. *Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ be a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ such that $f^-(\mathfrak{a}) \neq f^-(1)$ and $f^+(x) \neq f^+(1)$ for some $(\mathfrak{a}, x) \in L \times L$. If $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a maximal element of $(\mathcal{N}_F(L), \Subset)$, then it is described as follows:*

$$\begin{aligned} &f^- : L \to [-1, 0],\ \mathfrak{a} \mapsto \begin{cases} -1 & \text{if } \mathfrak{a} = 1, \\ 0 & \text{otherwise}, \end{cases} \\ &f^+ : L \to [0, 1],\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{4.1}$$

where $\mathcal{N}_F(L)$ is the set of all normal bipolar-valued fuzzy filters of $\mathcal{L} := (L, |)$.

Proof. Clearly, $(\mathcal{N}_F(L), \Subset)$ is a poset. Assume that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a maximal element of $(\mathcal{N}_F(L), \Subset)$. It is clear that $f^-(1) = -1$ and $f^+(1) = 1$ since $\mathfrak{f} := (L;\ f^-,\ f^+)$ is normal. Let $(\mathfrak{a}, x) \in L \times L$ be such that $f^-(\mathfrak{a}) \neq f^-(1)$ and $f^+(x) \neq f^+(1)$. If $f^-(\mathfrak{a}) \neq 0$ and $f^+(x) \neq 0$, then $-1 < f^-(\mathfrak{c}) < 0$ and $0 < f^+(z) < 1$ for some $(\mathfrak{c}, z) \in L \times L$. Let $\mathfrak{g} := (L;\ g^-,\ g^+)$ be a bipolar-valued fuzzy set in L defined by

$$g^- : L \to [-1, 0],\ \mathfrak{a} \mapsto \tfrac{1}{2}(f^-(\mathfrak{a}) + f^-(\mathfrak{c})),$$
$$g^+ : L \to [0, 1],\ x \mapsto \tfrac{1}{2}(f^+(x) + f^+(z)).$$

Let $x, y \in L$ be such that $x \leq_L y$. Then

$$g^-(x) = \tfrac{1}{2}(f^-(x) + f^-(\mathfrak{c})) \geq \tfrac{1}{2}(f^-(y) + f^-(\mathfrak{c})) = g^-(y)$$

and $g^+(x) = \frac{1}{2}(f^+(x)+f^+(z)) \leq \frac{1}{2}(f^+(y)+f^+(z)) = g^+(y)$. For every $x, y \in L$, we have

$$\begin{aligned} g^-((x|y)|(x|y)) &= \tfrac{1}{2}(f^-((x|y)|(x|y)) + f^-(\mathfrak{c})) \\ &\leq \tfrac{1}{2}(\max\{f^-(x), f^-(y)\} + f^-(\mathfrak{c})) \\ &= \tfrac{1}{2}\max\{f^-(x) + f^-(\mathfrak{c}), f^-(y) + f^-(\mathfrak{c})\} \\ &= \max\{\tfrac{1}{2}(f^-(x) + f^-(\mathfrak{c})), \tfrac{1}{2}(f^-(y) + f^-(\mathfrak{c}))\} \\ &= \max\{g^-(x), g^-(y)\} \end{aligned}$$

and

$$\begin{aligned} g^+((x|y)|(x|y)) &= \tfrac{1}{2}(f^+((x|y)|(x|y)) + f^+(z)) \\ &\geq \tfrac{1}{2}(\min\{f^+(x), f^+(y)\} + f^+(z)) \\ &= \tfrac{1}{2}\min\{f^+(x) + f^+(z), f^+(y) + f^+(z)\} \\ &= \min\{\tfrac{1}{2}(f^+(x) + f^+(z)), \tfrac{1}{2}(f^+(y) + f^+(z))\} \\ &= \min\{g^+(x), g^+(y)\}. \end{aligned}$$

Hence $\mathfrak{g} := (L;\ g^-,\ g^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 3.6, and $\mathfrak{g}_* := (L;\ g_*^-,\ g_*^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 4.4. We can observe that

$$\begin{aligned} g_*^-(x) &= g^-(x) - 1 - g^-(1) \\ &= \tfrac{1}{2}(f^-(x) + f^-(\mathfrak{c})) - 1 - \tfrac{1}{2}(f^-(1) + f^-(\mathfrak{c})) \\ &= \tfrac{1}{2}(f^-(x) - 1) \leq f^-(x) \end{aligned}$$

and

$$\begin{aligned} g_*^+(x) &= g^+(x) + 1 - g^+(1) \\ &= \tfrac{1}{2}(f^+(x) + f^+(z)) + 1 - \tfrac{1}{2}(f^+(1) + f^+(z)) \\ &= \tfrac{1}{2}(f^+(x) + 1) \geq f^+(x) \end{aligned}$$

for all $x \in L$. Hence $\mathfrak{f} \Subset \mathfrak{g}_*$, and so $\mathfrak{f} := (L;\ f^-,\ f^+)$ is not a maximal element of $(\mathcal{N}_F(L), \Subset)$. This is a contradiction, and therefore $(f^-(\mathfrak{a}), f^+(x)) = (0, 0)$ for all $(\mathfrak{a}, x) \in L \times L$ with $f^-(\mathfrak{a}) \neq -1$ and $f^+(x) \neq 1$. Consequently, $\mathfrak{f} := (L;\ f^-,\ f^+)$ is described as (4.1). □

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New Opial and Polya type inequalities over a spherical shell

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Abstract

Here we present general multivariate Opial and Polya type inequalities over spherical shells. The proofs derive by the use of some estimates coming out of some new trigonometric and hyperbolic Taylor's formulae ([1], [2]) and reducing the multivariate problem to a univariate one via general polar coordinates.

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1 Background

We need

Remark 1 *Let the spherical shell*

$$A := B(0, R_2) - \overline{B(0, R_1)},$$

$0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$; $r = |x|$, $r \in [R_1, R_2]$, $\forall\ x \in \overline{A}$, $|\cdot|$ *the Euclidean norm. Here* x *can be written uniquely as* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, *see ([3], pp. 149-150 and [5], p. 421).*

Furthermore for $F : \overline{A} \to \mathbb{R}$ *a Lebesgue integrable function we have that*

$$\int_A F(x)\, dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr \right) d\omega, \tag{1}$$

where $S^{N-1} := \left\{ x \in \mathbb{R}^N : |x| = 1 \right\}$.

1

Let $d\omega$ *be the element of surface measure on* $\mathbb{S}^{N-1}$ *with surface area*

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{2}$$

Here it is volume of A*,*

$$vol\,(A) = \frac{\omega_N\left(R_2^N - R_1^N\right)}{N}. \tag{3}$$

Above it is $B(0,r) := \left\{x \in \mathbb{R}^N : |x| < r\right\}$*,* $r > 0$*.*

Here K is either $\mathbb{R}$ or $\mathbb{C}$, and $C_K^n(I)$ denotes functions n-times continuously differentiable on an interval $I \subset \mathbb{R}$ with values in K.

From [2] we need to mention the following Opial type inequalities.

Theorem 2 *Let* $f \in C_K^2(I)$*, with interval* $I \subset \mathbb{R}$*,* $a, x \in I$*,* $a < x$*, and* $f(a) = f'(a) = 0$*, with* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_a^x |f(w)|\,|f''(w) + f(w)|\,dw \leq$$

$$2^{-\frac{1}{q}}\left(\int_a^x\left(\int_a^w |\sin(w-t)|^p\,dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x |f''(w) + f(w)|^q\,dw\right)^{\frac{2}{q}}. \tag{4}$$

Theorem 3 *Let* $f \in C_K^2(I)$*,* $a, x \in I$*,* $a < x$*, and* $f(a) = f'(a) = 0$*, with* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_a^x |f(w)|\,|f''(w) - f(w)|\,dw \leq$$

$$2^{-\frac{1}{q}}\left(\int_a^x\left(\int_a^w |\sinh(w-t)|^p\,dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x |f''(w) - f(w)|^q\,dw\right)^{\frac{2}{q}}. \tag{5}$$

Theorem 4 *Let* $f \in C_K^4(I)$*, interval* $I \subset \mathbb{R}$*, let* $a, x \in I$*,* $a < x$*,* $f(a) = f'(a) = f''(a) = f'''(a) = 0$*, with* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_a^x |f(w)|\left|f^{(iv)}(w) - f(w)\right|dw \leq$$

$$2^{-\left(1+\frac{1}{q}\right)}\left(\int_a^x\left(\int_a^w |\sinh(w-t) - \sin(w-t)|^p\,dt\right)dw\right)^{\frac{1}{p}} \tag{6}$$

$$\left(\int_a^x \left|f^{(iv)}(w) - f(w)\right|^q dw\right)^{\frac{2}{q}}.$$

2

Theorem 5 *All as in Theorem 4. Let* $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. *Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} |\alpha\beta| \left|\beta^2 - \alpha^2\right|} \left(\int_a^x \left(\int_a^w |\beta \sin(\alpha(w-t)) - \alpha \sin(\beta(w-t))|^p dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{7}$$

Theorem 6 *All as in Theorem 4. Let* $\alpha \in \mathbb{R}$, $\alpha \neq 0$. *Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1} |\alpha|^3} \left(\int_a^x \left(\int_a^w |\sin(\alpha(w-t)) - \alpha(w-t)\cos(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{8}$$

Theorem 7 *All as in Theorem 5. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} |\alpha\beta| \left|\beta^2 - \alpha^2\right|} \left(\int_a^x \left(\int_a^w |\alpha \sinh(\beta(w-t)) - \beta \sinh(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{9}$$

Theorem 8 *All as in Theorem 6. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1} |\alpha|^3} \left(\int_a^x \left(\int_a^w |\alpha(w-t)\cosh(\alpha(w-t)) - \sinh(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{10}$$

We will use the above Opial type inequalities in the case of $p = q = 2$.

The motivation came from the following famous Opial's inequality

Theorem 9 *(Z. Opial, 1960, [4]) Let $c > 0$ and $y(x)$ be real, continuously differentiable on $[0,c]$, with $y(0) = y(c) = 0$. Then*

$$\int_0^c |y(x)\,y'(x)|\,dx \le \frac{c}{4}\int_0^c (y'(x))^2\,dx. \tag{11}$$

Equality holds for the function $y(x) = x$ on $\left[0,\frac{c}{2}\right]$ and $y(x) = c - x$ on $\left[\frac{c}{2},c\right]$.

2 Results

First we present a collection of Opial type inequalities on the spherical shell A.

Theorem 10 *Let $F : \overline{A} \to \mathbb{R}$ Lebesgue integrable function with $F(\cdot\omega) \in C^2([R_1,R_2])$, with $F(R_1\omega) = \frac{\partial F}{\partial r}(R_1\omega) = 0$, $\forall\ \omega \in S^{N-1}$. Then*

$$\int_A |F(x)|\left|F(x) + \frac{\partial^2 F(x)}{\partial r^2}\right| dx \le$$

$$2^{-\frac{1}{2}}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}}\int_A\left(F(x) + \frac{\partial^2 F(x)}{\partial r^2}\right)^2 dx. \tag{12}$$

Proof. Here we apply Theorem 2 to $F(\cdot\omega)$ for $p = q = 2$. So for every $\omega \in S^{N-1}$ we have that

$$\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega)\right| dr \le$$

$$2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega)\right)^2 dr\right). \tag{13}$$

We have $R_1 \le r \le R_2$ and $R_1^{N-1} \le r^{N-1} \le R_2^{N-1}$, and $R_2^{1-N} \le r^{1-N} \le R_1^{1-N}$.

We observe the following

$$R_2^{1-N}\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega)\right| r^{N-1}dr \le \tag{14}$$

$$\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega)\right| r^{N-1}r^{1-N}dr =$$

$$\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega)\right| dr \overset{(13)}{\le}$$

$$2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}}$$
$$\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}r^{1-N}dr\right)\leq$$
$$R_1^{1-N}2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}} \tag{15}$$
$$\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}dr\right).$$

Therefore it holds

$$\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right|r^{N-1}dr\leq$$
$$\left(\frac{R_1}{R_2}\right)^{1-N}2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}} \tag{16}$$
$$\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}dr\right).$$

Consequently we obtain

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right|r^{N-1}dr\right)d\omega\leq$$
$$2^{-\frac{1}{2}}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}} \tag{17}$$
$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}dr\right)d\omega.$$

Applying (1) we obtain (12). ■

Next, we present more Opial type inequalities on spherical shell. Their proofs are similar to the proof of Theorem 10 and are based on Theorems 3-8. Use also of (1).

Theorem 11 *Same assumptions as in Theorem 10. Then*

$$\int_A |F(x)|\left|F(x)-\frac{\partial^2 F(x)}{\partial r^2}\right|dx\leq$$
$$2^{-\frac{1}{2}}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sinh(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}}\int_A\left(F(x)-\frac{\partial^2 F(x)}{\partial r^2}\right)^2 dx. \tag{18}$$

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Proof. Based on (5). ■

Theorem 12 *Let $F : \overline{A} \to \mathbb{R}$ Lebesgue integrable function with $F(\cdot\omega) \in C^4([R_1, R_2])$, with $F(R_1\omega) = \frac{\partial^{(i)} F}{\partial r^{(i)}}(R_1\omega) = 0$, $i = 1, 2, 3$; $\forall\ \omega \in S^{N-1}$. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} - F(x) \right| dx \leq$$

$$2^{-\frac{3}{2}} \left(\frac{R_2}{R_1}\right)^{N-1} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\sinh(r-t) - \sin(r-t))^2 \, dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} - F(x) \right)^2 dx. \tag{19}$$

Proof. Based on (6). ■

Theorem 13 *All as in Theorem 12. Let $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} + \left(\alpha^2 + \beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x) \right| dx \leq$$

$$\frac{1}{\sqrt{2}\left|\alpha\beta\left(\beta^2 - \alpha^2\right)\right|} \left(\frac{R_2}{R_1}\right)^{N-1}$$

$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\beta \sin(\alpha(r-t)) - \alpha \sin(\beta(r-t)))^2 \, dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} + \left(\alpha^2 + \beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x) \right)^2 dx. \tag{20}$$

Proof. Based on (7). ■

Theorem 14 *All as in Theorem 12. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} + 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right| dx \leq$$

$$\frac{1}{2^{\frac{3}{2}}\left|\alpha^3\right|} \left(\frac{R_2}{R_1}\right)^{N-1} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\sin(\alpha(r-t)) - \alpha(r-t)\cos(\alpha(r-t)))^2 \, dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} + 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right)^2 dx. \tag{21}$$

Proof. Based on (8). ■

Theorem 15 *All as in Theorem 13. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} - \left(\alpha^2+\beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x) \right| dx \le$$

$$\frac{1}{\sqrt{2}\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|} \left(\frac{R_2}{R_1}\right)^{N-1}$$

$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\alpha \sinh\left(\beta\left(r-t\right)\right) - \beta \sinh\left(\alpha\left(r-t\right)\right)\right)^2 dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} - \left(\alpha^2+\beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x) \right)^2 dx. \tag{22}$$

Proof. By (9). ■

Finally we give the following Opial type inequality.

Theorem 16 *All as in Theorem 14. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} - 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right| dx \le$$

$$\frac{1}{2^{\frac{3}{2}}|\alpha|^3} \left(\frac{R_2}{R_1}\right)^{N-1} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\alpha\left(r-t\right)\cosh\left(\alpha\left(r-t\right)\right) - \sinh\left(\alpha\left(r-t\right)\right)\right)^2 dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} - 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right)^2 dx. \tag{23}$$

Proof. Based on (10). ■

We need the following results.

Theorem 17 *([1]) For $f \in C_K^2([a,b])$ and $x \in [a,b] : f(a) = f'(a) = 0$, we have that*

$$f(x) = \int_a^x \left(f''(t) + f(t)\right) \sin(x-t)\, dt, \tag{24}$$

and

$$f(x) = \int_a^x \left(f''(t) - f(t)\right) \sinh(x-t)\, dt. \tag{25}$$

Theorem 18 *([1]) For $f \in C_K^4([a,b])$ and $x \in [a,b] : f(a) = f'(a) = f''(a) = f'''(a) = 0$, we have that*

$$f(x) = \int_a^x \left(f''''(t) - f(t)\right) \left(\frac{\sinh(x-t) - \sin(x-t)}{2} \right) dt. \tag{26}$$

Theorem 19 *([1]) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\beta^2-\alpha^2\right) \neq 0$, and $f \in C_K^4([a,b])$, $x \in [a,b] : f(a) = f'(a) = f''(a) = f'''(a) = 0$. Then*

$$f(x) = \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left(f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right)$$

$$(\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t)))\, dt. \tag{27}$$

Theorem 20 *([1]) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$, and $f \in C_K^4([a,b])$, $x \in [a,b] : f(a) = f'(a) = f''(a) = f'''(a) = 0$. Then*

$$f(x) = \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left(f''''(t) - \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right)$$

$$(\alpha \sinh(\beta(x-t)) - \beta \sinh(\alpha(x-t)) -)\, dt. \tag{28}$$

We will use

$$|\sin x| \leq |x|, \quad \forall\, x \in \mathbb{R}, \tag{29}$$

$$|\sinh x| \leq \cosh(b-a)\,|x|, \quad \forall\, x \in [-(b-a), b-a]. \tag{30}$$

Both of the above come by applications of mean value theorem.

We give the following Polya type univariate inequalities.

Theorem 21 *For $f \in C_K^2([a,b])$ and $x \in [a,b] : f(a) = f'(a) = 0$, it holds*

$$\int_a^b |f(x)|\, dx \leq \frac{(b-a)^2}{2} \int_a^b |f''(t) + f(t)|\, dt, \tag{31}$$

and

$$\int_a^b |f(x)|\, dx \leq \cosh(b-a) \frac{(b-a)^2}{2} \int_a^b |f''(t) - f(t)|\, dt. \tag{32}$$

Proof. (i) By (24) we have that

$$\begin{gathered} |f(x)| \leq \int_a^x |f''(t) + f(t)|\, |\sin(x-t)|\, dt \leq \\ \int_a^x |f''(t) + f(t)|\, (x-t)\, dt \leq \\ (x-a) \int_a^x |f''(t) + f(t)|\, dt \leq (x-a) \int_a^b |f''(t) + f(t)|\, dt. \end{gathered} \tag{33}$$

Therefore, it holds

$$\int_a^b |f(x)|\, dx \leq \left(\int_a^b (x-a)\, dx\right) \int_a^b |f''(t) + f(t)|\, dt = \tag{34}$$

$$\frac{(b-a)^2}{2}\int_a^b |f''(t)+f(t)|\,dt.$$

(ii) By (25) we have that

$$|f(x)| \leq \int_a^x |f''(t)-f(t)|\,|\sinh(x-t)|\,dt \leq$$

$$\cosh(b-a)\int_a^x |f''(t)-f(t)|\,(x-t)\,dt \leq \tag{35}$$

$$\cosh(b-a)\,(x-a)\int_a^x |f''(t)-f(t)|\,dt \leq$$

$$\cosh(b-a)\,(x-a)\int_a^b |f''(t)-f(t)|\,dt.$$

Therefore, we get

$$\int_a^b |f(x)|\,dx \leq \cosh(b-a)\left(\int_a^b (x-a)\,dx\right)\int_a^b |f''(t)-f(t)|\,dt = \tag{36}$$

$$\cosh(b-a)\,\frac{(b-a)^2}{2}\int_a^b |f''(t)-f(t)|\,dt.$$

■

Theorem 22 *All as in Theorem 18. Then*

$$\int_a^b |f(x)|\,dx \leq (\cosh(b-a)+1)\,\frac{(b-a)^2}{4}\int_a^b |f''''(t)-f(t)|\,dt. \tag{37}$$

Theorem 23 *All as in Theorem 19. Then*

$$\int_a^b |f(x)|\,dx \leq \frac{(b-a)^2}{|\beta^2-\alpha^2|}\int_a^b \left|f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2 f(t)\right|dt. \tag{38}$$

Theorem 24 *All as in Theorem 20, plus* $|\alpha|, |\beta| < 1$. *Then*

$$\int_a^b |f(x)|\,dx \leq \frac{\cosh(b-a)\,(b-a)^2}{|\beta^2-\alpha^2|}\int_a^b \left|f''''(t)-\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2 f(t)\right|dt. \tag{39}$$

Next comes a collection of Polya type inequalities on the spherical shell. Their proofs are based on Theorems 21-24, (1) and they are similar to the proof of Theorem 10, and as such details are omitted.

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Theorem 25 *Same assumptions as in Theorem 10. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \frac{(R_2-R_1)^2}{2} \int_A \left|\frac{\partial^2 F(x)}{\partial r^2} + F(x)\right| dx, \tag{40}$$

and

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \cosh(R_2-R_1) \frac{(R_2-R_1)^2}{2} \int_A \left|\frac{\partial^2 F(x)}{\partial r^2} - F(x)\right| dx. \tag{41}$$

Proof. Based on Theorem 21. ■

Theorem 26 *Same assumptions as in Theorem 12. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} (\cosh(R_2-R_1)+1) \frac{(R_2-R_1)^2}{4} \int_A \left|\frac{\partial^4 F(x)}{\partial r^4} - F(x)\right| dx. \tag{42}$$

Proof. Based on Theorem 22. ■

Theorem 27 *All as in Theorem 13. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \frac{(R_2-R_1)^2}{|\beta^2-\alpha^2|} \int_A \left|\frac{\partial^4 F(x)}{\partial r^4} + (\alpha^2+\beta^2) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x)\right| dx. \tag{43}$$

Proof. Based on Theorem 23. ■

We finish with

Theorem 28 *All as in Theorem 13, plus $|\alpha|, |\beta| < 1$. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \frac{\cosh(R_2-R_1)(R_2-R_1)^2}{|\beta^2-\alpha^2|} \int_A \left|\frac{\partial^4 F(x)}{\partial r^4} - (\alpha^2+\beta^2) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x)\right| dx. \tag{44}$$

Proof. Based on Theorem 24. ■

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Stable Attractors on a Certain Two-dimensional Piecewise Linear Map

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Abstract

In this article we study the behaviors of a piecewise linear map with initial condition in the second quadrant. There is a unique equilibrium point and two 4-cycles of the map. We found regions of initial condition that solutions become equilibrium point or 4-cycles. We divided the second quadrant into sub-regions and identify behaviors of solutions in each sub-region by direct calculations, and formulated inductive statements to explain the behaviors of the map without using stability theorems.

Key words: Coexisting attractors, Periodic solution, Equilibrium point, Piecewise linear map.
2010 Mathematics Subject Classification: 39A10 and 65Q10.

1 Introduction

Lozi map (Lozi, 1978) is a well known two dimensional piecewise linear map which is a simplified version of Hénon map and has a strange attractor. There are many applications of piecewise linear maps in models such as power electronic converters and switching circuits (Banerjee & Verghese, 2001; Zhusu baliyev &Mosekilde, 2003). We know that multistability (Simpson, 2010; Zhusubaliyev et al., 2008) can be found in piecewise linear map. Bifurcations sequence in a family of piecewise linear maps were cosidered in articles (Gardini & Tikjha, 2019; Tikjha & Gardini, 2020) and also a transition between invertibility and non-invertibility of piecewise linear map were studied in article (Gardini & Tikjha, 2020). A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a map is called *eventually periodic with prime period-p (or minimal period-p)* if there exists an integer $N > 0$

and a smallest positive integer p such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period p; that is,

$$(x_{n+p}, y_{n+p}) = (x_n, y_n) \text{ for all } n \geq N.$$

As we all known that piecewise linear function is not differentiable. In the case of system that can reduce to equation (one-dimensional map), we are unable to verify stability via stability theorem such as Schwazian derivative (D. Singer, 1978). An open problem about a piecewise linear map was mentioned in (Grove et al.,2012):

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases} \tag{1}$$

with initial condition $(x_0, y_0) \in \mathbb{R}^2$. Many papers studied the open problem for example: Gove et al. (2012) found that every solution is eventually prime period-3 solutions except for the unique equilibrium solution. In article (Tikjha et al., 2010; 2015; 2017) and (Tikjha & Lapiere, 2020), they studied some special cases of system (1), and showed that there are periodic attractors. They showed that every solution is eventually either periodic attractors or equilibrium point by using direct calculation and inductive statement. Recently in article (Aiewcharoen et al. 2021; Laoharenoo et al. 2023), they investigated a family of systems that contain absolute value similar to (1) and they showed that all solutions become the equilibrium point. Moreover, they also showed that there exist a prime period 5 when $b \leq -6$. In article (Lapiere & Tikjha, 2021), they also studied the special case of (1) with $a = b = d = -1$ and $c = 1$. Our goal is to continue investigate the special case of (1) with $a = c = -1$, $b = -3$ and $d = 1$ which Tikjha and Piasu (Tikjha & Piasu, 2020) reported the condition of solutions becoming equilibrium point or periodic with prime period 4. They investigated initial point only in region of the first quadrant. We aim to extend the initial condition in second quadrant and find all possible behaviors of solutions for this map and then characterize the coexisting attractors between equilibrium point and periodic with prime period 4 (4-cycle) and their basin of attractions.

2 Main Results

In this section we will study the following two dimensional map:

$$x_{n+1} = |x_n| - y_n - 3, y_{n+1} = x_n - |y_n| + 1 \tag{2}$$

with initial condition belonging to second quadrant. This map has the unique equilibrium point $(-1, -1)$ that can be computed by solving the system:

$$\begin{cases} \bar{x} = |\bar{x}| - \bar{y} - 3 \\ \bar{y} = \bar{x} - |\bar{y}| + 1 \end{cases}.$$

As in (Tikjha & Piasu, 2020), there are 4-cycles of the system (2) given by $P_{4.1} = \{((-5, -1), (3, -5), (5, -1), (3, 5))\}$ and $P_{4.2} = \{((1, -3), (1, -1), (-1, 1), (-3, -1))\}$.

The 4-cycles are found by numerical calculation. It is easy to verify that $P_{4.1}$ and $P_{4.2}$ are 4-cycles. Let (x_0, y_0) be in the second quadrant of xy plane, $Q_2 := \{(x, y) \in \mathbb{R}^2 | x < 0 \text{ and } y > 0\}$. We have the first iteration as the following:

$$\begin{cases} x_1 = |x_0| - y_0 - 3 & = -x_0 - y_0 - 3 \\ y_1 = x_0 - |y_0| + 1 & = x_0 - y_0 + 1 \end{cases} \tag{3}$$

Before we calculate the next iteration, we have to know the sign (negative or non-negative) of x_1 and y_1 which are the function of x_0 and y_0. The sign of x_1 will change when initial point (x_0, y_0) above or below the line $f(x) = -x - 3$ (resp. $g(x) = x + 1$ for y_1). Now we divide the second quadrant into three sub-regions as Fig. 1 that we will investigate in the next sub-section.

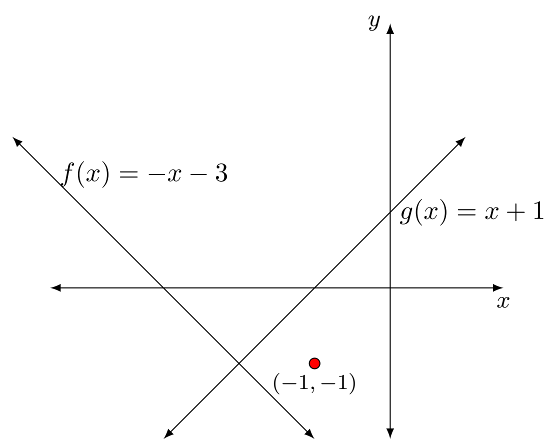

Figure 1: The second quadrant is separated into three sub-regions by the lines $f(x)$ and $g(x)$. The red point is the equilibrium point of system (2).

2.1 Stable equilibrium point

In this section we will investigate rightmost region of second quadrant that is when initial condition belonging to the green region as Fig.2 From (3), we have

$$\begin{cases} x_2 = 2y_0 - 1 \\ y_2 = -2x_0 - 3 < 0. \end{cases} \tag{4}$$

Firstly, we will investigate when $x_2 \geq 0$ that is initial condition $\frac{1}{2} \leq y_0 \leq 1$ as in Fig.3. So the next iteration can be written in the form:

$$\begin{cases} x_3 = 2x_0 + 2y_0 - 1 \\ y_3 = -2x_0 + 2y_0 - 3 < 0. \end{cases} \tag{5}$$

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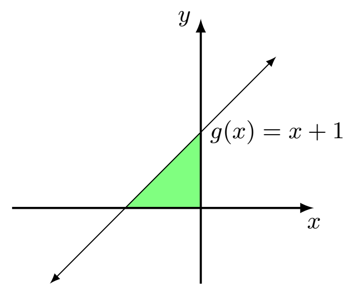

Figure 2: The region of initial points such that x_1 is negative and y_1 is positive.

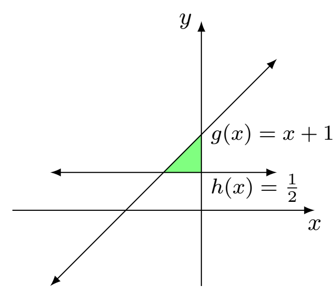

Figure 3: The region of initial points such that x_2 is non-negative.

Again we separate region in Fig.3 into two parts above and below a line $i(x) = -x + \frac{1}{2}$ as in Fig.4. For an initial condition above the line $i(x) = -x + \frac{1}{2}$, the forth iteration is in the form:

$$\begin{cases} x_4 = 4x_0 - 1 < 0 \\ y_4 = 4y_0 - 3. \end{cases} \tag{6}$$

If initial conditions are in green region in Fig.4 with above line $i(x)$ and $y_0 \in \left[\frac{1}{2}, \frac{3}{4}\right]$, we have $y_4 \leq 0$. By direct calculations we have: $\begin{cases} x_5 = -4x_0 - 4y_0 + 1 < 0 \\ y_5 = 4x_0 + 4y_0 - 3 < 0 \end{cases}$, and $\begin{cases} x_6 = -1 \\ y_6 = -1 \end{cases}$. The solution of this region is eventually equilibrium point within sixth iteration. For initial conditions are in green region in Fig. 4 with above line $i(x)$ and $y_0 \in \left(\frac{3}{4}, 1\right]$, we have the following closed form of solution: $\begin{cases} x_5 = -4x_0 - 4y_0 + 1 < 0 \\ y_5 = 4x_0 - 4y_0 + 3 < 0 \end{cases}$, and so

$$\begin{cases} x_6 = 8y_0 - 7 \\ y_6 = -8y_0 + 5 < 0 \end{cases}. \tag{7}$$

Note that the closed form of the sixth iteration with this region is independent from x_0. It is easy to verify that when $y_0 \in \left(\frac{3}{4}, \frac{7}{8}\right]$, $x_6 \leq 0$ and so $x_7 = y_7 = -1$.

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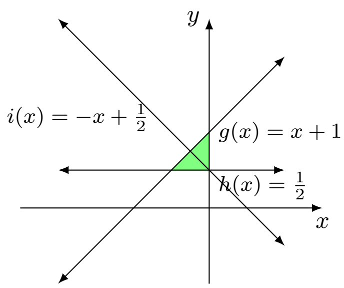

Figure 4: The third iteration of (5) x_3 change sign when initial point (x_0, y_0) crosses the line $i(x)$.

This means that the solution of this region is also eventually equilibrium point within seventh iteration. The remain region is when $y_0 \in \left(\frac{7}{8}, 1\right]$ which we have $x_6 > 0$. The following inductive statement will be used to prove that every solution is eventually equilibrium point for this remain region. Let $a_n = \frac{2^{2n+1}-1}{2^{2n+1}}, b_n = \frac{2^{2n+2}-1}{2^{2n+2}}, \delta_n = 2^{2n+2} - 1$ and $P(n)$ be the following statement : "$y_0 \in (a_n, 1]$, then

$$\begin{cases} x_{4n+3} = 2^{2n+2} y_0 - \delta_n \\ y_{4n+3} = -1 \end{cases}.$$

If $y_0 \in (a_n, b_n]$ then $x_{4n+3} \leq 0$ and so

$$\begin{cases} x_{4n+4} = -2^{2n+2} y_0 + \delta_n - 2 < 0 \\ y_{4n+4} = 2^{2n+2} y_0 - \delta_n \leq 0 \end{cases}, \begin{cases} x_{4n+5} = -1 \\ y_{4n+5} = -1 \end{cases}.$$

If $y_0 \in (b_n, 1]$ then $x_{4n+3} > 0$ and so

$$\begin{cases} x_{4n+4} = 2^{2n+2} y_0 - \delta_n - 2 < 0 \\ y_{4n+4} = 2^{2n+2} y_0 - \delta_n > 0 \end{cases}, \begin{cases} x_{4n+5} = -2^{2n+3} y_0 + 2\delta_n - 1 < 0 \\ y_{4n+5} = -1 \end{cases},$$

$$\begin{cases} x_{4n+6} = 2^{2n+3} y_0 - 2\delta_n - 1 \\ y_{4n+6} = -2^{2n+3} y_0 + 2\delta_n - 1 < 0 \end{cases}.$$

If $y_0 \in (b_n, a_{n+1}]$ then $x_{4n+6} \leq 0$, and so

$$\begin{cases} x_{4n+7} = -1 \\ y_{4n+7} = -1 \end{cases}.$$

If $y_0 \in (a_{n+1}, 1]$ then $x_{4n+6} > 0$."

We shall show that $P(1)$ is true. For $y_0 \in (a_1, 1] = \left(\frac{7}{8}, 1\right]$ and $\delta_1 = 15$, we have $x_6 = 8y_0 - 7 > 0, y_6 = -8y_0 + 5 < 0$ and so

$$\begin{cases} x_{4(1)+3} = x_7 = 16y_0 - 15 = 2^{2(1)+2} y_0 - \delta_1 \\ y_{4(1)+3} = y_7 = -1 \end{cases}.$$

If $y_0 \in (a_1, b_1] = (\frac{7}{8}, \frac{15}{16}]$ then $x_7 \leq 0$ and so

$$\begin{cases} x_{4(1)+4} = x_8 = -16y_0 + 13 = -2^{2(1)+2} y_0 + \delta_1 - 2 < 0 \\ y_{4(1)+4} = y_8 = 16y_0 - 15 = 2^{2(1)+2} y_0 - \delta_1 \leq 0 \end{cases},$$

$$\begin{cases} x_{4(1)+5} = x_9 = -1 \\ y_{4(1)+5} = y_9 = -1 \end{cases}.$$

If $y_0 \in (b_1, 1] = \left(\frac{15}{16}, 1\right]$ then $x_7 > 0$ and so

$$\begin{cases} x_{4(1)+4} = x_8 = 16y_0 - 17 = 2^{2(1)+2}y_0 - \delta_1 - 2 < 0 \\ y_{4(1)+4} = y_8 = 16y_0 - 15 = 2^{2(1)+2}y_0 - \delta_1 > 0 \end{cases},$$
$$\begin{cases} x_{4(1)+5} = x_9 = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\delta_1 - 1 < 0 \\ y_{4(1)+5} = y_9 = -1 \end{cases},$$
$$\begin{cases} x_{4(1)+6} = x_{10} = 32y_0 - 31 = 2^{2(1)+3}y_0 - 2\delta_1 - 1 \\ y_{4(1)+6} = y_{10} = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\delta_1 - 1 < 0 \end{cases}.$$

If $y_0 \in (b_1, a_2] = (\frac{15}{16}, \frac{31}{32}]$ then $x_{10} \leq 0$ and so
$$\begin{cases} x_{4(1)+7} = x_{11} = -1 \\ y_{4(1)+7} = y_{11} = -1 \end{cases}.$$
If $y_0 \in (a_2, 1] = (\frac{31}{32}, 1]$ then $x_{4(1)+6} = x_{10} = 32y_0 - 31 > 0$. Therefore $P(1)$ is true. It means that for this region and initial condition $y \in \left(\frac{7}{8}, \frac{31}{32}\right]$, the solution is eventually equilibrium point $(-1,-1)$. Next Suppose $P(k)$ is true. We shall show that $P(k+1)$ is true. For $y_0 \in (a_{k+1}, 1] = \left(\frac{2^{2k+3}-1}{2^{2k+3}}, 1\right]$, then
$$\begin{cases} x_{4k+6} = 2^{2k+3}y_0 - 2\delta_k - 1 > 0 \\ y_{4k+6} = -2^{2k+3}y_0 + 2\delta_k - 1 < 0 \end{cases}. \text{ Then}$$
$$\begin{cases} x_{4(k+1)+3} = 2^{2(k+1)+2}y_0 - (2^{2(k+1)+2} - 1) = 2^{2(k+1)+2}y_0 - \delta_{k+1} \\ y_{4(k+1)+3} = -1 \end{cases}.$$
If $y_0 \in (a_{k+1}, b_{k+1}] = (\frac{2^{2k+3}-1}{2^{2k+3}}, \frac{2^{2k+4}-1}{2^{2k+4}}]$ then $x_{4k+7} = x_{4(k+1)+3} \leq 0$ (by substituting boundary of y_0) and so
$$\begin{cases} x_{4(k+1)+4} = -2^{2(k+1)+2}y_0 + \delta_{k+1} - 2 < 0 \\ y_{4(k+1)+4} = 2^{2(k+1)+2}y_0 - \delta_{k+1} \leq 0 \end{cases},$$
$$\begin{cases} x_{4(k+1)+5} = -1 \\ y_{4(k+1)+5} = -1 \end{cases}.$$
If $y_0 \in (b_{k+1}, 1] = \left(\frac{2^{2k+4}-1}{2^{2k+4}}, 1\right]$ then $x_{4k+7} = x_{4(k+1)+3} > 0$ and so
$$\begin{cases} x_{4(k+1)+4} = 2^{2(k+1)+2}y_0 - \delta_{k+1} - 2 < 0 \\ y_{4(k+1)+4} = 2^{2(k+1)+2}y_0 - \delta_{k+1} > 0 \end{cases},$$
$$\begin{cases} x_{4(k+1)+5} = -2^{2(k+1)+3}y_0 + 2\delta_{k+1} - 1 < 0 \\ y_{4(k+1)+5} = -1 \end{cases},$$
$$\begin{cases} x_{4(k+1)+6} = 2^{2(k+1)+3}y_0 - 2\delta_{k+1} - 1 \\ y_{4(k+1)+6} = -2^{2(k+1)+3}y_0 + 2\delta_{k+1} - 1 < 0 \end{cases}.$$
If $y_0 \in (b_{k+1}, a_{k+2}] = (\frac{2^{2k+4}-1}{2^{2k+4}}, \frac{2^{2k+5}-1}{2^{2k+5}}]$ then $x_{4(k+1)+6} \leq 0$ and so $x_{4(k+1)+7} = -1$ and $y_{4(k+1)+7} = -1$.
If $y_0 \in (a_{k+2}, 1] = \left(\frac{2^{2k+5}-1}{2^{2k+5}}, 1\right]$ then $x_{4(k+1)+6} = x_{4k+10} = 2^{2(k+1)+3}y_0 - 2\delta_{k+1} - 1 > 0$. Hence $P(k+1)$ is also true. By mathematical induction $P(n)$ is true for any positive integer n. From the inductive statement we have that every solution with initial condition y_0 between a_n and b_n is eventually equilibrium point. It is easy to see that the limits of sequences a_n and b_n are equal to 1. Therefore we can confirm that with initial condition, the green region in Fig. 4 with above line $i(x)$, the solution is eventually equilibrium point.

For an initial condition below or in the line $i(x) = -x + \frac{1}{2}$, the initial condition satisfy $x_0 \leq -y_0 + \frac{1}{2}$ then $x_3 = 2x_0 + 2y_0 - 1 \leq 0$. We have the forth iteration as $x_4 = -4x_0 + 1 < 0$ and $y_4 = 4y_0 - 3$. In this green region below

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$i(x)$ of Fig. 4, y_0 is at most $\frac{3}{4}$. Then $y_4 = 4y_0 - 3 < 0$ and so $x_5 = y_5 = -1$. So we proved the following lemma.

Proposition 2.1. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of the map* (2) *and initial condition $(x_0, y_0) \in \{(x, y) \in Q_2 | y \leq x + 1$ and $\frac{1}{2} \leq y \leq 1\}$. Then every solution is eventually equilibrium point.*

Now we consider the below part of the Fig. 3, which (x_0, y_0) satisfies the following conditions: $x_1 = -x_0 - y_0 - 3 < 0, y_1 = x_0 - y_0 + 1 \geq 0$ and $x_0 < 0, y_0 > 0$. We have $x_2 = 2y_0 - 1$ and $y_2 = -2x_0 - 3$. In this case we consider when $0 < y_0 < \frac{1}{2}$. So $x_2 < 0$ and (x_0, y_0) belong to green portion of Fig. 5. The

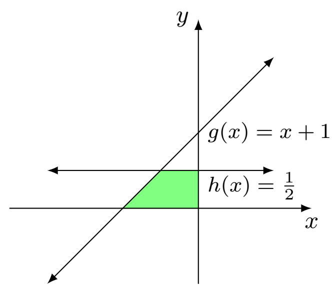

Figure 5: The green region of initial points such that x_2 is negative.

next iteration can be written in the form:

$$\begin{cases} x_3 = 2x_0 - 2y_0 + 1 \\ y_3 = -2x_0 + 2y_0 - 3 < 0. \end{cases} \tag{8}$$

We separate x_3 into two cases, above and below line $k(x) = x + \frac{1}{2}$ as in Fig. 6, when (x_0, y_0) is above $k(x)$ then $x_3 < 0$ while it is positive when (x_0, y_0) below $k(x)$.

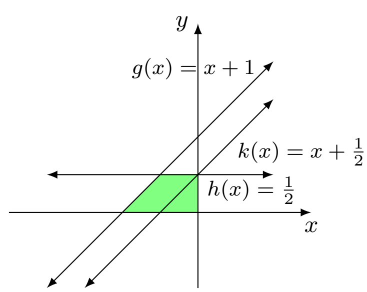

Figure 6: The x_3 of (8) change sign when initial point (x_0, y_0) crosses the line $k(x)$.

In the case of $x_3 \leq 0$ (above $k(x)$), we immediately have $x_4 = y_4 = -1$. For the case of $x_3 > 0$, we have

$$\begin{cases} x_4 = 4x_0 - 4y_0 + 1 \\ y_4 = -1 \end{cases}.$$

For $x_4 = 4x_0 - 4y_0 + 1 \leq 0$, we have

$$\begin{cases} x_5 = -4x_0 + 4y_0 - 3 < 0 \\ y_5 = 4x_0 - 4y_0 + 1 \leq 0 \end{cases}$$

and so $x_6 = y_6 = -1$. In the case of $x_4 = 4x_0 - 4y_0 + 1 > 0$, that is in remain region of initial condition in $\Delta = \{(x_0, y_0) \in Q_2 | 4x_0 - 4y_0 + 1 > 0\}$ as Fig.7. We will use an inductive statement to verify that the remain solution is even-

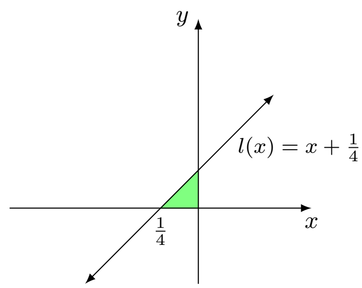

Figure 7: The green region is the initial points belonging to Δ.

tually equilibrium point. Let $\Delta_n = \{(x, y) \in Q_2 \mid 2^{2n}x - 2^{2n}y + 1 > 0\}, D_n = \{(x, y) \in Q_2 \mid 2^{2n+1}x - 2^{2n+1}y + 1 > 0\}$ and $\mathcal{Q}(n)$ be the following statement: "$(x_0, y_0) \in \Delta_n$ then

$$\begin{cases} x_{4n+1} = 2^{2n}x_0 - 2^{2n}y_0 - 1 < 0 \\ y_{4n+1} = 2^{2n}x_0 - 2^{2n}y_0 + 1 > 0 \end{cases}, \quad \begin{cases} x_{4n+2} = -2^{2n+1}x_0 + 2^{2n+1}y_0 - 3 < 0 \\ y_{4n+2} = -1 \end{cases},$$

$$\begin{cases} x_{4n+3} = 2^{2n+1}x_0 - 2^{2n+1}y_0 + 1 \\ y_{4n+3} = -2^{2n+1}x_0 + 2^{2n+1}y_0 - 3 < 0 \end{cases}.$$

If $(x_0, y_0) \in \Delta_n - D_n$ then $x_{4n+3} \leq 0$ and so $x_{4n+4} = y_{4n+4} = -1$.
If $(x_0, y_0) \in D_n$ then $x_{4n+3} > 0$ and so

$$\begin{cases} x_{4n+4} = 2^{2n+2}x_0 - 2^{2n+2}y_0 + 1 \\ y_{4n+4} = -1 \end{cases}.$$

If $(x_0, y_0) \in D_n - \Delta_{n+1}$ then $x_{4n+4} \leq 0$

$$\begin{cases} x_{4n+5} = -2^{2n+2}x_0 + 2^{2n+2}y_0 - 3 < 0 \\ y_{4n+5} = 2^{2n+2}x_0 - 2^{2n+2}y_0 + 1 \leq 0 \end{cases}, \text{ and so } x_{4n+6} = y_{4n+6} = -1.$$

If $(x_0, y_0) \in \Delta_{n+1}$ then $x_{4n+4} > 0$." We shall show that $\mathcal{Q}(1)$ is true. For $(x_0, y_0) \in \Delta_1 = \{(x, y) \in Q_2 \mid 4x - 4y + 1 > 0\}$. We have

$$\begin{cases} x_{4(1)+1} = 4x_0 - 4y_0 - 1 = 2^{2(1)}x_0 - 2^{2(1)}y_0 - 1 < 0 \\ y_{4(1)+1} = 4x_0 - 4y_0 + 1 = 2^{2(1)}x_0 - 2^{2(1)}y_0 + 1 > 0 \end{cases},$$

$$\begin{cases} x_{4(1)+2} = -8x_0 + 8y_0 - 3 = -2^{2(1)+1}x_0 + 2^{2(1)+1}y_0 - 3 < 0 \\ y_{4(1)+2} = -1 \end{cases},$$

$$\begin{cases} x_{4(1)+3} = 8x_0 - 8y_0 + 1 = 2^{2(1)+1}x_0 - 2^{2(1)+1}y_0 + 1 \\ y_{4(1)+3} = -8x_0 + 8y_0 - 3 = -2^{2(1)+1}x_0 + 2^{2(1)+1}y_0 - 3 < 0 \end{cases}.$$

If $(x_0, y_0) \in \Delta_1 - D_1 = \{(x,y) \in Q_2 \mid 0 < 4x - 4y + 1 \text{ and } 8x - 8y + 1 \leq 0\}$ then $x_7 \leq 0$ and so $x_{4(1)+4} = y_{4(1)+4} = -1$.
If $(x_0, y_0) \in D_1 = \{(x,y) \in Q_2 \mid 8x - 8y + 1 > 0\}$ then $x_7 > 0$ ans so
$$\begin{cases} x_{4(1)+4} = 16x_0 - 16y_0 + 1 = 2^{2(1)+2}x_0 - 2^{2(1)+2}y_0 + 1 \\ y_{4(1)+4} = -1 \end{cases}.$$
If $(x_0, y_0) \in D_1 - \Delta_2 = \{(x,y) \in Q_2 \mid 0 < 8x - 8y + 1 \text{ and } 16x - 16y + 1 \leq 0\}$ then $x_8 = 16x_0 - 16y_0 + 1 \leq 0$. Then
$$\begin{cases} x_{4(1)+5} = -16x_0 + 16y_0 - 3 = -2^{2(1)+2}x_0 + 2^{2(1)+2}y_0 - 3 < 0 \\ y_{4(1)+5} = 16x_0 - 16y_0 + 1 = 2^{2(1)+2}x_0 - 2^{2(1)+2}y_0 + 1 \leq 0 \end{cases},$$ and so $x_{4(1)+6} = y_{4(1)+6} = -1$.
If $(x_0, y_0) \in \Delta_2 = \{(x_0,y_0) \in Q_2 \mid 16x - 16y + 1 > 0\}$ then $x_{4(1)+4} > 0$. Hence $\mathcal{Q}(1)$ is true. Suppose $\mathcal{Q}(k)$ is true. Next, we show that $\mathcal{Q}(k+1)$ is true. Since $\mathcal{Q}(k)$ is true, we have $x_{4k+4} = 2^{2k+2}x_0 - 2^{2k+2}y_0 + 1 > 0$, and $y_{4k+4} = -1$ when $(x_0, y_0) \in \Delta_{k+1} = \{(x,y) \in Q_2 \mid 2^{2k+2}x - 2^{2k+2}y + 1 > 0\}$ and so
$$\begin{cases} x_{4(k+1)+1} = 2^{2(k+1)}x_0 - 2^{2(k+1)}y_0 - 1 < 0 \\ y_{4(k+1)+1} = 2^{2k+1}x_0 - 2^{2k+1}y_0 + 1 > 0 \end{cases},$$
$$\begin{cases} x_{4(k+1)+2} = -2^{2k+1+1}x_0 + 2^{2k+1+1}y_0 - 3 < 0 \\ x_{4(k+1)+2} = -1 \end{cases},$$
$$\begin{cases} x_{4(k+1)+3} = 2^{2(k+1)+1}x_0 - 2^{2(k+1)+1}y_0 + 1 \\ y_{4(k+1)+3} = -2^{2(k+1)+1}x_0 + 2^{2(k+1)+1}y_0 - 3 < 0 \end{cases}.$$
If $(x_0, y_0) \in \Delta_{k+1} - D_{k+1} = \{(x,y) \in Q_2 \mid 0 < -2^{2k+2}x + 2^{2k+2}y + 1 \text{ and } 2^{2(k+1)+1}x - 2^{2(k+1)+1}y + 1 \leq 0\}$ then $x_{4(k+1)+3} \leq 0$ and so
$$\begin{cases} x_{4(k+1)+4} = -1 \\ y_{4(k+1)+4} = -1 \end{cases}.$$
If $(x_0, y_0) \in D_{k+1} = \{(x,y) \in Q_2 \mid 2^{2(k+1)+1}x - 2^{2(k+1)+1}y + 1 > 0\}$ then $x_{4(k+1)+3} > 0$ and so
$$\begin{cases} x_{4(k+1)+4} = 2^{2(k+1)+2}x_0 - 2^{2(k+1)+2}y_0 + 1 \\ y_{4(k+1)+4} = -1 \end{cases}.$$
If $(x_0, y_0) \in D_{k+1} - \Delta_{k+2} = \{(x,y) \in Q_2 \mid 0 < 2^{2(k+1)+1}x - 2^{2(k+1)+1}y + 1 \text{ and } 2^{2(k+1)+2}x_0 - 2^{2(k+1)+2}y_0 + 1 \leq 0\}$ then $x_{4(k+1)+4} \leq 0$ and so
$$\begin{cases} x_{4(k+1)+5} = -2^{2(k+1)+2}x_0 + 2^{2(k+1)+2}y_0 - 3 < 0 \\ y_{4(k+1)+5} = 2^{2(k+1)+2}x_0 - 2^{2(k+1)+2}y_0 + 1 \leq 0 \end{cases}.$$
$$\begin{cases} x_{4(k+1)+6} = -1 \\ y_{4(k+1)+6} = -1 \end{cases}.$$
If $(x_0, y_0) \in \Delta_{k+2} = \{(x,y) \in Q_2 \mid 2^{2(k+1)+2}x - 2^{2(k+1)+2}y + 1 > 0\}$ then $x_{4(k+1)+4} > 0$. Hence $\mathcal{Q}(k+1)$ is also true. By mathematical induction $\mathcal{Q}(n)$ is true for any positive integer $n \geq 1$. So we proved the following lemma.

Proposition 2.2. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of the map (2) and initial condition $(x_0, y_0) \in \{(x,y) \in Q_2 | y \leq x + 1 \text{ and } 0 < y < \frac{1}{2}\}$. Then every solution is eventually equilibrium point.*

Now we complete the proof that every solution is eventually equilibrium point with initial point in the green region of Fig.2.

9

2.2 Coexisting attractors

This section we will consider the case that $x_1 = -x_0 - y_0 - 3 < 0$ and $y_1 = x_0 - y_0 + 1 < 0$ which means that initial point belong to cyan region of Fig.8. Then we have the next iteration in the form

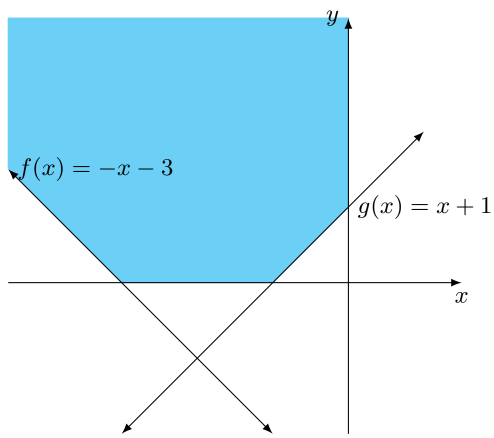

Figure 8: The region that x_1 and y_1 are negative, that (x_0, y_0) is in cyan.

$$\begin{cases} x_2 = 2y_0 - 1 \\ y_2 = -2y_0 - 1 < 0 \end{cases}.$$

That is the second iteration and the remain solutions are independent from x_0. If $y_0 \le \frac{1}{2}$ then $x_2 \le 0$ then $x_3 = y_3 = -1$. In the case of $\frac{1}{2} < y_0 \le \frac{3}{4}$, we have $x_2 > 0$ and so

$$\begin{cases} x_3 = 4y_0 - 3 \le 0 \\ y_3 = -1 \end{cases}, \quad \begin{cases} x_4 = -4y_0 + 1 < 0 \\ y_4 = 4y_0 - 3 \le 0 \end{cases}, \quad \begin{cases} x_5 = -1 \\ y_5 = -1 \end{cases}.$$

If $y_0 \ge \frac{5}{4}$ then

$$\begin{cases} x_3 = 4y_0 - 3 > 0 \\ y_3 = -1 \end{cases}, \quad \begin{cases} x_4 = 4y_0 - 5 \ge 0 \\ y_4 = 4y_0 - 3 > 0 \end{cases}, \quad \begin{cases} x_5 = -5 \\ y_5 = -1 \end{cases}.$$

If $\frac{3}{4} < y_0 \le \frac{7}{8}$ then

$$\begin{cases} x_3 = 4y_0 - 3 > 0 \\ y_3 = -1 \end{cases}, \quad \begin{cases} x_4 = 4y_0 - 5 < 0 \\ y_4 = 4y_0 - 3 > 0 \end{cases}, \quad \begin{cases} x_5 = -8y + 5 < 0 \\ y_5 = -1 \end{cases},$$
$$\begin{cases} x_6 = 8y - 7 < 0 \\ y_6 = -8y + 5 < 0 \end{cases}, \quad \begin{cases} x_7 = -1 \\ y_7 = -1 \end{cases}.$$

Now we can conclude that solutions with initial point in green portion of Fig. 9 become equilibrium point within seventh iteration while solutions with initial point in red portion of Fig. 9 become 4-cycle within fifth iteration. The remain region, cyan region of Fig. 9, is $\frac{7}{8} < y_0 < \frac{5}{4}$ which we have third iteration to fifth iteration are the same as in the case $\frac{3}{4} < y_0 \le \frac{7}{8}$ but the sixth iteration is $x_6 = 8y_0 - 7 > 0$ and $y_6 = -8y_0 + 5 < 0$. The remain iterations can be proved to become equilibrium point or 4-cycle by using induction. We will use the following inductive statement to verify. Let $A_n = \frac{2^{2n+2}-1}{2^{2n+2}}, l_n = \frac{2^{2n+1}-1}{2^{2n+1}}, u_n = \frac{2^{2n}+1}{2^{2n}}$

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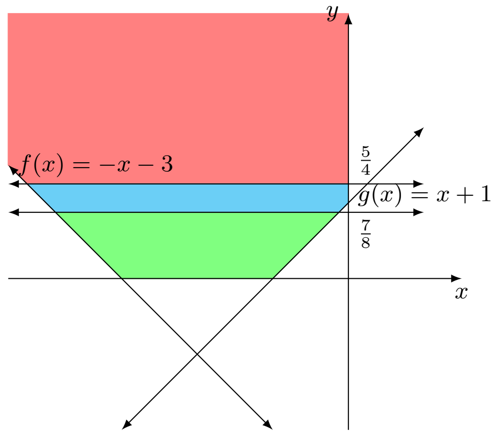

Figure 9: The red (green) region is initial points of solutions that are eventually 4-cycle $P_{4.1}$ (equilibrium point) while the remain region is in cyan.

and $\gamma_n = 2^{2n+2} - 1$ and $R(n)$ be the following statement : "for $y_0 \in (l_n, u_n)$, then $x_{4n+3} = 2^{2n+2}y_0 - \gamma_n, y_{4n+3} = -1$. If $y_0 \in (l_n, A_n]$ then $x_{4n+3} \le 0$ and so

$$\begin{cases} x_{4n+4} = -2^{2n+2}y_0 + \gamma_n - 2 < 0 \\ y_{4n+4} = 2^{2n+2}y_0 - \gamma_n \le 0 \end{cases}, \quad \begin{cases} x_{4n+5} = -1 \\ y_{4n+5} = -1 \end{cases}.$$

If $y_0 \in (A_n, u_n)$ then $x_{4n+3} > 0$ and so

$$\begin{cases} x_{4n+4} = 2^{2n+2}y_0 - \gamma_n - 2 \\ y_{4n+4} = 2^{2n+2}y_0 - \gamma_n > 0 \end{cases}.$$

If $y_0 \in [u_{n+1}, u_n)$ then $x_{4n+4} \ge 0$ and so $x_{4n+5} = -5$ and $y_{4n+5} = -1$.
If $y_0 \in (A_n, u_{n+1})$ then $x_{4n+4} < 0$ and so

$$\begin{cases} x_{4n+5} = -2^{2n+3}y_0 + 2\gamma_n - 1 < 0 \\ y_{4n+5} = -1 \end{cases}, \quad \begin{cases} x_{4n+6} = 2^{2n+3}y_0 - 2\gamma_n - 1 \\ y_{4n+6} = -2^{2n+3}y_0 + 2\gamma_n - 1 < 0 \end{cases}$$

If $y_0 \in (A_n, l_{n+1}]$ then $x_{4n+6} \le 0$ and so $x_{4n+7} = y_{4n+7} = -1$
If $y_0 \in (l_{n+1}, u_{n+1})$ then $x_{4n+6} > 0$."

We shall first show that $P(1)$ is true. For $y_0 \in (l_1, u_1) = (\frac{7}{8}, \frac{5}{4})$ and $\gamma_1 = 15$ we have $x_6 = 8y_0 - 7 > 0, y_6 = -8y_0 + 5 < 0$ and so

$$\begin{cases} x_{4(1)+3} = 16y_0 - 15 = 2^{2(1)+2}y_0 - \gamma_1 \\ y_{4(1)+3} = -1 \end{cases}$$

If $y_0 \in (l_1, A_1] = (\frac{7}{8}, \frac{15}{16}]$ then $x_7 \le 0$ and so

$$\begin{cases} x_{4(1)+4} = -16y_0 + 13 = -2^{2(1)+2}y_0 + \gamma_1 - 2 < 0 \\ y_{4(1)+4} = 16y_0 - 15 = 2^{2(1)+2}y_0 - \gamma_1 \le 0 \end{cases}, \quad \begin{cases} x_{4(1)+5} = -1 \\ y_{4(1)+5} = -1 \end{cases}.$$

If $y_0 \in (A_1, u_1) = (\frac{15}{16}, \frac{5}{4})$ then $x_7 > 0$ and so

$$\begin{cases} x_{4(1)+4} = 16y_0 - 17 = 2^{2(1)+2}y_0 - \gamma_1 - 2 \\ y_{4(1)+4} = 16y_0 - 15 = 2^{2(1)+2}y_0 - \gamma_1 > 0 \end{cases}.$$

If $y_0 \in [u_2, u_1) = [\frac{17}{16}, \frac{5}{4})$ then $x_8 \ge 0$ and so $x_{4(1)+5} = -5, y_{4(1)+5} = -1$.
If $y_0 \in (A_1, u_2) = (\frac{15}{16}, \frac{17}{16})$ then $x_8 < 0$ and so

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$$\begin{cases} x_{4(1)+5} = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\gamma_1 - 1 < 0 \\ y_{4(1)+5} = -1 \end{cases},$$
$$\begin{cases} x_{4(1)+6} = 32y_0 - 31 = 2^{2(1)+3}y_0 - 2\gamma_1 - 1 \\ y_{4(1)+6} = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\gamma_1 - 1 < 0 \end{cases}.$$

If $y_0 \in (A_1, l_2] = (\frac{15}{16}, \frac{31}{32}]$ then $x_{10} \le 0$ and so $x_{4(1)+7} = y_{4(1)+7} = -1$.
If $y_0 \in (l_2, u_2) = (\frac{31}{32}, \frac{17}{18})$ then $x_{4(1)+6} = 32y_0 - 31 = 2^{2(1)+3}y_0 - 2\gamma_1 - 1 > 0$. Thus $R(1)$ is true. So the base case of induction is done. Similar to $P(n)$, one can prove that step case is also true. By mathematical induction $R(n)$ is true for any positive integer $n \ge 1$. From inductive statement one can infer that solution will become 4-cycle ($P_{4.1}$) when $y_0 \in [u_{n+1}, u_n)$ while solution will become equilibrium point when $y_0 \in (l_n, A_n]$ and $y_0 \in (A_n, l_{n+1}]$. One can see that limit of sequence A_n, l_n, and u_n are 1. So the cyan region of Fig. 9 will collapse into a single line $L := \{(x, 1) | x \in [-4, 0]\}$. For $(x_0, y_0) \in L$ one can verify that $(x_2, y_2) = (1, -3) \in P_{4.2}$. It means the solution will become 4-cycle ($P_{4.2}$) when $y_0 \in L$.

Proposition 2.3. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of the map* (2) *and initial condition $(x_0, y_0) \in \{(x, y) \in Q_2 | y > x + 1$ and $y > -x - 3\}$. Then every solution is eventually equilibrium point.*

We can conclude that there are three attractors: equilibrium point, $P_{4.1}$ and $P_{4.2}$. The basin of attraction of equilibrium point is green portion of Fig.10 while $P_{4.1}$ has red portion of Fig. 10 and $P_{4.2}$ has L being the basin.

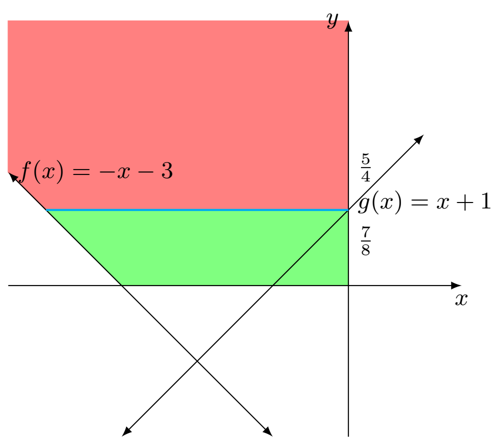

Figure 10: Basin of attraction of $P_{4.1}, P_{4.2}$ is in red and cyan respectively, while the basin of attraction of equilibrium point is in green.

3 Conclusion and discussion

We investigated the system of piecewise linear map (2) with initial condition in the second quadrant. By separating the second quadrant into three sub-regions as in Fig.1, we have the following behaviors of solutions. In the rightmost region of second quadrant (initial point below the line $g(x)$), every solution is eventually equilibrium point. For the middle region of second quadrant (initial point above the lines $f(x)$ and $g(x)$), the solution is eventually either equilibrium point or 4-cycle. We proved it by direct calculations and induction. For the last region of second quadrant (below the line $f(x)$) x_1 is positive and y_1 is negative. The behaviors of solution are more complicated than the other two sub-regions and interesting to study that we leave for future work. The behaviors of the map (2) are agree to Tikjha & Piasu (2020) that attractors are only equilibrium point and 4-cycles. It is possible to have equilibrium point and 4-cycles as attractors. But we do still not confirm that until knowing behaviors of solutions with initial condition (x_0, y_0) completely in $\mathbb{R}^2$.

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Some properties of the higher-order q-poly-tangent numbers and polynomials

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In this paper, we construct higher-order q-poly-tangent numbers and polynomials and give several properties, including addition formula and multiplication formula. Finally, we explore the distribution of roots of higher-order q-poly-tangent polynomials.

1 Introduction

In [7], we defined the tangent numbers and polynomials. The tangent polynomials are defined as the following generating function

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}\mathbf{T}_n(x)\frac{t^n}{n!}.$$

In [8], we constructed the poly-tangent numbers and polynomials. A modified poly-tangent numbers and polynomials different from the poly-tangent numbers and polynomials defined in [8] was introduced. In [9], we introduced tangent numbers and tangent polynomials of higher order. We also obtain interesting properties of these numbers and polynomials. For a nonnegative integer r, tangent polynomials of higher order are defined as the following generating function

$$\left(\frac{2}{e^{2t}+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\mathbf{T}_n^{(r)}(x)\frac{t^n}{n!}.$$

Definition 1.1. For any integer k, the modified poly-tangent polynomials are defined as the following generating function

$$\left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)e^{xt}=\sum_{n=0}^{\infty}T_n^{(k)}(x)\frac{t^n}{n!},$$

where $Li_k(t)=\sum_{n=1}^{\infty}\frac{t^n}{n^k}$ is polylogarithm function.

$T_n^{(k)}=T_n^{(k)}(0)$ are the called poly-tangent numbers when $x=0$. If we set $k=1$ in Definition 1.1, then the poly-tangent polynomials are reduced to classical tangent polynomials because of $Li_1(1-e^{-t})=t$. That is, $T_n^{(1)}(x)=\mathbf{T}_n(x)$.

2 Some properties of the higher-order q-poly-tangent numbers and polynomials

In this section, we define higher-order q-poly-tangent polynomials and study several properties, including addition formula and multiplication formula.

In [3], [2], [8], the q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q},\ (q \neq 1).$$

We note that $\lim_{q\to 1}[x]_q = x$. The q-factorial of n of order k is defined as

$$[n]_q^{\underline{(k)}} = [n]_q[n-1]_q \cdots [n-k+1]_q,\ \ (k = 1, 2, 3, \cdots),$$

where $[n]_q$ is q-number. Specially, when $k = n$, it is reduced the q-factorial

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q.$$

For $k \in \mathbb{Z}$, the q-analogue of polylogarithm function $Li_{k,q}$ is known by

$$Li_{k,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_q^k}.$$

Definition 2.1. For any integer k, a nonnegative integer r, higher-order q-poly-tangent polynomials are defined as the following generating function

$$\left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x)\frac{t^n}{n!}.$$

$T_{n,q}^{(k,r)} = T_{n,q}^{(k,r)}(0)$ are called higher-order q-poly-tangent numbers when $x = 0$. If we set $k = 1, q \to 1$ in Definition 2.1, then the higher-order q-poly-tangent polynomials are reduced to higher-order tangent polynomials.

Theorem 2.2. For any integer k and a nonnegative integer r, n, and m, we get

$$T_{n,q}^{(k,r)}(mx) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k,r)} m^{n-l} x^{n-l}.$$

Proof. From Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(mx)\frac{t^n}{n!} &= \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{mxt} \\
&= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(mx)^n\frac{t^n}{n!}\right) \qquad (1)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k,r)} m^{n-l}x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we finish the proof of Theorem 2.2 by comparing the coefficients of $\frac{t^n}{n!}$. $\square$

If $m = 1$ in Theorem 2.2, then we get the following corollary.

Corollary 2.3. For any integer k and a nonnegative integer r and n, we have

$$T_{n,q}^{(k,r)}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k,r)}\, x^{n-l}.$$

Theorem 2.4. For any integer k and a nonnegative integer r and n, we get

$$(1)\ \ T_{n,q}^{(k,r)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k,r)}(x) y^{n-l}.$$

$$(2)\ \ T_{n,q}^{(k,r+s)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k,r)}(x) T_{n-l,q}^{(k,s)}(y).$$

Proof. (1) Proof is omitted since it is a similar method of Theorem 2.2.
(2) From Definition 1.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,q}^{(k,r+s)}(x+y)\frac{t^n}{n!} \\
&= \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)}\right)^{r+s} e^{(x+y)t} \\
&= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_{n,q}^{(k,s)}(y)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} T_{l,q}^{(k,r)}(x) T_{n-l,q}^{(k,s)}(y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2}$$

Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (2). $\square$

Theorem 2.5. For any integer k and a nonnegative integer r, n, and m, we obtain

$$T_{n,q}^{(k,r)}(mx) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k,r)}(x)\,(m-1)^{n-l} x^{n-l}.$$

Proof. By utlizing Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(mx)\frac{t^n}{n!} &= \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)}\right)^{r} e^{xt} e^{(m-1)xt} \\
&= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(m-1)^n x^n\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} T_{l,q}^{(k,r)}(x)\,(m-1)^{n-l} x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3}$$

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Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (3). □

Theorem 2.6. For any integer k, a nonnegative integer r, and a positive integer n, we have

$$T_{n,q}^{(k,r)}(x+1) - T_{n,q}^{(k,r)}(x) = \sum_{l=0}^{n-1} \binom{n}{l} T_{l,q}^{(k,r)}(x).$$

Proof. By using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x)\frac{t^n}{n!} \\
&= \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{(x+1)t} - \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{xt} \\
&= \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{xt}\left(e^t - 1\right) \\
&= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} - 1\right) \\
&= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=1}^{\infty} \frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \binom{n+1}{l} T_{l,q}^{(k,r)}(x)\frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1} \binom{n}{l} T_{l,q}^{(k,r)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{4}$$

Then we compare the coefficients of $\frac{t^n}{n!}$ for $n \geq 1$. The reason both sides of the above equation (4) can be compared the coefficients is that $T_{0,q}^{(k,r)}(x+1) - T_{0,q}^{(k,r)}(x) = 0$. Thus, the proof is done. □

3 Polynomials and numbers related to higher-order q-poly-tangent polynomials and its symmtric property

In this section, we examine the association between higher-order q-poly-tangent numbers and poly-tangent polynomials using Cauchy product. We also explore relation of higher-order q-poly-tangent polynomials and Stirling numbers of the second kind. Furthermore, we study the symmetry properties of higher-order q-poly-tangent polynomials.

We recall a multinomial coefficient, which is

$$\binom{n}{m_1, m_2, \cdots, m_l} = \frac{n!}{m_1! m_2! \cdots m_l!}. \tag{5}$$

Let us consider the following equation using the equation (5) above. This equation is an equation expressed by applying Cauchy product continuously.

Theorem 3.1. For any integer k, a nonnegative integer n, and $r \geq 3$, we get

$$\begin{aligned} T_{n,q}^{(k,r)}(rx) = &\sum_{m_{r-1}=0}^{n} \sum_{m_{r-2}=0}^{m_{r-1}} \cdots \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \\ &\times \binom{n}{m_1, m_2 - m_1, \cdots, m_{r-1} - m_{r-2}, n - m_{r-1}} T_{m_1,q}^{(k)}(x) \\ &\times T_{m_2-m_1,q}^{(k)}(x) \cdots T_{m_{r-1}-m_{r-2},q}^{(k)}(x)\, T_{n-m_{r-1},q}^{(k)}(x), \end{aligned}$$

where $\binom{n}{m_1, m_2, \cdots, m_l}$ is multinomial coefficient.

Generating function of the Stirling numbers of the second kind $S_2(n,k)$ is defined as follows:

$$\sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}.$$

Theorem 3.2. For any integer k, a nonnegative integer r and a positive integer n, we obtain

$$T_{n,q}^{(k,r)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_2(l,m) T_{n-l,q}^{(k,r)},$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ is falling factorial.

Proof. From Definition 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x) \frac{t^n}{n!} &= \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)} \right)^r e^{xt} \\ &= \left(\frac{2Li_{k,q}(1-e^{-t})}{t(e^{2t}+1)} \right)^r \sum_{m=0}^{\infty} (x)_m \frac{(e^t-1)^m}{m!} \\ &= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (x)_m\, S_2(n,m) \frac{t^n}{n!} \right) \qquad (6) \\ &= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} (x)_m\, S_2(n,m) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_2(l,m) T_{n-l,q}^{(k,r)} \right) \frac{t^n}{n!}. \end{aligned}$$

Thus, we finish the proof by comparing the coefficients of $\frac{t^n}{n!}$. □

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Theorem 3.3. Let r and n be a nonnegative integer and $w_1, w_2 > 0\,(w_1 \neq w_2)$. Then we have

$$\sum_{l=0}^{n}\binom{n}{l}w_1^l w_2^{n-l}T_{l,q}^{(k,r)}(w_2x)T_{n-l,q}^{(k,r)}(w_1x) = \sum_{l=0}^{n}\binom{n}{l}w_2^l w_1^{n-l}T_{l,q}^{(k,r)}(w_1x)T_{n-l,q}^{(k,r)}(w_2x).$$

Proof. Let us consider the function

$$F(t) = \left(\frac{4Li_{k,q}(1-e^{-w_1t})Li_{k,q}(1-e^{-w_2t})}{t^2(e^{2w_1t}+1)(e^{2w_2t}+1)}\right)^r e^{2w_1w_2xt}. \tag{7}$$

Then we obtain

$$\begin{aligned} F(t) &= \left(\frac{2Li_{k,q}(1-e^{-w_1t})}{t(e^{2w_1t}+1)}\right)^r e^{w_1w_2xt}\left(\frac{2Li_{k,q}(1-e^{-w_2t})}{t(e^{2w_2t}+1)}\right)^r e^{w_1w_2xt} \\ &= \left(\sum_{n=0}^{\infty}w_1^{n+r}T_{n,q}^{(k,r)}(w_2x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_{n,q}^{(k,r)}(w_1x)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}w_1^{l+r}w_2^{n-l+r}T_{l,q}^{(k,r)}(w_2x)T_{n-l,q}^{(k,r)}(w_1x)\right)\frac{t^n}{n!}. \end{aligned} \tag{8}$$

By calculating in the same way as the above equation (8), we can get

$$F(t) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}w_2^{l+r}w_1^{n-l+r}T_{l,q}^{(k,r)}(w_1x)T_{n-l,q}^{(k,r)}(w_2x)\right)\frac{t^n}{n!}. \tag{9}$$

The proof is complete as a result of the equations (8) and (9). □

Let w is an odd number. Then we can easily see

$$\sum_{n=0}^{\infty}\tilde{A}_n(w)\frac{t^n}{n!} = \frac{e^{wt}+1}{e^t+1}, \tag{10}$$

where $\tilde{A}_n(w) = \sum_{l=0}^{w-1}(-1)^l\, l^n$ is called alternating power sum.

Theorem 3.4. Let w_1 and w_2 be an odd number and n be a nonnegative integer. Then we have

$$\begin{aligned} &\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}2^{n-j-l}w_1^{i+l+r}w_2^{2n-2j-i-l+r}T_{i,q}^{(k,r)} \\ &\qquad\times T_{n-j-i,q}^{(k,r)}\mathbf{T}_l(w_2x)\tilde{A}_{n-j-l}(w_1) \\ &= \sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}2^{n-j-l}w_2^{i+l+r}w_1^{2n-2j-i-l+r}T_{i,q}^{(k,r)} \\ &\qquad\times T_{n-j-i,q}^{(k,r)}\mathbf{T}_l(w_1x)\tilde{A}_{n-j-l}(w_2). \end{aligned}$$

Proof. First, let us assume that

$$G(t)=2\frac{4^r(Li_{k,q}(1-e^{-w_1t}))^r(Li_{k,q}(1-e^{-w_2t}))^r(e^{2w_1w_2t}+1)}{t^{2r}\left(e^{2w_1t}+1\right)^r\left(e^{2w_2t}+1\right)^r(e^{2w_1t}+1)(e^{2w_2t}+1)}e^{2w_1w_2xt}. \quad (11)$$

Then we calculate

$$\begin{aligned}
G(t)&=2\left(\frac{2Li_{k,q}(1-e^{-w_1t})}{t(e^{2w_1t}+1)}\right)^r\left(\frac{2Li_{k,q}(1-e^{-w_2t})}{t(e^{2w_2t}+1)}\right)^r\\
&\quad\times\frac{2}{(e^{2w_1t}+1)}e^{2w_1w_2xt}\frac{e^{2w_1w_2t}+1}{e^{2w_2t}+1}\\
&=\left(\sum_{n=0}^{\infty}w_1^{n+r}T_{n,q}^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_{n,q}^{(k.r)}\frac{t^n}{n!}\right)\\
&\quad\times\left(\sum_{n=0}^{\infty}w_1^{n}T_{n,q}(w_2x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}2^nw_2^{n}\tilde{A}_n(w_1)\frac{t^n}{n!}\right)\\
&=\left(\sum_{n=0}^{\infty}w_1^{n+r}T_{n,q}^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_{n,q}^{(k,r)}\frac{t^n}{n!}\right)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^{n-l}w_1^lw_2^{n-l}\mathbf{T}_l(w_2x)\tilde{A}_{n-l}(w_1)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}2^{n-j-l}w_1^{i+l+r}w_2^{2n-2j-i-l+r}\right.\\
&\qquad\qquad\left.\times\,T_{i,q}^{(k,r)}T_{n-j-i,q}^{(k,r)}\mathbf{T}_l(w_2x)\tilde{A}_{n-j-l}(w_1)\right)\frac{t^n}{n!}.
\end{aligned} \quad (12)$$

In a similar way to the above equation (12), we get

$$\begin{aligned}
G(t)&=\left(\sum_{n=0}^{\infty}w_1^{n+r}T_{n,q}^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_{n,q}^{(k,r)}\frac{t^n}{n!}\right)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^{n-l}w_2^lw_1^{n-l}\mathbf{T}_l(w_1x)\tilde{A}_{n-l}(w_2)\frac{t^n}{n!}.
\end{aligned} \quad (13)$$

Hence, by using Cauchy product, the proof is complete by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equations (12) and (13). □

4 Distribution of zeros of the higher-order q-poly-tangent polynomials

Using generating functions, the generalized forms of known polynomials such as the Bernoulli, Euler, falling factorial and tangent polynomials are studied. In particular, various properties of these polynomials were investigated through numerical experiments, see for example [1] , [4], [6], [7], [8], [9], [10], [11], [12].

In this section, we discover new interesting pattern of the zeros of the higher-order q-poly-tangent polynomials $T_{n,q}^{(k,3)}(x)$. We propose some conjectures by numerical experiments. The higher-order q-poly-tangent polynomials $T_{n,q}^{(k,3)}(x)$ can be determined explicitly.

A few of them are

$$T_{0,q}^{(k,3)}(x) = 1,$$

$$T_{1,q}^{(k,3)}(x) = -\frac{9}{2} + 3\left(\frac{1-q^2}{1-q}\right)^{-k} + x,$$

$$\begin{aligned}T_{2,q}^{(k,3)}(x) &= \frac{35}{2} + 6\left(\frac{1-q^2}{1-q}\right)^{-2k} - 30\left(\frac{1-q^2}{1-q}\right)^{-k} + 6\left(\frac{1-q^3}{1-q}\right)^{-k} - 9x \\ &\quad + 6\left(\frac{1-q^2}{1-q}\right)^{-k} x + x^2,\end{aligned}$$

$$\begin{aligned}T_{3,q}^{(k,3)}(x) &= -54 + 6\left(\frac{1-q^2}{1-q}\right)^{-3k} - 99\left(\frac{1-q^2}{1-q}\right)^{-2k} + 201\left(\frac{1-q^2}{1-q}\right)^{-k} \\ &\quad - 99\left(\frac{1-q^3}{1-q}\right)^{-k} + 36\left(\frac{1-q^2}{1-q}\right)^{-k}\left(\frac{1-q^3}{1-q}\right)^{-k} + 18\left(\frac{1-q^4}{1-q}\right)^{-k} \\ &\quad + \frac{105x}{2} + 18\left(\frac{1-q^2}{1-q}\right)^{-2k} x - 90\left(\frac{1-q^2}{1-q}\right)^{-k} x \\ &\quad + 18\left(\frac{1-q^3}{1-q}\right)^{-k} x - \frac{27x^2}{2} + 9\left(\frac{1-q^2}{1-q}\right)^{-k} x^2 + x^3,\end{aligned}$$

We investigate the beautiful zeros of the higher-order q-poly-tangent polynomials $T_{n,q}^{(k,r)}(x)$ by using a computer. We plot the zeros of higher-order q-poly-tangent polynomials $T_{n,q}^{(k,r)}(x)$ for $n = 30, r = 3$ and $x \in \mathbb{C}$(Figure 1).

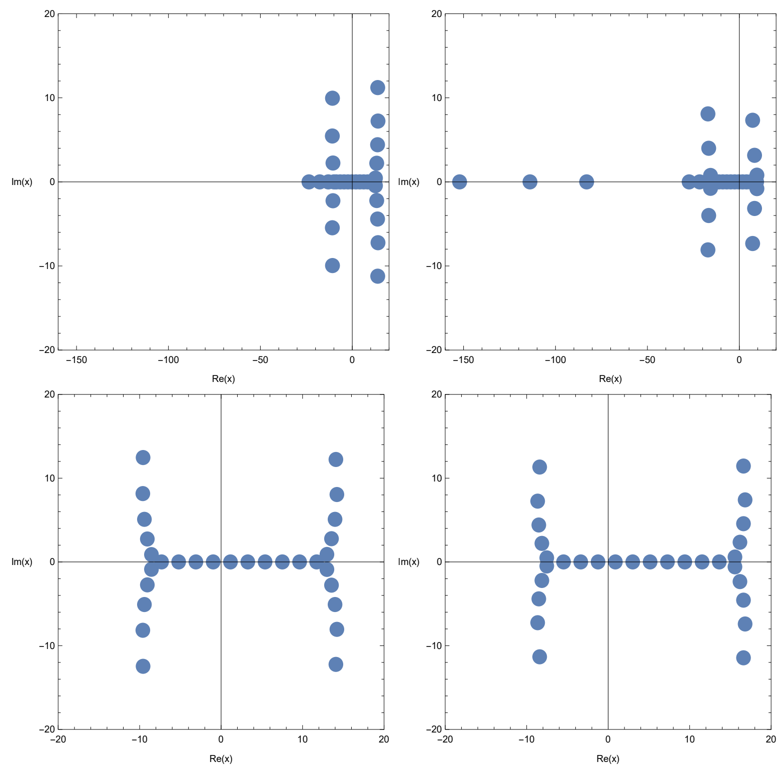

Figure 1: Zeros of $T_{n,q}^{(k,r)}(x)$

In Figure 1(top-left), we choose $n = 30, q = \frac{1}{10}$ and $k = -3$. In Figure 1(top-right), we choose $n = 30, q = \frac{9}{10}$ and $k = -3$. In Figure 1(bottom-left), we choose $n = 30, q = \frac{1}{10}$, and $k = 3$. In Figure 1(bottom-right), we choose $n = 30, q = \frac{9}{10}$ and $k = 3$.

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Stacks of zeros of $T_{n,q}^{(k,r)}(x)$ for $1 \leq n \leq 30$ from a 3-D structure are presented(Figure 2).

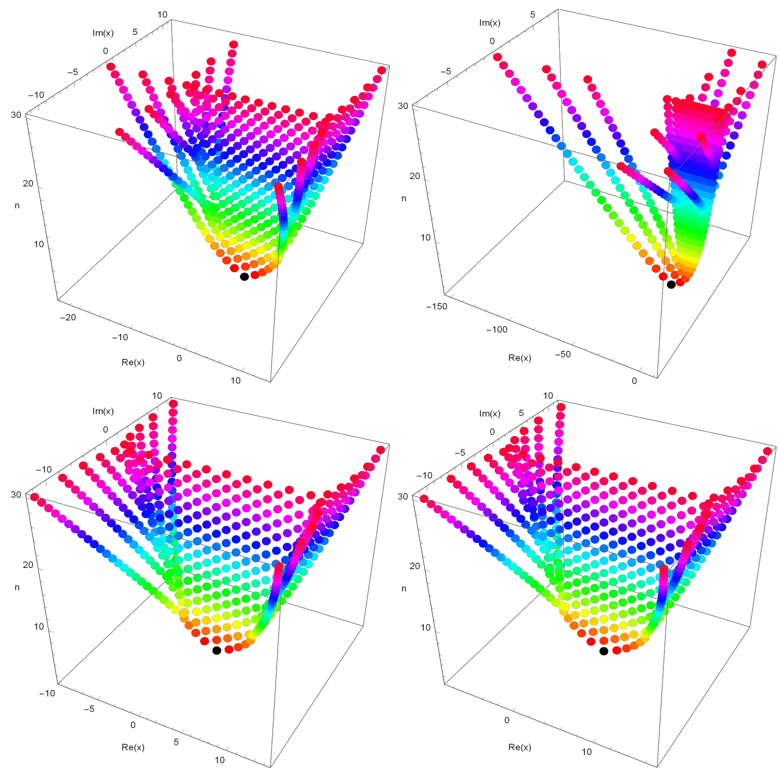

Figure 2: Stacks of zeros of $T_{n,q}^{(k,r)}(x)$ for $1 \leq n \leq 30$

In Figure 2(top-left), we choose $r = 3, q = \frac{1}{10}$ and $k = -3$. In Figure 2(top-right), we choose $r = 3, q = \frac{9}{10}$ and $k = -3$. In Figure 2(bottom-left), we choose $r = 3, q = \frac{1}{10}$, and $k = 3$. In Figure 2(bottom-right), we choose $r = 3, q = \frac{9}{10}$ and $k = 3$.

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We plot the real zeros of the higher-order q-poly-tangent polynomials $T_{n,q}^{(k,r)}(x)$ and $x \in \mathbb{C}$(Figure 3).

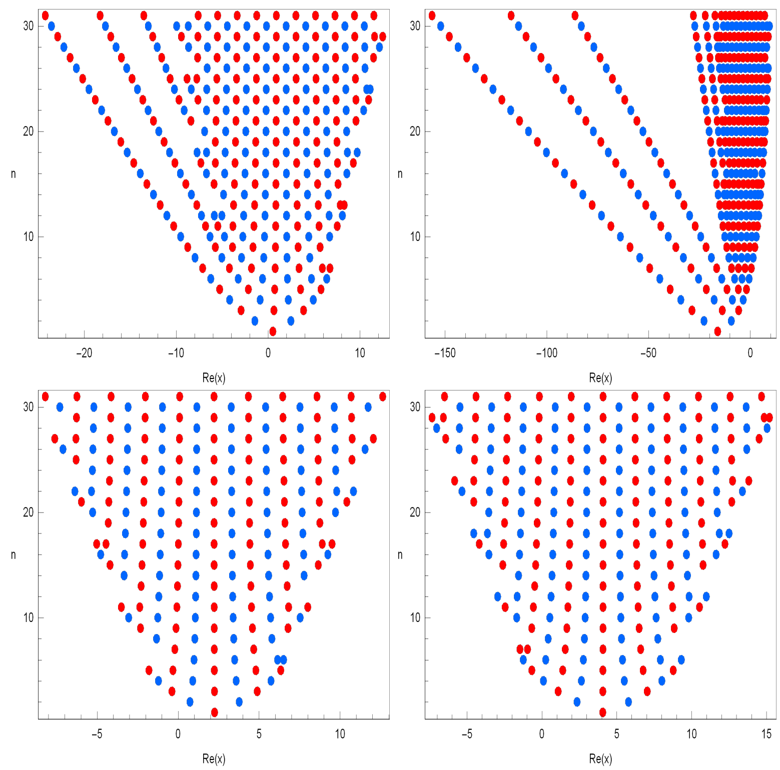

Figure 3: Real zeros of $T_{n,q}^{(k,r)}(x)$ for $1 \leq n \leq 30$

In Figure 3(top-left), we choose $r = 3, q = \frac{1}{10}$ and $k = -3$. In Figure 3(top-right), we choose $r = 3, q = \frac{9}{10}$ and $k = -3$. In Figure 3(bottom-left), we choose $r = 3, q = \frac{1}{10}$, and $k = 3$. In Figure 3(bottom-right), we choose $r = 3, q = \frac{9}{10}$ and $k = 3$.

Next, we calculated an approximate solution satisfying higher-order q-poly-tangent polynomials $T_{n,q}^{(k,r)}(x)$ for $x \in \mathbb{R}$. The results are given in Table 1 and Table 2.

Table 1. Approximate solutions of $T_{n,q}^{(k,r)}(x) = 0, k = -3, r = 3, q = \frac{1}{10}$

degree n	x
1	0.50700
2	-1.4556, 2.4696
3	-2.9508, 0.62706, 3.8447
4	-4.1946, -0.87747, 2.1935, 4.9066
5	-5.2759, -2.1561, 0.70762, 3.5182, 5.7412
6	-6.2440, -3.2614, -0.61966, 2.0917, 4.7059, 6.3694
7	-7.1317, -4.2202, -1.8162, 0.75900, 3.3518, 5.8907, 6.7156
8	-7.9630, -5.0461, -2.9002, -0.48398, 2.0429, 4.5281

Table 2. Approximate solutions of $T_{n,q}^{(k,r)}(x) = 0, k = 3, r = 3, q = \frac{1}{10}$

degree n	x
1	2.2461
2	0.72612, 3.7660
3	-0.38330, 2.2395, 4.8819
4	-1.2186, 0.89693, 3.5776, 5.7283
5	-1.8044, -0.32452, 2.2334, 4.7925, 6.3333
6	0.97798, 3.4837, 6.1347, 6.5052
7	-0.20813, 2.2289, 4.6632
8	-1.3362, 1.0256, 3.4282, 5.7835

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Generalized completely monotone functions on some types of white noise spaces

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Abstract: With this paper, we purpose to introduce and characterized some generalized classes of completely monotone functions on some types of white noise spaces.

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1. Introduction

The basic features of the completely monotone functions constructed on some forms of white noise spaces are provided in this study. if for each $\alpha \in \mathbb{Z}_+^n, (-1)^{|\alpha|} D^\alpha f(x) \geq 0$, then a function f is completely monotone on $\mathbb{R}_+^n$; see [3, 8, 12] for several features of completely monotone functions. According to Bernstein's theorem, f is completely monotone if and only if

$$f(x) = \int_{\mathbb{R}^n} e^{-x.t}\ d\mu(t) \qquad (1.1)$$

μ is a positive measure that is based on a subset of $\mathbb{R}_+^n$. Let Q stand for a locally compact basis on the space $\mathbb{R}^d$. $C_b(Q)$ is a linear space of continuous bounded complex-valued functions which is a complete normed space compared to the norm

$$\|f\|_\infty = \sup_{x \in Q} |f(x)| \qquad (1.2)$$

f defined on Q, where The space of infinitely differential and bounded functions on Q will be denoted by $C_b^\infty(Q)$,Moreover, by $S(Q)$, the linear subspace of $C_b^\infty(Q)$ created by the set which contains functions on Qlike that $x^\alpha D^\beta f(x) \leq C_{\alpha,\beta}$, with$\alpha, \beta \in \mathbb{Z}_+^n$and a constant $C_{\alpha,\beta}$.The space of tempered distributions is represented by $S^{'}(Q)$, Which is linear and continuous functional on $S(Q)$. There are numerous works that explore white noise spaces. Using the Wiener-Itô-Segal isomorphism and other Fock space riggings, some of these works are devoted to the building of test spaces, generalized functions, and operators having to act in these spaces [1,9].The study of PDEs and quantum field theory, where quantum fields are characterized as operator valued distributions, both depend heavily on distributions [5,11]. The works of Berezanskyi and Samoilenko [2] and Hida [9] are where the modern theory of generalized functions of infinitely many variables is derived. As infinite tensor products of one-dimensional spaces, the test and generalized function spaces in [2] were created. The theory of generalized functions was constructed using the classical method in [9], but all functions were functions of a point in the infinite-dimensional space on which the Gaussian measure was defined, which served the same purpose as the Lebesgue measure in the classical theory of generalized functions. The structure of this paper is as follows: section 2, we devoted to introduce and give the main properties of the class of double monotone functions defined on $S(Q)$. In section 3, the main properties of the class of weak monotone functions defined on $S(Q)$ are given.Section 4 introduces a novel method for creating spaces of generalized functions. Section 5 concludes by deriving the principal relationships between the creation of hypercomplex systems and the theory of white noise analysis.

2. Double completely monotone functions on $S(Q)$

Rabidly decreasing functions are the name given to the components of $S(Q)$, which has a family of seminorms for each $\alpha, \beta \in \mathbb{Z}_+^n$

$$\|f\|_{\alpha,\beta} = \sup_{x\in Q}\left|x^\alpha D^\beta f(x)\right| \tag{2.1}$$

Let $F : Q \to \mathbb{C}$ be a continuous double completely monotone function, i.e., $F = f_1 + if_2$ and f_1, f_2 are two completely monotone functions. We define

$$\langle \cdot , \cdot \rangle_F : C_0(Q) \times C_0(Q) \to \mathbb{C}$$

by

$$\langle \varphi, \psi \rangle_F := \int_Q \int_Q F(x-y)\, \varphi(x)\, \overline{\psi(y)}\, d\mu(x) d\nu(y) \tag{2.2}$$

where $\mu, \nu \in M(Q)$, the space of Radon measure on Q. The inner product $\langle \cdot , \cdot \rangle_{\mathrm{F}}$ satisfies the following conditions:

I. $\langle \cdot , \cdot \rangle_F$in the first coordinate is complex linear and in the second conjugate complex linear i.e., for any $\varphi, \psi \in C_0(Q)$ and any $c \in \mathbb{C}$

$$\langle c\varphi, \psi \rangle_F = c \langle \varphi, \psi \rangle_F \text{and} \langle \varphi, c\psi \rangle_F = \overline{c} \langle \varphi, \psi \rangle_F$$

II. $\langle \cdot , \cdot \rangle_F$ is conjugate symmetric i.e., for any $\varphi, \psi \in C_0(Q)$

$$\langle \varphi, \psi \rangle_F = \overline{\langle \psi, \varphi \rangle_F}$$

III. $\langle \cdot , \cdot \rangle_F$ is positive definite meaning that for any $\varphi \in C_0(Q)$

$$\langle \varphi, \varphi \rangle_F = L_F(\varphi) \geq 0$$

IV. For all $\varphi \in C_0(Q)$such that $\langle \varphi, \varphi \rangle_F = 0$, then $\varphi = 0$

Theorem 2.1. For any double completely monotone function F on Q, the inner product space $(C_0(Q), \langle \cdot , \cdot \rangle_F)$ is a complex Hilbert space.

Proof. We have that $\langle \cdot , \cdot \rangle_F$ is an inner product space and $C_0(Q)$ is an infinite space so all we need to prove is the completeness for that space , so we assume that we have a Cauchy sequence $\{\varphi_n\}$ and should prove that this Cauchy sequence converges to a limit in $(C_0(Q),\langle \cdot , \cdot \rangle_F)$.

Where

$$\langle \varphi,\psi \rangle_F = \int_Q \int_Q F(x-y)\, \varphi(x)\, \overline{\psi(y)}\, d\mu(x) d\nu(y)$$

$\varphi,\psi \in C_0(Q)$, $\mu,\nu \in M(Q)$ the space of Radon measure on .

$$\begin{aligned} \|\varphi_n - \varphi_m\|^2 &= \langle \varphi_n - \varphi_m, \varphi_n - \varphi_m \rangle \\ &= \int_Q \int_Q F(x-y)(\varphi_n - \varphi_m)(x)\, \overline{(\varphi_n - \varphi_m)(y)}\, d\mu(x) d\nu(y) \\ &= \int_Q \int_Q F(x-y) |\varphi_n(x) - \varphi_m(x)|^2 \, d\mu(x) d\nu(y) \\ &\to 0 \end{aligned}$$

$as\ n,m \to \infty$. This implies

$$|\varphi_n(x) - \varphi_m(x)|^2 \to 0$$

$as\ n,m \to \infty$. So

$$|\varphi_n(x) - \varphi_m(x)| \to 0$$

$as\ n,m \to \infty$.Since $\{\varphi_n\}$ is a Cauchy sequence and we have that $C_0(Q)$ is a complete space which means that $\lim_{n\to\infty} \varphi_n = \varphi\ as\ n \to \infty$ i.e $|\varphi_n(x) - \varphi(x)| \to 0\ as\ n \to \infty$, which tends to that φ belongs to$(C_0(Q),\langle \cdot , \cdot \rangle_F)$, so this space is complete.

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Corollary 2.2. For any double completely monotone function F on Q, the space $\mathcal{H}_F \equiv (C_0(Q),\langle \cdot , \cdot \rangle_F)$is a subspace of Hilbert space $L^2(\mu)$.

Proof. We want to prove $\mathcal{H}_F \subset L^2(\mu)$ so let $\varphi,\psi \in \mathcal{H}_F$ and we need to reach to these functions in $L^2(\mu)$.Assume that

$$\int_Q |F(x-y)|\, d\mu(x) \leq M_1 \quad for\ all\ y \in Q$$

and

$$\int_Q |F(x-y)|\, dv(y) \leq M_2 \quad for\ all\ x \in Q$$

and by using (2.2)

$$|\langle\varphi,\psi\rangle_F| = \left|\int_Q \int_Q F(x-y)\, \varphi(x)\, \overline{\psi(y)}\, d\mu(x) dv(y)\right|$$

Where by using (Cauchy – young inequality: If $\frac{1}{p}+\frac{1}{q}=1, then\ ab \leq \frac{a^p}{p}+\frac{b^q}{q}\ for\ \ a,b \geq 0)$

i.e.,

$$\left|\varphi(x)\overline{\psi(y)}\right| \leq \frac{|\,\varphi(x)|^2}{2}+\frac{|\psi(y)|^2}{2}$$

$$\leq \int_Q \int_Q \frac{|F(x-y)|}{2} dv(y) |\varphi(x)|^2\, d\mu(x) + \int_Q \int_Q \frac{|F(x-y)|}{2} d\mu(x) |\psi(y)|^2\, dv(y)$$

$$\leq \frac{M_2}{2} \|\varphi\|^2_{L_2(\mu)} + \frac{M_1}{2} \|\psi\|^2_{L_2(v)}$$

So ,$\psi \in\ L^2(\mu)$.

Let $\mathrm{M_c}$ stand for the set of all continuously real-valued functions ω on $\mathbb{R}^n$ that fulfill the requirements listed below:

1) $0\ =\ \omega(0)\ \leq\ \omega(\zeta+\eta)\ \leq\ \omega(\zeta)+\omega(\eta);\ \zeta,\eta\ \in \mathbb{R}^n$
2) $\int_{\mathbb{R}^n} \frac{\omega(\zeta)}{(1+|\zeta|)^{n+1}}\ d\zeta <\ \infty$
3) $\omega(\zeta) \geq\ a\ +\ b\ log(1+|\zeta|)$ for some constant a,b
4) $\omega(\zeta)$ is radial.

with the weight function ω in $\mathrm{M_c}$ and open set $\Omega\ \in\ \mathbb{R}^n$ Bj$\ddot{o}$rck extend the Schwartz space by the space $\mathrm{S}_\omega$of all $C^\infty -$ function $\varphi\ \in\ \mathrm{L}^1(\mathbb{R}^n)$:

$$\mathrm{P}_{\alpha,\lambda}(\varphi)\ =\ \sup_{x\in\ \mathbb{R}^n}\ e^{\lambda\omega(x)}\ |D^\alpha\varphi(x)| <\ \infty$$

And

$$\Pi_{\alpha,\lambda}(\varphi) = \sup_{\zeta \in \mathbb{R}^n} e^{\lambda\omega(\zeta)} |D^{\alpha}\widehat{\varphi}(\zeta)| < \infty$$

and S'_{ω} the dual space of S_{ω}. Let f be a double completely monotone function and

$$\omega_f(\zeta) = log(1 + |f(\zeta)|) \qquad (2.3)$$

for $s \in \mathbb{R}$ we denote by $\mathcal{H}_f^{\omega,s}$ the set of all generalized distributions $u \in S'_{\omega}$:

$$||u||_f^{\omega,s} = [\int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} |\hat{u}(\zeta)|^2 \, d\zeta]^{\frac{1}{2}} \qquad (2.4)$$

Theorem 2.3. The space $\mathcal{H}_f^{\omega,s}$is a Hilbert space with an inner product denoted by

$$\langle u, v\rangle_f^{\omega,s} = \int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} \hat{u}(\zeta)\overline{\hat{v}(\zeta)} d\zeta \qquad (2.5)$$

Proof. We need to prove that the space $\mathcal{H}_f^{\omega,s}$ is complete, so we assume that we have a Cauchy sequence $\{u_m\}$ in $\mathcal{H}_f^{\omega,s}$ and we want to prove that this Cauchy converges to a limit u in $\mathcal{H}_f^{\omega,s}$, where norm defined as:

$$\|u\|_f^{\omega,s} = \left[\int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} \, |\hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

So

$$\|u_m - u\|_f^{\omega,s} = \left[\int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} \, |\hat{u}_m(\zeta) - \hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

From (2.3), we have

$$\|u_m - u\|_f^{\omega,s} = \left[\int_{\mathbb{R}^n} (1 + |f(\zeta)|)^{2s} \, |\hat{u}_m(\zeta) - \hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

$$= \left[\int_{\mathbb{R}^n} (1 + |f(\zeta)|)^{2s+(n+1)} \, |\hat{u}_m(\zeta) - \hat{u}(\zeta)|^2 (1 + |f(\zeta)|)^{-(n+1)} d\zeta\right]^{\frac{1}{2}}$$

$$\leq \sup(1 + |f(\zeta)|)^{s+(n+1)/2} \, |\hat{u}_m(\zeta) - \hat{u}(\zeta)| \left[\int_{\mathbb{R}^n} (1 + |f(\zeta)|)^{-(n+1)} d\zeta\right]^{\frac{1}{2}}$$

$$\leq C\|\hat{u}_m - \hat{u}\|_p$$

$$\leq C\|u_m - u\|_{p+n+1}$$

Where $\geq s + (n+1)/2$, we find that $\|u_m - u\|_P \to 0$ as $m \to \infty \ \ \forall p \in N$, which come from that C_0^∞ is dense in S_ω, then $u \in \mathcal{H}_f^{\omega,s}$ which proves the completeness in it and so $\mathcal{H}_f^{\omega,s}$ is a Hilbert space.

Lemma 2.4. Let $u \in \mathcal{H}_f^{\omega,s}$, $< u, \cdot >_f^{\omega,s}$ is the conjugate linear functional on S_ω which uniquely extends to conjugate linear functional on $\mathcal{H}_f^{\omega,s}$ satisfying

1) $\langle\langle u,v \rangle\rangle_f^{\omega,s} = (2\Pi)^{-n} \int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} \hat{u}(\zeta)\overline{\hat{v}(\zeta)} d\zeta.$
2) $|\langle\langle u,v \rangle\rangle_f^{\omega,s}| \leq ||u||_f^{\omega,s} \, ||v||_f^{\omega,s}$, $u \in \mathcal{H}_f^{\omega,s}$,$v \in \mathcal{H}_f^{\omega,-s}$
3) $\langle\langle u,v \rangle\rangle_f^{\omega,s} = \overline{\langle\langle v, u \rangle\rangle}_f^{\omega,s}$

Theorem 2.5. The space S_ω is dense in $\mathcal{H}_f^{\omega,s}$for all $s \in \mathbb{R}$.

Proof. To prove that S_ω is dense in $\mathcal{H}_f^{\omega,s}$ we need to check two things the first is that $S_\omega \subset \mathcal{H}_f^{\omega,s}$ and the second is that $\overline{S_\omega} = \mathcal{H}_f^{\omega,s}$, for the first let we have a bijective map $g_s: S_\omega \to S_\omega$, $u \mapsto e^{s\omega_f(\zeta)}\hat{u}$**.** With (2.3) and f is a continuous double completely monotone function.

We have from the definition of map that $e^{s\omega_f(\zeta)}\hat{u} \in S_\omega \subset L_2$ which leads to $S_\omega \subset \mathcal{H}_f^{\omega,s}$.

Secondly we want to prove that $\overline{S_\omega} = \mathcal{H}_f^{\omega,s}$, so we must prove that $S_\omega^\perp = \{0\}$(orthogonal complement for S_ω). Where $S_\omega^\perp = \{u \in \mathcal{H}_f^{\omega,s} : \langle\langle u,\varphi \rangle\rangle_f^{\omega,s} = 0 \ \forall \varphi \in S_\omega\}$. We want to get to that $= 0$. i.e. $u \in \mathcal{H}_f^{\omega,s}$ with $u \in S_\omega^\perp$ lead to $\langle\langle u,\varphi \rangle\rangle_f^{\omega,s} = 0 \ \forall \varphi \in S_\omega$. We have

$$\langle\langle u,\varphi \rangle\rangle_f^{\omega,s} = \langle e^{s\omega_f(\zeta)}\hat{u}, e^{s\omega_f(\zeta)}\hat{\varphi}\rangle_{L_2} ,$$

Since g_s is bijective , $\forall \phi \in S_\omega$,we find $\langle e^{s\omega_f(\zeta)}\hat{u},\phi\rangle_{L_2} = 0$, since $S_\omega$is dense in L_p, $1 \leq p < \infty$.which mean that $e^{s\omega_f(\zeta)}\hat{u} = 0$, So $u = 0$, i.e. $S_\omega^\perp = \{0\}$, so $\overline{S_\omega} = \mathcal{H}_f^{\omega,s}$. Which complete the proof.

Note that $S_\omega$is dense in L_p comes from that $S_\omega \subset L_p$ .and that C_0^∞ is dense in L_p.

Corollary 2.6 $\mathcal{H}_f^{\omega,t} \subseteq \mathcal{H}_f^{\omega,s}$for $t > s$, the inclusion is continuous and has dense image.

3. Weak completely monotone functions

The purpose of this section is to discuss the idea of Weak completely monotone functions on the Schwartz spaces. Let $\Omega \subseteq \mathbb{R}$ be an open interval, $f \in C^1(\bar{\Omega})$ and $\varphi \in D(\Omega)$, where

$$D(\Omega) := \{\varphi \in C^\infty(\Omega, C) : supp(\varphi) := \overline{\{x;\ \varphi(x) \neq 0\}} \subseteq \Omega \text{ is compact}\}$$

Using integration by parts, we will get

$$\int_\Omega \acute{f}\,(x)\overline{\varphi(x)}dx = -\int_\Omega f\,(x)\overline{\acute{\varphi}(x)}dx$$

Since, $D(\Omega)$is the space of test functions which is dense in $L^P\,(\Omega)$ for $1 \leq p \leq \infty$, so we can rewrite the above equation using the scalar product of $L_2(\Omega)$ as

$$\langle \acute{f}\,|\varphi\,\rangle = -\langle\, f|\acute{\varphi}\rangle$$

We call a function g that satisfies $\langle\, g|\varphi\,\rangle = -\langle\, f|\acute{\varphi}\rangle$ a weak derivative of f. Let $\Omega \in \mathbb{R}^n$ open, $f \in C^1(\bar{\Omega})$ and $\varphi \in D(\Omega)$, then

$$\langle \frac{\partial}{\partial x_i} f|\varphi\,\rangle = -\,\langle\, f|\, \frac{\partial}{\partial x_i}\varphi\,\rangle$$

Applying Gauss Theorem, we similarly obtain

$$\langle D^\alpha f\,|\varphi\,\rangle = (-1\,)^{|\alpha|}\langle\, f\,|D^\alpha \varphi\,\rangle$$

Theorem 3.1. For any multi-index $\alpha \in \mathbb{R}$ the differential D^α is a continuous and linear operator from $\mathcal{H}_f^{\omega,s}$ to $\mathcal{H}_f^{\omega,s-|\alpha|}$.

Proof. Where the linearity of the operator is obvious, so all we need to prove is that

$$\|D^\alpha u\|_f^{\omega,s-|\alpha|} \leq c\|u\|_f^{\omega,s} \qquad (3.1)$$

From (2.4), we have,

$$\|u\|_f^{\omega,s-|\alpha|} = \left[\int_{\mathbb{R}^n} e^{2(s-|\alpha|)\omega_f(\zeta)}\,|\hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

So

$$\|D^\alpha u\|_f^{\omega,s-|\alpha|} = \left[\int_{\mathbb{R}^n} e^{2(s-|\alpha|)\omega_f(\zeta)}\,\left|\widehat{D^\alpha u}(\zeta)\right|^2 d\zeta\right]^{\frac{1}{2}}$$

Which equivalent to

$$\|D^\alpha u\|_f^{\omega,s-|\alpha|} = \left\| e^{(s-|\alpha|)\omega_f(\zeta)} \widehat{D^\alpha u}(\zeta) \right\|_{L_2}$$

For a particular case let $\alpha = 1$

$$\begin{aligned} \left\| e^{(s-1)\omega_f(\zeta)} \widehat{Du}(\zeta) \right\|_{L_2} &= \left\| e^{(s-1)\omega_f(\zeta)} \xi \, \hat{u}(\zeta) \right\|_{L_2} \\ &\leq C \left\| e^{s\omega_f(\zeta)} \hat{u}(\zeta) \right\|_{L_2} \\ &= c\|u\|_f^{\omega,s} \end{aligned}$$

i.e.,

$$\|D^\alpha u\|_f^{\omega,s-1} \leq c\|u\|_f^{\omega,s}$$

where by using induction on $|\alpha|$ we can generalize this for any multi index $\alpha \in \mathbb{R}$ .which follow from this that the linear operator D^α is continuous from $\mathcal{H}_f^{\omega,s}$ to $\mathcal{H}_f^{\omega,s-|\alpha|}$.

Theorem 3.2. The pairing $\langle\langle \cdot \,, \, \cdot \rangle\rangle_f^{\omega,s}$ identifies $\mathcal{H}_f^{\omega,-s}$isometrically with the antidual of $\mathcal{H}_f^{\omega,s}$. If $u \in \acute{D}_f^{\omega}$then $u \in \mathcal{H}_f^{\omega,s}$ if and only if there is a constant c such that $|u(\varphi)| \leq c\|\varphi\|_f^{\omega,-s}$ for $\in D_f^{\omega}$.

proof. Let the anti-dual of $\mathcal{H}_f^{\omega,s}$ be $\left(\mathcal{H}_f^{\omega,s}\right)'$ we will define a map $L: \mathcal{H}_f^{\omega,-s} \to \left(\mathcal{H}_f^{\omega,s}\right)'$ as

$$L_v(u) := \langle v, u \rangle = (2\pi)^{-n} \int \hat{v}(\zeta) \, \overline{\hat{u}(\zeta)} \, d\zeta$$

So we will show firstly that $L: v \to L_v$ is bijective. Let $L_v(u) = 0, so \ \langle v, u \rangle = 0 \ and$

$$(2\pi)^{-n} \int \hat{v}(\zeta) \, \overline{\hat{u}(\zeta)} d\zeta = 0$$

This implies

$$(2\pi)^{-n} \int e^{-s\omega_f(\zeta)} \hat{v}(\xi) \, e^{s\omega_f(\zeta)} \overline{\hat{u}(\zeta)} \, d\zeta = 0$$

and

$$(2\pi)^{-n} \int e^{-s\omega_f(\zeta)} \hat{v}(\zeta) \, \psi(\zeta) \, d\zeta = 0 \ \ for \ all \ \psi \in S_\omega$$

$so, v = 0$ in $\acute{S}_\omega and \ v = 0$ in $\mathcal{H}_f^{\omega,-s}$. So, L is one to one. Then we will show that L is surjective. Let $\psi \in \left(\mathcal{H}_f^{\omega,s}\right)' \ and \ \ \psi_1 \in \mathcal{H}_f^{\omega,s}$ we need to reach to $\psi_2 \in \mathcal{H}_f^{\omega,-s}$ such that $L_{\psi_2} = \psi$. So from Resize representation theorem we have, $\psi(u) = \langle\langle \psi_1, u \rangle\rangle_f^{\omega,s} \ for \ all \ u \in \mathcal{H}_f^{\omega,s}$. From the continuous linear function on s_ω then there exists $\psi_2 \in \mathcal{H}_f^{\omega,-s}$such that $\hat{\psi}_2(\zeta) = e^{2s\omega_f(\zeta)}\hat{\psi}_1(\zeta)$ at most, so this leads to

$$\begin{aligned}\psi(u) &= \langle\langle\psi_1,u\rangle\rangle_f^{\omega,s}\\ &= (2\pi)^{-n}\int e^{2s\omega_f(\zeta)}\hat{\psi}_1(\zeta)\,\overline{\hat{u}(\zeta)}\,d\zeta\\ &= (2\pi)^{-n}\int e^{2s\omega_f(\zeta)}\,e^{-2s\omega_f(\zeta)}\hat{\psi}_2(\zeta)\overline{\hat{u}(\zeta)}\,d\zeta\\ &= \langle\psi_2,u\rangle = L_{\psi_2}(u) \qquad for\ all\ \ u \in \mathcal{H}_f^{\omega,s}\end{aligned}$$

Hence,

$$\psi = L_{\psi_2}$$

So L is surjective. Next we will show the isometry of . Let $u \in \mathcal{H}_f^{\omega,s}$ and $v \in \mathcal{H}_f^{\omega,-s}$ such that

$$\hat{u}(\zeta) = e^{-2s\omega_f(\zeta)}\hat{v}(\zeta)$$

and

$$\begin{aligned}L_v(u) &= (2\pi)^{-n}\int \hat{u}(\zeta)\,\overline{\hat{v}(\zeta)}\,d\zeta\\ &= (2\pi)^{-n}\int e^{-2s\omega_f(\zeta)}\hat{v}(\zeta)\,\overline{\hat{v}(\zeta)}\,d\zeta\\ &= (2\pi)^{-n}\int e^{-2s\omega_f(\zeta)}|\hat{v}(\zeta)|^2\,d\zeta\\ &= \left[\|v\|_f^{\omega's}\right]^2\\ &= \ \|v\|_f^{\omega,-s}\|u\|_f^{\omega,s}\end{aligned}$$

Which means that $L_v$is isometry from $\mathcal{H}_f^{\omega,-s}$ to $\left(\mathcal{H}_f^{\omega,s}\right)'$.

The second part of the proof is that if $u \in \acute{D}_f^\omega$ then $u \in \mathcal{H}_f^{\omega,s}, so\ there\ exists\ a\ constant\ c$ such that

$$|u(\phi)| \le c\ \|\phi\|_f^{\omega,-s}.$$

So we will assume that $\in \acute{D}_f^\omega$, then $u \in \mathcal{H}_f^{\omega,s}$ and we want to prove that

$$|u(\phi)| \le c\ \|\phi\|_f^{\omega,-s}.$$

We have that

$$|L_u(\phi)| = |\langle u,\phi\rangle|\ \le \|u\|_f^{\omega,s}\|\phi\|_f^{\omega,-s} \le c\ \|\phi\|_f^{\omega,-s}$$

This implies

$$|u(\phi)| \le c\ \|\phi\|_f^{\omega,-s}.$$

Conversely, let $u \in \acute{D}_f^{\omega}$ and $|u(\phi)| \leq c\,\|\phi\|_f^{\omega,-s}$, we want to prove that $u \in \mathcal{H}_f^{\omega,s}$. Where $for\ all\ \ \phi \in D_f^{\omega}$ the map $\phi \mapsto u(\bar{\phi})$ can be extended uniquely to an element of a conjugate linear functional on $\mathcal{H}_f^{\omega,-s}$, with a bounded norm. So $there\ exists\ \psi \in \mathcal{H}_f^{\omega,s}$in sense that $L_{\psi}(\phi) = \langle\langle\psi,\phi\rangle\rangle = u(\bar{\phi}) = \langle u,\phi\rangle$. So $\psi = u\ and\ u \in \mathcal{H}_f^{\omega,s}$ which complete the proof.

4. Reproducing kernel Hilbert space $\mathcal{A}_F$

Let F be a continuous double completely monotone function on $\mathbb{R}^d$, set $F_y(x) := F(x - y)$ for all $x, y \in \mathbb{R}^d$. Define:

$$(\varphi * F)(x) := \int_{\mathbb{R}^d} \varphi(y)F_y(x)dy\ , \varphi \in S^d \tag{4.1}$$

and

$$\langle\, \varphi * F, \psi * F\,\rangle_{\mathcal{A}_F} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x - y)\varphi(x)\psi(y)dxdy \tag{4.2}$$

for all $\varphi, \psi \in S^d$. Then $\varphi * F;\ \varphi \in S^d$ forms a pre-Hilbert space $\mathcal{A}_F$ with inner-product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{A}_F}$

Lemma 4.1. A function $\varphi\ *\ F$ is in $\mathcal{A}_F$ if and only if $\widehat{\varphi}\ \in\ L^2(\mu)$ and

$$\|\varphi * F\|_{\mathcal{A}_F}^2 = \int_{\mathbb{R}^d} |\hat{\varphi}(\zeta)|^2\, d\mu(\zeta) \tag{4.3}$$

where μ is the tempered measure.

Proof. The first statement is obvious from the previous definitions in section 4., so we will prove (4.3) . Where we have that F is a continuous double completely monotone function, so we can use Bernstein's theorem

$$F(x) = \int_{\mathbb{R}^d} e^{-x\zeta}\, d\mu(\zeta)$$

So we have that, with $\phi \in S^d$ (Schwartz space on $\mathbb{R}^d$)

$$\begin{aligned}\int_{\mathbb{R}^d} F(x)\,\phi(x)\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-x\zeta}\,\phi(x)\,d\mu(\zeta)\,dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-x\zeta}\,\phi(x)dx\,d\mu(\zeta) \\ &= \int_{\mathbb{R}^d} \hat{\phi}(\zeta)\,d\mu(\zeta)\end{aligned}$$

And so we can conclude that

$$\|\phi * F\|^2_{\mathcal{A}_F} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)\,\phi(x)\overline{\phi(y)}\;dx\,dy$$

$$= \int_{\mathbb{R}^d} \left|\hat{\phi}(\zeta)\right|^2 d\mu(\zeta)$$

Theorem 4.2. Let $k \in N$, and $\Lambda \in D^k$ and set$F_\Lambda = \sum_{\zeta\in\Lambda} e^{-i\zeta x}$,$x \in \mathbb{R}^k$; as a tempered completely monotone distribution, and let$\mathcal{A}_{F_\Lambda}$be the generalized RKHS of Schwartz. Then a function h on $\mathbb{R}^k$is in $\mathcal{A}_{F_\Lambda}$if and only if it has a convolution factorization

$$h = \varphi * F_\Lambda$$

where ϕ is a measurable function such that $\hat{\phi}(\lambda)$ exists for all $\lambda \in \Lambda$, and $\hat{\phi}(\lambda)$, $\lambda \in \Lambda$ belongs to $l_2(\Lambda)$ and

$$\|h\|^2_{\mathcal{A}_F} = \sum_{\zeta\in\Lambda}\left|\hat{\phi}(\zeta)\right|^2 \qquad (4.4)$$

Proof. We have that $F_\Lambda(x) = \sum_{\zeta\in\Lambda} e^{-ix\zeta}$, $x \in \mathbb{R}^k$ as a tempered completely monotone distributions , we will prove that

$$\|\phi * F_\Lambda\|^2_{\mathcal{A}_F} = \sum_{\zeta\in\Lambda}\left|\hat{\phi}(\zeta)\right|^2$$

Where $\phi \in S^k$(the Schwartz space on $\mathbb{R}^k$) , $\hat{\phi}$ is the standard Fourier transform , from (4.1),

$$(\phi * F_\Lambda)(x) = \int_{\mathbb{R}^k} \phi(y) F_\Lambda(x-y)\,dy$$

$$= \int_{\mathbb{R}^k} \phi(y) \sum_{\zeta\in\Lambda} e^{-i\zeta(x-y)}\,dy$$

$$= \sum_{\zeta\in\Lambda} \int_{R^k} \phi(y) e^{-i\zeta(x-y)}\,dy$$

$$= \sum_{\zeta\in\Lambda} \int_{\mathbb{R}^k} \phi(y)\, e^{i\zeta y}\,dy\; e^{-i\zeta x}$$

$$= \sum_{\zeta\in\Lambda} \int_{\mathbb{R}^k} \overline{\phi(y) e^{-\imath\zeta y}}\,dy\; e^{-i\zeta x}$$

Hence

$$(\phi * F_\Lambda)(x) = \sum_{\zeta\in\Lambda} \overline{\hat{\phi}(\zeta)}\, e^{-i\zeta x}$$

And so,

$$\|\phi * F_\Lambda\|^2_{\mathcal{A}_F} = \langle \phi * F_\Lambda , \phi * F_\Lambda \rangle_{\mathcal{A}_{F_\Lambda}}$$
$$= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} F_\Lambda(x-y)\, \phi(x)\overline{\phi(y)}\ dx\, dy$$
$$= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \sum_{\zeta\in\Lambda} e^{-i\zeta(x-y)}\, \phi(x)\overline{\phi(y)}\ dx\, dy$$

Using Fubini's theorem:

$$\sum_{\zeta\in\Lambda} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} e^{-i\zeta(x-y)}\, \phi(x)\overline{\phi(y)}\ dx\, dy = \sum_{\zeta\in\Lambda} \left[\int_{\mathbb{R}^k} e^{-i\zeta x}\phi(x)\ dx \int_{\mathbb{R}^k} e^{i\zeta y}\overline{\phi(y)}\ dy \right]$$
$$= \sum_{\zeta\in\Lambda} \left[\int_{\mathbb{R}^k} e^{-i\zeta x}\phi(x)\ dx \int_{\mathbb{R}^k} \overline{e^{-i\zeta y}\phi(y)}\ dy \right]$$
$$= \sum_{\zeta\in\Lambda} \hat{\phi}(\zeta)\, \overline{\hat{\phi}(\zeta)}$$
$$= \sum_{\zeta\in\Lambda} \left|\hat{\phi}(\zeta)\right|^2$$

So,

$$\|\phi * F_\Lambda\|^2_{\mathcal{A}_F} = \sum_{\zeta\in\Lambda} \left|\hat{\phi}(\zeta)\right|^2$$

5. Concluding Remarks

In this work, we introduced and gave the main properties of the class of double monotone functions defined on $S(Q)$. Moreover, the main properties of the class of weak monotone functions defined on $S(Q)$ are given.Finally, a novel method for creating spaces of generalized functions are given. Tempered distributions refer to the set of all continuous linear functional on $S(Q)$, and it is represented by the symbol $\acute{S}(Q)$. suppose $l \in \acute{S}(Q)$ and $\alpha \in \mathbb{Z}^d_+$. The weak derivative $D^\alpha l$, often known as the derivative of the sense of distributions, is obtained by

$$(D^\alpha l)(f) = (-1)^\alpha\, l(D^\alpha f)$$

for $f \in (Q)$. This corresponds to $D^\alpha l\{g\} = l\{D^\alpha g\}$. Noting thatdistributions are always weakly derivative. If assume that $Q = \mathbb{R}^n$. So, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Let x^α be denote the product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\mathbb{Z}^n_+$ represents a set of n-tuples.$(\alpha_1, \cdots, \alpha_n)$where each α_i is an integer that is not negative , $|\alpha| = \sum_{i=1}^n \alpha_i$and D^αdenote the partial differential operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ .The particular case, which follows the space of rapidly decreasing function on $\mathbb{R}^n$ is denoted as $S(Q) = S(\mathbb{R}^n)$ (also known as the Schwartz space), and its dual space of a tempered distribution on $\mathbb{R}^n$ is denoted as $\acute{S}(Q) = \acute{S}(\mathbb{R}^n)$.

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Numerical investigation of zeros of the fully modified (p, q)-poly-Euler polynomials

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The aim of this paper is to introduce a fully modified (p,q)-poly-Euler polynomials and numbers of the first type. We investigate some properties that is related with (p, q)-Gaussian binomials coefficients. We also construct (p, q)-analogue of the Stirling numbers of the second kind and fully modified (p,q)-poly-Euler polynomials and numbers of the first type with two variables.

1 Introduction

Many researchers are interested in the applications of q-numbers and (p, q)-numbers. In areas of quantum mechanics, physics and mathematics, the applying theory is studied and extended actively. Especially, Mathematicians in the fields of combinatorics, number theory and special functions, frequently explorer that(cf [2], [3], [4], [7], [8],[9], [10], [11], [12], [13]). We also obtain the generalization of poly Bernoulli polynomials and poly tangent polynomials involving (p, q)-numbers. In this paper, we use the following notations. $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of complex numbers, respectively.

For $0 < q < p \leq 1$, the (p, q)-numbers are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q},$$

where $p = 1, [n]_{p,q} = [n]_q$ and $\lim_{q \to 1}[n]_q = n$.

The (p, q)-factorial of n of order k is defined as

$$[n]_{p,q}^{(\underline{k})} = [n]_{p,q}[n-1]_{p,q} \cdots [n-k+1]_{p,q},$$

for $k = 1, 2, 3, \cdots$. If $k = n$, it is denoted $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}$ that is called (p, q)-factorial of n.

The (p,q)-Gaussian binomial formula is defined by

$$(x+a)_{p,q}^{n}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}}a^{n-k}x^{k},$$

with the (p,q)-Gaussian binomial coefficient, $\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$ $(n\geq k)$.

In [13] , two type of the (p,q)-exponential functions are given as below

$$\begin{aligned}e_{p,q}(x)&=\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{x^n}{[n]_{p,q}!},\\E_{p,q}(x)&=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{x^n}{[n]_{p,q}!}.\end{aligned}\tag{1.1}$$

In [8], [10], the (p,q)-analogue of polylogarithm function $Li_{k,p,q}$ is known by

$$Li_{k,p,q}(x)=\sum_{n=1}^{\infty}\frac{x^n}{[n]_{p,q}^{k}},(k\in\mathbb{Z}).$$

In [5], we defined the fully modified q-poly-Bernoulli polynomials $\widetilde{B}_{n,q}^{(k)}(x)$ of the first type and the fully modified q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x)$ of the first type.

Definition 1.1. For $n\in\mathbb{Z}_+,k\in\mathbb{Z}$ and $0<q<1$, we define fully modified q-poly-Bernoulli polynomials $\widetilde{B}_{n,q}^{(k)}(x)$ of the first type and the fully modified q-poly-tangent polynomials $\widetilde{T}_{n,q}^{(k)}(x)$ of the first type by

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!}&=\frac{Li_{k,q}(1-e_q(-t))}{(e_q(t)-1)}e_q(xt),\\\sum_{n=0}^{\infty}\widetilde{E}_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!}&=\frac{[2]_qLi_{k,q}(1-e_q(-t))}{t(e_q(t)+1)}e_q(xt).\end{aligned}\tag{1.2}$$

When $x=0$, $\widetilde{B}_{n,q}^{(k)}=\widetilde{B}_{n,q}^{(k)}(0)$, $\widetilde{E}_{n,q}^{(k)}=\widetilde{E}_{n,q}^{(k)}(0)$ are called fully modified q-poly-Bernoulli numbers of the first type and fully modified q-poly-Euler numbers of the first type. If $q\to1$ in (1.2), we get the poly-Bernoulli polynomials $B_n^{(k)}(x)$ and poly-Euler polynomials $E_n^{(k)}(x)$, respectively.

Substitute $k=1,q\to1$ in (1.2), we have Bernoulli polynomials $B_n(x)$and Euler polynomials $E_n(x)$, respectively.

$$\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}=\left(\frac{t}{e^t-1}\right)e^{xt},\qquad\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=\left(\frac{2}{e^t+1}\right)e^{xt}.$$

2

2 Some properties of the fully modified (p,q)-poly-Euler polynomials of the first type

In this section, we introduce fully modified (p,q)-poly-Euler numbers and polynomials of the first type by the generating functions. We explore some identities of the polynomials and find a relation connected with (p,q)-analogue of the ordinary Euler polynomials.

Definition 2.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}, p, q \in \mathbb{R}$ such that $0 < q < p \leq 1$, we define a fully modified (p,q)-poly-tangent polynomials $\widetilde{E}^{(k)}_{n,p,q}(x)$ of the first type by

$$\sum_{n=0}^{\infty} \widetilde{E}^{(k)}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q} Li_{k,p,q}(1 - e_{p,q}(-t))}{t(e_{p,q}(t)+1)} e_{p,q}(xt).$$

When $x = 0$, $\widetilde{E}^{(k)}_{n,p,q} = \widetilde{E}^{(k)}_{n,p,q}(0)$ are called fully modified (p,q)-poly-Euler numbers of the first type. Note that $p = 1, [n]_{p,q} = [n]_q$, and $\widetilde{E}^{(k)}_{n,p,q}(x) = \widetilde{E}^{(k)}_{n,q}(x)$. If we set $k = 1, p = 1, q \to 1$ in Definition 2.1, then the Euler polynomials $E_n(x)$.

Theorem 2.2. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $p, q \in \mathbb{R}$ such that $0 < q < p \leq 1$, the following result holds*

$$\widetilde{E}^{(k)}_{n,p,q}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} p^{\binom{n-l}{2}} \widetilde{E}^{(k)}_{l,p,q} x^{n-l}.$$

In [5], the generating series of (p,q)-Stirling numbers of the second kind is defined by

$$\frac{(e_{p,q}(t)-1)^m}{[m]_{p,q}!} = \sum_{n=m}^{\infty} S_{p,q}(n,m) \frac{t^n}{[n]_{p,q}!}.$$

We also ontain

$$\frac{Li_{k,p,q}(1 - e_{p,q}(-t))}{t} = \sum_{n=0}^{\infty} \sum_{l=1}^{n+1} \frac{[l]_{p,q}!}{[l]^k_{p,q}[n+1]_{p,q}} (-1)^{l+n+1} S_{p,q}(n+1,l) \frac{t^n}{[n]_{p,q}!}.$$

Using the above identity, we derive the following result which is connected with (p,q)-Stirling numbers of the second kind and (p,q)-Euler polynomials.

Theorem 2.3. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $p, q \in \mathbb{R}$ such that $0 < q < p \leq 1$, the following identity holds*

$$\widetilde{E}^{(k)}_{n,p,q}(x) = \sum_{a=0}^{n} \sum_{l=1}^{a+1} \begin{bmatrix} n \\ a \end{bmatrix}_{p,q} \frac{[l]_{p,q}!}{[l]^k_{p,q}[a+1]_{p,q}} (-1)^{l+a+1} S_{p,q}(a+1,l) E_{n-a,p,q}(x).$$

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Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < p \leq 1$. By the recomposition of (p,q)-polylogarithm function in (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{E}_{n,p,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(t)+1)}e_{p,q}(xt)\\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+l}[l]_{p,q}!}{[l]_{p,q}^k[n+1]_{p,q}}S_{p,q}(n+1,l)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\\ &= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n\\ a\end{bmatrix}_{p,q}\frac{(-1)^{l+a+1}[l]_{p,q}!}{[l]_{p,q}^k[a+1]_{p,q}}S_{p,q}(a+1,l)E_{n-a,p,q}(x)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Comparing the coefficient both sides, we get

$$\widetilde{E}_{n,p,q}^{(k)}(x) = \sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n\\ a\end{bmatrix}_{p,q}\frac{[l]_{p,q}!}{[l]_{p,q}^k[a+1]_{p,q}}(-1)^{l+a+1}S_{p,q}(a+1,l)E_{n-a,p,q}(x).$$

Now, we introduce fully modified (p,q)-poly-Euler polynomials of the first type with two variables by using two generating functions.

Definition 2.4. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}, p,q \in \mathbb{R}$ and $0 < q < p \leq 1$, the fully modified (p,q)-poly-Euler polynomials $\widetilde{E}_{n,p,q}^{(k)}(x,y)$ of the first type with two variables by

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt).$$

Theorem 2.5. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. Then we have the addition theorem.*

$$\widetilde{E}_{n,p,q}^{(k)}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}\widetilde{E}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}.$$

Proof. Let n be a nonnegative integer and $k \in \mathbb{Z}$. Then we get

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{E}_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!} &= \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}\widetilde{E}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Thus, we have

$$\widetilde{E}_{n,p,q}^{(k)}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}\widetilde{E}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}.$$

Theorem 2.6. *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $p,q \in \mathbb{R}$ such that $0 < q < p \leq 1$. We have*

$$\widetilde{E}_{n,p,q}^{(k)}(x,y) - \widetilde{E}_{n,p,q}^{(k)}(x) = \sum_{l=0}^{n-1}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}q^{\binom{n-l}{2}}\ \widetilde{E}_{l,p,q}^{(k)}(x)\ y^{n-1}.$$

4

3 Distribution of zeros of the fully modified (p, q)-poly-Euler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the fully modified (p, q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,p,q}(x)$. The fully modified (p, q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{1,p,q}(x)$ can be determined explicitly. A few of them are

$$\widetilde{E}^{(k)}_{0,p,q}(x) = \frac{p+q}{2},$$

$$\widetilde{E}^{(k)}_{1,p,q}(x) = -\frac{3p}{4} - \frac{q}{4} + \frac{p+q}{2[2]^k_{p,q}} + \frac{px+qx}{2},$$

$$\begin{aligned}
\widetilde{E}^{(k)}_{2,p,q}(x) = {} & \frac{p^3}{8(p-q)} + \frac{p^2q}{8(p-q)} - \frac{pq^2}{8(p-q)} - \frac{q^3}{8(p-q)} \\
& - \frac{p^4}{4(p-q)^2[2]^k_{p,q}} + \frac{p^2q^2}{2(p-q)^2[2]^k_{p,q}} - \frac{q^4}{4(p-q)^2[2]^k_{p,q}} - \frac{p^5}{(p-q)^2(p+q)[2]^k_{p,q}} \\
& + \frac{2p^3q^2}{(p-q)^2(p+q)[2]^k_{p,q}} - \frac{pq^4}{(p-q)^2(p+q)} + \frac{p^5}{2(p-q)(p^2+pq+q^2)} \\
& - \frac{p^3q^2}{2(p-q)(p^2+pq+q^2)} + \frac{p^7}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}} \\
& - \frac{p^5q^2}{(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}} - \frac{p^4q^3}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}} \\
& + \frac{p^3q^4}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}} + \frac{p^2q^5}{(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}} \\
& - \frac{q^7}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}} - \frac{3p^3x}{4(p-q)} - \frac{p^2qx}{4(p-q)} \\
& + \frac{3pq^2x}{4(p-q)} + \frac{q^3x}{4(p-q)} + \frac{p^4x}{2(p-q)^2[2]^k_{p,q}} - \frac{p^2q^2x}{(p-q)^2[2]^k_{p,q}} \\
& + \frac{q^4x}{2(p-q)^2)[2]^k_{p,q}} + \frac{p^3x^2}{2(p-q)} - \frac{pq^2x^2}{2(p-q)}.
\end{aligned}$$

We investigate the zeros of the fully modified (p,q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,p,q}(x)$ by using a computer. We plot the zeros of the (p,q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,p,q}(x)$ for $n = 20$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we

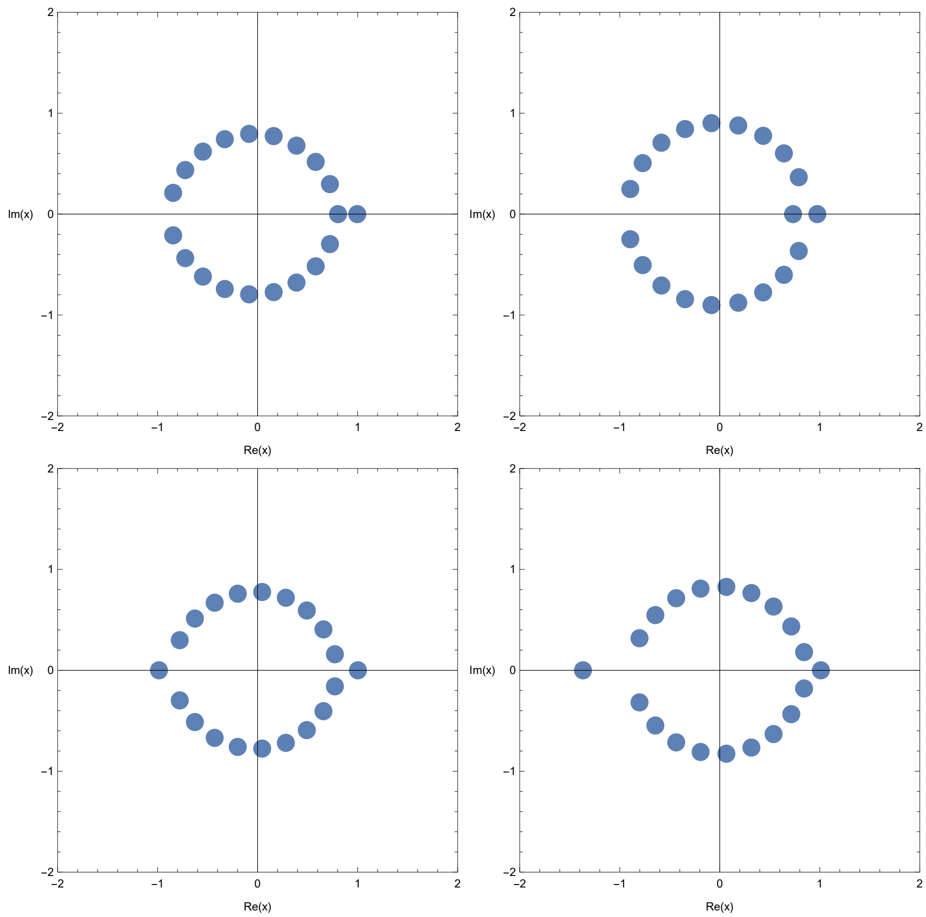

Figure 1: Zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$

choose $n = 20, p = 9/10, q = 1/10$, and $k = 1$. In Figure 1(top-right), we choose $n = 20, p = 9/10, q = 1/10$, and $k = 5$. In Figure 1(bottom-left), we choose $n = 20, p = 9/10, q = 1/10$, and $k = -1$. In Figure 1(bottom-right), we choose $n = 20, p = 9/10, q = 1/10$, and $k = -5$.

Stacks of zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$ for $1 \le n \le 20$ from a 3-D structure are presented(Figure 3). In Figure 3(top-left), we choose $p = 9/10, q = 1/10$, and

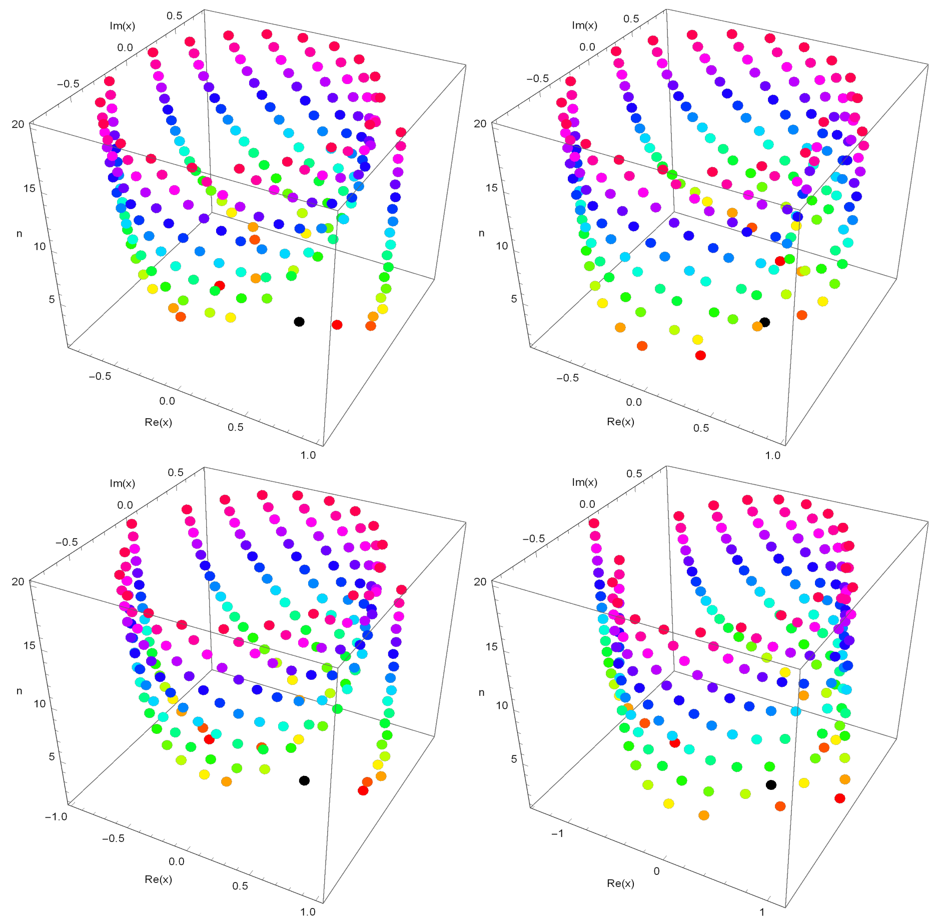

Figure 2: Stacks of zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$ for $1 \le n \le 20$

$k = 1$. In Figure 3(top-right), we choose $p = 9/10, q = 1/10$, and $k = 5$. In Figure 3(bottom-left), we choose $p = 9/10, q = 1/10$, and $k = -1$. In Figure 3(bottom-right), we choose $p = 9/10, q = 1/10$, and $k = -5$.

The plot of real zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$ for $1 \le n \le 20$ structure are presented(Figure 4).

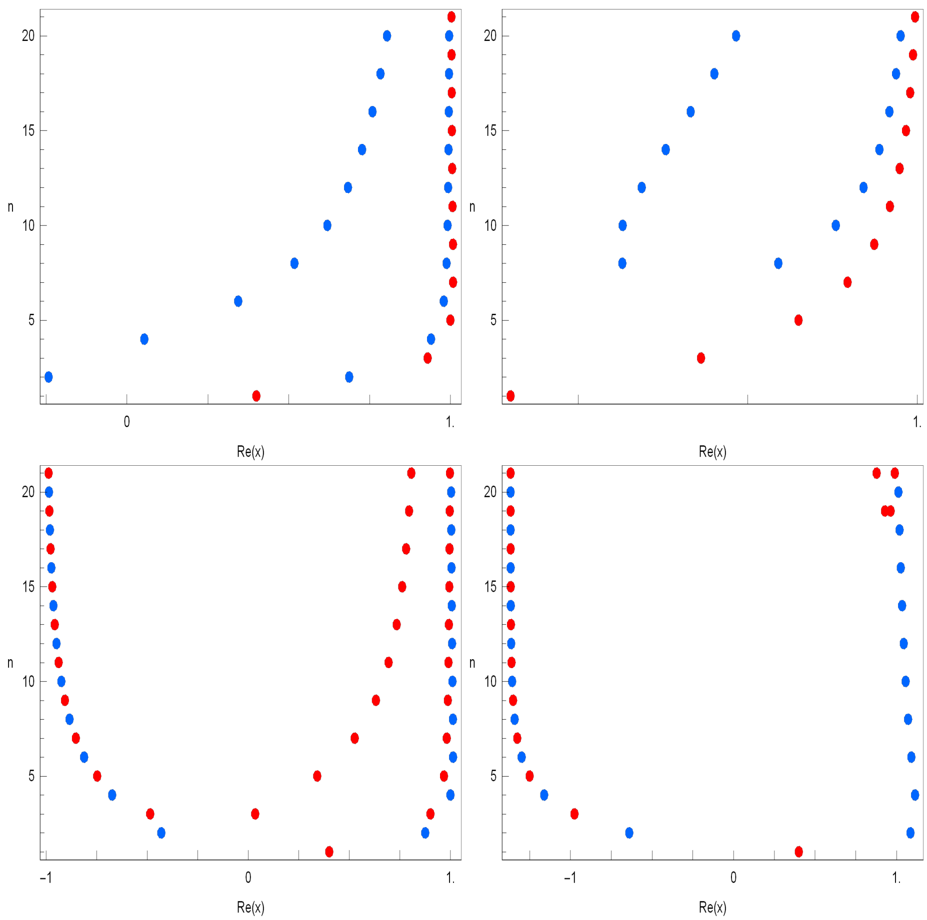

Figure 3: Real zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$ for $1 \le n \le 20$

In Figure 4(top-left), we choose $p = 9/10, q = 1/10$, and $k = 1$. In Figure 4(top-right), we choose $p = 9/10, q = 1/10$, and $k = 5$. In Figure 4(bottom-left), we choose $p = 9/10, q = 1/10$, and $k = -1$. In Figure 4(bottom-right), we choose $p = 9/10, q = 1/10$, and $k = -5$.

Next, we calculated an approximate solution satisfying (p,q)-poly-Eulert polynomials $\widetilde{E}_{n,p,q}^{(k)}(x) = 0$ for $x \in \mathbb{C}$. The results are given in Table 1 and Table 2.

Table 1. Approximate solutions of $\widetilde{E}_{n,p,q}^{(-5)}(x) = 0, p = 9/10, q = 1/10$

degree n	x
1	0.40000
2	$-0.64015, \quad 1.0846$
3	$-0.97595, \quad 0.71267 - 0.32117i, \quad 0.71267 + 0.32117i$
4	$-1.1619, \quad 0.24937 - 0.68250i, \quad 0.24937 + 0.68250i,$ 1.1131
5	$-1.2512, \quad -0.01718 - 0.76730i, \quad -0.01718 + 0.76730i,$ $0.86775 - 0.23513i, \quad 0.86775 + 0.23513i$
6	$-1.2998, \quad -0.22150 - 0.76057i, \quad -0.22150 + 0.76057i,$ $0.55131 - 0.56698i, \quad 0.55131 + 0.56698i, \quad 1.0902$

Table 2. Approximate solutions of $\widetilde{E}_{n,p,q}^{(5)}(x) = 0, p = 9/10, q = 1/10$

degree n	x
1	0.40000
2	$0.22222 - 0.58608i \quad 0.22222 + 0.58608i$
3	$-0.11575 - 0.76525i, \quad -0.11575 + 0.76525i, \quad 0.68089$
4	$-0.28591 - 0.75902i, \quad -0.28591 + 0.75902i,$ $0.51087 - 0.33499i, \quad 0.51087 + 0.33499i$
5	$-0.46789 - 0.68975i, \quad -0.46789 + 0.68975i,$ $0.28053 - 0.71282i, \quad 0.28053 + 0.71282i, \quad 0.82471$
6	$-0.54973 - 0.64739i, \quad -0.54973 + 0.64739i,$ $0.15514 - 0.78522i, \quad 0.15514 + 0.78522i,$ $0.61959 - 0.14392i, \quad 0.61959 + 0.14392i$

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Parametrized Gudermannian function induced Banach space valued ordinary and fractional neural networks approximations

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Abstract

Here we examine the univariate quantitative approximation, ordinary and fractional, of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative or fractional derivatives. Our operators are defined by using a density function generated by a parametrized Gudermannian sigmoid function. The approximations are pointwise and of the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

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1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact suport. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

The author inspired by [15], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3] - [7], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [8].

In this article we are greatly inspired by the related works [16], [17].

The author here performs parametrized Gudermannian function based neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with values to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X-valued high order derivative, or X-valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by a parametrized Gudermannian sigmoid function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \, s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation function is derived by the Gudermannian sigmoid functions. About neural networks in general read [18], [19], [21]. See also [9] for a complete study of real valued approximation by neural network operators.

2

2 Background

Here we consider the Gudermannian function ([23]) $gd(x)$ which is defined as follows

$$gd(x) := \int_0^x \frac{dt}{\cosh t} = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right), \forall\, x \in \mathbb{R}. \tag{1}$$

Let $\lambda > 0$, then

$$gd(\lambda x) = \int_0^{\lambda x} \frac{dt}{\cosh t} = 2\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right). \tag{2}$$

We will use the following normalized and parametrized function

$$f_\lambda(x) := \frac{2}{\pi} gd(\lambda x) = \frac{4}{\pi}\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right) = \tag{3}$$

$$\frac{2}{\pi}\int_0^{\lambda x} \frac{dt}{\cosh t} = \frac{4}{\pi}\int_0^{\lambda x} \frac{dt}{e^t + e^{-t}},\ x \in \mathbb{R}$$

We will prove that f_λ is a generator sigmoid function with the general properties as in [14]. When $0 < \lambda < 1$, f_λ is expected to outperform ReLu and Leaky ReLu activation functions.

We notice that

$$\left(\frac{2}{\pi} gd(x)\right)' = \frac{2}{\pi \cosh x} > 0,$$

and

$$f_\lambda'(x) = \left(\frac{2}{\pi} gd(\lambda x)\right)' = \frac{2\lambda}{\pi \cosh \lambda x} > 0, \quad \forall\, x \in \mathbb{R}. \tag{4}$$

Hence f_λ is strictly increasing on $\mathbb{R}$.

Furthermore we have

$$f_\lambda''(x) = -\frac{2\lambda^2}{\pi} \frac{\sinh \lambda x}{(\cosh \lambda x)^2}, \quad \forall\, x \in \mathbb{R}. \tag{5}$$

Notice that

$$f_\lambda''(x) > 0 \ \text{ for } x < 0, \text{ and}$$

$$f_\lambda''(x) < 0 \ \text{ for } x > 0, \text{ and}$$

$$f_\lambda''(0) = 0.$$

Therefore f_λ is stritly concave up for $x < 0$, and f_λ is striclty concave down for $x > 0$, and $f_\lambda(0) = 0$, with $(0,0)$ the inflection point.

Let $x \to +\infty$, then $\tanh\left(\frac{\lambda x}{2}\right) \to 1$ and $\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right) \to \frac{\pi}{4}$. Let $x \to -\infty$, then $\tanh\left(\frac{\lambda x}{2}\right) \to -1$ and $\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right) \to -\frac{\pi}{4}$.

Clearly, then $f_\lambda(+\infty) = 1$ and $f_\lambda(-\infty) = -1$, so that $y = \pm 1$ are horizontal asymptotes for f_λ.

Also it is $f_\lambda(x) \geq 0$ for $x \geq 0$, and $f_\lambda(x) < 0$ for $x < 0$. Obviously then $f_\lambda : \mathbb{R} \to [-1,1]$, with $f''_\lambda \in C(\mathbb{R})$.

Notice that $\tanh(-x) = -\tanh x$ and $\arctan(-x) = -\arctan x$, $x \in \mathbb{R}$.

We have that

$$f_\lambda(-x) = \frac{4}{\pi}\arctan\left(\tanh\left(-\frac{\lambda x}{2}\right)\right) = \frac{4}{\pi}\arctan\left(-\tanh\left(\frac{\lambda x}{2}\right)\right) =$$

$$-\frac{4}{\pi}\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right) = -f_\lambda(x),$$

proving

$$f_\lambda(-x) = -f_\lambda(x), \quad \forall\, x \in \mathbb{R}. \tag{6}$$

So, indeed, f_λ is a sigmoid function as in [14].

So, all the theory of [14] applies here for f_λ, etc.

We consider the activation function

$$\psi(x) := \frac{1}{4}\left(f_\lambda(x+1) - f_\lambda(x-1)\right), \quad x \in \mathbb{R}, \tag{7}$$

As in [13], p. 285, and [14], we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x+1 > x-1$, then $f_\lambda(x+1) > f_\lambda(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{f_\lambda(1)}{2} = \frac{gd(\lambda)}{\pi}. \tag{8}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(f'_\lambda(x+1) - f'_\lambda(x-1)\right) < 0,$$

by f'_λ being strictly decreasing over $[0,+\infty)$.

Let now $0 < x < 1$, then $1-x > 0$ and $0 < 1-x < 1+x$. It holds $f'_\lambda(x-1) = f'_\lambda(1-x) > f'_\lambda(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is stritly decreasing on $(0,+\infty)$.

Clearly, ψ is strictly increasing on $(-\infty,0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x\to+\infty}\psi(x) = \frac{1}{4}\left(f_\lambda(+\infty) - f_\lambda(+\infty)\right) = 0, \tag{9}$$

and

$$\lim_{x\to-\infty}\psi(x) = \frac{1}{4}\left(f_\lambda(-\infty) - f_\lambda(-\infty)\right) = 0. \tag{10}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) = \frac{gd(\lambda)}{\pi}.$$

We need

Theorem 1 *(by [14]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \quad \forall \ x \in \mathbb{R}. \tag{11}$$

Theorem 2 *(by [14]) It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{12}$$

Thus $\psi(x)$ is a density function on $\mathbb{R}$.

We give

Theorem 3 *(by [14]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) < \frac{\left(1 - f_{\lambda}\left(n^{1-\alpha}-2\right)\right)}{2} = \frac{\left(\pi - 2gd\left(\lambda\left(n^{1-\alpha}-2\right)\right)\right)}{2\pi}. \tag{13}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(\pi - 2gd\left(\lambda\left(n^{1-\alpha}-2\right)\right)\right)}{2\pi} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We further give

Theorem 4 *(by [14]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < \frac{1}{\psi(1)} = \frac{4}{f_{\lambda}(2)} = \frac{2\pi}{gd(2\lambda)}, \quad \forall \ x \in [a,b]. \tag{14}$$

Remark 5 *(by [14]) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1, \tag{15}$$

for at least some $x \in [a,b]$.

See also [13], p. 290, same reasoning.

Note 6 *For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (11))*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \leq 1. \tag{16}$$

5

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 7 *Let $f \in C([a,b], X)$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X-valued linear neural network operators*

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}, \quad x \in [a,b]. \tag{17}$$

Clearly here $A_n(f,x) \in C([a,b], X)$. For convenience we use the same A_n for real valued function when needed. We study here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

For convenience also we call

$$A_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k), \tag{18}$$

(similarly A_n^* can be defined for real valued function) that is

$$A_n(f,x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}. \tag{19}$$

So that

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} - f(x)$$

$$= \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}. \tag{20}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\| \leq \frac{2\pi}{gd(2\lambda)} \left\| A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right) \right\|. \tag{21}$$

That is

$$\|A_n(f,x) - f(x)\| \leq \frac{2\pi}{gd(2\lambda)} \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) \psi(nx-k) \right\|. \tag{22}$$

We will estimate the right hand side of (22).

For that we need, for $f \in C([a,b], X)$ the first modulus of continuity

$$\omega_1(f,\delta)_{[a,b]} := \omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \tag{23}$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X-valued) and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a,b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$, see [11].

Definition 8 *When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define*

$$\overline{A}_n(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx - k), \quad n \in \mathbb{N}, \ x \in \mathbb{R}, \tag{24}$$

the X-valued quasi-interpolation neural network operator.

Remark 9 *(by [14]) We have that the series $\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx - k)$ is absolutely convergent in X, hence it is convergent in X and $\overline{A}_n(f, x) \in X$.*

We denote by $\|f\|_\infty := \sup_{x \in [a,b]} \|f(x)\|$, for $f \in C([a,b], X)$, similarly is defined for $f \in C_B(\mathbb{R}, X)$.

3 Main Results

We present a series of X-valued neural network approximations to a function given with rates.

We first give

Theorem 10 *Let $f \in C([a,b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

i)

$$\|A_n(f, x) - f(x)\| \leq \frac{2\pi}{gd(2\lambda)} \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty\right] =: \rho, \tag{25}$$

and

ii)

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{26}$$

We notice $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right)\right)$.

Proof. As similar to [13], p. 293 is omitted, see also [14]. ∎

Next we give

Theorem 11 *Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then*

i)

$$\left\| \overline{A}_n(f,x) - f(x) \right\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty =: \mu, \tag{27}$$

and

ii)

$$\left\| \overline{A}_n(f) - f \right\|_\infty \leq \mu. \tag{28}$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n\to\infty} \overline{A}_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right)\right)$.

Proof. As similar to [13], p. 294 is omitted, see also [14]. ■

In the next we discuss high order neural network X-valued approximation by using the smoothness of f.

Theorem 12 *Let $f \in C^N([a,b], X)$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\|A_n(f,x) - f(x)\| \leq \frac{2\pi}{gd(2\lambda)} \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}(x)\right\|}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right)}{2} (b-a)^j \right] + \right. \tag{29}$$

$$\left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right) \left\|f^{(N)}\right\|_\infty (b-a)^N}{N!} \right] \right\},$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$\|A_n(f,x_0) - f(x_0)\| \leq \frac{2\pi}{gd(2\lambda)}$$

$$\left\{ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right) \left\|f^{(N)}\right\|_\infty (b-a)^N}{N!} \right\}, \tag{30}$$

and

iii)

$$\|A_n(f) - f\|_\infty \leq \frac{2\pi}{gd(2\lambda)} \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right)}{2} (b-a)^j \right] + \right.$$

$$\left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - f_\lambda\left(n^{1-\alpha} - 2\right)\right) \left\|f^{(N)}\right\|_\infty (b-a)^N}{N!} \right] \right\}. \tag{31}$$

Again we obtain $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.

Proof. As similar to [13], pp. 296-301 is omitted, see also [14]. ■

All integrals from now on are of Bochner type [20].

We need

Definition 13 *([12]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{32}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [22], p. 83), and also set $D_{*a}^{0} f := f$.

By [12], $\left(D_{*a}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, then by [12], $D_{*a}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{*a}^{\alpha} f \right\| \in C([a,b])$.

Definition 14 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b]. \tag{33}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We present the following X-valued fractional approximation result by neural networks.

Theorem 15 *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b], X)$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left\| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x) - f(x) \right\| \leq$$

$$\frac{2\pi}{gd(2\lambda)\,\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

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$$\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}, \tag{34}$$

ii) if $f^{(j)}(x)=0$, *for* $j=1,...,N-1$, *we have*

$$\|A_{n}(f,x)-f(x)\| \leq \frac{2\pi}{gd(2\lambda)\Gamma(\alpha+1)}$$

$$\Bigg\{\frac{\left(\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+$$

$$\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}, \tag{35}$$

iii)

$$\|A_{n}(f,x)-f(x)\| \leq \frac{2\pi}{gd(2\lambda)}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^{j}\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\Bigg\{\frac{\left(\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+$$

$$\left.\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\right\}, \tag{36}$$

$\forall\ x\in[a,b]$,

and

iv)

$$\|A_{n}f-f\|_{\infty} \leq \frac{2\pi}{gd(2\lambda)}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^{j}\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\Bigg\{\frac{\left(\sup_{x\in[a,b]}\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+$$

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$$\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\Bigg\}\Bigg\}. \tag{37}$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain X-valued fractionally type pointwise and uniform convergence with rates of $A_n \to I$ the unit operator, as $n\to\infty$.

Proof. It is very lengthy, as similar to [13], pp. 305-316, is omitted, see also [14]. ■

Next we apply Theorem 15 for $N=1$.

Theorem 16 *Let $0<\alpha,\beta<1$, $f\in C^{1}([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

i)

$$\left\|A_n(f,x)-f(x)\right\|\leq$$

$$\frac{2\pi}{gd(2\lambda)\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{38}$$

and

ii)

$$\left\|A_nf-f\right\|_{\infty}\leq\frac{2\pi}{gd(2\lambda)\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-f_{\lambda}\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \tag{39}$$

When $\alpha=\frac{1}{2}$ we derive

Corollary 17 *Let $0<\beta<1$, $f\in C^{1}([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

i)

$$\left\|A_n(f,x)-f(x)\right\|\leq$$

$$\frac{4\sqrt{\pi}}{gd(2\lambda)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\left(\frac{1-f_\lambda\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}\sqrt{(x-a)}+\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \tag{40}$$

and

ii)

$$\|A_nf-f\|_\infty\leq\frac{4\sqrt{\pi}}{gd(2\lambda)}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\left(\frac{1-f_\lambda\left(n^{1-\beta}-2\right)}{2}\right)\sqrt{(b-a)}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}<\infty. \tag{41}$$

We finish with

Remark 18 *Some convergence analysis follows:*

Let $0<\beta<1$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. *We elaborate on (41). Assume that*

$$\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[a,x]}\leq\frac{K_1}{n^\beta}, \tag{42}$$

and

$$\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,b]}\leq\frac{K_2}{n^\beta}, \tag{43}$$

$\forall\ x\in[a,b]$, $\forall\ n\in\mathbb{N}$, *where* $K_1,K_2>0$.

Then it holds

$$\frac{\left[\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}}\leq$$

$$\frac{\frac{(K_1+K_2)}{n^\beta}}{n^{\frac{\beta}{2}}}=\frac{(K_1+K_2)}{n^{\frac{3\beta}{2}}}=\frac{K}{n^{\frac{3\beta}{2}}}, \tag{44}$$

where $K:=K_1+K_2>0$.

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The other summand of the right hand side of (41), for large enough n, converges to zero at the speed $\left(\frac{1-f_\lambda\left(n^{1-\beta}-2\right)}{2}\right)$.

Then, for large enough $n \in \mathbb{N}$, by (41) and (44) and the last comment, we obtain that

$$\|A_n f - f\|_\infty \leq M \max\left(\frac{1}{n^{\frac{3\beta}{2}}}, \left(\frac{1-f_\lambda\left(n^{1-\beta}-2\right)}{2}\right)\right), \tag{45}$$

where $M > 0$.

If $\frac{1}{n^{\frac{3\beta}{2}}} \geq \left(\frac{1-f_\lambda\left(n^{1-\beta}-2\right)}{2}\right)$, *then* $\frac{1}{n^\beta} \geq \left(\frac{1-f_\lambda\left(n^{1-\beta}-2\right)}{2}\right)$, *and consequently* $\|A_n f - f\|_\infty$ *in (45) converges to zero faster than in Theorem 10. This because the differentiability of f.*

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Optimization of Adams-type difference formulas in Hilbert space $W_2^{(2,1)}(0,1)$

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In this paper, we consider the problem of constructing new optimal explicit and implicit Adams-type difference formulas for finding an approximate solution to the Cauchy problem for an ordinary differential equation in a Hilbert space. In this work, I minimize the norm of the error functional of the difference formula with respect to the coefficients, we obtain a system of linear algebraic equations for the coefficients of the difference formulas. This system of equations is reduced to a system of equations in convolution and the system of equations is completely solved using a discrete analog of a differential operator $d^2/dx^2 - 1$. Here we present an algorithm for constructing optimal explicit and implicit difference formulas in a specific Hilbert space. In addition, comparing the Euler method with optimal explicit and implicit difference formulas, numerical experiments are given. Experiments show that the optimal formulas give a good approximation compared to the Euler method.

Keywords: Hilbert space; initial-value problem; multistep method; the error functional; optimal difference formula.

1 Introduction

It is known that the solutions of many practical problems lead to solutions of differential equations or their systems. Although differential equations have so many applications and only a small number of them can be solved exactly using elementary functions and their combinations. Even in the analytical analysis of differential equations, their application can be inconvenient due to the complexity of the obtained solution. If it is very difficult to obtain or impossible to find an analytic solution to a differential equation, one can find an approximate solution.

In the present paper we consider the problem of approximate solution to the first order linear ordinary differential equation

$$y' = f(x, y),\ x \in [0, 1] \tag{1}$$

with the initial condition

$$y(0) = y_0. \tag{2}$$

We assume that $f(x, y)$ is a suitable function and the differential equation (1) with the initial condition (2) has a unique solution on the interval $[0, 1]$.

For approximate solution of problem (1)-(2) we divide the interval $[0, 1]$ into N pieces of the length $h = \frac{1}{N}$ and find approximate values y_n of the function $y(x)$ for $n = 0, 1, ..., N$ at nodes $x_n = nh$.

A classic method of approximate solution of the initial-value problem (1)-(2) is the Euler method. Using this method, the approximate solution of the differential equation is calculated as follows: to find an approximate value y_{n+1} of the function at the node x_{n+1}, it is used the approximate value y_n at the node x_n:

$$y_{n+1} = y_n + hy'_n, \tag{3}$$

where $y'_n = f(x_n, y_n)$, so that y_{n+1} is a linear combination of the values of the unknown function $y(x)$ and its first-order derivative at the node x_n.

Everyone are known that there are many methods for solving the initial-value problem for ordinary differential equation (1). For example, the initial-value problem can be solved using the Euler, Runge-Kutta, Adams-Bashforth and Adams-Moulton formulas of varying degrees [1]. In [2] by Ahmad Fadly Nurullah Rasedee, et al., research they discussed the order and stepsize strategies of the variable order stepsize algorithm. The stability and convergence estimations of the method are also established. In the work [3] by Adekoya Odunayo M. and Z.O.Ogunwobi, it was shown that the Adam-Bashforth-Moulton method is better than the Milne Simpson method in solving a second-order differential equation. Some studies have raised the question of whether Nordsieck's technique for changing the step size in the Adams-Bashforth method is equivalent to the explicit continuous Adams-Bashforth method. And in N.S.Hoang and R.B.Sidje's work [4] they provided a complete proof that the two approaches are indeed equivalent. In the works [5] and [6] there were shown the potential superiority of semi-explicit and semi-implicit methods over conventional linear multi-step algorithms.

However, it is very important to choose the right one among these formulas to solve the Initial-value problem and it is not always possible to do this. Also, in this work, in contrast to the above-mentioned works, exact estimates of the error of the formula is obtained.

Our aim, in this paper, is to construct new difference formulas that are exact for e^{-x} and optimal in the Hilbert space $W_2^{(2,1)}(0,1)$. Also these formulas can be used to solve certain classes of problems with great accuracy.

The rest of the work is organized as follows. In the first paragraph, an algorithm for constructing an explicit difference formula in the space is given. The above algorithm is used to obtain an analytical formula for the optimal coefficients of an explicit difference formula. In the second section, the same algorithm is used to obtain an analytical formula for the optimal coefficients of the implicit difference formula. In the third and fourth sections, respectively, exact formulas are given for the square of the norm of the error functionals of explicit and implicit difference formulas. Numerical experiments are presented at the end of the work.

2 Optimal explicit difference formulas of Adams-Bashforth type in the Hilbert space $W_2^{(2,1)}(0,1)$

We consider a difference formula of the following form for the approximate solution of the problem (1)-(2) [7, 8]

$$\sum_{\beta=0}^{k} C\left[\beta\right]\varphi\left[\beta\right]-h\sum_{\beta=0}^{k-1} C_1\left[\beta\right]\varphi'\left[\beta\right]\cong 0, \tag{1}$$

where $h=\frac{1}{N}$, N is a natural number, $C[\beta]$ and $C_1[\beta]$ are the coefficients, functions φ belong to the Hilbert space $W_2^{(2,1)}(0,1)$. The space $W_2^{(2,1)}(0,1)$ is defined as follows

$$W_2^{(2,1)}(0,1)=\{\varphi:[0,1]\to\mathbf{R}|\varphi' \text{ is abs.contunuous, } \varphi''\in L_2(0,1)\}$$

equipped with the norm [9, 10]

$$\|\varphi|W_2^{(2,1)}\|=\Big\{\int_0^1\Big(\varphi''(x)+\varphi'(x)\Big)^2dx\Big\}^{1/2}. \tag{2}$$

The following difference between the sums given in the formula (1) is called the error of the formula (1) [11]

$$(\ell,\varphi)=\sum_{\beta=0}^{k} C[\beta]\varphi\left(h\beta\right)-h\sum_{\beta=0}^{k-1} C_1[\beta]\varphi'\left(h\beta\right).$$

To this error corresponds the error functional [12]

$$\ell(x)=\sum_{\beta=0}^{k} C[\beta]\delta(x-h\beta)+h\sum_{\beta=0}^{k-1} C_1[\beta]\delta'(x-h\beta), \tag{3}$$

where $\delta(x)$ is Dirac's delta-function. We note that (ℓ,φ) is the value of the error functional ℓ at a function φ and it is defined as [13, 14]

$$(\ell,\varphi)=\int\limits_{-\infty}^{\infty}\ell(x)\varphi(x)dx.$$

It should be also noted that since the error functional ℓ is defined on the space $W_2^{(2,1)}(0,1)$ it satisfies the following conditions

$$(\ell,1)=0,\quad (\ell,e^{-x})=0.$$

These give us the following equations with respect to coefficients $C[\beta]$ and $C_1[\beta]$:

$$\sum_{\beta=0}^{k}C[\beta]=0, \tag{4}$$

$$\sum_{\beta=0}^{k}C[\beta]e^{-h\beta}+h\sum_{\beta=0}^{k-1}C_1[\beta]e^{-h\beta}=0. \tag{5}$$

Based on the Cauchy-Schwartz inequality for the absolute value of the error of the formula (1) we have the estimation

$$|(\ell,\varphi)|\leq\|\varphi|W_2^{(2,1)}\|\cdot\|\ell|W_2^{(2,1)*}\|.$$

Hence, the absolute error of the difference formula (1) in the space $W_2^{(2,1)}$ is estimated by the norm of the error functional ℓ on the conjugate space $W_2^{(2,1)*}$. From this we get the following[15].

Problem 1. Calculate the norm $\|\ell|W_2^{(2,1)*}\|$ of the error functional ℓ.

From the formula (3) one can see that the norm $\|\ell|W_2^{(2,1)*}\|$ depends on the coefficients $C[\beta]$ and $C_1[\beta]$.

Problem 2. Find such coefficients $C_1[\beta]=\mathring{C}_1[\beta]$ that satisfy the equality

$$\|\mathring{\ell}|W_2^{(2,1)*}\|=\inf_{\mathring{C}_1[\beta]}\sup_{\|\varphi|W_2^{(2,1)}\|\neq 0}\frac{|(\ell,\varphi)|}{\|\varphi|W_2^{(2,1)}\|}.$$

In this case $\mathring{C}_1[\beta]$ are called the optimal coefficients and the corresponding difference formula (1) is called the *optimal difference formula*.

A function ψ_ℓ satisfying the following equation is called the extremal function of the difference formula (1) [13]

$$(\ell,\psi_\ell)=\|\ell|W_2^{(2,1)*}\|\cdot\|\psi_\ell|W_2^{(2,1)}\|. \tag{6}$$

Since the space $W_2^{(2,1)}(0,1)$ is a Hilbert space, then from the Riesz theorem on the general form of a linear continuous functional on a Hilbert space there is a

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function ψ_ℓ (which is the extremal function) that satisfies the following equation [16, 17]

$$(\ell, \varphi) = \langle \varphi, \psi_\ell \rangle_{W_2^{(2,1)}} \tag{7}$$

and the equality $\|\ell|W_2^{(2,1)*}\| = \|\psi_\ell|W_2^{(2,1)}\|$ holds, here $\langle \varphi, \psi_\ell \rangle_{W_2^{(2,1)}}$ is the inner product in the space $W_2^{(2,1)}(0,1)$ and is defined as follows [18]

Theorem 2.1 *The solution of equation (7) has the form*

$$\psi_\ell(x) = \ell(x) * G_2(x) + de^{-x} + p_0 \tag{8}$$

and it is an extremal function for the difference formula (1), where $G_2(x) = \frac{\mathrm{sgn}(x)}{2}\left(\frac{e^x - e^{-x}}{2} - x\right)$, *$d$ and p_0 are real numbers.*

According to the above mentioned Riesz's theorem, the following equalities is fulfilled

$$\|\ell|W_2^{(2,1)*}\|^2 = (\ell, \psi_\ell) = \|\ell|W_2^{(2,1)*}\| \cdot \|\psi_\ell|W_2^{(2,1)}\|.$$

By direct calculation from the last equality for the norm of the error functional for the difference formula (1) we have the following result [18].

Theorem 2.2 *For the norm of the error functional of the difference formula (1) we have the following expression*

$$\|\ell|W_2^{(2,1)*}\|^2 = \sum_{\gamma=0}^{k}\sum_{\beta=0}^{k} C[\gamma]C[\beta]G_2(h\gamma - h\beta) - 2h\sum_{\gamma=0}^{k-1} C_1[\gamma]\sum_{\beta=0}^{k} C[\beta]G_2'(h\gamma - h\beta) -$$

$$-h^2\sum_{\gamma=0}^{k-1}\sum_{\beta=0}^{k-1} C_1[\gamma]C_1[\beta]G_2''(h\gamma - h\beta), \tag{9}$$

where $G_2'(x) = \frac{\mathrm{sgn}(x)}{2}\left(\frac{e^x + e^{-x}}{2} - 1\right)$ *and* $G_2''(x) = \frac{\mathrm{sgn}(x)}{2}\left(\frac{e^x - e^{-x}}{2}\right)$.

It is known that stability in the Dahlquist sense, just like strong stability, is determined only by the coefficients $C[\beta]$, $\beta = \overline{0,k}$. For this reason, our search for the optimal formula is only related to finding $C_1[\beta]$. Therefore, in this subsection we consider difference formulas of the Adams-Bashforth type, i.e. $C[k] = -C[k-1] = 1$ and $C[k-i] = 0$, $i = \overline{2,k}$, [19, 20]. Then is easy to check, that the coefficients satisfy the condition (4).

In this work, we find the minimum of the norm (9) by the coefficients $C_1[\beta]$ under the condition (5) in the space $W_2^{(2,1)}(0,1)$ [21]. Then using Lagrange method of undetermined multipliers we get the following system of linear equations with respect to the coefficients $C_1[\beta]$:

$$h\sum_{\gamma=0}^{k-1} C_1[\gamma]\, G_2''(h\beta - h\gamma) + de^{-h\beta} = -\sum_{\gamma=0}^{k} C[\gamma]\, G_2'(h\beta - h\gamma), \tag{10}$$

$\beta = \overline{0, k-1}$,

$$h\sum_{\gamma=0}^{k-1} C_1[\gamma]\, e^{-h\gamma} = -\sum_{\gamma=0}^{k} C[\gamma]\, e^{-h\gamma}. \tag{11}$$

It is easy to prove that the solution of this system gives the minimum value to the expression (9) under the condition (5). Here d is an unknown constant, $\overset{\circ}{C}_1[\beta]$ are optimal coefficient. Given that $C[k] = 1$, $C[k-1] = -1$, $C[k-i] = 0$, $i = \overline{2, k}$ the system (10),(11) is reduced to the form,

$$h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, G_2''(h\beta - h\gamma) + de^{-h\beta} = f[\beta], \qquad \beta = \overline{0, k-1} \tag{12}$$

$$h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} = g, \tag{13}$$

where

$$f[\beta] = \frac{1-e^h}{4}\left(e^{h\beta - hk} - e^{-h\beta + hk - h}\right), \tag{14}$$

$$g = e^{-hk+h} - e^{-hk}. \tag{15}$$

Assuming that $C_1[\beta] = 0$, for $\beta < 0$ and $\beta > k-1$, we rewrite the system (12), (13) in the convolution form

$$\begin{cases} h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = f[\beta] \;\; for\ \beta = \overline{0, k-1}, \\ h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} = g. \end{cases} \tag{16}$$

We denote first equation of the system (16) by U_{exp}

$$U_{exp}[\beta] = h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta}. \tag{17}$$

(12) implies that

$$U_{exp}[\beta] = f[\beta] \;\; for\, \beta = \overline{0, k-1}. \tag{18}$$

Now calculating the convolution we have

$$U_{exp}[\beta] = \overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, G_2''(h\beta - h\gamma) + de^{-h\beta}.$$

For $\beta < 0$ we get

$$U_{exp}[\beta] = h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1 \frac{sgn(h\beta - h\gamma)}{2}\left(\frac{e^{h\beta - h\gamma} - e^{-h\beta + h\gamma}}{2}\right) + de^{-h\beta}$$

$$= -\frac{e^{h\beta}}{4} h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} + \frac{e^{-h\beta}}{4} h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, e^{h\gamma} + de^{-h\beta} = -\frac{e^{h\beta}}{4} g + e^{-h\beta}(d+b).$$

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For $\beta > k-1$
$$U_{exp}[\beta] = \frac{e^{h\beta}}{4} g + e^{-h\beta}(d-b).$$
Then $d^+ = d+b$ and $d^- = d-b$ the function $U_{exp}[\beta]$ becomes
$$U_{exp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4} g + e^{-h\beta} d^+ & for\ \beta > k-1, \\ f[\beta] & for\ \beta = \overline{0, k-1}, \\ \frac{e^{h\beta}}{4} g + e^{-h\beta} d^- & for\ \ \beta < 0. \end{cases} \tag{19}$$
We use to find the unknowns d^+ and d^- from the discrete analogue of the differential operator $\frac{d^2}{dx^2} - \frac{d}{dx}$ which is given below [22]
$$D_1[\beta] = \frac{1}{1-e^{2h}} \begin{cases} -2e^h & for\ \ |\beta| = 1, \\ 2(1+e^{2h}) & for\ \beta = 0, \\ 0 & for\ \ |\beta| \geq 2. \end{cases} \tag{20}$$
The unknowns d^+ and d^- are determined from the conditions
$$\overset{\circ}{C}_1[\beta] = h^{-1} D_1[\beta] * U_{exp}[\beta] = 0\ for\ \beta < 0\ and\ \beta > k-1. \tag{21}$$
Calculate the convolution
$$h^{-1} D_1[\beta] * U_{exp}[\beta]$$
$$= h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta+\gamma] U_{exp}[-\gamma] + h^{-1} \sum_{\gamma=0}^{k-1} D_1[\beta-\gamma] U_{exp}[\gamma]$$
$$+ h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta-k-\gamma+1] U_{exp}[k+\gamma-1].$$
From (19) with $\beta = k$ and $\beta = -1$, we have
$$\begin{cases} h^{-1} D_1[0] U_{exp}[-1] + h^{-1} D_1[1] U_{exp}[-2] + h^{-1} D_1[-1] U_{exp}[0] = 0, \\ h^{-1} D_1[0] U_{exp}[k] + h^{-1} D_1[1] U_{exp}[k-1] + h^{-1} D_1[-1] U_{exp}[k+1] = 0. \end{cases}$$
Hence, due to (21), we get
$$\begin{cases} 2\left(1+e^{2h}\right)\left[-\frac{1}{4}e^{-h} g + e^h d^+\right] - 2e^h\left[-\frac{1}{4}e^{-2h} g + e^{2h} d^+\right] - 2e^h f[0] = 0 \\ 2\left(1+e^{2h}\right)\left[\frac{1}{4}e^{hk} g + e^{-hk} d^-\right] - 2e^h\left[\frac{1}{4}e^{hk+h} g + e^{-hk-h} d^-\right] - 2e^h f[hk-h] = 0. \end{cases}$$
From the first equation d^+ is equal to the following
$$d^+ = \frac{e^{hk} - e^{hk-h}}{4}.$$
From the second equation d^- is equal to the following
$$d^- = \frac{e^{hk} - 3e^{hk-h} + 2e^{hk-2h}}{4}.$$

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so

$$d = \frac{d_0^+ + d_0^-}{2} = \frac{e^{hk} - 2e^{hk-h} + e^{hk-2h}}{4} \quad and \quad b = \frac{d_0^+ - d_0^-}{2} = \frac{e^{hk-h} - e^{hk-2h}}{4}.$$

Now we calculate the optimal coefficients $\overset{\circ}{C}_1[\beta]$

$$\overset{\circ}{C}_1[\beta] = h^{-1} D_1[\beta] * U_{exp}[\beta] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[\beta-\gamma] U_{exp}[\gamma], \ \beta = \overline{0, k-1}.$$

Let $\beta = k-1$, then

$$\overset{\circ}{C}_1[k-1] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[k-1-\gamma] U_{exp}[\gamma]$$

$$= h^{-1} \{D_1[0] U_{exp}[k-1] + D_1[1] U_{exp}[k-2] + D_1[-1] U_{exp}[k]\}$$

$$= \frac{h^{-1}}{(1-e^{2h})} \cdot \left\{1 - e^{-h} + e^h - e^{2h}\right\} = \frac{e^h - 1}{he^h},$$

thus, $\overset{\circ}{C}_1[k-1] = \frac{e^h-1}{he^h}$ for $\beta = k-1$.

Compute $\overset{\circ}{C}_1[0]$

$$\overset{\circ}{C}_1[0] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[-\gamma] U_{exp}[\gamma]$$

$$= h^{-1} \{D_1[0] U_{exp}[0] + D_1[1] U_{exp}[-1] + D_1[-1] U_{exp}[1]\}$$

$$= \frac{h^{-1}}{2(1-e^{2h})} \cdot \left\{(1+e^{2h})(e^{-hk} - e^{hk-h} - e^{-hk+h} + e^{hk}) \right.$$

$$\left. - e^{-h}(-e^{-hk} + e^{-hk-h} - e^{hk+h} - e^{hk})\right\}$$

$$-\frac{h^{-1}}{2(1-e^{2h})} \cdot \left\{e^{-h}(e^{-hk+h} - e^{hk-2h} - e^{-hk+2h} + e^{hk-h})\right\} = \frac{h^{-1}}{2(1-e^{2h})} \cdot 0 = 0,$$

hence, $\overset{\circ}{C}_1[0] = 0$ for $\beta = 0$.

Now calculate $\overset{\circ}{C}_1[\beta]$ for $\beta = \overline{1, k-2}$

$$\overset{\circ}{C}_1[\beta] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[-\gamma] U_{exp}[\gamma]$$

$$= h^{-1} \{D_1[0] U_{exp}[\beta] + D_1[1] U_{exp}[\beta-1] + D_1[-1] U_{exp}[\beta+1]\}$$

$$= \frac{h^{-1}}{2(1-e^{2h})} \cdot \left\{(1+e^{2h})(1-e^h)(e^{-hk+h\beta} - e^{hk-h\beta-h})\right\}$$

$$-\frac{h^{-1}}{2(1-e^{2h})} \cdot \left\{e^{-h}(1-e^h)(e^{-hk+h\beta-h} - e^{hk-h\beta})\right\}$$

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$$-\frac{h^{-1}}{2(1-e^{2h})}\cdot\left\{e^{-h}(1-e^{h})(e^{-hk+h\beta+h}-e^{hk-h\beta-2h})\right\}=\frac{h^{-1}}{2(1-e^{2h})}\cdot 0=0,$$

thereby, $\overset{\circ}{C}_1[\beta]=0$ for $\beta=\overline{1,k-2}$.

Finally, we have proved the following theorem.

Theorem 2.3 *In the Hilbert space $W_2^{(2,1)}(0,1)$ there is a unique optimal explicit difference formula of the Adams-Bashforth type whose coefficients are determined by following expressions*

$$C[\beta]=\begin{cases}1 & for\ \ \beta=k,\\ -1 & for\ \ \beta=k-1,\\ 0 & for\ \ \beta=\overline{0,k-2},\end{cases}\tag{22}$$

$$\overset{\circ}{C}_1[\beta]=\begin{cases}\frac{e^h-1}{he^h} & for\ \ \beta=k-1,\\ 0 & for\ \ \beta=\overline{0,k-2}.\end{cases}\tag{23}$$

Thus, the optimal explicit difference formula in $W_2^{(2,1)}(0,1)$ has the form

$$\varphi_{n+k}=\varphi_{n+k-1}+\frac{e^h-1}{e^h}\varphi'_{n+k-1},\tag{24}$$

where $n=0,1,...,N-k,\ k\geq 1$.

3 Optimal implicit difference formulas of Adams-Moulton type in the Hilbert space $W_2^{(2,1)}(0,1)$

Consider an implicit difference formula of the form

$$\sum_{\beta=0}^{k}C[\beta]\varphi[\beta]-h\sum_{\beta=0}^{k}C_1[\beta]\varphi'[\beta]\cong 0\tag{1}$$

with the error function

$$\ell(x)=\sum_{\beta=0}^{k}C[\beta]\delta(x-h\beta)+h\sum_{\beta=0}^{k}C_1[\beta]\delta'(x-h\beta)\tag{2}$$

in the space $W_2^{(2,1)}(0,1)$.

In this section, we also consider the case $C[k]=-C[k-1]=1$, and $C[k-i]=0$, $i=\overline{2,k}$, i.e. Adams-Moulton type formula. Minimizing the norm of the error functional (2) of an implicit difference formula of the form (1) with respect to the coefficients $C_1[\beta]$, $\beta=\overline{0,k}$ in the space $W_2^{(2,1)}(0,1)$ we obtain a system of linear algebraic equations

$$\begin{cases}h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,G_2''(h\beta-h\gamma)+de^{-h\beta}=f[\beta], & \beta=\overline{0,k}\\ h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,e^{-h\gamma}=g.\end{cases}$$

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Here $\overset{\circ}{C}_1[\beta]$ are unknowns coefficients of the implicit difference formulas (1), $\beta = \overline{0,k}$ and d is an unknown constant,

$$f[\beta] = G_2'(h\beta - hk + h) - G_2'(h\beta - hk)$$

$$= \begin{cases} \frac{1}{4}\left(1 - e^h\right)\left(e^{-hk+h\beta} - e^{hk-h\beta-h}\right), & \beta = \overline{0,k-1}, \\ \frac{1}{4}\left(e^h + e^{-h} - 2\right), & \beta = k, \end{cases} \tag{3}$$

$$g = e^{-hk+h} - e^{-hk}. \tag{4}$$

Assuming, in general, that

$$\overset{\circ}{C}_1[\beta] = 0 \ \ , for \ \beta < 0 \ and \ \beta > k, \tag{5}$$

rewrite the system in the convolution form

$$\begin{cases} h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = f[\beta], & \beta = \overline{0,k}, \\ h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] e^{-h\gamma} = g. \end{cases}$$

Denote by $U_{imp}[\beta] = h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta}$. Shows that

$$U_{imp}[\beta] = f[\beta] \ for \ \beta = \overline{0,k} \tag{6}$$

Now we find $U_{imp}[\beta]$ for $\beta < 0$ and $\beta > k$. Let $\beta < 0$, then

$$U_{imp}[\beta] = h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] \frac{sgn(h\beta - h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2}\right) + de^{-h\beta}$$

$$= -\frac{e^{h\beta}}{4} h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] e^{-h\gamma} + \frac{e^{-h\beta}}{4} h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}.$$

Here d^+ is defined by the equality

$$d^+ = \frac{e^{-h\beta}}{4} h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}. \tag{7}$$

Similarly, for $\beta > k$ we have

$$U_{imp}[\beta] = h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] \frac{sgn(h\beta - h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2}\right) + de^{-h\beta}$$

$$= \frac{e^{h\beta}}{4} h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] e^{-h\gamma} - \frac{e^{-h\beta}}{4} h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}.$$

Here d^- is defined by the equality

$$d^- = -\frac{e^{-h\beta}}{4} h \sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}. \tag{8}$$

(7) and (8) immediately imply that

$$d = \frac{d^+ + d^-}{2}. \tag{9}$$

So $U_{imp}[\beta]$ for any $\beta \in Z$ is defined by the formula

$$U_{imp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4} g + e^{-h\beta} d^+ & for\ \beta > k, \\ f[\beta] & for\ \beta = \overline{0,k}, \\ \frac{e^{h\beta}}{4} g + e^{-h\beta} d^- & for\ \beta < 0. \end{cases} \tag{10}$$

If we operate operator (20) on expression $U_{imp}[\beta]$, we get

$$\overset{\circ}{C}_1[\beta] = h^{-1} D_1[\beta] * U_{imp}[\beta], \ \beta \in Z. \tag{11}$$

Assuming that $\overset{\circ}{C}_1[\beta] = 0$ for $\beta < 0$ and $\beta > k$, we get a system of linear equations for finding the unknowns d^+ and d^- in the formula (10). Indeed, calculating the convolution, we have

$$h^{-1} D_1[\beta] * U_{imp}[\beta] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[\beta - \gamma] U_{imp}[\gamma]$$

$$= h^{-1} \sum_{\gamma=-\infty}^{-1} D_1[\beta - \gamma] U_{imp}[\gamma] + h^{-1} \sum_{\gamma=0}^{k} D_1[\beta - \gamma] U_{imp}[\gamma]$$

$$+ h^{-1} \sum_{\gamma=k+1}^{\infty} D_1[\beta - \gamma] U_{imp}[\gamma]$$

$$= h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta + \gamma] U_{imp}[-\gamma] + h^{-1} \sum_{\gamma=0}^{k} D_1[\beta - \gamma] U_{imp}[\gamma]$$

$$+ h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta - k - \gamma] U_{imp}[k + \gamma]. \tag{12}$$

Equating the expression (12) to zero with $\beta = -1$, $\beta = k + 1$ and using the formulas (10), (20) we get

$$\begin{cases} h^{-1} D_1[0] U_{imp}[-1] + h^{-1} D_1[1] U_{imp}[-2] + h^{-1} D_1[-1] U_{imp}[0] = 0, \\ h^{-1} D_1[0] U_{imp}[k+1] + h^{-1} D_1[1] U_{imp}[k] + h^{-1} D_1[-1] U_{imp}[k+2] = 0 \end{cases}$$

or

$$\begin{cases} 2\left(1+e^{2h}\right)\left[-\frac{1}{4}e^{-h}g+e^{h}d^{+}\right]-2e^{h}\left[-\frac{1}{4}e^{-2h}g+e^{2h}d^{+}\right]-2e^{h}f[0]=0, \\ 2\left(1+e^{2h}\right)\left[\frac{1}{4}e^{hk+h}g+e^{-hk-h}d^{-}\right]-2e^{h}\left[\frac{1}{4}e^{hk+2h}g+e^{-hk-2h}d^{-}\right]-2e^{h}f[hk]=0. \end{cases}$$

By virtue of the formulas (3) and (4), finally, we find

$$d^{+}=\frac{1}{4}\left(e^{hk}-e^{hk-h}\right), \tag{13}$$

$$d^{-}=\frac{1}{4}\left(e^{hk-h}-e^{hk}\right). \tag{14}$$

Then from (9) we find that $d=0$.

As a result, we rewrite $U_{imp}[\beta]$ through the (13) and (14) as follows

$$U_{imp}[\beta]=\begin{cases} -\frac{e^{h\beta}}{4}g+\frac{e^{-h\beta}}{4}\left(e^{hk}-e^{hk-h}\right) & for\ \beta>k, \\ f[\beta] & for\ \beta=\overline{0,k}, \\ \frac{e^{h\beta}}{4}g+\frac{e^{-h\beta}}{4}\left(e^{hk-h}-e^{hk}\right) & for\ \ \beta<0. \end{cases} \tag{15}$$

Now we turn to calculating the optimal coefficients of implicit difference formulas $\overset{\circ}{C}_1[\beta]$, $\beta=\overline{0,k}$ according to the formula (11)

$$\overset{\circ}{C}_1[k]=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_1[k-\gamma]U_{imp}[\gamma]=$$

$$=h^{-1}\left\{D_1[0]U_{imp}[k]+D_1[1]U_{imp}[k-1]+D_1[-1]U_{imp}[k+1]\right\}=$$

$$=\frac{h^{-1}}{2\left(1-e^{2h}\right)}\left(-2e^{2h}+4e^{h}-2\right)=\frac{e^{h}-1}{h\left(e^{h}+1\right)}.$$

So $\overset{\circ}{C}_1[k]=\frac{e^{h}-1}{h(e^{h}+1)}$.

Calculate the next optimal coefficient

$$\overset{\circ}{C}_1[k-1]=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_1[k-\gamma-1]U_{imp}[\gamma]=$$

$$=h^{-1}\left\{D_1[0]U_{imp}[k-1]+D_1[1]U_{imp}[k-2]+D_1[-1]U_{imp}[k]\right\}=$$

$$=\frac{h^{-1}}{2\left(1-e^{2h}\right)}\left(-2e^{2h}+4e^{h}-2\right)=\frac{e^{h}-1}{h\left(e^{h}+1\right)}.$$

Thus $\overset{\circ}{C}_1[k-1]=\frac{e^{h}-1}{h(e^{h}+1)}$.

Go to computed $\overset{\circ}{C}_1[\beta]$ when $\beta=\overline{1,k-2}$

$$\overset{\circ}{C}_1[\beta]=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_1[\beta-\gamma]U_{imp}[\gamma]$$

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$$= h^{-1}\left\{D_1[0]U_{imp}[\beta] + D_1[1]U_{imp}[\beta-1] + D_1[-1]U_{imp}[\beta+1]\right\}$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)}\left\{\left(1+e^{2h}\right)\left(e^{-hk+h\beta} - e^{hk-h\beta-h} - e^{-hk+h\beta+h} + e^{hk-h\beta}\right)\right\}$$

$$-\frac{h^{-1}}{2\left(1-e^{2h}\right)}\left\{e^{h}\left(e^{-hk+h\beta-h} - e^{hk-h\beta} - e^{-hk+h\beta} + e^{hk-h\beta+h}\right)\right\}$$

$$-\frac{h^{-1}}{2\left(1-e^{2h}\right)}\left\{e^{h}\left(e^{-hk+h\beta+h} - e^{hk-h\beta-2h} - e^{-hk+h\beta+2h} + e^{hk-h\beta-h}\right)\right\}$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)}\cdot 0 = 0\,.$$

Thereby, $\overset{\circ}{C}_1[\beta] = 0$, for $\beta = \overline{1, k-2}$.
Then calculate $\overset{\circ}{C}_1[0]$

$$\overset{\circ}{C}_1[0] = h^{-1}\sum_{\gamma=-\infty}^{\infty} D_1[-\gamma]U_{imp}[\gamma]$$

$$= h^{-1}\left\{D_1[0]U_{imp}[0] + D_1[1]U_{imp}[-1] + D_1[-1]U_{imp}[1]\right\}$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)}\left\{\left(1+e^{2h}\right)\left(e^{-hk} - e^{hk-h} - e^{-hk+h} + e^{hk}\right)\right\}$$

$$-\frac{h^{-1}}{2\left(1-e^{2h}\right)}\left\{e^{h}\left(-e^{-hk} + e^{-hk-h} + e^{hk+h} - e^{hk}\right)\right\}$$

$$-\frac{h^{-1}}{2\left(1-e^{2h}\right)}\left\{e^{h}\left(e^{-hk+h} - e^{hk-2h} - e^{-hk+2h} + e^{hk-h}\right)\right\} = \frac{h^{-1}}{2\left(1-e^{2h}\right)}\cdot 0 = 0.$$

hence $\overset{\circ}{C}_1[0] = 0$.

Finally, we have proved the following.

Theorem 3.1 *In the Hilbert space $W_2^{(2,1)}(0,1)$, there exists a unique optimal implicit difference formula, of Adams-Moulton type, whose coefficients are determined by formulas*

$$C[\beta] = \begin{cases} 1 & for \quad \beta = k, \\ -1 & for \quad \beta = k-1, \\ 0 & for \quad \beta = \overline{0, k-2}, \end{cases} \tag{16}$$

$$\overset{\circ}{C}_1[\beta] = \begin{cases} \frac{e^h-1}{h(e^h+1)} & for \quad \beta = k, \\ \frac{e^h-1}{h(e^h+1)} & for \quad \beta = k-1, \\ 0 & for \quad \beta = \overline{0, k-2}\ . \end{cases} \tag{17}$$

Consequently, the optimal implicit difference formula in $W_2^{(2,1)}(0,1)$ has the form

$$\varphi_{n+k} = \varphi_{n+k-1} + \frac{e^h-1}{e^h+1}\left(\varphi'_{n+k} + \varphi'_{n+k-1}\right), \tag{18}$$

where $n = 0, 1, ..., N-k,\ k \geq 1$.

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4 Norm of the error functional of the optimal explicit difference formula

The square of the norm of an explicit Adams-Bashforth type difference formula is expressed by the equality

$$\left\|\ell\left|W_2^{(2,1)*}(0,1)\right\|^2=\sum_{\gamma=0}^{k}\sum_{\beta=0}^{k}C[\gamma]C[\beta]G_2\left[\gamma-\beta\right]-\right.$$

$$-2h\sum_{\gamma=0}^{k-1}C_1[\gamma]\sum_{\beta=0}^{k}C[\beta]G_2'\left[\gamma-\beta\right]-h^2\sum_{\gamma=0}^{k-1}\sum_{\beta=0}^{k-1}C_1[\gamma]C_1[\beta]G_2''\left[\gamma-\beta\right]. \tag{1}$$

In this section, we deal with the calculation of the squared norm (1) in the space $W_2^{(2,1)}(0,1)$. For this we use the coefficients $C[\beta]$ and $\overset{\circ}{C}_1[\beta]$, which is detected in the formulas (22) and (23).

Then we calculate (1) in sequence as follows.

$$\left\|\overset{\circ}{\ell}\left|W_2^{(2,1)*}(0,1)\right\|^2=\sum_{\gamma=0}^{k}C[\gamma]\left\{G_2\left[\gamma-k\right]-G_2\left[\gamma-k+1\right]\right\}\right.$$

$$-2h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}_1[\gamma]\left\{G_2'\left[\gamma-k\right]-G_2'\left[\gamma-k+1\right]\right\}$$

$$-h^2\frac{e^h-1}{he^h}\sum_{\gamma=0}^{k-1}\overset{\circ}{C}_1[\gamma]\left\{G_2''\left[\gamma-k+1\right]\right\}$$

$$=G_2\left[0\right]-G_2\left[1\right]-G_2\left[-1\right]+G_2\left[0\right]-\frac{2(e^h-1)}{e^h}\left\{G_2'\left[-1\right]-G_2'\left[0\right]\right\}-\frac{(e^h-1)^2}{e^{2h}}G_2''\left[0\right]$$

$$=-2G_2\left[1\right]+\frac{2(e^h-1)}{e^h}G_2'\left[1\right]=\frac{2(e^h-1)}{e^h}\cdot\frac{sgn(h)}{2}\left(\frac{e^h+e^h}{2}-1\right)$$

$$-2\cdot\frac{sgn(h)}{2}\left(\frac{e^h-e^h}{2}-h\right)=\frac{e^h-1}{e^h}\cdot\frac{e^{2h}-2e^h+1}{2e^h}-\frac{e^{2h}-1}{2e^h}+h$$

$$=h-\frac{\left(e^h-1\right)\left(3e^h-1\right)}{2e^{2h}}.$$

As a result, we get the following outcome.

Theorem 4.1 *The square of the norm of the optimal error functional of an explicit difference formula of the form (1) in the quotient space $W_2^{(2,1)}(0,1)$ is expressed as formula*

$$\left\|\overset{\circ}{\ell}\left|W_2^{(2,1)*}(0,1)\right\|^2=h-\frac{\left(e^h-1\right)\left(3e^h-1\right)}{2e^{2h}}\right..$$

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5 Norm of the error functional of the implicit optimal difference formula

In this case, the square of the norm of the error functional of an implicit Adams-Moulton type difference formula of the form (1) is expressed by the equality

$$\left\|\ell \left| W_2^{(2,1)*}(0,1)\right\|^2 = \sum_{\gamma=0}^{k}\sum_{\beta=0}^{k} C[\gamma]C[\beta]G_2\left[\gamma-\beta\right] - \right.$$

$$-2h\sum_{\gamma=0}^{k} C_1[\gamma]\sum_{\beta=0}^{k} C[\beta]G_2'\left[\gamma-\beta\right] - h^2\sum_{\gamma=0}^{k}\sum_{\beta=0}^{k} C_1[\gamma]C_1[\beta]G_2''\left[\gamma-\beta\right]. \qquad (1)$$

Here we use the optimal coefficients of an implicit difference formula of the form (1), which is detected in the formulas (16) and (17).
Then, we calculate (1) as follows

$$\left\|\mathring{\ell}\left|W_2^{(2,1)*}(0,1)\right\|^2 = \sum_{\gamma=0}^{k} C[\gamma]\left\{G_2\left[\gamma-k\right] - G_2\left[\gamma-k+1\right]\right\}\right.$$

$$-2h\sum_{\gamma=0}^{k}\mathring{C}_1[\gamma]\left\{G_2'\left[\gamma-k\right] - G_2'\left[\gamma-k+1\right]\right\}$$

$$-h^2\frac{e^h-1}{h\left(e^h+1\right)}\sum_{\gamma=0}^{k}\mathring{C}_1[\gamma]\left\{G_2''\left[\gamma-k\right] + G_2''\left[\gamma-k+1\right]\right\}$$

$$= G_2\left[0\right] - G_2\left[1\right] - G_2\left[-1\right] + G_2\left[0\right] - \frac{2h(e^h-1)}{h\left(e^h+1\right)}\left\{G_2'\left[0\right] - G_2'\left[1\right] + G_2'\left[-1\right] - G_2'\left[0\right]\right\}$$

$$-\frac{h^2\left(e^h-1\right)^2}{h^2\left(e^h+1\right)}\left\{G_2''\left[0\right] + G_2''\left[1\right] + G_2''\left[-1\right] + G_2''\left[0\right]\right\}$$

$$= -2G_2\left[1\right] + \frac{4(e^h-1)}{e^h+1}G_2'\left[1\right] - 2\frac{\left(e^h-1\right)^2}{\left(e^h+1\right)}G_2''\left[1\right]$$

$$= \frac{4(e^h-1)}{e^h+1}\cdot\frac{sgn(h)}{2}\left(\frac{e^h+e^h}{2}-1\right) - 2\cdot\frac{sgn(h)}{2}\left(\frac{e^h-e^h}{2}-h\right)$$

$$-2\frac{\left(e^h-1\right)^2}{\left(e^h+1\right)}\cdot\frac{sgn(h)}{2}\left(\frac{e^h-e^h}{2}\right)$$

$$= h - \frac{e^{2h}-1}{2e^h} + \frac{2\left(e^h-1\right)^2\left(e^h-1\right)}{2e^h\left(e^h+1\right)} - \frac{\left(e^h-1\right)^2\left(e^h-1\right)}{2e^h\left(e^h+1\right)}$$

$$= h + \frac{\left(e^h-1\right)}{2e^h}\left(\frac{\left(e^h-1\right)^2}{e^h+1} - e^h - 1\right) = h - \frac{2\left(e^h-1\right)}{e^h+1}\ .$$

Consequently, we get the following result.

Theorem 5.1 *Among all implicit difference formulas of the form (1) in the Hilbert space $W_2^{(2,1)}(0,1)$, there is a unique implicit optimal difference formula square the norms of the error functional of which is determined by the equality*

$$\left\| \overset{\circ}{\ell} \left| W_2^{(2,1)*}(0,1) \right\|^2 = h - \frac{2\left(e^h - 1\right)}{e^h + 1}.$$

6 Numerical results

In this section, we give some numerical results in order to show tables and graphs of solutions and errors of our optimal explicit difference formulas (24) and optimal implicit difference formulas (18), with coefficients given correspondingly in Theorem 2.3 and Theorem 3.1. We show the results of the created formulas in some examples in the form of tables and graphs. Here, of course, the results presented in the table are then shown in the graph.

```
Example: y'=xsin(-3x)-2, N = 10, h = 0.1, y_0=1; exact solution: 1/3xcos(-3x)-1/3cos(-3x)-2x+4/3
N=10
y_0=1
```

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal exp.diff .formula	Error of the Optimal exp.dif f.formula
0.0	1.0	1.0	0.0	1.0	0.0
0.1	0.846732386595652	0.8	0.046732386595651	0.809674836071919	0.037057550523732
0.2	0.713243836024086	0.597044797933387	0.116199038090699	0.616537425554994	0.096706410469092
0.3	0.588291007403512	0.385751948465486	0.202539058938026	0.415465694495946	0.172825312907565
0.4	0.460861782437999	0.162252141176661	0.298609641261337	0.202777507195246	0.258084275242752
0.5	0.321543799722049	-0.075029422262028	0.396573221984077	-0.023025755097653	0.344569554819702
0.6	0.163626945959078	-0.32490417159223	0.488531117551308	-0.260813018236448	0.424439964195526
0.7	-0.016182056206681	-0.583335029444922	0.567152973238241	-0.506742495160889	0.490560438954208
0.8	-0.217507085630584	-0.843759685110343	0.62625259947976	-0.754569321563065	0.537062235932482
0.9	-0.436530928599431	-1.097796739554436	0.661265810955004	-0.996317541717459	0.559786613118027
1.0	-0.666666666666667	-1.33626092877548	0.669594262108813	-1.22324622123982	0.556579554573153

Figure 1:

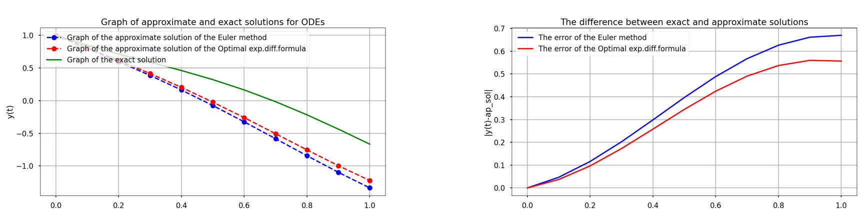

Figure 2:

```
Example: y'=xsin(-3x)-2, N = 10, h = 0.1, y_0=0; exact solution: 1/3xcos(-3x)-1/3cos(-3x)-2x+4/3
N=10
y_0=0
```

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal exp.diff. formula	Error of the Optimal exp.dif f.formula
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.177582570455131	0.2	0.022417429544869	0.190325163928081	0.01274259347295
0.2	0.318937005892713	0.35751574640716	0.038578740514447	0.340221215209065	0.021284209316352
0.3	0.433883226340232	0.48432171812668	0.050438491786449	0.460893051981951	0.02700982564172
0.4	0.529205824465697	0.588522961386072	0.059317136920375	0.560053645506219	0.030847821040522
0.5	0.609635519478213	0.675744617789102	0.066109098310888	0.643056025771146	0.033420506292933
0.6	0.67850887558362	0.749931212084026	0.071422336500406	0.713653904373384	0.035145028789763
0.7	0.738212180552286	0.81388417102237	0.075671990470084	0.774513191341514	0.036301010789229
0.8	0.790480484888187	0.869623707664033	0.079143222775846	0.827556373584513	0.037075888696326
0.9	0.836599409647578	0.918632824473592	0.082033414826014	0.874194714538262	0.037595304890684
1.0	0.877541649106374	0.962022162692381	0.084480513586007	0.915485129084372	0.037943479977998

Figure 3:

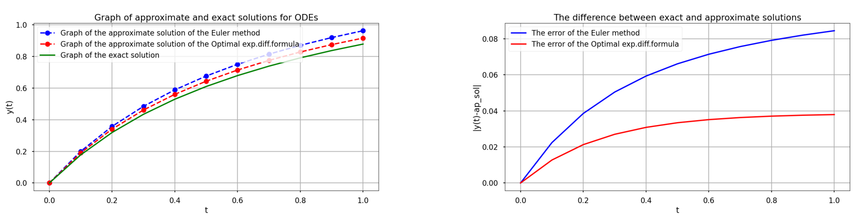

Figure 4:

The tables in Figures 1, 3, and 5 show the exact and approximate solutions and the differences between the exact and approximate solutions.

According to the tables in Figures 1 and 3 on the left side of these Figures 2 and 4 are graphs of approximate and exact solutions, and on the right side of these Figures 2 and 4 graphs of the difference between the actual and approximate solutions are shown. As can be seen from the results presented above, in a certain sense, the optimal explicit formula gives better results than the classical Euler formula.

In accordance with the table, shown in Figure 5, on the left side of these Figure 6, graphs of approximate and exact solutions are shown, and on the right side of these Figure 6, graphs of the difference between the authentic and approximate solutions. As can be seen from the results presented above, in a certain sense, optimal explicit and implicit difference formulas give better results than the classical Euler formula.

It should also be noted that with the help of newly constructed difference schemes, it is possible to obtain approximate solutions with good accuracy.

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Example: y'=y-x^2+1, N = 10, h = 0.1, y_0=0.5; exact solution: y(x)=(x+1)^2-0.5e^(x)
N=10
y_0=0.5

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal imp.diff. formula	Error of the Optimal imp.diff .formula
0.0	0.5	0.5	0.0	0.5	0.0
0.1	0.657414541	0.65	0.007414541	0.6572305225	0.0001840184
0.2	0.8292986209	0.814	0.0152986209	0.8288937051	0.0004049158
0.3	1.0150705962	0.9914	0.0236705962	1.0144040254	0.0006665708
0.4	1.2140876512	1.18154	0.0325476512	1.2131143814	0.0009732698
0.5	1.4256393646	1.383694	0.0419453646	1.4243096145	0.0013297501
0.6	1.6489405998	1.5970634	0.0518771998	1.6471993523	0.0017412475
0.7	1.8831236463	1.82076974	0.0623539063	1.8809100984	0.0022135479
0.8	2.1272295358	2.053846714	0.0733828218	2.1244764896	0.0027530461
0.9	2.3801984444	2.2952313854	0.084967059	2.3768316349	0.0033668095
1.0	2.6408590858	2.5437545239	0.0971045618	2.6367964373	0.0040626485

Figure 5:

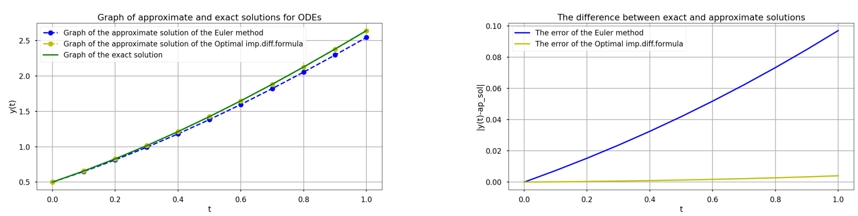

Figure 6:

Also, with the help of these methods, it is possible to solve problems in various fields of mechanics. The cited numerical results were obtained using the Python programming language.

7 Conclusion

In conclusion, In this paper, new Adams-type optimal difference formulas are constructed and exact expressions for the exact estimation of their error are obtained. Moreover, we have shown that the results obtained by the optimal explicit difference formulas constructed in the $W_2^{(2,1)}(0,1)$ Hilbert space are better than the results obtained by the Euler formula. In addition, the optimal implicit formula is more accurate than the optimal explicit formula and the effectiveness of the new optimal difference formulas was shown in the numerical results.

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Direct approach to the stability of various functional equations in Felbin's type non-archimedean fuzzy normed spaces

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Abstract. Using the direct approach, the authors find the Ulam stability of the septic functional equation and octic functional equation in Felbin's type non-Archimedean fuzzy normed space.

1. introduction and preliminaries

The emergence of functional equations coincided with the modern formulation of the function concept. The first publications regarding functional equations were authored by D'Alembert [1] during the period between 1747 and 1750. Due to their apparent simplicity and harmonic characteristics, functional equations have captured the interest of numerous renowned mathematicians. Notable figures such as Rassias [5], Aoki [4], Găvruta [6] and Jakhar [11, 12] have all engaged with this area of study.

The foundational concept of Hyers-Ulam stability for functional equations traces back to a renowned problem centered on group homomorphisms("Let G be a group and G' be a metric group with metric $d(.,.)$. Given $\epsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists homomorphism $H : G \to G'$ with $d(f(x), H(x)) < \epsilon$ for all $x \in G$?"), successfully addressed by Ulam [2] and Hyers [3]. Over the past decades, a substantial volume of literature has been devoted to addressing the stability problem in the context of functional equations, with significant focus on crucial issues within this domain (see [7, 8, 9, 10, 11, 12]). Consequently, numerous effective techniques have been detailed in various papers (such as [10, 18–24, 27]), encompassing approaches like the direct method, fixed point method. Notably, the

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direct method consistently emerges as the primary investigative tool for exploring functional equations of diverse kinds.

The idea of fuzzy criteria on a set of data is used in the field of fuzzy functional analysis. In 1984, Katsaras [13] was the first to suggest this concept while researching fuzzy topological vector spaces, his groundbreaking work [14, 15, 16] being a motivating factor for many mathematicians. The authors Cheng & Mordsen [19] introduced an alternative form of fuzzy norm for linear spaces utilizing a distinct technique . In a related context, Michalek and Kramosil [20] further investigated the associated fuzzy metric in 1994 . The concept of a fuzzy real number's criterion, as articulated by Gähler and Gähler [21], quantifies the discrepancy between its negative and positive components.

Interestingly, Samantha and Bag [22] identified an enigmatic criterion that diverged somewhat from Mordsen & Cheng's established criterion. They subsequently demonstrated an applicable decomposition theorem for this distinctive criterion. This concept has found utility in the advancement and execution of fuzzy functional analysis, leading to an array of publications from diverse researchers.

Of particular significance is the work conducted by Xiao and Zhu[26] in this domain . They explored into various aspects of fuzzy norm linear spaces, encompassing the consideration of Felbin-type fuzzy norms in their generalized manifestation. Bag and Samantha, in their contribution [27], presented a minor alteration to Felbin's concept of a fuzzy standardized linear space.

The functional equation

$$\begin{aligned}&g(u+4v)-7g(u+3v)+21g(u+2v)-35g(u+v)-21g(u-v)\\&+7g(u-2v)-g(u-3v)+35g(u)=5040g(v)\end{aligned}$$

is known as septic functional equation since cu^7 is the solution.

Similarly, the functional equation

$$\begin{aligned}&g(u+4v)-8g(u+3v)+28g(u+2v)-56g(u+v)\\&-56g(u-v)+28g(u-2v)-8g(u-3v)+g(u-4v)\\&+70g(u)=40320g(v)\end{aligned}$$

is known as octic functional equation since cu^8 is the solution. Each solution to a octic functional equation in particular is referred to as a octic mapping.

Now, the authors will address the definitions, notations, and fundamental characteristics of a non-Archimedean fuzzy normed linear space in the Felbin's type framework.

Definition 1.1. [26] A function $\sigma : R \to [0,1]$ is termed a fuzzy real number if its α-level set is represented as $[\sigma]_\alpha = \{s : \sigma(s) \geq \alpha\}$ and function satisfies two conditions:

(1) there exist $s_0 \in R$ such as $\sigma(s_0) = 1$.
(2) $[\sigma]_\alpha = [\sigma^1_\alpha, \sigma^2_\alpha]$ for each $\alpha \in (0,1]$
where $-\infty < [\sigma^1_\alpha] \leq [\sigma^2_\alpha] < +\infty$.

$\mathcal{F}$ denotes the set of all fuzzy real numbers.

Definition 1.2. [19] Let $\sigma, \varsigma \in \mathcal{F}$ and $[\sigma]_\alpha = [l^1_\alpha, m^1_\alpha]$, $[\varsigma]_\alpha = [l^2_\alpha, m^2_\alpha]$, $\alpha \in (0,1]$. Then $[\varsigma \oplus \sigma]_\alpha = [l^1_\alpha + l^2_\alpha, m^1_\alpha + m^2_\alpha]$.

Definition 1.3. [17] A partial order denoted by "$\preceq$" is established within the set $\mathcal{F}$ as follows: For any σ and ς in $\mathcal{F}$, $\sigma \preceq \varsigma$ holds if and only if, for all $\alpha \in (0,1]$, it satisfies $\sigma^1_\alpha \leq \varsigma^1_\alpha$ and $\sigma^2_\alpha \leq \varsigma^2_\alpha$, where $[\sigma]_\alpha = [\sigma^1_\alpha, \sigma^2_\alpha]$ and $[\varsigma]_\alpha = [\varsigma^1_\alpha, \varsigma^2_\alpha]$. Furthermore, a stricter inequality, denoted by " $<$ ", is defined within $\mathcal{F}$: $\sigma < \varsigma$ if and only if, for all $\alpha \in (0,1]$, the conditions $\sigma^1_\alpha < \varsigma^1_\alpha$ and $\sigma^2_\alpha < \varsigma^2_\alpha$ are satisfied.

Definition 1.4. [17] Consider a vector space U over R and $||.|| : U \to R^*(I)$(set of all upper semi continuous normal convex fuzzy real numbers) and let the mappings $\mathcal{L}, \mathcal{R} : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $\mathcal{L}(0,0) = 0$ and $\mathcal{R}(1,1) = 0$. Write

$$[||u||]_\alpha = [||u||^\alpha_1, ||u||^\alpha_2] \quad \forall \quad u \in U \quad \& \quad 0 < \alpha \leq 1$$

and suppose for all $u \in U, u \neq 0$, there exists $\alpha_0 \in (0,1]$ independent of u such that for all $\alpha \leq \alpha_0$
(I) $\quad ||u||^\alpha_2 < \infty$,
(II) $\quad ||u||^\alpha_2 > 0$. The quadruple $(U, ||.||, \mathcal{L}, \mathcal{U})$ is called a fuzzy normed linear space and $||.||$ is a fuzzy norm if

(1) $||u|| = \overline{0} \iff u = \underline{0}$.
(2) $||ru|| = |r|||u|| \quad \forall \quad u \in U, r \in R$.
(3) For all $u, v \in U$
  (a) whenever
$$p \leq ||u||^1_1, q \leq ||v||^1_1 \quad and \quad p + q \leq ||u+v||^1_1,$$
$$||u+v||(p+q) \geq \mathcal{L}(||u||(p), ||v||(q)),$$
  (b) whenever
$$p \geq ||u||^1_1, q \geq ||v||^1_1 \quad and \quad p + q \geq ||u+v||^1_1,$$
$$||u+v||(p+q) \leq \mathcal{U}(||u||(p), ||v||(q)).$$

Fuzzy norm on a linear space is defined in [17] as stated by C. Felbin. Now, as stated in [17], we define the fuzzy norm of modified Felbin's type on a linear space.

Definition 1.5. [17] Consider a linear space U over R. Let $|| \quad || : U \to \mathcal{F}^+$ be a mapping satisfying

(1) $||u|| = \overline{0} \iff u = \underline{0}$.
(2) $||ru|| = |r|||u|| \quad \forall \quad u \in U, r \in R$.
(3) $||v + u|| \preceq ||v|| \oplus ||u|| \ \forall \ v, u \in U$, and
$u \neq \underline{0} \Rightarrow ||u||(s) = 0 \quad \forall \quad s \leq 0$.

Then $(U, || \quad ||)$ is known as fuzzy normed linear space and $|| \quad ||$ is known as fuzzy norm on U.

Definition 1.6. [12] Suppose $\mathcal{K}$ be a field. An absolute value on $\mathcal{K}$ is classified as non-Archimedean field if it satisfies the following conditions for any elements a and b in $\mathcal{K}$:

(1) $|a| \geq 0$ and $|a| = 0 \iff a = 0$.
(2) $|a + b| \leq \max\{|a|, |b|\}$.
(3) $|ab| = |a||b|$.
(4) There exists $a_0 \in \mathcal{K}$ such that $|a_0| \neq 0, 1$.

The main objective of this study is to establish the generalized Hyers-Ulam stability for septic and octic functional equations within a modified Felbin-type fuzzy normed linear space. The article is organized into three sections. In section 2, we examine the stability analysis of the septic functional equation within a non-Archimedean fuzzy normed linear space of the Felbin type. Moving on to section 3, our focus shifts to the generalized Hyers-Ulam stability of the octic functional equation. This investigation takes place within a non-Archimedean fuzzy normed linear space of the Felbin type.

2. Stability of septic functional equation

The stability problems of various septic functional equations in several spaces such as intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean spaces, Banach spaces, orthogonal spaces and many other spaces have been broadly investigated by a number of mathematicians. Motivated by the approach of research by various mathematicians, an effort has been made in this paper to obtain the stability of the following functional equations.

$$\begin{aligned} &g(u + 4v) - 7g(u + 3v) + 21g(u + 2v) - 35g(u + v) - 21g(u - v) \\ &+7g(u - 2v) - g(u - 3v) + 35g(u) = 5040g(v). \qquad (2.1) \end{aligned}$$

To simplify notation, let us introduce the “difference operator ”denoted by $\Delta_{s'}$.

$$\begin{aligned}\Delta_{s'}g(u,v) &= g(u+4v)-7g(u+3v)+21g(u+2v)-35g(u+v)\\ &- 21g(u-v)+7g(u-2v)-g(u-3v)+35g(u)\\ &- 5040g(v).\end{aligned}$$

Theorem 2.1. *Suppose that U is a linear space and $(W,||.||^{\sim})$ is a fuzzy normed space. Consider $\psi: U^2 \to W$ be a mapping such that*

$$\lim_{n\to\infty}\frac{||\psi(2^n u,2^n v)||_{\alpha}^{\sim 1}}{|2^{7n}|}=\lim_{n\to\infty}\frac{||\psi(2^n u,2^n v)||_{\alpha}^{\sim 2}}{|2^{7n}|}=0, \tag{2.2}$$

for all $u,v\in U$ and $\alpha\in(0,1]$. Let $(V,||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g: U\to V$ is such that

$$||\Delta_{s'}g(u,v)||\preceq||\psi(u,v)||^{\sim} \tag{2.3}$$

for all $u,v\in U$, then there exists one and only one septic mapping $S: U\to V$ fulfilling the given condition

$$||S(u)-g(u)||\preceq\frac{1}{2^7}\max\left\{\frac{H(2^k u)}{2^{7k}};k\in N\cup 0\right\}, \tag{2.4}$$

where

$$\begin{aligned}H(2^k u) &= \left|\frac{1}{2520}\right|\Bigg[\left|\frac{1}{10080}\right|\Big(||\psi(0,6.2^k u)||^{\sim}\oplus||\psi(6.2^k u,-6.2^k u)||^{\sim}\Big)\\ &\oplus \left|\frac{1}{1440}\right|\Big(||\psi(0,4.2^k u)||^{\sim}\oplus||\psi(4.2^k u,-4.2^k u)||^{\sim}\Big)\\ &\oplus |\frac{1}{180}|\Big(||\psi(0,3.2^k u)||^{\sim}\oplus||\psi(3.2^k u,-3.2^k u)||^{\sim}\Big)\\ &\oplus \left|\frac{13}{288}\right|\Big(||\psi(0,2.2^k u)||^{\sim}\oplus||\psi(2.2^k u,-2.2^k u)||^{\sim}\Big)\\ &\oplus \left|\frac{373}{2520}\right|\Big(||\psi(0,2^k u)||^{\sim}\oplus||\psi(2^k u,-2^k u)||^{\sim}\Big)\\ &\oplus \left|\frac{1}{2}\right|||\psi(4.2^k u,2^k u)||^{\sim}\oplus\left|\frac{7}{2}\right|||\psi(3.2^k u,2^k u)||^{\sim}\\ &\oplus 11.||\psi(2.2^k u,2^k u)||^{\sim}\oplus 21.||\psi(2^k u,2^k u)||^{\sim}\\ &\oplus \left|\frac{1}{2}\right|||\psi(0,2.2^k u)||^{\sim}\oplus 28.||\psi(0,2^k u)||^{\sim}\oplus\left|\frac{217}{720}\right|||\psi(0,0)||^{\sim}\Bigg].\end{aligned}$$

Proof. Taking $u=0=v$ in (2.3), we obtain

$$||g(0)||\preceq\frac{||\psi(0,0)||^{\sim}}{|5040|}. \tag{2.5}$$

Taking $(u, v) = (0, u)$ in (2.3), the authors get

$$||g(4u) - 7g(3u) + 21g(2u) - 5075g(u) + 35g(0) - 21g(-u) \\ +7g(-2u) - g(-3u)|| \preceq ||\psi(0, u)||^{\sim}. \tag{2.6}$$

Putting $(u, v) = (u, -u)$ in (2.3), we get

$$||g(-3u) - 7g(-2u) - 5019g(-u) - 35g(0) + 35g(u) - 21g(2u) \\ +7g(3u) - g(4u)|| \preceq ||\psi(u, -u)||^{\sim}. \tag{2.7}$$

By (2.6) and (2.7), we obtain

$$||g(u) - g(-u)|| \preceq \frac{1}{|5040|}(||\psi(0, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}). \tag{2.8}$$

Putting $(u, v) = (4u, u)$ in (2.3), we get

$$||g(8u) - 7g(7u) + 21g(6u) - 35g(5u) + 35g(4u) - 21g(3u) \\ +7g(2u) - 5041g(u)|| \preceq ||\psi(4u, u)||^{\sim}. \tag{2.9}$$

Taking $(u, v) = (0, 2u)$ in (2.3), the authors get

$$||g(8u) - 7g(6u) + 21g(4u) - 5075g(2u) + 35g(0) - 21g(-2u) \\ +7g(-4u) - g(-6u)|| \preceq ||\psi(0, 2u)||^{\sim}. \tag{2.10}$$

By (2.9) and (2.10), we obtain

$$||7g(7u) - 28g(6u) + 35g(5u) - 14g(4u) + 21g(3u) \\ -5082g(2u) + 5041g(u) + 35g(0) - 21g(-2u) \\ +7g(-4u) - g(-6u)|| \preceq (||\psi(4u, u)||^{\sim} \oplus ||\psi(0, 2u)||^{\sim}). \tag{2.11}$$

Now, using (2.5), (2.8) and (2.11), we conclude

$$||7g(7u) - 27g(6u) + 35g(5u) - 21g(4u) + 21g(3u) - 5061g(2u) \\ +5041g(u)|| \preceq \frac{1}{|5040|}(||\psi(0, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \\ \oplus \frac{1}{|720|}(\psi(||0, 4u)||^{\sim} \oplus ||\psi(4u, -4u)||^{\sim}) \oplus \frac{1}{|240|}(||\psi(0, 2u)||^{\sim} \\ \oplus||\psi(2u, -2u)||^{\sim}) \oplus ||\psi(4u, u)||^{\sim} \oplus ||\psi(0, 2u)||^{\sim} \\ \oplus \frac{1}{144}||\psi(0, 0)||^{\sim}. \tag{2.12}$$

Putting $(u, v) = (3u, u)$ in (2.3), we get

$$||g(7u) - 7g(6u) + 21g(5u) - 35g(4u) + 35g(3u) - 21g(2u) \\ -5033g(u) - g(0)|| \preceq ||\psi(3u, u)||^{\sim}. \tag{2.13}$$

From (2.5), we obtain

$$||g(7u) - 7g(6u) + 21g(5u) - 35g(4u) + 35g(3u) - 21g(2u)$$
$$-5033g(u)|| \preceq ||\psi(3u, u)||^{\sim} \oplus \frac{1}{|5040|}||\psi(0, 0)||^{\sim}. \quad (2.14)$$

From (2.12) and (2.14), we obtain

$$||11g(6u) - 56g(5u) + 112g(4u) - 112g(3u) - 2457g(2u)$$
$$+20136g(u)|| \preceq \frac{1}{|10080|}(||\psi(0, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim})$$
$$\oplus\frac{1}{|1440|}(||\psi(0, 4u)||^{\sim} \oplus ||\psi(4u, -4u)||^{\sim}) \oplus \frac{1}{|480|}(||\psi(0, 2u)||^{\sim}$$
$$\oplus||\psi(2u, -2u)||^{\sim}) \oplus \frac{1}{2}||\psi(4u, u)||^{\sim} \oplus \frac{7}{2}||\psi(3u, u)||^{\sim}$$
$$\oplus\frac{1}{2}||\psi(0, 2u)||^{\sim} \oplus \frac{1}{240}||\psi(0, 0)||^{\sim}. \quad (2.15)$$

Putting $(u, v) = (2u, u)$ in (2.3), we obtain

$$||g(6u) - 7g(5u) + 21g(4u) - 35g(3u) + 35g(2u) - 5061g(u)$$
$$-g(-u) + 7g(0)|| \preceq ||\psi(2u, u)||^{\sim}. \quad (2.16)$$

Now, by using (2.5), (2.8) and (2.16), we get

$$||g(6u) - 7g(5u) + 21g(4u) - 35g(3u) + 35g(2u) - 5060g(u)||$$
$$\preceq ||\psi(2u, u)||^{\sim} \oplus \frac{1}{|5040|}(||\psi(0, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim})$$
$$\oplus\frac{1}{|720|}||\psi(0, 0)||^{\sim}. \quad (2.17)$$

From (2.15)and (2.17), the authors obtain

$$||21g(5u) - 119g(4u) + 273g(3u) - 2842g(2u) + 75796g(u)||$$
$$\preceq \frac{1}{|10080|}(||\psi(0, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \oplus \frac{1}{|1440|}(||\psi(0, 4u)||^{\sim}$$
$$\oplus||\psi(4u, -4u)||^{\sim}) \oplus \frac{1}{|480|}(||\psi(0, 2u)||^{\sim} \oplus |\psi(2u, -2u)||^{\sim})$$
$$\oplus|\frac{11}{5040}|(||\psi(0, u)||^{\sim} \oplus |\psi(u, -u)||^{\sim}) \oplus \frac{1}{|2|}|\psi(4u, u)||^{\sim}$$
$$\oplus|\frac{7}{2}|\psi(3u, u)||^{\sim} \oplus 11||\psi(2u, u)||^{\sim} \oplus \frac{1}{|2|}|\psi(0, 2u)||^{\sim}$$
$$\oplus|\frac{7}{360}|\psi(0, 0)||^{\sim}. \quad (2.18)$$

Putting $(u, v) = (u, u)$ in (2.3), we get

$$||g(5u) - 7g(4u) + 21g(3u) - 35g(2u) - g(-2u) - 5005g(u) \\ +7g(-u) - 21g(0)|| \preceq ||\psi(u, u)||^{\sim}. \tag{2.19}$$

With the help of (2.5), (2.8) and (2.19), we get

$$||g(5u) - 7g(4u) + 21g(3u) - 34g(2u) - 5012g(u)|| \preceq ||\psi(u, u)||^{\sim} \\ \oplus ||\psi(u, u)||^{\sim} \oplus \frac{1}{|5040|}(||\psi(0, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \\ \oplus \frac{1}{|720|}(||\psi(0, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}) \oplus \frac{1}{|240|}||\psi(0, 0)||^{\sim}). \tag{2.20}$$

Now, from (2.18) and (2.20), we conclude

$$||28g(4u) - 168g(3u) - 21328g(2u) + 181048g(u)|| \\ \preceq \frac{1}{|10080|}(||\psi(0, 6u)||^{\sim} \oplus |\psi(6u, -6u)||^{\sim}) \oplus \frac{1}{|1440|}(||\psi(0, 4u)||^{\sim} \\ \oplus ||\psi(4u, -4u)||^{\sim}) \oplus \frac{1}{|160|}(||\psi(0, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \\ \oplus \frac{79}{2520}(||\psi(0, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}) \oplus \frac{1}{2}||\psi(4u, u)||^{\sim} \\ \oplus \frac{7}{2}||\psi(3u, u)||^{\sim} \oplus 11||\psi(2u, u)||^{\sim} \oplus 21||\psi(u, u)||^{\sim} \\ \oplus \frac{1}{2}||\psi(0, 2u)||^{\sim} \oplus \frac{77}{720}||\psi(0, 0)||^{\sim}. \tag{2.21}$$

Now, by using (2.5), (2.6) and (2.8), we obtain

$$||g(4u) - 6g(3u) + 14g(2u) - 5054g(u)|| \preceq \frac{1}{|5040|}(||\psi(0, 3u)||^{\sim} \\ \oplus ||\psi(3u, -3u)||^{\sim}) \oplus \frac{1}{|720|}(||\psi(0, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \\ \oplus \frac{1}{|240|}(||\psi(0, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}) \oplus ||\psi(0, u)||^{\sim} \\ \oplus \frac{1}{|144|}||\psi(0, 0)||^{\sim}. \tag{2.22}$$

Now, by (2.21) and (2.22), the authors conclude that

$$||g(2u) - 2^7 g(u)|| \preceq H(u) \tag{2.23}$$

where

$$H(u) = \frac{1}{2520}\Bigg[\frac{1}{10080}(||\psi(0,6u)||^{\sim} \oplus ||\psi(6u,-6u)||^{\sim})$$
$$\oplus\frac{1}{1440}(||\psi(0,4u)||^{\sim} \oplus ||\psi(4u,-4u)||^{\sim}) \oplus \frac{1}{180}(||\psi(0,3u)||^{\sim}$$
$$\oplus||\psi(3u,-3u)||^{\sim}) \oplus \frac{13}{288}(||\psi(0,2u)||^{\sim} \oplus ||\psi(2u,-2u)||^{\sim})$$
$$\oplus\frac{373}{2520}(||\psi(0,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \oplus \frac{1}{2}||\psi(4u,u)||^{\sim}$$
$$\oplus\frac{7}{2}||\psi(3u,u)||^{\sim} \oplus 11||\psi(2u,u)||^{\sim} \oplus ||\psi(u,u)||^{\sim} \oplus \frac{1}{2}||\psi(0,2u)||^{\sim}$$
$$\oplus 28||\psi(0,u)||^{\sim} \oplus \frac{217}{720}||\psi(0,0)||^{\sim}\Bigg].$$

Hence from (2.23), we get

$$||g(2u) - 2^7 g(u)||_\alpha^1 \leq H_1(u), \tag{2.24}$$

and

$$||g(2u) - 2^7 g(u)||_\alpha^2 \leq H_2(u), \tag{2.25}$$

where for $\alpha \in (0,1]$, then

$$H_1(u) = \frac{1}{2520}\Bigg[\frac{1}{10080}(||\psi(0,6u)||_\alpha^{1\sim} \oplus ||\psi(6u,-6u)||_\alpha^{1\sim})$$
$$\oplus\frac{1}{1440}(||\psi(0,4u)||_\alpha^{1\sim} \oplus ||\psi(4u,-4u)||_\alpha^{1\sim}) \oplus \frac{1}{180}(||\psi(0,3u)||_\alpha^{1\sim}$$
$$\oplus||\psi(3u,-3u)||_\alpha^{1\sim}) \oplus \frac{13}{288}(||\psi(0,2u)||_\alpha^{1\sim} \oplus ||\psi(2u,-2u)||_\alpha^{1\sim})$$
$$\oplus\frac{373}{2520}(||\psi(0,u)||_\alpha^{1\sim} \oplus ||\psi(u,-u)||_\alpha^{1\sim}) \oplus \frac{1}{2}||\psi(4u,u)||_\alpha^{1\sim}$$
$$\oplus\frac{7}{2}||\psi(3u,u)||_\alpha^{1\sim} \oplus 11||\psi(2u,u)||_\alpha^{1\sim} \oplus ||\psi(u,u)||_\alpha^{1\sim} \oplus \frac{1}{2}||\psi(0,2u)||_\alpha^{1\sim}$$
$$\oplus 28||\psi(0,u)||_\alpha^{1\sim} \oplus \frac{217}{720}||\psi(0,0)||_\alpha^{1\sim}\Bigg] \tag{2.26}$$

and

$$H_2(u) = \frac{1}{2520}\Bigg[\frac{1}{10080}(||\psi(0,6u)||_\alpha^{2\sim} \oplus ||\psi(6u,-6u)||_\alpha^{2\sim})$$
$$\oplus\frac{1}{1440}(||\psi(0,4u)||_\alpha^{2\sim} \oplus ||\psi(4u,-4u)||_\alpha^{2\sim}) \oplus \frac{1}{180}(||\psi(0,3u)||_\alpha^{2\sim}$$
$$\oplus||\psi(3u,-3u)||_\alpha^{2\sim}) \oplus \frac{13}{288}(||\psi(0,2u)||_\alpha^{2\sim} \oplus ||\psi(2u,-2u)||_\alpha^{2\sim})$$
$$\oplus\frac{373}{2520}(||\psi(0,u)||_\alpha^{2\sim} \oplus ||\psi(u,-u)||_\alpha^{2\sim}) \oplus \frac{1}{2}||\psi(4u,u)||_\alpha^{2\sim}$$
$$\oplus\frac{7}{2}||\psi(3u,u)||_\alpha^{2\sim} \oplus 11||\psi(2u,u)||_\alpha^{2\sim} \oplus ||\psi(u,u)||_\alpha^{2\sim} \oplus \frac{1}{2}||\psi(0,2u)||_\alpha^{2\sim}$$
$$\oplus 28||\psi(0,u)||_\alpha^{2\sim} \oplus \frac{217}{720}||\psi(0,0)||_\alpha^{2\sim}\Bigg]. \tag{2.27}$$

From (2.25), we conclude

$$||\frac{g(2u)}{2^7} - g(u)||_\alpha^{1\sim} \leq \frac{H_1(u)}{|2^7|}. \tag{2.28}$$

Replacing u by $2^n u$ in (2.28) and dividing both side by 2^{7n}, we obtain

$$\left|\left|\frac{g(2^{n+1}u)}{2^{7(n+1)}} - \frac{g(2^n u)}{2^{7n}}\right|\right|_\alpha^1 \leq \frac{H_1(2^n u)}{|2^{7(n+1)}|} \tag{2.29}$$

for all non-negative integers n. Hence the sequence $\{\frac{g(2^n u)}{2^{7n}}\}$ is Cauchy. Every Cauchy sequence is convergent in Y, since Y is complete. So, the authors construct a mapping $S : U \to V$ such that

$$S(u) = \lim_{n\to\infty} \frac{g(2^n u)}{2^{7n}}, \tag{2.30}$$

i.e.,

$$\lim_{n\to\infty} \left|\left|\frac{g(2^n u)}{2^{7n}} - S(u)\right|\right| = \overline{0}. \tag{2.31}$$

Now for each non-negative integer n, the authors explore

$$\begin{aligned}\left|\left|\frac{g(2^n u)}{2^{7n}} - g(u)\right|\right|_\alpha^1 &= \left|\left|\sum_{k=0}^{n-1}\left(\frac{g(2^{k+1}u)}{2^{7(n+1)}} - \frac{g(2^k u)}{2^{7n}}\right)\right|\right|_\alpha^1 \\ &\leq \max\left\{\left|\left|\frac{g(2^{k+1}u)}{2^{7(n+1)}} - \frac{g(2^k u)}{2^{7n}}\right|\right|_\alpha^1 : 0 \leq k < n\right\} \\ &\leq \frac{1}{2^7}\max\left\{\frac{H_1(2^k u)}{|2^{7k}|} : 0 \leq k < n\right\}. \end{aligned}\tag{2.32}$$

Similarly, the authors can show that

$$\left|\left|\frac{g(2^n u)}{2^{7n}} - g(u)\right|\right|_\alpha^2 \leq \frac{1}{2^7} \max\left\{\frac{H_2(2^k u)}{|2^{7k}|} : 0 \leq k < n\right\}. \tag{2.33}$$

Taking $n \to \infty$ in (2.32) and (2.33), the authors see that inequality (2.4) holds. Next we prove that $S : U \to V$ is a cubic mapping. Replacing (u, v) by $(2^n u, 2^n v)$ and divide by $|2^{7n}|$ in (2.3), we get

$$\begin{aligned}
&\frac{1}{2^{7n}}||g(2^n(u+4v)) - 7g(2^n(u+3v)) + 21g(2^n(u+2v)) \\
&-35(2^n(u+v)) - 21g(2^n(u-v)) + 7g(2^n(u-2v)) \\
&-g(2^n(u-3v)) + 35g(2^n u) - 5040g(2^n v)|| \\
&\preceq \left|\left|\frac{\psi(2^n u, 2^n v)}{2^{7n}}\right|\right|^{\sim}
\end{aligned} \tag{2.34}$$

Taking $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
&||S(u+4v) - 3S(u+3v) + 21S(u+2v) - 35S(u+v) - 21S(u-v) \\
&+7S(u-2v) - S(u-3v) + 35S(u) - 5040S(v)|| \preceq \overline{0}
\end{aligned}$$

this implies that

$$\begin{aligned}
&S(u+4v) - 3S(u+3v) + 21S(u+2v) - 35S(u+v) - 21S(u-v) \\
&+7S(u-2v) - S(u-3v) + 35S(u) - 5040S(v) = \underline{0}.
\end{aligned}$$

□

Therefore, the mapping $S : U \to V$ is septic. Next we shall prove uniqueness of mapping S. Now, consider another septic mapping $S' : U \to V$ which satisfies (2.1) and (2.4). For fix $u \in U$, certainly $S(2^n u) = 2^{7n} S(u)$ and $S'(2^n u) = 2^{7n} S'(u)$ for all $n \in N$. Therefore,

$$\begin{aligned}
&||S(u) - S'(u)|| = \lim_{n\to\infty} \frac{1}{2^{7n}}||S(2^n u) - S'(2^n u)|| \\
&= \lim_{n\to\infty} \frac{1}{2^{7n}}||S(2^n u) - g(2^n u) + g(2^n u) - S'(2^n u)|| \\
&\preceq \lim_{n\to\infty} \max\left\{\frac{1}{2^{7n}}||S(2^n u) - g(2^n u)||, \frac{1}{2^{7n}}||g(2^n u) - S'(2^n u)||\right\} \\
&\preceq \lim_{k\to\infty} \lim_{n\to\infty} \max\left\{\max\left\{\frac{1}{2^{7(n+1)}}\left\{\frac{H(2^{k+n}u)}{2^{7k}}, \frac{H(2^{k+n}u)}{2^{7k}}\right\}\right\}\right\} \\
&= \overline{0}.
\end{aligned}$$

Therefore, $S(u) - S'(u) = \underline{0}$. So, $S(u) = S'(u)$ Hence, we deduced that S is unique mapping.

Corollary 2.2. *Suppose that* $(U, ||.||)$ *is a normed space,* $(W, || \quad ||^{\sim})$ *is a fuzzy normed space, and* $(V, || \quad ||)$ *is a non-Archimedean complete fuzzy normed space. Let* $w_0 \in W$ *and* $p < 7$ *be non-negative real numbers, respectively. If the mapping* $g : U \to V$ *is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||(||v||^p + ||u||^p)w_0||^{\sim} \tag{2.35}$$

for all $u, v \in U$*, then there exists one and only one septic mapping* $S : U \to V$ *fulfilling the given condition*

$$\begin{aligned} ||S(u) - g(u)|| \quad &\preceq \quad \frac{|| \, ||u||^p w_0||^{\sim}}{2^7} \left[\left| \frac{1}{2520} \right| \left(\left| \frac{50773}{840} \right| \oplus \left| \frac{61|2|^p}{96} \right| \oplus \left| \frac{|3|^p}{60} \right| \right. \right. \\ &\oplus \quad \left| \frac{|4|^p}{480} \right| \oplus \left| \frac{|6|^p}{3360} \right| \oplus \left| \frac{|4|^p + 1}{2} \right| \oplus \left| \frac{7(|3|^p + 1)}{2} \right| \\ &\oplus \quad \left. \left. |11(|2^p| + 1)| \right) \right]. \end{aligned}$$

Corollary 2.3. *Suppose that* $(U, ||.||)$ *is a normed space,* $(W, || \quad ||^{\sim})$ *is a fuzzy normed space, and* $(V, || \quad ||)$ *is a non-Archimedean complete fuzzy normed space. Let* $w_0 \in W$ *and* $p, q < 7$ *be non-negative real numbers, respectively. If the mapping* $g : U \to V$ *is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||(||v||^p ||u||^q)w_0||^{\sim}$$

for all $u, v \in U$*, then there exists one and only one septic mapping* $S : U \to V$ *fulfilling the given condition*

$$\begin{aligned} ||S(u) - g(u)|| \quad &\preceq \quad \frac{|| \, ||u||^{p+q} w_0||^{\sim}}{2^7} \left[\left| \frac{1}{2520} \right| \left(\left| \frac{52920}{2520} \right| \oplus \left| \frac{13|2|^{p+q}}{288} \right| \oplus \left| \frac{|3|^{p+q}}{180} \right| \right. \right. \\ &\oplus \quad \left| \frac{|4|^{p+q}}{1440} \right| \oplus \left| \frac{|6|^{p+q}}{10080} \right| \oplus \left| \frac{|4|^p}{2} \right| \oplus \left| \frac{7(|3|^p)}{2} \right| \\ &\oplus \quad \left. \left. |11(|2^p|)| \right) \right]. \end{aligned}$$

Corollary 2.4. *Suppose that* $(U, ||.||)$ *is a normed space,* $(W, || \quad ||^{\sim})$ *is a fuzzy normed space, and* $(V, || \quad ||)$ *is a non-Archimedean complete fuzzy normed space. Let* $w_0 \in W$ *and* $\lambda = s + r < 7$ *be non-negative real numbers, respectively. If the mapping* $g : U \to V$ *is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||[||v||^s ||u||^r + (||u||^{s+r} + ||v||^{s+r})]w_0||^{\sim} \tag{2.36}$$

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for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| &\preceq \frac{|| \, ||u||^{\lambda} w_0||^{\sim}}{2^7} \left[\left| \frac{1}{2520} \right| \left(\left| \frac{6733}{630} \right| \oplus \left| \frac{49|2|^{\lambda}}{72} \right| \oplus \left| \frac{|3|^{\lambda}}{45} \right| \right. \right. \\ &\oplus \left| \frac{|4|^{\lambda}}{380} \right| \oplus \left| \frac{|6|^{\lambda}}{2520} \right| \oplus \left| \frac{7(|3|^{r} + |3|^{\lambda} + 1)}{2} \right| \\ &\oplus \left. \left. \left| \frac{|4|^{r} + |4|^{\lambda} + 1}{2} \right| \oplus |11(|2|^{r} + 2^{\lambda} + 1)| \right) \right]. \end{aligned}$$

Theorem 2.5. *Suppose that U is a linear space and $(W, ||.||^{\sim})$ is a fuzzy normed space. Consider $\psi : U^2 \to W$ be a mapping such that*

$$\lim_{n\to\infty} |2^{7n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 1} = \lim_{n\to\infty} |2^{7n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 2} = 0, \quad (2.37)$$

for all $u \in U$ and $\alpha \in (0, 1]$. Let $(V, ||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g : U \to V$ is such that

$$||\Delta_{s'} g(u, v)|| \preceq ||\psi(u, v)||^{\sim} \quad (2.38)$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$||S(u) - g(u)|| \preceq \frac{1}{2^7} \max \left\{ |2^{7(k+1)}| H\left(\frac{u}{2^{k+1}}\right), k \in N \cup \{0\} \right\} \quad (2.39)$$

where

$$\begin{aligned}
H(\frac{u}{2^{k+1}}) &= \left|\frac{1}{2520}\right|\Bigg[\left|\frac{1}{10080}\right|\left(||\psi\left(0,\frac{6u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{6u}{2^{k+1}},\frac{-6u}{2^{k+1}}\right)||^{\sim}\right)\\
&\oplus \left|\frac{1}{1440}\right|\left(||\psi\left(0,\frac{4u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{4u}{2^{k+1}},\frac{-4u}{2^{k+1}}\right)||^{\sim}\right)\\
&\oplus |\frac{1}{180}|\left(||\psi\left(0,\frac{3u}{2^{k+1}}\right)||^{\sim}\oplus||\psi(\frac{3u}{2^{k+1}},\frac{-3u}{2^{k+1}})||^{\sim}\right)\\
&\oplus \left|\frac{13}{288}\right|\left(||\psi\left(0,\frac{2u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{2u}{2^{k+1}},\frac{-2u}{2^{k+1}}\right)||^{\sim}\right)\\
&\oplus \left|\frac{373}{2520}\right|\left(||\psi\left(0,\frac{u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{u}{2^{k+1}},\frac{-u}{2^{k+1}}\right)||^{\sim}\right)\\
&\oplus \left|\frac{1}{2}\right|||\psi\left(\frac{4u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\oplus\left|\frac{7}{2}\right|||\psi\left(\frac{3u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\\
&\oplus 11.||\psi\left(\frac{2u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\oplus 21.||\psi\left(\frac{u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\\
&\oplus \left|\frac{1}{2}\right|||\psi\left(0,\frac{2u}{2^{k+1}}\right)||^{\sim}\oplus 28.||\psi\left(0,\frac{u}{2^{k+1}}\right)||^{\sim}\\
&\oplus \left|\frac{217}{720}\right|||\psi(0,0)||^{\sim}\Bigg].
\end{aligned}$$

Proof. From (2.24), the authors get

$$\left|\left|g(u)-2^7g\left(\frac{u}{2}\right)\right|\right|_{\alpha}^{1}\le H_1\left(\frac{u}{2}\right) \tag{2.40}$$

for $\alpha\in(0,1]$. Replacing u by $\frac{u}{2^n}$ and multiplying both side by $|2^{7n}|$ in (2.40), we get

$$\left|\left|2^{7n}g\left(\frac{u}{2^n}\right)-2^{7(n+1)}g\left(\frac{u}{2^{n+1}}\right)\right|\right|_{\alpha}^{1}\le|2^{7n}|H_1\left(\frac{u}{2^{n+1}}\right) \tag{2.41}$$

for all negative integer n. Hence the sequence $g\{2^{7n}g(\frac{u}{2^n})\}$ is Cauchy by (2.37) and (2.41). Every Cauchy sequence is convergent in Y, since Y is complete. So, the authors construct a mapping $S:U\to V$ such that

$$S(u)=\lim_{n\to\infty}2^{7n}g\left(\frac{u}{2^n}\right)$$

for all $u\in U$. That is

$$\lim_{n\to\infty}||2^{7n}g\left(\frac{u}{2^n}\right)-S(u)||=\overline{0}$$

for all $u \in U$. Now, for each positive integer n, the authors have

$$\left|\left|2^{7n}g\left(\frac{u}{2^n}\right)-g(u)\right)\right|\right|_\alpha^1 = \left|\left|\sum_{k=0}^{n-1}\left(2^{7(k+1)}g\left(\frac{u}{2^{k+1}}\right)-2^{7k}g\left(\frac{u}{2^k}\right)\right)\right|\right|_\alpha^1$$
$$\leq \max\left\{\left|\left|\left(2^{7(k+1)}g\left(\frac{u}{2^{k+1}}\right)-2^{7k}g\left(\frac{u}{2^k}\right)\right)\right|\right|_\alpha^1 ; 0\leq k<n\right\}$$
$$\leq \frac{1}{|2^7|}\max\left\{|2^{7(k+1)}|H_1\left(\frac{u}{2^{k+1}}\right):0\leq k<n\right\}. \tag{2.42}$$

Similarly , it can be shown from (2.25)

$$\left|\left|2^{7n}g\left(\frac{u}{2^n}\right)-g(u)\right)\right|\right|_\alpha^2$$
$$\leq \frac{1}{|2^7|}\max\left\{|2^{7(k+1)}|H_2\left(\frac{u}{2^{k+1}}\right):0\leq k<n\right\}. \tag{2.43}$$

Taking $n\to\infty$ in (2.42) and (2.43), the authors see that inequality (2.39) holds. The authors conclude that $S(u)$ is a unique cubic mapping holding (2.39) using the same procedure as in the demonstration of theorem (2.1).

□

Corollary 2.6. *Suppose that $(U,||.||)$ is a normed space, $(W,||\quad||^\sim)$ is a fuzzy normed space, and $(V,||\quad||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0\in W$ and $p>7$ be non-negative real numbers, respectively. If the mapping $g:U\to V$ is such that*

$$||\Delta_s g(u,v)|| \preceq ||(||v||^p+||u||^p)w_0||^\sim \tag{2.44}$$

for all $u,v\in U$, then there exists one and only one septic mapping $S:U\to V$ fulfilling the given condition

$$\begin{aligned}||S(u)-g(u)|| \quad \preceq \quad & \frac{||\,||u||^p w_0||^\sim}{|2^p|}\left[\left|\frac{1}{2520}\right|\left(\left|\frac{50773}{840}\right|\oplus\left|\frac{61|2|^p}{96}\right|\oplus\left|\frac{|3|^p}{60}\right|\right.\right.\\ \oplus \quad & \left|\frac{|4|^p}{480}\right|\oplus\left|\frac{|6|^p}{3360}\right|\oplus\left|\frac{|4|^p+1}{2}\right|\oplus\left|\frac{7(|3|^p+1)}{2}\right|\\ \oplus \quad & \left.\left.|11(|2^p|+1)|\right)\right]\end{aligned}$$

for all $u\in U$.

Corollary 2.7. *Suppose that $(U,||.||)$ is a normed space, $(W,||\quad||^\sim)$ is a fuzzy normed space, and $(V,||\quad||)$ is a non-Archimedean complete* *fuzzy normed space. Let $w_0 \in W$ and $p, q > 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||(||v||^p ||u||^q) w_0||^{\sim}$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| \preceq & \frac{|| \, ||u||^{p+q} w_0||^{\sim}}{2^7} \Bigg[\left| \frac{1}{2520} \right| \Bigg(\left| \frac{52920}{2520} \right| \oplus \left| \frac{13|2|^{p+q}}{288} \right| \oplus \left| \frac{|3|^{p+q}}{180} \right| \\ \oplus & \left| \frac{|4|^{p+q}}{1440} \right| \oplus \left| \frac{|6|^{p+q}}{10080} \right| \oplus \left| \frac{|4|^p}{2} \right| \oplus \left| \frac{7(|3|^p)}{2} \right| \\ \oplus & |11(|2^p|)| \Bigg) \Bigg]. \end{aligned}$$

Corollary 2.8. *Suppose that $(U, ||.||)$ is a normed space, $(W, || \quad ||^{\sim})$ is a fuzzy normed space, and $(V, || \quad ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $\lambda = s + r > 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||[||v||^s ||u||^r + (||v||^{s+r} + ||u||^{s+r})] w_0||^{\sim} \tag{2.45}$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| \preceq & \frac{|| \, ||u||^{\lambda} w_0||^{\sim}}{|2|^{\lambda}} \Bigg[\left| \frac{1}{2520} \right| \Bigg(\left| \frac{6733}{630} \right| \oplus \left| \frac{49|2|^{\lambda}}{72} \right| \oplus \left| \frac{|3|^{\lambda}}{45} \right| \\ \oplus & \left| \frac{|4|^{\lambda}}{380} \right| \oplus \left| \frac{|6|^{\lambda}}{2520} \right| \oplus \left| \frac{7(|3|^r + |3|^{\lambda} + 1)}{2} \right| \\ \oplus & \left| \frac{|4|^r + |4|^{\lambda} + 1}{2} \right| \oplus |11(|2|^r + 2^{\lambda} + 1)| \Bigg) \Bigg] \end{aligned}$$

for all $u \in U$.

Counterexample 2.9. Consider a real Banach algebra $(U, ||.||)$ and a non-Archimedean complete fuzzy norm space $(U, || \quad ||^{\sim})$ in which

$$||u||^{\sim}(t) = \begin{cases} \frac{||u||^7}{t}, & when \ ||u||^7 < t, t \neq 0 \\ 1, & when \ ||u||^7 = t = 0 \\ 0, & otherwise. \end{cases}$$

whose α-level set is defined as $[||u||^{\sim}]_{\alpha} = [||u||^7, \frac{||u||^7}{\alpha}]$. Construct a mapping $g : U \to U$ such that $g(u) = u^7 + ||u||^7 u_0$, where u_0 is a unit vector and

$$||\Delta_o g(u,v)||^{\sim} \preceq ||(128||u||^7 + 42560||v||^7) u_0||^{\sim},$$

then there does not exist a septic mapping $S : U \to U$ fulfilling the given condition

$$||S(u) - g(u)||^{\sim} \preceq 2^7||||u||^7 u_0||^{\sim}.$$

3. Stability of octic functional equation

The stability problems of various octic functional equations in several spaces such as intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean spaces, Banach spaces, orthogonal spaces and many other spaces have been broadly investigated by a number of mathematicians. Motivated by the approach of research by various mathematicians, an effort has been made in this paper to obtain the stability of the following functional equations.

$$\begin{aligned}
&g(u+4v) - 8g(u+3v) + 28g(u+2v) - 56g(u+v) \\
&-56g(u-v) + 28g(u-2v) - 8g(u-3v) + g(u-4v) \\
&+70g(u) = 40320g(v). \qquad (3.1)
\end{aligned}$$

To simplify notation, let us introduce the "difference operator "denoted by Δ_o.

$$\begin{aligned}
\Delta_o g(u,v) &= g(u+4v) - 8g(u+3v) + 28g(u+2v) - 56g(u+v) \\
&- 56g(u-v) + 28g(u-2v) - 8g(u-3v) + g(u-4v) \\
&+ 70g(u) - 40320g(v).
\end{aligned}$$

Theorem 3.1. *Suppose that U is a linear space and $(W, ||.||^{\sim})$ is a fuzzy normed space. Consider $\psi : U^2 \to W$ be a mapping such that*

$$\lim_{n\to\infty} \frac{||\psi(2^n u, 2^n v)||^{\sim 1}_{\alpha}}{|2^{8n}|} = \lim_{n\to\infty} \frac{||\psi(2^n u, 2^n v)||^{\sim 2}_{\alpha}}{|2^{8n}|} = 0, \qquad (3.2)$$

for all $u, v \in U$ and $\alpha \in (0, 1]$. Let $(V, ||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g : U \to V$ is such that

$$||\Delta_o g(u,v)|| \preceq ||\psi(u,v)||^{\sim} \qquad (3.3)$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$||O(u) - g(u)|| \preceq \frac{1}{2^8} \max\left\{\frac{H(2^k u)}{2^{8k}}; k \in N \cup 0\right\}, \qquad (3.4)$$

where

$$\begin{aligned}
H(2^k u) &= \left|\frac{1}{20160}\right|\Bigg[\left|\frac{1}{80640}\right|\Big(||\psi(8.2^k u, 8.2^k u)||^\sim \oplus ||\psi(8.2^k u, -8.2^k u)||^\sim\Big) \\
&\oplus \left|\frac{1}{10080}\right|\Big(||\psi(6.2^k u, 6.2^k u)||^\sim \oplus ||\psi(6.2^k u, -6.2^k u)||^\sim\Big) \\
&\oplus |\frac{7}{960}|\Big(||\psi(4.2^k u, 4.2^k u)||^\sim \oplus ||\psi(4.2^k u, -4.2^k u)||^\sim\Big) \\
&\oplus \left|\frac{1}{15}\right|\Big(||\psi(3.2^k, 3.2^k u)||^\sim \oplus ||\psi(3.2^k u, -3.2^k u)||^\sim \\
&\oplus \left|\frac{139}{480}\right|\Big(||\psi(2.2^k u, 2.2^k u)||^\sim \oplus ||\psi(2.2^k u, -2.2^k u)||^\sim\Big) \\
&\oplus \left|\frac{417}{560}\right|\Big((||\psi(2^k u, 2^k u)||^\sim \oplus ||\psi(2^k u, -2^k u)||^\sim\Big) \\
&\oplus ||\psi(4.2^k u, 2^k u)||^\sim \oplus |8|.||\psi(3.2^k u, 2^k u)||^\sim \oplus |28|.||\psi(2.2^k u, 2^k u)||^\sim \\
&\oplus |56|.||\psi(2^k u, 2^k u)||^\sim \oplus |35|.||\psi(0, 2^k u)||^\sim \oplus \left|\frac{851}{672}\right|||\psi(0,0||^\sim\Bigg]
\end{aligned}$$

for all $u \in U$.

Proof. Taking $u = 0 = v$ in (3.3), we have

$$||g(0)|| \preceq \frac{1}{|40320|}||\psi(0,0)||^\sim. \tag{3.5}$$

Replacing (u, v) in (3.3) with (u,-v), we get

$$\begin{aligned}
&||g(u+4v) - 8g(u+3v) + 28g(u+2v) - 56g(u+v) \\
&-56g(u-v) + 28g(u-2v) - 8g(u-3v) + g(u-4v) \\
&+70g(u) - 40320g(-v)|| \preceq ||\psi(u,-v)||^\sim.
\end{aligned} \tag{3.6}$$

By using (3.3) and (3.6), we get

$$||g(u) - g(-u)|| \preceq \frac{1}{|40320|}(||\psi(u,u)||^\sim \oplus ||\psi(u,-u)||^\sim) \tag{3.7}$$

Replacing (u, v) in (3.3) with (0,2u), the authors get

$$\begin{aligned}
&||g(8u) - 8g(6u) + 28g(4u) - 40376g(2u) + 70g(0) - 56g(-2u) \\
&+28g(-4u) - 8g(-6u) + g(-8u)|| \preceq ||\psi(0,2u)||^\sim.
\end{aligned} \tag{3.8}$$

Now, by using (3.5), (3.7) and (3.8), we obtain

$$||g(8u) - 8g(6u) + 28g(4u) - 20216g(2u)|| \preceq \frac{1}{|80640|}(||\psi(8u, 8u)||^{\sim} \oplus||\psi(8u, -8u)||^{\sim}) \oplus \frac{1}{|10080|}(||\psi(6u, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \oplus\frac{1}{|2880|}(||\psi(4u, 4u)||^{\sim} \oplus ||\psi(4u, -4u)||^{\sim})\frac{1}{|1440|}(||\psi(2u, 2u)||^{\sim} \oplus||\psi(2u, -2u)||^{\sim}) \oplus \frac{1}{|1152|}||\psi(0, 0)||^{\sim}. \tag{3.9}$$

Replacing (u, v) in (3.3) with (4u,u), we get

$$||g(8u) - 8g(7u) + 28g(6u) - 56g(5u) + 70g(4u) - 56g(3u) +28g(2u) - 40328g(u) + g(0)|| \preceq ||\psi(4u, u)||^{\sim}. \tag{3.10}$$

By using (3.5), (3.9) and (3.10), we obtain

$$||8g(7u) - 36g(6u) + 56g(5u) - 42g(4u) + 56g(3u) - 20244g(2u) +40328g(u)|| \preceq \frac{1}{|80640|}(||\psi(8u, 8u)||^{\sim} \oplus ||\psi(8u, -8u)||^{\sim}) \oplus\frac{1}{|10080|}(||\psi(6u, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \oplus \frac{1}{|2880|}(||\psi(4u, 4u)||^{\sim} \oplus||\psi(4u, -4u)||^{\sim})\frac{1}{|1440|}(||\psi(2u, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \oplus||\psi(4u, u)||^{\sim} \oplus \frac{1}{|1120|}||\psi(0, 0)||^{\sim}. \tag{3.11}$$

Replacing (u, v) in (3.3) with (3u,u), we get

$$||g(7u) - 8g(6u) + 28g(5u) - 56g(2u) - 40292g(u) +g(-u) - 8g(0)|| \preceq ||\psi(3u, u)||^{\sim}. \tag{3.12}$$

Now, by using (3.5), (3.7) and (3.12), we obtain

$$||g(7u) - 8g(6u) + 28g(5u) - 56g(4u) + 70g(3u) - 56g(2u) -40291g(u)|| \preceq \frac{1}{40320}(||\psi(u, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}) \oplus ||\psi(3u, u)||^{\sim} \oplus\frac{1}{5040}||\psi(0, 0)||^{\sim}. \tag{3.13}$$

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By using (3.11) and (3.13), we get

$$\begin{aligned}
&||2g(6u) - 12g(5u) + 29g(4u) - 36g(3u) - 1414g(2u) \\
&+25904g(u)|| \preceq \frac{1}{14}\bigg[\frac{1}{80640}(||\psi(8u, 8u)||^{\sim} \oplus ||\psi(8u, -8u)||^{\sim}) \\
&\oplus\frac{1}{10080}(||\psi(6u, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \oplus \frac{1}{2880}(||\psi(4u, 4u)||^{\sim} \\
&\oplus||\psi(4u, -4u)||^{\sim}) \oplus \frac{1}{1440}(||\psi(2u, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \\
&\oplus\frac{1}{5040}(||\psi(u, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}) \oplus ||\psi(4u, u)||^{\sim} \oplus 8||\psi(3u, u)||^{\sim} \\
&\oplus\frac{5}{2016}||\psi(0, 0)||^{\sim}\bigg]. \qquad (3.14)
\end{aligned}$$

Replacing (u, v) in (3.3) with (2u,u), we get

$$\begin{aligned}
&||g(6u) - 8g(5u) + 28g(4u) - 56g(3u) + 70g(2u) + g(-2u) \\
&-40376g(u) - 8g(-u) + 28g(0)|| \preceq ||\psi(2u, u)||^{\sim}. \qquad (3.15)
\end{aligned}$$

By using (3.5), (3.7) and (3.15), we obtain

$$\begin{aligned}
&||g(6u) - 8g(5u) + 28g(4u) - 56g(3u) + 71g(2u) - 40384g(u)|| \\
&\preceq \frac{1}{5040}(||\psi(2u, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \oplus \frac{1}{630}(||\psi(u, u)||^{\sim} \\
&\oplus||\psi(u, -u)||^{\sim}) \oplus ||\psi(2u, u)||^{\sim} \oplus \frac{1}{180}||\psi(0, 0)||^{\sim}. \qquad (3.16)
\end{aligned}$$

Now, from (3.14) and (3.16), we obtain

$$\begin{aligned}
&||4g(5u) - 27g(4u) + 76g(3u) - 1556g(2u) + 106672g(u)|| \\
&\preceq \frac{1}{14}\bigg[\frac{1}{80640}(||\psi(8u, 8u)||^{\sim} \oplus ||\psi(8u, -8u)||^{\sim}) \\
&\oplus\frac{1}{10080}(||\psi(6u, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \oplus \frac{1}{2880}(||\psi(4u, 4u)||^{\sim} \\
&\oplus||\psi(4u, -4u)||^{\sim}) \oplus \frac{1}{160}(||\psi(2u, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \\
&\oplus\frac{5}{112}(||\psi(u, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}) \oplus ||\psi(4u, u)||^{\sim} \oplus 8||\psi(3u, u)||^{\sim} \\
&\oplus28||\psi(2u, u)||^{\sim} \oplus \frac{177}{1120}||\psi(0, 0)||^{\sim}\bigg] \qquad (3.17)
\end{aligned}$$

Replacing (u, v) in (3.3) with (u,u), we get

$$\begin{aligned}
&||g(5u) - 8g(4u) + 28g(3u) + g(-3u) - 56g(2u) - 8g(-2u) \\
&-40250g(u) + 28g(-u) - 56g(0)|| \preceq ||\psi(u, u)||^{\sim}. \qquad (3.18)
\end{aligned}$$

With the help of (3.5), (3.7) and (3.18), we get

$$\begin{aligned}
&||g(5u)-8g(4u)+29g(3u)-64g(2u)-40222g(u)||\\
&\preceq \frac{1}{5040}(||\psi(3u,3u)||^{\sim}\oplus||\psi(3u,-3u)||^{\sim})\oplus\frac{1}{630}(||\psi(2u,2u)||^{\sim}\\
&\oplus||\psi(2u,-2u)||^{\sim})\oplus\frac{1}{180}(||\psi(u,u)||^{\sim}\oplus||\psi(u,-u)||^{\sim})\\
&\oplus||\psi(u,u)||^{\sim}\oplus\frac{1}{90}||\psi(0,0)||^{\sim}. \qquad (3.19)
\end{aligned}$$

By using (3.19) and (3.17), we get

$$\begin{aligned}
&||g(4u)-8g(3u)-260g(2u)+53512g(u)||\\
&\preceq \frac{1}{70}\Bigg[\frac{1}{80640}(||\psi(8u,8u)||^{\sim}\oplus||\psi(8u,-8u)||^{\sim})\\
&\oplus\frac{1}{10080}(||\psi(6u,6u)||^{\sim}\oplus||\psi(6u,-6u)||^{\sim})\oplus\frac{1}{2880}(||\psi(4u,4u)||^{\sim}\\
&\oplus||\psi(4u,-4u)||^{\sim})\oplus\frac{1}{90}(||\psi(3u,3u)||^{\sim}\oplus||\psi(3u,-3u)||^{\sim})\\
&\oplus\frac{137}{1440}(||\psi(2u,2u)||^{\sim}\oplus||\psi(2u,-2u)||^{\sim})\\
&\oplus\frac{1793}{5040}(||\psi(u,u)||^{\sim}\oplus||\psi(u,-u)||^{\sim})\oplus||\psi(4u,u)||^{\sim}\oplus 8||\psi(3u,u)||^{\sim}\\
&\oplus 28||\psi(2u,u)||^{\sim}\oplus 56||\psi(u,u)||^{\sim}\oplus\frac{1573}{2016}||\psi(0,0)||^{\sim}\Bigg]. \qquad (3.20)
\end{aligned}$$

Replacing (u,v) in (3.3) with (0,u), we get

$$\begin{aligned}
&||g(4u)+g(-4u)-8g(3u)-8g(-3u)+28g(2u)+28g(-2u)\\
&-40376g(u)-56g(-u)+70g(0)||\preceq||\psi(0,u)||^{\sim}. \qquad (3.21)
\end{aligned}$$

Using (3.5), (3.7) and (3.21), we obtain

$$\begin{aligned}
&||g(4u)-8g(3u)+28g(2u)-20216g(u)||\preceq\frac{1}{2}\Bigg[\frac{1}{5040}(||\psi(4u,4u)||^{\sim}\\
&\oplus||\psi(4u,-4u)||^{\sim})\oplus\frac{1}{630}(||\psi(3u,3u)||^{\sim}\oplus||\psi(3u,-3u)||^{\sim})\\
&\oplus\frac{1}{180}(||\psi(2u,2u)||^{\sim}\oplus||\psi(2u,-2u)||^{\sim})\oplus\frac{1}{90}(||\psi(u,u)||^{\sim}\\
&\oplus||\psi(u,-u)||^{\sim})\oplus||\psi(0,u)||^{\sim}\oplus\frac{1}{72}||\psi(0,0)||^{\sim}\Bigg]. \qquad (3.22)
\end{aligned}$$

By (3.20) and (3.22), the authors conclude that

$$||g(2u)-2^{8}g(u)||\preceq H(u) \qquad (3.23)$$

where,

$$\begin{aligned}
H(u) &= \left|\frac{1}{20160}\right|\Bigg[\left|\frac{1}{80640}\right|(||\psi(8u,8u)||^{\sim} \oplus ||\psi(8u,-8u)||^{\sim}) \\
&\oplus \left|\frac{1}{10080}\right|(||\psi(6u,6u)||^{\sim} \oplus ||\psi(6u,-6u)||^{\sim}) \\
&\oplus \left|\frac{7}{960}\right|(||\psi(4u,4u)||^{\sim} \oplus ||\psi(4u,-4u)||^{\sim}) \\
&\oplus \left|\frac{1}{15}\right|(||\psi(3u,3u)||^{\sim} \oplus ||\psi(3u,-3u)||^{\sim}) \\
&\oplus \left|\frac{139}{480}\right|(||\psi(2u,2u)||^{\sim} \oplus ||\psi(2u,-2u)||^{\sim}) \\
&\oplus \left|\frac{417}{560}\right|(||\psi(u,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \oplus ||\psi(4u,u)||^{\sim} \\
&\oplus |8|.||\psi(3u,u)||^{\sim} \oplus |28|.||\psi(2u,u)||^{\sim} \oplus |56|.||\psi(u,u)||^{\sim} \\
&\oplus |35|.||\psi(0,u)||^{\sim} \oplus \frac{851}{672}||\psi(0,0)||^{\sim}\Bigg].
\end{aligned}$$

Hence from (3.23), we have

$$||g(2u) - 2^8 g(u)||_\alpha^1 \leq H_1(u), \tag{3.24}$$

and

$$||g(2u) - 2^8 g(u)||_\alpha^2 \leq H_2(u), \tag{3.25}$$

where for $\alpha \in (0,1]$

$$\begin{aligned}
H_1(u) &= \left|\frac{1}{20160}\right|\Bigg[\left|\frac{1}{80640}\right|(||\psi(8u,8u)||_\alpha^{1\sim} \oplus ||\psi(8u,-8u)||_\alpha^{1\sim}) \\
&\oplus \left|\frac{1}{10080}\right|(||\psi(6u,6u)||_\alpha^{1\sim} \oplus ||\psi(6u,-6u)||_\alpha^{1\sim}) \\
&\oplus \left|\frac{7}{960}\right|(||\psi(4u,4u)||_\alpha^{1\sim} \oplus ||\psi(4u,-4u)||_\alpha^{1\sim}) \\
&\oplus \left|\frac{1}{15}\right|(||\psi(3u,3u)||_\alpha^{1\sim} \oplus ||\psi(3u,-3u)||_\alpha^{1\sim}) \\
&\oplus \left|\frac{139}{480}\right|(||\psi(2u,2u)||_\alpha^{1\sim} \oplus ||\psi(2u,-2u)||_\alpha^{1\sim}) \\
&\oplus \left|\frac{417}{560}\right|(||\psi(u,u)||_\alpha^{1\sim} \oplus ||\psi(u,-u)||_\alpha^{1\sim}) \oplus ||\psi(4u,u)||_\alpha^{1\sim} \\
&\oplus |8|.||\psi(3u,u)||_\alpha^{1\sim} \oplus |28|.||\psi(2u,u)||_\alpha^{1\sim} \oplus |56|.||\psi(u,u)||_\alpha^{1\sim} \\
&\oplus |35|.||\psi(0,u)||_\alpha^{1\sim} \oplus \frac{851}{672}||\psi(0,0)||_\alpha^{1\sim}\Bigg] \qquad (3.26)
\end{aligned}$$

and

$$\begin{aligned}
H_2(u) &= \left|\frac{1}{20160}\right|\Bigg[\left|\frac{1}{80640}\right|(||\psi(8u,8u)||_\alpha^{2\sim} \oplus ||\psi(8u,-8u)||_\alpha^{2\sim}) \\
&\oplus \left|\frac{1}{10080}\right|(||\psi(6u,6u)||_\alpha^{2\sim} \oplus ||\psi(6u,-6u)||_\alpha^{2\sim}) \\
&\oplus \left|\frac{7}{960}\right|(||\psi(4u,4u)||_\alpha^{2\sim} \oplus ||\psi(4u,-4u)||_\alpha^{2\sim}) \\
&\oplus \left|\frac{1}{15}\right|(||\psi(3u,3u)||_\alpha^{2\sim} \oplus ||\psi(3u,-3u)||_\alpha^{2\sim}) \\
&\oplus \left|\frac{139}{480}\right|(||\psi(2u,2u)||_\alpha^{2\sim} \oplus ||\psi(2u,-2u)||_\alpha^{2\sim}) \\
&\oplus \left|\frac{417}{560}\right|(||\psi(u,u)||_\alpha^{2\sim} \oplus ||\psi(u,-u)||_\alpha^{2\sim}) \oplus ||\psi(4u,u)||_\alpha^{2\sim} \\
&\oplus |8|.||\psi(3u,u)||_\alpha^{2\sim} \oplus |28|.||\psi(2u,u)||_\alpha^{2\sim} \oplus |56|.||\psi(u,u)||_\alpha^{2\sim} \\
&\oplus |35|.||\psi(0,u)||_\alpha^{2\sim} \oplus \frac{851}{672}||\psi(0,0)||_\alpha^{2\sim}\Bigg]. \qquad (3.27)
\end{aligned}$$

From (3.24), we conclude

$$||\frac{g(2u)}{2^8} - g(u)||_\alpha^{1\sim} \leq \frac{H_1(u)}{|2^8|}. \qquad (3.28)$$

By substituting $2^n u$ for u in (3.28), then dividing both sides by 2^{8n}, we get

$$\left|\left|\frac{g(2^{n+1}u)}{2^{8(n+1)}} - \frac{g(2^n u)}{2^{8n}}\right|\right|_\alpha^1 \leq \frac{H_1(2^n u)}{|2^{8(n+1)}|} \tag{3.29}$$

for all non-negative integers n. Hence the sequence $\{\frac{g(2^n u)}{2^{8n}}\}$ is Cauchy by (3.2) and (3.29). Since Y is complete therefore, every Cauchy sequence is convergent in Y. So, the authors define a mapping $O : U \to V$ such that

$$O(u) = \lim_{n\to\infty} \frac{g(2^n u)}{2^{8n}}. \tag{3.30}$$

i.e.,

$$\lim_{n\to\infty} \left|\left|\frac{g(2^n u)}{2^{8n}} - O(u)\right|\right| = \overline{0}. \tag{3.31}$$

Now for each non-negative integer n, the authors have

$$\begin{aligned}
\left|\left|\frac{g(2^n u)}{2^{8n}} - g(u)\right|\right|_\alpha^1 &= \left|\left|\sum_{k=0}^{n-1}\left(\frac{g(2^{k+1}u)}{2^{8(n+1)}} - \frac{g(2^k u)}{2^{8n}}\right)\right|\right|_\alpha^1 \\
&\leq \max\left\{\left|\left|\frac{g(2^{k+1}u)}{2^{8(n+1)}} - \frac{g(2^k u)}{2^{8n}}\right|\right|_\alpha^1 : 0 \leq k < n\right\} \\
&\leq \frac{1}{2^8}\max\left\{\frac{H_1(2^k u)}{|2^{8k}|} : 0 \leq k < n\right\}.
\end{aligned} \tag{3.32}$$

Similarly, the authors can show that

$$\left|\left|\frac{g(2^n u)}{2^{8n}} - g(u)\right|\right|_\alpha^2 \leq \frac{1}{2^8}\max\left\{\frac{H_2(2^k u)}{|2^{8k}|} : 0 \leq k < n\right\}. \tag{3.33}$$

Taking $n \to \infty$ in (3.32) and (3.33), the authors see that inequality (3.4) holds. Next we prove that $O : U \to V$ is a octic mapping. Replacing (u, v) by $(2^n u, 2^n v)$ and divide by $|2^{8n}|$ in (3.3), we get

$$\begin{aligned}
&\frac{1}{|2^{8n}|}||g(2^n(u + 4v)) - 8g(u + 3v) + 28g(u + 2v) - 56g(u + v) \\
&-56g(u - v) + 28g(u - 2v) - 8g(u - 3v) + g(u - 4v) + 70g(u) \\
&-40320g(v)|| \preceq ||\frac{\psi(2^n u, 2^n v)}{2^{8n}}||^\sim.
\end{aligned}$$

Taking $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
&||O(u + 4v) - 8O(u + 3v) + 28O(u + 2v) - 56O(u + v) - 56O(u - v) \\
&+28O(u - 2v) - 8O(u - 3v) + O(u - 4v) + 70O(u) - 40320O(v)|| \preceq \overline{0}
\end{aligned}$$

this implies that

$$\begin{aligned}&O(u+4v)-8O(u+3v)+28O(u+2v)-56O(u+v)-56O(u-v)\\&+28O(u-2v)-8O(u-3v)+O(u-4v)+70O(u)-40320O(v)=\underline{0}.\end{aligned}$$

Therefore, the mapping $O : U \to V$ is octic. Next we shall prove uniqueness of mapping O. Now, consider another octic mapping $O' : U \to V$ which satisfies (3.1) and (3.4). For fix $u \in U$, certainly $O(2^n u) = 2^{8n}O(u)$ and $O'(2^n u) = 2^{8n}O'(u)$ for all $n \in N$. Therefore,

$$\begin{aligned}&||O(u)-O'(u)|| = \lim_{n\to\infty}\frac{1}{2^{8n}}||O(2^n u)-O'(2^n u)||\\&=\lim_{n\to\infty}\frac{1}{2^{8n}}||O(2^n u)-g(2^n u)+g(2^n u)-O'(2^n u)||\\&\preceq \lim_{n\to\infty}\max\left\{\frac{1}{2^{8n}}||O(2^n u)-g(2^n u)||,\frac{1}{2^{8n}}||g(2^n u)-O'(2^n u)||\right\}\\&\preceq \lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\max\left\{\frac{1}{2^{8(n+1)}}\left\{\frac{H(2^{k+n}u)}{2^{8k}},\frac{H(2^{k+n}u)}{2^{8k}}\right\}\right\}\right\}\\&=\overline{0}.\end{aligned}$$

Therefore, $O(u) - O'(u) = \underline{0}$. So, $O'(u) = O(u)$. Hence we deduced that O is unique mapping. □

Corollary 3.2. *Suppose that $(U, ||.||)$ is a normed space, $(W, ||\ \ ||^{\sim})$ is a fuzzy normed space, and $(V, ||\ \ ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $p < 8$ be non-negative real numbers. If the mapping $g : U \to V$ is such that*

$$||\Delta_o g(u,v)|| \preceq ||(||v||^p + ||u||^p)w_0||^{\sim} \tag{3.34}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned}||O(u)-g(u)|| \preceq\ & \frac{||\,||u||^p w_0||^{\sim}}{2^8}\left[\left|\frac{1}{20160}\right|\left(\left|\frac{8089}{560}\right|\oplus\left|\frac{139|2|^p}{120}\right|\oplus\left|\frac{4|3|^p}{15}\right|\right.\right.\\&\oplus\left|\frac{7|4|^p}{240}\right|\oplus\left|\frac{|6|^p}{2520}\right|\oplus\left|\frac{|8|^p}{20160}\right|\oplus\left||4|^p+1\right|\\&\left.\left.\oplus\left|8(|3|^p+1)\right|\oplus|28(|2^p|+1)|\right)\right]\end{aligned}$$

for all $u \in U$.

Corollary 3.3. *Suppose that $(U, ||.||)$ is a normed space, $(W, ||\ \ ||^{\sim})$ is a fuzzy normed space, and $(V, ||\ \ ||)$ is a non-Archimedean complete* *fuzzy normed space. Let $w_0 \in W$ and $p, q < 8$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||(||v||^p ||u||^q) w_0||^{\sim}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned} ||O(u) - g(u)|| \preceq \ & \frac{||\,||u||^{p+q} w_0||^{\sim}}{2^8} \Bigg[\left|\frac{1}{20160}\right| \Bigg(\left|\frac{2|8|^{p+q}}{80640}\right| \oplus \left|\frac{2|6|^{p+q}}{10080}\right| \oplus \left|\frac{14|4|^{p+q}}{960}\right| \\ \oplus \ & \left|\frac{2|3|^{p+q}}{15}\right| \oplus \left|\frac{278|2|^{p+q}}{480}\right| \oplus \left|\frac{32192}{560}\right| \oplus |(|4^p|)| \oplus |8(|3^p|)| \\ \oplus \ & |28(|2^p|)| \Bigg) \Bigg]. \end{aligned}$$

Corollary 3.4. *Assume that $(U, ||.||)$ is a normed space, $(W, || \quad ||^{\sim})$ is a fuzzy normed space, and $(V, || \quad ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $\lambda = s + r < 8$ be non-negative real numbers. If the mapping $g : U \to V$ is such that*

$$||\Delta_o g(u,v)|| \preceq ||[||v||^s ||u||^r + (||v||^{s+r} + ||u||^{s+r})] w_0||^{\sim} \tag{3.35}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned} ||O(u) - g(u)|| \preceq \ & \frac{||\,||u||^{\lambda} w_0||^{\sim}}{2^8} \Bigg[\left|\frac{1}{20160}\right| \Bigg(|13440|8|^{\lambda}| \oplus |1680|6|^{\lambda}| \\ \oplus \ & \left|\frac{7|4|^{\lambda}}{160}\right| \oplus \left|\frac{8|3|^{\lambda}}{5}\right| \oplus \left|\frac{139|2|^{\lambda}}{130}\right| \oplus \left|\frac{58091}{280}\right| \\ \oplus \ & |(|4|^r + |4|^{\lambda} + 1)| \oplus |8(|3|^r + |3|^{\lambda} + 1)| \\ \oplus \ & |(|2|^r + |2|^{\lambda} + 1) \Bigg) \Bigg] \end{aligned}$$

for all $u \in U$.

Theorem 3.5. *Assume that U is a linear space and $(W, ||.||^{\sim})$ is a fuzzy normed space. Consider $\psi : U^2 \to W$ be a function such that*

$$\lim_{n\to\infty} |2^{8n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 1} = \lim_{n\to\infty} |2^{8n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 2} = 0, \tag{3.36}$$

for all $u, v \in U$ and $\alpha \in (0, 1]$. Let $(V, ||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g : U \to V$ is such that

$$||\Delta_o g(u,v)|| \preceq ||\psi(u,v)||^{\sim} \tag{3.37}$$

for all $u \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$||O(u) - g(u)|| \preceq \frac{1}{2^8} \max\left\{|2^{8(k+1)}|H\left(\frac{u}{2^{k+1}}\right), k \in N \cup \{0\}\right\} \quad (3.38)$$

where

$$\begin{aligned} H\left(\frac{u}{2^{k+1}}\right) &= \left|\frac{1}{20160}\right|\left[\left|\frac{1}{80640}\right|\left(\left|\left|\psi\left(\frac{8u}{2^{k+1}}, \frac{8u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{8u}{2^{k+1}}, \frac{-8u}{2^{k+1}}\right)\right|\right|^{\sim}\right)\right. \\ &\oplus \left|\frac{1}{10080}\right|\left(\left|\left|\psi\left(\frac{6u}{2^{k+1}}, \frac{6u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{6u}{2^{k+1}}, \frac{-6u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{7}{960}\right|\left(\left|\left|\psi\left(\frac{4u}{2^{k+1}}, \frac{4u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{4u}{2^{k+1}}, \frac{-4u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{1}{15}\right|\left(\left|\left|\psi\left(\frac{3u}{2^{k+1}}, \frac{3u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{3u}{2^{k+1}}, \frac{-3u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{139}{480}\right|\left(\left|\left|\psi\left(\frac{2u}{2^{k+1}}, \frac{2u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{2u}{2^{k+1}}, \frac{-2u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{417}{560}\right|\left(\left|\left|\psi\left(\frac{u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{u}{2^{k+1}}, \frac{-u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\left|\psi\left(\frac{4u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus |8|.\left|\left|\psi\left(\frac{3u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \\ &\oplus |28|.\left|\left|\psi\left(\frac{2u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus |56|.\left|\left|\psi\left(\frac{u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \\ &\oplus \left.|35|.\left|\left|\psi\left(0, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\frac{851}{672}\right| ||\psi(0,0||^{\sim}\right] \end{aligned}$$

for all $u \in U$.

Proof. From (3.24), the authors get

$$\left|\left|g(u) - 2^8 g\left(\frac{u}{2}\right)\right|\right|_{\alpha}^{1} \leq H_1\left(\frac{u}{2}\right) \quad (3.39)$$

for $\alpha \in (0,1]$. Replacing u by $\frac{u}{2^n}$ and multiplying both side by $|2^{8n}|$ in (3.39), we get

$$\left|\left|2^{8n} g\left(\frac{u}{2^n}\right) - 2^{8(n+1)} g\left(\frac{u}{2^{n+1}}\right)\right|\right|_{\alpha}^{1} \leq |2^{8n}| H_1\left(\frac{u}{2^{n+1}}\right) \quad (3.40)$$

for all negative integer n. Hence the sequence $g\{2^{8n} g(\frac{u}{2^n})\}$ is Cauchy by (3.36) and (3.40). Every Cauchy sequence is convergent in Y since

Y is complete. So, the authors construct a mapping $O : U \to V$ such that

$$O(u) = \lim_{n\to\infty} 2^{8n} g\left(\frac{u}{2^n}\right)$$

for all $u \in U$. That is

$$\lim_{n\to\infty} ||2^{8n} g\left(\frac{u}{2^n}\right) - O(u)|| = \overline{0}$$

for all $u \in U$. Now, for each positive integer n, the authors have

$$\begin{aligned}
&\left|\left|2^{8n} g\left(\frac{u}{2^n}\right) - g(u)\right)\right|\right|_{\alpha}^{1} = \left|\left|\sum_{k=0}^{n-1}\left(2^{8(k+1)} g\left(\frac{u}{2^{k+1}}\right) - 2^{8k} g\left(\frac{u}{2^k}\right)\right)\right|\right|_{\alpha}^{1} \\
&\leq \max\left\{\left|\left|\left(2^{8(k+1)} g\left(\frac{u}{2^{k+1}}\right) - 2^{8k} g\left(\frac{u}{2^k}\right)\right)\right|\right|_{\alpha}^{1}; 0 \leq k < n\right\} \\
&\leq \frac{1}{|2^8|} \max\left\{|2^{8(k+1)}| H_1\left(\frac{u}{2^{k+1}}\right) : 0 \leq k < n\right\}. \qquad (3.41)
\end{aligned}$$

Similarly , it can be shown from (3.25)

$$\left|\left|2^{8n} g\left(\frac{u}{2^n}\right) - g(u)\right)\right|\right|_{\alpha}^{2} \leq \frac{1}{|2^8|} \max\left\{|2^{8(k+1)}| H_2\left(\frac{u}{2^{k+1}}\right); 0 \leq k < n\right\} \quad (3.42)$$

Taking $n \to \infty$ in (3.41) and (3.42), the authors see that inequality (3.39) holds. The authors conclude that $O(u)$ is a unique cubic mapping holding (3.38) using same procedure as in the demonstration of theorem (3.1).

□

Corollary 3.6. *Assume that $(U, ||.||)$ is a normed space, $(W, || \quad ||^{\sim})$ is a fuzzy normed space, and $(V, || \quad ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $p > 8$ be non-negative real numbers. If the mapping $g : U \to V$ is such that*

$$||\Delta_o g(u, v)|| \preceq ||(||v||^p + ||u||^p) w_0||^{\sim} \qquad (3.43)$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned}
||O(u) - g(u)|| \quad &\preceq \quad \frac{|| \, ||u||^p w_0||^{\sim}}{|2^p|}\left[\left|\frac{1}{20160}\right|\left(\left|\frac{8089}{560}\right| \oplus \left|\frac{139|2|^p}{120}\right| \oplus \left|\frac{4|3|^p}{15}\right|\right.\right. \\
&\oplus \quad \left|\frac{7|4|^p}{240}\right| \oplus \left|\frac{|6|^p}{2520}\right| \oplus \left|\frac{|8|^p}{20160}\right| \oplus \left||4|^p + 1\right| \\
&\oplus \quad \left.\left.\left|8(|3|^p + 1)\right| \oplus |28(|2^p| + 1)|\right)\right]
\end{aligned}$$

for all $u \in U$.

Corollary 3.7. *Suppose that $(U, ||.||)$ is a normed space, $(W, || \quad ||^{\sim})$ is a fuzzy normed space, and $(V, || \quad ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $p, q > 8$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u, v)|| \preceq ||(||v||^p ||u||^q) w_0||^{\sim}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned} ||O(u) - g(u)|| \quad \preceq \quad & \frac{|| \, ||u||^{p+q} w_0||^{\sim}}{2^8} \left[\left| \frac{1}{20160} \right| \left(\left| \frac{2|8|^{p+q}}{80640} \right| \oplus \left| \frac{2|6|^{p+q}}{10080} \right| \oplus \left| \frac{14|4|^{p+q}}{960} \right| \right. \right. \\ \oplus \quad & \left| \frac{2|3|^{p+q}}{15} \right| \oplus \left| \frac{278|2|^{p+q}}{480} \right| \oplus \left| \frac{32192}{560} \right| \oplus |(|4^p|)| \oplus |8(|3^p|)| \\ \oplus \quad & \left. \left. |28(|2^p|)| \right) \right]. \end{aligned}$$

Corollary 3.8. *Suppose that $(U, ||.||)$ is a normed space, $(W, || \quad ||^{\sim})$ is a fuzzy normed space, and $(V, || \quad ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $\lambda = s + r > 8$ be non-negative real numbers. If the mapping $g : U \to V$ is such that*

$$||\Delta_o g(u, v)|| \preceq ||[||v||^s ||u||^r + (||v||^{s+r} + ||u||^{s+r})] w_0||^{\sim} \tag{3.44}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned} ||O(u) - g(u)|| \quad \preceq \quad & \frac{|| \, ||u||^{\lambda} w_0||^{\sim}}{|2|^{\lambda}} \left[\left| \frac{1}{20160} \right| \left(|13440|8|^{\lambda}| \oplus |1680|6|^{\lambda}| \right. \right. \\ \oplus \quad & \left| \frac{7|4|^{\lambda}}{160} \right| \oplus \left| \frac{8|3|^{\lambda}}{5} \right| \oplus \left| \frac{139|2|^{\lambda}}{130} \right| \oplus \left| \frac{58091}{280} \right| \\ \oplus \quad & |(|4|^r + |4|^{\lambda} + 1)| \oplus |8(|3|^r + |3|^{\lambda} + 1)| \\ \oplus \quad & \left. \left. |(|2|^r + |2|^{\lambda} + 1) \right) \right] \end{aligned}$$

for all $u \in U$.

Counterexample 3.9. Consider a real Banach algebra $(U, ||.||)$ and a non-Archimedean complete fuzzy norm space $(U, || \quad ||^{\sim})$ in which

$$||u||^{\sim}(t) = \begin{cases} \frac{||u||^8}{t}, & \text{when } ||u||^8 < t, t \neq 0 \\ 1, & \text{when } ||u||^8 = t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

whose α-level set is defined as $[||u||^{\sim}]_\alpha = [||u||^8, \frac{||u||^8}{\alpha}]$. Construct a mapping $g : U \to U$ such that $g(u) = u^8 + ||u||^8 u_0$, where u_0 is a unit vector and

$$||\Delta_o g(u,v)||^{\sim} \preceq ||(256||u||^8 + 290816||v||^8)u_0||^{\sim},$$

then there does not exist an octic mapping $O : U \to U$ fulfilling the given condition

$$||O(u) - g(u)||^{\sim} \quad \preceq \quad 2^8||||u||^8 u_0||^{\sim}.$$

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General sigmoid based Banach space valued neural network multivariate approximations

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Abstract

Here we expose multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We treat also the case of approximation by iterated operators of the last four types. These approximations are derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by a general sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

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1 Introduction

The author in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

For this article the author is motivated by the article [13] of Z. Chen and F. Cao, also by [4], [5], [6], [7], [8], [9], [10], [11], [12], [15], [16].

The author here performs multivariate general sigmoid function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$. Also he does iterated approximation. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

The author here comes up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by a general sigmoid function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is a general sigmoid function. About neural networks read [17], [18], [19].

2 Basics

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and stricLty concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

We consider the activation function

$$\psi(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \quad x \in \mathbb{R}, \tag{1}$$

As in [11], p. 285, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x + 1 > x - 1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{2}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) < 0,$$

by h' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is stritly decreasing on $(0, +\infty)$.

Clearly, ψ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x \to +\infty} \psi(x) = \frac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0, \tag{3}$$

and

$$\lim_{x \to -\infty} \psi(x) = \frac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \tag{4}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

Theorem 1 *We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x - i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{5}$$

Proof. As exactly the same as in [11], p. 286 is omitted. ■

Theorem 2 *It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{6}$$

Proof. Similar to [11], p. 287. It is omitted. ■

Thus $\psi(x)$ is a density function on $\mathbb{R}$.

We give

Theorem 3 *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx - k) < \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}. \tag{7}$$

Notice that

$$\lim_{n\to+\infty}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}=0.$$

Proof. Let $x \geq 1$. That is $0 \leq x-1 < x+1$. Applying the mean value theorem we get

$$\psi(x) \overset{(1)}{=} \frac{1}{4}\cdot 2\cdot h'(\xi)=\frac{h'(\xi)}{2}, \tag{8}$$

for some $x-1<\xi<x+1$.

Since h' is strictly decreasing we obtain $h'(\xi)<h'(x-1)$ and

$$\psi(x)<\frac{h'(x-1)}{2},\ \ \forall\ x\geq 1. \tag{9}$$

Therefore we have

$$\sum_{\substack{k=-\infty \\ : |nx-k|\geq n^{1-\alpha}}}^{\infty}\psi(nx-k)=\sum_{\substack{k=-\infty \\ : |nx-k|\geq n^{1-\alpha}}}^{\infty}\psi(|nx-k|)<$$

$$\frac{1}{2}\sum_{\substack{k=-\infty \\ : |nx-k|\geq n^{1-\alpha}}}^{\infty}h'(|nx-k|-1)\leq\frac{1}{2}\int_{(n^{1-\alpha}-1)}^{+\infty}h'(x-1)\,d(x-1)=$$

$$\frac{1}{2}\left(h(x-1)\big|_{(n^{1-\alpha}-1)}^{+\infty}\right)=\frac{1}{2}\left[h(+\infty)-h\left(n^{1-\alpha}-2\right)\right]=\frac{1}{2}\left(1-h\left(n^{1-\alpha}-2\right)\right). \tag{10}$$

The claim is proved. ■

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

We further give

Theorem 4 *Let $x\in[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}<\frac{1}{\psi(1)},\quad\forall\ x\in[a,b]. \tag{11}$$

Proof. As similar to [11], p. 289 is omitted. ■

Remark 5 *([11], pp. 290-291)*

i) We have that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\neq 1, \tag{12}$$

for at least some $x\in[a,b]$.

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ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 1, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \leq 1. \tag{13}$$

We introduce

$$Z(x_1,...,x_N) := Z(x) := \prod_{i=1}^{N} \psi(x_i), \quad x=(x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{14}$$

It has the properties:

(i) $Z(x) > 0, \ \forall \, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1,...,x_N-k_N) = 1, \tag{15}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall \, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \tag{16}$$

$\forall \, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\,dx = 1, \tag{17}$$

that is Z is a multivariate density function.

Here denote $\|x\|_{\infty} := \max\{|x_1|,...,|x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,...,\infty)$, $-\infty := (-\infty,...,-\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil,...,\lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor,...,\lfloor nb_N \rfloor), \end{aligned} \tag{18}$$

where $a := (a_1,...,a_N)$, $b := (b_1,...,b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \psi(nx_i-k_i) \right) =$$

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$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}\left(\prod_{i=1}^{N}\psi\left(nx_i-k_i\right)\right)=\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi\left(nx_i-k_i\right)\right). \quad (19)$$

For $0<\beta<1$ and $n\in\mathbb{N}$, a fixed $x\in\mathbb{R}^N$, we have that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)=$$

$$\sum_{\begin{cases}k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z\left(nx-k\right)+\sum_{\begin{cases}k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z\left(nx-k\right). \quad (20)$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}$, where $r\in\{1,...,N\}$.

(v) As in [10], pp. 379-380, we derive that

$$\sum_{\begin{cases}k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z\left(nx-k\right)\overset{(7)}{<}\frac{1-h\left(n^{1-\beta}-2\right)}{2},\ \ 0<\beta<1, \quad (21)$$

with $n\in\mathbb{N}:n^{1-\beta}>2$, $x\in\prod_{i=1}^{N}\left[a_i,b_i\right]$.

(vi) By Theorem 4 we get that

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)}<\frac{1}{\left(\psi\left(1\right)\right)^{N}}, \quad (22)$$

$\forall\ x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right),\ \ n\in\mathbb{N}.$

It is also clear that

(vii)

$$\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\infty}Z\left(nx-k\right)<\frac{1-h\left(n^{1-\beta}-2\right)}{2}, \quad (23)$$

$0<\beta<1$, $n\in\mathbb{N}:n^{1-\beta}>2$, $x\in\mathbb{R}^N$.

Furthermore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)\neq 1, \quad (24)$$

for at least some $x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$.

6

Here $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^N [a_i, b_i]$, $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the following multivariate linear normalized neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i]\right)$):

$$A_n(f, x_1, ..., x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i)\right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i)\right)}. \quad (25)$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \quad (26)$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1, x) = 1, \ \forall \ x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$.

Furthermore it holds

$$\|A_n(f, x)\|_\gamma \leq \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} = \widetilde{A}_n\left(\|f\|_\gamma, x\right), \quad (27)$$

$\forall \ x \in \prod_{i=1}^N [a_i, b_i]$.

Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$.

So, we have that

$$\|A_n(f, x)\|_\gamma \leq \widetilde{A}_n\left(\|f\|_\gamma, x\right), \quad (28)$$

$\forall \ x \in \prod_{i=1}^N [a_i, b_i]$, $\forall \ n \in \mathbb{N}$, $\forall \ f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, then $cg \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Furthermore it holds

$$A_n(cg, x) = c\widetilde{A}_n(g, x), \ \ \forall \ x \in \prod_{i=1}^{N} [a_i, b_i]. \tag{29}$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$A_n(c) = c, \ \ \forall \ c \in X. \tag{30}$$

We call $\widetilde{A}_n$ the companion operator of A_n.

For convinience we call

$$A_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right), \tag{31}$$

$\forall \ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

That is

$$A_n(f, x) := \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}, \tag{32}$$

$\forall \ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f, x) - f(x) = \frac{A_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \tag{33}$$

Consequently we derive

$$\|A_n(f, x) - f(x)\|_\gamma \overset{(22)}{\leq} \frac{1}{(\psi(1))^N} \left\| A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \right\|_\gamma, \tag{34}$$

$\forall \ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

We will estimate the right hand side of (34).

For the last and others we need

Definition 6 *([11], p. 274) Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M, X)$. We define the first modulus of continuity of f as*

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in M: \\ \|x - y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \ \ 0 < \delta \leq diam(M). \tag{35}$$

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If $\delta > diam(M)$, *then*

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{36}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 7 *([11], p. 274) We have* $\omega_1(f,\delta) \to 0$ *as* $\delta \downarrow 0$, *iff* $f \in C(M,X)$, *where* M *is a convex compact subset of* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (35). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi(nx_i-k_i)\right), \tag{37}$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x) := C_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty}\left(n^N\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z(nx-k) =$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\left(n^N\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}...\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,...,t_N)\,dt_1...dt_N\right)$$

$$\cdot\left(\prod_{i=1}^{N}\psi(nx_i-k_i)\right), \tag{38}$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1,...,\theta_N) \in \mathbb{N}^N$, $r = (r_1,...,r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1,r_2,...r_N} \geq 0$, such that $\sum\limits_{r=0}^{\theta} w_r = \sum\limits_{r_1=0}^{\theta_1}\sum\limits_{r_2=0}^{\theta_2}...\sum\limits_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n}+\frac{r}{n\theta}\right) =$$

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$$\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},...,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right), \quad (39)$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x) := D_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx-k) = \quad (40)$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,k_2,...,k_N}(f)\left(\prod_{i=1}^{N}\psi(nx_i-k_i)\right),$$

$\forall\, x \in \mathbb{R}^N$.

In this article we study the approximation properties of $A_n, B_n, C_n,\ D_n$ neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate general sigmoid Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 8 *Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N,n \in \mathbb{N}$ with $n^{1-\beta}>2$. Then*

1)

$$\|A_n(f,x)-f(x)\|_\gamma \leq$$

$$\frac{1}{(\psi(1))^N}\left[\omega_1\left(f,\frac{1}{n^\beta}\right)+\left(1-h\left(n^{1-\beta}-2\right)\right)\left\|\|f\|_\gamma\right\|_\infty\right] =: \lambda_1(n), \quad (41)$$

and

2)

$$\left\|\|A_n(f)-f\|_\gamma\right\|_\infty \leq \lambda_1(n). \quad (42)$$

We notice that $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, pointwise and uniformly.

Above ω_1 is with respect to $p=\infty$ and the speed of convergnece is $\max\left(\frac{1}{n^\beta}, \left(1-h\left(n^{1-\beta}-2\right)\right)\right)$.

Proof. As similar to [12] is omitted. ∎

We make

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Remark 9 *([11], pp. 263-266) Let* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $N \in \mathbb{N}$*; where* $\|\cdot\|_p$ *is the* L_p*-norm,* $1 \leq p \leq \infty$. $\mathbb{R}^N$ *is a Banach space, and* $\left(\mathbb{R}^N\right)^j$ *denotes the* j*-fold product space* $\mathbb{R}^N \times ... \times \mathbb{R}^N$ *endowed with the max-norm* $\|x\|_{(\mathbb{R}^N)^j} := \max_{1 \leq \lambda \leq j} \|x_\lambda\|_p$*, where* $x := (x_1, ..., x_j) \in \left(\mathbb{R}^N\right)^j$.

Let $\left(X, \|\cdot\|_\gamma\right)$ *be a general Banach space. Then the space* $L_j := L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ *of all* j*-multilinear continuous maps* $g : \left(\mathbb{R}^N\right)^j \rightarrow X$, $j = 1, ..., m$*, is a Banach space with norm*

$$\|g\| := \|g\|_{L_j} := \sup_{\left(\|x\|_{(\mathbb{R}^N)^j} = 1\right)} \|g(x)\|_\gamma = \sup \frac{\|g(x)\|_\gamma}{\|x_1\|_p ... \|x_j\|_p}. \tag{43}$$

Let M *be a non-empty convex and compact subset of* $\mathbb{R}^k$ *and* $x_0 \in M$ *is fixed.*

Let O *be an open subset of* $\mathbb{R}^N : M \subset O$*. Let* $f : O \rightarrow X$ *be a continuous function, whose Fréchet derivatives (see [20])* $f^{(j)} : O \rightarrow L_j = L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ *exist and are continuous for* $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in \left(\mathbb{R}^N\right)^j$, $x \in M$.

We will work with $f|_M$.

Then, by Taylor's formula ([13]), ([20], p. 124), we get

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad \text{all } x \in M, \tag{44}$$

where the remainder is the Riemann integral

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!}\left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m du, \tag{45}$$

here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We consider

$$w := \omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x, y \in M: \\ \|x - y\|_p \leq h}} \left\|f^{(m)}(x) - f^{(m)}(y)\right\|, \tag{46}$$

$h > 0$.

We obtain

$$\left\|\left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m\right\|_\gamma \leq$$

$$\left\|f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right\| \cdot \|x - x_0\|_p^m \leq$$

$$w \|x - x_0\|_p^m \left\lceil \frac{u\|x - x_0\|_p}{h} \right\rceil, \tag{47}$$

by Lemma 7.1.1, [1], p. 208, where $\lceil \cdot \rceil$ *is the ceiling.*

Therefore for all $x \in M$ *(see [1], pp. 121-122):*

$$\left\| R_m\left(x, x_0\right)\right\|_\gamma \leq w \left\| x - x_0\right\|_p^m \int_0^1 \left\lceil \frac{u \left\| x - x_0\right\|_p}{h} \right\rceil \frac{(1-u)^{m-1}}{(m-1)!} du$$

$$= w \Phi_m \left(\left\| x - x_0 \right\|_p \right) \tag{48}$$

by a change of variable, where

$$\Phi_m(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t| - s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t| - jh)_+^m \right), \quad \forall\, t \in \mathbb{R}, \tag{49}$$

is a (polynomial) spline function, see [1], p. 210-211.

Also from there we get

$$\Phi_m(t) \leq \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h |t|^{m-1}}{8(m-1)!} \right), \quad \forall\, t \in \mathbb{R}, \tag{50}$$

with equality true only at $t = 0$*.*

Therefore it holds

$$\left\| R_m\left(x, x_0\right)\right\|_\gamma \leq w \left(\frac{\left\| x - x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\| x - x_0\right\|_p^m}{2m!} + \frac{h \left\| x - x_0\right\|_p^{m-1}}{8(m-1)!} \right), \quad \forall\, x \in M. \tag{51}$$

We have found that

$$\left\| f(x) - \sum_{j=0}^m \frac{f^{(j)}\left(x_0\right)\left(x - x_0\right)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1 \left(f^{(m)}, h \right) \left(\frac{\left\| x - x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\| x - x_0\right\|_p^m}{2m!} + \frac{h \left\| x - x_0\right\|_p^{m-1}}{8(m-1)!} \right) < \infty, \tag{52}$$

$\forall\, x, x_0 \in M$.

Here $0 < \omega_1\left(f^{(m)}, h\right) < \infty$*, by* M *being compact and* $f^{(m)}$ *being continuous on* M*.*

One can rewrite (52) as follows:

$$\left\| f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}\left(x_0\right)\left(\cdot - x_0\right)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1 \left(f^{(m)}, h \right) \left(\frac{\left\| \cdot - x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\| \cdot - x_0\right\|_p^m}{2m!} + \frac{h \left\| \cdot - x_0\right\|_p^{m-1}}{8(m-1)!} \right), \ \forall\, x_0 \in M, \tag{53}$$

a pointwise functional inequality on M.

Here $(\cdot - x_0)^j$ maps M into $(\mathbb{R}^N)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps $(\mathbb{R}^N)^j$ into X and it is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into X.

Clearly $f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, hence $\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \in C(M)$.

Let $\left\{ \widetilde{L}_N \right\}_{N \in \mathbb{N}}$ be a sequence of positive linear operators mapping $C(M)$ into $C(M)$.

Therefore we obtain

$$\left(\widetilde{L}_N \left(\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \right) \right)(x_0) \leq$$

$$\omega_1 \left(f^{(m)}, h \right) \left[\frac{\left(\widetilde{L}_N \left(\|\cdot - x_0\|_p^{m+1} \right) \right)(x_0)}{(m+1)!h} + \frac{\left(\widetilde{L}_N \left(\|\cdot - x_0\|_p^{m} \right) \right)(x_0)}{2m!} \right.$$

$$\left. + \frac{h \left(\widetilde{L}_N \left(\|\cdot - x_0\|_p^{m-1} \right) \right)(x_0)}{8(m-1)!} \right], \tag{54}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x_0 \in M$.

Clearly (54) is valid when $M = \prod_{i=1}^{N} [a_i, b_i]$ and $\widetilde{L}_n = \widetilde{A}_n$, see (26).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators A_n, $\widetilde{A}_n$ fulfill its assumptions, see (25), (26), (28), (29) and (30).

We present the following high order approximation results.

Theorem 10 *Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N} [a_i, b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$ and $r > 0$. Then*

1)

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq$$

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$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{55}$$

2) additionally if $f^{(j)}(x_0) = 0$, $j = 1, ..., m$, *we have*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{56}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

3)

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \sum_{j=1}^{m} \frac{1}{j!}\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma +$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{57}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

and

4)

$$\left\|\|A_n(f) - f\|_\gamma\right\|_{\infty, \prod_{i=1}^{N}[a_i, b_i]} \leq$$

$$\sum_{j=1}^{m} \frac{1}{j!}\left\|\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i, b_i]} +$$

$$\frac{\omega_1\left(f^{(m)}, r\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i, b_i]}^{\frac{1}{m+1}}\right)}{rm!}$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i, b_i]}^{\left(\frac{m}{m+1}\right)} \tag{58}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right].$$

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We need

Lemma 11 *The function* $\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^m\right)\right)(x_0)$ *is continuous in* $x_0 \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m \in \mathbb{N}$.

Proof. By Lemma 10.3, [11], p. 272. ■

We make

Remark 12 *By Remark 10.4, [11], p. 273, we get that*

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^k\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]} \leq$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{k}{m+1}\right)}, \tag{59}$$

for all $k = 1, ..., m$.

We give

Corollary 13 *(to Theorem 10, case of* $m = 1$*) Then*

1)

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \left\|\left(A_n\left(f^{(1)}(x_0)(\cdot - x_0)\right)\right)(x_0)\right\|_\gamma +$$

$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right)\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}} \tag{60}$$

$$\left[1 + r + \frac{r^2}{4}\right],$$

and

2)

$$\left\|\|(A_n(f)) - f\|_\gamma\right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq$$

$$\left\|\left\|\left(A_n\left(f^{(1)}(x_0)(\cdot - x_0)\right)\right)(x_0)\right\|_\gamma\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]} +$$

$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{2}}\right)$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{2}}\left[1 + r + \frac{r^2}{4}\right], \tag{61}$$

$r > 0$.

We make

Remark 14 *We estimate* ($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$),

$$\widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)} \overset{(22)}{<}$$

$$\frac{1}{(\psi(1))^N} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) = \tag{62}$$

$$\frac{1}{(\psi(1))^N} \left\{ \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \le \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) + \right.$$

$$\left. \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right\} \overset{(23)}{\le}$$

$$\frac{1}{(\psi(1))^N} \left\{ \frac{1}{n^{\alpha(m+1)}} + \left(\frac{1 - h(n^{1-\alpha} - 2)}{2} \right) \|b - a\|_\infty^{m+1} \right\}, \tag{63}$$

(where $b - a = (b_1 - a_1, ..., b_N - a_N)$*).*

We have proved that ($\forall\ x_0 \in \prod_{i=1}^{N} [a_i, b_i]$)

$$\widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) <$$

$$\frac{1}{(\psi(1))^N} \left\{ \frac{1}{n^{\alpha(m+1)}} + \left(\frac{1 - h(n^{1-\alpha} - 2)}{2} \right) \|b - a\|_\infty^{m+1} \right\} =: \varphi_1(n) \tag{64}$$

($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$).

And, consequently it holds

$$\left\| \widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} <$$

$$\frac{1}{(\psi(1))^N} \left\{ \frac{1}{n^{\alpha(m+1)}} + \left(\frac{1 - h(n^{1-\alpha} - 2)}{2} \right) \|b - a\|_\infty^{m+1} \right\} = \varphi_1(n) \to 0, \quad \text{as } n \to +\infty. \tag{65}$$

So, we have that $\varphi_1(n) \to 0$, as $n \to +\infty$. Thus, when $p \in [1,\infty]$, from Theorem 10 we have the convergence to zero in the right hand sides of parts (1), (2).

Next we estimate $\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma$.

We have that

$$\left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}. \tag{66}$$

When $p = \infty$, $j = 1, ..., m$, we obtain

$$\left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma \leq \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j. \tag{67}$$

We further have that

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \overset{(22)}{<}$$

$$\frac{1}{(\psi(1))^N} \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma Z(nx_0 - k) \right) \leq$$

$$\frac{1}{(\psi(1))^N} \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) = \tag{68}$$

$$\frac{1}{(\psi(1))^N} \left\| f^{(j)}(x_0) \right\| \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) =$$

$$\frac{1}{(\psi(1))^N} \left\| f^{(j)}(x_0) \right\| \left\{ \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right.$$

$$\left. + \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right\} \overset{(23)}{\leq} \tag{69}$$

$$\frac{1}{(\psi(1))^N} \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \left(\frac{1 - h\left(n^{1-\alpha} - 2\right)}{2} \right) \|b - a\|_\infty^j \right\} \to 0, \text{ as } n \to \infty.$$

That is

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \to 0, \text{ as } n \to \infty.$$

Therefore when $p = \infty$, for $j = 1, ..., m$, we have proved:

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma <$$

$$\frac{1}{(\psi(1))^N} \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \left(\frac{1 - h(n^{1-\alpha} - 2)}{2} \right) \|b - a\|_\infty^j \right\} \leq \tag{70}$$

$$\frac{1}{(\psi(1))^N} \left\| f^{(j)} \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + \left(\frac{1 - h(n^{1-\alpha} - 2)}{2} \right) \|b - a\|_\infty^j \right\} =: \varphi_{2j}(n) < \infty,$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 10, the right hand sides of (57) and (58) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 13, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Conclusion 15 *We have proved that the left hand sides of (55), (56), (57), (58) and (60), (61) converge to zero as $n \to \infty$, for $p \in [1, \infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (56). Higher speed of convergence happens also to the left hand side of (55).*

We give

Corollary 16 *(to Theorem 10) Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, such that $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^N [a_i, b_i]}$. Let $x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ and $r > 0$. Here $\varphi_1(n)$ as in (65) and $\varphi_{2j}(n)$ as in (70), where $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$, $j = 1, ..., m$. Then*

1)

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^m \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1 \left(f^{(m)}, r (\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{71}$$

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2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., m$, we have

$$\left\| \left(A_n(f) \right)(x_0) - f(x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\varphi_1(n)\right)^{\frac{1}{m+1}} \right)}{rm!} \left(\varphi_1(n)\right)^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{72}$$

3)

$$\left\| \left\| A_n(f) - f \right\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i, b_i]} \leq \sum_{j=1}^{m} \frac{\varphi_{2j}(n)}{j!} +$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\varphi_1(n)\right)^{\frac{1}{m+1}} \right)}{rm!} \left(\varphi_1(n)\right)^{\left(\frac{m}{m+1}\right)} \tag{73}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right] =: \varphi_3(n) \to 0, \text{ as } n \to \infty.$$

We continue with

Theorem 17 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\left\| B_n(f, x) - f(x) \right\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta} \right) + \left(1 - h\left(n^{1-\beta} - 2\right)\right) \left\| \left\| f \right\|_\gamma \right\|_\infty =: \lambda_2(n), \tag{74}$$

2)

$$\left\| \left\| B_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{75}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} B_n(f) = f$, uniformly. The speed of convergence above is $\max\left(\frac{1}{n^\beta}, \left(1 - h\left(n^{1-\beta} - 2\right)\right)\right)$.

Proof. As similar to [12] is omitted. ∎

We give

Theorem 18 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\left\| C_n(f, x) - f(x) \right\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \left(1 - h\left(n^{1-\beta} - 2\right)\right) \left\| \left\| f \right\|_\gamma \right\|_\infty =: \lambda_3(n), \tag{76}$$

2)

$$\left\| \left\| C_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_3(n). \tag{77}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} C_n(f) = f$, uniformly.

Proof. As similar to [12] is omitted. ■

We also present

Theorem 19 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta}>2$, ω_1 is for $p=\infty$. Then*

1)

$$\|D_n(f, x)-f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right)+\left(1-h\left(n^{1-\beta}-2\right)\right)\left\|\|f\|_\gamma\right\|_\infty=\lambda_4(n), \tag{78}$$

2)

$$\left\|\|D_n(f)-f\|_\gamma\right\|_\infty \leq \lambda_4(n). \tag{79}$$

Given that $f \in\left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n \rightarrow \infty} D_n(f)=f$, uniformly.

Proof. As similar to [12] is omitted. ■

We make

Definition 20 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, where $\left(X,\|\cdot\|_\gamma\right)$ is a Banach space. We define the general neural network operator*

$$F_n(f, x):=\sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx-k)=$$

$$\begin{cases} B_n(f, x), & \text{if } l_{nk}(f)=f\left(\frac{k}{n}\right), \\ C_n(f, x), & \text{if } l_{nk}(f)=n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \\ D_n(f, x), & \text{if } l_{nk}(f)=\delta_{nk}(f). \end{cases} \tag{80}$$

Clearly $l_{nk}(f)$ is an X-valued bounded linear functional such that $\|l_{nk}(f)\|_\gamma \leq \left\|\|f\|_\gamma\right\|_\infty$.

Hence $F_n(f)$ is a bounded linear operator with $\left\|\|F_n(f)\|_\gamma\right\|_\infty \leq \left\|\|f\|_\gamma\right\|_\infty$.

We need

Theorem 21 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \geq 1$. Then $F_n(f) \in C_B\left(\mathbb{R}^N, X\right)$.*

Proof. Very lengthy and as similar to [12] is omitted. ■

Remark 22 *By (25) it is obvious that $\left\|\|A_n(f)\|_\gamma\right\|_\infty \leq \left\|\|f\|_\gamma\right\|_\infty<\infty$, and $A_n(f) \in C\left(\prod_{i=1}^N\left[a_i, b_i\right], X\right)$, given that $f \in C\left(\prod_{i=1}^N\left[a_i, b_i\right], X\right)$.*

Call L_n any of the operators A_n, B_n, C_n, D_n.

Clearly then

$$\left\| \left\| L_n^2 (f) \right\|_\gamma \right\|_\infty = \left\| \left\| L_n (L_n (f)) \right\|_\gamma \right\|_\infty \leq \left\| \left\| L_n (f) \right\|_\gamma \right\|_\infty \leq \left\| \left\| f \right\|_\gamma \right\|_\infty, \tag{81}$$

etc.

Therefore we get

$$\left\| \left\| L_n^k (f) \right\|_\gamma \right\|_\infty \leq \left\| \left\| f \right\|_\gamma \right\|_\infty, \ \ \forall \ k \in \mathbb{N}, \tag{82}$$

the contraction property.

Also we see that

$$\left\| \left\| L_n^k (f) \right\|_\gamma \right\|_\infty \leq \left\| \left\| L_n^{k-1} (f) \right\|_\gamma \right\|_\infty \leq ... \leq \left\| \left\| L_n (f) \right\|_\gamma \right\|_\infty \leq \left\| \left\| f \right\|_\gamma \right\|_\infty. \tag{83}$$

Here L_n^k are bounded linear operators.

Notation 23 *Here $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by*

$$c_N := \begin{cases} (\psi (1))^{-N}, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{84}$$

$$\varphi (n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n, \ B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{85}$$

$$\Omega := \begin{cases} C \left(\prod_{i=1}^N [a_i, b_i], X \right), & \text{if } L_n = A_n, \\ C_B \left(\mathbb{R}^N, X \right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{86}$$

and

$$Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{87}$$

We give the condensed

Theorem 24 *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; n, $N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

(i)

$$\left\| L_n (f, x) - f (x) \right\|_\gamma \leq c_N \left[\omega_1 (f, \varphi (n)) + \left(1 - h \left(n^{1-\beta} - 2 \right) \right) \left\| \left\| f \right\|_\gamma \right\|_\infty \right] =: \tau (n), \tag{88}$$

where ω_1 is for $p = \infty$,

and

(ii)

$$\left\| \left\| L_n (f) - f \right\|_\gamma \right\|_\infty \leq \tau (n) \to 0, \text{ as } n \to \infty. \tag{89}$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n \to \infty} L_n (f) = f,$$

pointwise and uniformly.

Proof. By Theorems 8, 17, 18, 19. ■

Next we do iterated neural network approximation (see also [9]).

We make

Remark 25 *Let $r \in \mathbb{N}$ and L_n as above. We observe that*

$$L_n^r f - f = \left(L_n^r f - L_n^{r-1} f\right) + \left(L_n^{r-1} f - L_n^{r-2} f\right) +$$

$$\left(L_n^{r-2} f - L_n^{r-3} f\right) + ... + \left(L_n^2 f - L_n f\right) + \left(L_n f - f\right).$$

Then

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \leq \left\| \left\| L_n^r f - L_n^{r-1} f \right\|_\gamma \right\|_\infty + \left\| \left\| L_n^{r-1} f - L_n^{r-2} f \right\|_\gamma \right\|_\infty +$$

$$\left\| \left\| L_n^{r-2} f - L_n^{r-3} f \right\|_\gamma \right\|_\infty + ... + \left\| \left\| L_n^2 f - L_n f \right\|_\gamma \right\|_\infty + \left\| \left\| L_n f - f \right\|_\gamma \right\|_\infty =$$

$$\left\| \left\| L_n^{r-1} \left(L_n f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| L_n^{r-2} \left(L_n f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| L_n^{r-3} \left(L_n f - f\right) \right\|_\gamma \right\|_\infty$$

$$+ ... + \left\| \left\| L_n \left(L_n f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| L_n f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| L_n f - f \right\|_\gamma \right\|_\infty. \quad (90)$$

That is

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| L_n f - f \right\|_\gamma \right\|_\infty. \quad (91)$$

We give

Theorem 26 *All here as in Theorem 24 and $r \in \mathbb{N}$, $\tau(n)$ as in (88). Then*

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \leq r\tau(n). \quad (92)$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. By (91) and (89). ■

We make

Remark 27 *Let $m_1, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1$, $f \in \Omega$. Then $\varphi(m_1) \geq \varphi(m_2) \geq ... \geq \varphi(m_r)$, φ as in (85).*

Therefore

$$\omega_1(f, \varphi(m_1)) \geq \omega_1(f, \varphi(m_2)) \geq ... \geq \omega_1(f, \varphi(m_r)). \quad (93)$$

Assume further that $m_i^{1-\beta} > 2$, $i = 1, ..., r$. Then

$$\frac{1 - h\left(m_1^{1-\beta} - 2\right)}{2} \geq \frac{1 - h\left(m_2^{1-\beta} - 2\right)}{2} \geq ... \geq \frac{1 - h\left(m_r^{1-\beta} - 2\right)}{2}. \quad (94)$$

22

Let L_{m_i} as above, $i = 1, ..., r$, all of the same kind.

We write

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f =$$

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right) +$$

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right) +$$

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}f\right)\right) + ...+ \tag{95}$$

$$L_{m_r}\left(L_{m_{r-1}}f\right) - L_{m_r}f + L_{m_r}f - f =$$

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f - f\right) + L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f - f\right) +$$

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f - f\right) + ... + L_{m_r}\left(L_{m_{r-1}}f - f\right) + L_{m_r}f - f.$$

Hence by the triangle inequality property of $\left\|\|\cdot\|_\gamma\right\|_\infty$ we get

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f\right\|_\gamma\right\|_\infty \leq$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f - f\right)\right\|_\gamma\right\|_\infty +$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f - f\right)\right\|_\gamma\right\|_\infty +$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f - f\right)\right\|_\gamma\right\|_\infty + ...+$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}f - f\right)\right\|_\gamma\right\|_\infty + \left\|\left\|L_{m_r}f - f\right\|_\gamma\right\|_\infty$$

(repeatedly applying (81))

$$\leq \left\|\left\|L_{m_1}f - f\right\|_\gamma\right\|_\infty + \left\|\left\|L_{m_2}f - f\right\|_\gamma\right\|_\infty + \left\|\left\|L_{m_3}f - f\right\|_\gamma\right\|_\infty + ...+$$

$$\left\|\left\|L_{m_{r-1}}f - f\right\|_\gamma\right\|_\infty + \left\|\left\|L_{m_r}f - f\right\|_\gamma\right\|_\infty = \sum_{i=1}^{r}\left\|\left\|L_{m_i}f - f\right\|_\gamma\right\|_\infty. \tag{96}$$

That is, we proved

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f\right\|_\gamma\right\|_\infty \leq \sum_{i=1}^{r}\left\|\left\|L_{m_i}f - f\right\|_\gamma\right\|_\infty. \tag{97}$$

We give

Theorem 28 *Let $f \in \Omega$; N, $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1$; $m_i^{1-\beta} > 2$, $i = 1, ..., r$, $x \in Y$, and let $(L_{m_1}, ..., L_{m_r})$ as $(A_{m_1}, ..., A_{m_r})$ or $(B_{m_1}, ..., B_{m_r})$ or $(C_{m_1}, ..., C_{m_r})$ or $(D_{m_1}, ..., D_{m_r})$, $p = \infty$. Then*

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)(x) - f(x)\right\|_\gamma \leq$$

$$\left\| \left\| L_{m_r} \left(L_{m_{r-1}} \left(...L_{m_2} \left(L_{m_1} f \right) \right) \right) - f \right\|_\gamma \right\|_\infty \leq$$
$$\sum_{i=1}^{r} \left\| \left\| L_{m_i} f - f \right\|_\gamma \right\|_\infty \leq$$
$$c_N \sum_{i=1}^{r} \left[\omega_1 \left(f, \varphi \left(m_i \right) \right) + \left(1 - h \left(m_i^{1-\beta} - 2 \right) \right) \left\| \left\| f \right\|_\gamma \right\|_\infty \right] \leq$$
$$r c_N \left[\omega_1 \left(f, \varphi \left(m_1 \right) \right) + \left(1 - h \left(m_1^{1-\beta} - 2 \right) \right) \left\| \left\| f \right\|_\gamma \right\|_\infty \right]. \tag{98}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. Using (97), (93), (94) and (88), (89). ∎

We continue with

Theorem 29 *Let all as in Corollary 16, and* $r \in \mathbb{N}$. *Here* $\varphi_3(n)$ *is as in (73). Then*

$$\left\| \left\| A_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| A_n f - f \right\|_\gamma \right\|_\infty \leq r \varphi_3(n). \tag{99}$$

Proof. By (91) and (73). ∎

Application 30 *A typical application of all of our results is when* $\left(X, \|\cdot\|_\gamma \right) = (\mathbb{C}, |\cdot|)$, *where* $\mathbb{C}$ *are the complex numbers.*

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