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Journal of Computational Analysis and Applications

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Asymptotic behavior of equilibrium point for a system of fourth-order rational difference equations

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Abstract

Our aim in this paper is to investigate the dynamics of a system of fourth-order rational difference equations

$$x_{n+1}=\frac{x_{n-3}-y_{n-1}}{A+x_{n-3}y_{n-1}},\quad y_{n+1}=\frac{y_{n-3}-x_{n-1}}{A+y_{n-3}x_{n-1}},\quad n=0,1,\cdots,$$

where the parameter A is arbitrary positive real number and the initial conditions $x_{-3},x_{-2},x_{-1},x_0,\ y_{-3},y_{-2},y_{-1},y_0$ are arbitrary nonnegative real numbers. By using new iteration method for the more general nonlinear difference equations and inequality skills, we establish some sufficient conditions which guarantee the existence, unstability and global asymptotic stability of the equilibriums for this nonlinear system. Numerical examples to the difference system are given to verify our theoretical results.

Keywords: difference system; equilibrium point; asymptotical stability; unstability

1. Introduction

Because of the necessity for some techniques that can be used in mathematical models describing real situations, nonlinear difference equations have been studied in the fields of population biology, economics, probability theory, genetics, psychology etc (see, e.g., [1-4] and the references therein). In recent years, with the dramatically development of

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computer-based computational techniques, difference equations are found to be much appropriate mathematical representations for computer simulation and experiment (see, e.g., [5-8]). However, it is more interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the asymptotic stability of their equilibrium points (for example, see [9-19]).

Recently, Bajo and Liz [20] described the asymptotic behavior and the stability properties of the solution to the following nonlinear second-order difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + bx_n x_{n-1}}, \quad n = 0,1,\cdots. \tag{1.1}$$

for all values of the real parameters $a,b,$ and any initial condition $(x_{-1}, x_0) \in R^2$.

In [21], Kurbanli, Cinar, and Yalcinkaya investigated the positive solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}, \qquad n = 0,1,\cdots, \tag{1.2}$$

where $(x_k, y_k) \in [0,\infty)$ for $k = -1,0$.

Moreover, Touafek and Elsayed [22] deal with the periodic nature and the form of the solutions of the following systems of rational difference equations.

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}, \quad n = 0,1,\cdots, \tag{1.3}$$

with a nonzero real number's initial conditions.

As an extension of (1.3), Elsayed [23] continuously dealt with the existence of solutions and the periodicity character of the following systems of rational difference equations

$$x_{n+1} = \frac{x_n y_{n-1}}{y_n (x_n y_{n-1} \pm 1)}, \quad y_{n+1} = \frac{y_n x_{n-1}}{x_n (y_n x_{n-1} \pm 1)}, \quad n = 0,1,\cdots, \tag{1.4}$$

where the initial conditions x_{-1}, x_0, y_{-1} and y_0 are nonzero real numbers.

More recently, Khan and Qureshi [24] study the equilibrium points, local asymptotic stability of equilibrium point, unstability of equilibrium points, global character of equilibrium point, periodicity behavior of positive solutions and rate of convergence of positive solutions of the following systems

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}}, \quad y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}}, \quad n = 0,1,\cdots, \tag{1.5}$$

and

$$x_{n+1}=\frac{ay_{n-1}}{b-cx_nx_{n-1}},\quad y_{n+1}=\frac{a_1x_{n-1}}{b_1-c_1y_ny_{n-1}},\quad n=0,1,\cdots. \tag{1.6}$$

Especially, Yalçınkaya [25] investigated the sufficient condition for the global asymptotic stability of the following systems of difference equations

$$x_{n+1}=\frac{x_n+y_{n-1}}{x_ny_{n-1}-1},\quad y_{n+1}=\frac{y_n+x_{n-1}}{y_nx_{n-1}-1},\qquad n=0,1,\cdots. \tag{1.7}$$

where the initial conditions x_{-1},x_0,y_{-1} and y_0 are nonzero real numbers.

Motivated by works [20-25], our aim in this paper is to investigate the dynamics of a system of fourth-order rational difference equations

$$x_{n+1}=\frac{x_{n-3}-y_{n-1}}{A+x_{n-3}y_{n-1}},\quad y_{n+1}=\frac{y_{n-3}-x_{n-1}}{A+y_{n-3}x_{n-1}},\quad n=0,1,\cdots, \tag{1.8}$$

where $A\in(0,\infty)$ and $(x_n,y_n)\in[0,\infty)\times[0,\infty)$ for $n=-3,-2,-1,0,\cdots$.

For more related work, one can refer to [26-35] and references therein.

2. Some preliminary results

To prove the main results in this paper we first give some definitions and preliminary results [36-38] which are basically used throughout this paper.

Lemma 2.1 Let I_x, I_y be some intervals of real numbers and let $f:I_x^{k+1}\times I_y^{l+1}\to I_x$, $g:I_x^{k+1}\times I_y^{l+1}\to I_y$ be continuously differentiable functions. Then for every set of initial conditions $(x_i, y_j)\in I_x\times I_y$, $(i=-k,-k+1,\cdots,0,\ j=-l,-l+1,\cdots,0)$, the following system of difference equations

$$\begin{cases}x_{n+1}=f(x_n,x_{n-1},\cdots,x_{n-k},y_n,y_{n-1},\cdots,y_{n-l}),\\ y_{n+1}=g(x_n,x_{n-1},\cdots,x_{n-k},y_n,y_{n-1},\cdots,y_{n-l}),\end{cases}\quad n=0,1,2,\cdots, \tag{2.1}$$

has a unique solution $\{(x_i,y_j)\}_{i=-k,j=-l}^{+\infty,+\infty}$.

Definition 2.1 A point $(\bar{x},\bar{y})\in I_x\times I_y$ is called an equilibrium point of system (2.1) if

$$\bar{x}=f(\bar{x},\bar{x},\cdots,\bar{x},\bar{y},\bar{y},\cdots,\bar{y}),\quad \bar{y}=g\left(\bar{x}.\bar{x},\cdots,\bar{x},\bar{y},\bar{y},\cdots,\bar{y}\right).$$

That is, $(x_n,y_n)=(\bar{x},\bar{y})$ for $n\ge 0$ is the solution of difference system (2.1), or equivalently, $(\bar{x},\bar{y})$ is a fixed point of the vector map (f,g).

Definition 2.2 Assume that $(\bar{x},\bar{y})$ be an equilibrium point of the system (2.1). Then, we have

(i) An equilibrium point $(\bar{x},\bar{y})$ is called locally stable if for every $\varepsilon>0$, there exits $\delta>0$ such that for any initial conditions $(x_i,y_i)\in I_x\times I_y$ $(i=-k,\cdots,0,\ j=-l,\cdots,0)$, with $\sum_{i=-k}^{0}|x_i-\bar{x}|<\delta$, $\sum_{j=-l}^{0}|y_j-\bar{y}|<\delta$, we have $|x_n-\bar{x}|<\varepsilon, |y_n-\bar{y}|<\varepsilon$ for any $n>0$.

(ii) An equilibrium point $(\bar{x},\bar{y})$ is called attractor if $lim_{n\to\infty}x_n=\bar{x}$, $lim_{n\to\infty}y_n=\bar{y}$ for any initial conditions $(x_i,y_i)\in I_x\times I_y$ $(i=-k,\cdots,0,\ j=-l,\cdots,0)$.

(iii) An equilibrium point $(\bar{x},\bar{y})$ is called asymptotically stable if it is stable, and $(\bar{x},\bar{y})$ is also attractor.

(iv) An equilibrium point $(\bar{x},\bar{y})$ is called unstable if it is not locally stable.

Definition 2.3 Let $(\bar{x},\bar{y})$ be an equilibrium point of the vector map $F=(f,x_n,\cdots,x_{n-k},g,y_n,\cdots,y_{n-l})$, where f and g are continuously differentiable functions at $(\bar{x},\bar{y})$. The linearized system of (1.8) about the equilibrium point $(\bar{x},\bar{y})$ is $X_{n+1}=F(X_n)=F_j\cdot X_n$, where F_J is the Jacobian matrix of the system (1.8) about $(\bar{x},\bar{y})$ and $X_n=(x_n,\cdots,x_{n-k},y_n,\cdots,y_{n-l})^T$.

Definition 2.4 let p,q,s,t be four nonnegative integers such that $p+q=n, s+t=m$. Splitting $x=(x_1,x_2,\cdots,x_n)$ into $x=([x]_p,[x]_q)$ and $y=(y_1,y_2,\cdots,y_m)$ into $y=([y]_s,[y]_t)$, where $[x]_\sigma$ denotes a vector with σ-components of x. We say that the function $f(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_m)$ possesses a mixed monotone property in subsets $I_x^n\times I_y^m$ of $R^n\times R^m$ if $f([x]_p,[x]_q,[y]_s,[y]_t)$ is monotone non-decreasing in each component of $([x]_p,[y]_s)$, and is monotone non-increasing in each component of $([x]_q,[y]_t)$ for $(x,y)\in I_x^n\times I_y^m$. In particular, if $q=0,t=0$, then it is said to be monotone non-decreasing in $I_x^n\times I_y^m$.

Lemma 2.2 Assume that $X(n+1)=F(X(n))$, $n=0,1,\cdots$, is a system of difference equations and $\bar{X}$ is the equilibrium point of this system i.e., $F(\bar{X})=\bar{X}$. Then we have

(i) If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk

$|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable.

(ii) If one of eigenvalues of the Jacobian matrix J_F about $\bar{X}$ has norm greater than one, then $\bar{X}$ is unstable.

Lemma 2.3 Assume that $X(n+1) = F(X(n))$, $n = 0, 1, \cdots$, is a system of difference equations and $\bar{X}$ is the equilibrium point of this system, the characteristic polynomial of this system about the equilibrium point $\bar{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, with the real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lie inside the open unit disk $|\lambda| < 1$ if and only if

$$\Delta_k > 0 \quad \text{for} \quad k = 1, 2, \cdots, n, \tag{2.2}$$

where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

3. Main results and their proofs

In this section, we shall investigate the qualitative behavior of the system (1.8). Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1.8), then the system (1.8) has a unique equilibrium (0, 0) when $0 < A \le 2$, and the system (1.8) has following three equilibrium points $P_0 = (0,0)$, $P_1 = (\sqrt{A-2}, -\sqrt{A-2})$, and $P_2 = (-\sqrt{A-2}, \sqrt{A-2})$ if $A > 2$.

To construct corresponding linearlized form of the nonlinear system (1.8), we consider the transformation

$$(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) \mapsto (f, f_1, f_2, f_3, g, g_1, g_2, g_3), \tag{3.1}$$

where

$$f = \frac{x_{n-3} - y_{n-1}}{A + x_{n-3}y_{n-1}}, f_i = x_{n-i+1}, g = \frac{y_{n-3} - x_{n-1}}{A + y_{n-3}x_{n-1}}, g_i = y_{n-i+1}, i = 1, 2, 3.$$

The Jacobian matrix about the equilibrium point $(\overline{x},\overline{y})$ under the transformation (3.1) is given by

$$F_J(\overline{x},\overline{y})=\begin{bmatrix} 0 & 0 & 0 & \delta_1 & 0 & \delta_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_3 & 0 & 0 & 0 & 0 & 0 & \delta_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\delta_1=\frac{A+\overline{y}^2}{(A+\overline{x}\,\overline{y})^2}$, $\delta_2=\frac{-A-\overline{x}^2}{(A+\overline{x}\,\overline{y})^2}$, $\delta_2=\frac{-A-\overline{y}^2}{(A+\overline{x}\,\overline{y})^2}$, $\delta_4=\frac{A+\overline{x}^2}{(A+\overline{x}\,\overline{y})^2}$.

Theorem 3.1 If $A>1$, then the equilibrium point (0,0) of the system (1.8) is locally asymptotically stable.

Proof: We can easily obtain that the linearlized system of (1.8) about the equilibrium point (0,0) is

$$\varphi_{n+1}=D\varphi_n \tag{3.2}$$

where

$$\varphi_n=\begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix},\qquad D=\begin{bmatrix} 0 & 0 & 0 & \frac{1}{A} & 0 & -\frac{1}{A} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{A} & 0 & 0 & 0 & 0 & 0 & \frac{1}{A} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation of (3.2) is

$$f(\lambda)=(\lambda^4-\frac{1}{A})^2=0\,. \tag{3.3}$$

In view of $A>1$, it is clear that all roots of characteristic equation (3.3) lie inside unit disk. Hence the equilibrium (0,0) is locally asymptotically stable by Lemma 2.1.

Theorem 3.2 Let I_x, I_y be some intervals of real numbers and assume that $f: I_x^{k+1} \times I_y^{l+1} \to I_x$ and $g: I_x^{k+1} \times I_y^{l+1} \to I_y$ be continuously differentiable functions satisfying mixed monotone property. If there exits

$$\begin{cases} m_0 \le \min\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le \max\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le M_0, \\ n_0 \le \min\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le \max\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le N_0, \end{cases} \tag{3.4}$$

such that

$$\begin{cases} m_0 \le f([m_0]_p, [M_0]_q, [n_0]_s, [N_0]_t) \le f([M_0]_p, [m_0]_q, [N_0]_s, [n_0]_t) \le M_0, \\ n_0 \le g([m_0]_{p_1}, [M_0]_{q_1}, [n_0]_{s_1}, [N_0]_{t_1}) \le g([M_0]_{p_1}, [m_0]_{q_1}, [N_0]_{s_1}, [n_0]_{t_1}) \le N_0, \end{cases} \tag{3.5}$$

then there exit $(m, M) \in [m_0, M_0]^2$ and $(n, N) \in [n_0, N_0]^2$ satisfying

$$\begin{cases} M = f([M]_p, [m]_q, [N]_s, [n]_t), \quad m = f([m]_p, [M]_q, [n]_s, [N]_t), \\ N = g([M]_{p_1}, [m]_{q_1}, [N]_{s_1}, [n]_{t_1}), \quad n = g([m]_{p_1}, [M]_{q_1}, [n]_{s_1}, [N]_{t_1}). \end{cases} \tag{3.6}$$

Moreover, if $m = M, n = N$, then equation (2.1) has a unique equilibrium point $(\overline{x}, \overline{y}) \in [m_0, M_0] \times [n_0, N_0]$ and every solution of (2.1) converges to $(\overline{x}, \overline{y})$.

Proof. Using m_0, M_0, n_0 and N_0 as two couples of initial iterations, we construct four sequences $\{m_i\}, \{M_i\}, \{n_i\}$, and $\{N_i\}$ $(i = 1, 2, \cdots)$ from the following equations

$$\begin{cases} m_i = f([m_{i-1}]_p, [M_{i-1}]_q, [n_{i-1}]_s, [N_{i-1}]_t), \quad M_i = f([M_{i-1}]_p, [m_{i-1}]_q, [N_{i-1}]_s, [n_{i-1}]_t), \\ n_i = g([m_{i-1}]_{p_1}, [M_{i-1}]_{q_1}, [n_{i-1}]_{s_1}, [N_{i-1}]_{t_1}), \quad N_i = g([M_{i-1}]_{p_1}, [m_{i-1}]_{q_1}, [N_{i-1}]_{s_1}, [n_{i-1}]_{t_1}). \end{cases} \tag{3.7}$$

It is obvious from the mixed monotone property of f and g that the sequences $\{m_i\}, \{M_i\}$, $\{n_i\}$ and $\{N_i\}$ possess the following monotone property

$$\begin{cases} m_0 \le m_1 \le \cdots \le m_i \le \cdots \le M_i \le \cdots \le M_1 \le M_0, \\ n_0 \le n_1 \le \cdots \le n_i \le \cdots \le N_i \le \cdots \le N_1 \le N_0, \end{cases} \tag{3.8}$$

where i=0, 1, 2, $\cdots$, and

$$m_i \le x_u \le M_i, n_i \le y_v \le N_i, \text{ for } u \ge (k+1)i + 1, v \ge (l+1)i + 1, i = 0, 1, 2, \cdots. \tag{3.9}$$

Set

$$m = \lim_{i\to\infty} m_i, M = \lim_{i\to\infty} M_i, \quad n = \lim_{i\to\infty} n_i, N = \lim_{i\to\infty} N_i. \tag{3.10}$$

Then

$$m \le \liminf_{i\to\infty} x_i \le \limsup_{i\to\infty} x_i \le M, \quad n \le \liminf_{i\to\infty} y_i \le \limsup_{i\to\infty} y_i \le N. \tag{3.11}$$

By the continuity of f and g, one has

$$\begin{cases} M = f([M]_p,[m]_q,[N]_s,[n]_t), m = f([m]_p,[M]_q,[n]_s,[N]_t), \\ N = g([M]_{p_1},[m]_{q_1},[N]_{s_1},[n]_{t_1}), n = g([m]_{p_1},[M]_{q_1},[n]_{s_1},[N]_{t_1}). \end{cases} \tag{3.12}$$

Moreover, if $m = M, n = N$, then $m = M = \lim_{i\to\infty} x_i = \bar{x}, n = N = \lim_{i\to\infty} y_i = \bar{y}$, and then the proof is complete.

Theorem 3.3 If $1 < A$, then the equilibrium point $(0,0)$ of the system (1.8) is global attractor for any condition $(x_{-3}, x_{-2}x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in (0,\infty)^8$.

Proof: Let $(f,g):(0,\infty)^4 \times (0,\infty)^4 \to (0,\infty)\times(0,\infty)$ be a function defined by

$$f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) = \frac{x_{n-3} - y_{n-1}}{A + x_{n-3}y_{n-1}}, \tag{3.13}$$

and

$$g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) = \frac{y_{n-3} - x_{n-1}}{A + x_{n-1}y_{n-3}}. \tag{3.14}$$

Set

$$f = \frac{x-y}{A+xy}, \quad g = \frac{y-x}{A+xy}, \tag{3.15}$$

we can obtain that

$$\begin{aligned} f_x &= \frac{A+y^2}{(A+xy)^2} > 0, \quad g_x = \frac{-A-y^2}{(A+xy)^2} < 0, \\ f_y &= \frac{-A-x^2}{(A+xy)^2} < 0, \quad g_x = \frac{A+x^2}{(A+xy)^2} > 0, \end{aligned} \tag{3.16}$$

which implies that f and g possess a mixed monotone property.

Let $M_0 = N_0 = \max\{x_{-3}, x_{-2}x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0\}$ and $-A/M_0 < m_0 = n_0 < M_0/(1-A)$.

Thus, we have

$$m_0 < \frac{m_0 - N_0}{A + m_0 N_0} < \frac{M_0 - n_0}{A + M_0 n_0} < M_0, n_0 < \frac{n_0 - M_0}{A + n_0 M_0} < \frac{N_0 - m_0}{A + N_0 m_0} < N_0. \tag{3.17}$$

Moreover, from (1.8) and Theorem 3.2, one can derive that there exists $m, M \in [m_0, M_0]$,

$n, N \in [n_0, N_0]$ satisfying

$$m = \frac{m-N}{A+mN}, n = \frac{n-M}{A+nM}, M = \frac{M-n}{A+nM}, N = \frac{N-m}{A+Nm}. \tag{3.18}$$

Hence, we have

$$M = N = m = n = 0.$$

According to Lemma 2.2 and Theorem 3.2, If $1 < A$, the unique equilibrium $(0,0)$ is not only locally asymptotically stable, but also a global attractor. The proof is complete.

Theorem 3.4 If $A < 1$, then the equilibrium point $(0,0)$ is unstable.

Proof: It is easy to see that there exist roots of characteristic equation (3.3) lie outside unit disk when $A < 1$. According to Lemma 2.2, the equilibrium point (0, 0) is unstable.

Theorem 3.5 The equilibrium points $p_1 = (\sqrt{A-2}, -\sqrt{A-2})$, and $p_2 = (-\sqrt{A-2}, \sqrt{A-2})$ of the system (1.8) are locally asymptotically stable when $2 < A < 3$. And the equilibrium points p_1 and p_2 of the system (1.8) are unstable when $A > 3$.

Proof: We can easily obtain that the linear equations of the system (1.8) about the equilibrium point $p_1 = (\sqrt{A-2}, -\sqrt{A-2})$ is

$$\varphi_{n+1} = D^* \varphi_n,$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & \frac{A-1}{2} & 0 & \frac{1-A}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-A}{2} & 0 & 0 & 0 & 0 & 0 & \frac{A-1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.19}$$

The characteristic equation of (3.18) is

$$f(\lambda) = (\lambda^4 - \frac{A-1}{2})^2 = 0. \tag{3.20}$$

Hence, we have that the equilibrium point p_1 of the system (1.8) is locally asymptotically stable when $2 < A < 3$, and the equilibrium point p_1 of the system (1.8) is unstable when

$A > 3$. The stability and unstability of the equilibrium point p_2 can be proved similarly.

4. Numerical simulations

In this section some numerical examples are given in order to confirm the results of the previous sections and to support our theoretical discussions. These examples represent different types of qualitative behavior of solutions of the system (1.8). As examples, we consider the following difference equations

$$x_{n+1} = \frac{x_{n\text{-}3} - y_{n\text{-}1}}{3 + x_{n\text{-}3}y_{n\text{-}1}}, \quad y_{n+1} = \frac{y_{n\text{-}3} - x_{n\text{-}1}}{3 + y_{n\text{-}3}x_{n\text{-}1}}, \quad n = 0, 1, \cdots, \tag{4.1}$$

$$x_{n+1} = \frac{x_{n\text{-}3} - y_{n\text{-}1}}{5 + x_{n\text{-}3}y_{n\text{-}1}}, \quad y_{n+1} = \frac{y_{n\text{-}3} - x_{n\text{-}1}}{5 + y_{n\text{-}3}x_{n\text{-}1}}, \quad n = 0, 1, \cdots, \tag{4.2}$$

and

$$x_{n+1} = \frac{x_{n\text{-}3} - y_{n\text{-}1}}{0.5 + x_{n\text{-}3}y_{n\text{-}1}}, \quad y_{n+1} = \frac{y_{n\text{-}3} - x_{n\text{-}1}}{0.5 + y_{n\text{-}3}x_{n\text{-}1}}, \quad n = 0, 1, \cdots. \tag{4.3}$$

By employing the software package MATLAB7.0, we can solve the numerical solutions of the system (4.1), (4.2) and (4.3) which are shown respectively in Figures 4.1-Figure 4.4. More precisely, it is obvious that the equations (4.1) satisfy the conditions of Theorems 3.1 and Figure 4.1 shows that the solution of the difference system (1.8) is local stability if $A = 3$ and the initial conditions $x_{-3} = 6, x_{-2} = 4, x_{-1} = 2, x_0 = 8, y_{-3} = 1, y_{-2} = 4$, $y_{-1} = 2$ and $y_0 = 3$. We can also see that the equations (4.1) satisfy the conditions of Theorems 3.2 and Theorem 3.3, and Figure 4.2 shows that the solution of the difference system (1.8) is globally asymptotically stable where $A = 3$, $n_0 = m_0 = -0.9, N_0 = M_0 = 0.9$ and the initial conditions $x_{-3} = 0.01, x_{-2} = 0.02, x_{-1} = 0.01, x_0 = 0.03, y_{-3} = 0.2, y_{-2} = 0.4$, $y_{-1} = 0.8$ and $y_0 = 0.7$. It can be noticed that the equations (4.2) satisfy the conditions of Theorems 3.1, Theorems 3.2 and Theorem 3.3, and Figure 4.3 shows that the solution of the difference system (1.8) is globally asymptotically stable where $A = 5$, $n_0 = m_0 = -0.3, N_0 = M_0 = 0.5$ and the initial conditions $x_{-3} = 0.2, x_{-2} = 0.06, x_{-1} = 0.4, x_0 = 0.08, y_{-3} = 0.02, y_{-2} = 0.04$, $y_{-1} = 0.01$ and $y_0 = 0.1$. It is clear that the equations (4.3) satisfy the conditions of Theorem 3.4, and Figure 4.4 shows that the solution of the difference system (1.8) is unstable where $A = 0.5$, and the initial conditions $x_{-3} = 1.6, x_{-2} = 1, x_{-1} = 1.5, x_0 = 1.8, y_{-3} = 1, y_{-2} = 4$, $y_{-1} = 2$ and $y_0 = 3$.

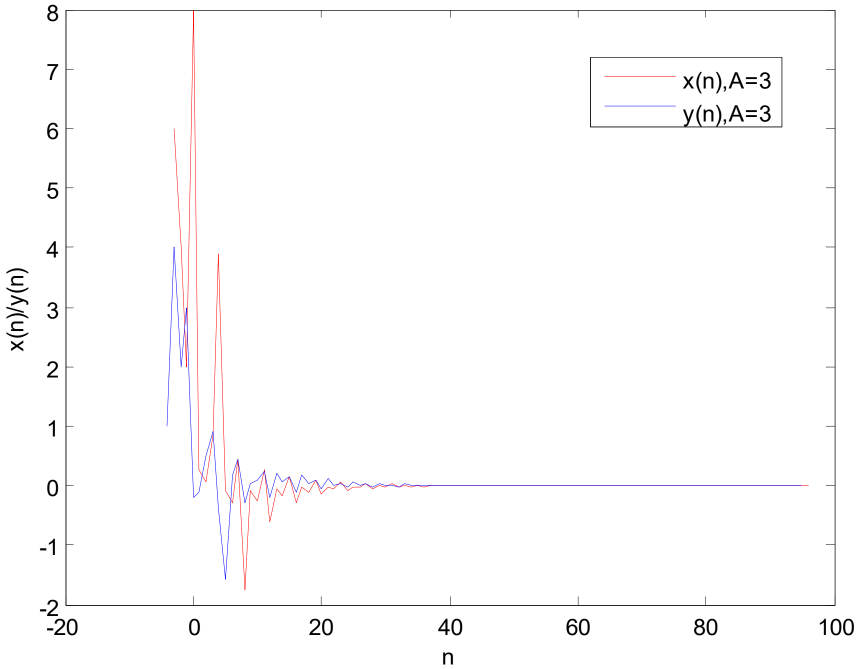

Figure 4.1. Solutions of (4.1) with A=3 and the initial conditions

$x_{-3}=6, x_{-2}=4, x_{-1}=2, x_0=8, y_{-3}=1, y_{-2}=4$, $y_{-1}=2$ and $y_0=3$

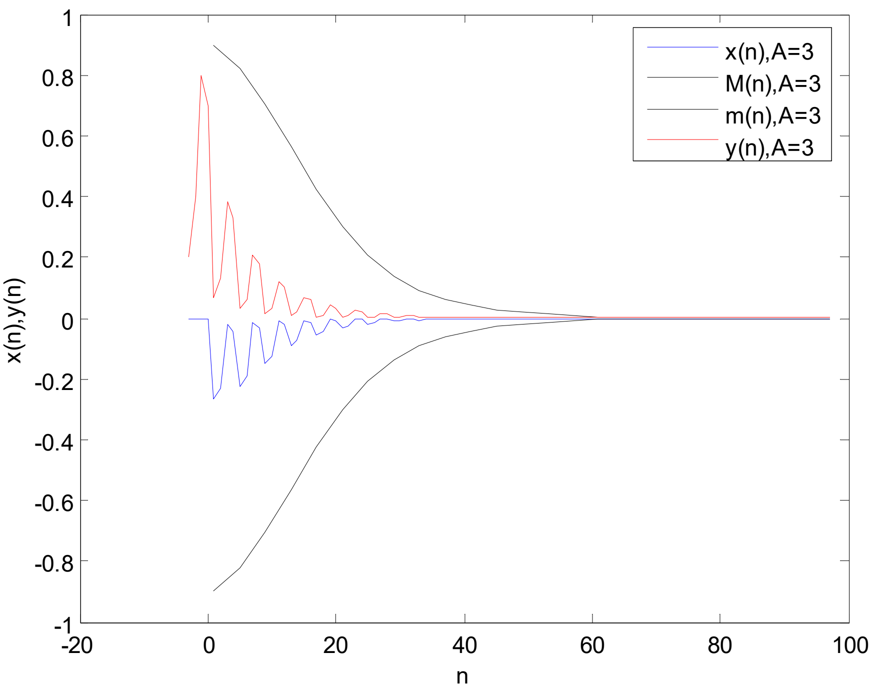

Figure 4.2. Solutions of (4.1) with $A=3$, $n_0=m_0=-0.9, N_0=M_0=0.9$ and the initial conditions

$x_{-3}=0.01, x_{-2}=0.02, x_{-1}=0.01, x_0=0.03, y_{-3}=0.2, y_{-2}=0.4$, $y_{-1}=0.8$ and $y_0=0.7$

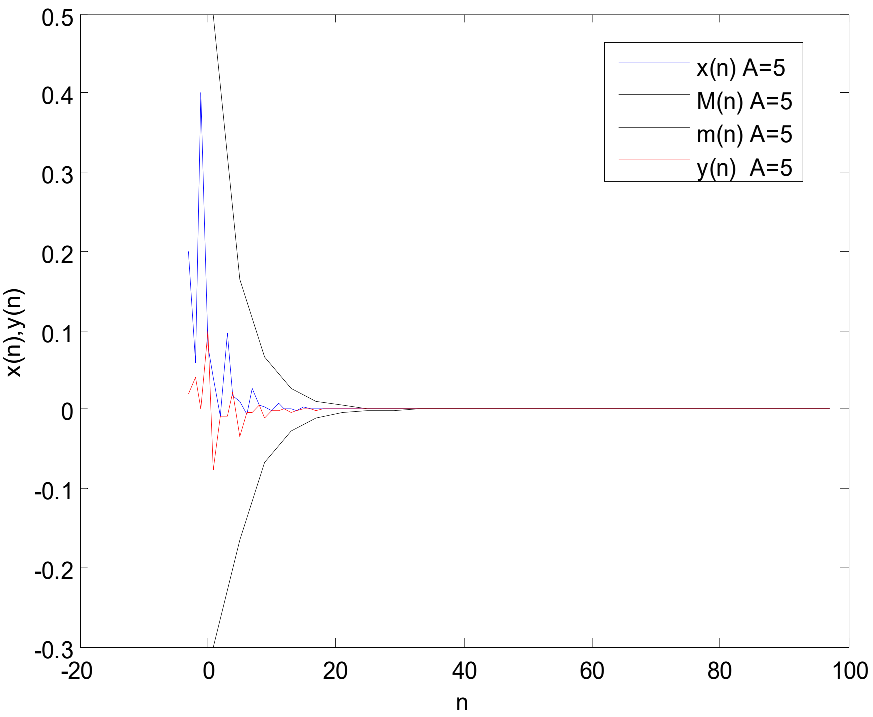

Figure 4.3. Solutions of (4.2) with $A = 5$, $n_0 = m_0 = -0.3, N_0 = M_0 = 0.5$ and the initial conditions

$x_{-3} = 0.2, x_{-2} = 0.06, x_{-1} = 0.4, x_0 = 0.08, y_{-3} = 0.02, y_{-2} = 0.04$, $y_{-1} = 0.01$ and $y_0 = 0.1$

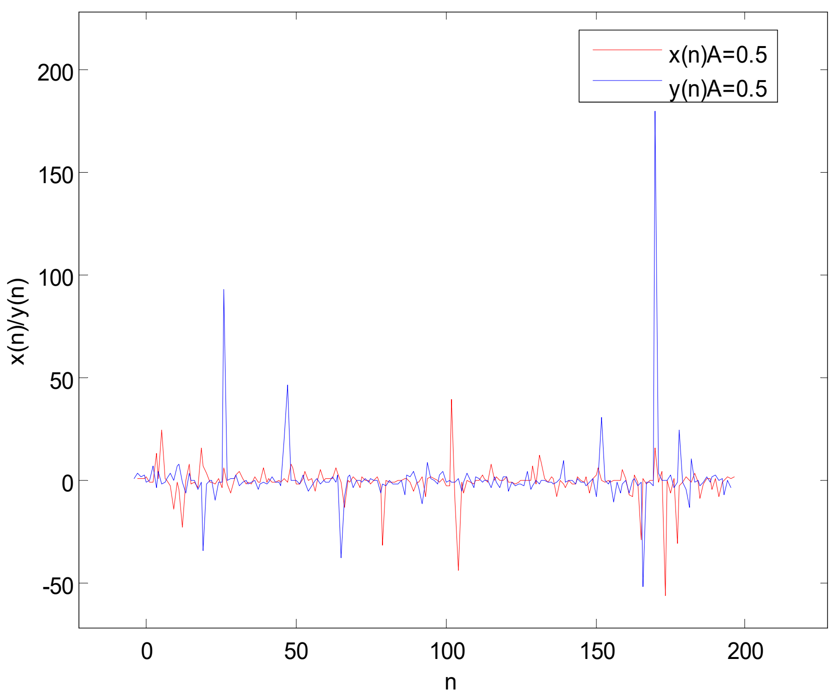

Figure 4.4. Solutions of (4.3) with $A = 0.5$, and the initial conditions

$x_{-3} = 1.6, x_{-2} = 1, x_{-1} = 1.5, x_0 = 1.8, y_{-3} = 1, y_{-2} = 4$, $y_{-1} = 2$ and $y_0 = 3$

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5. Conclusions

This paper presents the use of a variational iteration method for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. The numerical simulations show that this method is an effective and convenient one. The variational iteration method provides an efficient method to handle the nonlinear structure. Computations are performed using the software package MATLAB7.0.

We have dealt with the problem of global asymptotic stability analysis for a class of nonlinear high order difference equations. The general sufficient conditions have been obtained to ensure the existence, unstability and global asymptotic stability of the equilibrium point for the nonlinear difference equations. These criteria generalize and improve some known results. In particular, some illustrate examples are given to show the effectiveness of the obtained results. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equation.

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A version of the Hadamard inequality for Caputo fractional derivatives and related results

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Abstract

In this paper we are interested to give the Hadamard inequality for n-times differentiable convex functions via Caputo fractional derivatives. We also find bounds of a difference of this inequality.

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Key words and phrases: convex functions, Hadamard inequality, Caputo Fractional derivatives

1 introduction

Fractional calculus was mainly a study kept for the finest minds in mathematics. The history of fractional calculus is as old as the history of differential calculus. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. Fourier, Eulern and Laplace are among those mathematicians who showed a casual interest by fractional calculus and

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mathematical consequences. A lot of them established definitions by means of their own notion and style. Most renowned of these definitions are the Grunwald-Letnikov and Riemann-Liouville definitions [6–8].

In the following we give the definition of Caputo fractional derivatives [6].

Definition 1.1. Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having nth derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives of order α are defined as follows:

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, x > a \tag{1.1}$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, x < b. \tag{1.2}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D^{\alpha}_{a+} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^C D^{\alpha}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^0_{a+} f)(x) = ({}^C D^0_{b-} f)(x) = f(x)$$

where $n = 1$ and $\alpha = 0$.

Definition 1.2. ([7]) Let $f \in L[a, b]$. Then Riemann-Liouville fractional integrals of f of order α are defined as follows

$$I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \ x > a$$

and

$$I^{\alpha}_{b_-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \ x < b.$$

In [10], Sarikaya et al. proved following Hadamard-type inequalities for Riemann-Liouville fractional integrals:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[I^{\alpha}_{(\frac{a+b}{2})+} f(b) + I^{\alpha}_{(\frac{a+b}{2})-} f(a) \right] \\ &\le \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.3}$$

with $\alpha > 0$.

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[I^\alpha_{(\frac{a+b}{2})+}f(b)+I^\alpha_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[\left((\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left((\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{1.4}$$

with $\alpha>0$.

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following inequality for fractional integral holds*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[I^\alpha_{(\frac{a+b}{2})+}f(b)+I^\alpha_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(a)|^q+3|f'(b)|q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right) \\ &\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|], \end{aligned} \tag{1.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In recent days many researchers have focused their attention in establishing inequalities of Hadamard type via utilization of fractional integral operators, (see, [1–5, 9]) and references therein. In this paper we are interested to give versions of inequalities (1.3), (1.4) and (1.5) for n-times differentiable convex functions via Caputo fractional derivatives.

In the whole paper $C^n[a,b]$ denotes the space of n-times differentiable functions such that $f^{(n)}$ are continuous on $[a,b]$.

2 Hadamard-type inequalities for Caputo fractional derivatives

In this section we give a version of the Hadamard inequality via Caputo fractional derivatives. First we prove the following lemma.

Lemma 2.1. *Let $g:[a,b]\to\mathbb{R}$ be a function such that $g\in C^n[a,b]$, also let $g^{(n)}$ is integrable and symmetric to $\frac{a+b}{2}$. Then we have*

$$\begin{aligned} ({}^CD^\alpha_{a+}g)(b) &= (-1)^n({}^CD^\alpha_{b-}g)(a) \\ &= \frac{1}{2}[({}^CD^\alpha_{a+}g)(b)+(-1)^n({}^CD^\alpha_{b-}g)(a)]. \end{aligned}$$

Proof. By symmetricity of $g^{(n)}$ we have $g^{(n)}(a+b-x) = g^{(n)}(x)$, where $x \in [a,b]$. Replacing x with $a+b-x$ in the following integral we have

$$\begin{aligned}({}^C D^{\alpha}_{a+} g)(b) &= \frac{1}{\Gamma(n-\alpha)} \int_a^b \frac{g^{(n)}(x)}{(b-x)^{\alpha-n+1}} dx \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^b \frac{g^{(n)}(a+b-x)}{(x-a)^{\alpha-n+1}} dx \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^b \frac{g^{(n)}(x)}{(x-a)^{\alpha-n+1}} dx \\ &= (-1)^n ({}^C D^{\alpha}_{b-} g)(a).\end{aligned}$$

□

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in C^n[a,b]$. If $f^{(n)}$ is a convex function on $[a,b]$, then the following inequalities for Caputo fractional derivatives hold*

$$\begin{aligned}&f^{(n)}\left(\frac{a+b}{2}\right) \\ &\le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \left[({}^C D^{\alpha}_{(\frac{a+b}{2})+} f)(b) + (-1)^n ({}^C D^{\alpha}_{(\frac{a+b}{2})-} f)(a)\right] \\ &\le \frac{f^{(n)}(a) + f^{(n)}(b)}{2}.\end{aligned} \tag{2.1}$$

Proof. From convexity of $f^{(n)}$ we have

$$f^{(n)}\left(\frac{x+y}{2}\right) \le \frac{f^{(n)}(x) + f^{(n)}(y)}{2}.$$

Setting $x = \frac{t}{2}a + \frac{(2-t)}{2}b, y = \frac{(2-t)}{2}a + \frac{t}{2}b$ for $t \in [0,1]$. Then $x, y \in [a,b]$ and above inequality gives

$$2f^{(n)}\left(\frac{a+b}{2}\right) \le f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right),$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$ and integrating over $[0,1]$ we have

$$\begin{aligned}&2f^{(n)}\left(\frac{a+b}{2}\right) \int_0^1 t^{n-\alpha-1} dt \\ &\le \int_0^1 t^{n-\alpha-1} f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt + \int_0^1 t^{n-\alpha-1} f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right) dt \\ &= \frac{2^{n-\alpha}\Gamma(\alpha)}{(b-a)^{n-\alpha}} \left[({}^C D^{\alpha}_{(\frac{a+b}{2})+} f)(b) + (-1)^n ({}^C D^{\alpha}_{(\frac{a+b}{2})-} f)(a)\right],\end{aligned}$$

from which one can have

$$\begin{aligned}&f^{(n)}\left(\frac{a+b}{2}\right) \\ &\le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \left[({}^C D^{\alpha}_{(\frac{a+b}{2})+} f)(b) + (-1)^n ({}^C D^{\alpha}_{(\frac{a+b}{2})-} f)(a)\right].\end{aligned} \tag{2.2}$$

On the other hand convexity of $f^{(n)}$ gives

$$
\begin{aligned}
& f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f^{(n)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right) \\
& \leq \frac{t}{2}f^{(n)}(a)+\frac{2-t}{2}f^{(n)}(b)+\frac{2-t}{2}f^{(n)}(a)+\frac{t}{2}f^{(n)}(b),
\end{aligned}
$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$ and integrating over $[0,1]$ we have

$$
\begin{aligned}
& \int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
& \leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\alpha-1}dt,
\end{aligned}
$$

from which one can have

$$
\begin{aligned}
& \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right] \\
& \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.
\end{aligned} \tag{2.3}
$$

Combining inequality (2.2) and inequality (2.3) we get inequality (2.1). □

3 Caputo fractional inequalities related to the Hadamard inequality

In this section we give the bounds of a difference of the Hadamard inequality proved in previous section. For our results we use the following lemma.

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f\in C^{n+1}[a,b]$, then the following equality for Caputo fractional derivatives holds*

$$
\begin{aligned}
& \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right] \\
& \quad -f^{(n)}\left(\frac{a+b}{2}\right) \\
& = \frac{b-a}{4}\left[\int_0^1 t^{n-\alpha}f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right. \\
& \quad \left.-\int_0^1 t^{n-\alpha}f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right].
\end{aligned} \tag{3.1}
$$

Proof. One can note that

$$\begin{aligned}
&\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha} f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right] \\
&=\frac{b-a}{4}\left[t^{n-\alpha}\frac{2}{a-b}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)|_0^1\right. \\
&\left.-\int_0^1 \alpha t^{n-\alpha-1}\frac{2}{a-b}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right] \\
&=\frac{b-a}{4}\left[-\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)\right. \\
&\left.-\frac{2\alpha}{(a-b)}\int_b^{\frac{a+b}{2}}\left(\frac{2}{b-a}(b-x)\right)^{n-\alpha-1}\frac{2}{a-b}f^{(n)}(x)dx\right] \\
&=\frac{b-a}{4}\left[-\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)+\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}(-1)^n({}^C D^\alpha_{(\frac{a+b}{2})-}f)(b)\right].
\end{aligned} \tag{3.2}$$

Similarly

$$\begin{aligned}
&-\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha} f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right] \\
&=-\frac{b-a}{4}\left[\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)-\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}({}^C D^\alpha_{(\frac{a+b}{2})+}f)(a)\right].
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) one can have (3.1). □

Using the above lemma we give the following Caputo fractional Hadamard-type inequality.

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f\in C^{n+1}[a,b]$. If $|f^{(n+1)}|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^C D^\alpha_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^C D^\alpha_{(\frac{a+b}{2})-}f)(a)\right]\right. \\
&\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4(n-\alpha+1)}\left(\frac{1}{2(n-\alpha+2)}\right)^{\frac{1}{q}}\left[\left[(n-\alpha+1)\left|f^{(n+1)}(a)\right|^q\right.\right. \\
&\left.+(n-\alpha+3)\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}+\left[(n-\alpha+3)\left|f^{(n+1)}(a)\right|^q\right. \\
&\left.\left.+(n-\alpha+1)\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 3.1 and convexity of $|f^{(n+1)}|$ and for $q=1$ we have

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\int_0^1 t^{n-\alpha}\left(\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right)dt.\\
&=\frac{b-a}{4\,(n-\alpha+1)}\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right].
\end{aligned}$$

For $q>1$ using Lemma 3.1 we have

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)]\right.\\
&\quad\left.-f\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.\\
&\left.+\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}$$

Using power mean inequality we get

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{c}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Convexity of $|f^{(n+1)}|^q$ gives

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\\
&\quad -f^{(n)}\left(\frac{a+b}{2}\right)\Bigg|\\
&\leq \frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{n-\alpha}\left(\frac{t}{2}|f^{(n+1)}(a)|^q+\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.\\
&\quad \left.+\left[\int_0^1 t^{n-\alpha}\left(\frac{2-t}{2}|f^{(n+1)}(a)|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]\\
&= \frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f^{(n+1)}(a)|^q}{2(n-\alpha+2)}+\frac{|f^{(n+1)}(b)|^q}{n-\frac{\alpha}{k}+1}-\frac{|f^{(n+1)}(b)|^q}{2(n-\alpha+2)}\right]^{\frac{1}{q}}\right.\\
&\quad \left.+\left[\frac{|f^{(n+1)}(a)|^q}{n-\alpha+1}-\frac{|f^{(n+1)}(a)|^q}{2(n-\alpha+2)}+\frac{|f^{(n+1)}(b)|^q}{2(n-\alpha+2)}\right]^{\frac{1}{q}}\right],
\end{aligned}$$

which after a little computation gives the required result. □

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, $a<b$. If $|f^{(n+1)}|^q$ is convex on $[a,b]$ for $q>1$, then the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\\
&\quad -f^{(n)}\left(\frac{a+b}{2}\right)\Bigg|\\
&\leq \frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f^{(n+1)}(a)|^q+3|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right.\\
&\quad \left.+\left(\frac{3|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right]\\
&\leq \frac{b-a}{4}\left(\frac{4}{3(np-\alpha p+1)}\right)^{\frac{1}{p}}[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|],
\end{aligned} \tag{3.4}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1 we have

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.\\
&\quad\left.+\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}$$

From Hölder's inequality we get

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{b-a}{4}\left[\left[\int_0^1 t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\int_0^1 t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Convexity of $|f^{(n+1)}|^q$ gives

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1\left(\frac{t}{2}|f^{(n+1)}(a)|^q+\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\int_0^1\left(\frac{2-t}{2}|f^{(n+1)}(a)|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]\\
&=\frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f^{(n+1)}(a)|^q+3|f^{(n+1)}(b)|^q}{4}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\frac{3|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{4}\right]^{\frac{1}{q}}\right].
\end{aligned}$$

For second inequality of (3.4) we use Minkowski's inequality as follows

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\\
&\quad -f^{(n)}\left(\frac{a+b}{2}\right)\Bigg|\\
&\leq \frac{b-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[|f^{(n+1)}(a)|^q+3|f^{(n+1)}(b)|^q\right]^{\frac{1}{q}}\right.\\
&\quad \left.+\left[3|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q\right]^{\frac{1}{q}}\right]\\
&\leq \frac{b-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}(3^{\frac{1}{q}}+1)(|f^{(n+1)}(a)|+|f^{(n+1)}(b)|)\\
&\leq \frac{b-a}{4}\left(\frac{4}{3(np-\alpha p+1)}\right)^{\frac{1}{p}}(|f^{(n+1)}(a)|+|f^{(n+1)}(b)|).
\end{aligned}$$

□

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A hesitant fuzzy ordered information system

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Abstract

Hesitant fuzzy information systems are generalized types of traditional information systems. First, a dominance relation is defined by the score function of hesitant fuzzy value in hesitant fuzzy information systems. By introducing the dominance relation to hesitant fuzzy information systems, we then establish a dominance-based rough set model by replacing the indiscernibility relation in classic rough set theory with the dominance relation, and develop a ranking approach for all objects based on dominance classes. Furthermore, to simplify the knowledge representation, we provide an attribute reduction approach to eliminate the redundant information. And an example is provided to illustrate the validity of this approach.

Key words: Dominance relation; Dominance-based rough set; Hesitant fuzzy information systems; Reduction

1 Introduction

As a mathematical approach to handle imprecision, vagueness and uncertainty in data analysis, rough set theory introduced by Pawlak [22, 23] is a valid means of granular computing [24]. In Pawlak's rough set model, the equivalence relation is a key tool and can represent information systems or decision tables. However, the equivalence relation is a very stringent condition that may limit the application of rough sets in practical problems. Therefore many researchers have generalized the notion of Pawlak's rough set by replacing the equivalence relation with other binary relations. It may be a fuzzy, intuitionistic fuzzy, interval-valued fuzzy, hesitant fuzzy or other indiscernibility one within the generalized rough sets [1, 3, 4, 15, 21, 27, 31, 34, 39, 40, 42–51, 54, 55, 59].

The aforementioned rough sets, such as fuzzy rough set [1,3,21,27,34,39], intuitionistic fuzzy rough set [15, 54, 55], hesitant fuzzy rough set [4, 40, 46], and so on, do not consider

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attributes with preference-ordered domains. However, in many real-life situations, we are always faced with some problems in which the ordering of properties of the considered attributes plays a key role. In such case, to take into consideration the ordering properties of criteria, Greco et al. [8–11] generalized the notion of Pawlak's rough set and initiated the dominance-based rough sets approach (DRSA) by replacing the indiscernibility relation with a dominance relation. In DRSA, the knowledge approximated is a collection of upward and downward unions of classes and the dominance classes are sets of objects defined by a dominance relation in which condition attributes are the criteria and classes are preference ordered. Up to now, many fruitful results in DRSA have been achieved [5, 12, 13, 25, 29, 30, 52].

Hesitant fuzzy (HF) set theory, initiated by Torra and Narukawa [32] and Torra [33] as one of the extensions of Zadeh's fuzzy set [56], permits the membership degree of an element to a set having several possible different values. Because HF set can express the hesitant information more comprehensively than other extensions of fuzzy set, it has been applied in dealing with lots of decision making problems successfully [2, 6, 17, 18, 28, 35–38, 57]. Although rough sets and HF sets both capture particular facets of the same notion-imprecision, studies on the combination of rough set theory and HF set theory are rare. In [40], Yang et al. proposed the concept of HF rough sets by integrating HF sets with rough sets. However, Zhang et al. [46] pointed out that hesitant fuzzy subset based on the hesitant fuzzy rough sets is not necessarily antisymmetric. To remedy this defect, they introduced an HF rough set over two universes and give a new decision making approach in uncertainty environment using the model. Subsequently, Zhang et al. [47] extended the rough set into interval-valued hesitant fuzzy environment and introduced the concept of interval-valued hesitant fuzzy rough sets. In typical hesitant fuzzy background, Zhang and Yang [53] studied the constructive approach to rough set approximation operators and proposed a typical hesitant fuzzy rough set. By combining the hesitant fuzzy linguistic term set and rough set, Zhang et al. [41] developed a general framework for the study of hesitant fuzzy linguistic rough sets over two universes.

On the one hand, hybrid models integrating an HF set with a rough set are rarely developed despite the above mentioned research efforts. Knowledge reduction is also an important task in classic and generalized rough set theory. However, the issue has rarely been discussed under the hesitant fuzzy environment. On the other hand, it is well known that the rough set data analysis starts from information systems which contain data about objects of interest, characterized by a finite set of attributes. As an important type of data tables, information systems on decision problems have been widely studied [7, 14, 16, 19, 20, 25, 26]. However, in general, we may not have enough expertise or possess a sufficient level of knowledge to precisely express our preferences over the objects by using a value or a single term, and then, we may usually have a certain hesitancy between a few different values. In such a case, the traditional information system can not express our preferences or assessments by only a single term or value. Considering the facts, it is natural for us to investigate information systems in the context of hesitant fuzzy settings which is called hesitant fuzzy information systems. So how to make a decision by a dominance relation is an urgent need in hesitant fuzzy information systems. The aim of this paper is to introduce a dominance relation to hesitant fuzzy information systems and establish a rough set approach by replacing the indiscernibility relation with the dominance relation. Then we develop a reduction approach in hesitant fuzzy ordered information systems for eliminating redundant information from the perspective of the ordering of objects.

The rest of the paper is organized as follows. In Section 2, by reviewing some basic concepts, a dominance relation is introduced to hesitant fuzzy information systems and some properties are discussed. Section 3 establishes a dominance-based rough set approach in hesitant fuzzy ordered information systems by replacing the indiscernibility relation with a dominance relation. In Section 4, a ranking approach is established through the notions of dominance degree and whole dominance degree. Section 5 proposes a reduction approach in hesitant fuzzy ordered information system for eliminating redundant information from the perspective of the ordering of objects. Finally, we conclude the paper in Section 6.

2 Dominance relation in hesitant fuzzy information systems

In [32, 33], Torra and Narukawa introduced the notions related to HF sets.

Definition 2.1 ([32, 33]) *Let U be a fixed set, a hesitant fuzzy set $\mathbb{A}$ on U is in terms of a function $h_{\mathbb{A}}(x)$ that when applied to U returns a subset of [0,1].*

To be easily understood, Xia and Xu [35] denoted the HF set by a mathematical symbol:

$\mathbb{A} = \{< x, h_{\mathbb{A}}(x) > | x \in U\}$,

where $h_{\mathbb{A}}(x)$ is a set of some different values in [0,1], standing for the possible membership degrees of the element $x \in U$ to $\mathbb{A}$.

For convenience, Xia and Xu [35] called $h_{\mathbb{A}}(x)$ an HF element, and denoted the set of all HF sets on U by $HF(U)$.

To compare the HF elements, Xia and Xu [35] defined the following comparison laws:

Definition 2.2 ([35]) *For an HF element h, $s(h) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. For two HF elements h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 \succ h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.*

An HF information system is a quadruple $\mathcal{I} = (U, AT, V, f)$, where
- U is a non-empty finite set of objects called the universe;
- AT is a non-empty finite set of attributes;
- V is the domain of all attributes, i.e., $V = V_{AT} = \bigcup_{a\in AT} V_a$;
- $f : U \times AT \longrightarrow V$ is a total function such that $f(x, a) \in V_a$ for every $a \in AT, x \in U$, called

Table 1: An HF information system

U	a_1	a_2	a_3	a_4	a_5
x_1	{0.4,0.6,0.7}	{0.4,0.5,0.6}	{0.3,0.4,0.6}	{0.1,0.3,0.4}	{0.4,0.5,0.8}
x_2	{0.4,0.5,0.6}	{0.0,0.4,0.5}	{0.5,0.6,0.7}	{0.4,0.6,0.7}	{0.2,0.3,0.4}
x_3	{0.5,0.6,0.7}	{0.5,0.7,0.8}	{0.6,0.8,0.9}	{0.5,0.7,0.8}	{0.6,0.8,0.9}
x_4	{0.4,0.7,0.8}	{0.4,0.6,0.7}	{0.5,0.7,0.8}	{0.8,0.9,1.0}	{0.7,0.8,0.9}
x_5	{0.4,0.6,0.8}	{0.4,0.5,0.8}	{0.4,0.6,0.7}	{0.5,0.7,0.8}	{0.4,0.6,0.8}
x_6	{0.2,0.3,0.4}	{0.2,0.3,0.6}	{0.3,0.4,0.5}	{0.4,0.6,0.9}	{0.3,0.6,0.7}
x_7	{0.1,0.4,0.5}	{0.5,0.6,0.7}	{0.4,0.6,0.7}	{0.3,0.7,0.8}	{0.6,0.8,0.9}
x_8	{0.2,0.5,0.7}	{0.2,0.6,0.8}	{0.3,0.4,0.5}	{0.4,0.6,0.8}	{0.4,0.5,0.8}

an information function, where V_a is a set of HF elements. Denote as $f(x,a) = h_a(x)$, then we call it the HF value of x under the attribute a. In particular, if the information function $f(x,a)$ contains only one real number, the HF information system degenerates into a traditional information system [29].

Example 2.3 *An HF information system is given in Table 1, where* $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

In practical decision analysis, we always consider a dominance relation between objects that are possibly dominant in terms of values of an attributes set in an HF information system. Generally, an increasing preference and a decreasing preference can be considered by a decision maker. If the domain of an attribute is ordered by a decreasing or increasing preference, then the attribute is a criterion.

Definition 2.4 *An HF information system is called an HF ordered information system (HFOIS) if all attributes are criterions.*

On the basis of Definition 2.2, we develop an approach to rank two objects whose attribute characters are described by HF values.

Definition 2.5 *Let* $\mathcal{I} = (U, AT, V, f)$ *be an HFOIS. For* $x, y \in U$, *denote as*

$$x \succeq_A y \Longleftrightarrow \forall a \in A, f(x,a) \succeq f(y,a) \Longleftrightarrow \forall a \in A, f(x,a) \succ f(y,a) \vee f(x,a) = f(y,a),$$

then we say that x *dominates* y *with respect to* $A \subseteq AT$ *if* $x \succeq_A y$, *denoted by* $x\mathbb{R}_A^{\succeq}y$. *Where* $\mathbb{R}_A^{\succeq} = \{(y,x) \in U \times U | y \succeq_A x\}$ *is called a dominance relation in HFOIS. Analogously, we call the relation* $\mathbb{R}_A^{\preceq}$ *a dominated relation in HFOIS, which can be defined as follows:*

$$\mathbb{R}_A^{\preceq} = \{(y,x) \in U \times U | x \succeq_A y\}.$$

From Definitions 2.5 and 2.2, we can easily obtain the following theorem.

Theorem 2.6 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$, then*

(1) $\mathbb{R}_A^{\succeq}$ and $\mathbb{R}_A^{\preceq}$ are reflexive, transitive and unsymmetric;

(2) $\mathbb{R}_A^{\succeq} = \bigcap_{a \in A} \mathbb{R}_{\{a\}}^{\succeq}$, $\mathbb{R}_A^{\preceq} = \bigcap_{a \in A} \mathbb{R}_{\{a\}}^{\preceq}$.

The dominance class induced by the dominance relation $\mathbb{R}_A^{\succeq}$ is the set of objects dominating x, i.e.,

$$\begin{aligned}[x]_A^{\succeq} &= \{y \in U | f(y,a) \succ f(x,a) \vee f(y,a) = f(x,a) (\forall a \in A)\} \\ &= \{y \in U | (y,x) \in \mathbb{R}_A^{\succeq}\},\end{aligned}$$

where $[x]_A^{\succeq}$ describes the set of objects that may dominate x and is called the A-dominating set with respect to $x \in U$.

Similarly, the dominance class induced by the dominated relation $\mathbb{R}_A^{\preceq}$ is the set of objects dominated by x, i.e.,

$$\begin{aligned}[x]_A^{\preceq} &= \{y \in U | f(x,a) \succ f(y,a) \vee f(x,a) = f(y,a) (\forall a \in A)\} \\ &= \{y \in U | (y,x) \in \mathbb{R}_A^{\preceq}\},\end{aligned}$$

where $[x]_A^{\preceq}$ describes the set of objects that may be dominated by x and is called the A-dominated set with respect to $x \in U$.

Let $U/\mathbb{R}_A^{\succeq}$ denote classification on the universe, which is the family set $\{[x]_A^{\succeq} | x \in U\}$. Any element from $U/\mathbb{R}_A^{\succeq}$ is called a dominance class with respect to A. Dominance classes in $U/\mathbb{R}_A^{\succeq}$ do not constitute a partition of U, but constitute a covering of U.

In the text that follows, without loss of generality, we adopt the dominance relation $\mathbb{R}_A^{\succeq}$ for investigating HFOIS and consider attributes with increasing preference.

Theorem 2.7 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A, B \subseteq AT$.*

(1) If $B \subseteq A \subseteq AT$, then $\mathbb{R}_B^{\succeq} \supseteq \mathbb{R}_A^{\succeq} \supseteq \mathbb{R}_{AT}^{\succeq}$.

(2) If $B \subseteq A \subseteq AT$, then $[x]_B^{\succeq} \supseteq [x]_A^{\succeq} \supseteq [x]_{AT}^{\succeq}$.

(3) If $x_j \in [x_i]_A^{\succeq}$, then $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$ and $[x_i]_A^{\succeq} = \bigcup \{[x_j]_A^{\succeq} : x_j \in [x_i]_A^{\succeq}\}$.

(4) $[x_i]_A^{\succeq} = [x_j]_A^{\succeq}$ iff $f(x_i, a) = f(x_j, a) (\forall a \in A)$.

Proof. (1) and (2) are straightforward.

(3) If $x_j \in [x_i]_A^{\succeq}$, by Definition 2.5, then $f(x_j, a) \succeq f(x_i, a)$ for all $a \in A$. Similarly, for all $x \in [x_j]_A^{\succeq}$, we have $f(x, a) \succeq f(x_j, a)$. According to the transitivity of the dominance relation $\mathbb{R}_A^{\succeq}$, then $f(x, a) \succeq f(x_i, a)$, i.e. $x \in [x_i]_A^{\succeq}$. Thus $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$. Consequently, $[x_i]_A^{\succeq} = \bigcup \{[x_j]_A^{\succeq} : x_j \in [x_i]_A^{\succeq}\}$.

(4) "$\Rightarrow$" Assume that $[x_i]_A^{\succeq} = [x_j]_A^{\succeq}$, then $[x_i]_A^{\succeq} \subseteq [x_j]_A^{\succeq}$. Based on the result (3), for all $a \in A$, we have $f(x_i, a) \succeq f(x_j, a)$. Similarly, we can conclude that $f(x_j, a) \succeq f(x_i, a)$. Consequently, $f(x_i, a) = f(x_j, a) (\forall a \in A)$.

"$\Leftarrow$" It can be directly derived from the definition of the set of objects dominating x.
□

Example 2.8 (Continued from Example 2.3). Compute the classification induced by the dominance relation $\mathbb{R}^{\succeq}_{AT}$ in Table 1.

From Table 1, we have

$U/\mathbb{R}^{\succeq}_{AT} = \{[x_1]^{\succeq}_{AT}, [x_2]^{\succeq}_{AT}, \ldots, [x_8]^{\succeq}_{AT}\}$,

where

$[x_1]^{\succeq}_{AT} = \{x_1, x_3, x_4, x_5\}$, $[x_2]^{\succeq}_{AT} = \{x_2, x_3, x_4\}$, $[x_3]^{\succeq}_{AT} = \{x_3\}$, $[x_4]^{\succeq}_{AT} = \{x_4\}$,

$[x_5]^{\succeq}_{AT} = \{x_3, x_4, x_5\}$, $[x_6]^{\succeq}_{AT} = \{x_3, x_4, x_5, x_6\}$, $[x_7]^{\succeq}_{AT} = \{x_3, x_7\}$, $[x_8]^{\succeq}_{AT} = \{x_3, x_4, x_5, x_8\}$.

From Example 2.8, it is evident that dominance classes in $U/\mathbb{R}^{\succeq}_{AT}$ do not constitute a partition of U, but constitute a covering of U.

3 Rough set approach to HFOIS

In this section, we shall investigate the problems of set approximation and roughness measure with respect to the dominance relation $\mathbb{R}^{\succeq}_{A}$ in HFOIS.

Definition 3.1 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS. For any $X \subseteq U$ and $A \subseteq AT$, the lower and upper approximations of the set X with respect to the dominance relation $\mathbb{R}^{\succeq}_{A}$ are defined as follows:*

$\underline{\mathbb{R}^{\succeq}_{A}}(X) = \{x \in U \mid [x]^{\succeq}_{A} \subseteq X\}$,

$\overline{\mathbb{R}^{\succeq}_{A}}(X) = \{x \in U \mid [x]^{\succeq}_{A} \cap X \neq \emptyset\}$.

From Definition 3.1, we can easily obtain the following theorem.

Theorem 3.2 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $X, Y \subseteq U$, then*

(1) $\underline{\mathbb{R}^{\succeq}_{A}}(X) = \sim \overline{\mathbb{R}^{\succeq}_{A}}(\sim X)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X) = \sim \underline{\mathbb{R}^{\succeq}_{A}}(\sim X)$;

(2) $\underline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq X \subseteq \overline{\mathbb{R}^{\succeq}_{A}}(X)$;

(3) $A \subseteq AT \Longrightarrow \underline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq \underline{\mathbb{R}^{\succeq}_{AT}}(X)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X) \supseteq \overline{\mathbb{R}^{\succeq}_{AT}}(X)$;

(4) $X \subseteq Y \Longrightarrow \underline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq \underline{\mathbb{R}^{\succeq}_{A}}(Y)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq \overline{\mathbb{R}^{\succeq}_{A}}(Y)$;

(5) $\underline{\mathbb{R}^{\succeq}_{A}}(X \cap Y) = \underline{\mathbb{R}^{\succeq}_{A}}(X) \cap \underline{\mathbb{R}^{\succeq}_{A}}(Y)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X \cup Y) = \overline{\mathbb{R}^{\succeq}_{A}}(X) \cup \overline{\mathbb{R}^{\succeq}_{A}}(Y)$;

(6) $\underline{\mathbb{R}^{\succeq}_{A}}(X \cup Y) \supseteq \underline{\mathbb{R}^{\succeq}_{A}}(X) \cup \underline{\mathbb{R}^{\succeq}_{A}}(Y)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X \cap Y) \subseteq \overline{\mathbb{R}^{\succeq}_{A}}(X) \cap \overline{\mathbb{R}^{\succeq}_{A}}(Y)$;

(7) $\underline{\mathbb{R}^{\succeq}_{A}}(\emptyset) = \overline{\mathbb{R}^{\succeq}_{A}}(\emptyset) = \emptyset$, $\underline{\mathbb{R}^{\succeq}_{A}}(U) = \overline{\mathbb{R}^{\succeq}_{A}}(U) = U$;

(8) $\underline{\mathbb{R}^{\succeq}_{A}}(\underline{\mathbb{R}^{\succeq}_{A}}(X)) = \underline{\mathbb{R}^{\succeq}_{A}}(X)$, $\overline{\mathbb{R}^{\succeq}_{A}}(\overline{\mathbb{R}^{\succeq}_{A}}(X)) = \overline{\mathbb{R}^{\succeq}_{A}}(X)$.

Theorem 3.3 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. If $\mathbb{R}^{\succeq}_{A} = \mathbb{R}^{\succeq}_{AT}$, then $\underline{\mathbb{R}^{\succeq}_{A}}(X) = \underline{\mathbb{R}^{\succeq}_{AT}}(X)$ and $\overline{\mathbb{R}^{\succeq}_{A}}(X) = \overline{\mathbb{R}^{\succeq}_{AT}}(X)$.*

Proof. It is directly derived from Definitions 2.5 and 3.1.

□

Generally speaking, the uncertainty of a set is due to the existence of the borderline region. The wider the borderline region of a set is, the lower the accuracy of the set is. To express the idea precisely, some basic measures (accuracy and roughness) are defined to depict the quality of the rough approximation of a set. In the following, we introduce the concepts of roughness measure and accuracy measure to measure the imprecision of rough sets induced by dominance relation $\mathbb{R}_A^{\succeq}$ in HFOIS.

Definition 3.4 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS, $X \subseteq U$ and $A \subseteq AT$. Then the roughness measure $\rho_X^{\mathbb{R}_A^{\succeq}}$ of the set X with respect to the dominance relation $\mathbb{R}_A^{\succeq}$ is defined as follows:*

$$\rho_X^{\mathbb{R}_A^{\succeq}} = 1 - \frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|},$$

where $|\cdot|$ denotes the cardinality of a set. If $\overline{\mathbb{R}_A^{\succeq}}(X) = \emptyset$, we define $\rho_X^{\mathbb{R}_A^{\succeq}} = 0$. $\eta_X^{\mathbb{R}_A^{\succeq}} = \frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|}$ is referred to as the accuracy measure of X with respect to the dominance relation $\mathbb{R}_A^{\succeq}$.

According to Definition 3.4 and Theorem 3.2(2), we observe that $0 \le \rho_X^{\mathbb{R}_A^{\succeq}} \le 1$ and $0 \le \eta_X^{\mathbb{R}_A^{\succeq}} \le 1$.

Obviously, by Theorem 3.3 and Definition 3.4, we can draw the following conclusion.

Theorem 3.5 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. If $\mathbb{R}_A^{\succeq} = \mathbb{R}_{AT}^{\succeq}$, then $\rho_X^{\mathbb{R}_A^{\succeq}} = \rho_X^{\mathbb{R}_{AT}^{\succeq}}$ and $\eta_X^{\mathbb{R}_A^{\succeq}} = \eta_X^{\mathbb{R}_{AT}^{\succeq}}$.*

Theorem 3.6 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS, $X \subseteq U$ and $A \subseteq AT$, then the following holds:*

(1) $\rho_X^{\mathbb{R}_{AT}^{\succeq}} \le \rho_X^{\mathbb{R}_A^{\succeq}}$,

(2) $\eta_X^{\mathbb{R}_{AT}^{\succeq}} \ge \eta_X^{\mathbb{R}_A^{\succeq}}$.

Proof. (1) Since $A \subseteq AT$, by Theorem 3.2(3) we have $\underline{\mathbb{R}_A^{\succeq}}(X) \subseteq \underline{\mathbb{R}_{AT}^{\succeq}}(X)$ and $\overline{\mathbb{R}_A^{\succeq}}(X) \supseteq \overline{\mathbb{R}_{AT}^{\succeq}}(X)$. It implies that $\frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|} \le \frac{|\underline{\mathbb{R}_{AT}^{\succeq}}(X)|}{|\overline{\mathbb{R}_{AT}^{\succeq}}(X)|}$. According to Definition 3.4, then $\rho_X^{\mathbb{R}_A^{\succeq}} = 1 - \frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|} \ge 1 - \frac{|\underline{\mathbb{R}_{AT}^{\succeq}}(X)|}{|\overline{\mathbb{R}_{AT}^{\succeq}}(X)|} = \rho_X^{\mathbb{R}_{AT}^{\succeq}}$, i.e., $\rho_X^{\mathbb{R}_{AT}^{\succeq}} \le \rho_X^{\mathbb{R}_A^{\succeq}}$.

(2) It is directly derived from the result (1) and Definition 3.4. □

Example 3.7 *Consider HFOIS in Table 1. Let $A = \{a_1, a_4, a_5\} \subseteq AT$ and $X = \{x_2, x_3, x_5, x_7\}$. Now we compute the rough sets of X induced by $U/\mathbb{R}_{AT}^{\succeq}$ and $U/\mathbb{R}_A^{\succeq}$, respectively.*

By Definition 3.1 and Example 2.8, the rough set $(\underline{\mathbb{R}^{\succeq}_{AT}}(X), \overline{\mathbb{R}^{\succeq}_{AT}}(X))$ *can be obtained as follows:*

$\underline{\mathbb{R}^{\succeq}_{AT}}(X) = \{x_3, x_7\}$, $\overline{\mathbb{R}^{\succeq}_{AT}}(X)) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8\}$.

Then we compute the classification set induced by the dominance relation $U/\mathbb{R}^{\succeq}_{A}$*. By Table 1, we have*

$U/\mathbb{R}^{\succeq}_{A} = \{[x_1]^{\succeq}_{A}, [x_2]^{\succeq}_{A}, \ldots, [x_8]^{\succeq}_{A}\}$,

where

$[x_1]^{\succeq}_{A} = \{x_1, x_3, x_4, x_5\}$, $[x_2]^{\succeq}_{A} = \{x_2, x_3, x_4, x_5\}$, $[x_3]^{\succeq}_{A} = \{x_3, x_4\}$, $[x_4]^{\succeq}_{A} = \{x_4\}$,
$[x_5]^{\succeq}_{A} = \{x_3, x_4, x_5\}$, $[x_6]^{\succeq}_{A} = \{x_3, x_4, x_5, x_6\}$, $[x_7]^{\succeq}_{A} = \{x_3, x_4, x_7\}$, $[x_8]^{\succeq}_{A} = \{x_3, x_4, x_5, x_8\}$.

Similarly, by Definition 3.1, we calculate the rough set $(\underline{\mathbb{R}^{\succeq}_{A}}(X), \overline{\mathbb{R}^{\succeq}_{A}}(X))$ *as follows:*

$\underline{\mathbb{R}^{\succeq}_{A}}(X) = \emptyset$, $\overline{\mathbb{R}^{\succeq}_{A}}(X)) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8\}$.

Therefore, we have

$$\rho_X^{\mathbb{R}^{\succeq}_{A}} = 1 - \frac{|\underline{\mathbb{R}^{\succeq}_{A}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{A}}(X)|} = 1, \quad \rho_X^{\mathbb{R}^{\succeq}_{AT}} = 1 - \frac{|\underline{\mathbb{R}^{\succeq}_{AT}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{AT}}(X)|} = 1 - \frac{2}{7} = \frac{5}{7},$$

$$\eta_X^{\mathbb{R}^{\succeq}_{A}} = \frac{|\underline{\mathbb{R}^{\succeq}_{A}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{A}}(X)|} = 0, \quad \eta_X^{\mathbb{R}^{\succeq}_{AT}} = \frac{|\underline{\mathbb{R}^{\succeq}_{AT}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{AT}}(X)|} = \frac{2}{7}.$$

Thus, $\rho_X^{\mathbb{R}^{\succeq}_{AT}} \leq \rho_X^{\mathbb{R}^{\succeq}_{A}}$ *and* $\eta_X^{\mathbb{R}^{\succeq}_{AT}} \geq \eta_X^{\mathbb{R}^{\succeq}_{A}}$.

4 Ranking for all objects in HFOIS

In [58], Zhang et al. defined the concept of dominance degrees for ranking all objects in classical ordered information systems. Inspired by the idea, we introduce a dominance degree between two objects in HFOIS as follows:

Definition 4.1 *Let* $\mathcal{I} = (U, AT, V, f)$ *be an HFOIS and* $A \subseteq AT$*. Dominance degree between two objects with respect to the dominance relation* $\mathbb{R}^{\succeq}_{A}$ *is defined as*

$$\mathbb{D}_A(x_i, x_j) = \frac{|\sim [x_i]^{\succeq}_{A} \cup [x_j]^{\succeq}_{A}|}{|U|},$$

where $|\cdot|$ *denotes the cardinality of a set,* $x_i, x_j \in U$.

Theorem 4.2 *Dominance degree* $\mathbb{D}_A(x_i, x_j)$ *satisfies the following properties:*

(1) $\frac{1}{|U|} \leq \mathbb{D}_A(x_i, x_j) \leq 1$*;*

(2) if $(x_j, x_k) \in \mathbb{R}^{\succeq}_{A}$*, then* $\mathbb{D}_A(x_i, x_j) \leq \mathbb{D}_A(x_i, x_k)$ *and* $\mathbb{D}_A(x_j, x_i) \geq \mathbb{D}_A(x_k, x_i)$.

Proof. (1) It is straightforward.

(2) Assume that $(x_j, x_k) \in \mathbb{R}_A^{\succeq}$. By Theorem 2.7, then $[x_j]_A^{\succeq} \subseteq [x_k]_A^{\succeq}$. Therefore, we have

$$\begin{aligned}\mathbb{D}_A(x_i,x_j)-\mathbb{D}_A(x_i,x_k) &= \frac{1}{|U|}(|\sim[x_i]_A^{\succeq}\cup[x_j]_A^{\succeq}|-|\sim[x_i]_A^{\succeq}\cup[x_k]_A^{\succeq}|)\\ &\leq \frac{1}{|U|}(|\sim[x_i]_A^{\succeq}\cup[x_k]_A^{\succeq}|-|\sim[x_i]_A^{\succeq}\cup[x_k]_A^{\succeq}|)\\ &= 0,\end{aligned}$$

$$\begin{aligned}\mathbb{D}_A(x_j,x_i)-\mathbb{D}_A(x_k,x_i) &= \frac{1}{|U|}(|\sim[x_j]_A^{\succeq}\cup[x_i]_A^{\succeq}|-|\sim[x_k]_A^{\succeq}\cup[x_i]_A^{\succeq}|)\\ &\geq \frac{1}{|U|}(|\sim[x_k]_A^{\succeq}\cup[x_i]_A^{\succeq}|-|\sim[x_k]_A^{\succeq}\cup[x_i]_A^{\succeq}|)\\ &= 0.\end{aligned}$$

That is, $\mathbb{D}_A(x_i,x_j) \leq \mathbb{D}_A(x_i,x_k)$ and $\mathbb{D}_A(x_j,x_i) \geq \mathbb{D}_A(x_k,x_i)$. □

According to Definition 4.1, we may construct a dominance relation matrix with respect to A induced by the dominance relation $\mathbb{R}_A^{\succeq}$. Based on the dominance relation matrix, the whole dominance degree of each object can be calculated by the following formula

$$\mathbb{D}_A(x_i) = \frac{1}{|U|-1}\sum_{j\neq i}\mathbb{D}_A(x_i,x_j),\ x_i,x_j\in U. \tag{1}$$

Obviously, by the concepts of dominance degree and whole dominance degree, the following theorem holds.

Theorem 4.3 *Let $\mathcal{I}=(U,AT,V,f)$ be an HFOIS and $A\subseteq AT$. If $\mathbb{R}_A^{\succeq}=\mathbb{R}_{AT}^{\succeq}$, then $\mathbb{D}_A(x_i,x_j)=\mathbb{D}_{AT}(x_i,x_j)$ and $\mathbb{D}_A(x_i)=\mathbb{D}_{AT}(x_i)$.*

By employing the whole dominance degree of each object on the universe, we may rank all objects by the values of $\mathbb{D}_A(x_i)$. The following example is given to demonstrate the application of this method.

Example 4.4 *(Continued from Example 2.8). Rank all objects in U based on the dominance relation $\mathbb{R}_{AT}^{\succeq}$. By the concept of dominance degree, we obtain the dominance relation matrix as follows*

$$\begin{pmatrix} 1 & 0.75 & 0.625 & 0.625 & 0.875 & 0.875 & 0.625 & 0.875\\ 0.875 & 1 & 0.75 & 0.75 & 0.875 & 0.875 & 0.75 & 0.875\\ 1 & 1 & 1 & 0.875 & 1 & 1 & 1 & 1\\ 1 & 1 & 0.875 & 1 & 1 & 1 & 0.875 & 1\\ 1 & 0.875 & 0.75 & 0.75 & 1 & 1 & 0.75 & 1\\ 0.875 & 0.75 & 0.625 & 0.625 & 0.875 & 1 & 0.625 & 0.875\\ 0.875 & 0.875 & 0.875 & 0.75 & 0.875 & 0.875 & 1 & 0.875\\ 0.875 & 0.875 & 0.625 & 0.625 & 0.875 & 0.875 & 0.625 & 1 \end{pmatrix}.$$

Therefore, by Equation 1, the whole dominance degree of each object x_i can be calculated as follows:

$\mathbb{D}_{AT}(x_1) = 0.75$, $\mathbb{D}_{AT}(x_2) = 0.82$, $\mathbb{D}_{AT}(x_3) = 0.98$, $\mathbb{D}_{AT}(x_4) = 0.96$,

$\mathbb{D}_{AT}(x_5) = 0.875$, $\mathbb{D}_{AT}(x_6) = 0.75$, $\mathbb{D}_{AT}(x_7) = 0.857$, $\mathbb{D}_{AT}(x_8) = 0.768$.

An object with larger value implies a better object. Therefore, based on the values of $\mathbb{D}_{AT}(x_i)$, we can rank all objects as follows:

$$x_3 \succeq x_4 \succeq x_5 \succeq x_7 \succeq x_8 \succeq \begin{pmatrix} x_1 \\ x_6 \end{pmatrix}.$$

5 Attribute reduction in HFOIS

In order to simplify knowledge representation in HFOIS, it is necessary for us to reduce some dispensable attributes in the context of dominance relations. In this section, we will develop an approach to attribute reduction in a given HFOIS.

Definition 5.1 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. For any $B \subset A$, if $\mathbb{R}^{\succeq}_{A} = \mathbb{R}^{\succeq}_{AT}$ and $\mathbb{R}^{\succeq}_{B} \neq \mathbb{R}^{\succeq}_{AT}$, then we call A an attribute reduction of $\mathcal{I}$.*

By Definition 5.1, we can easily verify the following conclusion holds.

Theorem 5.2 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. If A is an attribute reduction of $\mathcal{I}$, then $\mathbb{D}_A(x_i, x_j) = \mathbb{D}_{AT}(x_i, x_j)$, $x_i, x_j \in U$.*

In what follows we define several special attributes in HFOIS as follows:

Definition 5.3 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS. If $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{(AT-\{a\})}$, an attribute $a \in AT$ is called dispensable with respect to the dominance relation $\mathbb{R}^{\succeq}_{AT}$; otherwise, a is called indispensable. The set of all indispensable attributes is called a core with respect to the dominance relation $\mathbb{R}^{\succeq}_{AT}$.*

Definition 5.4 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. Denote by $Dis(x, y) = \{a \in A | (x, y) \notin \mathbb{R}^{\succeq}_{\{a\}}\}$, then we call $Dis(x, y)$ a discernibility attribute set between x and y, and $DIS = (Dis(x, y) : x, y \in U)$ a discernibility matrix of the HFOIS.*

Theorem 5.5 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. Suppose that $Dis(x, y)$ is the discernibility attribute set of $\mathcal{I}$; then $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{A}$ iff $A \cap Dis(x, y) \neq \emptyset$ $(Dis(x, y) \neq \emptyset)$.*

Proof. "$\Longrightarrow$" Assume that $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{A}$, for any $y \in U$ then $[y]^{\succeq}_{AT} = [y]^{\succeq}_{A}$. If some $x \notin [y]^{\succeq}_{AT}$, then $x \notin [y]^{\succeq}_{A}$. Therefore, there exists $a \in A$ such that $(x, y) \notin \mathbb{R}^{\succeq}_{\{a\}}$. Thus, $a \in Dis(x, y)$. Consequently, if $Dis(x, y) \neq \emptyset$, we have $A \cap Dis(x, y) \neq \emptyset$.

"$\Longleftarrow$" Based on Definition 5.4, we can observe that if $(x, y) \notin \mathbb{R}^{\succeq}_{AT}$ for any $(x, y) \in U \times U$, then $Dis(x, y) \neq \emptyset$. Since $A \cap Dis(x, y) \neq \emptyset$, there exists $a \in A$ such that $a \in Dis(x, y)$, i.e., $(x, y) \notin \mathbb{R}^{\succeq}_{\{a\}}$. Thus $(x, y) \notin \mathbb{R}^{\succeq}_{A}$. Consequently, $\mathbb{R}^{\succeq}_{AT} \supseteq \mathbb{R}^{\succeq}_{A}$. On the other hand, note that $A \subseteq AT$, then we have $\mathbb{R}^{\succeq}_{AT} \subseteq \mathbb{R}^{\succeq}_{A}$. Hence, $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{A}$. □

Table 2: The discernibility matrix of Table 1

U	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	$\emptyset$	a_3a_4	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	a_4	$a_2a_3a_4a_5$	a_2a_4
x_2	$a_1a_2a_5$	$\emptyset$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_4a_5$	$a_2a_4a_5$	$a_2a_4a_5$	$a_2a_4a_5$
x_3	$\emptyset$	$\emptyset$	$\emptyset$	$a_1a_4a_5$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\emptyset$	a_2a_3	$\emptyset$	$\emptyset$	$\emptyset$	a_2	$\emptyset$
x_5	$\emptyset$	a_3	$a_2a_3a_5$	$a_1a_3a_4a_5$	$\emptyset$	$\emptyset$	a_2a_5	$\emptyset$
x_6	$a_1a_2a_3a_5$	a_1a_3	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$\emptyset$	$a_1a_2a_3a_5$	$a_1a_2a_5$
x_7	a_1	a_1a_3	$a_1a_2a_3a_4$	$a_1a_3a_4a_5$	a_1a_4	a_4	$\emptyset$	a_1
x_8	a_1a_3	a_1a_3	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	a_4	$a_2a_3a_5$	$\emptyset$

Definition 5.6 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS, $A \subseteq AT$ and $Dis(x, y)$ the discernibility attributes set of $\mathcal{I}$ with respect to $\mathbb{R}^{\succeq}_{AT}$. Denote as*

$$\mathcal{M} = \bigwedge \left\{ \bigvee \{a : a \in Dis(x, y) | x, y \in U\} \right\},$$

then we call $\mathcal{M}$ a discernibility function.

From the definition of minimal disjunctive normal form of the discernibility function and Theorem 5.5, we can easily verify the following conclusion.

Theorem 5.7 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS. The minimal disjunctive normal form of $\mathcal{M}$ is*

$$\mathcal{M} = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s} \right).$$

Denoted by $\mathcal{B}_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{\mathcal{B}_k : k = 1, 2, \ldots, t\}$ are the family of all attribute reductions of $\mathcal{I}$.

By Theorem 5.7, a practical approach to attribute reductions of HFOIS is provided. In the following, we shall illustrate how to obtain attribute reductions of an HFOIS by an example.

Example 5.8 *(Continued from Example 2.3). According to Definition 5.4, we obtain the discernibility matrix of Table 1 (see Table 2). Thus, we have*

$$\begin{aligned}\mathcal{M} = {} & (a_1 \vee a_2 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \wedge a_1 \wedge (a_1 \vee a_3) \wedge (a_3 \vee a_4) \wedge a_3 \\ & \wedge (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_2 \vee a_3) \wedge (a_2 \vee a_3 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_4) \\ & \wedge (a_1 \vee a_4 \vee a_5) \wedge (a_1 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_4 \vee a_5) \wedge (a_1 \vee a_4) \wedge a_4 \\ & \wedge (a_2 \vee a_4 \vee a_5) \wedge (a_2 \vee a_3 \vee a_4 \vee a_5) \wedge a_2 \wedge (a_2 \vee a_5) \wedge (a_2 \vee a_4) \\ = {} & a_1 \wedge a_2 \wedge a_3 \wedge a_4\end{aligned}$$

Therefore, there is only one attribute reduction for the HFOIS, which is $\{a_1, a_2, a_3, a_4\}$*. From the perspective of the ordering of objects, the attributes* a_1, a_2, a_3 *and* a_4 *are indispensable in Table 1.*

6 Conclusions

Although the conventional rough set theory is a powerful and useful mathematical tool to deal with uncertainty information, it can not deal with ordering objects instead of classifying objects. In this situation, we have investigated information systems in the context of hesitant fuzzy settings, which is called hesitant fuzzy information systems. The hesitant fuzzy information system is an important type of data tables, which is generalized from the traditional information systems. First, based on the score function of hesitant fuzzy value, a dominance relation has been introduced to hesitant fuzzy information systems. Then we have established a rough set approach in HFOIS by replacing the indiscernibility relation with the dominance relation, and given a ranking approach to all objects by employing the whole dominance degree of each object. Finally, from the perspective of the ordering of objects, we have also developed a reduction approach in HFOIS for eliminating redundant information.

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THE STABILITY OF CUBIC FUNCTIONAL EQUATIONS WITH INVOLUTION IN MODULAR SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability for the following cubic functional equation with involution

$$f(2x+y)+f(2x+\sigma(y))-2f(x+y)-2f(x+\sigma(y))-12f(x)=0$$

in modular spaces by using a fixed point theorem.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [21]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [6] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. A generalization of the Rassias' theorem was obtained by Găvruta [5] by replasing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

A problem that mathematicians has dealt with is "how to generalize the classical function space L^p". A first attempt was made by Birnhaum and Orlicz in 1931. This generalization found many applications in differential and intergral equations with kernls of nonpower types. The more abstract generalization was given by Nakano [14] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*. Since then, these have been thoroughly developed by several mathematicians, for example, Amemiya [1], Koshi and Shimogaki [9], Yamamuro [23], Orlicz [15], Mazur [11], Musielak [12], Luxemburg [10], Turpin [20]. This idea was refined and generalized by Musielak and Orlicz [13] in 1959.

Recently, Sadeghi [18] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition, Wongkum, Chaipunya, and Kumam [22] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular

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space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition, and Park, Bodaghi, and Kim [16] proved the generalized Hyers-Ulam stability for additive mappings in a modular space with $\triangle_2$-conditions.

Let X and Y be real vector spaces. For an additive mapping $\sigma : X \longrightarrow X$ with $\sigma(\sigma(x)) = x$ for all $x \in X$, σ is called *an involution* of X. For a given involution $\sigma : X \longrightarrow X$, the functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) \tag{1.1}$$

is called *an additive functional equation with involution* and a solution of (1.1) is called *an additive mapping with involution.* For a given involution $\sigma : X \longrightarrow X$, the functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \tag{1.2}$$

is called *the quadratic functional equation with involution* and a solution of (1.2) is called *a quadratic mapping with involution.* The functional equation (1.2) has been studied by Stetkær [19] and the generalized Hyers-Ulam stability for (1.2) has been obtained by Bouikhalene et al. [3, 4, 7].

In this paper, we prove the generalized Hyers-Ulam stability for the following cubic functional equation with involution

$$f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x) = 0 \tag{1.3}$$

in modular spaces without the $\triangle_2$-condition and the convexity by using a fixed point theorem. Unlike Banach spaces and F-spaces, due to the triangle inequlity in modular spaces, we need subtle calculation in the proofs of Theorem 2.1 and Theorem 2.2

Definition 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}, \mathbb{C}, \text{ or } \mathbb{N})$.
(1) A generalized functional $\rho : X \longrightarrow [0, \infty]$ is called *a modular* if
(M1) $\rho(x) = 0$ if and only if $x = 0$,
(M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
(M3) $\rho(z) \le \rho(x) + \rho(y)$ whenever z is a convex combination of x and y.
(2) If (M3) is replaced by
(M4) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in V$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$, then we say that ρ is *convex.*

For any modular ρ on X, the modular space X_ρ is defined by

$$X_\rho = \{x \in X \mid \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}$$

and the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\Big\{\lambda > 0 \mid \rho\Big(\frac{x}{\lambda}\Big) \le 1\Big\}.$$

Let X_ρ be a modular space and $\{x_n\}$ a sequence in X_ρ. Then (i) $\{x_n\}$ is called *ρ-Cauchy* if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, (ii) $\{x_n\}$ is called *ρ-convergent* to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, and (iii) a subset K of X_ρ is called *ρ-complete* if each ρ-Cauchy sequence is ρ-convergent to a point in K.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to x does not imply that $\{cx_n\}$ converges to cx for some $c \in \mathbb{K}$.

Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to $\triangle_2$-condition.

A modular space X_ρ is said to *satisfy the* $\triangle_2$*-condition* if there exists $k \geq 2$ such that $X_\rho(2x) \leq kX_\rho(x)$ for all $x \in X$. Some authors varied the notion so that only $k > 0$ is required and called it *the* $\triangle_2$*-type condition*. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\triangle_2$-condition. In [8], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\triangle_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

Example 1.2. A convex function ζ defined on the interval $[0, \infty)$, nondecreasing and continuous, such that $\zeta(0) = 0, \zeta(\alpha) > 0$ for $\alpha > 0$, $\zeta(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. Let (Ω, Σ, μ) be a measure space and $L^0(\mu)$ the set of all measurable real valued (or complex valued) functions on Ω. Deine the Orlicz modular ρ_ζ on $L^0(\mu)$ by the formula

$$\rho_\zeta(f) = \int_\Omega \zeta(|f|)d\mu.$$

The associated modular space with respect to this modular is called an Orlicz space, and will be denoted by (L^ζ, Ω, μ) or briefly L^ζ. In other words,

$$L^\zeta = \{f \in L^0(\mu) \mid \rho_\zeta(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

It is known that the Orlicz space L^ζ is ρ_ζ-complete. Moreover, $(L^\zeta, \|\cdot\|_{\rho_\zeta})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\zeta}$ is defined as follows

$$\|f\|_{\rho_\zeta} = \inf\Big\{\lambda > 0 \;\Big|\; \int_\Omega \zeta\Big(\frac{|f|}{\lambda}\Big)d\mu \leq 1\Big\}.$$

Further, if μ is the Lebesgue measure on $\mathbb{R}$ and $\zeta(t) = e^t - 1$, then ρ_ζ does not satisfy the $\triangle_2$-condition.

For a modular space X_ρ, a nonempty subset C of X_ρ, and a mapping $T : C \longrightarrow C$, *the orbit of* T *at* $x \in C$ is the set

$$\mathbb{O}(x) = \{x, Tx, T^2x, \cdots\}.$$

The quantity $\delta_\rho(x) = \sup\{\rho(u - v) \mid u, v \in \mathbb{O}(x)\}$ is called *the orbital diameter of* T *at* x and if $\delta_\rho(x) < \infty$, then one says that T has *a bounded orbit at* x.

Khamsi [8] proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\triangle_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

Lemma 1.3. [8] *Let* X_ρ *be a modular space whose induced modular is lower semi-continuous and let* $C \subseteq X_\rho$ *be a* ρ*-complete subset. If* $T : C \longrightarrow C$ *is a* ρ*-contraction, that is, there is a constant* $L \in [0, 1)$ *such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \ \forall x, y \in C$$

and T *has a bounded orbit at a point* $x_0 \in C$*, then the sequence* $\{T^nx_0\}$ *is* ρ*-convergent to a point* $w \in C$.

For any modular ρ on X and any linear space V, we define a set $\mathbb{M}$

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and the generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by for each $g \in \mathbb{M}$,

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\psi(x, 0),\ \forall x \in V\},$$

where $\psi : V^2 \longrightarrow [0, \infty)$ is a mapping. The proof of the following lemma is similar to the proof of Lemma 10 in [22].

Lemma 1.4. *Let V be a linear space, X_ρ a ρ-complete modular space where ρ is lower semi-continuous and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\psi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{8^n} = 0, \quad \psi(2x, 2x) \leq 8L\psi(x, x) \tag{1.4}$$

for all $x, y \in V$ and some L with $0 \leq L < 1$. Then we have the following :
(1) $\mathbb{M}$ is a linear space,
(2) $\widetilde{\rho}$ is a modular on $\mathbb{M}$,
(3) if ρ is convex, then $\widetilde{\rho}$ is convex,
(4) $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete, and
(5) $\widetilde{\rho}$ is lower semi-continuous.

Proof. (1), (2), and (3) are trivial.

(4) By the definition of $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$. Let $\epsilon > 0$. Take any $\widetilde{\rho}$-Cauchy sequence $\{g_n\}$ in $\mathbb{M}_{\widetilde{\rho}}$. Then there is an $l \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$ with $n, m \geq l$,

$$\rho(g_n(x) - g_m(x)) \leq \epsilon\psi(x, 0) \tag{1.5}$$

for all $x \in V$. Hence $\{g_n(x)\}$ is a ρ-Cauchy sequence in X_ρ for all $x \in V$. Since X_ρ is a ρ-complete modular space, there is a mapping $g : V \longrightarrow X_\rho$ such that $\rho(g_n(x) - g(x)) \longrightarrow 0$ as $n \to \infty$ for all $x \in V$. Since each $g_n \in \mathbb{M}$, there is an $m \in \mathbb{N}$ such that

$$\rho(g_m(0) - g(0)) = \rho(g(0)) \leq \epsilon$$

and hence $g \in \mathbb{M}_{\widetilde{\rho}}$. Since ρ is lower semi-continuous, by (1.5), we have

$$\rho(g_n(x) - g(x)) \leq \liminf_{m\to\infty} \rho(g_n(x) - g_m(x)) \leq \epsilon\psi(x, 0)$$

for all $x \in V$. Hence $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.

(5) Suppose that $\{g_n\}$ is a sequence in $\mathbb{M}_{\widetilde{\rho}}$ which is $\widetilde{\rho}$-convergent to $g \in \mathbb{M}_{\widetilde{\rho}}$. Let $\epsilon > 0$. Then for any $n \in \mathbb{N}$, there is a positive real number c_n such that

$$\widetilde{\rho}(g_n) \leq c_n \leq \widetilde{\rho}(g_n) + \epsilon$$

and so

$$\begin{aligned} \rho(g(x)) &\leq \liminf_{n\to\infty} \rho(g_n(x)) \\ &\leq \liminf_{n\to\infty} c_n\psi(x, 0) \leq \Big(\liminf_{n\to\infty} \rho(g_n(x)) + \epsilon\Big)\psi(x, 0) \end{aligned} \tag{1.6}$$

for all $x \in V$. Hence $\widetilde{\rho}$ is lower semi-continuous. □

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2. The generalized Hyers-Ulam stability for (1.3) in modular spaces

Throughout this section, we assume that every modular is lower semi-continuous. In this section, we prove the generalized Hyers-Ulam stability for (1.3).

For any $f : V \longrightarrow X_\rho$ and any involution $\sigma : V \longrightarrow V$, let

$$Df(x,y) = f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x).$$

Theorem 2.1. *Let V be a linear space, X_ρ a ρ-complete modular space and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\phi : V^2 \longrightarrow [0,\infty)$ be a mapping such that*

$$\phi(2x,2y) \le 8L\phi(x,y),\ \ \phi(x+\sigma(x), y+\sigma(y)) \le 8L\phi(x,y) \tag{2.1}$$

for all $x, y \in V$ and some L with $0 < L < \frac{1}{16}$ and

$$\rho(Df(x,y)) \le \phi(x,y) \tag{2.2}$$

for all $x, y \in V$. Then there exists a unique cubic mapping $F : V \longrightarrow X_\rho$ with involution such that

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \le \frac{2}{1-8L}\phi(x,0) \tag{2.3}$$

for all $x \in V$.

Proof. Let $\psi(x,y) = \phi(x,y) + \phi(y,x)$ for all $x, y \in V$. Then ψ satisfies (1.4) and hence, by Lemma 1.4, $\widetilde{\rho}$ is a lower semi-continuous convex modular on $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$, and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by

$$Tg(x) = \frac{1}{8}\Big(g(2x) + g(x+\sigma(x))\Big)$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g-h) \le c$ for some nonnegative real number c. Then by (2.1), we have

$$\begin{aligned}\rho(Tg(x) - Th(x)) &\le \rho\Big(\frac{1}{4}[g(2x) - h(2x)]\Big) + \rho\Big(\frac{1}{4}[g(x+\sigma(x)) - h(x+\sigma(x))]\Big)\\ &\le 16Lc\psi(x,0)\end{aligned}$$

for all $x \in V$ and so $\widetilde{\rho}(Tg - Th) \le 16L\widetilde{\rho}(g-h)$. Hence T is a $\widetilde{\rho}$-contraction. By (2.2), we get

$$\rho(f(x) + f(\sigma(x))) \le \phi(0,x) \tag{2.4}$$

and

$$\rho\Big(f(2x) - 8f(x)\Big) \le \rho(2f(2x) - 16f(x)) \le \phi(x,0) \tag{2.5}$$

for all $x \in X$. Letting $x = x + \sigma(x)$ in (2.4), by (M3), we have

$$\rho(f(x+\sigma(x))) \le \rho(2f(x+\sigma(x))) \le \phi(0, x+\sigma(x)) \le 8L\phi(0,x), \tag{2.6}$$

for all $x \in X$ and by (2.5) and (M3), we get

$$\rho\Big(\frac{1}{2^3}f(2x) - f(x)\Big) \le \rho(f(2x) - 8f(x)) \le \phi(x,0) \tag{2.7}$$

for all $x \in X$.

Now, we claim that T has a bounded orbit at $\frac{1}{4}f$. By the definition of T, we have

$$T^n f(x+\sigma(x)) = \frac{1}{2^{2n}}f(2^n(x+\sigma(x))) \tag{2.8}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Hence by (2.1), (2.6), and (2.8), we have

$$\rho(T^n f(x+\sigma(x))) \leq \rho(f(2^n(x+\sigma(x)))) \leq (8L)^{n+1}\phi(0,x)$$

for all $x \in V$ and for all $n \in \mathbb{N}$. By (2.7), for any nonnegative integer n, we obtain

$$\begin{aligned}
&\rho(\frac{1}{2}T^n f(x) - \frac{1}{2}f(x)) \\
&\leq \rho\Big(T^n f(x) - \frac{1}{2^3}f(2x)\Big) + \rho(\frac{1}{2^3}f(2x) - f(x)) \\
&\leq \rho\Big(\frac{1}{2}T^{n-1}f(2x) - \frac{1}{2}f(2x)\Big) + \rho\Big(T^{n-1}f(x+\sigma(x))\Big) + \phi(x,0) \\
&\leq \rho\Big(\frac{1}{2}T^{n-1}f(2x) - \frac{1}{2}f(2x)\Big) + (8L)^n\phi(0,x) + \phi(x,0)
\end{aligned}$$

for all $x \in V$ and by induction, we have

$$\begin{aligned}
\rho\Big(\frac{1}{2}T^n f(x) - \frac{1}{2}f(x)\Big) &\leq \sum_{i=0}^{n-1}(8L)^{n-i}\phi(0,2^i x) + \sum_{i=0}^{n-1}\phi(2^i x,0) \\
&\leq n(8L)^n\phi(0,x) + \frac{1}{1-8L}\phi(x,0)
\end{aligned} \tag{2.9}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Hence by (2.9), we get

$$\rho\Big(\frac{1}{4}T^n f(x) - \frac{1}{4}T^m f(x)\Big) \leq \frac{2}{1-8L}\phi(x,0) + [n(8L)^n + m(8L)^m]\phi(0,x) \tag{2.10}$$

for all $x \in V$ and all all nonnegative integers n, m and since $0 < L < \frac{1}{16}$, by (2.10), we have

$$\rho\Big(\frac{1}{4}T^n f(x) - \frac{1}{4}T^m f(x)\Big) \leq 4\phi(x,0) + \phi(0,x) \leq 4\psi(x,0)$$

for all $x \in V$ and all nonnegative integers n, m. Hence we have

$$\widetilde{\rho}\Big(T^n\frac{1}{4}f - T^m\frac{1}{4}f\Big) \leq 4$$

all nonnegative integers n, m and thus T has a bounded orbit at $\frac{1}{4}f$.

By Lemma 1.3, there is an $F \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$0 \leq \widetilde{\rho}(TF - F) \leq \liminf_{n\to\infty}\widetilde{\rho}\Big(TF - T^{n+1}\frac{1}{4}f\Big) \leq \liminf_{n\to\infty} 16L\widetilde{\rho}\Big(F - T^n\frac{1}{4}f\Big) = 0$$

and hence F is a fixed point of T in $\mathbb{M}_{\widetilde{\rho}}$. By induction, we can easily show that

$$\begin{aligned}
T^n f(x) &= \frac{1}{2^{3n}}f(2^n x) + \frac{1}{2^{3n}}\sum_{i=0}^{n-1}2^i f(2^{n-1}(x+\sigma(x))) \\
&= \frac{1}{2^{3n}}f(2^n x) + \frac{2^n-1}{2^{3n}}f(2^{n-1}(x+\sigma(x)))
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Moreover, we have

$$\rho\Big(\frac{1}{2^8}DF(x,y)\Big) \leq \rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) + \rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) \tag{2.11}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Note that

$$\begin{aligned}
&\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big)\\
&\leq \rho\Big(\frac{1}{2^6}\Big[F(2x+y) - T^n\frac{1}{4}f(2x+y)\Big]\Big) + \rho\Big(\frac{1}{2^4}\Big[F(2x+\sigma(y)) - T^n\frac{1}{4}f(2x+\sigma(y))\Big]\Big)\\
&+ \rho\Big(\frac{1}{2^4}\Big[2F(x+y) - T^n\frac{1}{4}2f(x+y)\Big]\Big) + \rho\Big(\frac{1}{2^4}\Big[2F(x+\sigma(y)) - T^n\frac{1}{4}2f(x+\sigma(y))\Big]\Big)\\
&+ \rho\Big(\frac{1}{2^4}\Big[12F(x) - T^n\frac{1}{4}12f(x)\Big]\Big)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Since $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F, we get

$$\lim_{n\to\infty}\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) = 0 \tag{2.12}$$

for all $x, y \in V$. Further, we have

$$\begin{aligned}
&\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = \rho\Big(\frac{1}{2^9}T^nDf(x,y)\Big)\\
&\leq \rho\Big(\frac{1}{2^{3n+8}}Df(2^nx,2^ny)\Big) + \rho\Big(\frac{2^n-1}{2^{3n+8}}Df(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\Big)\\
&\leq \phi(2^nx,2^ny) + \phi(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\\
&\leq 2(8L)^n\phi(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$\lim_{n\to\infty}\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = 0 \tag{2.13}$$

for all $x, y \in V$. By (2.11), (2.12), (2.13), and (M1), we obtain

$$DF(x,y) = 0$$

for all $x, y \in V$ and hence F is a cubic mapping with involution. Moreover, since ρ is lower semi-continuous, by (2.10), we get

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \leq \frac{2}{1-8L}\phi(x,0)$$

for all $x \in X$. □

If ρ is covex, then Theorem 2.1 can replaced by the following theorem.

Theorem 2.2. *All conditions of Theorem 2.1 are assumed. Further, suppose that ρ is a convex modular and $0 < L < \frac{1}{2}$. Then there exists a unique cubic mapping $F : V \longrightarrow X_\rho$ with involution such that*

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \leq \frac{1}{2^4(1-L)}\phi(x,0) \tag{2.14}$$

for all $x \in V$.

Proof. Let $\psi(x,y) = \phi(x,y) + \phi(y,x)$ for all $x, y \in V$. Then ψ satisfies (1.4) and hence, by Lemma 1.4, $\widetilde{\rho}$ is a lower semi-continuous convex modular on $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$, and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by

$$Tg(x) = \frac{1}{8}\Big(g(2x) + g(x+\sigma(x))\Big)$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g-h) \le c$ for some nonnegative real number c. Then by (2.1), we have

$$\begin{aligned}\rho(Tg(x)-Th(x)) &\le \frac{1}{2}\rho\Big(\frac{1}{4}[g(2x)-h(2x)]\Big)+\frac{1}{2}\rho\Big(\frac{1}{4}[g(x+\sigma(x))-h(x+\sigma(x))]\Big)\\ &\le 2Lc\psi(x,0)\end{aligned}$$

for all $x \in V$ and so $\widetilde{\rho}(Tg-Th) \le 2L\widetilde{\rho}(g-h)$. Hence T is a $\widetilde{\rho}$-contraction. By (2.2), we get

$$\rho(f(x)+f(\sigma(x))) \le \phi(0,x) \tag{2.15}$$

and

$$\rho\Big(f(2x)-8f(x)\Big) \le \frac{1}{2}\rho(2f(2x)-16f(x)) \le \frac{1}{2}\phi(x,0) \tag{2.16}$$

for all $x \in X$. Letting $x = x+\sigma(x)$ in (2.15), by (M3), we have

$$\rho(f(x+\sigma(x))) \le \frac{1}{2}\rho(2f(x+\sigma(x))) \le \frac{1}{2}\phi(0,x+\sigma(x)) \le 4L\phi(0,x), \tag{2.17}$$

for all $x \in X$ and by (2.16) and (M3), we get

$$\rho\Big(\frac{1}{2^3}f(2x)-f(x)\Big) \le \frac{1}{2^3}\rho(f(2x)-8f(x)) \le \frac{1}{2^4}\phi(x,0) \tag{2.18}$$

for all $x \in X$.

Now, we claim that T has a bounded orbit at $\frac{1}{4}f$. By the definition of T, we have

$$T^n f(x+\sigma(x)) = \frac{1}{2^{2n}}f(2^n(x+\sigma(x))) \tag{2.19}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Hence by (2.1), (2.15), and (2.19), we have

$$\rho(T^n f(x+\sigma(x))) \le \frac{1}{2^{2n}}\rho(f(2^n(x+\sigma(x)))) \le 2(2L)^{n+1}\phi(0,x)$$

for all $x \in V$ and for all $n \in \mathbb{N}$. By (2.18), for any nonnegative integer n, we obtain

$$\begin{aligned}&\rho\Big(\frac{1}{2}T^n f(x)-\frac{1}{2}f(x)\Big)\\ &\le \frac{1}{2}\rho\Big(T^n f(x)-\frac{1}{2^3}f(2x)\Big)+\frac{1}{2}\rho\Big(\frac{1}{2^3}f(2x)-f(x)\Big)\\ &\le \frac{1}{2^3}\rho\Big(\frac{1}{2}T^{n-1}f(2x)-\frac{1}{2}f(2x)\Big)+\frac{1}{2^4}\rho\Big(T^{n-1}f(x+\sigma(x))\Big)+\frac{1}{2^5}\phi(x,0)\\ &\le \frac{1}{2^3}\rho\Big(\frac{1}{2}T^{n-1}f(2x)-\frac{1}{2}f(2x)\Big)+\frac{(2L)^n}{2^3}\phi(0,x)+\frac{1}{2^5}\phi(x,0)\end{aligned}$$

for all $x \in V$ and by induction, we have

$$\begin{aligned}\rho\Big(\frac{1}{2}T^n f(x)-\frac{1}{2}f(x)\Big) &\le \sum_{i=0}^{n-1}\frac{(2L)^{n-i}}{2^{3(i+1)}}\phi(0,2^i x)+\frac{1}{2^5}\sum_{i=0}^{n-1}\frac{1}{2^{3i}}\phi(2^i x,0)\\ &\le \frac{(2L)^n}{4}\phi(0,x)+\frac{1}{2^5(1-L)}\phi(x,0)\end{aligned} \tag{2.20}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Hence by (2.20), we get

$$\rho\Big(\frac{1}{4}T^n f(x)-\frac{1}{4}T^m f(x)\Big) \le \frac{1}{2^4(1-L)}\phi(x,0)+\frac{1}{4}[(2L)^n+(2L)^m]\phi(0,x) \tag{2.21}$$

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for all $x \in V$ and all all nonnegative integers n, m and since $0 < L < \frac{1}{2}$, by (2.21), we have

$$\begin{aligned}\rho\Big(\frac{1}{4}T^n f(x) - \frac{1}{4}T^m f(x)\Big) &\le \frac{1}{2^4(1-L)}\phi(x,0) + \frac{1}{2}\phi(0,x)\\ &\le \frac{1}{8}\phi(x,0) + \frac{1}{2}\phi(0,x)\\ &\le \frac{1}{2}\psi(x,0)\end{aligned}$$

for all $x \in V$ and all nonnegative integers n, m. Hence we have

$$\widetilde{\rho}\Big(T^n\frac{1}{4}f - T^m\frac{1}{4}f\Big) \le \frac{1}{2}$$

all nonnegative integers n, m and thus T has a bounded orbit at $\frac{1}{4}f$.

By Lemma 1.3, there is an $F \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$0 \le \widetilde{\rho}(TF - F) \le \liminf_{n\to\infty} \widetilde{\rho}\Big(TF - T^{n+1}\frac{1}{4}f\Big) \le \liminf_{n\to\infty} 2L\widetilde{\rho}\Big(F - T^n\frac{1}{4}f\Big) = 0$$

and hence F is a fixed point of T in $\mathbb{M}_{\widetilde{\rho}}$. By induction, we can easily show that

$$\begin{aligned}T^n f(x) &= \frac{1}{2^{3n}}f(2^n x) + \frac{1}{2^{3n}}\sum_{i=0}^{n-1} 2^i f(2^{n-1}(x+\sigma(x)))\\ &= \frac{1}{2^{3n}}f(2^n x) + \frac{2^n-1}{2^{3n}}f(2^{n-1}(x+\sigma(x)))\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Moreover, we have

(2.22)

$$\rho\Big(\frac{1}{2^8}DF(x,y)\Big) \le \frac{1}{2}\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) + \frac{1}{2}\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big)$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Note that

$$\begin{aligned}&\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big)\\ &\le \frac{1}{2}\rho\Big(\frac{1}{2^6}\Big[F(2x+y) - T^n\frac{1}{4}f(2x+y)\Big]\Big) + \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[F(2x+\sigma(y)) - T^n\frac{1}{4}f(2x+\sigma(y))\Big]\Big)\\ &+ \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[2F(x+y) - T^n\frac{1}{4}2f(x+y)\Big]\Big) + \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[2F(x+\sigma(y)) - T^n\frac{1}{4}2f(x+\sigma(y))\Big]\Big)\\ &+ \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[12F(x) - T^n\frac{1}{4}12f(x)\Big]\Big)\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Since $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F, we get

$$\lim_{n\to\infty} \rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) = 0 \qquad (2.23)$$

for all $x, y \in V$. Further, we have

$$\begin{aligned}
&\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = \rho\Big(\frac{1}{2^9}T^nDf(x,y)\Big)\\
&\le \frac{1}{2}\rho\Big(\frac{1}{2^{3n+8}}Df(2^nx,2^ny)\Big) + \frac{1}{2}\rho\Big(\frac{2^n-1}{2^{3n+8}}Df(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\Big)\\
&\le \frac{1}{2^{3n+9}}\phi(2^nx,2^ny) + \frac{2^n-1}{2^{3n+9}}\phi(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\\
&\le \frac{(2L)^n}{2^9}\phi(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$\lim_{n\to\infty}\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = 0 \tag{2.24}$$

for all $x, y \in V$. By (2.22), (2.23), (2.24), and (M1), we obtain

$$DF(x,y) = 0$$

for all $x, y \in V$ and hence F is a cubic mapping with involution. Moreover, since ρ is lower semi-continuous, by (2.21), we get

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \le \frac{1}{2^4(1-L)}\phi(x,0)$$

for all $x \in X$. □

It is well-known that every normed space is a modular space with $\rho(x) = \|x\|$. Using Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let X and Y be normed spaces and ϵ, θ, and p be real numbers with $\epsilon \ge 0$, $\theta \ge 0$, and $0 < p < \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping with involution σ such that $f(0) = 0$ and*

$$\|D_f(x,y)\| \le \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

and

$$\|x + \sigma(x)\| \le 2\|x\|$$

for all $x, y \in X$. Then there is a cubic mapping $F : X \longrightarrow Y$ with involution such that

$$\|F(x) - f(x)\| \le \frac{1}{2(8-2^{2p})}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

Proof. Let $\rho(z) = \|z\|$ for all $y \in Y$ and $\phi(x,y) = \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$ for all $x, y \in V$. Then ρ is a convex modular on a normed space Y, $Y = Y_\rho$, and

$$\phi(2x,2y) \le 2^{2p}\phi(x,y), \;\; \phi(x+\sigma(x), y+\sigma(y)) \le 2^{2p}\phi(x,y)$$

for all $x, y \in V$. By Theorem 2.2, we have the results. □

Using Example 1.3, we get the following example.

Example 2.4. Let ϵ, θ, and p be real numbers with $\epsilon \geq 0$, $\theta \geq 0$, and $0 < p < \frac{3}{2}$. Let ζ be an Orlicz function and L^ζ the Orlicz space. Let $f : V \longrightarrow L^\zeta$ be a mapping with involution σ such that $f(0) = 0$ and

$$\begin{aligned}&\int_\Omega \zeta(|f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x)|)d\mu \\ &\leq \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)\end{aligned}$$

and

$$\|x + \sigma(x)\| \leq 2\|x\|$$

for all $x, y \in X$. Then there is a cubic mapping $F : X \longrightarrow Y$ with involution such that

$$\int_\Omega \zeta(|F(x) - \frac{1}{4}f(x)|)d\mu \leq \frac{1}{2(8 - 2^{2p})}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

Define a mapping $\rho_2 : \mathbb{R} \longrightarrow \mathbb{R}$ by $\rho_2(x) = |x|^{\frac{1}{2}}$. Then clearly, ρ_2 is a modular on $\mathbb{R}$ and $\mathbb{R}_{\rho_2} = \mathbb{R}$. Note that

$$\Big|\frac{1}{2} \times 2 + \frac{1}{2} \times 4\Big|^{\frac{1}{2}} = \sqrt{3} > \frac{\sqrt{2}}{2} + 1 = \frac{1}{2} \times |2|^{\frac{1}{2}} + \frac{1}{2} \times |4|^{\frac{1}{2}}.$$

Hence ρ_2 is not convex. Moreover, since $(\mathbb{R}, |\cdot|)$ is a complete normed space, we can easily show that $(\mathbb{R}, \rho_2)$ is a complete modular space. Using these and Theorem 2.1, we have the following example.

Example 2.5. Let ϵ, θ, and p be real numbers with $\epsilon \geq 0$, $\theta \geq 0$, and $0 < p < \frac{3}{2}$. Let $f : V \longrightarrow \mathbb{R}$ be a mapping with involution σ such that $f(0) = 0$ and

$$\begin{aligned}&|f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x)|^{\frac{1}{2}} \\ &\leq \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)\end{aligned}$$

and

$$\|x + \sigma(x)\| \leq 2\|x\|$$

for all $x, y \in X$. Then there is a cubic mapping $F : X \longrightarrow Y$ with involution such that

$$|F(x) - \frac{1}{4}f(x)|^{\frac{1}{2}} \leq \frac{2}{1 - 2^{2p}}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

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A nonstandard finite difference method applied to a mathematical cholera model with spatial diffusion

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Abstract

In this paper, we propose a nonstandard finite difference (NSFD) scheme to solve numerically a cholera epidemic model with spatial diffusion. Through constructing discrete Lyapunov functions, we prove the globally asymptotical stabilities of the disease-free equilibrium and the chronic infection equilibrium, which coincide with the corresponding continuous model . Finally, numerical simulations are provided to illustrate the theoretical results.

Cholera, partial differential equation, nonstandard finite difference scheme, Lyapunov function, global stability.

1 Introduction

Cholera is an infection of the intestines caused by the bacterium called Vibrio cholerae and can spread rapidly in areas with inadequate treatment of sewage and drinking water. The World Health Organization (WHO) has warned that there are an estimated 3-5 million infected cases and 28,800-130,000 deaths worldwide due to cholera every year. Since 1817, seven cholera pandemics have spread in many places, with periodic outbreaks such as the latest one in Yemen in October 2016, which is the worst cholera outbreak in the world. The total cases in Yemen have exceeded half a million, with nearly 2,000 deaths reported, since the outbreak began to spread rapidly at the end of April 2017 due to the deteriorating hygiene and sanitation conditions and the disrupted water supply across the country. There have been massive outbreaks of cholera in many developing countries of Africa and South-east Asia, including Congo (2008), Iraq (2008), Zimbabwe (2008-2009), Vietnam (2009), Nigeria (2010), Haiti (2010), Mexico (2013), South Sudan (2014), and Somalia (2017).

In recent years, many epidemic models have been proposed to a better understanding of the transmission of cholera. In 2001, Codeço [1] proposed a $SIRB$ epidemic model to study the transmission of cholera in which B represents the *V. cholerae* concentration in water. Hartley Morris and Smith [2] in 2006 discovered a representative hyperinfectious state of the pathogen, which is the 'explosive' infectivity of freshly shed *V. cholerae* based on the laboratory results. Tien and Earn [3] proposed a water-borne disease model with multiple transmission pathways: both direct human-to-human and indirect water-to-human transmissions, and identified how these transmission routes influence disease dynamics. Mukandavire *et al.* [4] simplified Hartley's model to understand transmission dynamics of cholera outbreak in Zimbabwe. Liao and Wang [5] in 2011 conducted a dynamical analysis of the Hartley's model to study the stability of both the disease-free and endemic equilibria so as to explore the complex epidemic and endemic dynamics of the disease. Bertuzzo *et al.* [6] based on the Codeco's work and developed a partial differential equation model for cholera epidemics. Their results suggested that cholera outbreaks may be triggered by time scales of disease dynamics. In a recently study, Safi *et al.* [7] designed a new two-strain model to assess the impact of basic control measures and dose-structured mass vaccination on cholera transmission dynamics in a population. More papers in the field of cholera epidemic models are presented in ([8–11]).

Nowadays, more and more researchers consider to discretize the continuous models for practical purposes. One of the reasons is that most numerical methods like traditional Euler, Runge-Kutta and some standard procedures of MATLAB software will fail to solve nonlinear systems generating oscillations, chaos, and unsteady states if the time step size increases to a critical size. The other reason is that the results of the discrete time models are more accurate and convenient to describe infectious diseases and can preserve as much as possible the qualitative properties of the corresponding continuous models. The nonstandard finite difference (NSFD) scheme developed by Mickens ([12–14]) performs well and has been applied to many articles. An NSFD discretization must satisfy one of the following two conditions ([15, 16]): nonlocal approximation is used and discretization of derivative must be a denominator function. Cui *et al.* [17] employed an NSFD scheme to discuss a class of SIR epidemic model with vaccination and treatment. The dynamical properties of their discretized model were analysed to demonstrate that the discretized epidemic model maintains essential properties of the corresponding continuous model, such as positivity property, boundness of solutions, equilibrium points and their local stability properties. Suryanto *et al.* [18] constructed an NSFD scheme to solve a SIR epidemic model with modified saturated incidence rate. From their numerical simulations, the NSFD scheme allowed large time step size to save the computational cost. Qin *et al.* [19] proposed an NSFD method for an epidemic model which described the hepatitis B virus infection with spatial dependence. They have shown that the NSFD method is unconditionally positive by using the M-matrix theory. Moveover, asymptotical stabilities of the steady-state solutions were fully determined by constructing discrete Lyapunov functions independent of the time and space step sizes. Manna and Chakrabarty [20] analysed a spatiotemporal model for HBV infection by using an NSFD scheme, and studied the global stability properties of the discretized model. The simulation results demonstrated the advantages of the usage of NSFD method over the other schemes. For more investigations on NSFD scheme can be found in ([21–24]).

In 2015, Wang and Wang [25] proposed a PDE model to simulate cholera infection with spatial diffusion, taking multiple transmission ways into account among the human host, the pathogen, and the environment. The model in their paper assumes that both the human population and the bacteria undergo a diffusion process and is given by the following system of PDEs:

$$\frac{\partial S}{dt} = \Lambda - \beta_W \frac{W(x,t)S(x,t)}{\kappa + W(x,t)} - \beta_h S(x,t)I(x,t) - \mu S(x,t) + D_1 \triangle S, \tag{1}$$

$$\frac{\partial I}{dt} = \beta_W \frac{W(x,t)S(x,t)}{\kappa + W(x,t)} + \beta_h S(x,t)I(x,t) - (\gamma + \mu + u_1)I(x,t) + D_2 \triangle I, \tag{2}$$

$$\frac{\partial W}{dt} = \xi I(x,t) - \delta W(x,t) + D_3 \triangle W, \tag{3}$$

$$\frac{\partial R}{dt} = \gamma I(x,t) - \mu R(x,t) + D_4 \triangle R, \tag{4}$$

where $S(x,t)$, $I(x,t)$, $R(x,t)$ and $W(x,t)$ denote the susceptible, the infected, the recovered populations and the density of *V. cholerae* at location x and time t, respectively. The parameters β_h and β_W denote the concentrations of the hyperinfectious (HI) and less-infectious (LI) vibrios, respectively. μ represents the natural death rate that is not related to the disease, u_1 defines the rate of disease-related death, κ is the concentration of vibrios in contaminated water, ξ the natural decay rate of *V. cholerae*, δ the bacterial death rate, γ the recovery rate, and D_i $(i = 1,2,3,4)$ are the diffusion coefficients.

Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\triangle$ is the Laplacian operator, that is $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ with n is the number of spatial dimensions of the domain Ω. The Neumann boundary conditions of the model system are:

$$\frac{\partial S}{dt} = \frac{\partial I}{dt} = \frac{\partial W}{dt} = \frac{\partial R}{dt} = 0, x \in \partial\Omega. \tag{5}$$

In the case that the diffusion coefficients D_i are all equal to zero, according to Wang and Wang's [25], we know that the basic reproduction number is given by:

$$R_0 = \frac{\Lambda}{\mu\delta\kappa(\gamma+\mu+u_1)}(\xi\beta_W + \delta\kappa\beta_h). \tag{6}$$

And the disease-free equilibrium $E_0(S_0, I_0, W_0, R_0)$ is $(\frac{\Lambda}{\mu}, 0, 0, 0)$, the endemic equilibrium $E^*(S^*, I^*, W^*, R^*)$ is determined by:

$$S^* = \frac{\Lambda}{\mu} - \frac{(\gamma+\mu+u_1)I^*}{\mu}, I^* = \frac{\beta_e S^*}{\gamma+\mu+u_1-\beta_h S^*} - \frac{\delta\kappa}{\xi}, W^* = \frac{\xi I^*}{\delta}, R^* = \frac{\gamma I^*}{\mu}.$$

Wang and Wang' paper [25] also established the following results:

Theorem 1 *Assume $D_i = 0$, then for model system (1-4), (1) the disease-free equilibrium E_0 is locally and globally asymptotically stable if $R_0 < 1$; and (2) if $R_0 > 1$, the unique chronic infection equilibrium E^* is globally asymptotically stable.*

In this paper, we consider the cholera spatially dependent model proposed in Wang and wang [25] and construct an NSFD scheme for this model. As far as we know, there are few studies on the continuous cholera models designed as discrete equations. The rest of the paper is organized as follows. In the next section, we construct a discretized cholera model with diffusion from the continuous model by using the nonstandard finite difference method. In Section 3 and Section 4, the global asymptotic stability analysis of the equilibria is performed by using discrete Lyapunov functions. In Section 5, we carry out the numerical study of the discrete model, which confirms our theoretical results. Finally, the conclusions are summarized in Section 6.

2 A discretized model

Assume $\Omega = [a, b]$ with $a, b \in R$, let $\triangle t$ be the time step size and $\triangle x = \frac{(b-a)}{N}$ be the space step size, $t_k = k\triangle t$ for $k \in N$ be the time mesh point, where N is the set of all non-negative integers. The space mesh point is $X_n = n\Delta x$ for $n \in \{0, 1, \cdots, N\}$. At each point, we denote approximations of $S(x_n, t_k)$, $I(x_n, t_k)$, $W(x_n, t_k)$ and $R(x_n, t_k)$ by S_n^k, I_n^k, W_n^k and R_n^k, respectively. For the sake of convenience, a $(N+1)-$ dimensional vector

$$U^k = (U_0^k, U_1^k, \cdots, U_N^k)^T \tag{7}$$

is used to represent all the approximation solutions at the time t_k. The notation $(\cdot)^T$ denotes the transposition of a vector, and all components of a vector U are non-negative.

We construct the following NSFD method for model system (1-4):

$$\frac{S_n^{k+1} - S_n^k}{\triangle t} = \Lambda - \beta_W \frac{S_n^{k+1} W_n^k}{\kappa + W_n^k} - \beta_h S_n^{k+1} I_n^k - \mu S_n^{k+1} + D_1 \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{(\triangle x)^2}, \tag{8}$$

$$\frac{I_n^{k+1} - I_n^k}{\triangle t} = \beta_W \frac{S_n^{k+1} W_n^k}{\kappa + W_n^k} + \beta_h S_n^{k+1} I_n^k - (\gamma + \mu + u_1) I_n^{k+1} + D_2 \frac{I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}}{(\triangle x)^2}, \tag{9}$$

$$\frac{W_n^{k+1} - W_n^k}{\triangle t} = \xi I_n^{k+1} - \delta W_n^{k+1} + D_3 \frac{W_{n+1}^{k+1} - 2W_n^{k+1} + W_{n-1}^{k+1}}{(\triangle x)^2}, \tag{10}$$

$$\frac{R_n^{k+1} - R_n^k}{\triangle t} = \gamma I_n^{k+1} - \mu R_n^{k+1} + D_4 \frac{R_{n+1}^{k+1} - 2R_n^{k+1} + R_{n-1}^{k+1}}{(\triangle x)^2}, \tag{11}$$

with discrete initial value conditions

$$S_n^0 = \psi_1(x_n), I_n^0 = \psi_2(x_n), W_n^0 = \psi_3(x_n), R_n^0 = \psi_4(x_n),$$

for $n \in \{0, 1, \cdots, N\}$, and discrete boundary condition is given as:

$$S_{-1}^k = S_0^k, S_N^k = S_{N+1}^k, I_{-1}^k = I_0^k, I_N^k = I_{N+1}^k, W_{-1}^k = W^k, W_N^k = W_{N+1}^k, R_{-1}^k = R_0^k, R_N^k = R_{N+1}^k.$$

It is easy to check that the solutions of the discrete system (8-11) are positive, and have the disease-free equilibrium E_0 and the chronic infection equilibrium E^*, which are the same as that of the model (1-4).

3 Global stability of the disease-free equilibrium

Since R does not appear in the first three equations of the system (8-11), we only need to study the system (8-10). In this section, we establish the global stability of the disease-free equilibrium of system (8-10) by constructing a discrete Lyapunov function.

Theorem 2 *If $R_0 < 1$, the disease-free equilibrium E_0 of the model system (8-10) is globally asymptotically stable.*

Proof Define a discrete Lyapunov function

$$L^k = \sum_{n=0}^{N} \frac{1}{\Delta t} [S_0 g(\frac{S_n^k}{S_0}) + I_n^k + \frac{(\gamma + \mu + u_1)(1 + \delta \Delta t)}{\xi} W_n^k], \tag{12}$$

where the function $g(x) = x - 1 - lnx$, $x \in R^+$, clearly, $g(x) \geq 0$ with equality only if $x = 1$. Thus we have $L^k \geq 0$ with equality if and only if $S_n^k = S_0$, $I_n^k = 0$ and $W_n^k = 0$ for all $n \in \{0, 1, \ldots, N\}$. Then, along the trajectory of (8-10), we have

$$\begin{aligned}
L^{k+1} - L^k &= \sum_{n=0}^{N} \frac{1}{\Delta t}[S_n^{k+1} - S_n^k + S_0 ln\frac{S_n^k}{S_n^{k+1}} + I_n^{k+1} - I_n^k + \frac{\gamma + \mu + u_1}{\xi}(W_n^{k+1} - W_n^k)] \\
&+ \frac{\delta(\gamma + \mu + u_1)}{\xi}(W_n^{k+1} - W_n^k) \\
&= \sum_{n=0}^{N}[2\Lambda - \frac{\beta_W S_n^{k+1} W_n^k}{\kappa + W_n^k} - \beta_h S_n^{k+1} I_n^k - \mu S_n^{k+1} + D_1 \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{(\triangle x)^2} \\
&- \frac{\Lambda^2}{\mu S_n^{k+1}} + \frac{\Lambda \beta_W W_n^k}{\mu(\kappa + W_n^k)} + \frac{\Lambda \beta_h I_n^k}{\mu} - \frac{\Lambda D_1}{\mu S_n^{k+1}} \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{(\triangle x)^2} \\
&+ \frac{\beta_W S_n^{k+1} W_n^k}{\kappa + W_n^k} + \beta_h S_n^{k+1} I_n^k - (\gamma + \mu + u_1) I_n^{k+1} + D_2 \frac{I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}}{(\triangle x)^2} \\
&+ (\gamma + \mu + u_1) I_n^{k+1} - \frac{\delta(\gamma + \mu + u_1)}{\xi} W_n^{k+1} + D_3 \frac{(\gamma + \mu + u_1)}{\xi} \frac{W_{n+1}^{k+1} - 2W_n^{k+1} + W_{n-1}^{k+1}}{(\triangle x)^2}] \\
&+ \frac{\delta(\gamma + \mu + u_1)}{\xi}(W_n^{k+1} - W_n^k) \\
&\leq \sum_{n=0}^{N}[\Lambda(2 - \frac{\Lambda}{\mu S_n^{k+1}} - \frac{\mu S_n^{k+1}}{\Lambda}) + (\gamma + \mu + u_1) I_n^k (R_0 - 1)] \\
&+ D_1 \frac{S_{N+1}^{k+1} - S_N^{k+1}}{(\triangle x)^2} + D_1 \frac{S_0^{k+1} - S_{-1}^{k+1}}{(\triangle x)^2} + D_2 \frac{I_{N+1}^{k+1} - I_N^{k+1}}{(\triangle x)^2} + D_2 \frac{I_0^{k+1} - I_{-1}^{k+1}}{(\triangle x)^2} \\
&+ D_3 \frac{(\gamma + \mu + u_1)}{\xi} \frac{W_{N+1}^{k+1} - W_N^{k+1}}{(\triangle x)^2} + D_3 \frac{(\gamma + \mu + u_1)}{\xi} \frac{W_0^{k+1} - W_{-1}^{k+1}}{(\triangle x)^2} \\
&= \sum_{n=0}^{N}[\Lambda(2 - \frac{\Lambda}{\mu S_n^{k+1}} - \frac{\mu S_n^{k+1}}{\Lambda}) + (\gamma + \mu + u_1) I_n^k (R_0 - 1)].
\end{aligned}$$

Since $2 - \frac{\Lambda}{\mu S_n^{k+1}} - \frac{\mu S_n^{k+1}}{\Lambda} \leq 0$ by the arithmetic-geometric inequality, it then follows that if $R_0 < 1$, $L^{k+1} - L^k < 0$, for all $k \in \mathbb{N}$ and the equality holds if and only if $S_n^{k+1} = \frac{\Lambda}{\mu}$. This yields that $\{L^k\}$ is a monotone decreasing sequence. Thus, there exists a constant L_0 such that $lim_{k \to +\infty}(L^{k+1} - L^k) = 0$. Therefore, we have $lim_{k \to +\infty} S_n^k = \frac{\Lambda}{\mu}$, $lim_{k \to +\infty} I_n^k = 0$, $lim_{k \to +\infty} W_n^k = 0$, for all $n \in \{0, 1, \ldots N\}$. Hence, E_0 is globally asymptotically stable when $R_0 < 1$. This completes the proof. ■

4 Global Stability of the chronic infection equilibrium

In this section we concern with the global stability of the chronic infection steady state of system (8-10) when $R_0 > 1$.

Theorem 3 *If $R_0 > 1$, the chronic infection equilibrium E^* of the model system (8-10) is globally asymptotically stable.*

Using the expression for S^* along with system (8-10) and discrete boundary conditions, we first have

$$\begin{aligned}
&\sum_{n=0}^{N}\frac{1}{\Delta t}[g(\frac{S_n^{k+1}}{S^*})-g(\frac{S_n^k}{S^*})]\\
&\leq\sum_{n=0}^{N}\frac{1}{\Delta t}[(S_n^{k+1}-S_n^k)(\frac{S_n^{k+1}-S^*}{S^*S_n^{k+1}})]\\
&=\sum_{n=0}^{N}\frac{1}{S^*}[(\Lambda-\frac{\beta_W S_n^{k+1}W_n^k}{\kappa+W_n^k}+\beta_h S_n^{k+1}I_n^k-\mu S_n^{k+1}+D_1\frac{S_{n+1}^{k+1}-2S_n^{k+1}+S_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{S^*}{S_n^{k+1}})]\\
&=\sum_{n=0}^{N}\frac{1}{S^*}[(\frac{\beta_W S^*W^*}{\kappa+W^*}+\beta_h S^*I^*+\mu S^*-\frac{\beta_W S_n^{k+1}W_n^k}{\kappa+W_n^k}-\beta_h S_n^{k+1}I_n^k-\mu S_n^{k+1})(1-\frac{S^*}{S_n^{k+1}})]\\
&+\sum_{n=0}^{N}\frac{1}{S^*}[(D_1\frac{S_{n+1}^{k+1}-2S_n^{k+1}+S_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{S^*}{S_n^{k+1}})]\\
&=\sum_{n=0}^{N}[-\frac{\mu(S_n^{k+1}-S^*)^2}{S^*S_n^{k+1}}+\frac{\beta_W W^*}{\kappa+W^*}(1-\frac{S^*}{S_n^{k+1}})(1-\frac{(\kappa+W^*)S_n^{k+1}W_n^k}{(\kappa+W_n^k)S^*W^*})\\
&+\beta_h I^*(1-\frac{S^*}{S_n^{k+1}})(1-\frac{S_n^{k+1}I_n^k}{S^*I^*})]-D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\triangle x)^2S_{n+1}^{k+1}S_n^{k+1}}.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\sum_{n=0}^{N}\frac{1}{\Delta t}[g(\frac{I_n^{k+1}}{I^*})-g(\frac{I_n^k}{I^*})]\\
&\leq\sum_{n=0}^{N}\frac{1}{\Delta t}[(I_n^{k+1}-I_n^k)(\frac{I_n^{k+1}-I^*}{I^*I_n^{k+1}})]\\
&=\sum_{n=0}^{N}\frac{1}{I^*}[\frac{\beta_W S_n^{k+1}W_n^k}{\kappa+W_n^k}+\beta_h S_n^{k+1}I_n^k-(\gamma+\mu+u_1)I_n^{k+1}\\
&+D_2(\frac{I_{n+1}^{k+1}-2I_n^{k+1}+I_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{I^*}{I_n^{k+1}})]\\
&=\sum_{n=0}^{N}[\frac{\beta_W}{I^*}(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}W_n^k}{\kappa+W_n^k}-\frac{I_n^{k+1}S^*W^*}{(\kappa+W^*)I^*})+\beta_h S^*(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}I_n^k}{S^*I^*}-\frac{I_n^{k+1}}{I^*})]\\
&+\frac{1}{I^*}\sum_{n=0}^{N}[(D_2\frac{I_{n+1}^{k+1}-2I_n^{k+1}+I_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{I^*}{I_n^{k+1}})]\\
&=\sum_{n=0}^{N}[\frac{\beta_W}{I^*}(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}W_n^k}{\kappa+W_n^k}-\frac{I_n^{k+1}S^*W^*}{(\kappa+W^*)I^*})+\beta_h S^*(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}I_n^k}{S^*I^*}-\frac{I_n^{k+1}}{I^*})]
\end{aligned}$$

$$- D_2 \sum_{n=0}^{N-1} \frac{(I_{n+1}^{k+1} - I_n^{k+1})^2}{(\triangle x)^2 I_{n+1}^{k+1} I_n^{k+1}}.$$

Similarly, by letting $\xi I^* = \delta W^*$, we obtain:

$$\begin{aligned}
&\sum_{n=0}^{N} \frac{1}{\Delta t}[g(\frac{W_n^{k+1}}{W^*}) - g(\frac{W_n^k}{W^*})] \\
&\leq \sum_{n=0}^{N} \frac{1}{\Delta t}[(W_n^{k+1} - W_n^k)(\frac{W_n^{k+1} - W^*}{W^* W_n^{k+1}})] \\
&= \sum_{n=0}^{N} \frac{1}{W^*}[(\xi I_n^{k+1} - \delta W_n^{k+1} + D_3 \frac{W_{n+1}^{k+1} - 2W_n^{k+1} + W_{n-1}^{k+1}}{(\triangle x)^2})(1 - \frac{W^*}{W_n^{k+1}})] \\
&= \sum_{n=0}^{N} [\frac{\delta}{W^*}(1 - \frac{W^*}{W_n^{k+1}})(\frac{W^* I_n^{k+1}}{I^*} - W_n^{k+1})] + \sum_{n=0}^{N} \frac{1}{W^*}[(\frac{W_{n+1}^{k+1} - 2W_n^{k+1} + W_{n-1}^{k+1}}{(\triangle x)^2})(1 - \frac{W^*}{W_n^{k+1}})] \\
&= \sum_{n=0}^{N} [\frac{\delta}{W^*}(1 - \frac{W^*}{W_n^{k+1}})(\frac{W^* I_n^{k+1}}{I^*} - W_n^{k+1})] - D_3 \sum_{n=0}^{N-1} \frac{(W_{n+1}^{k+1} - W_n^{k+1})^2}{(\triangle x)^2 W_{n+1}^{k+1} W_n^{k+1}}.
\end{aligned}$$

We then define the following Lyapunov function:

$$H^k = \sum_{n=0}^{N} \frac{1}{\Delta t}[\frac{1}{\beta_h I^*} g(\frac{S_n^k}{S^*}) + \frac{1}{\beta_h S^*} g(\frac{I_n^k}{I^*}) + \frac{\beta_W}{\beta_h \delta I^*} g(\frac{W_n^k}{W^*})]. \tag{13}$$

Thus, $H^k \geq 0$ for all $k \in \mathbb{N}$, with equality if and only if $S_n^k = S^*$, $I_n^k = I^*$ and $W_n^k = W^*$ for all $n \in \{0, 1, \dots N\}$. The difference of H^k is:

$$\begin{aligned}
H^{k+1} - H^k &= \sum_{n=0}^{N} [\frac{1}{\beta_h I^*}(\frac{S_n^{k+1} - S_n^k}{S^*} + ln\frac{S_n^k}{S_n^{k+1}}) + \frac{1}{\beta_h S^*}(\frac{I_n^{k+1} - I_n^k}{I^*} + ln\frac{I_n^k}{I_n^{k+1}}) \\
&+ \frac{\beta_W}{\delta \beta_h I^*}(\frac{W_n^{k+1} - W_n^k}{W^*} + ln\frac{W_n^k}{W_n^{k+1}})] - D_1 \sum_{n=0}^{N-1} \frac{(S_{n+1}^{k+1} - S_n^{k+1})^2}{(\Delta x)^2 S_{n+1}^{k+1} S_n^{k+1}} \\
&- D_2 \sum_{n=0}^{N-1} \frac{(I_{n+1}^{k+1} - I_n^{k+1})^2}{(\Delta x)^2 I_{n+1}^{k+1} I_n^{k+1}} - D_3 \sum_{n=0}^{N-1} \frac{(W_{n+1}^{k+1} - W_n^{k+1})^2}{(\Delta x)^2 W_{n+1}^{k+1} W_n^{k+1}} \\
&\leq \sum_{n=0}^{N} \{-\frac{\mu (S_n^{k+1} - S^*)^2}{\beta_h S_n^{k+1} S^* I^*} + (2 - \frac{S^*}{S_n^{k+1}} - \frac{I_n^{k+1}}{I^*} - \frac{S_n^{k+1} I_n^k}{I_n^{k+1} S^*} + \frac{I_n^{k+1}}{I^*}) \\
&- \frac{\beta_W W^*}{\beta_h I^* (\kappa + W^*)}[\frac{S^*}{S_n^{k+1}} + \frac{I_n^{k+1}}{I^*} + \frac{S_n^{k+1} W_n^k I^* (\kappa + W^*)}{I_n^{k+1} (\kappa + W_n^k) S^* W^*} - \frac{W_n^k (\kappa + W^*)}{(\kappa + W_n^k) W^*} - 2] \\
&- \frac{\beta_W W^*}{\beta_h I^* (\kappa + W^*)}(\frac{W_n^{k+1}}{W^*} + \frac{I_n^{k+1} W^*}{W_n^{k+1} I^*} - \frac{I_n^{k+1}}{I^*} - 1)\} \\
&- D_1 \sum_{n=0}^{N-1} \frac{(S_{n+1}^{k+1} - S_n^{k+1})^2}{(\Delta x)^2 S_{n+1}^{k+1} S_n^{k+1}} - D_2 \sum_{n=0}^{N-1} \frac{(I_{n+1}^{k+1} - I_n^{k+1})^2}{(\Delta x)^2 I_{n+1}^{k+1} I_n^{k+1}} - D_3 \sum_{n=0}^{N-1} \frac{(W_{n+1}^{k+1} - W_n^{k+1})^2}{(\Delta x)^2 W_{n+1}^{k+1} W_n^{k+1}}
\end{aligned}$$

$$\leq \sum_{n=0}^{N}\{-\frac{\mu(S_n^{k+1}-S^*)^2}{\beta_h S_n^{k+1}S^*I^*} - [g(\frac{S^*}{S_n^{k+1}}) + g(\frac{S_n^{k+1}I_n^k}{I_n^{k+1}S^*}) + \frac{I_n^{k+1}}{I^*} - ln\frac{I_n^{k+1}}{I_n^k}]$$
$$-\frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}[\frac{S^*}{S_n^{k+1}} + \frac{I_n^{k+1}}{I^*} + \frac{S_n^{k+1}I^*(\kappa+W^*)}{I_n^{k+1}S^*W^*} - \frac{\kappa+W^*}{W^*} - 2]$$
$$-\frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}(\frac{W_n^{k+1}}{W^*} + \frac{I_n^{k+1}W^*}{W_n^{k+1}I^*} - \frac{I_n^{k+1}}{I^*} - 1)\}$$
$$-D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\Delta x)^2 S_{n+1}^{k+1}S_n^{k+1}} - D_2\sum_{n=0}^{N-1}\frac{(I_{n+1}^{k+1}-I_n^{k+1})^2}{(\Delta x)^2 I_{n+1}^{k+1}I_n^{k+1}} - D_3\sum_{n=0}^{N-1}\frac{(W_{n+1}^{k+1}-W_n^{k+1})^2}{(\Delta x)^2 W_{n+1}^{k+1}W_n^{k+1}}$$
$$\leq \sum_{n=0}^{N}\{-\frac{\mu(S_n^{k+1}-S^*)^2}{\beta_h S_n^{k+1}S^*I^*} - g(\frac{S^*}{S_n^{k+1}}) - g(\frac{S_n^{k+1}I_n^k}{I_n^{k+1}S^*})$$
$$-\frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}[g(\frac{S^*}{S_n^{k+1}}) + g(\frac{S_n^{k+1}I_n^{k+1}(\kappa+W^*)}{I_n^{k+1}S^*W^*}) + g(\frac{W_n^{k+1}}{W^*}) + g(\frac{I_n^{k+1}W^*}{W_n^{k+1}I^*})]\}$$
$$-D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\Delta x)^2 S_{n+1}^{k+1}S_n^{k+1}} - D_2\sum_{n=0}^{N-1}\frac{(I_{n+1}^{k+1}-I_n^{k+1})^2}{(\Delta x)^2 I_{n+1}^{k+1}I_n^{k+1}} - D_3\sum_{n=0}^{N-1}\frac{(W_{n+1}^{k+1}-W_n^{k+1})^2}{(\Delta x)^2 W_{n+1}^{k+1}W_n^{k+1}}.$$

It is easy to see $H^{k+1} - H^k \leq 0$ for all $k \in \mathbb{N}$. Then there exists a constant H^* such that $lim_{k\to+\infty}(H^{k+1} - H^k) = 0$, which implies $lim_{k\to+\infty}S_n^k = S^*$. Combined with system (8-10), we have $lim_{k\to+\infty}I_n^k = I^*$ and $lim_{k\to+\infty}W_n^k = W^*$ as well, for all $n \in \{0, 1, \dots N\}$. Hence, E^* is globally asymptotically stable when $R_0 > 1$. This completes the proof.

5 Numerical results

In this section, we propose numerical simulations to verify the stability properties of the NSFD scheme. We use the data regarding the course of the cholera in Zimbabwe during 2008-2009, which is the worst outbreak in Africa in the past 30 years with over 100,000 humans have been infected and more than 4,300 killed. The total population in Zimbabwe is 12,347,240, for mathematical simplicity, we scale down all data numbers by a factor of 1,200. All epidemiological parameter values for cholera in literature are given as: $\Lambda = 4.5$, $\mu = 0.000442$, $\xi = 70$, $\delta = 0.2333$, $u_1 = 0.04$, $\gamma = 1.4$, $\kappa = 1000000$ ([2, 4, 5, 9]). In addition, the initial values are taken as $I(x, 0) = 10 \times \exp(-x)$, $S(x, 0) = 1000 \times \exp(-x)$, $W(x, 0) = 10 \times \exp(-x)$, and $R(x, 0) = 10 \times \exp(-x)$, where $x \in [0, 50]$.

Let the grid sizes used in the simulation are $\Delta x = 0.5$ and $\Delta t = 0.1$, respectively, and the diffusion coefficients D_i are all fixed as 0.01. The discussions in ([2,4,5,9]) indicate that parameters β_W and β_h are sensitive and vary from place to place, so we first set $\beta_W = 0.0001$ and $\beta_h = 0.0001$, which renders $R_0 = 0.7070 < 1$. Hence, model system has a disease-free equilibrium in this case, the number of infectious decreases quickly and the disease dies out. It can be observed from Figure 1, where the steady state approaches to $E_0 = (0.6, 0, 0)$. For the other case, we choose $\beta_W = 0.0001$, $\beta_h = 0.000236$ and do not change the other parameter values, which gives $R_0 = 1.6683 > 1$, the chronic infection steady state is $E^* = (0.5135, 1899.14, 8200.46)$ by calculation, the infected steady state is stable as can be observed numerically in Figure 2. We then examine the case with different sets of initial conditions when $R_0 > 1$, also obtain almost the same patterns.

Figure 3 compares the profile when we choose two different combinations of D_i, as, $(0.01, 0.05, 0.01, 0.05)$ and $(0.05, 0.1, 0.05, 0.1)$ for $R_0 > 1$. Only the distribution of the density of $I(x, t)$ is depicted, similar results for the other two variables $S(x, t)$ and $R(x, t)$ are not presented here. Comparing Fig. 3 and Fig 1.(a), we can find that diffusion coefficients have no effect on the convergence of solutions, but the larger diffusion coefficients will deduce the number of infected population and speed up the arrived time at the chronic infection equilibrium.

In a addition, we perform numerical simulations of a standard finite difference (SFD) scheme to compare the results with NSFD scheme using the same discrete boundary conditions and parameter values in Figure 4. The stronger competitiveness of NSFD scheme has been proved by its succsess in preserving the global stability of equilibrium and the failure of the SFD method.

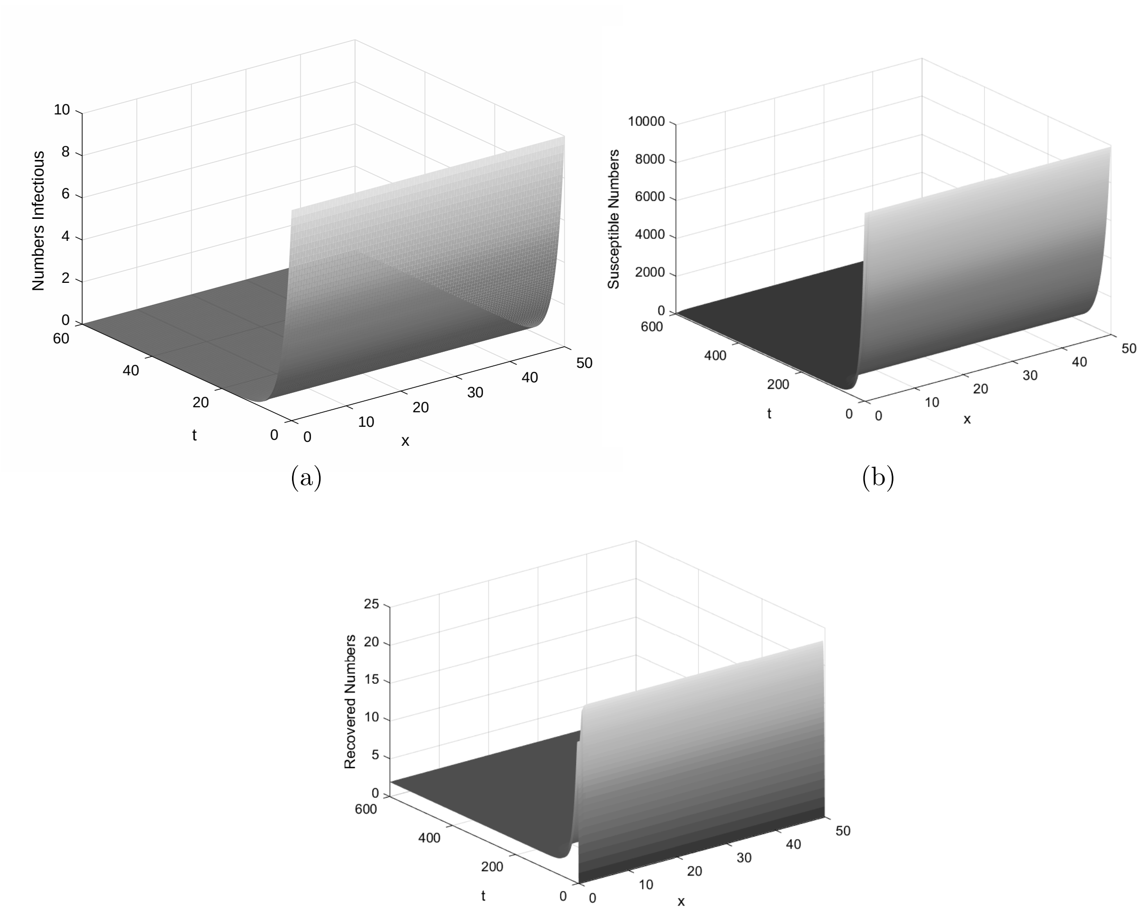


Figure 1: Graphs of the numerical solutions of the NSFD method when $R_0 < 1$.

6 Conclusions and discussions

In this article, we derive a discrete cholera infection model with spatial diffusion by using an NSFD method. We show that the disease-free steady state of the discrete model is globally

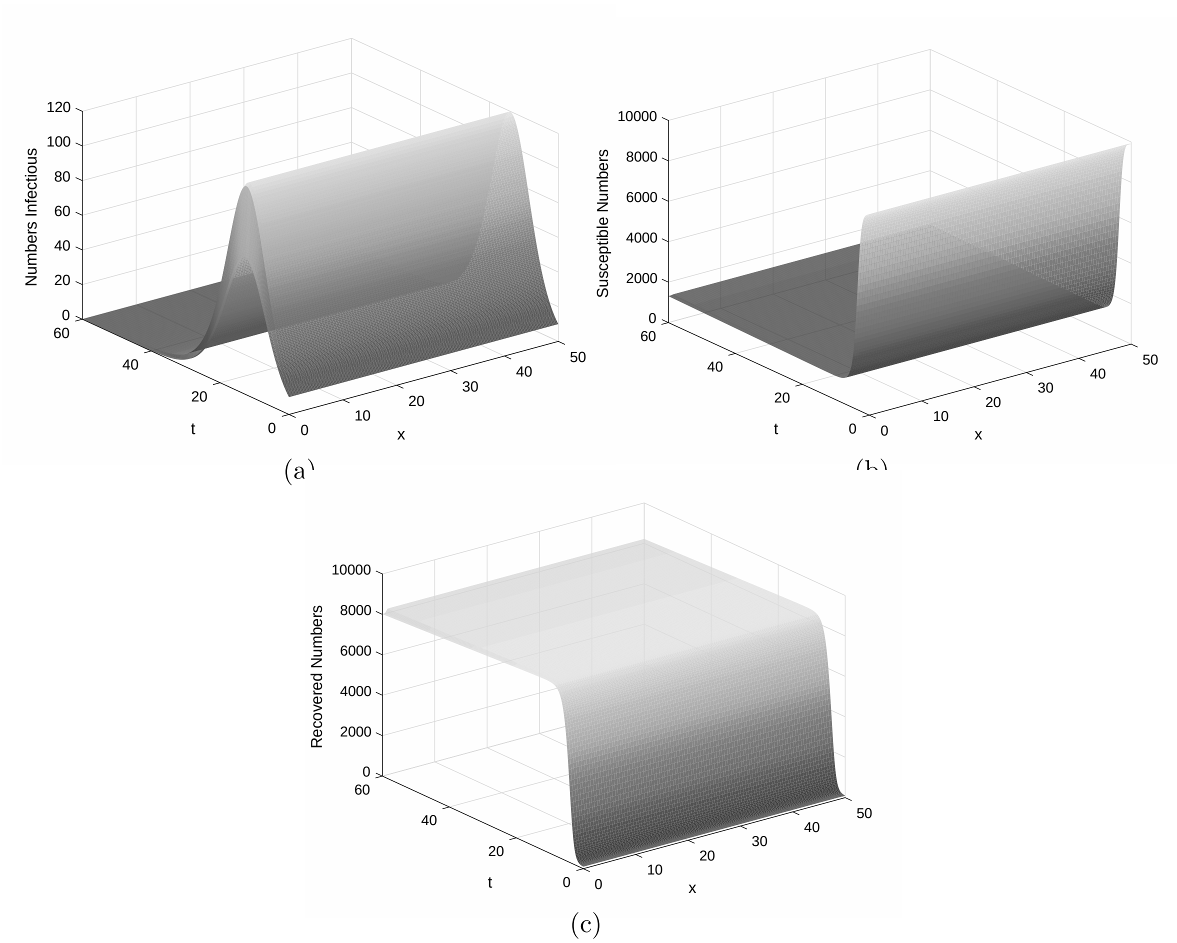
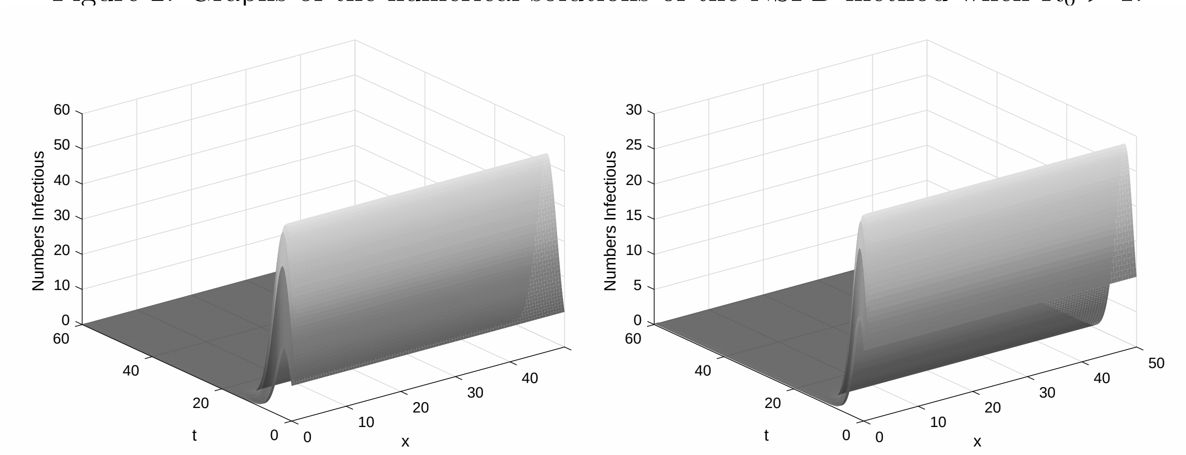


Figure 2: Graphs of the numerical solutions of the NSFD method when $R_0 > 1$.



(a) (b)

Figure 3: Dynamics of infected population when $R_0 > 1$ for two different sets of D_i.

asymptotically stable if the basic reproduction number $R_0 < 1$, and the chronic infection equilibrium is globally asymptotically stable when $R_0 > 1$. In a word, our results (Theorem 2 and Theorem 3) imply that the discretization scheme (8-11) is dynamically consistent with the continuous system with respect to the globally asymptotical stability of the steady-state solutions. Our simulation results also conclude that the diffusion coefficients have no relation to the global stability of such cholera epidemic. Finally, numerical results show the advantage of our method in comparison to an SFD method. Application of this method to the general delayed discrete epidemic models is our future work.

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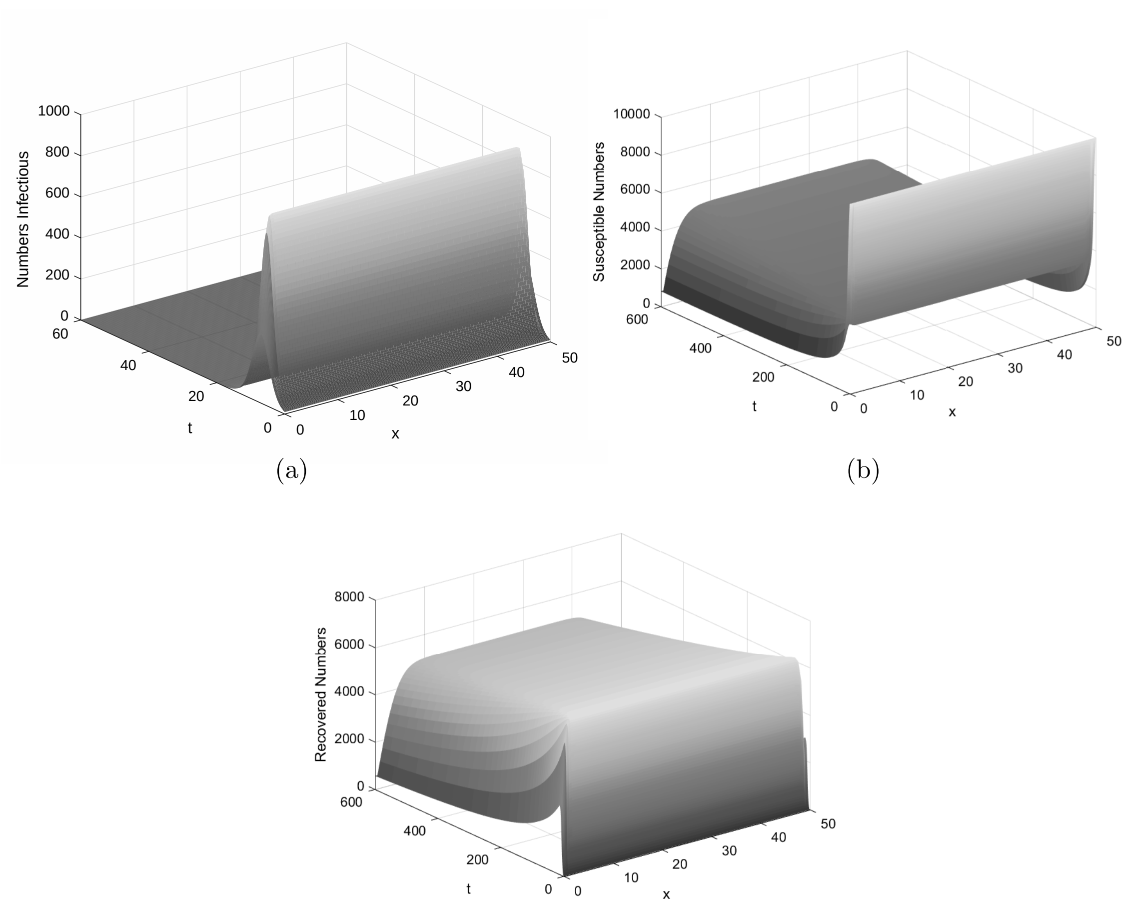

(a) (b)

(c)

Figure 4: Graphs of the numerical solutions of the SFD method.

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On the Higher Order Difference Equation

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_n x_{n-k}}{b x_n + c x_{n-l} + d x_{n-k}}$$

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ABSTRACT

The main objective of this paper is to investigate the global stability of the solutions, the boundedness and the periodic character of the nonlinear difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_n x_{n-k}}{b x_n + c x_{n-l} + d x_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c$ and d are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1}, ...,\ x_{-1}$, x_0 are positive real numbers where $s = max\{l,\ k\}$. Some numerical examples will be given to explicate our results.

Keywords: Difference equations, Stability, Global stability, Boundedness, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Our goal is to study some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_n x_{n-k}}{b x_n + c x_{n-l} + d x_{n-k}}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c$ and d are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1}, ...,\ x_{-1}$, x_0 are positive real numbers where $s = max\{l,\ k\}$.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1-270] and references therein.

Ibrahim [4] investigated the global attractivity of the positive solutions of the difference equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1+x_{n-k}x_{n-(2k+1)}}, \qquad n = 0, 1, ...$$

Zayed et al. et al. [5] studied the periodicity, the boundedness and the global stability of the positive solution of the difference equation,

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3}}{A x_n + B x_{n-1} + C x_{n-2} + D x_{n-3}}, \qquad n = 0, 1, ...$$

In [6] El-Dessoky investigated the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{a x_{n-l} + b x_{n-k}}{c + d x_{n-l} x_{n-k}}, \quad n = 0, 1, ...$$

Guo-Mei Tang et al. [7] obtained the global behavior of solutions of the following nonlinear difference equation

$$x_{n+1} = \frac{\alpha + x_n}{A + B x_n + x_{n-k}}, \qquad n = 0, 1, ...$$

Papaschinopoulos et al. [8] studied the asymptotic behavior and the periodicity of the positive solutions of the nonautonomous difference equation

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \qquad n = 0, 1, ...$$

El-Dessoky [9] obtained the global stability, the boundedness and the periodicity of the nonlinear difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}, \qquad n = 0,\ 1,\ ....$$

Nirmaladevi et al. [10] studied the periodicity solution and the global stability of nonlinear difference equation

$$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, \qquad n = 0, 1, ...$$

"Let I be some interval of real numbers and let

$$F : I^{s+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-s}, x_{-s+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-s}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-s}^{\infty}$.

DEFINITION 1.1. *(Equilibrium Point)*
A point $\overline{x} \in I$ is called an equilibrium point of the difference equation (2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of the difference equation (2), or equivalently, $\overline{x}$ is a fixed point of F.

DEFINITION 1.2. *(Stability)*
Let $\overline{x} \in (0,\ \infty)$ be an equilibrium point of the difference equation (2). Then, we have
(i) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-s}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-t} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -t.$$

(ii) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally asymptotically stable if $\overline{x}$ is locally stable solution of equation (2) and there exists $\gamma > 0$, such that for all $x_{-t}, ..., x_{-1},\ x_0 \in I$ with

$$|x_{-s} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of the difference equation (2) is called global attractor if for all $x_{-s}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of the difference equation (2) is called globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of the difference equation (2).

(v) The equilibrium point $\overline{x}$ of the difference equation (2) is called unstable if $\overline{x}$ is not locally stable.

Definition 1.3. *(Periodicity)*
A sequence $\{x_n\}_{n=-s}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -t$. *A sequence* $\{x_n\}_{n=-s}^{\infty}$ *is said to be periodic with prime period* p *if* p *is the smallest positive integer having this property.*

Definition 1.4. *Equation (2) is called permanent and bounded if there exists numbers* M *and* m *with* $0 < m < M < \infty$ *such that for any initial conditions* $x_{-s}, ..., x_{-1}, x_0 \in (0, \infty)$ *there exists a positive integer* N *which depends on these initial conditions such that*

$$m \leq x_n \leq M \qquad \text{for all } n > N.$$

Definition 1.5. *The linearized equation of the difference equation (2) about the equilibrium* $\overline{x}$ *is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{s} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0 \lambda^s + p_1 \lambda^{s-1} + ... + p_{s-1}\lambda + p_s = 0, \tag{4}$$

where

$$p_i = \tfrac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}.$$

Theorem 1.6. *[1]: Assume that* $p_i \in R$, $i = 1, 2, ..., s$ *and* s *is non-negative integer. Then*

$$\sum_{i=1}^{s} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+s} + p_1 y_{n+s-1} + ... + p_s y_n = 0, \;\; n = 0, 1, ... \, .$$

Theorem 1.7. *[1]: Consider the the difference equation (2) where* $F \in C(I^{t+1}, R)$ *and* I *is an open interval of real numbers. Let* $\overline{x}$ *be an equilibrium point of the difference equation (2). Finally, suppose that* F *satisfies the following two conditions:*

(i) F *is nondecreasing in each of its argements.*

(ii) F *satisfies the negative feedback property*

$$[F(x,\ x,\ ...,\ x) - x]\,(x - \overline{x}) < 0, \quad \text{for all } \ x \in I - \{0\}\,.$$

Then the equilibrium point $\overline{x}$ *isa global attractor of all solutions of the difference equation (2)."*

2. LOCAL STABILITY

In this section, we study the local stability character of the equilibrium point of equation (1).

Equation (1) has equilibrium point and is given by

$$\overline{x} = \alpha\overline{x} + \beta\overline{x} + \gamma\overline{x} + \tfrac{a\overline{x}^2}{b\overline{x} + c\overline{x} + d\overline{x}},$$

$$[(1 - \alpha - \beta - \gamma)\,(b + c + d) - a]\,\overline{x}^2 = 0.$$

If $(1 - \alpha - \beta - \gamma)\,(b + c + d) \neq a$, then the equilibrium point of the difference equation (1) is $\overline{x} = 0$.

Let $f:(0,\ \infty)^3 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$f(u,\ v,\ w)=\alpha u+\beta v+\gamma w+\frac{auw}{bu+cv+dw}.$$

Therefore, it follows that

$$\frac{\partial f(u,\ v,\ w)}{\partial u}=\alpha+\frac{aw(cv+dw)}{(bu+cv+dw)^2},\ \frac{\partial f(u,\ v,\ w)}{\partial v}=\beta-\frac{acuw}{(bu+cv+dw)^2}\text{ and }\frac{\partial f(u,\ v,\ w)}{\partial w}=\gamma+\frac{au(bu+cv)}{(bu+cv+dw)^2}.$$

THEOREM 2.1. *The zero equilibrium $\overline{x}$ of the difference equation (1) is locally asymptotically stable if*

$$(\alpha+\beta+\gamma)\,(b+c+d)+a<1. \tag{5}$$

Proof: So, we can write Eq. (6) at zero equilibrium point $\overline{x}=0$

$$\begin{aligned}\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x})}{\partial u} &= \alpha+\frac{a(c+d)}{(b+c+d)^2}=p_1,\ \frac{\partial f(\overline{x},\ \overline{x},\ \overline{x})}{\partial v}=\beta-\frac{ac}{(b+c+d)^2}=p_2\\ \text{and }\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x})}{\partial w} &= \gamma+\frac{a(b+c)}{(b+c+d)^2}=p_3.\end{aligned}$$

Then the linearized equation of equation (1) about $\overline{x}$ is

$$y_{n+1}-p_1y_{n-k}-p_2y_{n-l}-p_3y_{n-s}=0,$$

It follows by Theorem 1 that, equation (1) is asymptotically stable if and only if

$$|p_1|+|p_2|+|p_3|<1.$$

Thus,

$$\left|\alpha+\frac{a(c+d)}{(b+c+d)^2}\right|+\left|\beta-\frac{ac}{(b+c+d)^2}\right|+\left|\gamma+\frac{a(b+c)}{(b+c+d)^2}\right|<1,$$

and so

$$\alpha+\frac{a(c+d)}{(b+c+d)^2}+\beta-\frac{ac}{(b+c+d)^2}+\gamma+\frac{a(b+c)}{(b+c+d)^2}<1,$$

$$\begin{aligned}\alpha+\beta+\gamma+\frac{a(b+c+d)}{(b+c+d)^2} &< 1,\\ (\alpha+\beta+\gamma)\,(b+c+d)+a &< 1.\end{aligned}$$

The proof is complete.

Example 1. Consider $l=2$, $k=3$, $\alpha=0.3$, $\beta=0.02$, $\gamma=0.01$, $a=0.1$, $b=0.2$, $c=0.3$ and $d=0.7$ and the initial conditions $x_{-3}=0.2$, $x_{-2}=0.4$, $x_{-1}=0.6$ and $x_0=0.1$, the zero solution of the difference equation (1) is local stability (see Fig. 1).

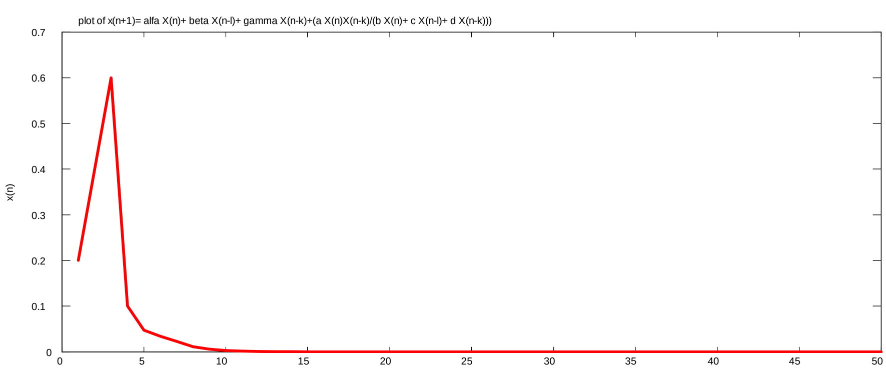

Figure 1. Sketch the behavior of zero solution of equation (1) is local stable.

Example 2. The solution of the difference equation (1) is unstable if $l = 2$, $k = 3$, $\alpha = 0.3$, $\beta = 0.2$, $\gamma = 0.1$, $a = 0.5$, $b = 0.2$, $c = 0.3$ and $d = 0.7$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 2).

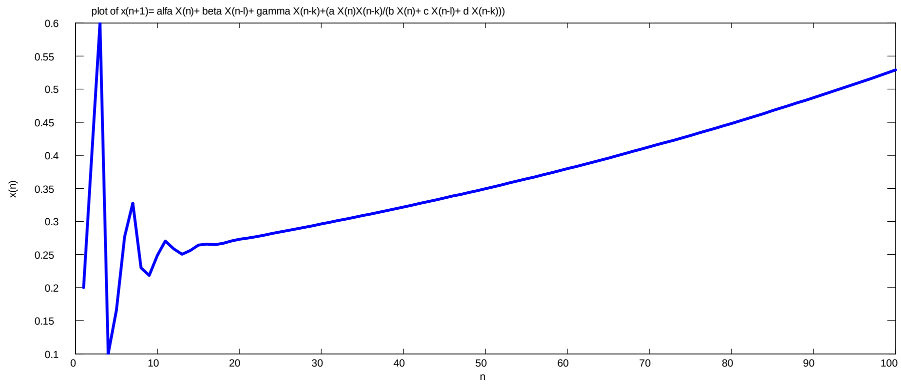

Figure 2. Draw the behavior of the solution of equation (1) is unstable.

3. GLOBAL STABILITY

In this section, the global asymptotic stability of equation (1) is studied.

THEOREM 3.1. *The equilibrium point* $\overline{x}$ *is a global attractor of Eq. (1) if* $\alpha + \beta + \gamma \neq 1$.

Proof: Suppose that ζ and η are real numbers and assume that $F : [\zeta, \eta]^3 \longrightarrow [\zeta, \eta]$ is a function defined by

$$F(x,\ y,\ z) = \alpha x + \beta y + \gamma z + \frac{axz}{bx+cy+dz}.$$

Then

$$\frac{\partial F(x,\ y,\ z)}{\partial x} = \alpha + \frac{az(cy+dz)}{(bx+cy+dz)^2},\ \frac{\partial F(x,\ y,\ z)}{\partial y} = \beta - \frac{acxz}{(bx+cy+dz)^2} \text{ and } \frac{\partial F(x,\ y,\ z)}{\partial z} = \gamma + \frac{ax(bx+cy)}{(bx+cy+dz)^2}.$$

Now, we can see that the function $F(x,\ y,\ z)$ nondecreasing in x, y and z. Then

$$\begin{aligned} [F(x,\ x,\ x) - x](x - \overline{x}) &= \left[\alpha x + \beta x + \gamma x + \frac{ax^2}{bx+cx+dx} - x\right](x - \overline{x}) \\ &= -\left[\left(1 - \alpha - \beta - \gamma - \frac{a}{b+c+d}\right)x\right](x - 0) \\ &= -\left(1 - \alpha - \beta - \gamma - \frac{a}{b+c+d}\right)x^2 < 0 \end{aligned}$$

If $\alpha + \beta + \gamma + \frac{a}{b+c+d} < 1$, then $F(x,\ y,\ z)$ satisfies the negative feedback property

$$[F(x,\ x,\ x) - x](x - \overline{x}_0) < 0, \quad \text{for } \overline{x}_0 = 0.$$

According to Theorem 2, then $\overline{x}$ is a global attractor of Eq. (1). This completes the proof.

Example 3. The solution of the difference equation (1) is global stability when $l = 2$, $k = 3$, $\alpha = 0.03$, $\beta = 0.02$, $\gamma = 0.01$, $a = 0.1$, $b = 0.2$, $c = 0.3$ and $d = 0.7$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 3).

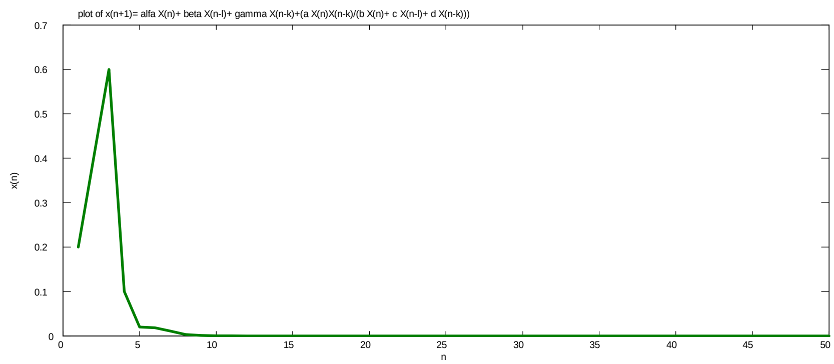

Figure 3. Plot the behavior of the solution of equation (1) is global stability.

4. BOUNDEDNESS OF THE SOLUTIONS

In this section, we investigate the boundedness nature of the positive solutions of equation (1).

THEOREM 4.1. *Every solution of Equation (1) is bounded if one of the following conditions holds:*

$$(i)\ \alpha+\frac{a}{d} < 1,\ \beta<1 \text{ and } \gamma<1. \tag{6}$$

$$(ii)\ \alpha < 1,\ \beta<1 \text{ and } \gamma+\frac{a}{b}<1. \tag{7}$$

Proof: First we prove every solution of Equation (1) is bounded if $\alpha+\frac{a}{d}<1,\ \beta<1$ and $\gamma<1$. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$\begin{aligned} x_{n+1} &= \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\frac{a x_n x_{n-k}}{b x_n+c x_{n-l}+d x_{n-k}}, \\ &\leqslant \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\frac{a x_n x_{n-k}}{d x_{n-k}} \\ &= \left(\alpha+\tfrac{a}{d}\right) x_n+\beta x_{n-l}+\gamma x_{n-k} \\ &< x_n+x_{n-l}+x_{n-k}. \end{aligned}$$

Then

$$x_{n+1}<x_n+x_{n-l}+x_{n-k} \quad \text{for all } n\geq 0.$$

So every solution of Eq. (1) is bounded from above by $M=x_0+x_{-l}+x_{-k}$.

Second we prove every solution of Equation (1) is bounded if $\alpha<1,\ \beta<1$ and $\gamma+\frac{a}{b}<1$. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$\begin{aligned} x_{n+1} &= \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\frac{a x_n x_{n-k}}{b x_n+c x_{n-l}+d x_{n-k}}, \\ &\leqslant \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\frac{a x_n x_{n-k}}{b x_n} \\ &= \alpha x_n+\beta x_{n-l}+\left(\gamma+\tfrac{a}{b}\right) x_{n-k} \\ &< x_n+x_{n-l}+x_{n-k}. \end{aligned}$$

Then

$$x_{n+1}<x_n+x_{n-l}+x_{n-k} \quad \text{for all } n\geq 0.$$

So every solution of Eq. (1) is bounded from above by $M=x_0+x_{-l}+x_{-k}$.

THEOREM 4.2. *Every solution of Equation (1) is unbounded if* $\alpha>1$*or* $\beta>1$ *or* $\gamma>1$.

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1).Then from Equation (1) we see that

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_n x_{n-k}}{b x_n + c x_{n-l} + d x_{n-k}} > \alpha x_n \quad \text{for all} \quad n \geq 0.$$

We see that the right hand side can be written as follows

$$z_{n+1} = \alpha z_{n-l}.$$

then

$$z_{ln+i} = \alpha^n z_{l+i} + const., \qquad i = 0, 1, ..., l,$$

and this equation is unstable because $\alpha > 1$, and $\lim_{n\to\infty} z_n = \infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above.

Similarly we can prof that every solution of Eq. (1) is unbounded if $\beta > 1$ or $\gamma > 1$. Thus, the proof is now completed.

Example 4. We assume $l = 2$, $k = 3$, $\alpha = 1.3$, $\beta = 0.2$, $\gamma = 0.1$, $a = 0.1$, $b = 0.2$, $c = 0.3$ and $d = 0.7$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the solution of the difference equation (1) is unbounded (see Fig. 4).

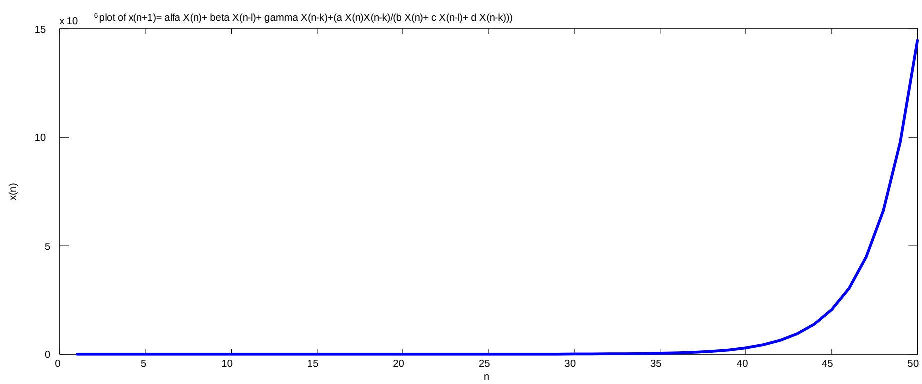

Figure 4. Plot the behavior of the solution of equation (1) is unbounded.

5. EXISTENCE OF PERIODIC SOLUTIONS

THEOREM 5.1. *Suppose that l and k are even positive integers, then equation (1) has no prime period two solutions.*

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,\ ...,$$

of equation (1). We see from equation (1) when l and k are an even, then $x_n = x_{n-l} = x_{n-k}$. It follows equation (1) that

$$P = \alpha Q + \beta Q + \gamma Q + \frac{aQ^2}{bQ + cQ + dQ},$$

and

$$Q = \alpha P + \beta P + \gamma P + \frac{aP^2}{bP + cP + dP}.$$

Therefore,

$$(b + c + d)\, P = (b + c + d)\, (\alpha + \beta + \gamma)Q + aQ, \tag{8}$$

$$(b + c + d)\, Q = (b + c + d)\, (\alpha + \beta + \gamma)P + aP, \tag{9}$$

Subtracting (9) from (8) gives

$$(b+c+d)(P-Q)=((b+c+d)(\alpha+\beta+\gamma)+a)(Q-P)$$

$$(P-Q)[(b+c+d)(1+\alpha+\beta+\gamma)+a]=0$$

Since $(b+c+d)(1+\alpha+\beta+\gamma)+a\neq 0$, then $p=q$. This is a contradiction. Thus, the proof is completed.

Theorem 5.2. *Let l is even and k is odd positive integers, then equation (1) has no positive prime period two solutions.*

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,...,$$

of equation (1). We see from equation (1) when l is an even and k is an odd, then $x_n = x_{n-l}$ and $x_{n+1} = x_{n-k}$. It follows equation (1) that

$$P=\alpha Q+\beta Q+\gamma P+\tfrac{aQP}{bQ+cQ+dP},$$

and

$$Q=\alpha P+\beta P+\gamma Q+\tfrac{aPQ}{bP+cP+dQ}.$$

Therefore,

$$(b+c)(1-\gamma)PQ+dP^2=(b+c)(\alpha+\beta)Q^2+d(\alpha+\beta)PQ+aQP, \tag{10}$$

$$(b+c)(1-\gamma)PQ+dQ^2=(b+c)(\alpha+\beta)P^2+d(\alpha+\beta)PQ+aPQ, \tag{11}$$

Subtracting (11) from (10) gives

$$d(P^2-Q^2)=(b+c)(\alpha+\beta)(Q^2-P^2)$$

$$(P^2-Q^2)(d+(b+c)(\alpha+\beta))=0$$

Then $P=\pm Q$. This is a contradiction. Thus, the proof is completed.

Theorem 5.3. *Suppose that l is odd and k is even positive integers, then equation (1) has no positive prime period two solutions.*

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,...,$$

of equation (1). We see from equation (1) when k is an even and l is an odd, then $x_n = x_{n-k}$ and $x_{n+1} = x_{n-l}$. It follows equation (1) that

$$P=\alpha Q+\beta P+\gamma Q+\tfrac{aQ^2}{bQ+cP+dQ},$$

and

$$Q=\alpha P+\beta Q+\gamma P+\tfrac{aP^2}{bP+cQ+dP}.$$

Therefore,

$$(b+d)(1-\beta)PQ+cP^2=(b+d)(\alpha+\gamma)Q^2+c(\alpha+\gamma)PQ+aQ^2, \tag{12}$$

$$(b+d)(1-\beta)PQ+cQ^2=(b+d)(\alpha+\gamma)P^2+c(\alpha+\gamma)PQ+aP^2, \tag{13}$$

Subtracting (13) from (12) gives

$$c(P^2-Q^2)=((b+d)(\alpha+\gamma)+a)(Q^2-P^2)$$

$$\left(P^2 - Q^2\right)\left[c\left(\left(b+d\right)\left(\alpha+\gamma\right)+a\right)\right] = 0$$

Then $P = \pm Q$. This is a contradiction. Thus, the proof is completed.

Theorem 5.4. *Let l, k are odd positive integers. If*

$$\left(1+\alpha-\beta-\gamma\right)\left(b+c+d\right) - a \neq 0,$$

then Eq. (1) has no prime period two solution.

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,\ ...,$$

of equation (1). We see from equation (1) when l and k are an odd, then $x_{n+1} = x_{n-l} = x_{n-k}$. It follows equation (1) that

$$P = \alpha Q + \beta P + \gamma P + \tfrac{aP^2}{bP+cP+dP},$$

and

$$Q = \alpha P + \beta Q + \gamma Q + \tfrac{aQ^2}{bQ+cQ+dQ}.$$

Therefore,

$$\left(1-\beta-\gamma\right)\left(b+c+d\right)P = \alpha\left(b+c+d\right)Q + aP, \tag{14}$$

$$\left(1-\beta-\gamma\right)\left(b+c+d\right)Q = \alpha\left(b+c+d\right)P + aQ, \tag{15}$$

Subtracting (15) from (14) gives

$$\left(1-\beta-\gamma\right)\left(b+c+d\right)\left(P-Q\right) = \alpha\left(b+c+d\right)\left(Q-P\right) + a(P-Q)$$

$$\left(P-Q\right)\left[\left(1+\alpha-\beta-\gamma\right)\left(b+c+d\right) - a\right] = 0$$

Since $\left(1+\alpha-\beta-\gamma\right)\left(b+c+d\right) - a \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

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BEST PROXIMITY POINT OF CONTRACTION TYPE MAPPING IN METRIC SPACE

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Abstract. The purpose of this article, we consider the existence of a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$ for generalized φ-weak contraction mapping $T : A \to B$, where $A, B(\neq \emptyset)$ are subsets of a metric space (X, d).

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X.$$

A mapping $T : X \to X$ is a *φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ and $\lim_{t\to\infty} \varphi(t) = \infty$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall\, x, y \in X. \tag{1.1}$$

If X is bounded, then the infinity condition can be omitted.

The concept of the φ-weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later Rhoades [14] in 2001, who extended the results of [1] to metric spaces.

Theorem 1.1. ([14]) *Let (X, d) be a complete metric space, $T : X \to X$ be a φ-weak contractive self-map on X. The T has a unique fixed point p in X.*

Remark 1.2. Theorem 1.1 is one of generalizations of the Banach contraction principle because it takes $\varphi(t) = (1 - \alpha)t$ for $\alpha \in (0, 1)$, then φ-weak contraction contains contraction as special cases.

Next, we present a brief discussion about best proximity point which is a interesting topic in best proximity theory.

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Let (X, d) be a metric space and $A(\neq \emptyset)$ be a subset of (X, d). Consider a mapping $T : A \to X$. The solutions to the fixed point equation $Tx = x$ are called *fixed points* of the mapping T. It is clear that $T(A) \cap A \neq \emptyset$ is a necessary (but not sufficient) condition for the existence of a fixed point for the mapping $T : A \to X$. If the necessary condition fails, then

$$d(x, Tx) > 0,$$

for all $x \in A$. This means that the mapping $T : A \to X$ does not have any fixed point, *i.e.*, $Tx = x$ has no solution. This point of view, it give us to think of a point $x \in A$ which is closest to Tx in some sense. Best approximation theory and best proximity point theory are relevant in this perspective. One of the most interesting best approximation theorem is due to Fan [3].

Theorem 1.3. ([3]) *Let $C(\neq \emptyset)$ be a compact convex subset of a normed linear space V and $F : C \to V$ be a continuous function. Then there exists a point $p \in C$ such that $\|p - Fp\| = d(Fp, C) = \inf\{\|Fp - c\| : c \in C\}$.*

Such an element $p \in C$ in Theorem 1.3 is called a *best approximant point* of T in C.

Although a best approximation point acts as an approximate solution of the equation $Fp = p$, the value $\|p - Fp\|$ need not be the optimum, *i.e.*, a best approximant point is not an optimal solution in the sense that

$$\min_{p \in A} \|p - Fp\|.$$

Naturally, let us consider nonempty subsets A, B of a metric space (X, d) and a mapping $T : A \to B$. Then one can think of finding an element $x^* \in A$ such that

$$d(x^*, Tx^*) = \min\{d(x, Tx) : x \in A\}.$$

Since

$$d(x, Tx) \geq dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

for all $x \in A$, the optimal solution of $\min_{x \in A} d(x, Tx)$ is one for which the value $dist(A, B)$ is attained. A point $x^* \in A$ is said to be a *best proximity point* of $T : A \to B$ if

$$d(x^*, Tx^*) = dist(A, B).$$

So a best proximity point of the mapping T is an approximate solution of the equation $Tx = x$ which is optimal solution in the sense that

$$\min_{x \in A} d(x, Tx).$$

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Remark 1.4. It is trivial that all best proximity point theorems work as a natural generalization of fixed point theorems if the mapping T is a self-mapping.

Recently Sultana and Vetrivel [15] obtained the following best proximity point theorem for mapping satisfies (1.1).

Theorem 1.5. ([15], Theorem 3.4) *Let $A, B(\neq \emptyset)$ be two closed subsets of a complete metric space (X, d) such that the pair (A, B) has the P-property and $A_0 \neq \emptyset$ and $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and it satisfies* (1.1)*. Then there exists a unique $p \in A$ such that $d(p, Tp) = dist(A, B)$.*

In 2016, Xue [16] introduced a new contraction type mapping as follows.

Definition 1.6. ([16]) A mapping $T : X \to X$ is a *generalized φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X \tag{1.2}$$

holds.

We notice immediately that if $T : X \to X$ is φ-weak contraction, then T satisfies the following inequality

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X.$$

However, the converse is not true in general.

Example 1.7. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x-y|$ and let $Tx = \frac{2}{5}x$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{4}{3}t$. Then T satisfies (1.2), but T does not satisfy inequality (1.1). Indeed,

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\leq |x-y| - \frac{4}{3} \cdot \frac{2}{5}|x-y| \\ &= d(x, y) - \varphi(d(Tx, Ty)) \end{aligned}$$

and

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\geq |x-y| - \frac{4}{3}|x-y| \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

for all $x, y \in X$.

Example 1.8. ([16]) Let $X = (-1, +\infty)$ be endowed by $d(x, y) = |x - y|$ and let $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t^2}{1+t}$. Then

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x}{1+x} - \frac{y}{1+y} \right| = \frac{|x-y|}{(1+x)(1+y)} \\ &\leq \frac{|x-y|}{1+|x-y|} = |x-y| - \frac{|x-y|^2}{1+|x-y|} \\ &= d(x,y) - \varphi(d(x,y)) \end{aligned}$$

holds for all $x, y \in X$. So T is a φ-weak contraction. However T is not a contraction.

Remark 1.9. The above examples show that the class of generalized φ-weak contractions properly includes the class of φ-weak contractions and the class of φ-weak contractions properly includes the class of contractions.

In fact, let $T : X \to X$ be a contraction, there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X.$$

Then

$$\begin{aligned} d(Tx, Ty) &\leq \alpha \cdot d(x, y) = d(x, y) - (1 - \alpha) d(x, y) \\ &= d(x, y) - \varphi(d(x, y)), \end{aligned}$$

where, $\varphi(d(x, y)) = (1 - \alpha)d(x, y)$. So, T is a φ-weak contraction. Moreover, let T be a φ-weak contraction, from property of φ, we have $d(Tx, Ty) \leq d(x, y)$ and

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)).$$

From (1.1),

$$\begin{aligned} d(Tx, Ty) &\leq d(x, y) - \varphi(d(x, y)) \\ &\leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X. \end{aligned}$$

Therefore, T is a generalized φ-weak contraction.

In the meantime, if T is a φ-weak contractive self mapping for one mapping φ so we do not expect that the φ-weak contractivity should be satisfied with the same function φ. Let us suppose that T is a φ-weak contractive self mapping and consider

$$\tilde{\varphi}(x) = \min \{\varphi(x/2); x/2\}.$$

Then, if $d(Tx, Ty) > \frac{1}{2} d(x, y)$, we have

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)) \leq d(x, y) - \varphi\left(\frac{1}{2} d(x, y)\right)$$

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on account of monotonocity of φ and finally

$$d(Tx,Ty) \le d(x,y) - \tilde{\varphi}(d(x,y)).$$

On the other hand, if $d(Tx,Ty) < \frac{1}{2}d(x,y)$, we get

$$d(Tx,Ty) < d(x,y) - \frac{1}{2}d(x,y) \le d(x,y) - \tilde{\varphi}(d(x,y)).$$

So T is just the $\tilde{\varphi}$-weak contractive mapping. The continuity and monotonocity of $\tilde{\varphi}$ follows directly from properties of min function, φ and the metric.

For related results, please see [9], [10], [11] and the references therein ([5], [6], [7], [8]).

The purpose of this article, we consider the existence of a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A,B)$ for generalized φ-weak contraction mapping $T : A \to B$, where $A, B(\neq \emptyset)$ are subsets of a metric space (X,d).

2. Preliminaries

Let A, B be two nonempty subsets of a metric space (X,d). Let us define the following notation which will be need throughout this article:

$$A_0 = \{x \in A : d(x,y) = dist(A,B) \quad \text{for some} \quad y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = dist(A,B) \quad \text{for some} \quad x \in A\}.$$

In [12], the authors discussed sufficient conditions which guarantee the nonemptiness of A_0 and B_0. Also, in [2], the authors proved that A_0 is contained in the boundary of A.

Let us define the notion of nonself generalized φ-weak contraction mapping as follows.

Definition 2.1. Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a *generalized φ-weak contraction* if

$$d(Tx,Ty) \le d(x,y) - \varphi(d(Tx,Ty)), \quad \forall\, x,y \in A, \tag{2.1}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that φ is positive on $(0,\infty)$, $\varphi^{-1}(0) = \{0\}$ and $\lim_{t\to\infty}\varphi(t) = \infty$. If A is bounded, then the infinity condition can be omitted.

The notion called the P-property was introduced in [13].

Definition 2.2. ([13]) Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to be *has the P-property*

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if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = dist(A, B) = d(x_2, y_2) \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2).$$

Now we recall the following results from [4] and [15].

Lemma 2.3. ([15]) *Let $\varphi : [0, \infty) \to [0, \infty)$ be a function such that $\varphi^{-1}(0) = \{0\}$ and φ is either nondecreasing or continuous. Then*

$$\varphi(\mu_n) \to 0 \quad \Rightarrow \quad \mu_n \to 0$$

for any bounded sequence $\{\mu_n\}$ of positive reals.

Lemma 2.4. ([4]) *For a given subset D of $\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, the following statements are equivalent:*

(i) *for any $\varepsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0, \varepsilon)$ such that*

$$u < \varepsilon + \delta \quad \Rightarrow \quad v \leq \gamma$$

for all $(u, v) \in D$,

(ii) *there exist a continuous and nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ such that*

$$\phi(u) < u, \ \forall\, u > 0 \quad \text{and} \quad v \leq \phi(u), \ \forall\, (u, v) \in D.$$

3. Main results

Lemma 3.1. *Let A and B be two nonempty subsets of a metric space (X, d) and $\varphi : [0, \infty) \to [0, \infty)$ be a function such that $\varphi^{-1}(0) = \{0\}$ and*

$$\varphi(t_n) \to 0 \quad \Rightarrow \quad t_n \to 0 \tag{3.1}$$

for any bounded sequence $\{t_n\}$ of positive reals. Let $T : A \to B$ be a genearlized φ-weak contraction mapping satisfying (2.1). Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0, \varepsilon)$ such that

$$d(x, y) < \varepsilon + \delta \quad \Rightarrow \quad d(Tx, Ty) \leq \gamma$$

for all $x, y \in A$.

Proof. Suppose that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$, $\gamma \in (0, \varepsilon_0)$ and there exist $x, y \in A$ such that

$$d(x, y) < \varepsilon_0 + \delta \quad \Rightarrow \quad d(Tx, Ty) > \gamma.$$

Let

$$\delta_n = \frac{1}{n^2} \quad \text{and} \quad \gamma_n = \frac{n^2}{1 + n^2}\varepsilon_0, \ \ \forall\, n \in \mathbb{N}.$$

Then there exist $\{x_n\}$ and $\{y_n\}$ in A such that

$$d(x_n, y_n) < \varepsilon_0 + \frac{1}{n^2} \quad \Rightarrow \quad d(Tx_n, Ty_n) > \frac{n^2}{1+n^2}\varepsilon_0. \tag{3.2}$$

From (2.1), we have

$$\begin{aligned}\frac{n^2}{1+n^2}\varepsilon_0 &< d(Tx_n, Ty_n)\\ &\leq d(x_n, y_n) - \varphi(d(Tx_n, Ty_n))\\ &< \varepsilon_0 + \frac{1}{n^2} - \varphi(d(Tx_n, Ty_n)).\end{aligned}$$

That is

$$\varphi(d(Tx_n, Ty_n)) < \varepsilon_0 + \frac{1}{n^2} - \frac{n^2}{1+n^2}\varepsilon_0 = \frac{1}{n^2} + \frac{\varepsilon_0}{1+n^2}.$$

Hence

$$\varphi(d(Tx_n, Ty_n)) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $d(Tx_n, Ty_n) \leq d(x_n, y_n)$ and $\{d(x_n, y_n)\}$ is bounded, we get $\{d(Tx_n, Ty_n)\}$ is bounded. By the given hypothesis (3.1),

$$d(Tx_n, Ty_n) \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, from (3.2),

$$\lim_{n\to\infty} d(Tx_n, Ty_n) \geq \varepsilon_0 > 0.$$

This is a contradiction. Thus Lemma 3.1 holds. □

The following theorem is main result which gives sufficient conditions for the existence of a unique best proximity point for generalized φ-weak contraction mapping.

Theorem 3.2. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a generalized φ-weak contraction mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$.*

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = dist(A, B).$$

Again, since $Tx_1 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = dist(A, B).$$

Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = dist(A, B), \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Since (A, B) has the P-property, from (3.3), we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n), \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

By the definition of T and (3.4), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le d(x_{n-1}, x_n) - \varphi(d(Tx_{n-1}, Tx_n)) \\ &\le d(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N}. \end{aligned}$$

Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is monotone nonincresing and bounded. Hence it converges. If we set $\lambda_n = d(x_n, x_{n+1})$ and L be the limit of λ_n, *i.e.*,

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = L \ge 0.$$

Now, we claim that $L = 0$. Suppose to the contrary that $L > 0$. Since $\{\lambda_n\}$ is nonincreasing sequence, *i.e.*,

$$\lambda_n \ge \lambda_{n+1} \ge \cdots \ge L > 0, \quad \forall\, n \in \mathbb{N}$$

and φ is nondecreasing, we obtain

$$\varphi(\lambda_n) \ge \varphi(L) > 0, \quad \forall\, n \in \mathbb{N}. \tag{3.5}$$

From the inequality

$$\begin{aligned} \lambda_n = d(x_n, x_{n+1}) &\le d(x_{n-1}, x) - \varphi(d(Tx_{n-1}, Tx_n)) \\ &= \lambda_{n-1} - \varphi(d(Tx_{n-1}, Tx_n)), \end{aligned}$$

(3.4) and (3.5), we have

$$\begin{aligned} \lambda_n &\le \lambda_{n-1} - \varphi(d(x_n, x_{n+1})) = \lambda_{n-1} - \varphi(\lambda_n) \\ &\le \lambda_{n-1} - \varphi(L), \quad \forall\, n \in \mathbb{N}. \end{aligned}$$

Since φ is continuous, we get $L \le L - \varphi(L)$. That is

$$\varphi(L) \le 0$$

which contradicts condition of φ. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = L = 0.$$

Now we apply Lemma 2.3 and Lemma 2.4 to the set $D = \{(d(x, y), d(Tx, Ty)) : x, y \in A\}$ on Lemma 3.1, one knows that there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is continuous and nondecreasing with

$$\phi(t) < t, \ \forall\, t > 0 \quad \text{and} \quad d(Tx, Ty) \le \phi(d(x, y)), \ \forall\, x, y \in A. \tag{3.6}$$

Thus for a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \le \varepsilon - \phi(\varepsilon), \quad \forall\, n \ge N. \tag{3.7}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.

Denotes the closed ball with center x and radius ε by $B[x, \varepsilon]$, we will claim the following relations.

Claim I. $T(B[x_N, \varepsilon] \cap A) \subseteq B[Tx_{N-1}, \varepsilon]$.

Let $x \in B[x_N, \varepsilon] \cap A$, *i.e.*, $d(x_N, x) \leq \varepsilon$, from (3.4), (3.6) and (3.7), then

$$\begin{aligned} d(Tx, Tx_{N-1}) &\leq d(Tx, Tx_N) + d(Tx_N, Tx_{N-1}) \\ &\leq \phi(d(x, x_N)) + d(x_{N+1}, x_N) \\ &\leq \phi(\varepsilon) + \{\varepsilon - \phi(\varepsilon)\} = \varepsilon, \end{aligned}$$

which implies that $Tx \in B[Tx_{N-1}, \varepsilon]$.

Claim II. $y \in B[Tx_{N-1}, \varepsilon]$ with $d(x, y) = dist(A, B)$ for some $x \in A_0$ implies $x \in B[x_N, \varepsilon] \cap A$.

Let $y \in B[Tx_{N-1}, \varepsilon]$ with $d(x, y) = dist(A, B)$ for some $x \in A_0$. From (3.3), we have $d(x_N, Tx_{N-1}) = dist(A, B)$. Therefore, by using the P-property of (A, B), we obtain that

$$d(x_N, x) = d(Tx_{N-1}, y) \leq \varepsilon.$$

Hence Claim II holds.

From (3.7), it is clear that

$$x_{N+1} \in B[x_N, \varepsilon] \cap A.$$

And by Claim I, we have $Tx_{N+1} \in B[Tx_{N-1}, \varepsilon]$. From (3.3), $d(x_{N+2}, Tx_{N+1}) = dist(A, B)$ with $x_{N+2} \in A_0$. From Claim II,

$$x_{N+2} \in B[x_N, \varepsilon] \cap A.$$

Again by Claim I, $Tx_{N+2} \in B[Tx_{N-1}, \varepsilon]$ and by (3.3), $d(x_{N+3}, Tx_{N+2}) = dist(A, B)$ with $x_{N+3} \in A_0$. Again by Claim II,

$$x_{N+3} \in B[x_N, \varepsilon] \cap A.$$

Continuing this process, we can conclude that

$$x_{N+m} \in B[x_N, \varepsilon] \cap A, \quad \forall\, m \in \mathbb{N},$$

i.e., $d(x_N, x_{N+m}) \leq \varepsilon$. Hence the sequence $\{x_n\}$ is a Cauchy sequence. Since A is closed subset of the complete metric space (X, d), there exists an element $x^* \in A$ such that $\lim_{n\to\infty} x_n = x^*$. By the definition of T, we have $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in A$ which implies that T is continuous in A. Therefore we obtain

$$\lim_{n\to\infty} Tx_n = Tx^*.$$

Also, from the continuity of the distance function d, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = d(x^*, Tx^*).$$

Equation (3.3), it means that the sequence $\{d(x_{n+1}, Tx_n)\}$ is a constant sequence with the value $dist(A, B)$. Hence

$$d(x^*, Tx^*) = dist(A, B),$$

i.e., x^* is a best proximity point of T.

Finally, we show that x^* is unique best proximity point of T. Suppose that x_1 and x_2 are two best proximity points of T in A with $x_1 \neq x_2$. Since x_1 and x_2 are two best proximity points of T, we have

$$d(x_1, Tx_1) = dist(A, B) = d(x_2, Tx_2).$$

By the P-property of (A, B), we obtain

$$d(x_1, x_2) = d(Tx_1, Tx_2).$$

Since x_1 and x_2 are distinct elements in A, one can have

$$\varphi(d(x_1, x_2)) > 0. \tag{3.8}$$

From the definition of T and (3.8),

$$\begin{aligned} d(x_1, x_2) = d(Tx_1, Tx_2) &\leq d(x_1, x_2) - \varphi(d(Tx_1, Tx_2)) \\ &= d(x_1, x_2) - \varphi(d(x_1, x_2)) \\ &< d(x_1, x_2). \end{aligned}$$

This is a contradiction. Therefore the uniqueness of the best proximity point follows. □

The following example illustrates that Theorem 3.2 holds.

Example 3.3. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Let $A = [-1, 1]$ and $B = [0, 2]$ be two subsets of (X, d). Define $T : A \to B$ by

$$Tx = \frac{2}{5}x$$

for each $x \in A$. Define $\varphi(t) : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{4}{3}t.$$

Then, by Example 1.7, T satisfies (1.2). It is easy to check that A and B are closed subsets of complete metric space (X, d), $\emptyset \neq A_0 = [0, 1] = B_0$ and $T(A_0) = [0, \frac{2}{5}] \subseteq [0, 1] = B_0$. Moreover (A, B) has the P-property. Indeed, let $d(x_1, Tx_1) = dist(A, B) = d(x_2, Tx_2)$. By

$$0 = dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

we have $x_1 = Tx_1$ and $x_2 = Tx_2$. Thus $d(x_1, x_2) = d(Tx_1, Tx_2)$. Hence (A, B) has the P-property. Therefore all the assumption of Theorem 3.2 hold and note that $x^* = 0$ is the unique best proximity point.

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Explicit viscosity rule of nonexpansive mappings in CAT(0) spaces

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Abstract

In this paper, we present a explicit viscosity technique of nonexpansive mappings in the framework of CAT(0) spaces. The strong convergence theorem of the proposed technique is proved under certain assumptions imposed on the sequence of parameters. The results presented in this paper extend and improve some recent announced in the current literature.

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1 Introduction

The study of spaces of nonpositive curvature originated with the discovery of hyperbolic spaces, and flourished by pioneering works of Hadamard and Cartan, etc. in the first decades of the twentieth century. The idea of nonpositive curvature geodesic metric spaces could be traced back to the work of Busemann and Alexandrov, etc. in the 50's. Later on Gromov [9] restated some features of global Riemannian geometry solely based on the so-called CAT(0) inequality. For through discussion of CAT(0) spaces and of fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [5].

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As we know, iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [1–3, 7, 14–17] and the references therein. One of the difficulties in carrying out results from Banach space to complete CAT(0) space setting lies in the heavy use of the linear structure of the Banach spaces. Berg and Nikolaev [4] introduce the notion of an inner product-like notion (quasi-linearization) in complete CAT(0) spaces to resolve these difficulties.

Fixed-point theory in CAT(0) spaces was first studied by Kirk [10, 11]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed.

In 2000, Moudaf's [13] introduce viscosity approximation methods as following

Theorem 1.1. *Let C be a nonempty closed convex subset of the real Hilbert space X. Let T be a nonexpansive mapping of C into itself such that $Fix(T)$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in [0, 1)$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \frac{\gamma_n}{1+\gamma_n} f(x_n) + \frac{1}{1+\gamma_n} T(x_n), \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(1) $\lim_{n\to\infty} \gamma_n = 0$,

(2) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(3) $\sum_{n=0}^{\infty} |\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the variational inequality

$$\langle x - f(x), x - y\rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $P_{Fix(T)}f(x^*) = x^*$.

Shi and Chen [15] studied the convergence theorems of the following Moudaf's viscosity iterations for a nonexpansive mapping in CAT(0) spaces.

$$x_{n+1} = tf(x_n) \oplus (1-t)T(x_n), \tag{1.1}$$

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n). \tag{1.2}$$

They proved that $\{x_n\}$ defned by (1.1) and $\{x_n\}$ defined by (1.2) converged strongly to a fxed point of T in the framework of CAT(0) space.

Zhao et al. [18] applied viscosity approximation methods for the implicit midpoint rule for nonexpansive mappings

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

Motivated and inspired by the idea of Kwun et al. [12], in this paper, we extend and study the explicit viscosity rules of nonexpansive mappings in CAT(0) spaces

$$\begin{cases} x_{n+1} = (1-\alpha_n)f(x_n) \oplus \alpha_n T(y_n), \\ y_n = (1-\beta_n)x_n \oplus \beta_n T(x_n). \end{cases} \tag{1.3}$$

2 Preliminaries

Let(X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0)= x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2,and x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}d(x_i, x_j) = d(x_i, x_j)$for $i, j = 1, 2, 3$.
A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) = d_{\mathbb{E}^2}(\overline{x}, \overline{y}) \tag{2.1}$$

Let $x, y \in X$ and by the Lemma 2.1(iv) of [8] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y). \tag{2.2}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique fixed point z satisfying the above equation.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1. ([8]) *Let X be a $CAT(0)$ spaces.*

(a) For any $x, y, z \in X$ and $t \in [0, 1]$,

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z). \tag{2.3}$$

(b) For any $x, y, z \in X$ and $t \in [0, 1]$,

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)^2 d(x, z) + td^2(y, z) - t(1 - t)d^2(x, y). \tag{2.4}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [5]). If x, y_1, y_2 are points of a CAT(0) spaces and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denoted by $\frac{(y_1 \oplus y_2)}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.5}$$

This inequality is the (CN) inequality of Bruhat and Tits [6]. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (cf. [5], page 163).

Definition 2.2. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *nonexpensive* if

$$d(T(x), T(y)) \leq d(x, y), \quad x, y \in C$$

Definition 2.3. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *contraction* if

$$d(T(x), T(y)) \leq \theta d(x, y), \quad x, y \in C, \, \theta \in [0, 1)$$

Berg and Nikolaev [4] introduce the concept of quasi-linearization as follow. Let us denote the pair $(a, b) \in X \times X$ by the $\overrightarrow{ab}$ and call it a vector. Then, quasi-linearization is defined as a mapping

$$\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$$

defined as

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \tag{2.6}$$

it is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(a, c)$$

for all $a, b, c, d \in X$. It is well-known [4] that a geodesically connected metric space is a CAT(0) space of and only if it satisfy the Cauchy-Schwarz inequality.
Let C be a non-empty closed convex subset of a complete CAT(0) space X. The metric projection $P_c : X \to C$ is defined by

$$u = P_c(x) \Longleftrightarrow \inf\{d(y, x) : y \in C\}, \quad \forall x \in X$$

Definition 2.4. Let $P_c : X \to C$ is called the *metric projection* if for every $x \in X$ there exist a unique nearest point in C, denoted by $P_c x$, such that

$$d(x, P_c x) \leq d(x, y), \quad y \in C$$

The following theorem gives you the conditions for a projection mapping to be nonexpensive.

Theorem 2.5. *Let C be a non-empty closed convex subset of a real $CAT(0)$ space X and $P_c : X \to X$ a metric projection. Then*
(1) $d(P_c x, P_c y) \leq \langle \overrightarrow{xy}, \overrightarrow{P_c x P_c y} \rangle$ *for all $x, y \in X$,*
(2) P_c *is nonexpensive mapping, that is, $d(x, p_c x) \leq d(x, y)$ for all $y \in C$,*
(3) $\langle \overrightarrow{xP_c x}, \overrightarrow{yP_c y} \rangle \leq 0$ *for all $x \in X$ and $y \in C$.*

Further if, in addition, C is bounded, then $Fix(T)$ is nonempty.
The following lemmas are very useful for proving our main results:

Lemma 2.6. (*The demiclosedness principle*) *Let C be a nonempty closed convex subset of the real $CAT(0)$ space X and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \text{and} \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$. Here $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequences.

Lemma 2.7. *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \forall n \geq 0$, *where* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence with*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n \to 0$.

3 The main result

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X*. Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \phi$ *and* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0,1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be a sequence generated by* (1.3), *where*$\{\alpha_n\}$ *and* $\{\beta_n\}$ *are the sequence in* $(0,1)$ *satisfying the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 1$ *and* $\lim_{n\to\infty} \beta_n = 1$,

(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty \forall n \geq 0$,

(4) $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the nonexpansive mapping* T *which is also the unique solution of the variational inequality*

$$\langle \overrightarrow{xf(x)}, \overrightarrow{yx} \rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* *is the unique fixed point of the contraction* $P_{Fix(T)}f$, *that is,* $P_{Fix(T)}f(x^*) = x^*$.

Proof. We divide the proof into the following five steps.

Step 1. First, we show that x_n is bounded. Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)f(x_n) \oplus \alpha_n T(y_n), p) \\ &\leq (1-\alpha_n)d(f(x_n), p) + \alpha_n d(T(y_n), p) \\ &\leq (1-\alpha_n)d(f(x_n), f(p)) + (1-\alpha_n)d(f(p), p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)\theta d(x_n, p) + (1-\alpha_n)d(f(p), p) + \alpha_n d(y_n, p). \end{aligned} \tag{3.1}$$

Now consider

$$\begin{aligned} d(y_n, p) &= ((1-\beta_n)x_n \oplus \beta_n T(x_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(T(x_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

Using this in (3.1) we have

$$\begin{aligned} d(x_{n+1},p) &\leq (1-\alpha_n)\theta d(x_n,p) + (1-\alpha_n)d(f(p),p) + \alpha_n d(x_n,p) \\ &= [(1-\alpha_n)\theta + \alpha_n]d(x_n,p) + (1-\alpha_n)d(f(p),p) \\ &= [1-1+\alpha+(1-\alpha_n)\theta]d(x_n,p) + (1-\alpha_n)d(f(p),p) \\ &= [1-(1-\alpha)+(1-\alpha_n)\theta]d(x_n,p) + (1-\alpha_n)d(f(p),p) \\ &= [1-(1-\alpha_n)(1-\theta)]d(x_n,p) + (1-\alpha_n)(1-\theta)\left(\frac{1}{1-\theta}d(f(p),p)\right), \end{aligned}$$

thus we have

$$d(x_{n+1},p) \leq \max\left\{d(x_n,p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\},$$

similarly

$$d(x_n,p) \leq \max\left\{d(x_{n-1},p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\}.$$

From this

$$\begin{aligned} &d(x_{n+1},p) \\ &\leq \max\left\{d(x_n,p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\} \\ &\leq \max\left\{d(x_{n-1},p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\} \\ &\;\;\vdots \\ &\leq \max\left\{d(x_0,p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\}, \end{aligned}$$

which shows that $\{x_n\}$ is bounded. From this we deduce immediately that $\{f(x_n)\}$, $\{T(x_n)\}$ are bounded.

Step 2. Next, we want to prove that $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$.

For this consider

$$\begin{aligned} &d(x_{n+1},x_n) \\ &= d((1-\alpha_n)f(x_n) \oplus \alpha_n T(y_n), (1-\alpha_{n-1})f(x_{n-1}) \oplus \alpha_{n-1}T(y_{n-1})) \\ &\leq (1-\alpha_n)\theta d(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(T(y_{n-1}), f(x_{n-1})) + \alpha_n d(y_n,y_{n-1}). \end{aligned} \tag{3.2}$$

Now consider

$$\begin{aligned} &d(y_n,y_{n-1}) \\ &= d((1-\beta_n)x_n \oplus \beta_n T(x_n), (1-\beta_{n-1})x_{n-1} \oplus \beta_{n-1}T(x_{n-1})) \\ &\leq (1-\beta_n)d(x_n,x_{n-1}) + |\beta_n - \beta_{n-1}|d(T(x_{n-1}),x_{n-1}) + \beta_n d(x_n,x_{n-1}) \\ &\leq d(x_n,x_{n-1}) + |\beta_n - \beta_{n-1}|d(T(x_{n-1}),x_{n-1}). \end{aligned}$$

Using this in (3.2) we get

$$\begin{aligned} &d(x_{n+1},x_n) \\ &\leq (1-\alpha_n)\theta d(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(T(y_{n-1}), f(x_{n-1})) \\ &\quad + \alpha_n d(x_n,x_{n-1}) + \alpha_n|\beta_n - \beta_{n-1}|d(T(x_{n-1}),x_{n-1}) \\ &= [(1-\alpha_n)\theta + \alpha_n]d(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(T(y_{n-1}), f(x_{n-1})) \\ &\quad + \alpha_n|\beta_n - \beta_{n-1}|d(T(x_{n-1}),x_{n-1}). \end{aligned}$$

Let $\lambda_n = (1-\alpha_n)$ so $\lambda_n \in (0,1)$, since $\alpha_n \in (0,1)$ $\sum_{n=0}^{\infty} \lambda = \infty$, $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}|$. By using Lemma 2.7, we get $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

STEP 3. Now, we want to prove that$\lim_{n\to\infty} d(x_n, T(y_n)) \to 0$

$$\begin{aligned} d(x_n, T(y_n)) &\le d(x_n, T(x_n)) + d(T(x_n), T(y_n)) \\ &\le d(x_n, T(x_n)) + d(x_n, y_n) \\ &= d(x_n, T(x_n)) + d(x_n, (1-\beta_n)x_n \oplus \beta_n T(x_n)) \\ &\le d(x_n, T(x_n)) + \beta_n d(x_n, T(x_n)) \\ &\le (1+\beta_n) d(x_n, T(x_n)) \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

STEP 4. In this step, we claim that $\limsup_{n\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle \le 0$, where $x^* = P_{F(T)} f(x^*)$.

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and Lemma 2.6 we have $p = T(p)$. This together with the property of the metric projection implies that

$$\lim_{n\to\infty} \sup \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle = \lim_{n\to\infty} \sup \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_{n_i}} \rangle = \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* p} \rangle \le 0.$$

STEP 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$. Here again $x^* \in Fix(T)$ is the unique fixed point of the contraction $P_{Fix(T)} f$. Consider

$$\begin{aligned} & d^2(x_{n+1}, x^*) \\ &= d^2((1-\alpha_n) f(x_n) \oplus \alpha_n T(y_n), x^*) \\ &= (1-\alpha_n)_n^2 d^2(f(x_n), x^*) + (1-\alpha_n) d^2(T(x_n), x^*) \\ &\quad + 2\alpha_n (1-\alpha_n) \langle \overrightarrow{f(x_n) x^*}, \overrightarrow{T(y_n) x^*} \rangle \\ &\le \alpha_n^2 d^2(y_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{f(x_n) f(x^*)}, \overrightarrow{T(y_n) x^*} \rangle \\ &\quad + 2\alpha_n (1-\alpha_n) \langle \overrightarrow{f(x^*) x^*}, \overrightarrow{T(y_n) x^*} \rangle \\ &\le \alpha_n^2 d^2(y_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n) d(f(x_n), f(x^*)) d(T(y_n), x^*) \\ &\quad + 2\alpha_n (1-\alpha_n) \langle \overrightarrow{f(x^*) x^*}, \overrightarrow{T(y_n) x^*} \rangle \\ &\le \alpha_n^2 d^2(y_n, x^*) + 2\alpha_n(1-\alpha_n)\theta d(x_n, x^*) d(y_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\ &\quad + 2\alpha_n (1-\alpha_n) \langle \overrightarrow{f(x^*) x^*}, \overrightarrow{T(y_n) x^*} \rangle, \end{aligned} \tag{3.3}$$

now consider

$$\begin{aligned} d(y_n, x^*) &= d((1-\beta_n)x_n \oplus \beta_n T(x_n), x^*) \\ &\le (1-\beta_n) d(x_n, x^*) + \beta_n d(T(x_n), x^*) \\ &\le (1-\beta_n) d(x_n, x^*) + \beta_n d(x_n, x^*) \\ &\le d(x_n, x^*), \end{aligned} \tag{3.4}$$

using (3.2) in (3.3) we get

$$\begin{aligned}
&d^2(x_{n+1}, x^*) \\
&\leq \alpha_n^2 d^2(x_n, x^*) + 2\alpha_n(1-\alpha_n)\theta d(x_n, x^*) d(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\leq \alpha_n^2 d^2(x_n, x^*) + 2\alpha_n(1-\alpha_n)\theta d(x_n, x^*) d(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\leq [\alpha_n^2 + 2\alpha_n(1-\alpha_n)\theta] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^* \rangle.
\end{aligned} \tag{3.5}$$

Note that $\alpha_n\theta < \alpha_n$ since $\alpha_n \in (0,1)$ and $\theta \in [0,1)$

$$2\alpha_n\theta < 2\alpha_n,$$

which implies that

$$\alpha_n^2 + 2\alpha_n\theta(1-\alpha_n) < \alpha_n^2 + 2\alpha_n(1-\alpha_n),$$

therefore, we have

$$\begin{aligned}
&d^2(x_{n+1}, x^*) \\
&\leq [\alpha_n^2 + 2\alpha_n(1-\alpha_n)] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\leq [2\alpha_n - \alpha_n^2)] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\leq 2\alpha_n d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\leq 2[1-(1-\alpha_n)] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle,
\end{aligned} \tag{3.6}$$

as by $\lim_{n\to\infty} \alpha_n = 1$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \sup \frac{(1-\alpha_n)^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle}{(1-\alpha_n)} \\
&= \lim_{n\to\infty} \sup[(1-\alpha_n) d^2(f(x_n), x^*) + 2\alpha_n \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle] \\
&\leq 0.
\end{aligned} \tag{3.7}$$

From (3.6), (3.7), and Lemma 2.7 we have

$$\lim_{n\to\infty} d^2(x_{n+1}, x_n) = 0,$$

which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

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The generalized viscosity implicit rules of asymptotically nonexpansive mappings in CAT(0) spaces

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Abstract

In this paper, we establish the generalized viscosity implicit rules of asymptotically nonexpansive mappings in CAT(0) spaces. The strong convergence theorems of the implicit rules proposed are proved under certain assumptions imposed on the control parameters. The results presented in this paper improve and extend some recent corresponding results announced.

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1 Introduction

The study of spaces of nonpositive curvature originated with the discovery of hyperbolic spaces, and flourished by pioneering works of Hadamard and Cartan, etc. in the first decades of the twentieth century. The idea of nonpositive curvature geodesic metric spaces could be traced back to the work of Busemann and Alexandrov, etc. in the 50's. Later on Gromov [11] restated some features of global Riemannian geometry solely based on the so-called CAT(0) inequality. For through discussion of CAT(0) spaces and of fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [6].

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As we know, iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [1–4, 8, 9, 16, 18–21] and the references therein. One of the difficulties in carrying out results from Banach space to complete CAT(0) space setting lies in the heavy use of the linear structure of the Banach spaces. Berg and Nikolaev [5] introduce the notion of an inner product-like notion(quasilinearization) in complete CAT(0) spaces to resolve these difficulties.

Fixed-point theory in CAT(0) spaces was frsc studied by Kirk [13,14]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fxed point. Since then, the fxed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed.

In 2000, Moudaf's [15] introduce viscosity approximation methods as following

Theorem 1.1. *Let C be a nonempty closed convex subset of the real Hilbert space X. Let T be a nonexpansive mapping of C into itself such that $Fix(T) = \{x : T(x) = x\}$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in [0, 1)$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \frac{\gamma_n}{1+\gamma_n} f(x_n) + \frac{1}{1+\gamma_n} T(x_n), \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(1) $\lim_{n\to\infty} \gamma_n = 0$,

(2) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(3) $\sum_{n=0}^{\infty} \left| \frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n} \right| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the variational inequality

$$\langle x - f(x), x - y\rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)} f$, that is, $P_{Fix(T)} f(x^*) = x^*$.

Shi and Chen [17] studied the convergence theorems of the following Moudaf's viscosity iterations for a nonexpansive mapping in CAT(0) spaces.

$$x_{n+1} = tf(x_n) \oplus (1 - t)T(x_n), \tag{1.1}$$

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n). \tag{1.2}$$

They proved that $\{x_n\}$ defned by (1.1) and $\{x_n\}$ defned by (1.2) converged strongly to a fxed point of T in the framework of CAT(0) space.

Zhao et al. [22] applied viscosity approximation methods for the implicit midpoint rule for nonexpansive mappings

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \forall n \geq 0.$$

Motivated by He et al. [12], in this paper, we study the generalized viscosity implicit rules of asymptotically nonexpansive mappings in the framework of CAT(0) spaces.

More precisely, we consider the following implicit iterative algorithm

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}) \tag{1.3}$$

under suitable conditions, we proved that the sequence $\{x_n\}$ converge strongly to a fixed point of the asymptotically nonexpansive mapping T.

2 Preliminaries

Let(X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0)= x$, $c(l) = y$, and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2,and x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3$ in (X,d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2} d(x_i, x_j) = d(x_i, x_j)$for $i, j = 1, 2, 3$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the *$CAT(0)$ inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) = d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$ and by the Lemma 2.1(iv) of [10] for each $t \in [0,1]$, there exist a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y), \quad d(y,z) = (1-t)d(x,y). \tag{2.2}$$

From now on, we will use the notation $(1-t)x \oplus ty$ for the unique fixed point z satisfying the above equation.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1. ([10]) *Let X be a $CAT(0)$ spaces.*

(*a*) *For any $x, y, z \in X$ and $t \in [0,1]$,*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z). \tag{2.3}$$

(*b*) *For any $x, y, z \in X$ and $t \in [0,1]$,*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)^2 d(x,z) + td^2(y,z) - t(1-t)d^2(x,y). \tag{2.4}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [6]). If x, y_1, y_2 are points of a CAT(0) spaces and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \le \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.5}$$

This inequality is the (CN) inequality of Bruhat and Tits [7]. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (cf. [6], page 163).

Definition 2.2. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *nonexpensive* if

$$d(T(x), T(y)) \le d(x, y), \quad x, y \in C.$$

Definition 2.3. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *contraction* if

$$d(T(x), T(y)) \le \theta d(x, y), \quad x, y \in C\, \theta \in [0, 1).$$

Berg and Nikolaev [5] introduce the concept of quasi-linearization as follow. Let us denote the pair $(a, b) \in X \times X$ by the $\overrightarrow{ab}$ and call it a vector. Then, quasi-linearization is defined as a mapping

$$\langle \cdot, \cdot \rangle : (x \times X) \times (X \times X) \longrightarrow \mathbb{R}$$

defined as

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \tag{2.6}$$

it is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a, b)d(a, c)$$

for all $a, b, c, d \in X$. It is well-known [5] that a geodesically connected metric space is a CAT(0) space of and only if it satisfy the Cauchy-Schwarz inequality.
Let C be a non-empty closed convex subset of a complete CAT(0) space X. The metric projection $P_c : X \to C$ is defined by

$$u = P_c(x) \Longleftrightarrow \inf\{d(y, x) : y \in C\}, \quad \forall x \in X$$

Definition 2.4. Let $P_c : X \to C$ is called the *metric projection* if for every $x \in X$ there exist a unique nearest point in C, denoted by $P_c x$, such that

$$d(x, P_c x) \le d(x, y), \quad y \in C.$$

The following theorem gives you the conditions for a projection mapping to be nonexpensive.

Theorem 2.5. *Let C be a non-empty closed convex subset of a real CAT(0) space X and $P_c : X \to X$ a metric projection. Then*
(1) $d(P_c x, P_c y) \le \langle \overrightarrow{xy}, \overrightarrow{P_c x P_c y} \rangle$ *for all* $x, y \in X$,
(2) P_c *is nonexpensive mapping, that is,* $d(x, p_c x) \le d(x, y)$ *for all* $y \in C$,
(3) $\langle \overrightarrow{xP_c x}, \overrightarrow{yP_c y} \rangle \le 0$ *for all* $x \in X$ *and* $y \in C$.

Definition 2.6. A mapping $T : C \to C$ is called *asymptotically nonexpensive* if there exist a sequence a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x - T^n y) \leq k_n d(x, y), \quad \forall x, y \in C, \, n \geq 1. \tag{2.7}$$

It is well known that if T is an asymptotically nonexpansive, then $Fix(T)$ is always closed and convex. Further if, in addition, C is bounded, then $Fix(T)$ is nonempty.

The following lemmas are very useful for proving our main results:

Lemma 2.7. (*The demiclosedness principle*) *Let C be a nonempty closed convex subset of the real $CAT(0)$ space X and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \textit{and} \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$. Here $\to$ and $\rightharpoonup$) denote strong and weak) convergence, respectively.

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

Lemma 2.8. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \beta_n)a_n + \delta_n, \forall n \geq 0$, where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \beta_n = \infty$,

(2) $\limsup_{n\to\infty} \sup \frac{\delta_n}{\beta_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} a_n \to 0$.

3 The main results

Theorem 3.1. *Let C be a non-empty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be an asymptotically nonexpensive mapping with sequence $\{k_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $Fix(T) \neq \emptyset$. Let $f : C \longrightarrow C$ be a contraction with coefficient $\theta \in [0, 1)$. For arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (1.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in $(0, 1)$ satisfying the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\lim_{n\to\infty} \frac{k_n^2 - 1}{\alpha_n} = 0$,

(3) $0 < \tau < \beta_n < \beta_{n+1} < 1$, *for all* $n \geq 0$,

(4) $\lim_{n\to\infty} d(T^n(x_n), (x_n)) = 0$.

Then $\{x_n\}$ converges strongly to the point $x^* = P_{Fix(T)} f(x^*)$ of the mapping T, which is also the unique solution of the variational inequality

$$\langle \overrightarrow{xf(x)}, \overrightarrow{xy} \rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)} f$, that is, $P_{Fix(T)} f(x^*) = x^*$.

Proof. We have divided the proof into four steps.

Step 1: First, we show that the generalized viscosity implicit rule (**??**) is well-defined

$$S_n(x) = \alpha_n f(x_n) \oplus (1 - \alpha_n) T^n(\beta_n x_n \oplus (1 - \beta_n) x_{n+1}).$$

Consider

$$\begin{aligned}
&d(S_n(x), S_n(y))\\
&= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x),\\
&\quad \alpha_n f(y_n) \oplus (1-\alpha_n)T^n(\beta_n y_n \oplus (1-\beta_n)y))\\
&= (1-\alpha_n)d(T^n(\beta_n x_n \oplus (1-\beta_n)x), T^n(\beta_n y_n \oplus (1-\beta_n)y))\\
&\le (1-\alpha_n)k_n(1-\beta_n)d(x,y).
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{k_n^2-1}{\alpha_n} = 0$, or $n \ge 0$. We may assume that $(1-\alpha_n)k_n(1-\beta_n) \le 1-\tau$ for all $n \ge 0$. This implies that S_n is a contraction for each n. Therefore there exists a unique fixed point for S_n by contraction principle, which also implies that (1.3) is well-defined.

Step 2: Now, we show that the sequence $\{x_n\}$ is bounded. Indeed take $p \in Fix(T)$ arbitrary, we have

$$\begin{aligned}
&d(x_{n+1}, p)\\
&= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p)\\
&\le \alpha_n d((f(x_n), p) + (1-\alpha_n)d((\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p)\\
&\le \alpha_n d((f(x_n), f(p)) + \alpha_n d((f(p), p) + (1-\alpha_n)k_n d((\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p)\\
&\le \alpha_n \theta d(x_n, p) + \alpha_n d((f(p), p) + (1-\alpha_n)k_n\beta_n d(x_n, p)\\
&\quad + (1-\beta_n)k_n(1-\beta_n)d(x_{n+1}, p)\\
&\le (\alpha_n\theta + (1-\alpha_n)k_n\beta_n)d(x_n, p) + \alpha_n d((f(p), p)\\
&\quad + (1-\beta_n)k_n(1-\beta_n)d(x_{n+1}, p),
\end{aligned}$$

it follows that

$$\begin{aligned}
&[1-(1-\alpha_n)k_n(1-\beta_n)]d(x_{n+1}, p)\\
&= (\alpha_n\theta + (1-\alpha_n)k_n\beta_n)d(x_n, p) + \alpha_n d((f(p), p).
\end{aligned} \tag{3.1}$$

Since $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in $(0,1)$

$$\liminf_{n\to\infty}(1-\alpha_n)k_n(1-\beta_n) \le 1$$

for any given positive number ϵ, $0 < \epsilon < 1-\theta$, there exists a sufficient large positive integer n_0, such that for any $n > n_0$, we have

$$k_n^2 - 1 \le \beta_n \epsilon \alpha_n$$

and

$$k_n - 1 \le \frac{k_n+1}{\beta_n}(k_n-1) \le \frac{k_n^2-1}{\beta_n} \le \epsilon\alpha_n.$$

Moreover, by (3.1)

$$\begin{aligned}
d(x_{n+1},p) &= \frac{\alpha_n\theta+(1-\alpha_n)k_n\beta_n}{1-(1-\alpha_n)k_n(1-\beta_n)}d(x_n,p)+\frac{\alpha_n}{1-(1-\alpha_n)k_n(1-\beta_n)}d(f(p),p)\\
&= \left[1-\frac{\alpha_n(k_n-\theta)-(k_n-1)}{1-(1-\alpha_n)k_n(1-\beta_n)}\right]d(x_n,p)\\
&\quad+\frac{\alpha_n}{1-(1-\alpha_n)k_n(1-\beta_n)}d(f(p),p)\\
&\le \left[1-\frac{\alpha_n(k_n-\theta-\epsilon)}{1-(1-\alpha_n)k_n(1-\beta_n)}\right]d(x_n,p)\\
&\quad+\frac{\alpha_n(k_n-\theta-\epsilon)}{1-(1-\alpha_n)k_n(1-\beta_n)}\left(\frac{1}{(k_n-\theta-\epsilon)}d(f(p),p)\right)\\
&\le \max\left\{d(x_n,p),\frac{1}{k_n-\theta-\epsilon}d(f(p),p)\right\}\\
&\le \max\left\{d(x_n,p),\frac{1}{1-\theta-\epsilon}d(f(p),p)\right\}.
\end{aligned}$$

By applying induction, we obtain

$$d(x_{n+1},p)\le \max\left\{d(x_0,p),\frac{1}{1-\theta-\epsilon}d(f(p),p)\right\}.$$

Hence, we conclude that $\{x_n\}$ is bounded. Consequently, we deduce immediately from it that $\{f(x_n)\}$ and $\{T^n(\beta_n x_n\oplus(1-\beta_n)x_{n+1}\}$ are bounded.

STEP 3: Now, we prove that $\lim_{n\to\infty} d(x_{n+1},x_n)=0$

$$\begin{aligned}
d(x_{n+1},x_n) &\le d(x_{n+1},T^nx_n)+d(T^nx_n,x_n)\\
&= d(\alpha_n f(x_n)\oplus(1-\alpha_n)T^n(\beta_n x_n\oplus(1-\beta_n)x_{n+1}),T^nx_n)+d(T^nx_n,x_n)\\
&\le \alpha_n d((f(x_n),T^nx_n)+(1-\alpha_n)d(T^n(x_n(\beta_n x_n\oplus(1-\beta_n)x_{n+1})),T^nx_n)\\
&\quad+d(T^nx_n,x_n)\\
&\le \alpha_n d((f(x_n),T^nx_n)+(1-\alpha_n)k_n d((\beta_n x_n\oplus(1-\beta_n)x_{n+1}),x_n)\\
&\quad+d(T^nx_n,x_n)\\
&\le \alpha_n d((f(x_n),T^nx_n)+(1-\alpha_n)k_n(1-\beta_n)d(x_{n+1},x_n)+d(T^nx_n,x_n)\\
&\le \alpha_n M_1+(1-\alpha_n)k_n(1-\beta_n)d(x_{n+1},x_n)+d(T^nx_n,x_n),
\end{aligned}$$

where $M_1=\sup\{d((f(x_n),T^nx_n),n\ge 1\}$ is constant such that

$$1-(1-\alpha_n)k_n(1-\beta_n)d(x_{n+1},x_n)\le \alpha_n M_1+d(T^nx_n,x_n)$$

It gives

$$\begin{aligned}
d(x_{n+1},x_n) &\le \frac{\alpha_n M_1}{1-(1-\alpha_n)k_n(1-\beta_n)}\\
&\quad+\frac{1}{1-(1-\alpha_n)k_n(1-\beta_n)}d(T^nx_n,x_n)
\end{aligned}$$

Since $1-(1-\alpha_n)k_n(1-\beta_n)\ge\tau$ by virtue of the conditions (1) and (4), we have

$$\lim_{n\to\infty} d(x_{n+1},x_n)=0. \tag{3.2}$$

Step 4: Now we show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

$$\begin{aligned}
&d(x_n, T^{n-1}x_n)\\
&= d(\alpha_{n-1}f(x_{n-1}) \oplus (1-\alpha_{n-1})T^{n-1}(\beta_{n-1}x_{n-1} \oplus (1-\beta_{n-1})x_n), T^{n-1}x_n)\\
&\leq \alpha_{n-1}d((f(x_{n-1}), T^{n-1}x_n) + (1-\alpha_{n-1})k_n d((\beta_{n-1}x_{n-1} \oplus (1-\beta_{n-1})x_n), x_n)\\
&\leq \alpha_{n-1}d((f(x_{n-1}), T^{n-1}x_n) + (1-\alpha_{n-1})k_n\beta_{n-1}d(x_n, x_{n-1})\\
&\leq \alpha_{n-1}M_1 + (1-\alpha_{n-1})k_n\beta_{n-1}d(x_n, x_{n-1})
\end{aligned}$$

by condition (1) and (3.2) we have

$$\lim_{n\to\infty} d(x_n, T^{n-1}x_n) = 0.$$

Hence we get

$$\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, Tx_n)\\
&\leq d(x_n, T^n x_n) + k_1 d(T^{n-1}x_n, x_n)\\
&\to 0 \quad (n\to\infty)
\end{aligned} \tag{3.3}$$

Then, it follows from (3.2) and (3.3) that

$$\begin{aligned}
d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}, x_n) &\leq d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}, Tx_n) + d(Tx_n, x_n)\\
&\leq k_n d((\beta_n x_n \oplus (1-\beta_n)x_{n+1}, x_n) + d(Tx_n, x_n)\\
&\leq k_n(1-\beta_n)d(x_{n+1}, x_n) + d(Tx_n, x_n)\\
&\leq k_n d(x_{n+1}, x_n) + d(Tx_n, x_n)\\
&\to 0 \quad (n\to\infty).
\end{aligned}$$

Step 5: In this step, we claim that

$$\limsup_{x\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle \leq 0,$$

where $x^* = P_{Fix(T)}f(x^*)$. Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty} d(x_n, T(x_n) = 0$ and the Lemma 2.7 we have $p = T(p)$. This together, with the property of metric projection implies that

$$\begin{aligned}
\limsup_{x\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle &= \limsup_{x\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_{n_i}} \rangle\\
&= \limsup_{x\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* p} \rangle\\
&\leq 0.
\end{aligned}$$

Step 6: Finally, we show that $x_n \to x^*$ as $n \to \infty$.

Now, we prove that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Now, we again take $x^* \in Fix(T)$ is the unique fixed point of the contraction $P_{Fix(T)}f$. Consider

$$\begin{aligned}
d^2(x_n, x_n) &= d^2(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*)\\
&= \alpha_n^2 d^2(f(x_n), x^*) + (1-\alpha_n^2)d^2(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*)\\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle\\
&\le \alpha_n^2 d^2(f(x_n), x^*) + (1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*)\\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)f(x^*)}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle\\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle\\
&\le (1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*)\\
&\quad + 2\alpha_n(1-\alpha_n)d(f(x_n)f(x^*))d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*) + K_n\\
&\le (1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*)\\
&\quad + 2\theta\alpha_n(1-\alpha_n)k_n d(x_n, x^*)d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*) + K_n,
\end{aligned}$$

where

$$K_n = \alpha_n^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle,$$

it becomes

$$\begin{aligned}
&(1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*)\\
&\quad + 2\theta\alpha_n(1-\alpha_n)k_n d(x_n, x^*)d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*) + K_n + d^2(x_n, x_n)\\
&\ge 0.
\end{aligned}$$

Solving this quadratic inequality for $d((\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*)$ yields

$$\begin{aligned}
&d((\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*)\\
&\ge \frac{1}{2(1-\alpha_n)^2k_n^2}\Big\{ -2\theta\alpha_n(1-\alpha_n)k_n d(x_n, x^*)\\
&\qquad + \sqrt{4\theta^2\alpha_n^2(1-\alpha_n)^2k_n^2 d^2(x_n, x^*) - 4(1-\alpha_n)^2k_n^2(K_n - d^2(x_n, x^*))}\Big\}\\
&= \frac{-\theta\alpha_n d(x_n, x^*) + \sqrt{\theta^2\alpha_n^2 d^2(x_n, x^*) - K_n + d^2(x_{n+1}, x^*)}}{(1-\alpha_n)k_n}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\beta_n d(x_n, x^*) + (1-\beta_n)d(x_{n+1}, x^*)\\
&\ge \frac{-\theta\alpha_n d(x_n, x^*) + \sqrt{\theta^2\alpha_n^2 d^2(x_n, x^*) - K_n + d^2(x_{n+1}, x^*)}}{(1-\alpha_n)k_n},
\end{aligned}$$

namely,

$$\begin{aligned}
&[(1-\alpha_n)k_n\beta_n]d(x_n, x^*) + (1-\alpha_n)k_n(1-\beta_n)d(x_{n+1}, x^*)\\
&\ge \sqrt{\theta^2\alpha_n^2 d^2(x_n, x^*) - K_n + d^2(x_{n+1}, x^*)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\theta^2\alpha_n^2 d^2(x_n,x^*) - K_n + d^2(x_{n+1},x^*)\\
&\le [(1-\alpha_n)k_n\beta_n + \theta\alpha_n]^2 d^2(x_n,x^*)\\
&\quad + (1-\alpha_n)^2k_n^2(1-\beta_n)^2 d^2(x_{n+1},x^*)\\
&\quad + 2[(1-\alpha_n)k_n\beta_n+\theta\alpha_n](1-\alpha_n)k_n(1-\beta_n)d(x_n,x^*)d(x_{n+1},x^*)\\
&\le [(1-\alpha_n)k_n\beta_n + \theta\alpha_n]^2 d^2(x_n,x^*)\\
&\quad + (1-\alpha_n)^2k_n^2(1-\beta_n)^2 d^2(x_{n+1},x^*)\\
&\quad + ((1-\alpha_n)k_n\beta_n+\theta\alpha_n)(1-\alpha_n)k_n(1-\beta_n)(d^2(x_n,x^*)+d^2(x_{n+1},x^*)),
\end{aligned}$$

which is reduced to the inequality

$$\begin{aligned}
&[1-(1-\alpha_n)^2k_n^2(1-\beta_n)^2 - ((1-\alpha_n)k_n\beta_n+\theta\alpha_n)(1-\alpha_n)k_n(1-\beta_n)]d^2(x_{n+1},x^*)\\
&\le [((1-\alpha_n)k_n(1-\beta_n))^2 + (1-\alpha_n)k_n(1-\beta_n)(1-\alpha_n)k_n(1-\beta_n)\\
&\quad - \theta^2\alpha_n^2]d^2(x_n,x^*) + K_n,
\end{aligned}$$

that is,

$$\begin{aligned}
&[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]d^2(x_{n+1},x^*)\\
&\le [((1-\alpha_n)k_n\beta_n+\theta\alpha_n)(k_n+\alpha_n(\theta-k_n)) - \theta^2\alpha_n^2]d^2(x_n,x^*) + K_n
\end{aligned} \tag{3.4}$$

it follows from (3.4) that

$$\begin{aligned}
&d^2(x_{n+1},x^*)\\
&\le \frac{[((1-\alpha_n)k_n\beta_n+\theta\alpha_n)(k_n+\alpha_n(\theta-k_n)) - \theta^2\alpha_n^2]d^2(x_n,x^*)}{[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]}\\
&\quad + \frac{K_n}{[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]}.
\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}
w_n &= \frac{1}{\alpha_n}\left\{1 - \frac{((1-\alpha_n)k_n\beta_n+\theta\alpha_n)(k_n+\alpha_n(\theta-k_n)) - \theta^2\alpha_n^2}{1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n}\right\}\\
&= \frac{1}{\alpha_n}\frac{1-k_n^2-2\alpha_nk_n(\theta-k_n)-\alpha_n^2(\theta-k_n)-\theta^2\alpha_n^2}{1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n}\\
&\le \frac{-\beta_n\epsilon-2k_n(\theta-k_n)-\alpha_n(\theta-k_n)^2+\theta^2\alpha_n}{1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n}
\end{aligned}$$

since $0<\epsilon<1-\theta$ and the sequence $\{\beta_n\}$ satisfies $0<\tau\le\beta_n\le\beta_{n+1}<1$ for all $n\ge 0$ and $\lim_{n\to\infty}\beta_n$ exists, assume that

$$\lim_{n\to\infty}\beta_n = \beta^* > 0.$$

Then

$$\lim_{n\to\infty} w_n \le \frac{(2-\beta^*)(1-\theta)}{\beta^*} > 0.$$

Let $0 < \lambda_1 < \frac{(2-\beta^*)(1-\theta)}{\beta^*}$. Then there exists an sufficiently large integer N_1 such that $w_n > \lambda_1$ for all $n > N_1$. Hence, we have

$$\frac{((1-\alpha_n)k_n\beta_n + \theta\alpha_n)(k_n + \alpha_n(\theta - k_n)) - \theta^2\alpha_n^2}{1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n} \leq 1 - \lambda_1\alpha_n, \quad \forall n \geq N_1. \tag{3.6}$$

It turns out from (3.5) that

$$d^2(x_{n+1}, x^*) \leq (1-\lambda_1\alpha_n)d^2(x_n, x^*) + \frac{K_n}{[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]}. \tag{3.7}$$

From (3.5), $\lim_{n\to\infty}\alpha_n = 0$ and Step 4 we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{K_n}{\alpha_n\lambda_1[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]} \\
&= \limsup_{n\to\infty} \frac{\alpha_n^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle}{\alpha_n\lambda_1[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]} \\
&= \limsup_{n\to\infty} \frac{\alpha_n d^2(f(x_n), x^*) + 2(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle}{\lambda_1[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]} \\
&\leq 0.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) and the Lemma 2.8 we have

$$\lim_{n\to\infty} d(x_{n+1}, x^*) = 0.$$

This implies that $x_n \to x^*$ as $n \to \infty$. This complete the proof. □

The following result is an immediate consequence of the Theorem 3.1.

Theorem 3.2. *Let C be a non-empty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be an nonexpensive mapping with $Fix(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$ and for arbitrary initial point $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), \tag{3.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in $(0,1)$ satisfying the condition of Theorem 3.1.

Then $\{x_n\}$ converges strongly to the point $x^* = P_{Fix(T)}f(x^*)$ of the mapping T, which is also the unique solution of the variational inequality

$$\langle \overrightarrow{xf(x)}, \overrightarrow{xy}\rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $P_{Fix(T)}f(x^*) = x^*$.

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On some sixth-order rational recursive sequences

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Abstract

We study the sixth-order recursive sequences of the form

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}(a_n + b_n x_{n-5}x_n)},$$

where a_n and b_n are sequences of real numbers, via the technique of Lie group analysis. Symmetry generators associated with the group of transformations that map solutions onto themselves are obtained and exact solutions derived. The 'final constraint' when finding the symmetries, is used to split the solution into different categories. The result of this work generalizes a recent work by Elsayed et al.

Keywords Difference equation; Symmetry; Group invariant solutions
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1 Introduction

Among the numerous well-known techniques for solving differential equations, is the powerful Lie symmetry approach. In the nineteenth century, the Norwegian mathematician Sophus Lie [12] developed a systematic algorithm based on the invariance of the ordinary differential equations under a group of transformations (symmetry). In the twentieth century, Maeda [13, 14] demonstrated that this approach can be extended to ordinary difference equations and recently, Hydon [6] used a similar approach to come up with some interesting results. It is now known that Lie's method can be implemented to find symmetries, first integrals (conservation laws) and closed form solutions of difference equations, even in the context of variational equations.

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In this paper, we obtain symmetry generators admitted by the difference equations of the form

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}(a_n + b_n x_{n-5}x_n)}, \tag{1}$$

where a_n and b_n are random sequences, and then proceed to find the solutions in closed form via the invariance of the group of transformations admitted by (2). We first present the solutions in a unified manner and then split them into different categories based on some properties of the 'final constraint'.
This work generalizes the work by Elsayed et. al. [3], where the authors obtained the formulas of the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}(\pm + \pm x_{n-5}x_n)}, \quad n = 0, 1, \ldots, \tag{2}$$

in which the initial conditions $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary non-zero real numbers.
For similar work on the symmetry approach, see [4, 5, 7, 15, 16] and on different methods, see [1, 2, 8, 10, 19].

1.1 Preliminaries

In this section, we shortly present key elements of Lie group analysis of difference equations. For more understanding of the concepts and notation, we refer the reader to [6, 17] where our definitions and most of our notation are taken from.
Let

$$x^\star = X(x; \varepsilon) \tag{3}$$

be a one parameter Lie group of transformations.

Definition 1.1 *An infinitely differentiable function F is an invariant function of the Lie group of point transformation* (3) *if and only if, for any group transformations,*

$$F(x) = F(x^\star). \tag{4}$$

Definition 1.2 *The infinitesimal generator of the one-parameter Lie group of point transformation* (3) *is the operator*

$$X = X(x) = \xi(x) \times \Delta = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{5}$$

where Δ is the gradient operator.

Theorem 1.1 *$F(x)$ is invariant under the Lie group of transformations* (3) *if and only if*

$$XF(x) = 0. \tag{6}$$

Now, consider a general kth-order difference equation

$$u_{n+k} = \omega(n, u_n, u_{n+1}, \dots, u_{n+k-1}) \tag{7}$$

for some smooth function ω. We are seeking a one-parameter Lie group of point transformations

$$n^* = n, \tag{8a}$$
$$u_n^* = u_n + \varepsilon\xi(n, u_n) +)(\varepsilon^2), \tag{8b}$$
$$\vdots$$
$$u_{n+k}^* = u_{n+k} + \varepsilon S^k \xi(n, u_n) +)(\varepsilon^2), \tag{8c}$$

where ξ denotes the characteristic, ε (ε is small enough) is the group parameter and $S : n \mapsto n+1$ stands for the shift forward operator . The symmetry criterion is given by

$$u_{n+k}^* = \omega(n, u_n^*, u_{n+1}^*, \dots, u_{n+k-1}^*), \tag{9}$$

whenever (7) holds, and further the substitution of (8) in (9) yields the linearized symmetry condition:

$$S^k \xi(n, u_n) - X\omega = 0 \tag{10}$$

where X, the corresponding prolonged symmetry operator of the group of transformations (8), is given by

$$X = \xi(n, u_n)\frac{\partial}{\partial u_n} + S\xi(n, u_n)\frac{\partial}{\partial u_{n+1}} + \dots + S^{k-1}\xi(n, u_n)\frac{\partial}{\partial u_{n+k-1}}. \tag{11}$$

The characteristics are obtained by solving the functional equation (10). As simple as (10) may look, its solution is found after a series of steps that require a set of cumbersome calculations.
In this work, we employ the well-known choice of canonical coordinate [9]

$$S_n = \int \frac{du_n}{\xi(n, u_n)} \tag{12}$$

to reduce the order of the difference equation under investigation.

2 Main results

Consider the sixth-order difference equations of the form (2). Let

$$u_{n+6} = \omega = \frac{u_n u_{n+5}}{u_{n+1}(A_n + B_n u_n u_{n+5})}, \tag{13}$$

where A_n and B_n are random sequences, be the forward difference equation equivalent to (2).
The linearized symmetry condition (10) imposed on (13) leads to

$$\begin{aligned} &\xi(n+6,\omega) - \frac{A_n u_n \xi(n+5,u_{n+5})}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} + \frac{u_n u_{n+5}\xi(n+1,u_{n+1})}{u_{n+1}^2(A_n + B_n u_n u_{n+5})} \\ &- \frac{A_n u_{n+5}\xi(n,u_n)}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} = 0. \end{aligned} \tag{14}$$

By the means of the first-order partial differential operator

$$L = \frac{\partial}{\partial u_n} - \frac{\omega_{u_n}}{\omega_{u_{n+5}}}\frac{\partial}{\partial u_{n+5}},$$

we can get rid of the first term in (14). This yields the following:

$$\begin{aligned} &\frac{A_n u_{n+5}\xi'(n+5,u_{n+5})}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} - \frac{A_n u_{n+5}\xi'(n,u_n)}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} - \frac{A_n \xi(n+5,u_{n+5})}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} \\ &+ \frac{A_n u_{n+5}\xi(n,u_n)}{u_n u_{n+1}(A_n + B_n u_n u_{n+5})^2} = 0. \end{aligned} \tag{15}$$

Here, it is important to simplify the equation in order to minimize the number of derivations. Thus, we clear fractions in (15) and divide the resulting equation by $u_n u_{n+5}$ to get

$$\xi'(n+5,u_{n+5}) - \frac{1}{u_{n+5}}\xi(n+5,u_{n+5}) - \xi'(n,u_n) + \frac{1}{u_n}\xi(n,u_n) = 0. \tag{16}$$

Differentiating (16) with respect to u_n, keeping u_{n+5} fixed, leads to

$$\frac{d}{du_n}\left[-\xi'(n,u_n) + \frac{1}{u_n}\xi(n,u_n)\right] = 0. \tag{17}$$

Clearly, the solution of (17) is

$$\xi(n,u_n) = f(n)u_n + g(n)u_n \ln u_n \tag{18}$$

for some arbitrary functions f and g of n. To ease the computation we shall assume that g is zero. Using the expression of the characteristic given in (18), equation (14) becomes

$$B_n f(n+1)u_n u_{n+5} + B_n f(n+6)u_n u_{n+5} - A_n f(n) + A_n f(n+1) - A_n f(n+5) + A_n f(n+6) = 0. \tag{19}$$

which splits into

$$1: \qquad f(n) + f(n+5) = 0 \tag{20}$$

$$u_n u_{n+5}: \qquad f(n+1) + f(n+6) = 0. \tag{21}$$

The system above reduces to the final constraint:

$$f(n) + f(n+5) = 0. \tag{22}$$

Solving (22) for f, we obtain five independent solutions given by $(-1)^n$, $\exp(\pm n\pi/5)$ and $\exp(\pm 3n\pi/5)$. Therefore, the characteristics are

$$\xi_1 = (-1)^n u_n,\ \xi_2 = \beta^n u_n,\ \xi_3 = \bar{\beta}^n u_n,\ \xi_4 = \theta^n u_n,\ \xi_5 = \bar{\theta}^n u_n, \tag{23}$$

and so the prolonged infinitesimal generators admitted by (13) are

$$X_1 = (-1)^n u_n \partial_{u_n} + (-1)^{n+1} u_{n+1}\partial_{u_{n+1}} + (-1)^{n+2} u_{n+2}\partial_{u_{n+2}} + (-1)^{n+3} u_{n+3}\partial_{u_{n+3}} + (-1)^{n+4} u_{n+4}\partial_{u_{n+4}} + (-1)^{n+5} u_{n+5}\partial_{u_{n+5}}, \tag{24a}$$

$$X_2 = \beta^n u_n \partial_{u_n} + \beta^{n+1} u_{n+1}\partial_{u_{n+1}} + \beta^{n+2} u_{n+2}\partial_{u_{n+2}} + \beta^{n+3} u_{n+3}\partial_{u_{n+3}} + \beta^{n+4} u_{n+4}\partial_{u_{n+4}} + \beta^{n+5} u_{n+5}\partial_{u_{n+5}}, \tag{24b}$$

$$X_3 = \bar{\beta}^n u_n \partial_{u_n} + \bar{\beta}^{n+1} u_{n+1}\partial_{u_{n+1}} + \bar{\beta}^{n+2} u_{n+2}\partial_{u_{n+2}} + \bar{\beta}^{n+3} u_{n+3}\partial_{u_{n+3}} + \bar{\beta}^{n+4} u_{n+4}\partial_{u_{n+4}} + \bar{\beta}^{n+5} u_{n+5}\partial_{u_{n+5}}, \tag{24c}$$

$$X_4 = \theta^n u_n \partial_{u_n} + \theta^{n+1} u_{n+1}\partial_{u_{n+1}} + \theta^{n+2} u_{n+2}\partial_{u_{n+2}} + \theta^{n+3} u_{n+3}\partial_{u_{n+3}} + \theta^{n+4} u_{n+4}\partial_{u_{n+4}} + \theta^{n+5} u_{n+5}\partial_{u_{n+5}}, \tag{24d}$$

$$X_5 = \bar{\theta}^n u_n \partial_{u_n} + \bar{\theta}^{n+1} u_{n+1}\partial_{u_{n+1}} + \bar{\theta}^{n+2} u_{n+2}\partial_{u_{n+2}} + \bar{\theta}^{n+3} u_{n+3}\partial_{u_{n+3}} + \bar{\theta}^{n+4} u_{n+4}\partial_{u_{n+4}} + \bar{\theta}^{n+5} u_{n+5}\partial_{u_{n+5}}. \tag{24e}$$

Note that $\beta = \exp(\pi/5)$ and $\theta = \exp(3\pi/5)$ Using the generator X_2, we have the canonical coordinate

$$S_n = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n|. \tag{25}$$

Taking advantage of the form of the relation (22), we construct the invariant function $\tilde{V}_n$

$$\tilde{V}_n = S_n \beta^n + S_{n+5} \beta^{n+5} \tag{26}$$

in view of the fact that

$$X_1 \tilde{V}_n = (-1)^n + (-1)^{n+5} = 0, \tag{27a}$$
$$X_2 \tilde{V}_n = \beta^n + \beta^{n+5} = 0, \tag{27b}$$
$$X_3 \tilde{V}_n = \bar{\beta}^n + \bar{\beta}^{n+5} = 0, \tag{27c}$$
$$X_4 \tilde{V}_n = \theta^n + \theta^{n+5} = 0, \tag{27d}$$
$$\tag{27e}$$

and

$$X_5 \tilde{V}_n = \bar{\theta}^n + \bar{\theta}^{n+5} = 0. \tag{27f}$$

For rational difference equations, it is convenience to use

$$|V_n| = \exp\{-\tilde{V}_n\}, \tag{28}$$

i.e., $V_n = \pm 1/(u_n u_{n+5})$ but we will be using the plus sign. Substituting (28) into equation (13), we reduce it to

$$V_{n+1} = A_n V_n + B_n. \tag{29}$$

We iterate (29) to get its solution in closed form as

$$V_j = V_0 \left(\prod_{k_1=0}^{j-1} A_{k_1} \right) + \sum_{l=0}^{j-1} \left(B_l \prod_{k_2=l+1}^{j-1} A_{k_2} \right). \tag{30}$$

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From (25), (26) and (28), we have

$$
\begin{aligned}
|u_n| =& \exp\left(\beta_n S_n\right) \\
=& \exp\Bigg[(-1)^n c_1 + \beta^n c_2 + \bar{\beta}^n c_3 + \theta^n c_4 + \bar{\theta}^n c_5 - \frac{1}{5}\sum_{k_1=0}^{n-1}(-1)^{n-k_1}|\tilde{V}_{k_1}| \\
& - \frac{1}{5}\sum_{k_2=0}^{n-1}\beta^n\bar{\beta}^{k_2}|\tilde{V}_{k_2}| - \frac{1}{5}\sum_{k_3=0}^{n-1}\bar{\beta}^n\beta^{k_3}|\tilde{V}_{k_3}| - \frac{1}{5}\sum_{k_4=0}^{n-1}\theta^n\bar{\theta}^{k_4}|\tilde{V}_{k_4}| \\
& - \frac{1}{5}\sum_{k_5=0}^{n-1}\bar{\theta}^n\theta^{k_5}|\tilde{V}_{k_5}|\Bigg] \\
=& \exp\Bigg[(-1)^n c_1 + \beta^n c_2 + \bar{\beta}^n c_3 + \theta^n c_4 + \bar{\theta}^n c_5 + \frac{1}{5}\sum_{k_1=0}^{n-1}(-1)^{n-k_1}\ln|V_{k_1}| \\
& + \frac{1}{5}\sum_{k_2=0}^{n-1}\beta^n\bar{\beta}^{k_2}\ln|V_{k_2}| + \frac{1}{5}\sum_{k_3=0}^{n-1}\bar{\beta}^n\beta^{k_3}\ln|V_{k_3}| + \frac{1}{5}\sum_{k_4=0}^{n-1}\theta^n\bar{\theta}^{k_4}\ln|V_{k_4}| \\
& + \frac{1}{5}\sum_{k_5=0}^{n-1}\bar{\theta}^n\theta^{k_5}\ln|\mathcal{V}_{k_5}|\Bigg]
\end{aligned}
$$

$$
= \exp\left[H_n + \frac{1}{5}\sum_{k=0}^{n-1}\left[(-1)^{n-k} + 2\mathrm{Re}(\gamma_1(n,k)+\gamma_2(n,k))\right]\ln|V_k|\right], \quad (31)
$$

where $H_n = (-1)^n c_1 + \beta^n c_2 + \bar{\beta}^n c_3 + \theta^n c_4 + \bar{\theta}^n c_5$, $\gamma_1(n,k) = \beta^n\bar{\beta}^k$ and $\gamma_2(n,k) = \theta^n\bar{\theta}^k$.
The following properties hold:

$$
\begin{aligned}
&\gamma_1(0,1) = \bar{\beta}, \gamma_1(0,3) = \bar{\theta}, \gamma_1(0,5) = -1, \gamma_1(0,7) = \theta, \gamma_1(1,0) = \beta, \\
&\gamma_1(3,0) = \theta, \gamma_1(5,0) = -1, \gamma_1(7,0) = \bar{\theta}, \gamma_1(n+9,k) = \gamma_1(n,k+1), \\
&\gamma_1(n,k+9) = \gamma_1(n+1,k), \gamma_1(10n,k) = \gamma_1(0,k), \gamma_1(n,10k) = \gamma_1(n,0); \\
&\gamma_2(0,1) = \bar{\theta}, \gamma_2(0,3) = \beta, \gamma_2(0,5) = -1, \gamma_2(0,7) = \bar{\beta}, \gamma_2(1,0) = \theta, \\
&\gamma_2(3,0) = \bar{\beta}, \gamma_2(5,0) = -1, \gamma_2(7,0) = \beta, \gamma_2(n+9,k) = \gamma_2(n,k+1), \\
&\gamma_2(n,k+9) = \gamma_2(n+1,k), \gamma_2(10n,k) = \gamma_2(0,k), \gamma_2(n,10k) = \gamma_2(n,0). \quad (32)
\end{aligned}
$$

From the expression of u_n given in (31) and from the above properties (32), it is clear that

$$|u_{10n+j}| = \exp\left(H_j + \frac{1}{5}\sum_{k_1=0}^{10n+j-1}\left[(-1)^k + 2\mathrm{Re}(\gamma_1(0,k)+\gamma_2(0,k))\right]\ln|V_{k_1}|\right). \tag{33}$$

For $j = 0$, we have that

$$\begin{aligned}|u_{10n}| &= \exp(H_0 + \ln|V_0| - \ln|V_5| + \ldots + \ln|V_{10n-10}| - \ln|V_{10n-5}|)\\ &= \exp(H_0)\prod_{s=0}^{n-1}\left|\frac{V_{10s}}{V_{10s+5}}\right|.\end{aligned} \tag{34}$$

By setting $n = 0$ in (31), we get $\exp(H_0) = u_0$ and so

$$|u_{10n}| = |u_0|\prod_{s=0}^{n-1}\left|\frac{V_{10s}}{V_{10s+5}}\right|. \tag{35}$$

It can be shown, using (28), that we need not the absolute value function in (36). Similarly, for any $j = 0, 1, \ldots, 9$, we obtain the following:

$$u_{10n+j} = u_j\prod_{s=0}^{n-1}\frac{V_{10s+j}}{V_{10s+j+5}}. \tag{36}$$

Thus, using (30),

$$u_{10n+j} = u_j\prod_{s=0}^{n-1}\frac{V_0\left(\prod\limits_{k_1=0}^{10s+j-1}A_{k_1}\right) + \sum\limits_{l=0}^{10s+j-1}\left(B_l\prod\limits_{k_2=l+1}^{10s+j-1}A_{k_2}\right)}{V_0\left(\prod\limits_{k_1=0}^{10s+j+4}A_{k_1}\right) + \sum\limits_{l=0}^{10s+j+4}\left(B_l\prod\limits_{k_2=l+1}^{10s+j+4}A_{k_2}\right)}$$

$$= u_j\prod_{s=0}^{n-1}\frac{\left(\prod\limits_{k_1=0}^{10s+j-1}A_{k_1}\right) + u_0u_5\sum\limits_{l=0}^{10s+j-1}\left(B_l\prod\limits_{k_2=l+1}^{10s+j-1}A_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{10s+j+4}A_{k_1}\right) + u_0u_5\sum\limits_{l=0}^{10s+j+4}\left(B_l\prod\limits_{k_2=l+1}^{10s+j+4}A_{k_2}\right)}.$$

Hence, the solution to our equation (2) is

$$x_{10n+j-5} = x_{j-5}\prod_{s=0}^{n-1}\frac{\left(\prod\limits_{k_1=0}^{10s+j-1}a_{k_1}\right) + x_{-5}x_0\sum\limits_{l=0}^{10s+j-1}\left(b_l\prod\limits_{k_2=l+1}^{10s+j-1}a_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{10s+j+4}a_{k_1}\right) + x_{-5}x_0\sum\limits_{l=0}^{10s+j+4}\left(b_l\prod\limits_{k_2=l+1}^{10s+j+4}a_{k_2}\right)} \tag{37}$$

where $j = 0, 1, 2, \ldots, 9$, whenever the denominators do not vanish. In the following section, we turn to the special case where a_n and b_n are constant sequences.

3 The case when a_n and b_n are constant sequences

In this case, let $a_n = a$ and $b_n = b$ where $a, b \in \mathbb{R}$.

3.1 The case $a \neq 1$

Using (37), the solution is given by

$$x_{10n+j-5} = \bar{x}_j \prod_{s=0}^{n-1} \frac{a^{10s+j} + bx_{-5}x_0 \frac{1-a^{10s+j}}{1-a}}{a^{10s+j+5} + bx_{-5}x_0 \frac{1-a^{10s+j+5}}{1-a}}, \tag{38}$$

where $j = 0, 1, 2, 3, \ldots, 9$, $\bar{x}_j$ is defined as

$$\bar{x}_j = \begin{cases} x_{j-5}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}\left(a^{j-5} + x_{-5}x_0 b \frac{1-a^{j-5}}{1-a}\right)}, & 6 \leq j \leq 9, \end{cases}$$

and for all $(j, s) \in \{0, 1, 2, \ldots, 9\} \times \{0, 1, 2, \ldots, n-1\}$,

$$(1-a)a^{10s+j} + bx_{-5}x_0(1 - a^{10s+j}) \neq 0.$$

3.1.1 The case $a = -1$

In this case, we have

$$x_{10n+j-5} = \bar{x}_j \prod_{s=0}^{n-1} \frac{(-1)^j + bx_{-5}x_0 \frac{1-(-1)^j}{2}}{(-1)^{j+1} + bx_{-5}x_0 \frac{1-(-1)^{j+1}}{2}},$$

where

$$\bar{x}_j = \begin{cases} x_{j-5}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}\left((-1)^{j+1} + x_{-5}x_0 b \frac{1-(-1)^{j+1}}{2}\right)}, & 6 \leq j \leq 9. \end{cases}$$

Evaluating the above, we obtain the following solution which, for $b = \pm 1$, appears in [3] (see Theorems 3.1 and 5.1).

$$x_{10n-5} = x_{-5}(-1 + bx_{-5}x_0)^{-n}, \qquad x_{10n-4} = x_{-4}(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n-3} = x_{-3}(-1 + bx_{-5}x_0)^{-n}, \qquad x_{10n-2} = x_{-2}(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n-1} = x_{-1}(-1 + bx_{-5}x_0)^{-n}, \qquad x_{10n} = x_0(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n+1} = \frac{x_{-5}x_0}{x_{-4}(-1 + bx_{-5}x_0)^{n+1}}, \qquad x_{10n+2} = \frac{x_{-5}x_0}{x_{-3}}(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n+3} = \frac{x_{-5}x_0}{x_{-2}(-1 + x_{-5}x_0 b)^{n+1}}, \qquad x_{10n+4} = \frac{x_{-5}x_0}{x_{-1}}(-1 + bx_{-5}x_0)^{n},$$

where $bx_{-5}x_0 \neq 1$.

However, the solution can be written in a more compact form, i.e.,

$$x_{10n-j+5} = \begin{cases} x_{j-5}(-1 + bx_{-5}x_0)^{(-1)^{j+1}n}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}}(-1 + bx_{-5}x_0)^{\frac{1-(-1)^j}{2} + (-1)^{j+1}n}, & 6 \leq j \leq 9; \end{cases}$$

as long as $bx_{-5}x_0 \neq 1$.

3.2 The case $a = 1$

Using (37), the solution, which for $b = \pm 1$ appears in [3] (see Theorems 2.1 and 4.1), is given by

$$x_{10n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{1 + 10sbx_{-5}x_0}{1 + (10s + 5)bx_{-5}x_0}, \quad x_{10n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{1 + (10s + 1)bx_{-5}x_0}{1 + (10s + 6)bx_{-5}x_0},$$

$$x_{10n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1 + (10s + 2)bx_{-5}x_0}{1 + (10s + 7)bx_{-5}x_0}, \quad x_{10n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1 + (10s + 3)bx_{-5}x_0}{1 + (10s + 8)bx_{-5}x_0},$$

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$$x_{10n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1+(10s+4)bx_{-5}x_0}{1+(10s+9)bx_{-5}x_0}, \quad x_{10n} = x_0 \prod_{s=0}^{n-1} \frac{1+(10s+5)bx_{-5}x_0}{1+(10s+10)bx_{-5}x_0},$$

$$x_{10n+1} = \frac{x_{-5}x_0}{x_{-4}(1+bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+6)bx_{-5}x_0}{1+(10s+11)bx_{-5}x_0},$$

$$x_{10n+2} = \frac{x_{-5}x_0}{x_{-3}(1+2bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+7)bx_{-5}x_0}{1+(10s+12)bx_{-5}x_0},$$

$$x_{10n+3} = \frac{x_{-5}x_0}{x_{-2}(1+3bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+8)bx_{-5}x_0}{1+(10s+13)bx_{-5}x_0},$$

$$x_{10n+4} = \frac{x_{-5}x_0}{x_{-1}(1+4bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+9)bx_{-5}x_0}{1+(10s+14)bx_{-5}x_0},$$

where $jbx_{-5}x_0 \neq -1$ for all $j = 5, 6, 7, \ldots, 10n+4$.

More compactly, the solution can be written as

$$x_{10n+j-5} = \begin{cases} x_{j-5} \prod_{s=0}^{n-1} \frac{1+(10s+j)bx_{-5}x_0}{1+(10s+j+5)bx_{-5}x_0}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}(1+b(j-5)x_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+j)bx_{-5}x_0}{1+(10s+j+5)bx_{-5}x_0}, & 6 \leq j \leq 9. \end{cases}$$

4 Conclusion

In this paper, we derived symmetry generators for the difference equations (2) and explicit formulas for the solutions of the equations were obtained. As a recent result, Theorems 2.1, 3.1, 4.1 and 5.1 of Elsayed et al. [3] were generalized.

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