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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
<http://www.msci.memphis.edu/~ganastss/jocaaa>

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Francesco Altomare

Dipartimento di Matematica
Universita' di Bari
Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional
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Ravi P. Agarwal

Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference
Equations, Inequalities

George A. Anastassiou

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, U.S.A
Tel. 901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real
Analysis,
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J. Marshall Ash

Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu

Department of Mathematics and
Computer Sciences,
Cankaya University, Faculty of Art
and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr

Fractional Differential Equations
Nonlinear Analysis, Fractional
Dynamics

Carlo Bardaro

Dipartimento di Matematica e
Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
<http://www.unipg.it/~bardaro/>
Functional Analysis and
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Martin Bohner

Department of Mathematics and
Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential
equations, dynamic equations on
time scale, applications in
economics, finance, biology.

Jerry L. Bona

Department of Mathematics
The University of Illinois at
Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail: bona@math.uic.edu
Partial Differential Equations,
Fluid Dynamics

Luis A. Caffarelli

Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko

Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir

School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman

TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi

Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

J .A. Goldstein

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

H. H. Gonska

Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany

011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef

Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han

Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He

Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann

Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive
Linear Operators)

Xing-Biao Hu

Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim

Department of Mathematics
Kyungnam University
Masan Kyungnam, 631-701, Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
Variational Inequalities, Nonlinear
Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic

Department of Mathematics
University of Rhode Island
Kingston, RI 02881, USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze

Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar

Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra

Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata

Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost
Automorphy

M. Zuhair Nashed

Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama

Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou

Department of Mathematics
National Technical University of
Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Choonkil Park

Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,

Professor of Finance, College of
Business, and Director of
Quantitative Finance Program,
Department of Applied Mathematics &
Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY
11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm

Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems,
Scattering Theory, Operator Theory,
Theoretical Numerical Analysis,
Wave Propagation, Signal Processing
and Tomography

Tomasz Rychlik

Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics,
Probabilistic Inequalities

Boris Shekhtman

Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach
spaces, Classical Analysis

T. E. Simos

Department of Computer
Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.

Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava

Department of Mathematics and
Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-
6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,
Fractional Calculus and Appl.,
Integral Equations and Transforms,
Higher Transcendental Functions and
Appl., q-Series and q-Polynomials,
Analytic Number Th.

I. P. Stavroulakis

Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche

Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-
rostock.de
Numerical Fourier Analysis, Fourier
Analysis, Harmonic Analysis, Signal
Analysis, Spectral Methods,
Wavelets, Splines, Approximation
Theory

Roberto Triggiani

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, rtrggiani@memphis.edu

Juan J. Trujillo

University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

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Ram Verma

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1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis,
Numerical Analysis, Variational
Inequalities, Optimization Theory,
Computational Mathematics, Operator
Theory

Xiang Ming Yu

Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory,
Wavelets

Xiao-Jun Yang

*State Key Laboratory for Geomechanics
and Deep Underground Engineering,
China University of Mining and Technology,
Xuzhou 221116, China*
*Local Fractional Calculus and Applications,
Fractional Calculus and Applications,
General Fractional Calculus and
Applications,
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Viscoelasticity and Computational methods
for Mathematical
Physics.*
dyangxiaojun@163.com

Richard A. Zalik

Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev
Systems, Wavelet Theory

Ahmed I. Zayed

Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic
analysis and wavelets, Special
functions and orthogonal
polynomials, Integral transforms

Ding-Xuan Zhou

Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708, Fax: 852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline
functions, Wavelets

Xin-long Zhou

Fachbereich Mathematik, Fachgebiet
Informatik
Gerhard-Mercator-Universitat
Duisburg
Lotharstr.65, D-47048 Duisburg,
Germany
e-mail: [Xzhou@informatik.uni-
duisburg.de](mailto:Xzhou@informatik.uni-
duisburg.de)
Fourier Analysis, Computer-Aided
Geometric Design, Computational
Complexity, Multivariate
Approximation Theory, Approximation
and Interpolation Theory

Jessada Tariboon

Department of Mathematics,
King Mongkut's University of
Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales,
Differential/Difference Equations,
Fractional Differential Equations

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Journal of Computational Analysis and Applications

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA.
Tel. 901.678.3144
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Some Fixed Point Results of Caristi Type in G –Metric Spaces

Hamed M. Obiedat¹ and Ameer A. Jaber²

^{1,2}Department of Mathematics

Hashemite University

P.O.Box150459

Zarqa13115-Jordan

email¹: hobiedat@hu.edu.jo, email²: ameerj@hu.edu.jo,

September 4, 2017

Abstract

In this paper, we prove several fixed point results for mappings of Caristi type in the setting of G –metric spaces.

1 Introduction

The class of G –metric spaces introduced by Z. Mustafa and B. Sims (See [7]) was to provide a new class of generalized metric spaces and to extend the fixed point theory for a variety of mappings. Moreover, many theorems were proved in this new setting with most of them recognizable as counterparts of well-known metric space theorems (See [6], [8], [9]).

Caristi's fixed point theorem provides a generalization of Banach's contraction mapping principle (See [2]). Due to the importance of Caristi's fixed point theorem, it has been improved, generalized, extended and used in many application (See [1], [3], [4], [5]). In this paper, we prove several fixed point results for mappings of Caristi type in the setting of G –metric spaces.

⁰2000 *Mathematics Subject Classification.* 47H10, 54E50.

Key words and phrases. Caristi's Fixed Point Theorem; G-Metric Spaces; Lower semi-Continuous Functions.

Definition 1 ([7]) *G-metric space is a pair (X, G) , where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$, we have:*

- (G1) $G(x, y, z) = 0$ if $x = y = z$;
- (G2) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$;
- (G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;
- (G4) $G(x, y, z) = G(p\{x, z, y\})$ (symmetry in all three variables);
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, (rectangle inequality).

The function G is called a G -metric on X .

Definition 2 ([7]) *A sequence (x_n) in a G -metric space X is said to converge if there exists $x \in X$ such that $\lim_{n,m \rightarrow \infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is G -convergent to x .*

Proposition 3 ([7]) *Let X be G -metric space. Then the following statements are equivalent.*

1. (x_n) is G -convergent to x .
2. $G(x_n, x_n, x) \rightarrow 0$, as $n \rightarrow \infty$.
3. $G(x_n, x, x) \rightarrow 0$, as $n \rightarrow \infty$.
4. $G(x_m, x_n, x) \rightarrow 0$, as $m, n \rightarrow \infty$.

In a G -metric space X , a sequence (x_n) is said to be G -Cauchy if given $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N_\varepsilon$.

Proposition 4 ([7]) *In a G -metric space X , the following statements are equivalent.*

1. The sequence (x_n) is G -Cauchy.
2. For every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Definition 5 ([7]) Let (X, G) and (X', G') be two G -metric spaces, and let $f : (X, G) \rightarrow (X', G')$ be a function, then f is said to be G -continuous at a point $a \in X$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$.

A function f is G -continuous on X if it is G -continuous at all $a \in X$.

Proposition 6 ([7]) Let (X, G) and (X', G') be two G -metric spaces. Then a function $f : (X, G) \rightarrow (X', G')$ is G -continuous at a point $x \in X$ if and only if it is G -sequentially continuous at x ; that is, whenever (x_n) is G -convergent to x we have $(f(x_n))$ is G -convergent to $f(x)$.

A G -metric space (X, G) is called symmetric G -metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$, and called nonsymmetric if it is not symmetric.

Proposition 7 ([7]) Let X be a G -metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables. A G -metric space X is said to be complete if every G -Cauchy sequence in X is G -convergent in X .

Definition 8 With \mathcal{M} we indicate the space of functions ρ , where

1. $\rho : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing, continuous and concave,
2. $\rho(0) = 0$.

Lemma 9 Let (X, G) be a complete G -metric space and let $\rho \in \mathcal{M}$. Then $(X, \rho \circ G)$ is a complete G -metric space.

Proof. First let us prove that ρ is subadditive. To do so, let $x, y \in [0, \infty)$ and set $k = \frac{\rho(x)}{\rho(x) + \rho(y)}$. Then since ρ is increasing, we have

$$\rho(x + y) = \rho\left(\frac{k}{k}x + \frac{1 - k}{1 - k}y\right) \leq \max\left\{\rho\left(\frac{x}{k}\right), \rho\left(\frac{y}{1 - k}\right)\right\}.$$

Since ρ is concave and $\rho(0) = 0$, we have

$$\rho(x) = \rho\left(\frac{k}{k}x\right) = \rho\left(\frac{k}{k}x + (1 - k).0\right) \geq k\rho\left(\frac{1}{k}x\right) + (1 - k)\rho(0) = k\rho\left(\frac{1}{k}x\right)$$

which implies that $\frac{1}{k}\rho(x) \geq \rho(\frac{1}{k}x)$. Similarly $\frac{1}{1-k}\rho(y) \geq \rho(\frac{1}{1-k}x)$. Therefore,

$$\rho(x + y) \leq \max\{\rho(\frac{x}{k}), \rho(\frac{y}{1-k})\} \leq \max\{\frac{1}{k}\rho(x), \frac{1}{1-k}\rho(y)\} \leq \rho(x) + \rho(y).$$

This completes the proof that ρ is subadditive. Now to prove that $\rho \circ G$ defines G -metric on X , we let $x, y, z, a \in X$. Then

G1) Since ρ is strictly increasing and $\rho(0) = 0$ then $\rho \circ G(x, y, z) = 0$ implies $G(x, y, z) = 0$ which means $x = y = z$;

G2) Since $0 < G(x, x, y)$; with $x \neq y$ and ρ is strictly increasing with $\rho(0) = 0$, then $0 < \rho \circ G(x, x, y)$; with $x \neq y$;

G3) Since $G(x, x, y) \leq G(x, y, z)$, with $z \neq y$ and ρ is strictly increasing then $\rho \circ G(x, x, y) \leq \rho \circ G(x, y, z)$, with $z \neq y$,

G4) Since $G(x, y, z) = G(p\{x, z, y\})$ and ρ is strictly increasing(injective) then $\rho \circ G(x, y, z) = \rho \circ G(p\{x, z, y\})$ (symmetry in all three variables); ■

G5) Since $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ and ρ is strictly increasing and subadditive then

$$\rho \circ G(x, y, z) \leq \rho(G(x, a, a) + G(a, y, z)) \leq \rho \circ G(x, a, a) + \rho \circ G(a, y, z),$$

which proves that $\rho \circ G$ defines G -metric on X . We still need to prove that $(X, \rho \circ G)$ is complete, so let $\{x_n\}$ be a Cauchy sequence in $(X, \rho \circ G)$. Then $\lim_{n,m \rightarrow \infty} \rho \circ G(x_n, x_m, x_m) = 0$. Since ρ is continuous and strictly increasing with $\rho(0) = 0$, we have $\rho(\lim_{n,m \rightarrow \infty} G(x_n, x_m, x_m)) = 0$. This implies

$\lim_{n,m \rightarrow \infty} G(x_n, x_m, x_m) = 0$, which means that $\{x_n\}$ is Cauchy sequence in the complete G -metric space (X, G) . Therefore, there exists $x^* \in X$ such that $\{x_n\}$ G -converges to $x^* \in X$. Hence $\lim_{n \rightarrow \infty} G(x_n, x^*, x^*) = 0$, which implies $\rho(\lim_{n \rightarrow \infty} G(x_n, x^*, x^*)) = \rho(0) = 0$. By continuity of ρ , we have $\lim_{n \rightarrow \infty} \rho(G(x_n, x^*, x^*)) = 0$, which implies $\{x_n\}$ is $\rho \circ G$ -convergent in $(X, \rho \circ G)$. This completes the proof of Lemma 9.

Definition 10 With $\mathcal{L}(X)$ we indicate the space of functions ϕ , where $\phi : X \rightarrow \mathbb{R}^+$ is lower semi-continuous.

Remark 11 Let (X, G) be a G -metric space and $\phi \in \mathcal{L}(X)$. Define \preceq on X by

$$x \preceq y \iff G(x, y, y) \leq \phi(x) - \phi(y) \quad \forall x, y \in X,$$

then (X, G, \preceq) is partially ordered G -metric space. In fact, $\forall x, y, z \in X$ the following conditions are satisfied

- i) since $0 = G(x, x, x) \leq \phi(x) - \phi(x) = 0$, we have that $x \preceq x$
- ii) if $x \preceq y$ and $y \preceq x$, then $0 \leq G(x, y, y) \leq \phi(x) - \phi(y) = -(\phi(y) - \phi(x)) \leq -G(y, x, x) \leq 0$. This implies that $G(x, y, y) = G(y, x, x) = 0$. Hence, $x = y$.
- iii) if $x \preceq y$ and $y \preceq z$, then

$$\begin{aligned} G(x, z, z) &\leq G(x, y, y) + G(y, z, z) \text{ by rectangle inequality} \\ &\leq \phi(x) - \phi(y) + \phi(y) - \phi(z) \\ &= \phi(x) - \phi(z), \end{aligned}$$

which implies $x \preceq z$.

2 Main Results

In this section, we introduce several fixed point results for mappings of Caristi type in the setting of G -metric spaces. We use the existence of a maximal element to prove Caristi's fixed point theorem in the setting of G -metric spaces.

Theorem 12 *Let (X, G, \preceq) be a partially ordered G -metric space with \preceq as defined in Remark 11. Then the following statements are equivalent:*

- 1 Any selfmapping T on X satisfies $G(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$ has a fixed point.
- 2 X has a maximal element.

Proof. 1 \implies 2) Suppose that $T : X \rightarrow X$ has a fixed point, say x^* , and $x_1 \preceq x_2 \preceq \dots \preceq$ be a chain in X . Fix x_j , then $G(x_j, x^*, x^*) = G(x_j, Tx^*, Tx^*) \leq \phi(x_j) - \phi(Tx^*) = \phi(x_j) - \phi(x^*)$ which implies that $x_j \preceq x^*$. Hence X has x^* as the maximal element.

2 \implies 1) Suppose X has x^* as a maximal element, then $Tx^* \preceq x^*$. Since T satisfies $G(x^*, Tx^*, Tx^*) \leq \phi(x^*) - \phi(Tx^*)$ which implies that $x^* \preceq Tx^*$. Therefore, $Tx^* = x^*$. ■

Theorem 13 *Let (X, G, \preceq) be a partially ordered complete G -metric space, $\phi : X \rightarrow \mathbb{R}^+$ be a lower semi-continuous and $T : X \rightarrow X$ be selfmapping satisfying the inequality; $G(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$. Then T has a fixed point.*

Proof. Let $\mathcal{C} = \{x_t : t \in \Delta\} \subseteq X$ be any chain in X and let $\{t_n\}$ be any increasing sequence of elements of Δ . We prove first that $\phi(\mathcal{C})$ is a decreasing net. To do so, let c_t and c_s be any pair of elements in \mathcal{C} with $x_t \preceq x_s$ for $t, s \in \Delta$. Then $G(x_t, x_s, x_s) \leq \phi(x_t) - \phi(x_s)$, which implies that $\phi(x_s) \leq \phi(x_t) - G(x_t, x_s, x_s)$. Therefore, $\{\phi(x_t)\}_{t \in \Delta}$ is a decreasing net of positive real numbers. Thus $\inf\{\phi(x_t) : t \in \Delta\}$ exists by completeness property of \mathbb{R} . Now choose $\{t_n\}_{n \in \mathbb{N}}$ to be an increasing sequence of Δ such that $\lim_{n \rightarrow \infty} \phi(x_{t_n}) = \inf\{\phi(x_t) : t \in \Delta\}$. Then $\{x_{t_n}\}$ is G -Cauchy since for $n, m \in \mathbb{N}$, we have

$$G(x_{t_n}, x_{t_m}, x_{t_m}) \leq \phi(x_{t_n}) - \phi(x_{t_m}). \tag{1}$$

Thus passing to the limit in the inequality (1) implies $G(x_{t_n}, x_{t_m}, x_{t_m}) = 0$ as $n, m \rightarrow \infty$. Since (X, G, \preceq) is G -complete then there exists $x^* \in X$ such that $\{x_{t_n}\}$ converges to x^* . To prove that x^* is an upper bound of the set \mathcal{C} , let $m, n \in \mathbb{N}$ since $\{x_{t_n}\}$ converges to x^* and $\{x_{t_n}\}$ is increasing imply $x_{t_n} \preceq x^* \forall n \geq 1$. Therefore,

$$\begin{aligned} G(x_{t_n}, x^*, x^*) &= \lim_{m \rightarrow \infty} G(x_{t_n}, x_{t_m}, x_{t_m}) \\ &\leq \phi(x_{t_n}) - \lim_{m \rightarrow \infty} \phi(x_{t_m}) \\ &\leq \phi(x_{t_n}) - \inf_{m \rightarrow \infty} \phi(x_{t_m}) \\ &\leq \phi(x_{t_n}) - \phi(x^*). \end{aligned}$$

Then $\phi(x^*) \leq \phi(x_{t_n}) \forall n \geq 1$ which implies that $\phi(x^*) \leq \inf\{\phi(x_t) : t \in \Delta\}$. Hence $x_t \preceq x^* \forall t \in \Delta$ since ϕ is decreasing which means that x^* is an upper bound of the chain \mathcal{C} . Therefore Zorn's lemma implies that (X, \preceq) has a maximal element. By Theorem 12 any selfmapping $T : X \rightarrow X$ satisfies the inequality $G(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$ has a fixed point. ■

Corollary 14 *Let (X, G) be a partially ordered G -metric space. Suppose $f : X \rightarrow X$ is any function and $T : X \rightarrow X$ be G -continuous. If there exists a real number $r < 0$ such that for all $x \in X$*

$$G(f(x), Tf(x), Tf(x)) \leq G(x, Tx, Tx) + rG(x, f(x), f(x)),$$

then f has a fixed point.

Proof. Define $\phi : X \rightarrow \mathbb{R}^+$ by $\phi(x) = -\frac{G(x, Tx, Tx)}{r}$. Then the lower semi continuity of ϕ follows from the G -continuity of T . Now

$$G(f(x), Tf(x), Tf(x)) = -r\phi(f(x)) \leq -r\phi(x) + rG(x, f(x), f(x)).$$

Then

$$\phi(f(x)) \leq \phi(x) - G(x, f(x), f(x)),$$

which implies

$$G(x, f(x), f(x)) \leq \phi(x) - \phi(f(x)).$$

Define \preceq on X by

$$x \preceq y \iff G(x, y, y) \leq \phi(x) - \phi(y) \quad \forall x, y \in X.$$

Then by Theorem 13, there exists $x^* \in X$ such that $f(x^*) = x^*$. ■

Corollary 15 Let (X, G) be a complete G -metric space and let $\rho \in \mathcal{M}$. Then $(X, \rho \circ G)$ is a complete G -metric space. Then any selfmapping T on X satisfies $\rho \circ G(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$ has a fixed point.

Corollary 16 Let (X, G) be a complete G -metric space and let $\rho \in \mathcal{M}$. Suppose $f : X \rightarrow X$ is any function and $T : X \rightarrow X$ is G -continuous. If for all $x \in X$

$$G(f(x), Tf(x), Tf(x)) \leq G(x, Tx, Tx) - \rho \circ G(x, f(x), f(x)).$$

Then f has a fixed point.

Proof. Define $\phi(x) = \rho^{-1} \circ G(x, Tx, Tx)$. Then lower semi continuity of ϕ follows from the G -continuity of T and continuity of ρ^{-1} . Now

$$G(f(x), Tf(x), Tf(x)) = \rho(\phi(f(x))) \leq \rho(\phi(x)) - \rho \circ G(x, f(x), f(x)).$$

Then

$$\rho(\phi(f(x))) \leq \rho(\phi(x)) - \rho \circ G(x, f(x), f(x)).$$

By the subadditivity of ρ , the above inequality becomes

$$\begin{aligned} \rho(\phi(f(x)) + G(x, f(x), f(x))) &\leq \rho(\phi(f(x))) + \rho \circ G(x, f(x), f(x)) \\ &\leq \rho(\phi(x)). \end{aligned}$$

Now since ρ is increasing, we obtain

$$\phi(f(x)) + G(x, f(x), f(x)) \leq \phi(x).$$

Hence

$$G(x, f(x), f(x)) \leq \phi(x) - \phi(f(x)).$$

Define \preceq on X by

$$x \preceq y \iff G(x, y, y) \leq \phi(x) - \phi(y) \quad \forall x, y \in X.$$

Then by Theorem 13 there exists $x^* \in X$ such that $f(x^*) = x^*$. ■

Corollary 17 *Let (X, G) be a complete G -metric space and Suppose $f : X \rightarrow X$ is any function and $T : X \rightarrow X$ is G -continuous. If there exist a real number $r < 0$ and $n \in \mathbb{N}$ such that for all $x, y \in X$*

$$G(f(x), Tf(x), T^n f(x)) \leq G(x, Tx, T^n x) + rG(x, f(x), f(x))$$

then f has a fixed point.

Proof. Define $\phi : X \rightarrow \mathbb{R}^+$ by $\phi(x) = -\frac{G(x, Tx, T^n x)}{r}$. Then lower semi continuity of ϕ follows from the G -continuity of T . Then

$$\begin{aligned} -r\phi(f(x)) &= G(x, Tf(x), T^n f(x)) \\ &\leq -r\phi(x) + rG(x, f(x), f(x)). \end{aligned}$$

Then we obtain

$$G(x, f(x), f(x)) \leq \phi(x) - \phi(f(x)).$$

Define \preceq on X by

$$x \preceq y \iff G(x, y, y) \leq \phi(x) - \phi(y) \quad \forall x, y \in X.$$

Then by Theorem 13 there exists $x^* \in X$ such that $f(x^*) = x^*$.

Corollary 18 *Let (X, G) be a complete G -metric space and let $\rho \in \mathcal{M}$. Suppose $f : X \rightarrow X$ is any function and $T : X \rightarrow X$ is G -continuous. If there exist $\rho \in \mathcal{M}$ and $n \in \mathbb{N}$ such that for all $x \in X$*

$$G(f(x), Tf(x), T^n f(x)) \leq G(x, Tx, T^n x) - \rho \circ G(x, f(x), f(x)),$$

then f has a fixed point.

■
 The following theorem gives a natural generalization of Caristi type mapping in the setting of G -metric spaces.

Theorem 19 *Let (X, G) be a complete G -metric space. Suppose $T : X \rightarrow X$ is G -continuous. If there exists $\phi_y \in \mathcal{L}(X)$ for all $y \in X$ such that for all $x \in X$*

$$G(Tx, T^2x, Ty) \leq \phi_y(x) - \phi_y(Tx),$$

then T has a fixed point.

Proof. Fix $x_0 \in X$ and let $x_n = T^n x_0$ $n = 1, 2, 3, \dots$. Then

$$\begin{aligned} G(x_n, x_{n+1}, Ty) &= G(Tx_{n-1}, Tx_n, Ty) = G(Tx_{n-1}, T^2x_{n-1}, Ty) \\ &\leq \phi_y(x_{n-1}) - \phi_y(Tx_{n-1}) \\ &= \phi_y(x_{n-1}) - \phi_y(x_n). \end{aligned}$$

Then for each $y \in X$,

$$\begin{aligned} \sum_{j=1}^n G(x_n, x_{n+1}, Ty) &= G(x_1, x_2, Ty) + G(x_2, x_3, Ty) + \dots + G(x_n, x_{n+1}, Ty) \\ &\leq \phi_y(x_0) - \phi_y(x_1) + \phi_y(x_1) - \phi_y(x_2) + \dots - \phi_y(x_{n-1}) + \phi_y(x_{n-1}) - \phi_y(x_n) \\ &\leq \phi_y(x_0) - \phi_y(x_n) \\ &\leq \phi_y(x_0) + C, \end{aligned}$$

where $C > 0$, which implies that

$$\sum_{j=1}^{\infty} G(x_n, x_{n+1}, Ty) \leq \phi_y(x_0) + C < \infty.$$

Then $\sum_{j=1}^{\infty} G(x_n, x_{n+1}, Ty)$ is a convergent series. Hence $\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, Ty) =$

0. Therefore,

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, Ty, Ty) + G(Ty, x_m, x_m) \\ &\leq G(x_n, x_m, Ty) + G(x_n, x_m, Ty) \\ &\leq 2 \sum_{j=n}^m G(x_j, x_{j+1}, Ty) \rightarrow 0 \text{ as } m, n \rightarrow \infty \end{aligned}$$

which implies that $\{x_n\}$ is Cauchy in complete G -metric space. That is, there exists $x^* \in X$ such that $\{x_n\}$ converges to x^* . Now

$$G(Tx^*, T^2x^*, x^*) = \lim_{n \rightarrow \infty} G(Tx_n, T^2x_n, x_n) = \lim_{n \rightarrow \infty} G(x_{n+1}, x_{n+2}, x_n) = 0,$$

which implies that x^* is a fixed point of T . ■

Corollary 20 *Let (X, G) be a complete G -metric space. Suppose $T : X \rightarrow X$ is G -continuous and for all $x \in X$*

$$G(Tf(x), T^2f(x), Ty) \leq G(x, Tx, Ty) - G(f(x), f^2(x), Ty).$$

Then f has a fixed point.

Proof. For each $y \in X$, choose $\phi_y(x) = G(x, Tx, Ty)$ then the result follows by applying Theorem 19 ■

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Meir-Keeler contraction mappings in M_b -metric Spaces

N. Mlaiki¹, N. Souayah², K. Abodayeh¹, T. Abdeljawad¹

Department of Math & General Sciences, Prince Sultan University¹

Department of Natural Sciences, Community College, King Saud University²

Riyadh 11586, Saudi Arabia

E-mail: nmlaiki@psu.edu.sa

nsouayah@ksu.edu.sa

kamal@psu.edu.sa

tabdeljawad@psu.edu.sa

Abstract

In this paper, we generalize the notion of Meir-Keeler contraction condition in M_b -metric spaces. We prove some fixed point theorems for this class of contractions which enables us to extend and generalize the recent results of Gholmian and Khanehgir [2].

1 Introduction and preliminaries

First of all, we would like to mention that this work is inspired by the work of Gholmian and Khanehgir [2]. In 1922 Banach established one of the most important theorem in fixed point theory known as the "Banach contraction principle". Subsequently, many authors have extended this theorem in many different ways. For example, in 1969, Meir and Keeler [3] generalize the Banach's theorem using the weakly uniformly strict contraction and proved the following theorem:

Theorem 1. *Let (X, d) be a complete metric space and f a mapping of X into itself satisfying the following condition:*

given $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq d(x, y) < \epsilon + \delta$ implies $d(f(x), f(y)) < \epsilon$.

Then f has a unique fixed point ξ . Moreover, For any $x \in X$, $\lim_{n \rightarrow \infty} f^n(x) = \xi$.

The Theorem 1 has been extended in many different metric spaces under several contractive definitions, see [2], [5].

On the other hand, several types of generalized metric spaces are proposed and a series of fixed point theorems for various classes of mapping are obtained, see [4], [6], [8], [9], [10], [11], [12].

M -metric spaces was introduced by Asadi see [1], which is an extension of partial metric spaces. So, first we remind the reader of the definition of an M -metric spaces along with some other notations.

Notation 1. [1]

1. $m_{x,y} := \min\{m(x, x), m(y, y)\}$
2. $M_{x,y} := \max\{m(x, x), m(y, y)\}$

Definition 1. [1] Let X be a nonempty set, if the function $m : X^2 \rightarrow R^+$ satisfies the following conditions: for all $x, y, z \in X$

- (1) $m(x, x) = m(y, y) = m(x, y)$ if and only if $x = y$,
- (2) $m_{x,y} \leq m(x, y)$,
- (3) $m(x, y) = m(y, x)$,
- (4) $(m(x, y) - m_{x,y}) \leq (m(x, z) - m_{x,z}) + (m(z, y) - m_{z,y})$.

Then the pair (X, m) is called an M -metric space.

Recently, Mlaiki et al. [7] developed the concept of M_b -metric spaces which extends the M -metric spaces and some fixed point theorems are established. Motivated by the properties of this original metric space, we introduce the notion of generalized Meir-Keeler contraction mappings in the M_b -metric spaces.

Now, let's recall some definitions and notations of M_b -metric spaces.

Notation 2. [7]

1. $m_{bx,y} := \min\{m_b(x, x), m_b(y, y)\}$
2. $M_{bx,y} := \max\{m_b(x, x), m_b(y, y)\}$

Definition 2. [7] An M_b -metric space on a nonempty set X is a function $m_b : X^2 \rightarrow R^+$ that satisfies the following conditions, for all $x, y, z \in X$ we have

- (1) $m_b(x, x) = m_b(y, y) = m_b(x, y)$ if and only if $x = y$,
- (2) $m_{bx,y} \leq m_b(x, y)$,
- (3) $m_b(x, y) = m_b(y, x)$,
- (4) There exists a real number $s \geq 1$ such that for all $x, y, z \in X$ we have

$$(m_b(x, y) - m_{bx,y}) \leq s[(m_b(x, z) - m_{bx,z}) + (m_b(z, y) - m_{bz,y})] - m_b(z, z).$$

The number s is called the coefficient of the M_b -metric space (X, m_b) .

Now, we give an example of an M_b -metric which is not an M -metric space.

Example 1. [7] Let $X = [0, \infty)$ and $p > 1$ be constant and $m_b : X^2 \rightarrow [0, \infty)$ defined by for all $x, y \in X$ we have

$$m_b(x, y) = \max\{x, y\}^p + |x - y|^p.$$

Note that (X, m_b) is an M_b -metric with coefficient $s = 2^p$. Now, we show that (X, m_b) is not an M -metric space. Take $x = 5, y = 1$ and $z = 4$, we get $m_b(x, y) - m_{bx,y} = 5^p + 4^p - 1$ and $(m_b(x, z) - m_{bx,z}) + (m_b(z, y) - m_{bz,y}) = 5^p + 1 - 4^p + 4^p + 3^p - 1 = 5^p + 3^p$. Therefore,

$$m_b(x, y) - m_{bx,y} > (m_b(x, z) - m_{bx,z}) + (m_b(z, y) - m_{bz,y}),$$

as required.

Definition 3. [7] Let (X, m_b) be a M_b -metric space. Then:

1) A sequence $\{x_n\}$ in X converges to a point x if and only if

$$\lim_{n \rightarrow \infty} (m_b(x_n, x) - m_{bx_n, x}) = 0.$$

2) A sequence $\{x_n\}$ in X is said to be m_b -Cauchy sequence if and only if

$$\lim_{n, m \rightarrow \infty} (m_b(x_n, x_m) - m_{bx_n, x_m}), \text{ and } \lim_{n \rightarrow \infty} (M_{bx_n, x_m} - m_{bx_n, x_m})$$

exist and finite.

3) An M_b -metric space is said to be complete if every m_b -Cauchy sequence $\{x_n\}$ converges to a point x such that

$$\lim_{n \rightarrow \infty} (m_b(x_n, x) - m_{bx_n, x}) = 0 \text{ and } \lim_{n \rightarrow \infty} (M_{bx_n, x} - m_{bx_n, x}) = 0.$$

Definition 4. Each m_b -metric generates a topology τ_{m_b} on X whose base is the family of open m_b -balls $\{B_{m_b}(x, \epsilon) \mid x \in X, \epsilon > 0\}$ where $B_{m_b}(x, \epsilon) = \{y \in X \mid m_b(x, y) - m_{bx, y} < \epsilon\}$.

Definition 5. Let X be a nonempty set, $T : X \rightarrow X$ be a mapping and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Then, T is said to be α -admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1. \tag{1}$$

Definition 6. A mapping $T : X \rightarrow X$ is called triangular α -admissible if it is α -admissible and it satisfies the following condition:

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1, \text{ then } \alpha(x, z) \geq 1 \text{ where } x, y, z \in X.$$

Definition 7. Let (X, m_b) be an m_b -metric space with coefficient s , an α -admissible mapping $T : X \rightarrow X$ is said to be generalized Meir-Keeler contraction of type (I) if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \beta(m_b(x, y))M(x, y) < \epsilon + \delta \text{ implies } \alpha(x, y)m_b(Tx, Ty) < \epsilon \tag{2}$$

where

$$M(x, y) = \max\{m_b(x, y), m_b(Tx, x), m_b(Ty, y)\}, \text{ for all } x, y \in \mathbb{N} \tag{3}$$

and $\beta : [0, \infty) \rightarrow (0, \frac{1}{s})$ is a given function.

Definition 8. Let (X, m_b) be an m_b -metric space with coefficient s . A triangular α -admissible mapping $T : X \rightarrow X$ is said to be generalized Meir-Keeler contraction of type (II) if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \beta(m_b(x, y))N(x, y) < \epsilon + \delta \text{ implies } \alpha(x, y)m_b(Tx, Ty) < \epsilon \tag{4}$$

where

$$N(x, y) = \max\{m_b(x, y), \frac{1}{2}[m_b(Tx, x) + m_b(Ty, y)]\}, \text{ for all } x, y \in \mathbb{N} \tag{5}$$

and $\beta : [0, \infty) \rightarrow (0, \frac{1}{s})$ is a given function.

Remark 1. 1. Suppose that $T : X \rightarrow X$ is a generalized Meir-Keeler contraction of type (I). Then

$$\alpha(x, y)m_b(Tx, Ty) < \beta(m_b(x, y))M(x, y) \tag{6}$$

for all $x, y \in X$ when $M(x, y) > 0$.

2. Note that for all $x, y \in X$, we have $N(x, y) \leq M(x, y)$.

2 Main Results

Theorem 2. *Let (X, m_b) be a complete M_b metric space and $T : X \rightarrow X$ be a triangular α -admissible mapping. Suppose that the following conditions hold:*

- (a) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(Tx_0, x_0) \geq 1$.*
- (b) *If $\{x_n\}$ is a sequence in X that converges to z as $n \rightarrow \infty$, and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}$.*
- (c) *If for each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$2s\epsilon \leq m_b(y, Ty) \frac{1 + m_b(x, Tx)}{1 + M(x, y)} + N(x, y) < s(2\epsilon + \delta),$$

then we have $\alpha(x, y)m_b(Tx, Ty) < \epsilon$.

Then, T has a fixed point in X .

Proof. Note that condition (c) implies that

$$\alpha(x, y)m_b(Tx, Ty) < \frac{1}{2s}m_b(y, Ty) \frac{1 + m_b(x, Tx)}{1 + M(x, y)} + \frac{1}{2s}N(x, y).$$

Let $x_0 \in X$ that satisfies condition (a) and define the sequence $\{x_n\}$ by $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists an n such that $x_{n+1} = x_n$, then we are done. Without loss of generality, we may assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.

Since T is α -admissible, we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ and thus $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. By repeating the same argument, we get $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} m_b(x_{n+1}, x_{n+2}) &= m_b(Tx_n, Tx_{n+1}) \\ &\leq \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) \\ &< \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) \frac{1 + m_b(x_n, x_{n+1})}{1 + M(x_n, x_{n+1})} + \frac{1}{2s}N(x_n, x_{n+1}). \end{aligned}$$

Note that $M(x_n, x_{n+1}) = \max\{m_b(x_n, x_{n+1}), m_b(x_{n+1}, x_{n+2})\}$. So, if $M(x_n, x_{n+1}) = m_b(x_{n+1}, x_{n+2})$ then we have

$$\begin{aligned} m_b(x_{n+1}, x_{n+2}) = m_b(Tx_n, Tx_{n+1}) &\leq \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) \\ &< \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) \frac{1 + m_b(x_n, x_{n+1})}{1 + m_b(x_{n+1}, x_{n+2})} + \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) \\ &< \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) + \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) \\ &= \frac{1}{s}m_b(x_{n+1}, x_{n+2}) \leq m_b(x_{n+1}, x_{n+2}), \end{aligned}$$

which leads to a contradiction. Therefore, $M(x_n, x_{n+1}) = m_b(x_n, x_{n+1})$. Also note that $N(x_n, x_{n+1}) \leq M(x_n, x_{n+1})$ and hence

$$\begin{aligned} m_b(x_{n+1}, x_{n+2}) &< \frac{1}{2s}m_b(x_{n+1}, x_{n+2})\frac{1 + m_b(x_n, x_{n+1})}{1 + m_b(x_n, x_{n+1})} + \frac{1}{2s}m_b(x_n, x_{n+1}) \\ &= \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) + \frac{1}{2s}m_b(x_n, x_{n+1}) \\ &\leq \frac{1}{2s}m_b(x_n, x_{n+1}) + \frac{1}{2s}m_b(x_n, x_{n+1}) \\ &= \frac{1}{s}m_b(x_n, x_{n+1}) \leq m_b(x_n, x_{n+1}). \end{aligned}$$

Therefore, $m_b(x_{n+1}, x_{n+2}) < m_b(x_n, x_{n+1})$ and thus the sequence $\{m_b(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence that converges to some number say $r \geq 0$. By Condition (c) in the hypothesis of the theorem, choose $\epsilon = \frac{r}{s}$. Note that, $\lim_{n \rightarrow \infty} [m_b(x_{n+1}, x_{n+2}) + m_b(x_n, x_{n+1})] = 2r$. Hence, there exists $N_0 \in \mathbb{N}$ such that

$$2r < m_b(x_{N_0+1}, x_{N_0+2}) + m_b(x_{N_0}, x_{N_0+1}) < 2r + \delta.$$

Therefore,

$$\begin{aligned} 2s\epsilon &< m_b(x_{N_0+1}, x_{N_0+2}) + m_b(x_{N_0}, x_{N_0+1}) \\ &= m_b(x_{N_0+1}, Tx_{N_0+1}) \left[\frac{1 + m_b(x_{N_0}, Tx_{N_0})}{1 + M(x_{N_0}, x_{N_0+1})} \right] + N(x_{N_0}, x_{N_0+1}) \\ &< 2s\epsilon + \delta < s(2\epsilon + \delta). \end{aligned}$$

Using the fact that $\alpha(x_{N_0}, x_{N_0+1}) \geq 1$ and condition (c), we deduce that

$$m_b(x_{N_0+1}, x_{N_0+2}) \leq \alpha(x_{N_0}, x_{N_0+1})m_b(Tx_{N_0}, Tx_{N_0+1}) < \epsilon = \frac{r}{s} \leq r.$$

But we know that for all $n \in \mathbb{N}$, $r \leq m_b(x_n, x_{n+1})$ which leads to a contradiction. Thus $r = 0$; that is, $\lim_{n \rightarrow \infty} m_b(x_n, x_{n+1}) = 0$. Now let $\epsilon > 0$ and $\delta' = \min\{\delta, \epsilon, 1\}$. Since $\lim_{n \rightarrow \infty} m_b(x_n, x_{n+1}) = 0$, there exists $k \in \mathbb{N}$ such that $m_b(x_m, x_{m+1}) < \frac{\delta'}{4}$, for all $m \geq k$. Let $\eta = s(2\epsilon + \frac{\delta'}{2})$ and consider the set

$$B[x_k, \eta] = \{x_i \mid i \geq k, m_b(x_i, x_k) - m_{bx_i, x_k} < \eta\}.$$

We prove that T maps $B[x_k, \eta]$ to itself. Let $x_l \in B[x_k, \eta]$. Then we have $m_b(x_l, x_k) - m_{bx_l, x_k} < \eta$. If $l = k$, then we have $Tx_l = Tx_k = x_{k+1} \in B[x_k, \eta]$. So we may assume that $l > k$. Suppose that $2s\epsilon \leq m_b(x_l, x_k)$, so that

$$2s\epsilon \leq m_b(x_l, x_k) - m_{bx_l, x_k} < \eta.$$

Note $m_b(x_l, x_k) \leq N(x_l, x_k)$. Hence, $2s\epsilon \leq m_b(x_l, x_k)$ and this implies that $\epsilon \leq \frac{1}{2s}m_b(x_l, x_k)$. Thus,

$$\epsilon \leq \frac{1}{2s}m_b(x_k, x_{k+1})\frac{1 + m_b(x_l, x_{l+1})}{1 + M(x_l, x_k)} + \frac{1}{2s}N(x_l, x_k).$$

Therefore,

$$\frac{1}{2s}m_b(x_k, x_{k+1})\frac{1 + m_b(x_l, x_{l+1})}{1 + M(x_l, x_k)} + \frac{1}{2s}N(x_l, x_k) < \epsilon + \frac{\delta'}{2},$$

and this implies that

$$2s\epsilon \leq m_b(x_k, Tx_k)\frac{1 + m_b(x_l, Tx_l)}{1 + M(x_l, x_k)} + N(x_l, x_k) < s(2\epsilon + \delta').$$

Thus, by part (c) of the theorem, we have $m_b(Tx_l, Tx_k) \leq \alpha(x_l, x_k)m_b(Tx_l, Tx_k) < \epsilon$.
Therefore,

$$\begin{aligned} m_b(Tx_l, x_k) - m_{bTx_l, x_k} &\leq m_b(Tx_l, x_k) \\ &\leq s [(m_b(Tx_l, Tx_k) - m_{bTx_l, Tx_k}) + (m_b(Tx_k, x_k) - m_{bTx_k, x_k})] \\ &\leq s [m_b(Tx_l, Tx_k) + m_b(Tx_k, x_k)] \\ &< s[\epsilon + \frac{\delta'}{4}] \\ &< s[2\epsilon + \frac{\delta'}{2}] \end{aligned}$$

which implies that $x_{l+1} \in B[x_k, \eta]$ as desired. Now assume that $m_b(x_l, x_k) < 2s\epsilon$. Then we have

$$\begin{aligned} m_b(Tx_l, x_k) - m_{bTx_l, x_k} &\leq m_b(Tx_l, x_k) \\ &\leq s [(m_b(Tx_l, Tx_k) - m_{bTx_l, Tx_k}) + (m_b(Tx_l, x_k) - m_{bTx_l, x_k})] \\ &\leq s [m_b(Tx_l, Tx_k) + m_b(Tx_k, x_k)] \\ &\leq s\alpha(x_l, x_k)m_b(Tx_l, Tx_k) + sm_b(Tx_k, x_k) \\ &< s \left[\frac{1}{2s}m_b(x_k, x_{k+1})\frac{1 + m_b(x_l, Tx_{l+1})}{1 + M(x_l, x_k)} + \frac{1}{2s}N(x_l, x_k) \right] + sm_b(x_{k+1}, x_k) \\ &\leq \frac{1}{2}m_b(x_k, x_{k+1}) + \frac{m_b(x_k, x_{k+1})m_b(x_l, x_{l+1})}{2(1 + m_b(x_l, x_k))} + \frac{1}{2}N(x_l, x_k) + sm_b(x_{k+1}, x_k) \\ &\leq \frac{\delta'}{8} + \frac{m_b(x_k, x_{k+1})m_b(x_l, x_{l+1})}{2(1 + m_b(x_l, x_k))} + \frac{1}{2}N(x_l, x_k) + s\frac{\delta'}{4}. \end{aligned}$$

On the other hand, note that

$$\frac{m_b(x_k, x_{k+1})}{1 + m_b(x_l, x_k)} \leq m_b(x_k, x_{k+1}) < \frac{\delta'}{4} < 1.$$

Hence,

$$\begin{aligned} m_b(Tx_l, x_l) - m_{bTx_l, x_l} &\leq m_b(Tx_l, x_k) \\ &= \frac{\delta'}{8} + \frac{1}{2}m_b(x_l, x_{l+1}) + \frac{1}{2}N(Tx_l, x_k) + s\frac{\delta'}{4} \\ &< \left[\frac{\delta'}{8} + \frac{\delta'}{8} + s\epsilon \right] + s\frac{\delta'}{4} \\ &\leq s\left(\frac{\delta'}{2} + 2\epsilon\right). \end{aligned}$$

Therefore, for all $m > k$, we have

$$m_b(x_m, x_k) - m_{bx_m, x_k} < s \left(\frac{\delta'}{2} + 2\epsilon \right).$$

Now, for every $m, n \in \mathbb{N}$ such that $m > n > k$, we have

$$\begin{aligned} m_b(x_m, x_n) - m_{bx_m, x_n} &\leq s [(m_b(x_m, x_k) - m_{bx_m, x_k}) + (m_b(x_k, x_n) - m_{bx_k, x_n})] \\ &\leq sm_b(x_m, x_k) + sm_b(x_k, x_n) \\ &< s.s\left(\frac{\delta'}{2} + 2\epsilon\right) + s.s\left(\frac{\delta'}{2} + 2\epsilon\right) \\ &= s^2(4\epsilon + \delta') \leq 5s^2\epsilon \end{aligned}$$

which implies that $\lim_{n,m \rightarrow \infty} m_b(x_m, x_n) - m_{bx_m, x_n}$ exists and finite. Using the same argument it is not difficult to show that $\lim_{n,m \rightarrow \infty} M_b(x_m, x_n) - m_{bx_m, x_n}$ exists and finite. Therefore, the sequence $\{x_n\}$ is an m_b -Cauchy sequence and since X is complete, there exists $u \in X$ such that $\lim_{n \rightarrow \infty} M_{bx_n, u} - m_{bx_n, u} = 0$.

Finally, we show that u is a fixed point for T ; that is $Tu = u$.

$$\begin{aligned} \lim_{n \rightarrow \infty} (M_{bx_n, u} - m_{bx_n, u}) &= 0 \\ \lim_{n \rightarrow \infty} (M_{bx_{n+1}, u} - m_{bx_{n+1}, u}) &= 0 \\ \lim_{n \rightarrow \infty} (M_{bTx_n, u} - m_{bTx_n, u}) &= 0 \\ M_b(Tu, u) - m_{bTu, u} &= 0 \end{aligned}$$

Then, $M_{bTu, u} = m_{bTu, u}$, and similarly by the convergence of x_n we obtain that $m_b(Tu, u) = m_{bTu, u}$, which implies that $Tu = u$ as required. \square

Definition 9. Let (X, m_b) be an m_b -metric space and let T be a self mapping on X . T is called m_b -orbitally continuous if whenever

$$\lim_{n \rightarrow +\infty} m_b(T^n x, z) = m_b(z, z) \Rightarrow \lim_{n \rightarrow +\infty} m_b(TT^n x, Tz) = m_b(Tz, Tz) \forall x, z \in X. \tag{7}$$

Remark 2. Note that, continuous mappings are m_b -orbitally continuous. But the converse is not necessary true, for example, consider the m_b -metric space defined by $m_b(x, y) = [\max(x, y)]^q$ ($q \geq 1$) for all $x, y \in X$ where $X = [0, 1]$ and the map $T : X \rightarrow X$ defined by

$$T = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

It is not difficult to see that T is not continuous, but T is m_b -orbitally continuous.

Theorem 3. Let (X, m_b) be a complete m_b -metric space with coefficient s and $T : X \rightarrow X$ be a mapping. Suppose that the following conditions hold:

- a) T is an m_b -orbitally continuous generalized Meir-Keeler contraction of type (I),
- b) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1,$
- c) if $\{x_n\}$ is a sequence in X such that $x_n \rightarrow z$ as $n \rightarrow \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1,$
- d) $s > 1$ or β is a continuous function.
then, T has a fixed point in X .

Proof. Let $x_0 \in X$ be such that condition b) holds and define $\{x_n\}$ in X so that $x_1 = Tx_0, x_{n+1} = Tx_n \forall n \in \mathbb{N}$. Without loss of generality, we may suppose that $x_{n+1} \neq x_n \forall n \in \mathbb{N} \cup 0$. Since T is α -admissible, then $\alpha(x_n, x_{n+1}) \geq 1 \forall n \in \mathbb{N}$.

As T is a generalized Meir-Keeler contraction of type (I), then by replacing x by x_n and y by x_{n+1} in (4), we observe that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \beta(m_b(x_n, x_{n+1}))M(x_n, x_{n+1}) < \epsilon + \delta \implies \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) < \epsilon \quad (8)$$

where

$$M(x_n, x_{n+1}) = \max[m_b(x_n, x_{n+1}), m_b(x_{n+1}, x_{n+2})]. \quad (9)$$

Next, we distinguish two following cases:

Case 1. Assume that $M(x_n, x_{n+1}) = m_b(x_{n+1}, x_{n+2})$.

In this case, equation (8) becomes

$$\epsilon \leq \beta(m_b(x_n, x_{n+1}))m_b(x_{n+1}, x_{n+2}) < \epsilon + \delta \implies \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) < \epsilon$$

and using that $\alpha(x_n, x_{n+1}) \geq 1 \forall n \in \mathbb{N}$, we have

$$m_b(Tx_n, Tx_{n+1})m_b(x_{n+1}, x_{n+2}) < \epsilon \leq \beta(m_b(x_n, x_{n+1}))m_b(x_{n+1}, x_{n+2}).$$

Then $m_b(x_{n+1}, x_{n+2}) < m_b(x_{n+1}, x_{n+2}) \forall n \in \mathbb{N}$ which gives a contradiction.

Case 2. Assume that $M(x_n, x_{n+1}) = m_b(x_n, x_{n+1})$.

Since $M(x_n, x_{n+1}) > 0 \forall n \in \mathbb{N}$ due to Remark 1, we get

$$\begin{aligned} m_b(x_{n+1}, x_{n+2}) &\leq \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) \\ &< \epsilon \\ &\leq \beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1}) \\ &< \frac{1}{s}m_b(x_n, x_{n+1}) \leq m_b(x_n, x_{n+1}). \end{aligned} \quad (10)$$

That is $\{m_b(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence in \mathbb{R}^+ and it converges to some $r \geq 0$. Let prove that $r = 0$.

Let be untrue, then we have $r > 0$. We assert that $0 < r \leq m_b(x_n, x_{n+1}) \forall n \in \mathbb{N}$.

First, suppose that $s > 1$. Applying equation (10), we have $m_b(x_{n+1}, x_{n+2}) < \frac{1}{s}m_b(x_n, x_{n+1})$.

By taking the limit as n tends to infinity, we get $r \leq \frac{1}{s}r < r$ which is a contradiction and so $r = 0$.

Next, suppose that β is a continuous function. We prove in the following claim that $\{\beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence in \mathbb{R}^+ .

Claim 1. Let $\beta : [0, \infty[\rightarrow [0, \frac{1}{s})$ a continuous function. Then, $\{\beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1})\}$ is strictly decreasing positive sequence in \mathbb{R}^+ .

First, note that

$$\begin{aligned} \beta(m_b(x_{n+1}, x_{n+2}))m_b(x_{n+1}, x_{n+2}) &< m_b(x_{n+1}, x_{n+2}) \\ &\leq \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) \\ &< \beta(m_b(x_n, x_{n+1}))M(x_n, x_{n+1}). \end{aligned}$$

If $M(x_n, x_{n+1}) = m_b(x_n, x_{n+1})$, we obtain

$$\beta(m_b(x_{n+1}, x_{n+2}))m_b(x_{n+1}, x_{n+2}) < \beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1}).$$

If $M(x_n, x_{n+1}) = m_b(x_{n+2}, x_{n+1})$, we have $m_b(x_{n+2}, x_{n+1}) < m_b(x_n, x_{n+1})$ (as $m_b(x_n, x_{n+1})$ is a strictly decreasing). Then, $\beta(m_b(x_{n+1}, x_{n+2}))m_b(x_{n+1}, x_{n+2}) < \beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1})$. Thus, $\{\beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1})\}$ is strictly decreasing positive sequence in \mathbb{R}^+ which prove our claim as desired.

From Claim 1, we have $\{\beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1})\}$ converges to some $r' \geq 0$. We consider the two following cases:

Case 1. $r' = 0$

Since $\lim_{n \rightarrow \infty} m_b(x_n, x_{n+1}) \neq 0$ so we have

$$\exists \epsilon > 0, \forall k \in \mathbb{N}, \exists n_k \geq k, m_b(x_{n_k}, x_{n_{k+1}}) \geq \epsilon.$$

Now, let $\epsilon' > 0$ be given. Since $\lim_{n \rightarrow \infty} \beta(m_b(x_{n_k}, x_{n_{k+1}}))m_b(x_{n_k}, x_{n_{k+1}}) = 0$.

Therefore, using (4), we derive

$$\exists k' \in \mathbb{N}, \forall k \geq k', \epsilon \beta(m_b(x_{n_k}, x_{n_{k+1}})) \leq \beta(m_b(x_{n_k}, x_{n_{k+1}}))m_b(x_{n_k}, x_{n_{k+1}}) < \epsilon'.$$

It enforces that $\lim_{n \rightarrow \infty} \beta(m_b(x_{n_k}, x_{n_{k+1}})) = 0$.

By continuity of β , we obtain $\beta(r) = 0 \implies r = 0$ which is a contradiction.

Case 2. $r' > 0$

we can distinguish two subcases: $r < r'$ and $r > r'$.

If $r < r'$, then $\beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1}) < \frac{1}{s}m_b(x_n, x_{n+1})$ and by taking the limit as n tends to infinity we get $r' \leq \frac{r}{s} \leq r$ which is a contradiction with $r' > 0$.

If $r > r'$, let $\delta > 0$ be such that satisfying (4) whenever $\epsilon = r'$. We know there exists $N_0 \in \mathbb{N}$ such that

$$r' \leq \beta(m_b(x_{N_0}, x_{N_0+1}))m_b(x_{N_0}, x_{N_0+1}) < r' + \delta.$$

Thus

$$\begin{aligned} r < m_b(x_{N_0+1}, x_{N_0+2}) &\leq \alpha(x_{N_0}, x_{N_0+1})m_b(Tx_{N_0}, Tx_{N_0+1}) \\ &< r' \leq r \end{aligned}$$

which leads to contradiction with $0 < r \leq m_b(x_n, x_{n+1}) \forall n \in \mathbb{N}$.

Thus, $r = 0$ and so $\lim_{n \rightarrow \infty} m_b(x_n, x_{n+1})$.

Next, we intend to show that the sequence $\{x_n\}$ is an m_b -Cauchy sequence. For this purpose, we will prove that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\lim_{n, m \rightarrow \infty} m_b(x_n, x_m) - m_{bx_{n,m}} < \infty$. We will prove that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$m_b(x_l, x_{l+k}) - m_{bx_{l,l+k}} < \epsilon. \tag{11}$$

Since the sequence $\{m_b(x_n, x_{n+1})\} \rightarrow 0, n \rightarrow \infty$, for every $\delta > 0$ there exists $N \in \mathbb{N}$ such that $m_b(x_n, x_{n+1}) < \delta$ for all $n \geq N$. Choose $\delta < \epsilon$. We will prove equation (11) by using induction on k .

- for $k = 1$, we have $m_b(x_l, x_{l+1}) < \epsilon \Rightarrow m_b(x_l, x_{l+1}) - m_{bx_{l,l+1}} < \epsilon$ so, (11) it clearly holds for all $l \geq N$ (due to the choice of δ).
- Assume that the inequality (11) holds for some $k = m$, that is $m_b(x_l, x_{l+m}) - m_{bx_{l,l+m}} < \epsilon \forall l \geq N$.

For $k = m + 1$, we have to show that

$$m_b(x_l, x_{l+m+1}) - m_{bx_{l,l+m+1}} < \epsilon \quad \forall l \geq N \tag{12}$$

Employing condition (4) of the definition fo the M_b -metric space, we get

$$\begin{aligned} m_b(x_{l-1}, x_{l+m}) - m_{bx_{l-1,l+m}} &< s[m_b(x_{l-1}, x_l) - m_{bx_{l-1,l}} + m_b(x_l, x_{l+m}) - m_{bx_{l,l+m}} - m_b(x_l, x_l)] \\ &\leq s[m_b(x_{l-1}, x_l) + m_b(x_l, x_{l+m})] \\ &\leq s[\delta + \epsilon] \quad \forall l \geq N. \end{aligned}$$

If $\beta(m_b(x_{l-1}, x_{l+m}))m_b(x_{l-1}, x_{l+m}) \geq \epsilon$, then we deduce

$$\begin{aligned} \epsilon &\leq \beta(m_b(x_{l-1}, x_{l+m}))m_b(x_{l-1}, x_{l+m}) \\ &\leq \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) \\ &= \beta(m_b(x_{l-1}, x_{l+m}))\max[m_b(x_{l-1}, x_{l+m}), m_b(x_l, x_{l-1}), m_b(x_{l+m+1}, x_{l+m})] \\ &< \beta(m_b(x_{l-1}, x_{l+m}))\max[s(\delta + \epsilon), \delta, \delta] \\ &< \delta + \epsilon. \end{aligned}$$

Using (8) with $x = x_{l-1}, y = x_{l+m}$, we find

$$\epsilon \leq \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) < \epsilon + \delta,$$

then

$$\alpha(x_{l-1}, x_{l+m})m_b(Tx_{l-1}, Tx_{l+m}) < \epsilon$$

which gives $m_b(x_l, x_{l+m+1}) < \epsilon$. Hence, (8) holds for $k = m + 1$.

If $\beta(m_b(x_{l-1}, x_{l+m}))m_b(x_{l-1}, x_{l+m}) < \epsilon$, then

$$\begin{aligned} \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) &= \beta(m_b(x_{l-1}, x_{l+m}))\max[m_b(x_{l-1}, x_{l+m}), \\ &\quad m_b(x_l, x_{l-1}), m_b(x_{l+m+1}, x_{l+m})] \\ &< \beta(m_b(x_{l-1}, x_{l+m}))\max[m_b(x_{l-1}, x_{l+m}), \delta, \delta] \\ &< \epsilon. \end{aligned}$$

From Remark 1, we get

$$\alpha(x_{l-1}, x_{l+m})m_b(Tx_{l-1}, Tx_{l+m}) < \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) < \epsilon$$

then

$$\alpha(x_{l-1}, x_{l+m})m_b(x_l, x_{l+m+1}) < \epsilon.$$

So

$$m_b(x_l, x_{l+m+1}) < \alpha(x_{l-1}, x_{l+m})m_b(x_l, x_{l+m+1}) < \epsilon$$

that is (11) holds for $k = m + 1$.

Note that $M(x_{l-1}, x_{l+m}) > 0$, otherwise $m_b(x_l, x_{l-1}) = 0$ and hence $x_l = x_{l-1}$, which is contradiction. Thus, $m_b(x_l, x_{l+k}) < \epsilon \forall l \geq N$ and $k \geq 1$, it means

$$m_b(x_n, x_m) < \epsilon \quad \forall \quad m \geq n \geq N. \tag{13}$$

Hence, it is easy to deduce that $\{x_n\}$ is an m_b -Cauchy sequence. Since X is a complete m_b -metric space, there exists $u \in X$ such that $\lim_{n \rightarrow \infty} (M_{bx_n, u} - m_{bx_n, u}) = 0$.

Now, we will show that $Tu = u$ for any $n \in \mathbb{N}$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} (M_{bx_n, u} - m_{bx_n, u}) &= 0 \\ \lim_{n \rightarrow \infty} (M_{bx_{n+1}, u} - m_{bx_{n+1}, u}) &= 0 \\ \lim_{n \rightarrow \infty} (M_{bTx_n, u} - m_{bTx_n, u}) &= 0 \\ M_b(Tu, u) - m_{bTu, u} &= 0 \end{aligned}$$

Then, $M_{bTu, u} = m_{bTu, u}$, and similarly by the convergence we obtain that $m_b(Tu, u) = m_{bTu, u}$, which implies that $Tu = u$ as desired. \square

Next, we prove the same result for a self mapping T on X which is an m_b -orbitally continuous generalized Meir-Keeler contraction of type (II).

Theorem 4. *Let (X, m_b) be a complete m_b -metric space, $T : X \rightarrow X$ be a mapping. Assume that the following conditions are satisfied:*

- a) T is an m_b -orbitally continuous generalized Meir-Keeler contraction of type (II),
- b) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0)$,
- c) If $\{x_n\}$ is a sequence in X such that $x_n \rightarrow z$ as $n \rightarrow \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1$,
- d) $s > 1$ or β is a continuous function, then T has a unique fixed point in X .

Proof. By remark 1, we have $N(x, y) \leq M(x, y)$. Hence, similarly to the proof of theorem 3, the result of our theorem will follow as desired. \square

Theorem 5. *Let (X, m_b) be a complete m_b -metric space with coefficient s and satisfies the following conditions:*

- a) if $\{x_n\}$ is a sequence in X which converges to z with respect to τ_{m_b} and satisfies $\alpha(x_{n+1}, x_n) \geq 1$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n , then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_z, x_{n_k}) \geq 1$ and $\alpha(x_{n_k}, x_z) \geq 1$ for all k ,
- b) $T : X \rightarrow X$ is a generalized Meir-Keeler contraction of type (II),
- c) there exists $x_0 \in X$ such that $\alpha(x_0, T_{x_0}) \geq 1$, $\alpha(T_{x_0}, x_0) \geq 1$
- d) $s > 1$ or β is a continuous function then, T has a fixed point in X .

Proof. By the proof of theorem 2, one can easily deduce that $\{x_n\}$ defined by $x_1 = T_{x_0}$ and $x_{n+1} = T_{x_n}$ ($n \in \mathbb{N}$) converges to some $z \in X$ with $m_b(z, z) = 0$, by condition a), there exist a subsequence $\{x_{n_k}\}$ of x_n such that $\alpha(z, x_{n_k}) \geq 1$ and $\alpha(x_{n_k}, z) \geq 1$ for all k .

Note that, if $N(z, x_{n_k}) = 0$, then $T_z = z$ and we are done.

Now, by remark 1 for all $k \in \mathbb{N}$ we have

$$\begin{aligned} m_b(T_z, x_{n+1}) = m_b(T_z, T_{x_n}) &\leq \alpha(z, x_{n_k})m_b(T_z, T_{x_{n_k}}) \\ &< \beta(m_b(z, x_{n_k}))N(z, x_{n_k}). \end{aligned}$$

Taking the limit $k \rightarrow \infty$ we obtain $\lim_{k \rightarrow \infty} N(z, x_{n_k}) = \max\{0, \frac{1}{2}m_b(T_z, z)\} = \frac{1}{2}m_b(T_z, z)$.

Thus, $\lim_{n \rightarrow \infty} m_b(T_z, x_{n_{k+1}}) \leq \frac{1}{2s}m_b(T_z, z)$. Hence,

$$m_b(T_z, z) \leq sm_b(T_z, x_{n_{k+1}}) + sm_b(x_{n_{k+1}}, z).$$

Taking the limit $k \rightarrow \infty$ we obtain

$$m_b(T_z, z) \leq \frac{1}{2}m_b(T_z, z).$$

which implies $m_b(T_z, z) = 0$, similarly we can show that $M_{bT_z, z} = 0$ and therefore, $T_z = z$ as desired. □

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Generalized Ulam-Hyers Stability for Generalized types of $(\gamma - \psi)$ -Meir-Keeler Mappings via Fixed Point Theory in S -metric spaces

Mi Zhou¹, Xiao-lan Liu^{2,3*}, Arslan Hojat Ansari⁴, Yeol Je Cho⁵, Stojan Radenović⁶

Abstract: In this paper, we introduce several extensions of Meir-Keeler contractive mappings in the structure of S -metric spaces. Then we investigate some existence, uniqueness, and generalized Ulam-Hyers stability results for the classes of MKC mappings via fixed point theory. Besides the theoretical results, we also present some illustrative examples to verify the effectiveness and applicability of our main results.

MSC: 47H10;54H25

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1. Introduction

1.1. S -metric spaces

Very recently, Sedghi et al.[1] have introduced the notion of an S -metric space and proved that this notion is a generalization of a G -metric space and D^* -metric space. Also, they have proved some properties of an S -metric and some fixed point results for a self-map on S -metric spaces. After that, many interesting results were obtained by transporting certain results in metric spaces and known generalizes metric spaces to S -metric spaces, see ([2]-[10]).

First, we recall the definition of an S -metric space and some useful notions and lemmas for the following discussion.

In the sequel, the letters \mathbb{N}, \mathbb{R}^+ and \mathbb{R} will denote the sets of positive integers, nonnegative real numbers and real numbers, respectively.

Definition 1.1. [1] Let X be a nonempty set. An S -metric on X is a function $S : X^3 \mapsto [0, \infty)$ that satisfies the following conditions for $\forall x, y, z, a \in X$:

- (S1) $S(x, y, z) = 0$ if and only if $x = y = z$;
- (S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

*Correspondence: stellalwp@163.com

² College of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China

³ Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, Zigong, Sichuan 643000, China

Full list of author information is available at the end of the article

The pair (X, S) is called an S -metric space.

Immediate examples of such S -metric spaces are:

- (1) Let $X = \mathbb{R}^+$ and $\|\cdot\|$ be a norm on X , then $S(x, y, z) = \|2x + y - 3z\| + \|x - z\|$ is an S -metric on X , for $\forall x, y, z \in X$.
- (2) Let X be a nonempty set, d is ordinary metric on X , the $S_d(x, y, z) = d(x, z) + d(y, z)$ is an S -metric on X , for $\forall x, y, z \in X$.

Lemma 1.1. [1] Let (X, S) be an S -metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z), \text{ and } S(x, x, z) \leq 2S(x, x, y) + S(z, z, y),$$

for $\forall x, y, z \in X$.

Lemma 1.2. [1] Let (X, S) be an S -metric space. Then $S(x, x, y) = S(y, y, x)$, for $\forall x, y \in X$.

Lemma 1.3. Let (X, S) be an S -metric space. Then, for $\forall x, y, z \in X$, it follows that

- (1) $S(x, y, y) \leq S(x, x, y)$.
- (2) $S(x, y, x) \leq S(x, x, y)$.
- (3) $S(x, y, z) \leq S(x, x, z) + S(y, y, z)$.
- (4) $S(x, y, z) \leq S(y, y, z) + S(x, x, z)$.
- (5) $S(x, y, z) \leq S(y, y, x) + S(z, z, x)$.
- (6) $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$.
- (7) $S(x, y, z) \leq \frac{2}{3}[S(x, x, y) + S(y, y, z) + S(z, z, x)]$.

Proof. It follows from (S2) and Lemma 1.2, one can easily obtain (1) – (5). Now, we prove (6) and (7) also hold true.

By Lemma 1.1 and Lemma 1.2, we have

$$\begin{aligned} 2S(x, x, z) &= S(x, x, z) + S(z, z, x) \\ &\leq [2S(x, x, y) + S(y, y, z)] + [2S(z, z, y) + S(x, x, y)] \\ &= 3[S(y, y, z) + S(y, y, x)]. \end{aligned}$$

Hence, $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$. Then (6) holds true.

By virtue of (3) – (5) and Lemma 1.2, we have $3S(x, y, z) = 2[S(x, x, y) + S(y, y, z) + S(z, z, x)]$, which implies (7) holds true. □

Definition 1.2. [1] Let (X, S) be an S -metric space.

- (1) A sequence $\{x_n\} \subset X$ is said to convergent to $x \in X$ if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall n \geq n_0$, we have $S(x_n, x_n, x) < \epsilon$.
- (2) A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if $S(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall n, m \geq n_0$, we have $S(x_n, x_n, x_m) < \epsilon$, or for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall l, m, n \geq n_0$, we have $S(x_l, x_m, x_n) < \epsilon$.

(3) The S -metric space (X, S) is said to be complete if every Cauchy sequence is a convergent sequence.

(4) A mapping $T : X \mapsto X$ is said to be S -continuous if $\{Tx_n\}$ is S -convergent to Tx , where $\{x_n\}$ is an S -convergent sequence converging to x .

Lemma 1.4. [1] Let (X, S) be an S -metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, then $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$.

Lemma 1.5. [1] Let (X, S) be an S -metric space. If the sequences $\{x_n\}$ in X such that $x_n \rightarrow x$, then x is unique.

1.2. The generalized Ulam-Hyers Stability

The stability problem of functional equations, originated from a question of Ulam [11], in 1940, concerns the stability of group homomorphism which stated as follows:

Let G_1 be a group and G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if a function $h : G_1 \mapsto G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta,$$

$\forall x, y \in G_1$, then there is a homomorphism $H : G_1 \mapsto G_2$ with $d(h(x), H(x)) < \epsilon, \forall x \in G_1$?

If the answer is affirmative, then we say that the equation of homomorphism $H(xy) = H(x)H(y)$ is stable. The first affirmative partial answer to the equation of Ulam for Banach spaces was given by Hyers [12] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability and has attracted attentions of many mathematicians.

In particular, Ulam-Hyers stability results in fixed point theory and remarkable results on the stability of certain classes of functional equation via fixed point approach have been studied densely, see ([13]-[16]).

Definition 1.3. Let (X, S) be an S -metric space and $T : X \mapsto X$ be a mapping. By definition, the fixed point equation

$$x = Tx, \quad x \in X \tag{1}$$

is said to be generalized Ulam-Hyers stable in the framework of an S -metric space if there exists an increasing operator $\varphi : [0, \infty) \mapsto [0, \infty)$, continuous at 0 and $\varphi(0) = 0$, such that for each $\epsilon > 0$ and an ϵ -solution $w^* \in X$, that is

$$S(w^*, Tw^*, Tw^*) \leq \epsilon, \tag{2}$$

there is a solution $x^* \in X$ of the fixed point equation (1) such that

$$S(w^*, x^*, x^*) \leq \varphi(\epsilon). \tag{3}$$

If $\varphi(t) = ct, \forall t \geq 0$, where $c > 0$, then (1) is said to be Ulam-Hyers stable in the framework of an S -metric space.

1.3. The generalized $(\gamma - \psi)$ -Meir-Keeler contractive mappings

In 1969, Meir and Keeler [17] established a fixed point theorem in a metric space (X, d) for mappings satisfying the condition that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \epsilon, \tag{4}$$

$\forall x, y \in X$. This condition is called the Meir-Keeler contractive (*MKC*, for short) type condition. Since then, many authors extended and improved this condition and established fixed point results for new generalized conditions, see Maiti and Pal [18], Park and Rhoades [19], Mongkolkeha and Kuman [20] and so on. On the other hand, Samet et al.[21] introduced the concepts of $\alpha - \psi$ -contractive mapping and α -admissible mapping in metric spaces. Also they proved a fixed point theorem for $\alpha - \psi$ contractive mappings in complete metric spaces using the concept of α -admissible mappings. Motivated by Samet’s work, Latif et al.[22] introduced a new type of a generalized $(\alpha - \psi)$ -Meir-Keeler contractive mapping and established some interesting theorems on the existence of fixed points for such mappings via admissible mappings.

Admissible mappings in the setting of S -metric spaces can be defined as follows.

Definition 1.4. A mapping $T : X \mapsto X$ is called γ -admissible if for $\forall x, y, z \in X$, we have

$$\gamma(x, y, z) \geq 1 \Rightarrow \gamma(Tx, Ty, Tz) \geq 1,$$

where, $\gamma : X^3 \mapsto [0, \infty)$ is a given function. If in addition,

$$\begin{cases} \gamma(x, y, y) \geq 1 \\ \gamma(y, z, z) \geq 1 \end{cases} \text{ implies } \gamma(x, z, z) \geq 1, \quad \forall x, y, z \in X. \text{ Then } T \text{ is called triangular } \gamma\text{-admissible.}$$

Example 1.1. Let $X = [1, \infty)$ and $T : X \mapsto X$. Define $Tx = x^2$ and $\gamma(x, y, z) = \begin{cases} 2, & \text{if } x \geq y \geq z; \\ 0, & \text{otherwise.} \end{cases}$

Then T is γ -admissible.

Definition 1.5. We say that:

- (1) A sequence $\{x_n\}$ in X is (T, γ) -orbital if $x_n = T^n x_0$ and $\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \forall n \in \{0\} \cup \mathbb{N}$.
- (2) T is γ -orbital continuous if, for every (T, γ) -orbital sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \rightarrow Tx$ as $k \rightarrow \infty$.
- (3) X is (T, γ) -regular if, for every (T, γ) -orbital sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geq 1, \forall k \in \mathbb{N}$.
- (4) X is γ -regular if, for every sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \forall n \in \{0\} \cup \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geq 1, \forall k \in \mathbb{N}$.
- (5) X is (T, γ) -limit if, for every sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \forall n \in \{0\} \cup \mathbb{N}$, then $\gamma(x, Tx, Tx) \geq 1$.

Remark 1.1. (1) If T is continuous, then T is γ -orbital continuous (for any γ).

(2) If X is γ -regular, then X is also (T, γ) -regular (for any γ).

Lemma 1.6. Let $\gamma : X^3 \mapsto [0, \infty)$ and $T : X \mapsto X$ be γ -admissible with triangular admissibility. Assume that there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then $\gamma(x_m, x_n, x_n) \geq 1$, for $\forall m, n \in \mathbb{N}$ with $m < n$.

Proof. Since there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$, then from the definition of γ -admissibility, we deduce that $\gamma(x_1, x_2, x_2) = \gamma(Tx_0, Tx_1, Tx_1) \geq 1$.

By continuing this process, we get $\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \forall n \in 0 \cup \mathbb{N}$.

Suppose that $m < n$. Since $\begin{cases} \gamma(x_m, x_{m+1}, x_{m+1}) \geq 1 \\ \gamma(x_{m+1}, x_{m+2}, x_{m+2}) \geq 1, \end{cases}$

by the definition of triangular γ -admissibility, we deduce that $\gamma(x_m, x_{m+2}, x_{m+2}) \geq 1$. By continuing this process, we get $\gamma(x_m, x_n, x_n) \geq 1, \forall m, n \in \mathbb{N}$ with $m < n$. □

Let Ψ stand for the family of nondecreasing functions $\psi : [0, \infty) \mapsto [0, \infty)$ satisfying conditions:

(Ψ 1) $\sum_{n=1}^{\infty} \psi^n(t) < \infty, \forall t > 0$, where ψ^n is the n^{th} iterate of ψ ;

(Ψ 2) $\psi(0) = 0$.

Remark 1.2. For every function $\psi : [0, \infty) \mapsto [0, \infty)$ the following holds:

if ψ is nondecreasing, then for each $t > 0$,

$$\lim_{n \rightarrow \infty} \psi^n(t) = 0 \Rightarrow \psi(t) < t \Rightarrow \psi(0) = 0.$$

Therefore, if $\psi \in \Psi$, then for every $t > 0, \psi(t) < t$ and ψ is continuous at 0.

Definition 1.6. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a $(\gamma - \psi)$ -Meir-Keeler contractive mapping if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(S(x, y, y)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, y, y)S(Tx, Ty, Ty) < \epsilon, \forall x, y \in X.$$

Remark 1.3. It is easily shown that if $T : X \mapsto X$ is a $(\gamma - \psi)$ -Meir-Keeler contractive mapping, then

$$\gamma(x, y, y)S(Tx, Ty, Ty) < \psi(S(x, y, y)),$$

$\forall x, y \in X$, when $x \neq y$.

Definition 1.7. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a $(\gamma - \psi)$ -Meir-Keeler contractive mapping of dim3 if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(S(x, y, z)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \epsilon.$$

Remark 1.4. It is easily shown that if $T : X \mapsto X$ is a $(\gamma - \psi)$ -Meir-Keeler contractive mapping of dim3, then

$$\gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \psi(S(x, y, z)),$$

$\forall x, y, z \in X$ when $x \neq y \neq z$.

Definition 1.8. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of type A if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M_1(x, y)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, y, y)S(Tx, Ty, Ty) < \epsilon,$$

where $M_1(x, y) = \max\{S(x, y, y), S(x, Tx, Tx), S(y, Ty, Ty)\}, \forall x, y \in X$.

Definition 1.9. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of type B if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M_2(x, y)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, y, y)S(Tx, Ty, Ty) < \epsilon,$$

where $M_2(x, y) = \max\{S(x, y, y), \frac{1}{2}[S(x, Tx, Tx) + S(y, Ty, Ty)]\}, \forall x, y \in X$.

Remark 1.5. (1) It is obviously that $M_2(x, y) \leq M_1(x, y), \forall x, y \in X$, where $M_1(x, y), M_2(x, y)$ are defined in Definition 1.8 and Definition 1.9, respectively.

(2) Let $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of type A (resp., type B). Then $\gamma(x, y, y)S(Tx, Ty, Ty) < \psi(M_1(x, y))$, (resp., $\psi(M_2(x, y))$), $\forall x, y \in X$.

Definition 1.10. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of dim3 of type A if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M'_1(x, y, z)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \epsilon,$$

where

$$M'_1(x, y, z) = \max\{S(x, y, y), S(y, z, z), S(z, x, x), S(x, Tx, Tx), S(y, Ty, Ty)S(z, Tz, Tz)\}, \forall x, y, z \in X.$$

Definition 1.11. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of dim3 of type B if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M'_2(x, y, z)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Ty) < \epsilon,$$

where

$$M'_2(x, y, z) = \max\{S(x, y, y), S(y, z, z), S(z, x, x), \frac{1}{2}[S(x, Tx, Tx) + S(y, Ty, Ty)], \frac{1}{2}[S(y, Ty, Ty) + S(z, Tz, Tz)], \frac{1}{2}[S(z, Tz, Tz) + S(x, Tx, Tx)]\},$$

$\forall x, y, z \in X$.

Remark 1.6. (1) It is obviously that $M'_2(x, y, z) \leq M'_1(x, y, z), \forall x, y, z \in X$, where $M'_1(x, y, z), M'_2(x, y, z)$ are defined in Definition 1.10 and Definition 1.11, respectively.

(2) Let $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of dim3 of type A (resp., type B). Then $\gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \psi(M'_1(x, y, z))$, (resp., $\psi(M'_2(x, y, z))$), $\forall x, y, z \in X$.

2. Fixed point theorems for several types of $(\gamma - \psi)$ -Meir-Keeler contractive mappings in S -metric spaces

In this section, by introducing the class of $(\gamma - \psi)$ -Meir-Keeler contractive mapping and the classes of generalized $(\gamma - \psi)$ -Meir-Keeler contractive mappings, we study the existence and uniqueness of fixed points for these contractive mappings via γ -admissible mappings.

Proposition 2.1. Assume that T is γ -admissible and $(\gamma - \psi)$ -Meir-Keeler contractive. Let $x, y \in X$ such that $\gamma(x, y, y) \geq 1$. Then

$$\gamma(T^n x, T^n y, T^n y) \geq 1, \quad \forall n \in \mathbb{N}, \tag{5}$$

the sequence $\{S(T^n x, T^n y, T^n y)\}$ is non-increasing, bounded and $S(T^n x, T^n y, T^n y) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Since T is γ -admissible and $\gamma(x, y, y) \geq 1$, then (5) follows directly by induction on n .

Next, let $n \in \mathbb{N}$. If $T^n x \neq T^n y$, by (5) and Remark 1.3, it follows that

$$\begin{aligned} S(T^{n+1} x, T^{n+1} y, T^{n+1} y) &\leq \gamma(T^n x, T^n y, T^n y) S(T^{n+1} x, T^{n+1} y, T^{n+1} y) \\ &= \gamma(T^n x, T^n y, T^n y) S(T(T^n x), T(T^n y), T(T^n y)) \\ &< \psi(S(T^n x, T^n y, T^n y)) \\ &< S(T^n x, T^n y, T^n y). \end{aligned}$$

Else, if $T^n x = T^n y$, then $S(T^n x, T^n y, T^n y) = S(T^{n+1} x, T^{n+1} y, T^{n+1} y)$.

Eventually, we conclude that $\{S(T^n x, T^n y, T^n y)\}$ is a non-increasing and bounded sequence.

Hence, there exists $r \in [0, \infty)$ such that $\lim_{n \rightarrow \infty} S(T^n x, T^n y, T^n y) = r$.

In what follows, we will prove that $r = 0$. Suppose, on the contrary, that $r > 0$. Since T is a $(\gamma - \psi)$ -Meir-Keeler contractive mapping, for $\epsilon = \psi(r) > 0$, there exists $\delta > 0$ and a $p \in \mathbb{N}$ such that

$$\epsilon \leq \psi(S(T^p x, T^p y, T^p y)) < \epsilon + \delta \text{ implies } \gamma(T^p x, T^p y, T^p y) S(T^{p+1} x, T^{p+1} y, T^{p+1} y) < \epsilon.$$

By taking (5) into account, we get that

$$S(T^{p+1} x, T^{p+1} y, T^{p+1} y) < \epsilon = \psi(r) < r,$$

which is a contradiction, since $r = \inf\{S(T^n x, T^n y, T^n y)\}_{n=1}^{\infty}$.

Consequently, we have $\lim_{n \rightarrow \infty} S(T^n x, T^n y, T^n y) = 0$. □

Proposition 2.2. Assume that T is γ -admissible and $(\gamma - \psi)$ -Meir-Keeler contractive of dim3. Let $x, y, z \in X$ such that $\gamma(x, Tx, Tx) \geq 1, \gamma(y, Ty, Ty) \geq 1, \gamma(z, Tz, Tz) \geq 1$. Then

$$\gamma(T^n x, T^n y, T^n z) \geq 1, \quad \forall n \in \mathbb{N}, \tag{6}$$

the sequence $\{S(T^n x, T^n y, T^n z)\}$ is non-increasing, bounded and $S(T^n x, T^n y, T^n z) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Using similar process to the proof of Proposition 2.1, one can safely draw the conclusion. □

Theorem 2.1. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a $(\gamma - \psi)$ -MKC mapping. Assume that

(A1) T is γ -admissible;

(A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

(A3) T is γ -orbital continuous.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Due to assumption (A2), there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n, \forall n \in \{0\} \cup \mathbb{N}$. Note that if $x_{n_0} = x_{n_0+1}$ for some n_0 , then $x^* = x_{n_0}$ is a fixed point of T . So we suppose that $x_n \neq x_{n+1}$ for $\forall n \in \{0\} \cup \mathbb{N}$. Since T is γ -admissible, we have that

$$\gamma(x_0, x_1, x_1) = \gamma(x_0, Tx_0, Tx_0) \geq 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(x_1, x_2, x_2) \geq 1.$$

By induction, we get that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{7}$$

From (7) together with the assumption of the theorem that T is a $(\gamma - \psi)$ -MKC mapping, it follows that for $\forall n \in \mathbb{N}$, we have that

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(S(x_{n-1}, x_n, x_n)). \end{aligned}$$

Since $\psi \in \Psi$, by induction, we have that

$$S(x_n, x_{n+1}, x_{n+1}) < \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{8}$$

Using (S2) and (8), for $\forall m, n \in \mathbb{N}$ with $m < n$, we have that

$$\begin{aligned} S(x_m, x_n, x_n) &\leq 2 \sum_{k=m}^{n-2} S(x_k, x_{k+1}, x_{k+1}) + S(x_{n-1}, x_n, x_n) \\ &\leq 2 \sum_{k=m}^{n-2} \psi^k(S(x_0, x_1, x_1)) + \psi^{n-1}(S(x_0, x_1, x_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_1, x_1) > 0$, by Remark 1.2, we get that

$$\lim_{n,m \rightarrow \infty} S(x_m, x_n, x_n) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the S -metric space (X, S) .

As (X, S) is complete, then there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x^*) = 0. \tag{9}$$

Since T is γ -orbital continuous, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that Tx_{n_k} converges to Tx^* as $k \rightarrow \infty$. By the uniqueness of this limit, we get $x^* = Tx^*$, that is x^* is a fixed point of T . □

Theorem 2.2. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a $(\gamma - \psi)$ -MKC mapping of dim3. Assume that

- (A1) T is γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) T is γ -orbital continuous.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Due to assumption (A2), there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \{0\} \cup \mathbb{N}$. Note that if $x_{n_0} = x_{n_0+1}$ for some n_0 , then $x^* = x_{n_0}$ is a fixed point of T . So we suppose that $x_n \neq x_{n+1}$ for all $n \in \{0\} \cup \mathbb{N}$. Since T is γ -admissible, we have that

$$\gamma(x_0, x_1, x_1) = \gamma(x_0, Tx_0, Tx_0) \geq 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(x_1, x_2, x_2) \geq 1.$$

By induction, we get that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{10}$$

From (10) together with the assumption of the theorem that T is a $(\gamma - \psi)$ -MKC mapping of dim3, it follows that for $\forall n \in \mathbb{N}$, we have that

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)\gamma(x_n, x_{n+1}, x_{n+1})\gamma(x_n, x_{n+1}, x_{n+1})S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(S(x_{n-1}, x_n, x_n)). \end{aligned}$$

Since $\psi \in \Psi$, by induction, we have that

$$S(x_n, x_{n+1}, x_{n+1}) < \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}.$$

Using Lemma 1.3 and (10), for $l, m, n \in \mathbb{N}$ with $l < m < n$, we have that

$$\begin{aligned} S(x_l, x_m, x_n) &\leq S(x_l, x_l, x_m) + S(x_m, x_m, x_n) \\ &\leq 2 \sum_{k=l}^{m-2} S(x_k, x_{k+1}, x_{k+1}) + S(x_{m-1}, x_m, x_m) + 2 \sum_{k=m}^{n-2} S(x_k, x_{k+1}, x_{k+1}) + S(x_{n-1}, x_n, x_n) \\ &\leq 2 \sum_{k=l}^{m-2} \psi^k(S(x_0, x_1, x_1)) + \psi^{m-1}(S(x_0, x_1, x_1)) + 2 \sum_{k=m}^{n-2} \psi^k(S(x_0, x_1, x_1)) + \psi^{n-1}(S(x_0, x_1, x_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_1, x_1) > 0$, by Remark 1.2, we get that

$$\lim_{l, m, n \rightarrow \infty} S(x_l, x_m, x_n) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the S -metric space (X, S) .

As (X, S) is complete, then there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x^*) = 0.$$

Since T is γ -orbital continuous, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that Tx_{n_k} converges to Tx^* as $k \rightarrow \infty$. By the uniqueness of this limit, we get $x^* = Tx^*$, that is x^* is a fixed point of T . □

In the next theorems, we replace the γ -orbital continuity of T by a regularity condition or $(T - \gamma)$ -limit condition over the S -metric spaces (X, S) .

Theorem 2.3. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a $(\gamma - \psi)$ -MKC mapping. Assume that

- (A1) T is γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -regular.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Following the line of the proof of Theorem 2.1, it follows that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n, \forall n \in \{0\} \cup \mathbb{N}$ is a Cauchy sequence in the complete S -metric space (X, S) , that is convergent to $x^* \in X$.

Since $\{x_n\}$ is a (T, γ) -orbital sequence, by (A3), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x^*, x^*) \geq 1, \quad \forall k \in \mathbb{N}. \tag{11}$$

Using Remark 1.3 and (11), we have that

$$\begin{aligned} S(x_{n_k+1}, Tx^*, Tx^*) &= S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leq \gamma(x_{n_k}, x^*, x^*)S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leq \psi(S(x_{n_k}, x^*, x^*)). \end{aligned}$$

Letting $k \rightarrow \infty$, since ψ is continuous at $t = 0$, it follows that $S(x^*, Tx^*, Tx^*) = 0$, then $x^* = Tx^*$. \square

Theorem 2.4. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a $(\gamma - \psi)$ -MKC mapping of dim3. Assume that

- (A1) T is γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -limit.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Following the line of the proof of Theorem 2.1, it follows that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \in \{0\} \cup \mathbb{N}$ is a Cauchy sequence in the complete S -metric space (X, S) , that is convergent to $x^* \in X$.

By (A3), we have

$$\gamma(x^*, Tx^*, Tx^*) \geq 1. \tag{12}$$

Using Remark 1.4 and (12), we have that

$$\begin{aligned} S(x_{n+1}, Tx^*, Tx^*) &= S(Tx_n, Tx^*, Tx^*) \\ &\leq \gamma(x_n, x_{n+1}, x_{n+1})\gamma(x^*, Tx^*, Tx^*)\gamma(x^*, Tx^*, Tx^*)S(Tx_n, Tx^*, Tx^*) \\ &\leq \psi(S(x_n, x^*, x^*)). \end{aligned}$$

Letting $n \rightarrow \infty$, since ψ is continuous at $t = 0$, it follows that $S(x^*, Tx^*, Tx^*) = 0$, then $x^* = Tx^*$. \square

Example 2.1. Let $X = [0, \infty)$ be an S -metric space with the S -metric defined by $S(x, y, z) = |x - z| + |y - z|, \forall x, y, z \in X$. For $\forall k > 1$, consider the self-mapping $T : X \mapsto X$ given by

$$Tx = \begin{cases} e^{x-1}, & x \geq 1, \\ \frac{x^2}{4}, & 0 \leq x < 1. \end{cases}$$

Also, define $\gamma : X^3 \mapsto [0, 1)$ as

$$\gamma(x, y, z) = \begin{cases} 1, & x, y, z \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$.

Clearly, T is not continuous at $x = 1$. Then we will claim that T is a $(\gamma - \psi)$ -MKC.

Let $\epsilon > 0$ be given. Take $\delta = \epsilon$ and suppose that $\epsilon \leq \frac{1}{2}|x - y| < \epsilon + \delta$, we want to show that $\gamma(x, y, y)S(Tx, Ty, Ty) < \epsilon$.

Suppose that $\gamma(x, y, y) = 1$, then $x, y \in [0, \infty)$ and $|x + y| < 2$. So $Tx = \frac{x^2}{4} \in [0, 1)$, $Ty = \frac{y^2}{4} \in [0, 1)$. Hence, $S(Tx, Ty, Ty) = |\frac{x^2}{4} - \frac{y^2}{4}| = \frac{|x^2 - y^2|}{4} = \frac{|x+y||x-y|}{4} < \frac{|x-y|}{2} < \frac{\epsilon + \delta}{2} < \epsilon$.

Also, T is γ -admissible. To see that, let $x, y, z \in X$ such that $\gamma(x, y, z) \geq 1$, which implies that $x, y, z \in [0, 1)$. Due to the definitions of γ and T , we have that

$$Tx = \frac{x^2}{4} \in [0, 1), \quad Ty = \frac{y^2}{4} \in [0, 1), \quad Tz = \frac{z^2}{4} \in [0, 1).$$

Hence, $\gamma(Tx, Ty, Ty) \geq 1$. Moreover, there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Indeed, for any $x_0 \in [0, 1)$, we have $\gamma(x_0, \frac{x_0^2}{4}, \frac{x_0^2}{4}) \geq 1$.

Finally, let $\{x_n\}$ be a (T, γ) -orbital sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$. By the definition of γ , we have that $x_n \in [0, 1)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geq 1, \forall k \in \mathbb{N}$.

So we conclude that all the hypotheses of Theorem 2.3 are fulfilled. In fact, 0 and 1 are two fixed points of T .

Example 2.2. Let $X = [0, \infty)$ be an S -metric space with the S -metric defined by $S(x, y, z) = |x - z| + |y - z|, \forall x, y, z \in X$. For $\forall k > 1$, consider the self-mapping $T : X \mapsto X$ given by

$$Tx = \begin{cases} x^{x-1}, & x \geq 1, \\ \frac{x^2}{4}, & 0 \leq x < 1. \end{cases}$$

Also, define $\gamma : X^3 \mapsto [0, 1)$ as

$$\gamma(x, y, z) = \begin{cases} 1, & x, y, z \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$.

Clearly, T is not continuous at $x = 1$. Then we will claim that T is a $(\gamma - \psi)$ -MKC mapping of dim3.

Let $\epsilon > 0$ be given. Take $\delta = \epsilon$ and suppose that $\epsilon \leq \frac{1}{2}(|x - y| + |y - z|) < \epsilon + \delta$, we want to show that $\gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \epsilon$.

Suppose that $\gamma(x, Tx, Tx) = \gamma(y, Ty, Ty) = \gamma(z, Tz, Tz) = 1$, then $x, y, z, Tx, Ty, Tz \in [0, 1)$ and $|x + y| < 2, |y + z| < 2$. So $Tx = \frac{x^2}{4} \in [0, 1), Ty = \frac{y^2}{4} \in [0, 1), Tz = \frac{z^2}{4} \in [0, 1)$.

Hence,

$$\begin{aligned}
 S(Tx, Ty, Tz) &= \left| \frac{x^2}{4} - \frac{y^2}{4} \right| + \left| \frac{y^2}{4} - \frac{z^2}{4} \right| \\
 &= \frac{|x^2 - y^2|}{4} + \frac{|y^2 - z^2|}{4} \\
 &= \frac{|x + y||x - y|}{4} + \frac{|y + z||y - z|}{4} \\
 &< \frac{|x - y|}{2} + \frac{|y - z|}{2} \\
 &< \frac{\epsilon + \delta}{2} \\
 &= \epsilon.
 \end{aligned}$$

Also, T is γ -admissible. To see that, let $x, y, z \in X$ such that $\gamma(x, y, z) \geq 1$, which implies that $x, y, z \in [0, 1)$. Due to the definitions of γ and T , we have that

$$Tx = \frac{x^2}{4} \in [0, 1), \quad Ty = \frac{y^2}{4} \in [0, 1), \quad Tz = \frac{z^2}{4} \in [0, 1).$$

Hence, $\gamma(Tx, Ty, Tz) \geq 1$. Moreover, there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Indeed, for any $x_0 \in [0, 1)$, we have $\gamma(x_0, \frac{x_0^2}{4}, \frac{x_0^2}{4}) \geq 1$.

Finally, let $\{x_n\}$ be a sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$ with $\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1$. By the definition of γ , we have that $x, Tx \in [0, 1)$. Then $\gamma(x, Tx, Tx) \geq 1$.

So we conclude that all the hypotheses of Theorem 2.4 are fulfilled. In fact, 0 and 1 are two fixed points of T .

Now, we propose the following conditions for the uniqueness of a fixed point of a $(\gamma - \psi)$ -MKC mapping and a $(\gamma - \psi)$ -MKC mapping of dim3. Let $Fix(T)$ denote the set of fixed points of the mapping T .

(U1) For $\forall x, y \in Fix(T)$, there exists $z \in X$ such that $\gamma(x, z, z) \geq 1$ and $\gamma(y, z, z) \geq 1$.

Theorem 2.5. Adding the condition (U1) to the hypotheses of Theorem 2.1 (resp. Theorem 2.3), we obtain the uniqueness of a fixed point T .

Proof. Let $u, v \in X$ be two fixed points of T . By (U1), there exists $z \in X$ such that $\gamma(u, z, z) \geq 1$ and $\gamma(v, z, z) \geq 1$.

Since T is γ -admissible, we get by induction that

$$\gamma(u, u, T^n z) \geq 1 \quad \text{and} \quad \gamma(v, v, T^n z) \geq 1, \quad \forall n \in \mathbb{N}. \tag{13}$$

From (13), we have that

$$\begin{aligned}
 S(u, u, T^n z) &= S(Tu, Tu, T(T^{n-1}z)) \\
 &\leq \gamma(u, u, T^{n-1}z) S(Tu, Tu, T(T^{n-1}z)) \\
 &< \psi(S(u, u, T^{n-1}z)).
 \end{aligned}$$

Iteratively, we get

$$S(u, u, T^n z) < \psi^n(S(u, u, z)).$$

Letting $n \rightarrow \infty$, and since $\psi \in \Psi$, we have that

$$\lim_{n \rightarrow \infty} S(u, u, T^n z) = 0. \tag{14}$$

Similarly, we also can get

$$\lim_{n \rightarrow \infty} S(v, v, T^n z) = 0. \tag{15}$$

Combining (14) and (15), it follows that $T^n z \rightarrow u$ and $T^n z \rightarrow v$, as $n \rightarrow \infty$.

By Lemma 1.5, we get $u = v$, that is, fixed point of T is unique. □

As an alternative uniqueness condition for fixed points of $(\gamma - \psi)$ -MKC mappings, we suggest the following hypothesis:

(U2) For $\forall x, y \in Fix(T)$, then $\gamma(x, y, y) \geq 1$.

Theorem 2.6. Adding the condition (U2) to the hypotheses of Theorem 2.1(resp.Theorem 2.3), we obtain the uniqueness of a fixed point T .

Proof. Let u, v be two distinct fixed points of T . Then $\gamma(u, v, v) > 0$.

Due to the property of ψ , we get that

$$\psi(S(u, v, v)) > 0.$$

Let $\epsilon = \psi(S(u, v, v)) > 0$; then, for any $\delta > 0$, we find that

$$\epsilon = \psi(S(u, v, v)) < \epsilon + \delta.$$

Considering (U2) and the assumption of theorem that T is a $(\gamma - \psi)$ -MKC mapping, we obtain that

$$S(u, v, v) \leq \gamma(u, v, v)S(Tu, Tv, Tv) < \psi(S(u, v, v)) < S(u, v, v),$$

which is a contradiction. Then $u = v$. □

As a uniqueness condition for fixed points of $(\gamma - \psi)$ -MKC mappings of dim3, we suggest the following hypothesis:

(U3) For $\forall x \in Fix(T)$, then $\gamma(x, x, x) \geq 1$.

Theorem 2.7. Adding the condition (U3) to the hypotheses of Theorem 2.2(resp.Theorem 2.4), we obtain the uniqueness of a fixed point T .

Proof. Let u, v be two distinct fixed points of T .

Due to the property of ψ , we get that $\psi(S(u, v, v)) > 0$.

Let $\epsilon = \psi(S(u, v, v)) > 0$; then, for any $\delta > 0$, we find that

$$\epsilon = \psi(S(u, v, v)) < \epsilon + \delta.$$

Considering (U3) and the assumption of theorem that T is a $(\gamma - \psi)$ -MKC mapping of dim3, we obtain that

$$S(u, v, v) \leq \gamma(u, Tu, Tu)\gamma(v, Tv, Tv)\gamma(v, Tv, Tv)S(Tu, Tv, Tv) < \psi(S(u, v, v)) < S(u, v, v),$$

which is a contradiction. Then $u = v$. □

Theorem 2.8. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -MKC mapping of type A . Assume also that:

- (A1) T is triangular γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -regular.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption (A2), let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n, \forall n \in \{0\} \cup \mathbb{N}$. Without loss of generality, we assume that $x_n \neq x_{n+1}$, for $\forall n \in \{0\} \cup \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{16}$$

Indeed, if there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_{n_0} = Tx^*$. Since T is triangular γ -admissible, by Lemma 1.6, we have that

$$\gamma(x_n, x_m, x_m) \geq 1, \quad \forall n, m \in \mathbb{N} \text{ with } n < m. \tag{17}$$

Step1. We will prove that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{18}$$

Taking (16) and (17) into account together with the fact that T is generalized $(\gamma - \psi)$ -MKC mapping of type A , for each $n \in \{0\} \cup \mathbb{N}$, we get

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M_1(x_{n-1}, x_n)) \\ &< \psi(M_1(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} M_1(x_{n-1}, x_n) &= \max\{S(x_{n-1}, x_n, x_n), S(x_{n-1}, Tx_{n-1}, Tx_{n-1}), S(x_n, Tx_n, Tx_n)\} \\ &= \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_{n+1})\}. \end{aligned}$$

If $M_1(x_{n-1}, x_n) = S(x_n, x_{n+1}, x_{n+1})$. Since ψ is nondecreasing, from the inequality above, we have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_n, x_{n+1}, x_{n+1})) < S(x_n, x_{n+1}, x_{n+1}), \quad \forall n \in \mathbb{N},$$

which is a contradiction. Thus, $M_1(x_{n-1}, x_n) = S(x_{n-1}, x_n, x_n)$ and we also have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_{n-1}, x_n, x_n)) < S(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}. \tag{19}$$

So, we deduce that the sequence $\{S(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = t. \tag{20}$$

Iteratively, we derive from (19) that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{21}$$

On account of (21) and Remark 1.2, we obtain

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{22}$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.

Suppose, on the contrary, that there exist $\epsilon > 0$ and a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that

$$S(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) > 2\epsilon. \tag{23}$$

First, we will show that the existence of $k \in \mathbb{N}$ such that $n(i) < k \leq n(i+1)$. Later, we will prove that for given $\epsilon > 0$ above, there exists $\delta > 0$ such that

$$\frac{\epsilon}{2} < \psi(M_1(x_{n(i)}, x_k)) < \frac{\epsilon + \delta}{2},$$

but

$$\gamma(x_{n(i)}, x_k, x_k)S(Tx_{n(i)}, Tx_k, Tx_k) \geq \epsilon,$$

which contradicts (23), where $M_1(x_{n(i)}, x_k) = \max\{S(x_{n(i)}, x_k, x_k), S(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}), S(x_k, x_{k+1}, x_{k+1})\}$

Let $r = \min\{\epsilon, \frac{\delta}{2}\}$. Taking Step1 into account, we will choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{r}{8}, \tag{24}$$

for all $n > n_0$. Let $n(i) > n_0$. According to our construction, we have $n(i) \leq n(i+1) - 1$.

If $S(x_{n(i)}, x_{n(i+1)-1}, x_{n(i+1)-1}) < \frac{\epsilon+r}{2}$, then by Lemma 1.1, we have

$$\begin{aligned} S(x_{n(i)}, x_{n(i+1)}, n_{n(i+1)}) &\leq 2S(x_{n(i)}, x_{n(i+1)-1}, n_{n(i+1)-1}) + S(x_{n(i+1)-1}, x_{n(i+1)}, n_{n(i+1)}) \\ &\leq \epsilon + r + \frac{r}{8} \\ &= \epsilon + \frac{7r}{8} \\ &< 2\epsilon, \end{aligned}$$

which contradicts (23). Consequently, there exist values of $k \in \mathbb{N}$ such that $n(i) \leq k \leq n(i+1)$ and $S(x_{n(i)}, x_k, x_k) > \frac{\epsilon+r}{2}$.

Indeed, if $S(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) \geq \frac{\epsilon+r}{2}$, then we have $S(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) \geq \frac{r}{8}$, which contradicts (24).

Hence, we can choose the smallest integer $k > n(i)$ such that

$$S(x_{n(i)}, x_k, x_k) \geq \frac{\epsilon + r}{2}.$$

So, necessarily, we also have $S(x_{n(i)}, x_{k-1}, x_{k-1}) < \frac{\epsilon+r}{2}$.

Therefore, we find that

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &\leq 2S(x_k, x_{k-1}, x_{k-1}) + S(x_{n(i)}, x_{k-1}, x_{k-1}) \\ &< 2 \cdot \frac{r}{8} + \frac{\epsilon + r}{2} \\ &= \frac{\epsilon}{2} + \frac{3r}{4}. \end{aligned}$$

Hence, we get the following approximation:

$$\frac{\epsilon + r}{2} \leq S(x_{n(i)}, x_k, x_k) \leq \frac{\epsilon}{2} + \frac{3r}{4}, \tag{25}$$

for a integer k satisfying $n(i) \leq k \leq n(i + 1)$.

On the other hand, the three terms of $M_1(x_{n(i)}, x_k)$ are bounded above by $\frac{\epsilon}{2} + r$, that is

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &< \frac{\epsilon}{2} + \frac{3r}{4} < \frac{\epsilon}{2} + r. \\ S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) &< \frac{r}{8} < \frac{\epsilon}{2} + r. \\ S(x_k, x_{k+1}, x_{k+1}) &< \frac{r}{8} < \frac{\epsilon}{2} + r. \end{aligned}$$

Combing these estimations presented above, we conclude that

$$\psi(M_1(x_{n(i)}, x_k)) < M_1(x_{n(i)}, x_k) < \frac{\epsilon}{2} + r < \frac{\epsilon + \delta}{2}.$$

Since T is generalized $(\gamma - \psi)$ -MKC mapping of type of A and it is γ -triangular admissible mapping, we have that

$$S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \leq \gamma(x_{n(i)}, x_k, x_k)S(x_{n(i)+1}, x_{k+1}, x_{k+1}) < \frac{\epsilon}{2}.$$

At the same time, by Lemma 1.1, we have that

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + S(x_{n(i)+1}, x_k, x_k) \\ &\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2S(x_k, x_{k+1}, x_{k+1}) + S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \\ &< 2 \cdot \frac{r}{8} + 2 \cdot \frac{r}{8} + \frac{\epsilon}{2} \\ &= \frac{\epsilon + r}{2}, \end{aligned}$$

which contradicts (25).

Thus, claim (23) is false and the sequence $\{x_n\}$ is a Cauchy sequence, that is

$$\lim_{n,m \rightarrow \infty} S(x_n, x_m, x_m) = 0. \tag{26}$$

Since (X, S) is a complete S -metric space, then there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x^*, x^*) = \lim_{n,m \rightarrow \infty} S(x_n, x_m, x^*) = 0. \tag{27}$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$.

From (27) and assumption (A3), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x^*, x^*) \geq 1, \quad \forall k \in \mathbb{N}. \tag{28}$$

By using Lemma 1.1 and (28) together with the assumption of the theorem that T is a generalized $(\gamma - \psi)$ -MKC mapping of type A , we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2S(Tx_{n_k}, Tx^*, Tx^*) + S(Tx_{n_k}, x^*, x^*) \\ &\leq 2\gamma(x_{n_k}, x^*, x^*)S(Tx_{n_k}, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*) \\ &\leq \psi(M_1(x_{n_k}, x^*)) + S(x_{n_k+1}, x^*, x^*), \end{aligned}$$

where, $M_1(x_{n_k}, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), S(x^*, Tx^*, Tx^*)\}$.
 Suppose that $M_1(x_{n_k}, x^*) = S(x_{n_k}, x^*, x^*)$, then from the above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*) \\ &< S(x_{n_k}, x^*, x^*) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, we have

$$S(x^*, Tx^*, Tx^*) < 2S(x^*, x^*, x^*) = 0,$$

which is a contradiction.

Next, we suppose that $M_1(x_{n_k}, x^*) = S(x_{n_k}, x_{n_k+1}, x_{n_k+1})$, then we have that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x_{n_k+1}, x_{n_k+1})) + S(x_{n_k+1}, x^*, x^*) \\ &< S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, this implies that

$$S(x^*, Tx^*, Tx^*) < \lim_{k \rightarrow \infty} [S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*)] = 0,$$

which is again a contradiction.

Finally, we suppose that $M_1(x_{n_k}, x^*) = S(x^*, Tx^*, Tx^*)$, then we obtain that

$$S(x^*, Tx^*, Tx^*) < \psi(S(x^*, Tx^*, Tx^*)) + S(x_{n_k+1}, x^*, x^*). \tag{29}$$

Letting $k \rightarrow \infty$ in (29), we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &< \psi(S(x^*, Tx^*, Tx^*)) + S(x^*, x^*, x^*) \\ &< S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*) \\ &= S(x^*, Tx^*, Tx^*), \end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by (S1) in Definition 1.1, we have $x^* = Tx^*$. □

Theorem 2.9. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -MKC mapping of dim3 of type A. Assume also that:

- (A1) T is triangular γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -limit.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption (A2), let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \{0\} \cup \mathbb{N}$. Without loss of generality, we assume that $x_n \neq x_{n+1}$, for $\forall n \in \{0\} \cup \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{30}$$

Indeed, if there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_{n_0} = Tx^*$. Since T is triangular γ -admissible, by Lemma 1.6, we have that

$$\gamma(x_m, x_n, x_n) \geq 1, \quad \forall m, n \in \mathbb{N} \text{ with } m < n. \tag{31}$$

Step1. We will prove that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Taking (30) and (31) into account together with the fact that T is generalized $(\gamma - \psi)$ -MKC mapping of dim3 of type A, for each $n \in \{0\} \cup \mathbb{N}$, we get

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)\gamma(x_n, x_{n+1}, x_{n+1})\gamma(x_n, x_{n+1}, x_{n+1})S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M'_1(x_{n-1}, x_n, x_n)) \\ &< M'_1(x_{n-1}, x_n, x_n), \end{aligned}$$

where

$$\begin{aligned} M'_1(x_{n-1}, x_n, x_n) &= \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_n), S(x_n, x_{n-1}, x_{n-1}), \\ &\quad S(x_{n-1}, Tx_{n-1}, Tx_{n-1}), S(x_n, Tx_n, Tx_n), S(x_n, Tx_n, Tx_n)\} \\ &= \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_{n+1})\}. \end{aligned}$$

If $M'_1(x_{n-1}, x_n, x_n) = S(x_n, x_{n+1}, x_{n+1})$. Since ψ is nondecreasing, from the inequality above, we have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_n, x_{n+1}, x_{n+1})) < S(x_n, x_{n+1}, x_{n+1}), \quad \forall n \in \mathbb{N},$$

which is a contradiction. Thus, $M'_1(x_{n-1}, x_n, x_n) = S(x_{n-1}, x_n, x_n)$ and we also have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_{n-1}, x_n, x_n)) < S(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}. \tag{32}$$

So, we deduce that the sequence $\{S(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = t.$$

Iteratively, we derive from (32) that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{33}$$

On account of (33) and Remark 1.2, we obtain

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.

We will prove that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall m, n \geq n_0$,

$$S(x_m, x_m, x_n) < \epsilon. \tag{34}$$

Taking Step1 into account, for each $\epsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{\epsilon - \psi(\epsilon)}{2}, \forall n \geq n_0. \tag{35}$$

We prove (34) by induction on n . (34) holds for $m = n$ and $n = n + 1$ by using (35) and the fact that $\frac{\epsilon - \psi(\epsilon)}{2} < \epsilon$. Assume (34) holds for $n = k$. For $n = k + 1$, we have

$$\begin{aligned} S(x_m, x_{k+1}, x_{k+1}) &\leq 2S(x_m, x_{m+1}, x_{m+1}) + S(x_{m+1}, x_{k+1}, x_{k+1}) \\ &\leq \epsilon - \psi(\epsilon) + \psi(S(x_m, x_k, x_k)) \\ &\leq \epsilon - \psi(\epsilon) + \psi(\epsilon) \\ &= \epsilon. \end{aligned}$$

By induction on n , we conclude that (34) holds for all $n \geq m \geq n_0$. So $\{x_n\}$ is a Cauchy sequence that is

$$\lim_{n, m \rightarrow \infty} S(x_n, x_m, x_m) = 0.$$

Since (X, S) is a complete S -metric space, then there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x^*, x^*) = \lim_{n, m \rightarrow \infty} S(x_n, x_m, x^*) = 0.$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$. From assumption (A3), we have that

$$\gamma(x^*, Tx^*, Tx^*) \geq 1, \quad \forall k \in \mathbb{N}.$$

By using Lemma 1.1 and above inequality together with the assumption of the theorem that T is a generalized $(\gamma - \psi)$ -MKC mapping of dim3 of type A , we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2S(Tx_{n_k}, Tx^*, Tx^*) + S(Tx_{n_k}, x^*, x^*) \\ &\leq 2\gamma(x_{n_k}, Tx_{n_k}, Tx_{n_k})\gamma(x^*, Tx^*, Tx^*)\gamma(x^*, Tx^*, Tx^*)S(Tx_{n_k}, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*) \\ &\leq 2\psi(M'_1(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*), \end{aligned}$$

where, $M'_1(x_{n_k}, x^*, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), S(x^*, Tx^*, Tx^*)\}$. Suppose that $M'_1(x_{n_k}, x^*, x^*) = S(x_{n_k}, x^*, x^*)$, then from the above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2\psi(S(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*) \\ &< 2S(x_{n_k}, x^*, x^*) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, we have

$$S(x^*, Tx^*, Tx^*) < 3S(x^*, x^*, x^*) = 0,$$

which is a contradiction.

Next, we suppose that $M'_1(x_{n_k}, x^*, x^*) = S(x_{n_k}, x_{n_k+1}, x_{n_k+1})$, then we have that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x_{n_k+1}, x_{n_k+1})) + S(x_{n_k+1}, x^*, x^*) \\ &< S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, this implies that

$$S(x^*, Tx^*, Tx^*) < \lim_{k \rightarrow \infty} [S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*)] = 0,$$

which is again a contradiction.

Finally, we suppose that $M_1'(x_{n_k}, x^*, x^*) = S(x^*, Tx^*, Tx^*)$, then we obtain that

$$S(x^*, Tx^*, Tx^*) < \psi(S(x^*, Tx^*, Tx^*)) + S(x_{n_{k+1}}, x^*, x^*).$$

Letting $k \rightarrow \infty$ in above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &< \psi(S(x^*, Tx^*, Tx^*)) + S(x^*, x^*, x^*) \\ &< S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*) \\ &= S(x^*, Tx^*, Tx^*), \end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by (S1) in Definition 1.1, we have $x^* = Tx^*$. □

Example 2.3. Let $X = [0, \infty)$ and $S(x, y, z) = |x - y| + |x - z|, \forall x, y, z \in X$. Then (X, S) is a complete S -metric space.

Define $T : X \mapsto X$ and $\gamma : X^3 \mapsto [0, \infty)$ as follow:

$$Tx = \begin{cases} kx - (k - 1), & k > 1, \quad x \geq 1; \\ \frac{x}{4}, & x \in [0, 1). \end{cases} \quad \text{and } \gamma(x, y, z) = \begin{cases} 1, & \text{if } x, y, z \in [0, 1); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}, t \geq 0$.

We first show that T is a triangular γ -admissible mapping. Let $x, y, z \in X$, if $\gamma(x, y, z) \geq 1$, the $x, y, z \in [0, 1)$. On the other hand, for $\forall x, y, z \in [0, 1)$, we have $Tx = \frac{x}{4} \in [0, 1), Ty = \frac{y}{4} \in [0, 1), Tz = \frac{z}{4} \in [0, 1)$. It follows that $\gamma(Tx, Ty, Tz) \geq 1$. Also, if $\gamma(x, y, y) \geq 1$ and $\gamma(y, y, z) \geq 1$, then $x, y, z \in [0, 1)$ and hence $\gamma(x, z, z) \geq 1$. Thus, the first assertion holds. Notice that $\gamma(0, 0, 0) = 1$.

Next, if $\{x_n\}$ is a (T, γ) -orbital sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$. By the definition of γ , we have that $x_n \in [0, 1)$ and $x \in [0, 1)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geq 1, \forall k \in \mathbb{N}$.

Finally, we will show that T is generalized $(\gamma - \psi)$ -MKC mapping of type A.

If $\gamma(x, y, y) = 0$, it is obviously to verify the assertion.

If $\gamma(x, y, y) \neq 0$, it follows that $x, y \in [0, 1)$ and $\gamma(x, y, y) = 1$.

For $\epsilon > 0$,

Case 1. If $M_1(x, y) = 2|x - y|$, taking $\delta = \epsilon$, then $\epsilon \leq \psi(M_1(x, y)) = |x - y| < 2\epsilon$ implies that $\gamma(x, y, y)S(Tx, Ty, Ty) = \frac{|x-y|}{2} < \epsilon$.

Case 2. If $M_1(x, y) = \frac{3|x|}{2}$, taking $\delta = \frac{\epsilon}{3}$, then $\epsilon \leq \psi(M_1(x, y)) = \frac{3|x|}{4} < \epsilon + \frac{\epsilon}{3}$ implies that $\gamma(x, y, y)S(Tx, Ty, Ty) = \frac{1}{2}|x - y| \leq \frac{1}{2}(|x| + |y|) < \frac{1}{2}(|x| + |x|) = |x| < \epsilon$.

Case 3. If $M_1(x, y) = \frac{3|y|}{2}$, taking $\delta = \frac{\epsilon}{3}$, then $\epsilon \leq \psi(M_1(x, y)) = \frac{3|y|}{4} < \epsilon + \frac{\epsilon}{3}$ implies that $\gamma(x, y, y)S(Tx, Ty, Ty) = \frac{1}{2}|x - y| \leq \frac{1}{2}(|x| + |y|) < \frac{1}{2}(|y| + |y|) = |y| < \epsilon$.

Therefore, conditions of Theorem 2.8 hold and T has a fixed point. Indeed, $x^* = 0$ and $x^* = 1$ are two fixed points.

In what follows, we present an existence theorem for fixed point of a generalized $(\gamma - \psi)$ -MKC mapping of type B and a generalized $(\gamma - \psi)$ -MKC mapping of type B. Taking Remark 1.5 and Remark 1.6 into account, we observe that the proof of this theorem is similar to the proof of Theorem 2.8 and Theorem 2.9.

Theorem 2.10. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -MKC mapping of type B. Assume also that:

- (A1) T is triangular γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -regular.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 2.11. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -MKC mapping of type B . Assume also that:

- (A1) T is triangular γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -limit.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Definition 2.1. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of type C if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M_3(x, y)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, y, y)S(Tx, Ty, Ty) < \epsilon, \tag{36}$$

where $M_3(x, y) = \max\{S(x, y, y), S(x, Tx, Tx), S(y, Ty, Ty), \frac{1}{8}[S(x, Ty, Ty) + S(y, Tx, Tx)]\}, \forall x, y \in X$.

Theorem 2.12. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -MKC mapping of type C . Assume also that:

- (A1) T is triangular γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -regular.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption (A2), let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$.

Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n, \forall n \in \{0\} \cup \mathbb{N}$.

Since T is triangular γ -admissible, by Lemma 1.6, we have that

$$\gamma(x_n, x_m, x_m) \geq 1, \quad \forall n, m \in \mathbb{N} \text{ with } n < m.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_0 = Tx^*$. For this, we assume that $x_n \neq x_{n+1}, \forall n \in \{0\} \cup \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{37}$$

Step1. We will prove that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{38}$$

Taking (36) and (38) into account together with the fact that T is generalized $(\gamma - \psi)$ -MKC mapping of type C , for each $n \in \{0\} \cup \mathbb{N}$, we get

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M_3(x_{n-1}, x_n)) \\ &< \psi(M_3(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} M_3(x_{n-1}, x_n) &= \max\{S(x_{n-1}, x_n, x_n), S(x_{n-1}, Tx_{n-1}, Tx_{n-1}), S(x_n, Tx_n, Tx_n), \\ &\quad \frac{1}{8}[S(x_{n-1}, Tx_n, Tx_n) + S(x_n, Tx_{n-1}, Tx_{n-1})]\} \\ &= \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_{n+1}), \frac{1}{8}[S(x_{n-1}, x_{n+1}, x_{n+1}) + S(x_n, x_n, x_n)]\}. \end{aligned}$$

Regarding Lemma 1.1, we estimate the last term in the expression of $M_3(x_{n-1}, x_n)$ as follows:

$$\begin{aligned} &\frac{1}{8}[S(x_{n-1}, x_{n+1}, x_{n+1}) + S(x_n, x_n, x_n)] \\ &= \frac{1}{8}S(x_{n-1}, x_{n+1}, x_{n+1}) \\ &\leq \frac{1}{8}[2S(x_{n-1}, x_n, x_n) + S(x_n, x_{n+1}, x_{n+1})] \\ &= \frac{1}{4}S(x_{n-1}, x_n, x_n) + \frac{1}{8}S(x_n, x_{n+1}, x_{n+1}) \\ &\leq \max\{S(x_{n-1}, x_{n+1}, x_{n+1}), S(x_n, x_n, x_n)\}. \end{aligned}$$

Consequently, we get that

$$M_3(x_{n-1}, x_n) = \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_{n+1}, x_{n+1})\}. \tag{39}$$

Let us consider the two cases. If $M_3(x_{n-1}, x_n) = S(x_n, x_{n+1}, x_{n+1})$. Since ψ is nondecreasing, then we have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_n, x_{n+1}, x_{n+1})) < S(x_n, x_{n+1}, x_{n+1}), \tag{40}$$

which is a contradiction. Thus, $M_3(x_{n-1}, x_n) = S(x_{n-1}, x_n, x_n)$ and we also have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_{n-1}, x_n, x_n)) < S(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}. \tag{41}$$

So, we derive that the sequence $\{S(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = t. \tag{42}$$

Recursively, we deduce from (41) that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{43}$$

On account of (43) and Remark 1.2, we obtain

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{44}$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.

Suppose, on the contrary, that there exist $\epsilon > 0$ and a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that

$$S(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) > 2\epsilon. \tag{45}$$

First, we will show that the existence of $k \in \mathbb{N}$ such that $n(i) < k \leq n(i+1)$. Later, we will prove that for given $\epsilon > 0$ above, there exists $\delta > 0$ such that

$$\frac{\epsilon}{2} < \psi(M_3(x_{n(i)}, x_k)) < \frac{\epsilon + \delta}{2},$$

but

$$\gamma(x_{n(i)}, x_k, x_k)S(Tx_{n(i)}, Tx_k, Tx_k) \geq \epsilon,$$

which contradicts (45), where

$$M_3(x_{n(i)}, x_k) = \max\{S(x_{n(i)}, x_k, x_k), S(x_{n(i)}, Tx_{n(i)}, Tx_{n(i)}), S(x_k, Tx_k, Tx_k), \\ \frac{1}{8}[S(x_{n(i)}, Tx_k, Tx_k) + S(x_k, Tx_{n(i)}, Tx_{n(i)})]\}.$$

Let $r = \min\{\epsilon, \frac{\delta}{2}\}$. Taking Step1 into account, we will choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{r}{8}, \tag{46}$$

for all $n > n_0$. Let $n(i) > n_0$. According to our construction, we have $n(i) \leq n(i+1) - 1$.

If $S(x_{n(i)}, x_{n(i+1)-1}, x_{n(i+1)-1}) < \frac{\epsilon+r}{2}$, then by Lemma 1.1, we have

$$S(x_{n(i)}, x_{n(i+1)}, n_{n(i+1)}) \leq 2S(x_{n(i)}, x_{n(i+1)-1}, n_{n(i+1)-1}) + S(x_{n(i+1)-1}, x_{n(i+1)}, n_{n(i+1)}) \\ \leq \epsilon + r + \frac{r}{8} \\ = \epsilon + \frac{7r}{8} \\ < 2\epsilon,$$

which contradicts (45). Therefore, there exist values of $k \in \mathbb{N}$ such that $n(i) \leq k \leq n(i+1)$ and $S(x_{n(i)}, x_k, x_k) > \frac{\epsilon+r}{2}$.

Indeed, if $S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) \geq \frac{\epsilon+r}{2}$, then we have $S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) \geq \frac{r}{8}$, which contradicts (46).

Hence, we can choose the smallest integer $k > n(i)$ such that

$$S(x_{n(i)}, x_k, x_k) \geq \frac{\epsilon + r}{2}.$$

So, necessarily, we also have $S(x_{n(i)}, x_{k-1}, x_{k-1}) < \frac{\epsilon+r}{2}$.

Therefore, we find that

$$S(x_{n(i)}, x_k, x_k) \leq 2S(x_k, x_{k-1}, x_{k-1}) + S(x_{n(i)}, x_{k-1}, x_{k-1}) \\ < 2 \cdot \frac{r}{8} + \frac{\epsilon + r}{2} \\ = \frac{\epsilon}{2} + \frac{3r}{4}.$$

Hence, we get the following approximation:

$$\frac{\epsilon + r}{2} \leq S(x_{n(i)}, x_k, x_k) \leq \frac{\epsilon}{2} + \frac{3r}{4}, \tag{47}$$

for a integer k satisfying $n(i) \leq k \leq n(i + 1)$.

On the other hand, the first three terms of $M_3(x_{n(i)}, x_k)$ are bounded above by $\frac{\epsilon}{2} + r$, that is

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &< \frac{\epsilon}{2} + \frac{3r}{4} < \frac{\epsilon}{2} + r. \\ S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) &< \frac{r}{8} < \frac{\epsilon}{2} + r. \\ S(x_k, x_{k+1}, x_{k+1}) &< \frac{r}{8} < \frac{\epsilon}{2} + r. \end{aligned}$$

Eventually, the last term of $M_3(x_{n(i)}, x_k)$ can be estimated as follows:

$$\begin{aligned} &\frac{1}{8}[S(x_{n(i)}, Tx_k, Tx_k) + S(x_k, Tx_{n(i)}, Tx_{n(i)})] \\ &\leq \frac{1}{8}[2S(x_{n(i)}, x_k, x_k) + S(x_k, x_{k+1}, x_{k+1}) + 2S(x_{n(i)}, x_k, x_k) + S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1})] \\ &= \frac{1}{8}[4S(x_{n(i)}, x_k, x_k) + S(x_k, x_{k+1}, x_{k+1}) + S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1})] \\ &< \frac{\epsilon}{4} + \frac{3r}{8} + \frac{r}{32} \\ &= \frac{\epsilon}{4} + \frac{13r}{32} \\ &< \frac{\epsilon}{2} + r. \end{aligned}$$

Combing these estimations presented above, we conclude that

$$\psi(M_3(x_{n(i)}, x_k)) < M_3(x_{n(i)}, x_k) < \frac{\epsilon}{2} + r < \frac{\epsilon + \delta}{2}.$$

Since T is generalized $(\gamma - \psi)$ -MKC mapping of type of C and it is γ -triangular admissible mapping, we have that

$$S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \leq \gamma(x_{n(i)}, x_k, x_k)S(x_{n(i)+1}, x_{k+1}, x_{k+1}) < \frac{\epsilon}{2}.$$

At the same time, by Lemma 1.1, we have that

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + S(x_{n(i)+1}, x_k, x_k) \\ &\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2S(x_k, x_{k+1}, x_{k+1}) + S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \\ &< 2 \cdot \frac{r}{8} + 2 \cdot \frac{r}{8} + \frac{\epsilon}{2} \\ &= \frac{\epsilon + r}{2}, \end{aligned}$$

which contradicts (47).

Thus, claim (45) is false and the sequence $\{x_n\}$ is a Cauchy sequence, that is

$$\lim_{n,m \rightarrow \infty} S(x_n, x_m, x_m) = 0. \tag{48}$$

Since (X, S) is a complete S -metric space, then there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x^*, x^*) = \lim_{n,m \rightarrow \infty} S(x_n, x_m, x^*) = 0.$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$.

From (42) and assumption (A3), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x^*, x^*) \geq 1, \quad \forall k \in \mathbb{N}. \tag{49}$$

By using Lemma 1.1 and (48) together with the assumption of the theorem that T is a generalized $(\gamma - \psi)$ -MKC mapping of type C , we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2S(Tx_{n_k}, Tx^*, Tx^*) + S(Tx_{n_k}, x^*, x^*) \\ &\leq 2\gamma(x_{n_k}, x^*, x^*)S(Tx_{n_k}, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*) \\ &\leq \psi(M_3(x_{n_k}, x^*)) + S(x_{n_k+1}, x^*, x^*), \end{aligned}$$

where,

$$\begin{aligned} M_3(x_{n_k}, x^*) &= \max\{S(x_{n_k}, x^*, x^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), S(x^*, Tx^*, Tx^*), \\ &\quad \frac{1}{8}[S(x_{n_k}, Tx^*, Tx^*) + S(x^*, x_{n_k+1}, x_{n_k+1})]\}. \end{aligned}$$

Notice that as $S(x^*, Tx^*, Tx^*) > 0$, then we have that $M_3(x_{n_k}, x^*) > 0$.

From Lemma 1.1, it follows that

$$\begin{aligned} &\frac{1}{8}[S(x_{n_k}, Tx^*, Tx^*) + S(x^*, x_{n_k+1}, x_{n_k+1})] \\ &\leq \frac{1}{8}[2S(x_{n_k}, x^*, x^*) + S(x^*, Tx^*, Tx^*) + 2S(x^*, x_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k+1}, x_{n_k+1})] \\ &= \frac{1}{2}S(x_{n_k}, x^*, x^*) + \frac{1}{8}S(x^*, Tx^*, Tx^*) + \frac{1}{8}S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) \\ &\leq \max\{S(x_{n_k}, x^*, x^*), S(x^*, Tx^*, Tx^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1})\}. \end{aligned}$$

By the above inequality, we have that

$$M_3(x_{n_k}, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x^*, Tx^*, Tx^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1})\}.$$

Suppose that $M_3(x_{n_k}, x^*) = S(x_{n_k}, x^*, x^*)$, then, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*) \\ &< S(x_{n_k}, x^*, x^*) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, we have

$$S(x^*, Tx^*, Tx^*) < 2S(x^*, x^*, x^*) = 0,$$

which is a contradiction.

Next, we suppose that $M_3(x_{n_k}, x^*) = S(x_{n_k}, x_{n_k+1}, x_{n_k+1})$, then we have that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x_{n_k+1}, x_{n_k+1})) + S(x_{n_k+1}, x^*, x^*) \\ &< S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, this implies that

$$S(x^*, Tx^*, Tx^*) < \lim_{k \rightarrow \infty} [S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*)] = 0,$$

which is again a contradiction.

Finally, we suppose that $M_3(x_{n_k}, x^*) = S(x^*, Tx^*, Tx^*)$, then we obtain that

$$S(x^*, Tx^*, Tx^*) < \psi(S(x^*, Tx^*, Tx^*)) + S(x_{n_k+1}, x^*, x^*) < S(x^*, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*).$$

Letting $k \rightarrow \infty$ in above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &< S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*) \\ &= S(x^*, Tx^*, Tx^*), \end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by (S1) in Definition 1.1, we have $x^* = Tx^*$. \square

Definition 2.2. Let (X, S) be an S -metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)$ -Meir-Keeler contractive mapping of dim3 of type C if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M'_3(x, y, z)) < \epsilon + \delta(\epsilon) \text{ implies } \gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \epsilon,$$

where

$$\begin{aligned} M'_3(x, y, z) &= \max\{S(x, y, y), S(y, z, z), S(z, x, x), S(x, Tx, Tx), S(y, Ty, Ty), S(z, Tz, Tz), \\ &\quad \frac{1}{8}[S(x, Ty, Ty) + S(y, Tx, Tx)], \frac{1}{8}[S(y, Tz, Tz) + S(z, Ty, Ty)], \\ &\quad \frac{1}{8}[S(z, Tx, Tx) + S(x, Tz, Tz)]\}, \end{aligned}$$

$\forall x, y, z \in X$.

Theorem 2.13. Let (X, S) be a complete S -metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)$ -MKC mapping of dim3 of type C . Assume also that:

- (A1) T is γ -admissible;
- (A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;
- (A3) (X, S) is (T, γ) -limit.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption (A2), let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$.

Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \{0\} \cup \mathbb{N}$.

Since T is triangular γ -admissible, by Lemma 1.6, we have that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} . \tag{50}$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_{n_0} = Tx^*$. For this, we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{51}$$

Step1. We will prove that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Taking (50) and (51) into account together with the fact that T is generalized $(\gamma - \psi)$ -MKC mapping of dim3 of type C , for each $n \in \mathbb{N}$, we get

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)\gamma(x_n, x_{n+1}, x_{n+1})\gamma(x_n, x_{n+1}, x_{n+1})S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M'_3(x_{n-1}, x_n, x_n)) \\ &< M'_3(x_{n-1}, x_n, x_n), \end{aligned}$$

where

$$\begin{aligned} M'_3(x_{n-1}, x_n, x_n) &= \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_n), S(x_n, x_{n-1}, x_{n-1}), S(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ &S(x_n, Tx_n, Tx_n), S(x_n, Tx_n, Tx_n), \frac{1}{8}[S(x_{n-1}, Tx_n, Tx_n) + S(x_n, Tx_{n-1}, Tx_{n-1})], \\ &\frac{1}{8}[S(x_n, Tx_n, Tx_n) + S(x_n, Tx_n, Tx_n)], \frac{1}{8}[S(x_n, Tx_{n-1}, Tx_{n-1}) + S(x_{n-1}, Tx_n, Tx_n)]\} \\ &= \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_{n+1}), \frac{1}{8}[S(x_{n-1}, x_{n+1}, x_{n+1}) + S(x_n, x_n, x_n)], \\ &\frac{1}{8}[2S(x_n, x_{n+1}, x_{n+1})], \frac{1}{8}[S(x_{n-1}, x_{n+1}, x_{n+1}) + S(x_n, x_n, x_n)]\}. \end{aligned}$$

Regarding Lemma 1.1, we estimate the last term in the expression of $M'_3(x_{n-1}, x_n, x_n)$ as follows:

$$\begin{aligned} &\frac{1}{8}[S(x_{n-1}, x_{n+1}, x_{n+1}) + S(x_n, x_n, x_n)] \\ &= \frac{1}{8}S(x_{n-1}, x_{n+1}, x_{n+1}) \\ &\leq \frac{1}{8}[2S(x_{n-1}, x_n, x_n) + S(x_n, x_{n+1}, x_{n+1})] \\ &= \frac{1}{4}S(x_{n-1}, x_n, x_n) + \frac{1}{8}S(x_n, x_{n+1}, x_{n+1}) \\ &\leq \max\{S(x_{n-1}, x_{n+1}, x_{n+1}), S(x_n, x_n, x_n)\}. \end{aligned}$$

Consequently, we get that

$$M_3(x_{n-1}, x_n, x_n) = \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_{n+1}, x_{n+1})\}.$$

Let us consider the two cases. If $M'_3(x_{n-1}, x_n, x_n) = S(x_n, x_{n+1}, x_{n+1})$. Since ψ is nondecreasing, then we have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_n, x_{n+1}, x_{n+1})) < S(x_n, x_{n+1}, x_{n+1}),$$

which is a contradiction. Thus, $M'_3(x_{n-1}, x_n, x_n) = S(x_{n-1}, x_n, x_n)$ and we also have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_{n-1}, x_n, x_n)) < S(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}. \tag{52}$$

So, we derive that the sequence $\{S(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = t.$$

Recursively, we deduce from (52) that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{53}$$

On account of (53) and Remark 1.2, we obtain

$$\lim_{n \rightarrow \infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.

We will prove that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall m, n \geq n_0$,

$$S(x_m, x_m, x_n) < \epsilon. \tag{54}$$

Taking Step1 into account, for each $\epsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{\epsilon - \psi(\epsilon)}{2}, \forall n \geq n_0. \tag{55}$$

We prove (54) by induction on n . (54) holds for $m = n$ and $n = n + 1$ by using (55) and the fact that $\frac{\epsilon - \psi(\epsilon)}{2} < \epsilon$. Assume (54) holds for $n = k$. For $n = k + 1$, we have

$$\begin{aligned} S(x_m, x_{k+1}, x_{k+1}) &\leq 2S(x_m, x_{m+1}, x_{m+1}) + S(x_{m+1}, x_{k+1}, x_{k+1}) \\ &\leq \epsilon - \psi(\epsilon) + \psi(S(x_m, x_k, x_k)) \\ &\leq \epsilon - \psi(\epsilon) + \psi(\epsilon) \\ &= \epsilon. \end{aligned}$$

By induction on n , we conclude that (54) holds for all $n \geq m \geq n_0$. So $\{x_n\}$ is a Cauchy sequence that is

$$\lim_{n, m \rightarrow \infty} S(x_n, x_m, x_m) = 0.$$

Since (X, S) is a complete S -metric space, then there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} S(x_n, x^*, x^*) = \lim_{n, m \rightarrow \infty} S(x_n, x_m, x^*) = 0.$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$.

From assumption (A3), we have that

$$\gamma(x^*, Tx^*, Tx^*) \geq 1, \quad \forall k \in \mathbb{N}.$$

By using Lemma 1.1 and above inequality together with the assumption of the theorem that T is a generalized $(\gamma - \psi)$ -MKC mapping of dim3 of type C ,

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2S(Tx_{n_k}, Tx^*, Tx^*) + S(Tx_{n_k}, x^*, x^*) \\ &\leq 2\gamma(x_{n_k}, Tx_{n_k}, Tx_{n_k})\gamma(x^*, Tx^*, Tx^*)\gamma(x^*, Tx^*, Tx^*)S(Tx_{n_k}, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*) \\ &\leq \psi(M'_3(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*), \end{aligned}$$

where,

$$\begin{aligned} M'_3(x_{n_k}, x^*, x^*) &= \max\{S(x_{n_k}, x^*, x^*), S(x^*, x^*, x^*), S(x^*, x_{n_k}, x_{n_k}), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), \\ &S(x^*, Tx^*, Tx^*), S(x^*, Tx^*, Tx^*), S(x^*, Tx^*, Tx^*), \\ &\frac{1}{8}[S(x_{n_k}, Tx^*, Tx^*) + S(x^*, x_{n_k+1}, x_{n_k+1})]\}. \end{aligned}$$

Notice that as $S(x^*, Tx^*, Tx^*) > 0$, then we have that $M'_3(x_{n_k}, x^*, x^*) > 0$.

From Lemma 1.1, it follows that

$$\begin{aligned} & \frac{1}{8}[S(x_{n_k}, Tx^*, Tx^*) + S(x^*, x_{n_k+1}, x_{n_k+1})] \\ & \leq \frac{1}{8}[2S(x_{n_k}, x^*, x^*) + S(x^*, Tx^*, Tx^*) + 2S(x^*, x_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k+1}, x_{n_k+1})] \\ & = \frac{1}{2}S(x_{n_k}, x^*, x^*) + \frac{1}{8}S(x^*, Tx^*, Tx^*) + \frac{1}{8}S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) \\ & \leq \max\{S(x_{n_k}, x^*, x^*), S(x^*, Tx^*, Tx^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1})\}. \end{aligned}$$

By the above inequality, we have that

$$M'_3(x_{n_k}, x^*, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x^*, Tx^*, Tx^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1})\}.$$

Suppose that $M'_3(x_{n_k}, x^*, x^*) = S(x_{n_k}, x^*, x^*)$, then, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) & \leq \psi(S(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*) \\ & < S(x_{n_k}, x^*, x^*) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, we have

$$S(x^*, Tx^*, Tx^*) < 2S(x^*, x^*, x^*) = 0,$$

which is a contradiction.

Next, we suppose that $M'_3(x_{n_k}, x^*, x^*) = S(x_{n_k}, x_{n_k+1}, x_{n_k+1})$, then we have that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) & \leq \psi(S(x_{n_k}, x_{n_k+1}, x_{n_k+1})) + S(x_{n_k+1}, x^*, x^*) \\ & < S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \rightarrow \infty$ in the inequality above, this implies that

$$S(x^*, Tx^*, Tx^*) < \lim_{k \rightarrow \infty} [S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*)] = 0,$$

which is again a contradiction.

Finally, we suppose that $M'_3(x_{n_k}, x^*, x^*) = S(x^*, Tx^*, Tx^*)$, then we obtain that

$$S(x^*, Tx^*, Tx^*) < \psi(S(x^*, Tx^*, Tx^*)) + S(x_{n_k+1}, x^*, x^*) < S(x^*, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*).$$

Letting $k \rightarrow \infty$ in above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) & < S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*) \\ & = S(x^*, Tx^*, Tx^*), \end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by (S1) in Definition 1.1, we have $x^* = Tx^*$. □

In what follows, we propose the condition for the uniqueness of a fixed point of a generalized $(\gamma - \psi)$ -MKC of type C mappings:

(U1') For $\forall x^*, y^* \in \text{Fix}(T)$, there exists $z^* \in X$ such that $\gamma(x^*, z^*, z^*) \geq 1$, $\gamma(y^*, z^*, z^*) \geq 1$ and $\gamma(z^*, Tz^*, Tz^*) \geq 1$.

(U2') Let $x^*, y^* \in \text{Fix}(T)$. If there exists a sequence $\{x_n\}$ in X such that $\gamma(x^*, x_n, x_n) \geq 1$, $\gamma(y^*, x_n, x_n) \geq 1$, then $S(x_n, x_{n+1}, x_{n+1}) \leq \inf\{S(x^*, x_n, x_n), S(y^*, x_n, x_n)\}$.

Theorem 2.14. Adding conditions $(U1')$, $(U2')$ to the statements of Theorem 2.12, one has that T has the unique fixed point.

Proof. Let x^*, y^* be two distinct fixed points of T . Form condition $(U1')$, there exists $z^* \in X$ such that

$$\gamma(x^*, z^*, z^*) \geq 1, \gamma(y^*, z^*, z^*) \geq 1, \gamma(z^*, Tz^*, Tz^*) \geq 1.$$

Owing to the fact that T is triangular γ -admissible and $\gamma(z^*, Tz^*, Tz^*) \geq 1$, we have

$$\gamma(Tz^*, T^2z^*, T^2z^*) \geq 1.$$

Inductively, we find

$$\gamma(T^{n-1}z^*, T^n z^*, T^n z^*) \geq 1, \quad \forall n \in \mathbb{N}.$$

Since $\gamma(x^*, z^*, z^*) \geq 1$ and $\gamma(z^*, Tz^*, Tz^*) \geq 1$, then by the triangular γ -admissibility of T , we have

$$\gamma(x^*, Tz^*, Tz^*) \geq 1.$$

Again, since $\gamma(x^*, Tz^*, Tz^*) \geq 1$ and $\gamma(Tz^*, T^2z^*, T^2z^*) \geq 1$, we derive

$$\gamma(x^*, T^2z^*, T^2z^*) \geq 1.$$

Inductively, we get

$$\gamma(x^*, T^n z^*, T^n z^*) \geq 1, \quad \forall n \in \mathbb{N}. \tag{56}$$

In the similar way, we also have that

$$\gamma(y^*, T^n z^*, T^n z^*) \geq 1, \quad \forall n \in \mathbb{N}. \tag{57}$$

Define an iterative sequence $\{z_n\}$ by $z_{n+1} = Tz_n, \forall n \in \{0\} \cup \mathbb{N}$ and $z_0 = z^*$.

Step1. We will prove that $\lim_{n \rightarrow \infty} S(x^*, z_n, z_n) = 0$.

By (56) and the statement of the theorem that T is generalized $(\gamma - \psi)$ -MKC mapping of type C . we have

$$\begin{aligned} S(x^*, z_{n+1}, z_{n+1}) &\leq \gamma(x^*, z_n, z_n)S(Tx^*, Tz_n, Tz_n) \\ &\leq \psi(M_3(x^*, z_n)). \end{aligned}$$

If $\psi(M_3(x^*, z_n)) = 0$, then

$$\lim_{n \rightarrow \infty} S(x^*, z_n, z_n) = 0.$$

Now, suppose that $\psi(M_3(x^*, z_n)) > 0$, then $M_3(x^*, z_n) > 0$.

Since T is a generalized $(\gamma - \psi)$ -MKC mapping of type C , we get

$$\begin{aligned} S(x^*, z_{n+1}, z_{n+1}) &\leq \gamma(x^*, z_n, z_n)S(Tx^*, Tz_n, Tz_n) \\ &\leq \psi(M_3(x^*, z_n)) \\ &< M_3(x^*, z_n), \end{aligned}$$

where $M_3(x^*, z_n) = \max\{S(x^*, z_n, z_n), S(x^*, Tx^*, Tx^*), S(z_n, Tz_n, Tz_n), \frac{1}{8}[S(x^*, Tz_n, Tz_n) + S(z_n, Tx^*, Tx^*)]\}$.
 Taking (U2') and Lemma 1.1 into account, we have

$$M_3(x^*, z_n) = S(x^*, z_n, z_n).$$

Thus, $S(x^*, z_{n+1}, z_{n+1}) < S(x^*, z_n, z_n)$.

Letting $n \rightarrow \infty$ in the inequality above, we obtain

$$\lim_{n \rightarrow \infty} S(x^*, z_{n+1}, z_{n+1}) < \lim_{n \rightarrow \infty} S(x^*, z_n, z_n),$$

which is a contradiction. Then,

$$M_3(x^*, z_n) = S(x^*, z_n, z_n) = 0.$$

Hence, we get that

$$\lim_{n \rightarrow \infty} S(x^*, z_n, z_n) = 0.$$

Step2. We will prove that $\lim_{n \rightarrow \infty} S(y^*, z_n, z_n) = 0$.

In a analogous way of Step1., we can complete the proof of $\lim_{n \rightarrow \infty} S(y^*, z_n, z_n) = 0$.

By Lemma 1.1,

$$S(x^*, y^*, y^*) \leq 2S(x^*, z_n, z_n) + S(y^*, z_n, z_n).$$

Letting $n \rightarrow \infty$ in the above inequality, we get

$$S(x^*, y^*, y^*) = 0,$$

therefore, we have $x^* = y^*$. □

As a uniqueness condition for fixed points of $(\gamma - \psi)$ -MKC mappings of dim3 of type C , we suggest the following hypothesis:

(U3') For $\forall x^*, y^* \in \text{Fix}(T)$, $\gamma(x^*, x^*, x^*) \geq 1$, $\gamma(y^*, y^*, y^*) \geq 1$.

Theorem 2.15. Adding condition (U3') to the statements of Theorem 2.13, one has that T has the unique fixed point.

Proof. Let x^*, y^* be two distinct fixed points of T . Form condition (U3')

$$\gamma(x^*, x^*, x^*) \geq 1, \gamma(y^*, y^*, y^*) \geq 1. \tag{58}$$

By (58) and the statement of the theorem that T is generalized $(\gamma - \psi)$ -MKC mapping of dim3 of type C . we have

$$\begin{aligned} S(x^*, y^*, y^*) &\leq \gamma(x^*, Tx^*, Tx^*)\gamma(y^*, Ty^*, Ty^*)\gamma(y^*, Ty^*, Ty^*)S(Tx^*, Ty^*, Ty^*) \\ &\leq \psi(M'_3(x^*, y^*, y^*)) < M'_3(x^*, y^*, y^*). \end{aligned}$$

but

$$\begin{aligned} M'_3(x^*, y^*, y^*) &= \max\{S(x^*, y^*, y^*), S(y^*, y^*, y^*), S(y^*, x^*, x^*), S(x^*, Tx^*, Tx^*), S(y^*, Ty^*, Ty^*), \\ &S(y^*, Ty^*, Ty^*), \frac{1}{8}[S(x^*, Ty^*, Ty^*) + S(y^*, Tx^*, Tx^*)], \\ &\frac{1}{8}[S(y^*, Ty^*, Ty^*) + S(y^*, Ty^*, Ty^*)], \frac{1}{8}[S(y^*, Tx^*, Tx^*) + S(x^*, Ty^*, Ty^*)]\} \\ &= S(x^*, y^*, y^*). \end{aligned}$$

so,

$$S(x^*, y^*, y^*) < S(x^*, y^*, y^*)$$

which is again a contradiction. therefore, we have $x^* = y^*$. □

3. Generalized Ulam-Hyers Stability for MKC mappings

In the following section, by introducing the generalized Ulam-Hyers stability in the framework of S -metric spaces, we study the stability for MKC mappings.

Theorem 3.1. Let (X, S) be a complete S -metric and $T : X \rightarrow X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.12 hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any ϵ -solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \geq 1$, where $x^* \in \text{Fix}(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. From the conclusion of Theorem 2.12, it follows that there exists $x^* \in \text{Fix}(T)$ such that $S(x^*, Tx^*, Tx^*) = 0$. Let $\epsilon > 0$ and w^* be a ϵ -solution of (2).

From (A2), we have $\gamma(x^*, w^*, w^*) \geq 1$. Since T is triangular γ -admissible, we can obtain that $\gamma(Tx^*, Tw^*, Tw^*) = \gamma(x^*, Tw^*, Tw^*) \geq 1$.

Thus, we also get that

$$\begin{aligned} S(x^*, w^*, w^*) &= S(Tx^*, w^*, w^*) \\ &\leq S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\ &\leq \gamma(Tx^*, Tw^*, Tw^*)S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\ &< \psi(M_3(x^*, w^*)) + 2\epsilon, \end{aligned}$$

where $M_3(x^*, w^*) = \max\{S(x^*, w^*, w^*), S(x^*, Tx^*, Tx^*), S(w^*, Tw^*, Tw^*), \frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)]\}$.

We also get

$$\begin{aligned} &\frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)] \\ &\leq \frac{1}{8}[2S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*) + 2S(w^*, x^*, x^*) + S(x^*, Tx^*, Tx^*)] \\ &= \frac{1}{8}[4S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*)] \\ &= \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}S(w^*, Tw^*, Tw^*) \\ &\leq \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}\epsilon \\ &< \max\{S(x^*, w^*, w^*), \epsilon\}. \end{aligned}$$

From the inequality above, we have that

$$M_3(x^*, w^*) < \max\{S(x^*, w^*, w^*), \epsilon\}.$$

It is obviously that if $S(x^*, w^*, w^*) < \epsilon$, then the proof is complete.

Suppose that $\max\{S(x^*, w^*, w^*), \epsilon\} = S(x^*, w^*, w^*)$. Then, we have

$$M_3(x^*, w^*) < S(x^*, w^*, w^*).$$

So, we can deduce that

$$\begin{aligned} S(x^*, w^*, w^*) &\leq \psi(S(x^*, w^*, w^*)) + 2\epsilon, \\ S(x^*, w^*, w^*) - \psi(S(x^*, w^*, w^*)) &\leq 2\epsilon. \end{aligned}$$

From assumption (A1), we get that

$$\beta(S(x^*, w^*, w^*)) \leq 2\epsilon.$$

Hence, $S(x^*, w^*, w^*) \leq \beta^{-1}(2\epsilon)$.

Therefore, (1) is generalized Ulam-Hyers stable. □

Theorem 3.2. Let (X, S) be a complete S -metric and $T : X \rightarrow X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.13 hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any ϵ -solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \geq 1$, where $x^* \in \text{Fix}(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. From the conclusion of Theorem 2.13, it follows that there exists $x^* \in \text{Fix}(T)$ such that $S(x^*, Tx^*, Tx^*) = 0$. Let $\epsilon > 0$ and w^* be a ϵ -solution of (2).

From (A2), we have $\gamma(x^*, w^*, w^*) \geq 1$. Since T is triangular γ -admissible, we can obtain that $\gamma(Tx^*, Tw^*, Tw^*) = \gamma(x^*, Tw^*, Tw^*) \geq 1$.

Thus, we also get that

$$\begin{aligned} S(x^*, w^*, w^*) &= S(Tx^*, w^*, w^*) \\ &\leq S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\ &\leq \gamma(Tx^*, Tw^*, Tw^*)S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\ &< \psi(M'_3(x^*, w^*, w^*)) + 2\epsilon, \end{aligned}$$

where

$$\begin{aligned} M'_3(x^*, w^*) &= \max\{S(x^*, w^*, w^*), S(w^*, w^*, w^*), S(w^*, x^*, x^*), \\ &S(x^*, Tx^*, Tx^*), S(w^*, Tw^*, Tw^*), \frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)] \\ &\frac{1}{8}[S(w^*, Tw^*, Tw^*) + S(w^*, Tw^*, Tw^*)], \frac{1}{8}[S(w^*, Tx^*, Tx^*) + S(x^*, Tw^*, Tw^*)]\} \\ &= \max\{S(x^*, w^*, w^*), S(w^*, Tw^*, Tw^*), \frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)]\} \end{aligned}$$

We also get

$$\begin{aligned} &\frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)] \\ &\leq \frac{1}{8}[2S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*) + 2S(w^*, x^*, x^*) + S(x^*, Tx^*, Tx^*)] \\ &= \frac{1}{8}[4S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*)] \\ &= \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}S(w^*, Tw^*, Tw^*) \\ &\leq \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}\epsilon \\ &< \max\{S(x^*, w^*, w^*), \epsilon\}. \end{aligned}$$

From the inequality above, we have that

$$M'_3(x^*, w^*, w^*) < \max\{S(x^*, w^*, w^*), \epsilon\}.$$

It is obviously that if $S(x^*, w^*, w^*) < \epsilon$, then the proof is complete. Suppose that $\max\{S(x^*, w^*, w^*), \epsilon\} = S(x^*, w^*, w^*)$. Then, we have

$$M'_3(x^*, w^*, w^*) < S(x^*, w^*, w^*).$$

So, we can deduce that

$$\begin{aligned} S(x^*, w^*, w^*) &\leq \psi(S(x^*, w^*, w^*)) + 2\epsilon, \\ S(x^*, w^*, w^*) - \psi(S(x^*, w^*, w^*)) &\leq 2\epsilon. \end{aligned}$$

From assumption (A1), we get that

$$\beta(S(x^*, w^*, w^*)) \leq 2\epsilon.$$

Hence, $S(x^*, w^*, w^*) \leq \beta^{-1}(2\epsilon)$.

Therefore, (1) is generalized Ulam-Hyers stable. □

Corollary 3.1. Let (X, S) be a complete S -metric and $T : X \rightarrow X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.8(resp., Theorem 2.10) hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any ϵ -solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \geq 1$, where $x^* \in \text{Fix}(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. The proof is an analog of the proof of Theorem 3.1. □

Corollary 3.2. Let (X, S) be a complete S -metric and $T : X \rightarrow X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.9(resp., Theorem 2.11) hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any ϵ -solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \geq 1$, where $x^* \in \text{Fix}(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. The proof is an analog of the proof of Theorem 3.2. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details

¹ School of Science and Technology, Sanya University, Sanya, Hainan 572000, China.

² College of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China.

³ Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, Zigong, Sichuan 643000, China.

⁴ Department of Mathematics, Karaj Branch Islamic Azad University, Karaj, Iran.

⁵ Department of Mathematics Education and RINS, Gyeongsang National University, Gajwa-dong,900, 52828 Jinju, Korea.

⁶ Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade, Serbia.

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New oscillation criteria for second-order nonlinear delay dynamic equations with nonpositive neutral coefficients on time scales

Ming Zhang^{a,b*}, Wei Chen^{a†}, M.M.A. El-Sheikh^{c‡}, R.A. Sallam^{c§},
A.M. Hassan^{d¶} and Tongxing Li^{b||}

^a*School of Information Engineering, Wuhan University of Technology, Wuhan, Hubei 430070, P. R. China*

^b*School of Information Science and Engineering, Linyi University, Linyi, Shandong 276005, P. R. China*

^c*Department of Mathematics, Faculty of Science, Menoufia University, Shebin El-Koom 32511, Egypt*

^d*Department of Mathematics, Faculty of Science, Benha University, Benha-Kalubia 13518, Egypt*

Abstract

We analyze the oscillatory behavior of solutions to a nonlinear second-order neutral delay dynamic equation with a nonpositive neutral coefficient under the assumptions that allow applications to Emden–Fowler type dynamic equations. New theorems complement and improve related contributions to the subject. An example is included.

Keywords: Oscillation, second-order delay dynamic equation, neutral type equation, Emden–Fowler type equation.

Mathematics Subject Classification 2010: 34K11, 34N05.

1 Introduction

In this paper, we study the oscillation of a class of second-order neutral dynamic equations

$$[r(t)(z^\Delta(t))^\alpha]^\Delta + q(t)f(x(\delta(t))) = 0. \tag{1.1}$$

Here $t \in [t_0, \infty)_{\mathbb{T}}$, $\alpha \geq 1$ is a quotient of odd natural numbers, and $z(t) = x(t) - p(t)x(\tau(t))$. The increasing interest in oscillation of solutions to various classes of equations is motivated by their applications in natural sciences, engineering, and control; see, for instance, [1–30] and the references cited therein. Analysis of qualitative properties of (1.1) is important not only for the sake of further development of the oscillation theory, but for practical reasons too. As a matter of fact, a particular case of (1.1), an Emden–Fowler dynamic equation

$$[r(t)(x^\Delta(t))^\alpha]^\Delta + q(t)x^\beta(\delta(t)) = 0,$$

has applications in mathematical, theoretical, and chemical physics; see Li and Rogovchenko [15–18].

Throughout the paper, we assume that the following assumptions are satisfied:

*e-mail: zhangming@lyu.edu.cn

†e-mail: greatchen@whut.edu.cn

‡e-mail: msheikh_1999@yahoo.com

§e-mail: ragaasallam@yahoo.com

¶e-mail: ahmed.mohamed@fsc.bu.edu.eg

||e-mail: litongx2007@163.com (Corresponding author)

- (H₁) $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)\Delta t = \infty$, $R(t) = \int_{t_1}^t r^{-\frac{1}{\alpha}}(s)\Delta s$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ is sufficiently large;
- (H₂) $p, q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $0 \leq p(t) \leq p_0 < 1$, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t ;
- (H₃) $\tau, \delta \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t) \leq t$, $\delta(t) \leq t$, and $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \delta(t) = \infty$;
- (H₄) $f \in C(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$ for all $u \neq 0$, and there exists a positive constant k such that $f(u)/u^\beta \geq k$ for all $u \neq 0$, where $\beta \geq \alpha$ is a quotient of odd natural numbers.

By a solution to (1.1) we mean a function $x \in C_{rd}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [t_0, \infty)_{\mathbb{T}}$, such that $r(z^\Delta)^\alpha \in C_{rd}^1[T_x, \infty)_{\mathbb{T}}$ and x satisfies (1.1) on $[T_x, \infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \in [T, \infty)_{\mathbb{T}}\} > 0$ for all $T \in [T_x, \infty)_{\mathbb{T}}$ and assume that (1.1) possesses such solutions. As usual, a solution of (1.1) is said to be oscillatory if it is not of the same sign eventually; otherwise, it is called nonoscillatory.

Recently, a great deal of interest in oscillatory properties of solutions to various classes of equations with nonnegative neutral coefficients has been shown; see, for instance, [2, 4, 5, 14–17, 19, 20, 22, 27, 28] and the references cited therein. However, there are relatively fewer results for equations with nonpositive neutral coefficients; see [3, 4, 7, 13, 21, 23–25, 29]. In the papers by Arul and Shobha [3] and Li et al. [21], a particular case of (1.1), a neutral differential equation

$$[r(t)(z'(t))^\alpha]' + q(t)f(x(\delta(t))) = 0$$

was studied. Seghar et al. [23] investigated the neutral difference equation

$$\Delta(a_n \Delta(x_n - p_n x_{n-k})) + q_n f(x_{n-l}) = 0.$$

Bohner and Li [7] and Karpuz [13] established oscillation results for neutral dynamic equations

$$(r(t)|z^\Delta(t)|^{p-2}z^\Delta(t))^\Delta + q(t)|x(\delta(t))|^{p-2}x(\delta(t)) = 0, \quad z(t) = x(t) - p(t)x(\tau(t))$$

and

$$(x(t) - p(t)x(\tau(t)))^{\Delta\Delta} + q(t)x(\delta(t)) = 0,$$

whereas Zhang et al. [29] explored (1.1) assuming that $\alpha = \beta$.

It should be noted that research in this paper was strongly motivated by the paper [29]. Our principal goal is to analyze the oscillatory behavior of solutions to (1.1) in the case where $\beta \geq \alpha$. As customary for papers on oscillation, all functional inequalities are supposed to hold eventually. Without loss of generality, we can deal only with positive solutions of (1.1).

2 Main results

For the proofs of our oscillation criteria we need the following lemmas. The first lemma is extracted from the monograph by Bohner and Peterson [9, Theorem 1.93], and the latter lemmas can be obtained by similar techniques to those used in [3, 21].

Lemma 2.1. *Assume that $v : \mathbb{T} \rightarrow \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $y : \tilde{\mathbb{T}} \rightarrow \mathbb{R}$. If $v^\Delta(t)$ and $y^{\tilde{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(y(v(t)))^\Delta = y^{\tilde{\Delta}}(v(t))v^\Delta(t).$$

Lemma 2.2. *Let x be a positive solution of (1.1). Then z has the following two possible cases:*

- (I) $z(t) > 0$, $z^\Delta(t) > 0$, $(r(t)(z^\Delta(t))^\alpha)^\Delta \leq 0$;
- (II) $z(t) < 0$, $z^\Delta(t) > 0$, $(r(t)(z^\Delta(t))^\alpha)^\Delta \leq 0$

for $t \in [t_1, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ is sufficiently large.

Lemma 2.3. *Let x be a positive solution of (1.1) and assume that the corresponding z has property (II) of Lemma 2.2. Then*

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Lemma 2.4. *If x is a positive solution of (1.1) such that case (I) of Lemma 2.2 is satisfied, then $x(t) \geq z(t)$ and $z(t)/R(t)$ is strictly decreasing for large t .*

Theorem 2.1. *Assume that $\delta([t_0, \infty)_{\mathbb{T}}) = [\delta(t_0), \infty)_{\mathbb{T}}$ and $\delta^\Delta(t) > 0$. If for any $M > 0$,*

$$\limsup_{t \rightarrow \infty} \left[Q(t) + \alpha \int_t^\infty \delta^\Delta(s) r^{-\frac{1}{\alpha}}(\delta(s)) Q^{\frac{\alpha+1}{\alpha}}(\sigma(s)) \Delta s \right] \left(\int_{t_0}^{\delta(t)} r^{-\frac{1}{\alpha}}(s) \Delta s \right)^\alpha > 1, \tag{2.1}$$

where $Q(t) = kM^{\beta-\alpha} \int_t^\infty q(u) \Delta u$, then solutions of (1.1) are either oscillatory or converge to zero as $t \rightarrow \infty$.

Proof. Let x be a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. It follows from Lemma 2.2 that z satisfies either (I) or (II) for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1. Suppose first that z satisfies case (I). By virtue of the definition of z ,

$$x(t) = z(t) + p(t)x(\tau(t)) \geq z(t)$$

and so we can write (1.1) in the form

$$[r(t)(z^\Delta(t))^\alpha]^\Delta \leq -kq(t)z^\beta(\delta(t)).$$

Defining the Riccati transformation

$$\nu(t) = \frac{r(t)(z^\Delta(t))^\alpha}{z^\alpha(\delta(t))}, \tag{2.2}$$

then $\nu(t) > 0$ and there exists a constant $M > 0$ such that

$$\begin{aligned} \nu^\Delta(t) &= \frac{[r(t)(z^\Delta(t))^\alpha]^\Delta}{z^\alpha(\delta(t))} + [r(t)(z^\Delta(t))^\alpha]^\sigma \left[\frac{1}{z^\alpha(\delta(t))} \right]^\Delta \\ &\leq -kM^{\beta-\alpha}q(t) - \alpha\delta^\Delta(t)\nu(\sigma(t)) \frac{z^\Delta(\delta(t))}{z(\delta(\sigma(t)))}. \end{aligned} \tag{2.3}$$

Taking into account that $\nu^{\frac{1}{\alpha}}(\sigma(t)) = r^{\frac{1}{\alpha}}(\sigma(t))z^\Delta(\sigma(t))/z(\delta(\sigma(t)))$, $r(t)(z^\Delta(t))^\alpha \leq 0$, and $\delta(t) \leq t \leq \sigma(t)$, we conclude that

$$\frac{z^\Delta(\delta(t))}{z(\delta(\sigma(t)))} \geq \frac{\nu^{\frac{1}{\alpha}}(\sigma(t))}{r^{\frac{1}{\alpha}}(\delta(t))}. \tag{2.4}$$

Combining (2.3) and (2.4), we arrive at

$$\nu^\Delta(t) \leq -kM^{\beta-\alpha}q(t) - \alpha\delta^\Delta(t)r^{-\frac{1}{\alpha}}(\delta(t))\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{2.5}$$

Integrating (2.5) from t to s , we deduce that

$$\nu(s) - \nu(t) \leq -kM^{\beta-\alpha} \int_t^s q(u) \Delta u - \alpha \int_t^s \delta^\Delta(u) r^{-\frac{1}{\alpha}}(\delta(u)) \nu^{\frac{\alpha+1}{\alpha}}(\sigma(u)) \Delta u,$$

which yields

$$\nu(t) \geq kM^{\beta-\alpha} \int_t^s q(u) \Delta u + \alpha \int_t^s \delta^\Delta(u) r^{-\frac{1}{\alpha}}(\delta(u)) \nu^{\frac{\alpha+1}{\alpha}}(\sigma(u)) \Delta u.$$

Passing to the limit as $s \rightarrow \infty$, we have

$$\nu(t) \geq Q(t) + \alpha \int_t^\infty \delta^\Delta(u) r^{-\frac{1}{\alpha}}(\delta(u)) \nu^{\frac{\alpha+1}{\alpha}}(\sigma(u)) \Delta u. \tag{2.6}$$

An application of (2.6) implies that

$$\nu(t) \geq Q(t) + \alpha \int_t^\infty \delta^\Delta(u) r^{-\frac{1}{\alpha}}(\delta(u)) Q^{\frac{\alpha+1}{\alpha}}(\sigma(u)) \Delta u. \tag{2.7}$$

By virtue of (2.2), we conclude that

$$\begin{aligned} \frac{1}{\nu(t)} &= \frac{1}{r(t)} \left(\frac{z(\delta(t))}{z^\Delta(t)} \right)^\alpha \\ &= \frac{1}{r(t)} \left(\frac{z(t_2) + \int_{t_2}^{\delta(t)} r^{\frac{1}{\alpha}}(s) z^\Delta(s) r^{-\frac{1}{\alpha}}(s) \Delta s}{z^\Delta(t)} \right)^\alpha \\ &\geq \frac{1}{r(t)} \left(\frac{r^{\frac{1}{\alpha}}(t) z^\Delta(t) \int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s) \Delta s}{z^\Delta(t)} \right)^\alpha, \end{aligned}$$

that is,

$$\nu(t) \left(\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s) \Delta s \right)^\alpha \leq 1. \tag{2.8}$$

Using (2.7) and (2.8), we deduce that

$$\limsup_{t \rightarrow \infty} \left[Q(t) + \alpha \int_t^\infty \delta^\Delta(s) r^{-\frac{1}{\alpha}}(\delta(s)) Q^{\frac{\alpha+1}{\alpha}}(\sigma(s)) \Delta s \right] \left(\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s) \Delta s \right)^\alpha \leq 1,$$

which contradicts (2.1).

Case 2. Suppose now that z satisfies case (II). It follows from Lemma 2.3 that $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof. \square

Theorem 2.2. *If there exists a positive function $\beta \in C_{\text{rd}}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$, and for any $M > 0$,*

$$\int_{t_2}^\infty \left[kM^{\beta-\alpha} q(s) \beta(s) \left(\frac{R(\delta(s))}{R(s)} \right)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\beta^\Delta(s))^{\alpha+1} r(s)}{\beta^\alpha(s)} \right] \Delta s = \infty, \tag{2.9}$$

then conclusion of Theorem 2.1 remains intact.

Proof. Assume that x is a nonoscillatory solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$ that satisfies $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By virtue of Lemma 2.2, z satisfies either (I) or (II) for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1. Suppose that z satisfies case (I). Define the Riccati transformation

$$\omega(t) = \beta(t) \frac{r(t)(z^\Delta(t))^\alpha}{z^\alpha(t)}.$$

Then $\omega(t) > 0$ and there exists a constant $M > 0$ such that

$$\begin{aligned} \omega^\Delta(t) &= [r(t)(z^\Delta(t))^\alpha]^\Delta \frac{\beta(t)}{z^\alpha(t)} + [r(t)(z^\Delta(t))^\alpha]^\sigma \left[\frac{\beta(t)}{z^\alpha(t)} \right]^\Delta \\ &\leq -kM^{\beta-\alpha} q(t) \beta(t) \frac{z^\alpha(\delta(t))}{z^\alpha(t)} + \frac{\beta^\Delta(t)}{\beta(\sigma(t))} \omega(\sigma(t)) - \alpha \frac{\beta(t)}{\beta^\sigma(t)} \frac{z^\Delta(t)}{z(t)} \omega(\sigma(t)) \\ &\leq -kM^{\beta-\alpha} q(t) \beta(t) \frac{z^\alpha(\delta(t))}{z^\alpha(t)} + \frac{\beta^\Delta(t)}{\beta(\sigma(t))} \omega(\sigma(t)) - \alpha \frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t)) r^{\frac{1}{\alpha}}(t)} \omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \end{aligned}$$

In view of Lemma 2.4, we obtain

$$\omega^\Delta(t) \leq -kM^{\beta-\alpha} q(t) \beta(t) \left(\frac{R(\delta(t))}{R(t)} \right)^\alpha + \frac{\beta^\Delta(t)}{\beta(\sigma(t))} \omega(\sigma(t)) - \alpha \frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t)) r^{\frac{1}{\alpha}}(t)} \omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{2.10}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A > 0$$

with

$$B = \frac{\beta^\Delta(t)}{\beta(\sigma(t))} \quad \text{and} \quad A = \alpha \frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t)) r^{\frac{1}{\alpha}}(t)},$$

and using (2.10), we deduce that

$$\omega^\Delta(t) \leq -kM^{\beta-\alpha}q(t)\beta(t)\left(\frac{R(\delta(t))}{R(t)}\right)^\alpha + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\beta^\Delta(t))^{\alpha+1}r(t)}{\beta^\alpha(t)}. \tag{2.11}$$

Integrating (2.11) from t_2 ($t_2 \in [t_1, \infty)_{\mathbb{T}}$) to t , we arrive at

$$\int_{t_2}^t \left[kM^{\beta-\alpha}q(s)\beta(s)\left(\frac{R(\delta(s))}{R(s)}\right)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\beta^\Delta(s))^{\alpha+1}r(s)}{\beta^\alpha(s)} \right] \Delta s \leq \omega(t_2),$$

which contradicts (2.9).

Case 2. If z satisfies case (II), then $\lim_{t \rightarrow \infty} x(t) = 0$ due to Lemma 2.3. The proof is complete. □

Remark 2.1. *On the basis of Theorem 2.2, one can obtain Philos-type oscillation criteria for equation (1.1). The details are left to the reader.*

Example 2.1. For $t \in [1, \infty)_{\mathbb{T}}$, consider the second-order superlinear Emden–Fowler neutral delay dynamic equation

$$\left(x(t) - \frac{1}{3}x\left(\frac{t}{2}\right)\right)^{\Delta\Delta} + \frac{\gamma}{t}x^\beta\left(\frac{t}{4}\right) = 0, \quad \beta > 1, \gamma > 0. \tag{2.12}$$

Let $\beta(t) = 1$. It follows from Theorem 2.2 that every solution x of equation (2.12) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Remark 2.2. *For a class of second-order neutral delay dynamic equations (1.1), we derived two new oscillation results which complement and improve those obtained by Zhang et al. [29]. A distinguishing feature of our criteria is that we do not impose specific restriction $\alpha = \beta$. Since the sign of the derivative z^Δ is not known, it is difficult to establish sufficient conditions which ensure that every solution x of (1.1) is just oscillatory and does not satisfy $\lim_{t \rightarrow \infty} x(t) = 0$. Neither is it possible to use the technique exploited in this paper for proving that all solutions of (1.1) approach zero at infinity. As mentioned in the paper by Zhang et al. [29], it would be of interest to study (1.1) in the case where $\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(t)\Delta t < \infty$.*

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A Consistency Reaching Approach for Probability-interval Valued Hesitant Fuzzy Preference Relations

Jiuping Xu^{1,*}, Kang Xu^{1,2}, Zhibin Wu¹

1. Business School, Sichuan University, Chengdu 610065, P R China

2. School of Economics and Management, Hubei University of Automotive Technology, Shiyan, Hubei, 442002, P R China

Abstract

In a group decision making (GDM) situation with qualitative settings and complex environments, experts may require intervals with corresponding possibility values, rather than only interval-valued hesitant fuzzy sets (IVHFSs) or probability-hesitant fuzzy sets (P-HFSs), to express their preferences. In this paper, in line with such situations, probability-interval valued hesitant fuzzy sets (P-IVHFSs) are presented to address GDM problems with hesitant fuzzy intervals and the corresponding possibility values. A P-IVHFS can serve as an extension of both a P-HFS and an IVHFS. As important tools in GDM, P-IVHFSs can describe the actual preferences of decision-makers and better reflect their uncertainty, hesitancy, and inconsistency, thus enhancing the modeling abilities of HFSs. Firstly, the concept of P-IVHFSs is defined, and then some properties of P-IVHFSs are presented. Furthermore, probability-interval valued hesitant fuzzy preference relations (P-IVHFPRs) are defined and the consistency of P-IVHFPRs is discussed. Then, based on related research, a decomposition method is developed to deal with the consistency of P-IVHFPRs. Finally an example is provided to illustrate the proposed approach.

Keywords:

Decision making, Fuzzy sets, P-IVHFS, Preference relation, Consistency

1. Introduction

Torra initiated the notable concept of HFSs, which represented a new generalization for fuzzy sets, as this method permits an element to have not just one but a set of several possible membership values. Consequently, HFSs can describe the hesitancy experienced by decision makers (DMs) in the decision-making process. As a result of this innovation, the HFS has attracted an increasing amount of attention in academia since its introduction. In recent years, there have been a number of developments regarding the theory of HFSs. For example, Xu and Xia defined the concept of the hesitant fuzzy element (HFE), which can be considered to be the basic unit of a HFS. Moreover, Rodríguez et al. proposed the hesitant fuzzy linguistic term set to deal with linguistic decision making. Chen et al. extended HFSs to IVHFSs, which represent the membership degrees of an element to a set with several possible interval values. Farhadinia proposed a series of score functions for HFSs and Wei, Zhang, Yu, and Ai et al. studied their aggregation operators. Farhadinia, Xu and Xia, Peng et al., and Chen et al. discussed the information measures of HFSs. Wang et al. studied the interval-valued hesitant fuzzy linguistic set, which can serve as an extension of both a linguistic term set and an interval-valued hesitant fuzzy set. Finally, Wu and Xu presented the concept of possibility distribution for a hesitant fuzzy linguistic term set and Zhu and Xu extended HFSs to P-HFSs.

* Corresponding author. Tel: +86 28 85418191; Fax: +86 28 85415143.
E-mail address: xujiuping@scu.edu.cn(J. Xu)

In group decision making (GDM) problems with fuzzy preference relations, some of the experts' preference properties are often assumed and it is desirable to avoid contradictions or, in other words, inconsistent opinions. One of these properties is associated with the pairwise comparison transitivity between any three alternatives. For fuzzy preference relations, the transitivity has been modeled in many different ways depending on the role of the preference intensities. The purpose of consistency control is to measure the level of consistency of each individual preference relation so as to identify the expert, alternative and preference values that are the most inconsistent within the GDM problem. This inconsistency identification is also used to suggest possible new consistent preference values.

In the process of GDM, preference relations are very popular tools for expressing the DM's preferences when they compare a set of alternatives. Various types of preference relations have been suggested for different environments. For example, Orlovsky proposed the concept of fuzzy preference relations, and Xu introduced the concept of interval fuzzy preference relations to express uncertainty and vagueness. In many practical decision making problems, due to a lack of available information, it may be difficult for DMs to quantify their opinions precisely with a crisp number; however they can be represented by an interval number within $[0, 1]$. This means that it is vital to introduce the concept of IVHFSs, which permit the membership degrees of an element to a given set to have some different interval values. Chen et al. introduced interval-valued hesitant preference relations and their applications to GDM. Moreover, Farhadinia discussed the information measures of IVHFSs and Wang et al. developed interval-valued hesitant fuzzy linguistic sets, and discussed their applications in multi-criteria decision-making problems.

However, in a GDM situation with qualitative settings and in complex environments, experts may require intervals with corresponding possibility values rather than only IVHFSs or P-HFSs, to express their preferences. Consider the following case for example: the DMs of a large organization discuss the membership of x into a set A ; forty percent of them want to assign values between 0.3 and 0.4, while the remaining sixty percent wish to assign values between 0.5 and 0.6. At this point, interval numbers with probability values can be used, i.e., $\{[0.3, 0.4](40\%), [0.5, 0.6](60\%)\}$, or $\{[0.3, 0.4](0.4), [0.5, 0.6](0.6)\}$, to represent the preferences of the large organization. In accordance with such cases, in this paper, P-IVHFSs are presented to address GDM problems with hesitant fuzzy intervals and the corresponding possibility values. A P-IVHFS can serve as an extension of both a P-HFS and an IVHFS. Furthermore, as a powerful tool in GDM, P-IVHFSs can describe the actual preferences of decision-makers flexibly and better reflect their uncertainty, hesitancy, and inconsistency, and thus enhance the modeling abilities of HFSs. The consistency of preference relations has become a research topic of great interest in recent years. For example, Liao et al. defined the concept of the multiplicative consistent hesitant fuzzy preference relation. Furthermore, Wu and Xu developed separate consistency and consensus processes to deal with the hesitant fuzzy linguistic preference relations of individual rationality and group rationality. Zhu and Xu proposed the concept of the probability-hesitant fuzzy preference relation. As mentioned earlier, to date there has been a great deal of research into preference relations and interval preference relations. Nevertheless, in a probability-interval valued hesitant fuzzy environment, it is still not known how to calculate or improve the consistency of preference relations. Therefore, this study focuses on resolving this problem.

In this paper, based on the P-HFS and IVHFS, a definition of P-IVHFS is provided, and the relationship between the P-HFS, IVHFS and P-IVHFS is illustrated. Furthermore, motivated by the comparison method of HFES, the comparison method of P-IVHFEs is defined. Additionally, inspired by the operations of IVHFEs, the complement, union and intersection and operational laws of P-IVHFEs are provided. Moreover, based on related studies, the definition of P-IVHFPRs is also provided. Subsequently, the consistency of P-IVHFPRs is discussed, using the multiplicative transitivity to verify the consistency of a P-IVHFPR. Finally, based on the method in a hesitant fuzzy environment, some definitions related to multiplicative consistent P-

IVHFPRs are provided, and a decomposition method to repair the consistency of P-IVHFPRs is proposed.

The rest of this paper is organized as follows. In Section 2, some concepts and properties associated with the topic are briefly reviewed. In Section 3, P-IVHFSs are proposed and some of their properties are discussed. In Section 4, P-IVHFPRs are proposed and in Section 5, the consistency of P-IVHFPRs is discussed. In Section 6, based on the multiplicative consistency of hesitant fuzzy preference relations, a decomposition method to deal with the consistency of P-IVHFPRs is proposed. Finally an example is provided to illustrate the algorithm.

2. Preliminaries

In this section, some concepts and properties associated with the topic are briefly reviewed.

Definition 1. Let $\tilde{a} = [a^L, a^U] = \{x | a^L \leq x \leq a^U\}$, and then \tilde{a} is called an interval number. For convenience, interval numbers are sometimes also called interval values. In particular, if $a^L = a^U$, \tilde{a} is a real number. If $a^L \geq 0$, then \tilde{a} is called a positive interval number.

For any two positive interval numbers $\tilde{a} = [a^L, a^U]$, $\tilde{b} = [b^L, b^U]$ and $\lambda \geq 0, \delta > 0$, then

- (1) $\tilde{a} = \tilde{b}$ if $a^L = b^L$ and $a^U = b^U$;
- (2) $\tilde{a} + \tilde{b} = [a^L + b^L, a^U + b^U]$;
- (3) $\tilde{a} \cdot \tilde{b} = [a^L \cdot b^L, a^U \cdot b^U]$;
- (4) $\lambda \tilde{a} = [\lambda a^L, \lambda a^U]$;
- (5) $\tilde{a}^\delta = [a^L, a^U]^\delta = [(a^L)^\delta, (a^U)^\delta]$;

$$(6) \delta \tilde{a} = \delta^{[a^L, a^U]} = [\min\{\delta^{a^L}, \delta^{a^U}\}, \max\{\delta^{a^L}, \delta^{a^U}\}] = \begin{cases} [\delta^{a^U}, \delta^{a^L}], & \text{if } 0 < \delta < 1; \\ [\delta^{a^L}, \delta^{a^U}], & \text{if } \delta \geq 1. \end{cases}$$

Definition 2. [29] Let $\tilde{a}_1 = [a_1^L, a_1^U]$ and $\tilde{a}_2 = [a_2^L, a_2^U]$ be two interval numbers, and $len(\tilde{a}_1) = a_1^U - a_1^L$, $len(\tilde{a}_2) = a_2^U - a_2^L$, then the degree of possibility of $\tilde{a}_1 \geq \tilde{a}_2$ is defined as follows:

$$p(\tilde{a}_1 \geq \tilde{a}_2) = \max\{1 - \max\{\frac{a_2^U - a_1^L}{len(\tilde{a}_1) + len(\tilde{a}_2)}, 0\}, 0\} \tag{1}$$

Similarly, the degree of possibility of $\tilde{a}_2 \geq \tilde{a}_1$ is defined as follows:

$$p(\tilde{a}_2 \geq \tilde{a}_1) = \max\{1 - \max\{\frac{a_1^U - a_2^L}{len(\tilde{a}_1) + len(\tilde{a}_2)}, 0\}, 0\} \tag{2}$$

Based on Definition 2, the following results hold:

- (1) $0 \leq p(\tilde{a}_1 \geq \tilde{a}_2) \leq 1, 0 \leq p(\tilde{a}_2 \geq \tilde{a}_1) \leq 1$.
- (2) $p(\tilde{a}_1 \geq \tilde{a}_2) + p(\tilde{a}_2 \geq \tilde{a}_1) = 1$. Especially, $p(\tilde{a}_1 \geq \tilde{a}_1) = p(\tilde{a}_2 \geq \tilde{a}_2) = 1$.

Definition 3. [15, 16] Let X be a universal set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0, 1]$.

To be easily understood, the HFS can be expressed by a mathematical symbol [21]:

$$\tilde{A} = \left\{ \left\langle x, \tilde{h}_{\tilde{A}}(x) \right\rangle \mid x \in X \right\}$$

where $\tilde{h}_{\tilde{A}}(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set \tilde{A} . $\tilde{h}_{\tilde{A}}(x)$ is called a hesitant fuzzy element (HFE) and Θ the set of all HFEs [22].

For three HFEs h, h_1 and h_2 , Torra and Narukawa [15, 16] defined the corresponding complement, union and intersection, namely

- (1) $h^c = \cup_{\gamma \in h} \{1 - \gamma\}$;
- (2) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$;
- (3) $h_1 \cap h_2 = \cap_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$.

Operational laws on the HFEs h , h_1 and h_2 have been given as follows [22]:

- (1) $h^\lambda = \cup_{\gamma \in h} \{\gamma^\lambda\}, \lambda > 0;$
- (2) $\lambda h = \cup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}, \lambda > 0;$
- (3) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\};$
- (4) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}.$

Definition 4. [2] Let X be a universal set, and $D[0, 1]$ be the set of all closed subintervals of $[0, 1]$. An interval-valued hesitant fuzzy set (IVHFS) on X is

$$\tilde{A} = \left\{ \left\langle x_i, \tilde{h}_{\tilde{A}}(x_i) \right\rangle \mid x_i \in X, i = 1, 2, \dots, n \right\}$$

where $\tilde{h}_{\tilde{A}}(x_i) : X \rightarrow D[0, 1]$ denotes all possible interval-valued membership degrees of the element $x_i \in X$ to the set \tilde{A} . For convenience, we call $\tilde{h}_{\tilde{A}}(x_i)$ an interval-valued hesitant fuzzy element (IVHFE), which is denoted by

$$\tilde{h}_{\tilde{A}}(x_i) = \left\{ \tilde{\gamma} \mid \tilde{\gamma} \in \tilde{h}_{\tilde{A}}(x_i) \right\}$$

Here $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$ is an interval number. $\tilde{\gamma}^L = \inf \tilde{\gamma}$ and $\tilde{\gamma}^U = \sup \tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively. When the lower and upper limits of the interval numbers are identical, IVHFS reduces to HFS [15]. Namely HFS is a special case of IVHFS.

Example 1. Let $X = \{x_1, x_2\}$ be a universal set, and the two IVHFEs $\tilde{h}_{\tilde{A}}(x_1) = \{[0.1, 0.3], [0.4, 0.5]\}$ and $\tilde{h}_{\tilde{A}}(x_2) = \{[0.1, 0.2], [0.4, 0.6], [0.7, 0.8]\}$ denote the membership degrees of $x_i (i = 1, 2)$ to the set \tilde{A} . \tilde{A} is an IVHFS, where

$$\tilde{A} = \{ \langle x_1, \{[0.1, 0.3], [0.4, 0.5]\} \rangle, \langle x_2, \{[0.1, 0.2], [0.4, 0.6], [0.7, 0.8]\} \rangle \}$$

Definition 5. [2] For an IVHFE \tilde{h} , $s(\tilde{h}) = \frac{1}{l_{\tilde{h}}} \sum_{\tilde{\gamma} \in \tilde{h}} \tilde{\gamma}$ is called the score function of \tilde{h} where $l_{\tilde{h}}$ is the number of the interval values in \tilde{h} , and $s(\tilde{h})$ is an interval value belonging to $[0, 1]$. For two IVHFEs \tilde{h}_1 and \tilde{h}_2 , if $s(\tilde{h}_1) \geq s(\tilde{h}_2)$, then $\tilde{h}_1 \geq \tilde{h}_2$.

Definition 6. [2] For three IVHFEs h , h_1 and h_2 , the corresponding complement, union and intersection and operational laws have been given as follows. If $\tilde{\gamma}^L = \tilde{\gamma}^U$, then the following operations reduce to those of HFEs:

- (1) $\tilde{h}^c = \{[1 - \tilde{\gamma}^U, 1 - \tilde{\gamma}^L] \mid \tilde{\gamma} \in \tilde{h}\};$
- (2) $\tilde{h}_1 \cup \tilde{h}_2 = \{[\max(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\};$
- (3) $\tilde{h}_1 \cap \tilde{h}_2 = \{[\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\};$
- (4) $\tilde{h}^\lambda = \{[(\tilde{\gamma}^L)^\lambda, (\tilde{\gamma}^U)^\lambda] \mid \tilde{\gamma} \in \tilde{h}\}, \lambda > 0;$
- (5) $\lambda \tilde{h} = \{[1 - (1 - \tilde{\gamma}^L)^\lambda, 1 - (1 - \tilde{\gamma}^U)^\lambda] \mid \tilde{\gamma} \in \tilde{h}\}, \lambda > 0;$
- (6) $\tilde{h}_1 \oplus \tilde{h}_2 = \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\};$
- (7) $\tilde{h}_1 \otimes \tilde{h}_2 = \{[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}.$

3. P-IVHFS

Inspired by the P-HFS [19, 37] and IVHFS [2], the definition of a P-IVHFS is provided.

Definition 7. Let X be a universal set, and $D[0, 1]$ be the set of all closed subintervals of $[0, 1]$. A P -IVHFS on X is

$$\tilde{A} = \left\{ \left\langle x_i, \tilde{h}_{\tilde{A}}(x_i, p_{ij}) \right\rangle \mid x_i \in X, i = 1, 2, \dots, n, j = 1, 2, \dots, m_i \right\}$$

where $\sum_{j=1}^{m_i} p_{ij} = 1$, m_i denotes the number of the interval values in $\tilde{h}_{\tilde{A}}(x_i, p_{ij})$, p_{ij} denotes the corresponding probability of the j th interval value in $\tilde{h}_{\tilde{A}}(x_i, p_{ij})$, and $\tilde{h}_{\tilde{A}}(x_i, p_{ij}) : X \rightarrow D[0, 1]$ denotes all possible interval-valued membership degrees of the element $x_i \in X$ to the set \tilde{A} . For convenience, $\tilde{h}_{\tilde{A}}(x_i, p_{ij})$ is called a probability-interval valued hesitant fuzzy element (P -IVHFE), which is denoted by

$$\tilde{h}_{\tilde{A}}(x_i, p_{ij}) = \left\{ \tilde{\gamma} \mid \tilde{\gamma} \in \tilde{h}_{\tilde{A}}(x_i, p_{ij}) \right\}$$

Here $\tilde{\gamma}$ is an interval number with a corresponding possibility. For simplicity, P -IVHFE can be denoted by \tilde{h}_i , $i = 1, 2, \dots$. A P -IVHFE is the basic unit of a P -IVHFS, and the former can be considered as a special case of the latter. The relationship between a P -IVHFE and a P -IVHFS is similar to that between an IVHFE and an IVHFS [2].

Suppose that $\tilde{\gamma}^L = \inf \tilde{\gamma}$ and $\tilde{\gamma}^U = \sup \tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively. When the lower and upper limits of the interval numbers are identical, the interval numbers are reduced to crisp numbers, and a P -IVHFS is reduced to a P -HFS. Thus, a P -HFS is a special case of a P -IVHFS. Meanwhile, it is clear that without the probability description p_{ij} , that is the probability values $p_{ij} (j = 1, 2, \dots)$ are identical, a P -IVHFE is reduced to an IVHFE, and a P -IVHFS is reduced to an IVHFS. Thus, an IVHFE is a special case of a P -IVHFE, and an IVHFS is a special case of a P -IVHFS.

Example 2. Let $X = \{x_1, x_2\}$ be a universal set, and the two P -IVHFEs

$$\begin{aligned} \tilde{h}_{\tilde{A}}(x_1, p_{1j}) &= \{[0.2, 0.3](p_{11} = 0.4), [0.5, 0.6](p_{12} = 0.6)\} \\ \tilde{h}_{\tilde{A}}(x_2, p_{2j}) &= \{[0.1, 0.2](p_{21} = 0.3), [0.3, 0.5](p_{22} = 0.5), [0.6, 0.7](p_{23} = 0.2)\} \end{aligned}$$

denote the membership degrees of $x_i (i = 1, 2)$ to the set \tilde{A} . \tilde{A} is a P -IVHFS, where

$$\tilde{A} = \left\{ \begin{aligned} &\langle x_1, \{[0.2, 0.3](p_{11} = 0.4), [0.5, 0.6](p_{12} = 0.6)\} \rangle, \\ &\langle x_2, \{[0.1, 0.2](p_{21} = 0.3), [0.3, 0.5](p_{22} = 0.5), [0.6, 0.7](p_{23} = 0.2)\} \rangle \end{aligned} \right\}$$

Based on the comparison method of HFEs [21], the following comparison method of P -IVHFEs is defined:

Definition 8. For a P -IVHFE, $s(\tilde{h}) = \sum_{\tilde{\gamma} \in \tilde{h}} \tilde{\gamma} p_{\tilde{\gamma}}$ is called the score of \tilde{h} , where $p_{\tilde{\gamma}}$ is the corresponding probability of $\tilde{\gamma}$. It is clear that $s(\tilde{h})$ is also an interval number.

Then by Eqs. (1) and (2), we can get the possibilities of $s(\tilde{h}_1) \geq s(\tilde{h}_2)$ and $s(\tilde{h}_2) \geq s(\tilde{h}_1)$, namely $p(\tilde{h}_1 \geq \tilde{h}_2)$ and $p(\tilde{h}_2 \geq \tilde{h}_1)$.

If $p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) > 0.5$, then $s(\tilde{h}_1)$ is superior to $s(\tilde{h}_2)$, and thus \tilde{h}_1 is superior to \tilde{h}_2 , denoted by $\tilde{h}_1 > \tilde{h}_2$ or $\tilde{h}_2 < \tilde{h}_1$.

If $p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) < 0.5$, then $s(\tilde{h}_2)$ is superior to $s(\tilde{h}_1)$, and thus \tilde{h}_2 is superior to \tilde{h}_1 , denoted by $\tilde{h}_2 > \tilde{h}_1$ or $\tilde{h}_1 < \tilde{h}_2$.

In particular, if $p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) = 0.5$, then \tilde{h}_1 is indifferent to \tilde{h}_2 , denoted by $\tilde{h}_1 \sim \tilde{h}_2$.

Example 3. In Example 2, for the two P -IVHFEs

$$\begin{aligned} \tilde{h}_1 &= \{[0.2, 0.3](p_{11} = 0.4), [0.5, 0.6](p_{12} = 0.6)\} \\ \tilde{h}_2 &= \{[0.1, 0.2](p_{21} = 0.3), [0.3, 0.5](p_{22} = 0.5), [0.6, 0.7](p_{23} = 0.2)\} \end{aligned}$$

according to Definition 1, we have

$$s(\tilde{h}_1) = [0.2, 0.3] \times 0.4 + [0.5, 0.6] \times 0.6 = [0.38, 0.48]$$

$$s(\tilde{h}_2) = [0.1, 0.2] \times 0.3 + [0.3, 0.5] \times 0.5 + [0.6, 0.7] \times 0.2 = [0.3, 0.45]$$

Using Definition 2, we obtain

$$p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) = \max\{1 - \max\{\frac{0.45 - 0.38}{0.15 + 0.1}, 0\}, 0\} = 0.72$$

which indicates that $\tilde{h}_1 > \tilde{h}_2$.

To be easily formulated, a P-IVHFE can be denoted by $\tilde{h} = \{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U](p_{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U]}) \mid \tilde{\gamma}_i \in \tilde{h}\}$, for simplicity, denoted by $\tilde{h} = \{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U](p_{\tilde{\gamma}_i}) \mid \tilde{\gamma}_i \in \tilde{h}\}$, where $p_{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U]}$ (or $p_{\tilde{\gamma}_i}$) denotes the corresponding probability value of $[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U]$ (i.e., $\tilde{\gamma}_i$). Based on the operations of IVHFEs [2], the complement, union and intersection and operational laws of P-IVHFEs can be provided as follows:

Definition 9. Let \tilde{h}, \tilde{h}_1 and \tilde{h}_2 be three P-IVHFEs, then

- (1) $\tilde{h}^c = \{[1 - \tilde{\gamma}_i^U, 1 - \tilde{\gamma}_i^L](p_{\tilde{\gamma}_i}) \mid \tilde{\gamma}_i \in \tilde{h}\};$
- (2) $\tilde{h}_1 \cup \tilde{h}_2 = \{[\max(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\};$
- (3) $\tilde{h}_1 \cap \tilde{h}_2 = \{[\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\};$
- (4) $\tilde{h}^\lambda = \{[(\tilde{\gamma}_i^L)^\lambda, (\tilde{\gamma}_i^U)^\lambda](p_{\tilde{\gamma}_i}) \mid \tilde{\gamma}_i \in \tilde{h}\};$
- (5) $\lambda \tilde{h} = \{[1 - (1 - \tilde{\gamma}_i^L)^\lambda, 1 - (1 - \tilde{\gamma}_i^U)^\lambda](p_{\tilde{\gamma}_i}) \mid \tilde{\gamma}_i \in \tilde{h}, \lambda > 0\};$
- (6) $\tilde{h}_1 \oplus \tilde{h}_2 = \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\};$
- (7) $\tilde{h}_1 \otimes \tilde{h}_2 = \{[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \mid \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}.$

It is clear that without the probability description p_{ij} , that is the probability values $p_{ij}(j = 1, 2, \dots)$ are identical, then the operational laws of P-IVHFEs are reduced to those of the IVHFEs.

Theorem 1. When IVHFEs are extended to P-IVHFEs, the following operational laws [2] still are true in the P-IVHFS environment. Let \tilde{h}, \tilde{h}_1 and \tilde{h}_2 be three P-IVHFEs, then

- (1) $\tilde{h}_1 \oplus \tilde{h}_2 = \tilde{h}_2 \oplus \tilde{h}_1;$
- (2) $\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1;$
- (3) $\lambda(\tilde{h}_1 \oplus \tilde{h}_2) = \lambda \tilde{h}_1 \oplus \lambda \tilde{h}_2, \lambda > 0;$
- (4) $(\tilde{h}_1 \otimes \tilde{h}_2)^\lambda = \tilde{h}_1^\lambda \otimes \tilde{h}_2^\lambda, \lambda > 0;$
- (5) $\lambda_1 \tilde{h} \oplus \lambda_2 \tilde{h} = (\lambda_1 + \lambda_2) \tilde{h}, \lambda_1, \lambda_2 > 0;$
- (6) $\tilde{h}^{\lambda_1} \oplus \tilde{h}^{\lambda_2} = \tilde{h}^{(\lambda_1 + \lambda_2)}, \lambda_1, \lambda_2 > 0.$
- (7) $\tilde{h}_1^c \cup \tilde{h}_2^c = (\tilde{h}_1 \cap \tilde{h}_2)^c;$
- (8) $\tilde{h}_1^c \cap \tilde{h}_2^c = (\tilde{h}_1 \cup \tilde{h}_2)^c;$
- (9) $(\tilde{h}^c)^\lambda = (\lambda \tilde{h})^c;$
- (10) $\lambda(\tilde{h}^c) = (\tilde{h}^\lambda)^c;$
- (11) $\tilde{h}_1^c \oplus \tilde{h}_2^c = (\tilde{h}_1 \otimes \tilde{h}_2)^c;$
- (12) $\tilde{h}_1^c \otimes \tilde{h}_2^c = (\tilde{h}_1 \oplus \tilde{h}_2)^c.$

Since they can be proven analogously, like those in an IVHFS environment, they are just listed without any proof. Meanwhile, according to Definition 9, the following operational laws also hold:

Theorem 2. Let \tilde{h} , \tilde{h}_1 and \tilde{h}_2 be three P-IVHFEs, then

- (1) $(\tilde{h} \cup \tilde{h}_1) \cup \tilde{h}_2 = \tilde{h} \cup (\tilde{h}_1 \cup \tilde{h}_2)$;
- (2) $(\tilde{h} \cap \tilde{h}_1) \cap \tilde{h}_2 = \tilde{h} \cap (\tilde{h}_1 \cap \tilde{h}_2)$;
- (3) $(\tilde{h} \oplus \tilde{h}_1) \oplus \tilde{h}_2 = \tilde{h} \oplus (\tilde{h}_1 \oplus \tilde{h}_2)$;
- (4) $(\tilde{h} \otimes \tilde{h}_1) \otimes \tilde{h}_2 = \tilde{h} \otimes (\tilde{h}_1 \otimes \tilde{h}_2)$.

Proof. In the following, only (3) is proven; others can be obtained directly by Definition 9. Suppose that

$$\tilde{h} = \{[\tilde{\gamma}^L, \tilde{\gamma}^U](p_{\tilde{\gamma}}) \mid \tilde{\gamma} \in \tilde{h}\}; \tilde{h}_1 = \{[\tilde{\gamma}_1^L, \tilde{\gamma}_1^U](p_{\tilde{\gamma}_1}) \mid \tilde{\gamma}_1 \in \tilde{h}_1\}; \tilde{h}_2 = \{[\tilde{\gamma}_2^L, \tilde{\gamma}_2^U](p_{\tilde{\gamma}_2}) \mid \tilde{\gamma}_2 \in \tilde{h}_2\},$$

then according to Definition 9,

$$\tilde{h} \oplus \tilde{h}_1 = \{[\tilde{\gamma}^L + \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L, \tilde{\gamma}^U + \tilde{\gamma}_1^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U](p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1}) \mid \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1\}$$

is obtained, therefore,

$$\begin{aligned} & (\tilde{h} \oplus \tilde{h}_1) \oplus \tilde{h}_2 \\ &= \{[(\tilde{\gamma}^L + \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L) + \tilde{\gamma}_2^L - (\tilde{\gamma}^L + \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L) \cdot \tilde{\gamma}_2^L, (\tilde{\gamma}^U + \tilde{\gamma}_1^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U) + \tilde{\gamma}_2^U \\ & \quad - (\tilde{\gamma}^U + \tilde{\gamma}_1^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U) \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \mid \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\} \\ &= \{[\tilde{\gamma}^L + \tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L + \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}^U + \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \\ & \quad - \tilde{\gamma}^U \cdot \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U + \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \mid \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\} \end{aligned}$$

Likewise,

$$\begin{aligned} & \tilde{h} \oplus (\tilde{h}_1 \oplus \tilde{h}_2) \\ &= \{[\tilde{\gamma}^L + (\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L) - \tilde{\gamma}^L \cdot (\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L), \tilde{\gamma}^U + (\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U) \\ & \quad - \tilde{\gamma}^U \cdot (\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U)](p_{\tilde{\gamma}} \cdot (p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2})) \mid \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\} \\ &= \{[\tilde{\gamma}^L + \tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L + \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}^U + \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \\ & \quad - \tilde{\gamma}^U \cdot \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U + \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \mid \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\} \end{aligned}$$

can be obtained. Therefore, we have

$$(\tilde{h} \oplus \tilde{h}_1) \oplus \tilde{h}_2 = \tilde{h} \oplus (\tilde{h}_1 \oplus \tilde{h}_2)$$

which completes the proof. □

4. P-IVHFPRs and Consistency

In this section, we present P-IVHFPRs and discuss their consistency.

4.1. P-IVHFPRs

In the GDM process, preference relations are very popular tools for expressing the DM's preferences when they compare a set of alternatives. Various types of preference relations have been given for different environments [2].

In order to represent preference relations more objectively, suppose that DMs are allowed to provide several possible interval fuzzy preference values and the associated probability values when they compare two alternatives, then we get the following P-IVHFPR:

Definition 10. Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal set. A P-IVHFPR on X is denoted by a matrix $\tilde{R} = (\tilde{h}_{ij})_{n \times n} \subset X \times X$, where $\tilde{h}_{ij} = \{\tilde{h}_{ij}^t(p_{ij}^t), t = 1, 2, \dots, m_{ij}\}$ is a P-IVHFE, indicating all possible degrees to which x_i is preferred to x_j and the corresponding probability values with m_{ij} representing the number of intervals in the P-IVHFE. In addition, \tilde{h}_{ij}^t should satisfy

$$\begin{aligned} \inf \tilde{h}_{ij}^{\sigma(t)} + \sup \tilde{h}_{ji}^{\sigma(m_{ij}+1-t)} &= \sup \tilde{h}_{ij}^{\sigma(t)} + \inf \tilde{h}_{ji}^{\sigma(m_{ij}+1-t)} = 1, \\ p_{ij}^{\sigma(t)} &= p_{ji}^{\sigma(m_{ij}+1-t)}, \\ \tilde{h}_{ii} &= \{[0.5, 0.5](p = 1)\}, \quad i, j = 1, 2, \dots, n \end{aligned} \tag{3}$$

where we arrange the intervals in \tilde{h}_{ij} in an increasing order, and let $\tilde{h}_{ij}^{\sigma(t)}$ be the t th smallest interval in \tilde{h}_{ij} . $\inf \tilde{h}_{ij}^{\sigma(t)}$ and $\sup \tilde{h}_{ij}^{\sigma(t)}$ denote the lower and upper limits of $\tilde{h}_{ij}^{\sigma(t)}$ respectively, $p_{ij}^{\sigma(t)}$ and $p_{ji}^{\sigma(m_{ij}+1-t)}$ denote the corresponding values of $\tilde{h}_{ij}^{\sigma(t)}$ and $\tilde{h}_{ji}^{\sigma(m_{ij}+1-t)}$ respectively, $p = 1$ denotes the corresponding value is equal to 1.

Example 4. The following matrix in which every element is a P-IVHFE can represent a probability-interval valued hesitant fuzzy preference relation:

$$\tilde{R}_e = (\tilde{h}_{ij})_{3 \times 3} = \begin{pmatrix} \{[0.5, 0.5](1)\} & \{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\} & \{[0.5, 0.6](1)\} \\ \{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\} & \{[0.5, 0.5](1)\} & \{[0.3, 0.4](0.2), [0.5, 0.7](0.5), [0.8, 0.9](0.3)\} \\ \{[0.4, 0.5](1)\} & \{[0.1, 0.2](0.3), [0.3, 0.5](0.5), [0.6, 0.7](0.2)\} & \{[0.5, 0.5](1)\} \end{pmatrix}$$

where \tilde{h}_{ij} denotes the group preference degree that the alternative x_i is superior to the alternative x_j .

Motivated by [2, 35], it can be explained how the elements in the matrix are obtained. Take \tilde{h}_{23} as an example. Since \tilde{h}_{23} represents all possible probability-interval valued preference degrees to which x_3 is preferred to x_2 , its values come from $\tilde{h}_{23}^1 = [0.1, 0.2]$, $\tilde{h}_{23}^2 = [0.3, 0.5]$, $\tilde{h}_{23}^3 = [0.6, 0.7]$ which is provided by a DM. The DM is sure that the preference value is the interval $[0.1, 0.2]$ with a probability of 30%, and the interval $[0.3, 0.5]$ with a probability of 50%, and the interval $[0.6, 0.7]$ with a probability of 20%. Therefore, the \tilde{h}_{23} can be denoted by $\{[0.1, 0.2](0.3), [0.3, 0.5](0.5), [0.6, 0.7](0.2)\}$. Similarly the symmetric element of \tilde{h}_{23} , i.e., \tilde{h}_{32} can be denoted by $\{[0.3, 0.4](0.2), [0.5, 0.7](0.5), [0.8, 0.9](0.3)\}$. Other symmetric elements \tilde{h}_{ij} and \tilde{h}_{ji} in \tilde{R}_e are obtained in an analogous way, and satisfy the complementary properties defined in Eq.(3). In addition, when $i = j$, \tilde{h}_{ij} represents the preference degree to which x_i is preferred to itself; namely, it is preferred equally, therefore $\tilde{h}_{ii} = \{[0.5, 0.5](1)\} (i = 1, 2, 3)$. Through the above procedure, the aforementioned matrix \tilde{R}_e is obtained.

4.2. The Consistency of P-IVHFPRs

Cardinal consistency is a stronger concept than ordinal consistency. In the analytic hierarchy process, Saaty [13] first addressed the issue of consistency, and developed the notions of perfect consistency and acceptable consistency. Ordinal consistency is based on the notion of transitivity, meaning that if A is preferred to B and B is preferred to C, it perceives A to be preferred to C, which is normally referred to as weak transitivity [4, 26]. The weak transitivity is the minimum requirement condition to ensure that the hesitant fuzzy preference relation is consistent. There are further two conditions, named additive transitivity and multiplicative transitivity [14] which are more restrictive than weak transitivity and can imply reciprocity. Even though both additive transitivity and multiplicative transitivity can be used to measure consistency, the additive consistency may produce infeasible results [27]. Thus, the multiplicative transitivity is also used to verify the consistency of a P-IVHFPR.

Let $U = (u_{ij})_{n \times n}$, where u_{ij} denotes a ratio of preference intensity for the alternative A_i to that for A_j . Then the condition of multiplicative transitivity can be rewritten as follows:

$$u_{ij}u_{jk}u_{ki} = u_{ik}u_{kj}u_{ji} \tag{4}$$

Under the assumption of reciprocity, and in the case where $(u_{ik}, u_{kj}) \notin \{(0, 1), (1, 0)\}$, Eq.(4) can be expressed as follows [4]:

$$u_{ij} = \frac{u_{ik}u_{kj}}{u_{ik}u_{kj} + (1 - u_{ik})(1 - u_{kj})} \tag{5}$$

in the case where $(u_{ik}, u_{kj}) \in \{(0, 1), (1, 0)\}$, stipulating $u_{ij} = 0$.

Based on the multiplicative consistency of hesitant fuzzy preference relations, and using a decomposition method, the following definition is obtained:

Definition 11. Let $\tilde{R} = (\tilde{h}_{ij})_{n \times n}$ be a P-IVHFP on a fixed set $X = \{x_1, x_2, \dots, x_n\}$ and $\tilde{h}_{ij} = \{\tilde{h}_{ij}^t(p_{ij}^t), t = 1, 2, \dots, m_{ij}\}$ be a P-IVHFE; suppose that $\tilde{h}_{ij}^t = [\#\tilde{h}_{ij}^t(x), \dagger\tilde{h}_{ij}^t(x)]$, where $\#\tilde{h}_{ij}^t(x)$ is the left endpoint of \tilde{h}_{ij}^t , and $\dagger\tilde{h}_{ij}^t(x)$ is the right endpoint of \tilde{h}_{ij}^t , let

$$\tilde{R}^A = [\tilde{R}_{ij}^t(p_{ij}^t)]_{n \times n} = \begin{bmatrix} \{0.5(1)\} & \{\#\tilde{h}_{12}^t(p_{12}^t)\} & \cdots & \{\#\tilde{h}_{1n}^t(p_{1n}^t)\} \\ \{\dagger\tilde{h}_{21}^t(p_{21}^t)\} & \{0.5(1)\} & \cdots & \{\#\tilde{h}_{2n}^t(p_{2n}^t)\} \\ \cdots & \cdots & \cdots & \cdots \\ \{\dagger\tilde{h}_{n1}^t(p_{n1}^t)\} & \{\dagger\tilde{h}_{n2}^t(p_{n2}^t)\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

namely, $\tilde{R}_{ij}^t = \begin{cases} \#\tilde{h}_{ij}^t, & \text{if } i < j, \\ 0.5, & \text{if } i = j, \\ \dagger\tilde{h}_{ij}^t, & \text{if } i > j \end{cases}$ which means if $i < j$, taking the left endpoint of \tilde{h}_{ij}^t , while if $i > j$, taking the right endpoint of \tilde{h}_{ij}^t . And let

$$\tilde{R}^B = [\tilde{r}_{ij}^t(p_{ij}^t)]_{n \times n} = \begin{bmatrix} \{0.5(1)\} & \{\dagger\tilde{h}_{12}^t(p_{12}^t)\} & \cdots & \{\dagger\tilde{h}_{1n}^t(p_{1n}^t)\} \\ \{\#\tilde{h}_{21}^t(p_{21}^t)\} & \{0.5(1)\} & \cdots & \{\dagger\tilde{h}_{2n}^t(p_{2n}^t)\} \\ \cdots & \cdots & \cdots & \cdots \\ \{\#\tilde{h}_{n1}^t(p_{n1}^t)\} & \{\#\tilde{h}_{n2}^t(p_{n2}^t)\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

which means if $i < j$, taking the right endpoint of \tilde{h}_{ij}^t , while if $i > j$, taking the left endpoint of \tilde{h}_{ij}^t , namely,

$$\tilde{r}_{ij}^t = \begin{cases} \dagger\tilde{h}_{ij}^t, & \text{if } i < j, \\ 0.5, & \text{if } i = j, \\ \#\tilde{h}_{ij}^t, & \text{if } i > j \end{cases}$$

for convenience, \tilde{R}^A and \tilde{R}^B are called the decomposition of \tilde{R} , while \tilde{R} is the composition of \tilde{R}^A and \tilde{R}^B . Then $\tilde{R} = (\tilde{h}_{ij})_{n \times n}$ is multiplicative consistent if and only if \tilde{R}^A and \tilde{R}^B are both multiplicative consistent, i.e., the following two conditions are satisfied simultaneously:

$$(1) \tilde{R}_{ij}^{\sigma(s)} = \begin{cases} 0, & \text{if } (\tilde{R}_{ik}, \tilde{R}_{kj}) \in \{(\{0\}, \{1\}), (\{1\}, \{0\})\} \\ \frac{\tilde{R}_{ik}^{\sigma(s)} p_{ik}^{\sigma(s)} \tilde{R}_{kj}^{\sigma(s)} p_{kj}^{\sigma(s)}}{\tilde{R}_{ik}^{\sigma(s)} p_{ik}^{\sigma(s)} \tilde{R}_{kj}^{\sigma(s)} p_{kj}^{\sigma(s)} + (1 - \tilde{R}_{ik}^{\sigma(s)}) p_{ik}^{\sigma(s)} (1 - \tilde{R}_{kj}^{\sigma(s)}) p_{kj}^{\sigma(s)}}, & \text{otherwise,} \end{cases}$$

for all $i \leq k \leq j$

$$\text{i.e., } \tilde{R}_{ij}^{\sigma(s)} = \begin{cases} 0, & \text{if } (\tilde{R}_{ik}, \tilde{R}_{kj}) \in \{(\{0\}, \{1\}), (\{1\}, \{0\})\} \\ \frac{\tilde{R}_{ik}^{\sigma(s)} \tilde{R}_{kj}^{\sigma(s)}}{\tilde{R}_{ik}^{\sigma(s)} \tilde{R}_{kj}^{\sigma(s)} + (1 - \tilde{R}_{ik}^{\sigma(s)}) (1 - \tilde{R}_{kj}^{\sigma(s)})}, & \text{otherwise,} \end{cases}$$

for all $i \leq k \leq j$

$$(2) \tilde{r}_{ij}^{\sigma(s)} = \begin{cases} 0, & \text{if } (\tilde{r}_{ik}, \tilde{r}_{kj}) \in \{(\{0\}, \{1\}), (\{1\}, \{0\})\} \\ \frac{\tilde{r}_{ik}^{\sigma(s)} \tilde{r}_{kj}^{\sigma(s)}}{\tilde{r}_{ik}^{\sigma(s)} \tilde{r}_{kj}^{\sigma(s)} + (1 - \tilde{r}_{ik}^{\sigma(s)}) (1 - \tilde{r}_{kj}^{\sigma(s)})}, & \text{otherwise,} \end{cases}$$

for all $i \leq k \leq j$

$$p_{ij}^{\sigma(s)} = p_{ik}^{\sigma(s)} p_{kj}^{\sigma(s)}$$

where $\tilde{R}_{ij}^{\sigma(s)}$, $\tilde{R}_{ik}^{\sigma(s)}$ and $\tilde{R}_{kj}^{\sigma(s)}$ are the s th smallest values in \tilde{R}_{ij} , \tilde{R}_{ik} and \tilde{R}_{kj} respectively; $p_{ij}^{\sigma(s)}$, $p_{ik}^{\sigma(s)}$ and $p_{kj}^{\sigma(s)}$ are their corresponding probability values, respectively; $\tilde{r}_{ij}^{\sigma(s)}$, $\tilde{r}_{ik}^{\sigma(s)}$ and $\tilde{r}_{kj}^{\sigma(s)}$ are the s th smallest values in \tilde{r}_{ij} , \tilde{r}_{ik} and \tilde{r}_{kj} respectively; and $p_{ij}^{\sigma(s)}$, $p_{ik}^{\sigma(s)}$ and $p_{kj}^{\sigma(s)}$ are their corresponding probability values, respectively.

If without the probability description and $\#h_{ik}^t = \dagger h_{ik}^t$, $\#h_{kj}^t = \dagger h_{kj}^t$, then Definition 11 is reduced to that of a hesitant fuzzy preference relation.

It can be proven that any P-IVHFPR $\tilde{R} = (\tilde{h}_{ij})_{2 \times 2}$ is multiplicative consistent.

By extending the definitions in a hesitant fuzzy environment, the following definitions in a probability-interval valued hesitant fuzzy environment are obtained:

Definition 12. Let $\tilde{R} = (\tilde{h}_{ij})_{n \times n}$ be a P-IVHFPR on a fixed set $X = \{x_1, x_2, \dots, x_n\}$ and $\tilde{h}_{ij} = \{\tilde{h}_{ij}^t(p_{ij}^t)\}$, $t = 1, 2, \dots, m_{ij}$ be a P-IVHFE in which m_{ij} represents the number of intervals suppose that $\tilde{h}_{ij}^t = [\# \tilde{h}_{ij}^t(x), \dagger \tilde{h}_{ij}^t(x)]$, let

$$\tilde{R}^A = [\tilde{R}_{ij}^t(p_{ij}^t)]_{n \times n} = \begin{bmatrix} \{0.5(1)\} & \{\#\tilde{h}_{12}^t(p_{12}^t)\} & \cdots & \{\#\tilde{h}_{1n}^t(p_{1n}^t)\} \\ \{\dagger\tilde{h}_{21}^t(p_{21}^t)\} & \{0.5(1)\} & \cdots & \{\#\tilde{h}_{2n}^t(p_{2n}^t)\} \\ \dots & \dots & \dots & \dots \\ \{\dagger\tilde{h}_{n1}^t(p_{n1}^t)\} & \{\dagger\tilde{h}_{n2}^t(p_{n2}^t)\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

$$\tilde{R}^B = [\tilde{r}_{ij}^t(p_{ij}^t)]_{n \times n} = \begin{bmatrix} \{0.5(1)\} & \{\dagger\tilde{h}_{12}^t(p_{12}^t)\} & \cdots & \{\dagger\tilde{h}_{1n}^t(p_{1n}^t)\} \\ \{\#\tilde{h}_{21}^t(p_{21}^t)\} & \{0.5(1)\} & \cdots & \{\dagger\tilde{h}_{2n}^t(p_{2n}^t)\} \\ \dots & \dots & \dots & \dots \\ \{\#\tilde{h}_{n1}^t(p_{n1}^t)\} & \{\#\tilde{h}_{n2}^t(p_{n2}^t)\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

then we call \bar{R} a perfect multiplicative consistent P-IVHFPR, if \bar{R} is the composition of \bar{R}^A and \bar{R}^B , $\bar{R}^A = (\bar{R}_{ij}(x))_{n \times n}(p_{ij}^t)$, $\bar{R}^B = (\bar{r}_{ij}(x))_{n \times n}(p_{ij}^t)$, and

$$\bar{R}_{ij}^{\sigma(s)}(x) = \begin{cases} \frac{1}{j-i-1} \sum_{k=i+1}^{j-1} \frac{\tilde{R}_{ik}^{\sigma(s)}(x)\tilde{R}_{kj}^{\sigma(s)}(x)}{\tilde{R}_{ik}^{\sigma(s)}(x)\tilde{R}_{kj}^{\sigma(s)}(x)+(1-\tilde{R}_{ik}^{\sigma(s)}(x))(1-\tilde{R}_{kj}^{\sigma(s)}(x))}, & i+1 < j \\ \tilde{R}_{ij}^{\sigma(s)}(x), & i+1 = j \\ \{0.5\}, & i = j \\ 1 - \bar{R}_{ji}^{\sigma(s)}(x), & i > j \end{cases} \tag{6}$$

$$\bar{r}_{ij}^{\sigma(s)}(x) = \begin{cases} \frac{1}{j-i-1} \sum_{k=i+1}^{j-1} \frac{\tilde{r}_{ik}^{\sigma(s)}(x)\tilde{r}_{kj}^{\sigma(s)}(x)}{\tilde{r}_{ik}^{\sigma(s)}(x)\tilde{r}_{kj}^{\sigma(s)}(x)+(1-\tilde{r}_{ik}^{\sigma(s)}(x))(1-\tilde{r}_{kj}^{\sigma(s)}(x))}, & i+1 < j \\ \tilde{r}_{ij}^{\sigma(s)}(x), & i+1 = j \\ \{0.5\}, & i = j \\ 1 - \bar{r}_{ji}^{\sigma(s)}(x), & i > j \end{cases} \tag{7}$$

where $\bar{R}_{ij}^{\sigma(s)}(x)$, $\tilde{R}_{ik}^{\sigma(s)}(x)$, $\tilde{R}_{kj}^{\sigma(s)}(x)$, $\tilde{r}_{ij}^{\sigma(s)}(x)$, $\tilde{r}_{ik}^{\sigma(s)}(x)$, $\tilde{r}_{kj}^{\sigma(s)}(x)$ denote the s th smallest values in $\bar{R}_{ij}(x)$, $\tilde{R}_{ik}(x)$, $\tilde{R}_{kj}(x)$, $\tilde{r}_{ij}(x)$, $\tilde{r}_{ik}(x)$, $\tilde{r}_{kj}(x)$ respectively, and $s = 1, 2, \dots, l$, $l = \max\{m_{ik}, m_{kj}\}$, in which m_{ik} , m_{kj} represent the number of intervals in \tilde{h}_{ik} and \tilde{h}_{kj} respectively.

If the two endpoints of the intervals are considered, the two corresponding probability-hesitant fuzzy preference relations are multiplicative consistent, thus it is believed that the P-IVHFPR is multiplicative consistent.

Definition 13. Let $\tilde{R} = (\tilde{h}_{ij})_{n \times n}$, \tilde{R}^A , \tilde{R}^B , \bar{R}^A , \bar{R}^B be as before, then we call $\tilde{R} = (\tilde{h}_{ij})_{n \times n}$ an acceptable multiplicative consistent P-IVHFPR if

$$\begin{cases} d(\tilde{R}^A, \bar{R}^A) < \theta_0 \\ d(\tilde{R}^B, \bar{R}^B) < \theta_0 \end{cases}$$

where $d(\tilde{R}^A, \bar{R}^A)$ is the distance measure between \tilde{R}^A and \bar{R}^A , $d(\tilde{R}^B, \bar{R}^B)$ is the distance measure between \tilde{R}^B and \bar{R}^B . $d(\tilde{R}^A, \bar{R}^A)$ and $d(\tilde{R}^B, \bar{R}^B)$ can be calculated by the following Eqs.(8) and (9). θ_0 is the consistency level. We usually take $\theta_0 = 0.1$ in practice.

4.3. An Iterative Algorithm for Improving the Consistency of P-IVHFPR

In general, the P-IVHFPR constructed by the decision maker often has an unacceptable multiplicative consistency which means $d(\tilde{R}^A, \bar{R}^A) \geq \theta_0$ or $d(\tilde{R}^B, \bar{R}^B) \geq \theta_0$. At this time, it is necessary to adjust the elements in the P-IVHFPR in order to improve the consistency. Based on the algorithm in a hesitant fuzzy environment [9], An iterative algorithm is proposed as follows to repair the consistency of the P-IVHFPR.

An Iterative Algorithm for Improving the Consistency of P-IVHFPR

Input: P-IVHFPR $\tilde{R} = (\tilde{h}_{ij})_{n \times n}$; k , the number of iterations; δ , the step size, $0 \leq \lambda = k\delta \leq 1$; θ_0 , the consistency level. Hereby we take $\theta_0 = 0.1$.

Output: P-IVHFPR $\tilde{R}^{(k)}$, with satisfactory consistency.

Step 1. Let $k = 1$, and construct a perfect multiplicative consistent P-IVHFPR \bar{R} , where \bar{R} is the composition of \bar{R}^A and \bar{R}^B , $\bar{R}^A = (\bar{R}_{ij}(x))_{n \times n}$, $\bar{R}^B = (\bar{r}_{ij}(x))_{n \times n}$. \bar{R}^A and \bar{R}^B are defined in Definition 12.

Step 2. Calculate the deviations $d(\tilde{R}^{(k)A}, \bar{R}^A)$ and $d(\tilde{R}^{(k)B}, \bar{R}^B)$. Eqs.(8) and (9) are given as follows:

$$\begin{cases} d_{Ham\ min\ g}(\tilde{R}^{(k)A}, \bar{R}^A) = \frac{1}{(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1}^n \left[\sum_{s=1}^{m_{ij}} |R_{ij}^{(k)\sigma(s)} - \bar{R}_{ij}^{\sigma(s)}| p_{ij}^{\sigma(s)} \right] \\ d_{Ham\ min\ g}(\tilde{R}^{(k)B}, \bar{R}^B) = \frac{1}{(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1}^n \left[\sum_{s=1}^{m_{ij}} |r_{ij}^{(k)\sigma(s)} - \bar{r}_{ij}^{\sigma(s)}| p_{ij}^{\sigma(s)} \right] \end{cases} \quad (8)$$

or

$$\begin{cases} d_{Euclidean}(\tilde{R}^{(k)A}, \bar{R}^A) = \left[\frac{1}{(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1}^n \left(\sum_{s=1}^{m_{ij}} |R_{ij}^{(k)\sigma(s)} - \bar{R}_{ij}^{\sigma(s)}| p_{ij}^{\sigma(s)} \right)^2 \right]^{\frac{1}{2}} \\ d_{Euclidean}(\tilde{R}^{(k)B}, \bar{R}^B) = \left[\frac{1}{(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1}^n \left(\sum_{s=1}^{m_{ij}} |r_{ij}^{(k)\sigma(s)} - \bar{r}_{ij}^{\sigma(s)}| p_{ij}^{\sigma(s)} \right)^2 \right]^{\frac{1}{2}} \end{cases} \quad (9)$$

where $R_{ij}^{(k)\sigma(s)}$, $\bar{R}_{ij}^{\sigma(s)}$, $r_{ij}^{(k)\sigma(s)}$, $\bar{r}_{ij}^{\sigma(s)}$ are the s th smallest values in $\tilde{R}^{(k)A}$, \bar{R}^A , $\tilde{R}^{(k)B}$, \bar{R}^B respectively. $\tilde{R}^{(k)A}$ and $\tilde{R}^{(k)B}$ are the resolution of $\tilde{R}^{(k)}$. \bar{R}^A and \bar{R}^B are the resolution of \bar{R} , which is the corresponding perfect multiplicative relation of \bar{R} . If $d(\tilde{R}^{(k)A}, \bar{R}^A) < \theta_0$ and $d(\tilde{R}^{(k)B}, \bar{R}^B) < \theta_0$, then go to Step 4; Otherwise, go to Step 3.

Step 3. Repair the inconsistent multiplicative P-IVHFPR, transforming $\tilde{R}^{(k)A}$ to $\hat{R}^{(k)A}$ and $\tilde{R}^{(k)B}$ to $\hat{R}^{(k)B}$ by using the following equations. We give Eqs.(10) and (11).

$$\hat{R}_{ij}^{(k)\sigma(s)} = \frac{(R_{ij}^{(k)\sigma(s)})^{1-\lambda} (\bar{R}_{ij}^{\sigma(s)})^\lambda}{(R_{ij}^{(k)\sigma(s)})^{1-\lambda} (\bar{R}_{ij}^{\sigma(s)})^\lambda + (1-R_{ij}^{(k)\sigma(s)})^{1-\lambda} (1-\bar{R}_{ij}^{\sigma(s)})^\lambda} \quad i, j = 1, 2, \dots, n \quad (10)$$

$$\hat{r}_{ij}^{(k)\sigma(s)} = \frac{(r_{ij}^{(k)\sigma(s)})^{1-\lambda} (\bar{r}_{ij}^{\sigma(s)})^\lambda}{(r_{ij}^{(k)\sigma(s)})^{1-\lambda} (\bar{r}_{ij}^{\sigma(s)})^\lambda + (1-r_{ij}^{(k)\sigma(s)})^{1-\lambda} (1-\bar{r}_{ij}^{\sigma(s)})^\lambda} \quad i, j = 1, 2, \dots, n \quad (11)$$

where $\hat{R}_{ij}^{(k)\sigma(s)}$, $R_{ij}^{(k)\sigma(s)}$, $\bar{R}_{ij}^{\sigma(s)}$ are the s th smallest values in $\hat{R}_{ij}^{(k)}$, $R_{ij}^{(k)}$, $\bar{R}_{ij}^{\sigma(s)}$ respectively, $\hat{r}_{ij}^{(k)\sigma(s)}$, $r_{ij}^{(k)\sigma(s)}$, $\bar{r}_{ij}^{\sigma(s)}$ are the s th smallest values in $\hat{r}_{ij}^{(k)}$, $r_{ij}^{(k)}$, $\bar{r}_{ij}^{\sigma(s)}$ respectively. Let $R^{(k+1)A} = \hat{R}^{(k)A}$, $R^{(k+1)B} = \hat{R}^{(k)B}$ and $k = k + 1$, then go to Step 2.

Step 4. Output $\tilde{R}^{(k)}$.

Step 5. End.

From the calculation process, it can be seen that the iterative process is convergent; for example when we take $\lambda = 1$. Therefore, only the steps are listed without providing any proof.

5. An Illustrative Example and Discussion

In this section, an example is used to illustrate the algorithm.

5.1. Illustrative Example

A large project of Jiudianxia reservoir operation [2, 25] is employed to demonstrate the validity of our approach. The reservoir is designed for many purposes, such as power generation, irrigation, total water supply for industry, agriculture, residents and environment. Because of different requirements for the partition of the amount of water, four reservoir operation schemes x_1, x_2, x_3 and x_4 are suggested.

x_1 : maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy;

x_2 : maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy, lower supply for ecosystem;

x_3 : maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy, total supply for ecosystem and environment, whose 90% is used for flushing sands at low water period;

x_4 : maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy, total supply for ecosystem and environment, whose 50% is used for flushing sands at low water period.

To select the best scheme, the government assigns a large consultancy organization to evaluate four competing schemes. Due to uncertainties, the DMs give their preference information regarding alternatives in the form of interval values with probabilities. Take schemes x_1 and x_2 as an example; the DMs evaluate the degrees to which x_1 is preferred to x_2 , where 40% give a rating of [0.2,0.3] and the remaining 60% give [0.5,0.6]. Assume that these DMs in the consultancy firm cannot be persuaded each other to change their minds, the preference information that x_1 is preferred to x_2 provided by the organization can be considered as a P-IVHFE, i.e., $\{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\}$. The preference information of the organization is listed as a P-IVHFPR \tilde{R} .

$$\tilde{R} = (\tilde{h}_{ij})_{4 \times 4} = \begin{bmatrix} \{[0.5, 0.5](1)\} & \{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\} & & & \\ \{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\} & \{[0.5, 0.5](1)\} & & & \\ \{[0.4, 0.5](1)\} & \{[0.3, 0.4](1)\} & & & \\ \{[0.5, 0.6](1)\} & \{[0.3, 0.5](0.6), [0.5, 0.6](0.4)\} & & & \\ & \{[0.5, 0.6](1)\} & & \{[0.4, 0.5](1)\} & \\ & \{[0.6, 0.7](1)\} & & \{[0.4, 0.5](0.4), [0.5, 0.7](0.6)\} & \\ & \{[0.5, 0.5](1)\} & & \{[0.1, 0.2](0.3), [0.3, 0.5](0.5), [0.6, 0.7](0.2)\} & \\ \{[0.3, 0.4](0.2), [0.5, 0.7](0.5), [0.8, 0.9](0.3)\} & & & \{[0.5, 0.5](1)\} & \end{bmatrix}$$

To get the optimal alternative, the following steps are adopted.

Step 1. First of all, let $k = 1$ and construct the perfect multiplicative consistent P-IVHFPR \bar{R} . By Definition 12, we get

$$\bar{R}^A = \begin{bmatrix} \{0.5(1)\} & \{0.4(0.6), 0.7(0.4)\} & \{0.5(1)\} & \{0.4(1)\} \\ \{0.3(0.4), 0.6(0.6)\} & \{0.5(1)\} & \{0.6(1)\} & \{0.4(0.4), 0.5(0.6)\} \\ \{0.5(1)\} & \{0.4(1)\} & \{0.5(1)\} & \{0.1(0.3), 0.3(0.5), 0.6(0.2)\} \\ \{0.6(1)\} & \{0.5(0.6), 0.6(0.4)\} & \{0.4(0.2), 0.7(0.5), 0.9(0.3)\} & \{0.5(1)\} \end{bmatrix}$$

$$\tilde{R}^B = \begin{bmatrix} \{0.5(1)\} & \{0.5(0.6), 0.8(0.4)\} & \{0.6(1)\} & \{0.5(1)\} \\ \{0.2(0.4), 0.5(0.6)\} & \{0.5(1)\} & \{0.7(1)\} & \{0.5(0.4), 0.7(0.6)\} \\ \{0.4(1)\} & \{0.3(1)\} & \{0.5(1)\} & \{0.2(0.3), 0.5(0.5), 0.7(0.2)\} \\ \{0.5(1)\} & \{0.3(0.6), 0.5(0.4)\} & \{0.3(0.2), 0.5(0.5), 0.8(0.3)\} & \{0.5(1)\} \end{bmatrix}$$

Therefore, according to Eq.(6), we have

$$\bar{R}_{13}^{\sigma(1)} = \frac{\tilde{R}_{12}^{\sigma(1)} \tilde{R}_{23}^{\sigma(1)}}{\tilde{R}_{12}^{\sigma(1)} \tilde{R}_{23}^{\sigma(1)} + (1 - \tilde{R}_{12}^{\sigma(1)})(1 - \tilde{R}_{23}^{\sigma(1)})} = \frac{0.4 \times 0.6}{0.4 \times 0.6 + (1 - 0.4) \times (1 - 0.6)} = 0.5$$

$$p_{13}^{\sigma(1)} = 0.6$$

$$\bar{R}_{13}^{\sigma(2)} = \frac{\tilde{R}_{12}^{\sigma(2)} \tilde{R}_{23}^{\sigma(2)}}{\tilde{R}_{12}^{\sigma(2)} \tilde{R}_{23}^{\sigma(2)} + (1 - \tilde{R}_{12}^{\sigma(2)})(1 - \tilde{R}_{23}^{\sigma(2)})} = \frac{0.7 \times 0.6}{0.7 \times 0.6 + (1 - 0.7) \times (1 - 0.6)} = 0.778$$

$$p_{13}^{\sigma(2)} = 0.4$$

where \tilde{R}_{23} , i.e., $\{0.6(1)\}$ can be regarded as $\{0.6(0.6), 0.6(0.4)\}$. So,

$$\bar{R}_{13} = \{0.5(0.6), 0.778(0.4)\}$$

hence,

$$\bar{R}_{31} = \{0.222(0.4), 0.5(0.6)\}$$

Analogously, by Eq.(6), we have

$$\bar{R}_{14}^{\sigma(s)} = \frac{1}{2} \left(\frac{\tilde{R}_{12}^{\sigma(s)} \tilde{R}_{24}^{\sigma(s)}}{\tilde{R}_{12}^{\sigma(s)} \tilde{R}_{24}^{\sigma(s)} + (1 - \tilde{R}_{12}^{\sigma(s)})(1 - \tilde{R}_{24}^{\sigma(s)})} + \frac{\tilde{R}_{13}^{\sigma(s)} \tilde{R}_{34}^{\sigma(s)}}{\tilde{R}_{13}^{\sigma(s)} \tilde{R}_{34}^{\sigma(s)} + (1 - \tilde{R}_{13}^{\sigma(s)})(1 - \tilde{R}_{34}^{\sigma(s)})} \right)$$

$s = 1, 2, \dots$

Similar to the previous method to deal with P-IVHFE $\{0.6(1)\}$, in order to facilitate observing the probability values, $\tilde{R}_{34} = \{0.1(0.3), 0.3(0.5), 0.6(0.2)\}$ can be regarded as, or in other words,

$$\tilde{R}_{34} = \{0.1(0.3), 0.3(0.5), 0.6(0.2)\}$$

$$= \{0.1(0.3), 0.3(0.1), 0.3(0.2), 0.3(0.2), 0.6(0.2)\}$$

Similarly,

$$\tilde{R}_{12} = \{0.4(0.6), 0.7(0.4)\}$$

$$= \{0.4(0.3), 0.4(0.1), 0.4(0.2), 0.7(0.2), 0.7(0.2)\}$$

$$\tilde{R}_{24} = \{0.4(0.4), 0.5(0.6)\}$$

$$= \{0.4(0.3), 0.4(0.1), 0.5(0.2), 0.5(0.2), 0.5(0.2)\}$$

$$\tilde{R}_{13} = \{0.5(1)\}$$

$$= \{0.5(0.3), 0.5(0.1), 0.5(0.2), 0.5(0.2), 0.5(0.2)\}$$

therefore,

$$\bar{R}_{14}^{\sigma(1)} = \frac{1}{2} \left(\frac{\tilde{R}_{12}^{\sigma(1)} \tilde{R}_{24}^{\sigma(1)}}{\tilde{R}_{12}^{\sigma(1)} \tilde{R}_{24}^{\sigma(1)} + (1 - \tilde{R}_{12}^{\sigma(1)})(1 - \tilde{R}_{24}^{\sigma(1)})} + \frac{\tilde{R}_{13}^{\sigma(1)} \tilde{R}_{34}^{\sigma(1)}}{\tilde{R}_{13}^{\sigma(1)} \tilde{R}_{34}^{\sigma(1)} + (1 - \tilde{R}_{13}^{\sigma(1)})(1 - \tilde{R}_{34}^{\sigma(1)})} \right)$$

$$= \frac{1}{2} \left(\frac{0.4 \times 0.4}{0.4 \times 0.4 + (1 - 0.4) \times (1 - 0.4)} + \frac{0.5 \times 0.1}{0.5 \times 0.1 + (1 - 0.5) \times (1 - 0.1)} \right)$$

$$= 0.204$$

$$p_{14}^{\sigma(1)} = 0.3$$

Similarly, we have

$$\bar{R}_{14}^{\sigma(2)} = 0.304, p_{14}^{\sigma(2)} = 0.1,$$

$$\bar{R}_{14}^{\sigma(3)} = 0.35, p_{14}^{\sigma(3)} = 0.2,$$

$$\bar{R}_{14}^{\sigma(4)} = 0.5, p_{14}^{\sigma(4)} = 0.2,$$

$$\bar{R}_{14}^{\sigma(5)} = 0.65, p_{14}^{\sigma(5)} = 0.2$$

thus,

$$\begin{aligned} \bar{R}_{14} &= \{0.204(0.3), 0.304(0.1), 0.35(0.2), 0.5(0.2), 0.65(0.2)\} \\ \bar{R}_{41} &= \{0.35(0.2), 0.5(0.2), 0.65(0.2), 0.696(0.1), 0.796(0.3)\} \end{aligned}$$

Analogously, we get

$$\begin{aligned} \bar{R}_{24} &= \{0.143(0.3), 0.391(0.5), 0.692(0.2)\} \\ \bar{R}_{42} &= \{0.308(0.2), 0.609(0.5), 0.857(0.3)\} \end{aligned}$$

hence,

$$\bar{R}^A = \begin{bmatrix} & \{0.5(1)\} & & \{0.4(0.6), 0.7(0.4)\} \\ & \{0.3(0.4), 0.6(0.6)\} & & \{0.5(1)\} \\ & \{0.222(0.4), 0.5(0.6)\} & & \{0.4(1)\} \\ \{0.35(0.2), 0.5(0.2), 0.65(0.2), 0.696(0.1), 0.796(0.3)\} & & \{0.308(0.2), 0.609(0.5), 0.857(0.3)\} & \\ \{0.5(0.6), 0.778(0.4)\} & & \{0.204(0.3), 0.304(0.1), 0.35(0.2), 0.5(0.2), 0.65(0.2)\} & \\ \{0.4(0.4), 0.5(0.6)\} & & \{0.143(0.3), 0.391(0.5), 0.692(0.2)\} & \\ & \{0.5(1)\} & & \{0.1(0.3), 0.3(0.5), 0.6(0.2)\} \\ \{0.4(0.2), 0.7(0.5), 0.9(0.3)\} & & & \{0.5(1)\} \end{bmatrix}$$

In the similar way, according to Eq.(7), we can obtain

$$\bar{R}^B = \begin{bmatrix} & \{0.5(1)\} & & \{0.5(0.6), 0.8(0.4)\} \\ & \{0.2(0.4), 0.5(0.6)\} & & \{0.5(1)\} \\ & \{0.097(0.4), 0.3(0.6)\} & & \{0.3(1)\} \\ \{0.159(0.2), 0.248(0.2), 0.35(0.2), 0.45(0.1), 0.614(0.3)\} & & \{0.155(0.2), 0.3(0.5), 0.632(0.3)\} & \\ \{0.7(0.6), 0.903(0.4)\} & & \{0.386(0.3), 0.55(0.1), 0.65(0.2), 0.752(0.2), 0.841(0.2)\} & \\ & \{0.7(1)\} & & \{0.368(0.3), 0.7(0.5), 0.845(0.2)\} \\ & \{0.5(1)\} & & \{0.2(0.3), 0.5(0.5), 0.7(0.2)\} \\ \{0.3(0.2), 0.5(0.5), 0.8(0.3)\} & & & \{0.5(1)\} \end{bmatrix}$$

Step 2. Calculate the deviations $d(\tilde{R}^{(k)A}, \bar{R}^A)$ and $d(\tilde{R}^{(k)B}, \bar{R}^B)$.

Using Eq.(8), we get

$$\begin{aligned} d_{Ham \min g}(\tilde{R}^A, \bar{R}^A) &= \frac{1}{(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1}^n \left[\sum_{s=1}^{m_{ij}} \left| R_{ij}^{(k)\sigma(s)} - \bar{R}_{ij}^{\sigma(s)} \right| p_{ij}^{\sigma(s)} \right] \\ &= \frac{1}{6} \sum_{i=1}^4 \sum_{j=1}^4 \left[\sum_{s=1}^{m_{ij}} \left| R_{ij}^{\sigma(s)} - \bar{R}_{ij}^{\sigma(s)} \right| p_{ij}^{\sigma(s)} \right] \\ &= \frac{1}{6} [(|0.5 - 0.5| \times 0.6 + |0.5 - 0.778| \times 0.4) + (|0.204 - 0.4| \times 0.3 + |0.304 - 0.4| \times 0.1 \\ &+ |0.35 - 0.4| \times 0.2 + |0.5 - 0.4| \times 0.2 + |0.65 - 0.4| \times 0.2) + (|0.143 - 0.4| \times 0.3 \\ &+ |0.391 - 0.4| \times 0.1 + |0.391 - 0.5| \times 0.4 + |0.692 - 0.5| \times 0.2) + (|0.5 - 0.222| \times 0.4 \\ &+ |0.5 - 0.5| \times 0.6) + (|0.35 - 0.6| \times 0.2 + |0.5 - 0.6| \times 0.1 + |0.65 - 0.6| \times 0.2 \\ &+ |0.696 - 0.6| \times 0.1 + |0.796 - 0.6| \times 0.3) + (|0.308 - 0.5| \times 0.2 + |0.609 - 0.5| \times 0.4 \\ &+ |0.609 - 0.6| \times 0.1 + |0.857 - 0.6| \times 0.3)] \\ &= \frac{0.8092}{6} = 0.135 > \theta_0 = 0.1 \end{aligned}$$

Analogously, by Eq.(8), we can obtain

$$d_{Ham \min g}(\tilde{R}^B, \bar{R}^B) = \frac{0.9152}{6} = 0.153 > \theta_0 = 0.1$$

Therefore, \tilde{R}^A and \tilde{R}^B are both not multiplicative consistent P-IVHFPR. \tilde{R}^A and \tilde{R}^B need to be repaired by Eqs.(10) and (11).

Step 3. Repair the inconsistent multiplicative P-IVHFPR.

Hereby we try to assign a value, such as let $\lambda = 0.7$, then

$$\widehat{R}^{(1)A} = \begin{bmatrix} \{0.5(1)\} & \{0.4(0.6), 0.7(0.4)\} \\ \{0.3(0.4), 0.6(0.6)\} & \{0.5(1)\} \\ \{0.294(0.4), 0.5(0.6)\} & \{0.4(1)\} \\ \{0.423(0.2), 0.53(0.2), 0.635(0.2), 0.669(0.1), 0.745(0.3)\} & \{0.362(0.2), 0.577(0.4), 0.606(0.1), 0.798(0.3)\} \\ \{0.5(0.6), 0.706(0.4)\} & \{0.255(0.3), 0.331(0.1), 0.365(0.2), 0.47(0.2), 0.577(0.2)\} \\ \{0.4(0.4), 0.5(0.6)\} & \{0.202(0.3), 0.394(0.1), 0.423(0.4), 0.638(0.2)\} \\ \{0.5(1)\} & \{0.1(0.3), 0.3(0.5), 0.6(0.2)\} \\ \{0.4(0.2), 0.7(0.5), 0.9(0.3)\} & \{0.5(1)\} \end{bmatrix}$$

$$\widehat{R}^{(1)B} = \begin{bmatrix} \{0.5(1)\} & \{0.5(0.6), 0.8(0.4)\} \\ \{0.2(0.4), 0.5(0.6)\} & \{0.5(1)\} \\ \{0.157(0.4), 0.329(0.6)\} & \{0.3(1)\} \\ \{0.238(0.2), 0.315(0.2), 0.393(0.2), 0.465(0.1), 0.581(0.3)\} & \{0.191(0.2), 0.3(0.4), 0.356(0.1), 0.594(0.3)\} \\ \{0.671(0.6), 0.843(0.4)\} & \{0.419(0.3), 0.535(0.1), 0.607(0.2), 0.685(0.2), 0.762(0.2)\} \\ \{0.7(1)\} & \{0.406(0.3), 0.644(0.1), 0.7(0.4), 0.809(0.2)\} \\ \{0.5(1)\} & \{0.2(0.3), 0.5(0.5), 0.7(0.2)\} \\ \{0.3(0.2), 0.5(0.5), 0.8(0.3)\} & \{0.5(1)\} \end{bmatrix}$$

It follows that the normalized Hamming distance

$$d_{Ham \min g}(\widehat{R}^{(1)A}, \bar{R}^A) = 0.037 < \theta_0 = 0.1$$

$$d_{Ham \min g}(\widehat{R}^{(1)B}, \bar{R}^B) = 0.038 < \theta_0 = 0.1$$

Let $R^{(2)A} = \widehat{R}^{(1)A}$, $R^{(2)B} = \widehat{R}^{(1)B}$, then we have

$$d_{Ham \min g}(R^{(2)A}, \bar{R}^A) = 0.037 < 0.1$$

$$d_{Ham \min g}(R^{(2)B}, \bar{R}^B) = 0.038 < 0.1$$

the normalized Hamming distances are less than the consistency level 0.1, so $R^{(2)A}$ and $R^{(2)B}$ are the repaired \tilde{R}^A and \tilde{R}^B respectively.

Step 4. Output $\tilde{R}^{(k)}$.

The composition of $R^{(2)A}$ and $R^{(2)B}$, i.e.,

$$\tilde{R}^{(2)} = \begin{bmatrix} \{[0.5, 0.5](1)\} \\ \{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\} \\ \{[0.157, 0.294](0.4), [0.329, 0.5](0.6)\} \\ \{[0.238, 0.423](0.2), [0.315, 0.53](0.2), [0.393, 0.635](0.2), [0.465, 0.669](0.1), [0.581, 0.745](0.3)\} \\ \{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\} & \{[0.5, 0.671](0.6), [0.706, 0.843](0.4)\} \\ \{[0.5, 0.5](1)\} & \{[0.6, 0.7](1)\} \\ \{[0.3, 0.4](1)\} & \{[0.5, 0.5](1)\} \\ \{[0.191, 0.362](0.2), [0.3, 0.577](0.4), [0.356, 0.606](0.1), [0.594, 0.798](0.3)\} & \{[0.3, 0.4](0.2), [0.5, 0.7](0.5), [0.8, 0.9](0.3)\} \\ \{[0.255, 0.419](0.3), [0.33, 0.535](0.1), [0.365, 0.607](0.2), [0.47, 0.685](0.2), [0.577, 0.762](0.2)\} \\ \{[0.202, 0.406](0.3), [0.394, 0.644](0.1), [0.423, 0.7](0.4), [0.638, 0.809](0.2)\} \\ \{[0.1, 0.2](0.3), [0.3, 0.5](0.5), [0.6, 0.7](0.2)\} \\ \{[0.5, 0.5](1)\} \end{bmatrix}$$

is the repaired multiplicative P-IVHFPR of \tilde{R} .

Step 5. The last step is to sort the four schemes (alternatives).

Using Definition 8, let $p_{ij} = p(\tilde{R}_{ij}^{(2)} \geq \tilde{R}_{ji}^{(2)})$, then we get the following complementary matrix:

$$P = \begin{bmatrix} 0.5 & 1 & 1 & 0.454 \\ 0 & 0.5 & 1 & 0.554 \\ 0 & 0 & 0.5 & 0 \\ 0.546 & 0.446 & 1 & 0.5 \end{bmatrix}$$

If critical value λ is allowed to be an appropriate value, such as a value between the largest and the second largest value of p_{ij} , $i, j = 1, 2, 3, 4$ (not including 1), e.g., $\lambda = 0.55$, and

$$\tilde{p}_{ij} = \begin{cases} 1, & \text{if } p_{ij} \geq \lambda, \\ 0, & \text{if } p_{ij} < \lambda \end{cases}$$

then further we get

$$\tilde{P} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

According to \tilde{p} , we have

$$x_1 \succ x_2, x_1 \succ x_3, x_2 \succ x_3, x_2 \succ x_4, x_4 \succ x_3$$

namely,

$$x_1 \succ x_2 \succ x_4 \succ x_3$$

which indicates that the first scheme is the most desirable according to the opinion of the large consultancy firm.

5.2. Discussion and Comparison

Having carefully analyzed the calculation process and results, the following conclusions can be drawn:

(1) Since there are probability values in a probability-interval valued hesitant fuzzy environment, after the multiplications, there are decimals which are not the integer multiples of 0.1 in the calculated results, such as 0.696, 0.857, 0.391... If one searches the relevant documents on interval-valued preference relations, it can be found that this inevitably happens in the calculation process. Therefore, future research could try and explain this phenomenon.

(2) It can be seen that there are overlapping intervals in the calculated results. Such as a P-IVHFE,

$$\tilde{R}_{12}^{(2)} = \{[0.238, 0.423](0.2), [0.315, 0.53](0.2), [0.393, 0.635](0.2), [0.465, 0.669](0.1), [0.581, 0.745](0.3)\}$$

where between the intervals $[0.238, 0.423]$ and $[0.315, 0.53]$, there is an overlapping interval $[0.315, 0.423]$. To deal with this problem, without a loss of generality, it is assumed that all the interval values have a uniform distribution, then they can be changed into an equivalent expression in which the intervals are not overlapping. For example, as for $\tilde{R}_{12}^{(2)}$, we have

$$\begin{aligned} & [0.238, 0.423](0.2) \\ & = \{[0.238, 0.315](0.2 \times \frac{0.315-0.238}{0.423-0.238}), [0.315, 0.393](0.2 \times \frac{0.393-0.315}{0.423-0.238}), [0.393, 0.423](0.2 \times \frac{0.423-0.393}{0.423-0.238})\} \\ & = \{[0.238, 0.315](0.083), [0.315, 0.393](0.084), [0.393, 0.423](0.033)\} \end{aligned}$$

In a similar way, we get

$$[0.315, 0.53](0.2) = \{[0.315, 0.393](0.073), [0.393, 0.465](0.067), [0.465, 0.53](0.060)\}$$

$$[0.393, 0.635](0.2) = \{[0.393, 0.465](0.059), [0.465, 0.581](0.096), [0.581, 0.635](0.045)\}$$

$$[0.465, 0.669](0.1) = \{[0.465, 0.581](0.057), [0.581, 0.669](0.043)\}$$

therefore,

$$\begin{aligned} \tilde{R}_{12}^{(2)} & = \{[0.238, 0.315](0.083), [0.315, 0.393](0.084 + 0.073), [0.393, 0.465](0.033 + 0.067 + 0.059), \\ & \quad [0.465, 0.581](0.060 + 0.096 + 0.057), [0.581, 0.745](0.045 + 0.043 + 0.3)\} \\ & = \{[0.238, 0.315](0.083), [0.315, 0.393](0.157), [0.393, 0.465](0.159), \\ & \quad [0.465, 0.581](0.213), [0.581, 0.745](0.388)\} \end{aligned}$$

There are not any overlapping intervals in this new expression.

It can be seen from the above example that P-IVHFPRs are useful in resolving large GDM problems, because they express intuitively the uncertain and hesitant preference information provided by each DM in a decision-making organization. This differs from the approach of interval-valued fuzzy sets for GDM, where the opinions of the DMs based on a pairwise comparison of alternatives, are first aggregated and, correspondingly, only the average interval-valued preference information is obtained. However, the use of, P-IVHFPRs does not need to perform such an aggregation and, hence, provides a more comprehensive description of the opinions of these DMs [2]. In the above example, if the probability-interval valued preference information is firstly aggregated at the beginning of the calculation, with regard to the probability values as the corresponding weights, using Definition 8, e.g.,

$$\begin{aligned}
 s(\tilde{h}_{12}) &= (\{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\}) \\
 &= [0.4 \times 0.6 + 0.7 \times 0.4, 0.5 \times 0.6 + 0.8 \times 0.4] = [0.52, 0.62]
 \end{aligned}$$

then we get

$$\begin{aligned}
 \tilde{R} &= (\tilde{h}_{ij})_{4 \times 4} \\
 &= \begin{bmatrix}
 \{[0.5, 0.5](1)\} & \{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\} \\
 \{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\} & \{[0.5, 0.5](1)\} \\
 \{[0.4, 0.5](1)\} & \{[0.3, 0.4](1)\} \\
 \{[0.5, 0.6](1)\} & \{[0.3, 0.5](0.6), [0.5, 0.6](0.4)\} \\
 \{[0.5, 0.6](1)\} & \{[0.4, 0.5](1)\} \\
 \{[0.6, 0.7](1)\} & \{[0.4, 0.5](0.4), [0.5, 0.7](0.6)\} \\
 \{[0.5, 0.5](1)\} & \{[0.1, 0.2](0.3), [0.3, 0.5](0.5), [0.6, 0.7](0.2)\} \\
 \{[0.3, 0.4](0.2), [0.5, 0.7](0.5), [0.8, 0.9](0.3)\} & \{[0.5, 0.5](1)\}
 \end{bmatrix} \\
 \rightarrow [s(\tilde{h}_{ij})]_{4 \times 4} &= \begin{bmatrix}
 [0.5, 0.5] & [0.52, 0.62] & [0.5, 0.6] & [0.4, 0.5] \\
 [0.38, 0.48] & [0.5, 0.5] & [0.6, 0.7] & [0.46, 0.62] \\
 [0.4, 0.5] & [0.3, 0.4] & [0.5, 0.5] & [0.3, 0.45] \\
 [0.5, 0.6] & [0.38, 0.54] & [0.55, 0.7] & [0.5, 0.5]
 \end{bmatrix}
 \end{aligned}$$

Further, in the same way as before, let $p_{ij} = (s(\tilde{h}_{ij}) \geq s(\tilde{h}_{ji}))$, then the following complementary matrix is obtained:

$$P' = \begin{bmatrix}
 0.5 & 1 & 1 & 0 \\
 0 & 0.5 & 1 & 0.75 \\
 0 & 0 & 0.5 & 0 \\
 1 & 0.25 & 1 & 0.5
 \end{bmatrix}$$

indicating that

$$x_1 \succ x_2, x_1 \succ x_3, x_2 \succ x_3, x_2 \succ x_4, x_4 \succ x_1, x_4 \succ x_3$$

which is heavily inconsistent. Let

$$\tilde{p}'_{ij} = \begin{cases} 1, & \text{if } p'_{ij} \geq \lambda, \\ 0, & \text{if } p'_{ij} < \lambda \end{cases}$$

Only when we let critical value $\lambda > 0.75$, can a consistent result be obtained. At this time,

$$\tilde{P}' = \begin{bmatrix}
 0 & 1 & 1 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0
 \end{bmatrix}$$

which indicates that

$$x_1 \succ x_2, x_1 \succ x_3, x_2 \succ x_3, x_4 \succ x_1, x_4 \succ x_3$$

namely,

$$x_4 \succ x_1 \succ x_2 \succ x_3$$

From the results of the calculations, one can find a difference in the ranking results derived in these two approaches. The reason is that for each decision-making organization composed of multiple DMs, a group's preference value is obtained by aggregating (namely, averaging) individual preference values. Such an aggregation actually amounts to implementing a transformation of P-IVHFEs into an interval-valued fuzzy number. As a result, it leads to the loss of information, which affects the final ranking results. Thus, the comparison clearly shows the benefits of the proposed GDM approach based on P-IVHFPRs [2].

Compared with that in a hesitant fuzzy environment, this method's implementation could be far more sophisticated in a probability-interval valued hesitant fuzzy environment, but has led to some new problems. For example, in order to get the equivalent expression in which the intervals are not overlapping, it is assumed that all the interval values have a uniform distribution. If they are not have a uniform distribution, but some other type, e.g., a normal distribution, it is not known what would happen. Therefore this should be a topic for future research.

In spite of what has been mentioned above, compared with P-HFSs, IVHFSs and a possibility-hesitant fuzzy linguistic term set, P-IVHFSs can describe the actual preferences of decision-makers and better reflect their uncertainty, hesitancy, and inconsistency, and thus enhance the modeling abilities of HFSs. The proposed method using P-IVHFSs has the following advantages.

First, compared with P-HFSs, P-IVHFSs can better depict uncertainty.

Second, compared with IVHFSs, P-IVHFSs can depict hesitancy more accurately and differentiate intervals according to their possibilities.

Third, compared with a possibility-hesitant fuzzy linguistic term set, P-IVHFSs can express the evaluation information more flexibly. Possibility-hesitant fuzzy linguistic term sets can therefore be regarded as a special case of P-IVHFSs.

Although the representation of P-IVHFSs looks complex, they can depict fuzzy information clearly and retain the completeness of original data or the inherent thoughts of decision-makers, which is a prerequisite of guaranteeing the accuracy of final outcomes. Additionally, as far as the applicability of P-IVHFSs is concerned, decision-makers can make a trade-off between the features of P-IVHFSs and the relative computational cost. Moreover, the complexity and amount of computation can be clearly reduced with the assistance of programming software [17].

6. Conclusion

In this paper, P-HFSs and IVHFSs have been extended to P-IVHFSs. As an important tool in GDM, P-IVHFSs can describe the actual preferences of decision-makers and better reflect their uncertainty, hesitancy, and inconsistency, and thus enhance the modeling abilities of HFSs. Based on related research, a decomposition method has been proposed to deal with the consistency of P-IVHFPRs. A simulated example has also been provided to illustrate the use of the proposed approach. The main contributions of this paper are summarized as follows.

- (1) The concept of P-IVHFSs has been defined and some desirable properties of P-IVHFSs have been discussed. P-IVHFSs are a natural development to manage the possible preferences in decision making following the introduction of P-HFSs and IVHFSs.
- (2) P-IVHFPRs have been proposed and the consistency of P-IVHFPRs has been discussed, using the multiplicative transitivity to verify the consistency of a P-IVHFPR. Moreover, a decomposition method has been proposed to deal with the consistency of P-IVHFPRs.
- (3) Based on the multiplicative consistency of hesitant fuzzy preference relations, an iterative algorithm has been proposed for improving the consistency of P-IVHFPR.

In future research, the developed theoretical structure could be extended to the probability distributions of preferences on the intervals. Another potential area of research would be to analyze the hesitant fuzzy information in P-IVHFPRs.

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Dynamics and Solutions of Some Recursive Sequences of Higher Order

Asim Asiri¹ and E. M. Elsayed^{1,2}

¹King Abdulaziz University, Faculty of Science,
Mathematics Department, P. O. Box 80203,
Jeddah 21589, Saudi Arabia.

²Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mail: amkasiri@kau.edu.sa, emmelsayed@yahoo.com.

ABSTRACT

In this article we study the existence of solutions and some of their qualitative behavior of the following rational nonlinear difference equation

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{b + cx_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, \dots,$$

where a, b and c are real numbers, k is a non-negative integer number and the initial conditions $x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0$ are arbitrary non-negative real numbers. Also, the solutions of some special cases of the equation under consideration will be obtained.

Keywords: recursive sequence, periodicity, solutions of difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

During the last decade, the research on difference equations has been increasing. The fact that difference equations demonstrate themselves as mathematical models representing some real life phenomena is a significant reason of this concern. For example, they are used in probability theory, economics, genetics in biology, geometry, electrical network, quanta in radiation, psychology, sociology, etc. Actually, no doubt that the difference equations play and will play a remarkable role in applicable analysis and in mathematics generally.

Recently, many authors' attention was on studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. Now, we write some results in this area: Cinar [3–4] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}.$$

Cinar et al. [5] discussed the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [8–9] looked at the global stability, periodicity character and derive the solution of some special cases of the following difference equations

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^k x_{n-i}}.$$

Elsayed [13] examined the behavior and found the form of solution of the nonlinear difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n + dx_{n-2}}.$$

In [2], Belhannache et al. investigated the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-2k-1}}{C + D \prod_{i=1}^k x_{n-2i}^{m_i}}.$$

Karatas et al. [29] achieved the solution of the following difference equation

$$x_{n+1} = \frac{ax_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}.$$

In [35] Simsek and Abdullayev found the solution of the recursive sequence

$$x_{n+1} = \frac{x_{n-(4k+3)}}{1 + \prod_{t=0}^2 x_{n-(k+1)t-k}}.$$

Other related results on rational difference equations can be found in the references. [1-52].

Our aim in this paper is to investigate the dynamics of the solution of the following nonlinear difference equation of higher order

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{b + cx_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, \dots, \tag{1}$$

where a, b and c are real numbers, k a is non negative integer number and the initial conditions $x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0$ are arbitrary non-negative real numbers. Also, we obtain the solutions of some special cases of Eq.(1).

Suppose that I is an interval of real numbers and let $f : I^{k+1} \rightarrow I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^\infty$.

Definition 1. (Equilibrium Point)

A point $\bar{x} \in I$ is called an equilibrium point of Eq.(2) if $\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x})$. That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, \bar{x} is a fixed point of f .

Definition 2. (Periodicity)

A sequence $\{x_n\}_{n=-k}^\infty$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3. (Stability)

(i) The equilibrium point \bar{x} of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point \bar{x} of Eq.(2) is locally asymptotically stable if \bar{x} is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \gamma,$$

we have $\lim_{n \rightarrow \infty} x_n = \bar{x}$.

(iii) The equilibrium point \bar{x} of Eq.(2) is a global attractor if for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$, we have $\lim_{n \rightarrow \infty} x_n = \bar{x}$.

(iv) The equilibrium point \bar{x} of Eq.(2) is globally asymptotically stable if \bar{x} is locally stable, and \bar{x} is also a global attractor of Eq.(2).

(v) The equilibrium point \bar{x} of Eq.(2) is unstable if \bar{x} is not locally stable.

The linearized equation of Eq.(2) about the equilibrium \bar{x} is the linear difference equation

$$y_{n+1} = \sum_{i=0}^k \frac{\partial f(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [32]: Assume that $p_i \in R$, $i = 1, 2, \dots, k$ and $k \in \{0, 1, 2, \dots\}$. Then

$$\sum_{i=1}^k |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \dots + p_k x_n = 0, \quad n = 0, 1, \dots .$$

2. DYNAMICS OF SOLUTIONS OF EQ.(1)

In this section we look at some qualitative behavior of Eq.(1) such as local stability, periodicity and boundedness character of solutions of Eq.(1) when the constants a , b and c are positive real numbers.

2.1. Local Stability of the Equilibrium Points

We now investigate the local stability character of the solutions of Eq.(1).

The equilibrium points of Eq.(1) are given by the relation $\bar{x} = \frac{a\bar{x}}{b+c\bar{x}^2}$, which gives

$$\bar{x} = 0 \quad \text{or} \quad \bar{x} = \sqrt{\frac{a-b}{c}}.$$

Note that if $a > b$, then Eq.(1) has a unique positive equilibrium point.

Let $f : (0, \infty)^2 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v) = \frac{au}{b + cv}. \tag{4}$$

Therefore it follows that

$$\frac{\partial f(u, v)}{\partial u} = \frac{ab}{(b + cv)^2}, \quad \frac{\partial f(u, v)}{\partial v} = \frac{-acu^2}{(b + cv)^2}.$$

THEOREM 2.1. *The following statements are true:*

- (1) If $a \leq b$, then $\bar{x} = 0$ is the only equilibrium point of Eq.(1) and it is locally stable.
- (2) If $a > b$, then the equilibrium points $\bar{x} = 0$ and $\bar{x} = \sqrt{\frac{a-b}{c}}$ of Eq.(1) are unstable.

Proof. (1) If $a \leq b$, then we see from Eq.(4) that

$$\frac{\partial f(0, 0)}{\partial u} = \frac{a}{b}, \quad \frac{\partial f(0, 0)}{\partial v} = 0.$$

Then the linearized equation associated with Eq.(1) about $\bar{x} = 0$ is

$$y_{n+1} - \frac{a}{b} y_{n-2k-1} = 0, \tag{5}$$

and whose characteristic equation is

$$\lambda^{2k+2} - \frac{a}{b} = 0. \tag{6}$$

It follows by Theorem A that, Eq.(5) is asymptotically stable. Then the equilibrium point $\bar{x} = 0$ of Eq.(1) is locally stable.

(2) Assume that $a > b$. (i) At $\bar{x} = 0$ it follows again from Eq.(6) and Theorem A that $\bar{x} = 0$ is unstable.

(ii) At $\bar{x} = \sqrt{\frac{a-b}{c}}$ we see from Eq.(4) that

$$\frac{\partial f(\bar{x}, \bar{x})}{\partial u} = \frac{b}{a}, \quad \frac{\partial f(\bar{x}, \bar{x})}{\partial v} = \frac{-(a-b)}{a}.$$

Then the linearized equation of Eq.(1) about $\bar{x} = \sqrt{\frac{a-b}{c}}$ is

$$y_{n+1} + \frac{a-b}{a}y_{n-k} - \frac{b}{a}y_{n-2k-1} = 0, \tag{7}$$

and whose characteristic equation is

$$\lambda^{2k+2} + \frac{a-b}{a}\lambda^{k+1} - \frac{b}{a} = 0. \tag{8}$$

Therefore $\lambda^{k+1} = -1$ or $\lambda^{k+1} = \frac{b}{a}$. Then it follows by Theorem A that the equilibrium point $\bar{x} = \sqrt{\frac{a-b}{c}}$ of Eq.(1) is unstable. The proof is complete.

2.2. Existence of Period (2k+2) Solutions

In this section we look at the existence of period $(2k + 2)$ solutions of Eq.(1).

Remark: The initial values $\{x_{-2k-1}, x_{-2k}, x_{-2k+1}, \dots, x_{-1}, x_0\}$ of Eq.(1) have not to be equal zero at the same time, otherwise Eq.(1) will have only the zero solution.

In the sequel we assume that any element of the set $\{x_{-2k-1}, x_{-2k}, x_{-2k+1}, \dots, x_{-1}, x_0\}$ doesn't equal zero.

THEOREM 2.2. *Eq.(1) has positive prime period $(2k + 2)$ solutions if and only if*

$$(b + cA_i - a) = 0, \tag{9}$$

where $A_i = x_{-k+i}x_{-2k-1+i}$ (for $i = 0, 1, 2, \dots, k$) and $A_{k+1+i} = A_i$.

Proof. Firstly, we suppose that there exists a prime period $(2k + 2)$ solution of Eq.(1) of the form

$$\dots, x_{-2k-1}, x_{-2k}, x_{-2k+1}, \dots, x_{-1}, x_0, x_{-2k-1}, x_{-2k}, x_{-2k+1}, \dots, x_{-1}, x_0, \dots$$

That is $x_{N+1} = x_{N-2k-1}$ for $N \geq 0$. We now will show that (9) holds. We see from Eq.(1) that

$$\begin{aligned} x_{-2k-1} &= x_1 = \frac{ax_{-2k-1}}{b + cA_0}, & x_{-2k} &= x_2 = \frac{ax_{-2k}}{b + cA_1}, & x_{-2k+1} &= x_3 = \frac{ax_{-2k+1}}{b + cA_2}, \dots, \\ x_{-k-2} &= x_k = \frac{ax_{-k-2}}{b + cA_{k-1}}, & x_{-k-1} &= x_{k+1} = \frac{ax_{-k-1}}{b + cA_k}, \\ x_{-k} &= x_{k+2} = \frac{ax_{-k}}{b + cA_{k+1}} = \frac{ax_{-k}}{b + cA_0}, \dots, & x_{-2} &= x_{2k} = \frac{ax_{-2}}{b + cA_{2k-1}} = \frac{ax_{-2}}{b + cA_{k-2}}, \\ x_{-1} &= x_{2k+1} = \frac{ax_{-1}}{b + cA_{2k}} = \frac{ax_{-1}}{b + cA_{k-1}}, & x_0 &= x_{2k+2} = \frac{ax_0}{b + cA_{2k+1}} = \frac{ax_0}{b + cA_k}. \end{aligned}$$

Then it is easy to see that

$$\begin{aligned} x_{-2k-1}(b + cA_0) &= ax_{-2k-1} \Rightarrow x_{-2k-1}(b + cA_0 - a) = 0, \\ x_{-2k}(b + cA_1) &= ax_{-2k} \Rightarrow x_{-2k}(b + cA_1 - a) = 0, \\ x_{-2k+1}(b + cA_2) &= ax_{-2k+1} \Rightarrow x_{-2k+1}(b + cA_2 - a) = 0, \dots, \\ x_{-1}(b + cA_{k-1}) &= ax_{-1} \Rightarrow x_{-1}(b + cA_{k-1} - a) = 0, \\ x_0(b + cA_k) &= ax_0 \Rightarrow x_0(b + cA_k - a) = 0. \end{aligned}$$

Since $x_j \neq 0$ for all $-(2k + 1) \leq j \leq 0$, then Condition (9) is satisfied.

Secondly, we suppose that (9) is true. We will prove that Eq.(1) has a prime period $(2k + 2)$ solution. It follows from Eq.(1) and Eq.(9) that

$$\begin{aligned} x_1 &= \frac{ax_{-2k-1}}{b + cA_0} = x_{-2k-1}, \quad x_2 = \frac{ax_{-2k}}{b + cA_1} = x_{-2k}, \quad x_3 = \frac{ax_{-2k+1}}{b + cA_2} = x_{-2k+1}, \dots, \\ x_k &= \frac{ax_{-k-2}}{b + cA_{k-1}} = x_{-k-2}, \quad x_{k+1} = \frac{ax_{-k-1}}{b + cA_k} = x_{-k-1}, \quad x_{k+2} = \frac{ax_{-k}}{b + cA_{k+1}} = x_{-k}, \dots, \\ x_{2k} &= \frac{ax_{-2}}{b + cA_{2k-1}} = x_{-2}, \quad x_{2k+1} = \frac{ax_{-1}}{b + cA_{2k}} = x_{-1}, \quad x_{2k+2} = \frac{ax_0}{b + cA_{2k+1}} = x_0, \end{aligned}$$

which completes the proof.

2.3. Boundedness and Global Stability of Solutions

Here we examine the boundedness nature of the solutions of Eq.(1). In addition, we deal with the global stability of the equilibrium point $\bar{x} = 0$.

THEOREM 2.3. *Every solution of Eq.(1) is bounded.*

Proof. Let $\{x_n\}_{n=-2k-1}^\infty$ be a solution of Eq.(1), we have to look at the following two cases

(1) If $a \leq b$. It follows from Eq.(1) that

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{b + cx_{n-k}x_{n-(2k+1)}} \leq \frac{ax_{n-(2k+1)}}{b} \leq x_{n-(2k+1)}.$$

Then the subsequences $\{x_{(2k+2)n-2k-1}\}_{n=0}^\infty, \{x_{(2k+2)n-2k}\}_{n=0}^\infty, \dots, \{x_{(2k+2)n-1}\}_{n=0}^\infty, \{x_{(2k+2)n}\}_{n=0}^\infty$ are decreasing and so are bounded from above by $M = \max \left\{ x_{-2k-1}, x_{-2k}, x_{-2k+1}, \dots, x_{-1}, x_0, \sqrt{\frac{a}{c}} \right\}$.

(2) If $a > b$. For the sake of contradiction, we suppose that there exists a subsequence $\{x_{(2k+2)n-2k-1}\}_{n=0}^\infty$ and it is not bounded from above. Then we obtain from Eq.(1), for sufficiently large n , that

$$\begin{aligned} \infty &= \lim_{n \rightarrow \infty} x_{(2k+2)n+1} = \lim_{n \rightarrow \infty} \frac{ax_{(2k+2)n-(2k+1)}}{b + cx_{(2k+2)n-k}x_{(2k+2)n-(2k+1)}} \\ &< \lim_{n \rightarrow \infty} \frac{ax_{(2k+2)n-(2k+1)}}{cx_{(2k+2)n-k}x_{(2k+2)n-(2k+1)}} = \lim_{n \rightarrow \infty} \frac{a}{cx_{(2k+2)n-k}}. \end{aligned} \tag{10}$$

It follows that the limit of the right hand side of (10) is bounded which is a contradiction, and so the proof of the theorem is complete.

THEOREM 2.4. *If $a \leq b$, then every solution of Eq.(1) converges to the equilibrium point $\bar{x} = 0$.*

Proof. It was shown in Theorem 2.1 that $\bar{x} = 0$ is local stable and then it suffices to show that $\bar{x} = 0$ is global attractor of the solutions of Eq.(1).

We claim that each one of the subsequences $\{x_{(2k+2)n-2k-1}\}_{n=0}^\infty, \{x_{(2k+2)n-2k}\}_{n=0}^\infty, \dots, \{x_{(2k+2)n-1}\}_{n=0}^\infty, \{x_{(2k+2)n}\}_{n=0}^\infty$ has limit equal to zero. For the sake of contradiction, suppose that there exists a subsequence $\{x_{(2k+2)n-2k-1}\}_{n=0}^\infty$ with limit doesn't zero. Now we see from Eq.(1) that

$$bx_{(2k+2)n+1} + cx_{(2k+2)n-k}x_{(2k+2)n-(2k+1)} = ax_{(2k+2)n-(2k+1)},$$

or

$$x_{(2k+2)n-(2k+1)} = \frac{bx_{(2k+2)n+1}}{a - cx_{(2k+2)n+1}x_{(2k+2)n-k}}.$$

Now it follows from the boundedness of the solution that

$$\lim_{n \rightarrow \infty} x_{(2k+2)n-(2k+1)} = \lim_{n \rightarrow \infty} \frac{bx_{(2k+2)n+1}}{a - cx_{(2k+2)n+1}x_{(2k+2)n-k}} < \frac{bM}{a - cM^2} < 0,$$

where $M \geq \sqrt{\frac{a}{c}}$ which is a contradiction and this completes the proof of the theorem.

Numerical Examples

For confirming the results of this section, we present some numerical examples which show the behavior of solutions of Eq.(1). See Figures 1, 2 and 3 below.

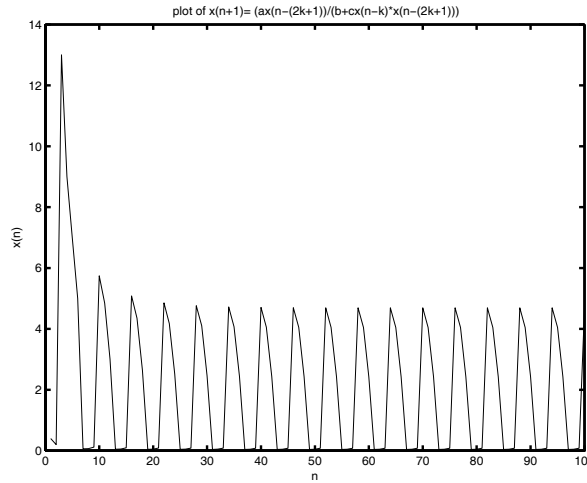


Figure 1: $a = 3, b = 2, c = 5, k = 2, x_{-5} = 0.4, x_{-4} = 0.2, x_{-3} = 13, x_{-2} = 9, x_{-1} = 7, x_0 = 5$.

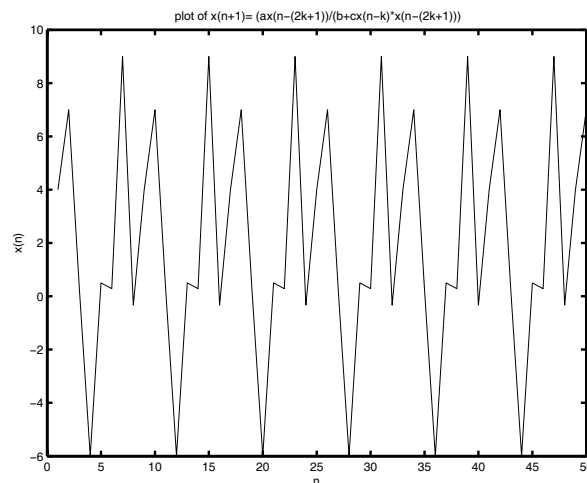


Figure 2: $a = 10, b = 6, c = 2, k = 3, x_{-7} = 4, x_{-6} = 7, x_{-5} = 2/9, x_{-4} = -6, x_{-3} = 0.5, x_{-2} = 2/7, x_{-1} = 9, x_0 = -2/6$.

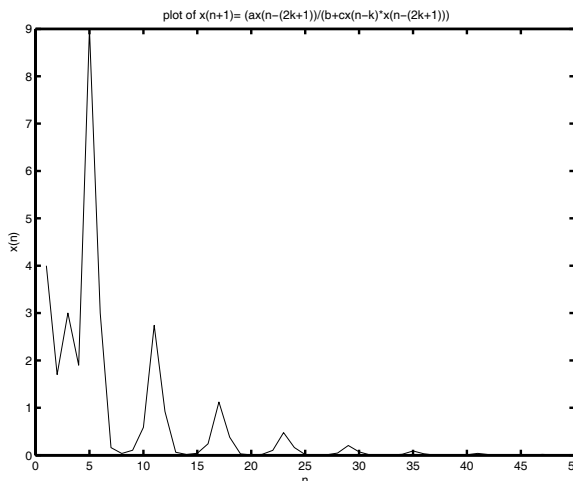


Figure 3: $a = 3, b = 7, c = 9, k = 2, x_{-5} = 4, x_{-4} = 1.7, x_{-3} = 3, x_{-2} = 1.9, x_{-1} = 9, x_0 = 3.$

3. THE SOLUTIONS FORM OF SOME SPECIAL CASES OF EQ.(1)

Our goal in this section is to find a specific form of the solutions of some special cases of Eq.(1) and give numerical examples in each case when the constants a, b and c are integer numbers.

3.1. On the Difference Equation $x_{n+1} = \frac{x_{n-(2k+1)}}{-1 + x_{n-k}x_{n-(2k+1)}}$

In this section we obtain the solution of the following equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{-1 + x_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, \dots, \tag{11}$$

where the initial values are arbitrary non zero real numbers with $x_{-k+i}x_{-2k-1+i} \neq 1$ (for $i = 0, 1, 2, \dots, k$).

THEOREM 3.1. *Let $\{x_n\}_{n=-2k-1}^\infty$ be a solution of Eq.(11). Then for $n = 1, 2, \dots$*

$$\begin{aligned} x_{(2k+2)n-2k-1} &= \frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^n}, & x_{(2k+2)n-2k} &= \frac{x_{-2k}}{(-1 + x_{-k+1}x_{-2k})^n}, \\ x_{(2k+2)n-2k+1} &= \frac{x_{-2k+1}}{(-1 + x_{-k+2}x_{-2k+1})^n}, & \dots, \\ x_{(2k+2)n-k-1} &= \frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^n}, & x_{(2k+2)n-k} &= x_{-k}(-1 + x_{-k}x_{-2k-1})^n, \\ x_{(2k+2)n-k+1} &= x_{-k+1}(-1 + x_{-k+1}x_{-2k})^n, & \dots, \\ x_{(2k+2)n-1} &= x_{-1}(-1 + x_{-1}x_{-k-2})^n, & x_{(2k+2)n} &= x_0(-1 + x_0x_{-k-1})^n. \end{aligned}$$

Proof: For $n = 1$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned} x_{(2k+2)n-4k-3} &= \frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^{n-1}}, & x_{(2k+2)n-4k-2} &= \frac{x_{-2k}}{(-1 + x_{-k+1}x_{-2k})^{n-1}}, \\ x_{(2k+2)n-4k-1} &= \frac{x_{-2k+1}}{(-1 + x_{-k+2}x_{-2k+1})^{n-1}}, & \dots, \\ x_{(2k+2)n-3k-3} &= \frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^{n-1}}, & x_{(2k+2)n-3k-2} &= x_{-k}(-1 + x_{-k}x_{-2k-1})^{n-1}, \\ x_{(2k+2)n-3k-1} &= x_{-k+1}(-1 + x_{-k+1}x_{-2k})^{n-1}, & \dots, \end{aligned}$$

$$x_{(2k+2)n-2k-3} = x_{-1} (-1 + x_{-1}x_{-k-2})^{n-1}, \quad x_{(2k+2)n-2k-2} = x_0 (-1 + x_0x_{-k-1})^{n-1}.$$

Now, it follows from Eq.(11) that

$$\begin{aligned} x_{(2k+2)n-2k-1} &= \frac{x_{(2k+2)n-(4k+3)}}{-1 + x_{(2k+2)n-3k-2}x_{(2k+2)n-(4k+3)}} \\ &= \frac{\frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^{n-1}}}{-1+x_{-k}(-1+x_{-k}x_{-2k-1})^{n-1} \frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^{n-1}}} = \frac{x_{-2k-1}}{-1 + x_{-k}x_{-2k-1}}. \end{aligned}$$

Hence, we have

$$x_{(2k+2)n-2k-1} = \frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^n}.$$

Also, we see from Eq.(11) that

$$\begin{aligned} x_{(2k+2)n-k-1} &= \frac{x_{(2k+2)n-(3k+3)}}{-1 + x_{(2k+2)n-2k-2}x_{(2k+2)n-(3k+3)}} \\ &= \frac{\frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^{n-1}}}{-1+x_0(-1+x_0x_{-k-1})^{n-1} \frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^{n-1}}} = \frac{x_{-k-1}}{1 + x_0x_{-k-1}}. \end{aligned}$$

Thus

$$x_{(2k+2)n-k-1} = \frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^n}.$$

Similarly

$$\begin{aligned} x_{(2k+2)n-1} &= \frac{x_{(2k+2)n-(2k+3)}}{-1 + x_{(2k+2)n-k-2}x_{(2k+2)n-(2k+3)}} = \frac{\frac{x_{-1}(-1+x_{-1}x_{-k-2})^{n-1}}{x_{-k-2}}}{-1+\frac{x_{-k-2}}{(-1 + x_{-1}x_{-k-2})^n}x_{-1}(-1+x_{-1}x_{-k-2})^{n-1}} \\ &= \frac{\frac{x_{-1}(-1+x_{-1}x_{-k-2})^{n-1}}{x_{-1}x_{-k-2}}}{-1+\frac{x_{-1}x_{-k-2}}{(-1 + x_{-1}x_{-k-2})^n}} \left(\frac{-1+x_{-1}x_{-k-2}}{-1+x_{-1}x_{-k-2}} \right) = x_{-1} (-1 + x_{-1}x_{-k-2})^{n-1} (-1 + x_{-1}x_{-k-2}). \end{aligned}$$

Then, we get

$$x_{(2k+2)n-1} = x_{-1} (-1 + x_{-1}x_{-k-2})^n.$$

Similarly, one can obtain the other relations. Thus, the proof is completed.

Note that the equilibrium points of Eq.(11) are given by the equation $\bar{x} = \frac{\bar{x}}{-1 + \bar{x}^2}$. Then we have $\bar{x}(\bar{x}^2 - 2) = 0$. Thus Eq.(11) has the equilibrium points $0, \sqrt{2}, -\sqrt{2}$.

THEOREM 3.2. *The following statements are true:*

- (a) If $x_{-k+i}x_{-2k-1+i} \neq 2$ (for $i = 0, 1, 2, \dots, k$), then all the solutions of Eq.(11) are unbounded.
- (b) Eq.(11) has a periodic solutions of period $(2k + 2)$ iff $x_{-k+i}x_{-2k-1+i} = 2$ (for $i = 0, 1, 2, \dots, k$) and will be take the form $\{x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0, x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0, \dots\}$.

Proof: (a) The proof in this case follows directly from the form of the solution as given in Theorem 3.1.

(b) First suppose that there exists a prime period $(2k + 2)$ solution of Eq.(11) of the form

$$x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0, x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0, \dots .$$

Then we see from the form of solution of Eq.(11) that

$$\begin{aligned} x_{-2k-1} &= \frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^n}, & x_{-2k} &= \frac{x_{-2k}}{(-1 + x_{-k+1}x_{-2k})^n}, \\ x_{-2k+1} &= \frac{x_{-2k+1}}{(-1 + x_{-k+2}x_{-2k+1})^n}, & \dots, \\ x_{-k-1} &= \frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^n}, & x_{-k} &= x_{-k}(-1 + x_{-k}x_{-2k-1})^n, \\ x_{-k+1} &= x_{-k+1}(-1 + x_{-k+1}x_{-2k})^n, & \dots, \\ x_{-1} &= x_{-1}(-1 + x_{-1}x_{-k-2})^n, & x_0 &= x_0(-1 + x_0x_{-k-1})^n, \end{aligned}$$

Then

$$\begin{aligned} x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k} = x_{-k+2}x_{-2k+1} = \dots = -1 + x_{-k}x_{-2k-1} = \\ x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k} = \dots = x_0x_{-k-1} = 2, \end{aligned}$$

or $x_{-k+i}x_{-2k-1+i} = 2$. (for $i = 0, 1, 2, \dots, k$).

Second suppose that

$$\begin{aligned} x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k} = x_{-k+2}x_{-2k+1} = \dots = -1 + x_{-k}x_{-2k-1} = \\ x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k} = \dots = x_0x_{-k-1} = 2. \end{aligned}$$

Then we see from Eq.(11) that

$$\begin{aligned} x_{(2k+2)n-2k-1} &= x_{-2k-1}, & x_{(2k+2)n-2k} &= x_{-2k}, & x_{(2k+2)n-2k+1} &= x_{-2k+1}, & \dots, \\ x_{(2k+2)n-k-1} &= x_{-k-1}, & x_{(2k+2)n-k} &= x_{-k}, & x_{(2k+2)n-k+1} &= x_{-k+1}, & \dots, \\ x_{(2k+2)n-1} &= x_{-1}, & x_{(2k+2)n} &= x_0. \end{aligned}$$

Thus we have a period $(2k + 2)$ solution and the proof is complete.

In the following we give some numerical examples to confirm the obtained results for Eq.(11). See Figures 4 and 5 below.

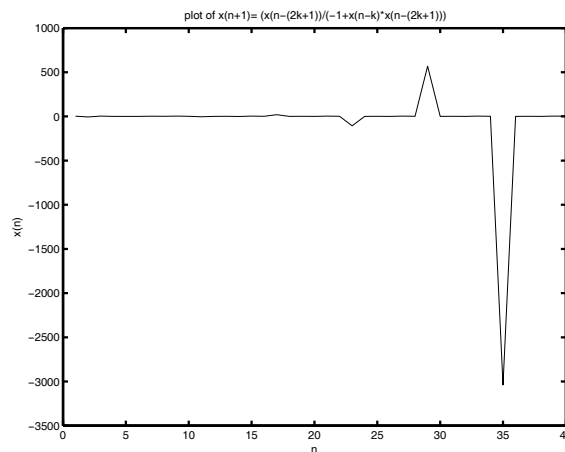


Figure 4: $k = 2, x_{-5} = 2.4, x_{-4} = -6.2, x_{-3} = 4, x_{-2} = 0.9, x_{-1} = 0.7, x_0 = 0.5$.

Now, it follows from Eq.(12) that

$$\begin{aligned}
 x_{(2k+2)n-2k-1} &= \frac{x_{(2k+2)n-(4k+3)}}{1 - x_{(2k+2)n-3k-2}x_{(2k+2)n-(4k+3)}} \\
 &= \frac{x_{-2k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_{-k}x_{-2k-1}}{1 - (2i+1)x_{-k}x_{-2k-1}} \right)}{1 - x_{-k} \prod_{i=0}^{n-2} \left(\frac{1 - (2i+1)x_{-k}x_{-2k-1}}{1 - (2i+2)x_{-k}x_{-2k-1}} \right) x_{-2k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_{-k}x_{-2k-1}}{1 - (2i+1)x_{-k}x_{-2k-1}} \right)} \\
 &= \frac{x_{-2k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_{-k}x_{-2k-1}}{1 - (2i+1)x_{-k}x_{-2k-1}} \right)}{1 - \left(\frac{x_{-k}x_{-2k-1}}{1 - (2n-2)x_{-k}x_{-2k-1}} \right)} \left(\frac{1 - (2n-2)x_{-k}x_{-2k-1}}{1 - (2n-2)x_{-k}x_{-2k-1}} \right) \\
 &= x_{-2k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_{-k}x_{-2k-1}}{1 - (2i+1)x_{-k}x_{-2k-1}} \right) \left(\frac{1 - (2n-2)x_{-k}x_{-2k-1}}{1 - (2n-1)x_{-k}x_{-2k-1}} \right).
 \end{aligned}$$

Hence, we have

$$x_{(2k+2)n-2k-1} = x_{-2k-1} \prod_{i=0}^{n-1} \left(\frac{1 - 2ix_{-k}x_{-2k-1}}{1 - (2i+1)x_{-k}x_{-2k-1}} \right).$$

Similarly

$$\begin{aligned}
 x_{(2k+2)n-k-1} &= \frac{x_{(2k+2)n-(3k+3)}}{1 - x_{(2k+2)n-2k-2}x_{(2k+2)n-(3k+3)}} \\
 &= \frac{x_{-k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_0x_{-k-1}}{1 - (2i+1)x_0x_{-k-1}} \right)}{1 - x_0 \prod_{i=0}^{n-2} \left(\frac{1 - (2i+1)x_0x_{-k-1}}{1 - (2i+2)x_0x_{-k-1}} \right) x_{-k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_0x_{-k-1}}{1 - (2i+1)x_0x_{-k-1}} \right)} \\
 &= \frac{x_{-k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_0x_{-k-1}}{1 - (2i+1)x_0x_{-k-1}} \right)}{1 - x_0x_{-k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_0x_{-k-1}}{1 - (2i+2)x_0x_{-k-1}} \right)} \\
 &= x_{-k-1} \prod_{i=0}^{n-2} \left(\frac{1 - 2ix_0x_{-k-1}}{1 - (2i+1)x_0x_{-k-1}} \right) \left(\frac{1 - (2n-2)x_0x_{-k-1}}{1 - (2n-1)x_0x_{-k-1}} \right).
 \end{aligned}$$

Hence, we have

$$x_{(2k+2)n-k-1} = x_{-k-1} \prod_{i=0}^{n-1} \left(\frac{1 - 2ix_0x_{-k-1}}{1 - (2i+1)x_0x_{-k-1}} \right).$$

Similarly, we can easily get the other relations. Thus, the proof is completed.

THEOREM 3.4. *Eq.(12) has the unique equilibrium point $\bar{x} = 0$.*

Proof: For the equilibrium points of Eq.(12), we can write $\bar{x} = \frac{\bar{x}}{1 - \bar{x}^2}$. Then we have $\bar{x}^3 = 0$. Thus the equilibrium point of Eq.(12) is $\bar{x} = 0$.

The following figures show the behavior of the solutions of Eq.(12) with a fixed order and some numerical values of the initial values.

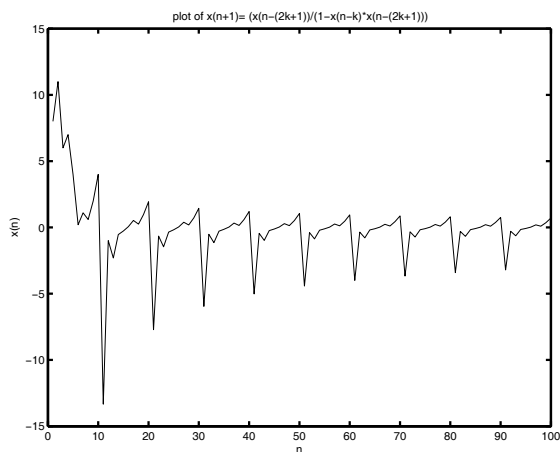


Figure 6: $k = 4, x_{-9} = 8, x_{-8} = 11, x_{-7} = 6, x_{-6} = 7, x_{-5} = 4, x_{-4} = 0.2, x_{-3} = 1.1, x_{-2} = 0.6, x_{-1} = 2, x_0 = 4.$

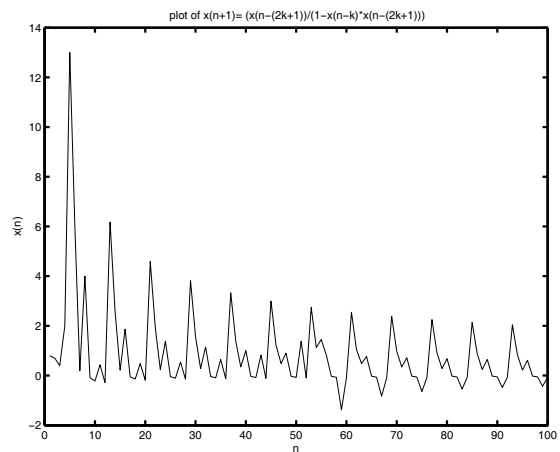


Figure 7: $k = 3, x_{-7} = 0.8, x_{-6} = 0.7, x_{-5} = 0.4, x_{-4} = 2, x_{-3} = 13, x_{-2} = 6, x_{-1} = 0.2, x_0 = 4.$

Notice: The proofs of the theorems in the following section are similar to that are presented in the previous sections and so they will be omitted.

3.3. On the Difference Equation
$$x_{n+1} = \frac{x_{n-(2k+1)}}{-1 - x_{n-k}x_{n-(2k+1)}}$$

Here we obtain the form of the solutions of the following equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{-1 - x_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, \dots, \tag{13}$$

where the initial values are arbitrary non zero real numbers with $x_{-k+i}x_{-2k-1+i} \neq -1$ (for $i = 0, 1, 2, \dots, k$).

THEOREM 3.5. Let $\{x_n\}_{n=-2k-1}^\infty$ be a solution of Eq.(13). Then for $n = 1, 2, \dots$

$$\begin{aligned} x_{(2k+2)n-2k-1} &= \frac{(-1)^n x_{-2k-1}}{(1 + x_{-k}x_{-2k-1})^n}, & x_{(2k+2)n-2k} &= \frac{(-1)^n x_{-2k}}{(1 + x_{-k+1}x_{-2k})^n}, \\ x_{(2k+2)n-2k+1} &= \frac{(-1)^n x_{-2k+1}}{(1 + x_{-k+2}x_{-2k+1})^n}, \dots, \\ x_{(2k+2)n-k-1} &= \frac{(-1)^n x_{-k-1}}{(1 + x_0x_{-k-1})^n}, & x_{(2k+2)n-k} &= (-1)^n x_{-k} (1 + x_{-k}x_{-2k-1})^n, \\ x_{(2k+2)n-k+1} &= (-1)^n x_{-k+1} (1 + x_{-k+1}x_{-2k})^n, \dots, \\ x_{(2k+2)n-1} &= (-1)^n x_{-1} (1 + x_{-1}x_{-k-2})^n, & x_{(2k+2)n} &= (-1)^n x_0 (1 + x_0x_{-k-1})^n. \end{aligned}$$

THEOREM 3.6. Eq.(13) has a unique equilibrium point which is zero.

THEOREM 3.7. Let $\{x_n\}_{n=-2k-1}^\infty$ be a solution of Eq.(13). Then the following statements are true:

- (1) If $x_{-k+i}x_{-2k-1+i} \neq -2$ (for $i = 0, 1, 2, \dots, k$), Then $\{x_n\}_{n=-2k-1}^\infty$ is unbounded.
- (2) Eq.(13) has a periodic solutions of period $(2k + 2)$ iff $x_{-k+i}x_{-2k-1+i} = -2$ (for $i = 0, 1, 2, \dots, k$) and will take the form $\{x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0, x_{-2k-1}, x_{-2k}, \dots, x_{-1}, x_0, \dots\}$.

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Extremal solutions for a coupled system of nonlinear fractional differential equations with p-Laplacian operator *

Ying He[†]

School of Mathematics and Statistics, Northeast Petroleum University, Daqing163318, P.R.China.

Abstract. This paper studies the existence of extremal solutions for nonlinear fractional differential coupled systems with p-Laplacian operator. The monotone iterative method combined with lower and upper solutions is applied. As an application, an example is presented to illustrate the main result.

Key words. Fractional differential system; p-Laplacian operator; Extremal solution; Monotone iterative technique;

MR(2000) Subject Classifications: 34B15.

1. Introduction

In recent years, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself and its applications. The study of coupled systems involving fractional-order differential equation is also very significant as such systems appear in a variety of problems of applied nature, especially in bioscience. For details and examples the reader is referred to the papers [1 – 4] and the reference therein.

In addition, much effort has been made towards the study of the existence of solutions for fractional differential equations involving the p-Laplacian operator based on different fractional derivatives[5 – 9]. In [10], Li and Lin considered a Hadamard fractional boundary value problem with p-Laplacian operator as below:

$$\begin{cases} D^\beta(\varphi_p(D^\alpha x(t))) = f(t, x(t)), & 0 < t < e, \\ x(1) = x'(1) = x'(e) = 0, D^\alpha x(1) = D^\alpha x(e) = 0 \end{cases}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $\varphi_p(s) = |s|^{p-2}s, p > 1$, and $f : [1, e] \times [0, +\infty) \rightarrow [0, +\infty)$ is a positive continuous function. By using the Leray-Schauder type alternative and the Guo-

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[†]Corresponding author. E-mail adress:heyng65332015@163.com;

Krasnoselskii fixed point theorem, the existence and the uniqueness of the positive solutions were established.

To best of our knowledge, only few papers considered the method of upper and lower solutions for a coupled system of fractional p-Laplacian equation. Motivated by [11 – 12], in this paper, we use the monotone iterative technique, combined with the method of upper and lower solution to study the coupled system of fractional differential equations with p-Laplacian operator, which is given by

$$\begin{cases} D^\beta(\phi_p(D^\alpha x(t))) = f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)), & t \in [0, 1], \\ D^\beta(\phi_p(D^\alpha y(t))) = g(t, y(t), x(t), D^\alpha y(t), D^\alpha x(t)), & t \in [0, 1], \\ D^\alpha x(t)|_{t=0} = 0, & t^{1-\alpha}x(t)|_{t=0} = r_1, \\ D^\alpha y(t)|_{t=0} = 0, & t^{1-\alpha}y(t)|_{t=0} = r_2, \end{cases} \quad (1.1)$$

where $J = [0, 1]$, $f, g \in C(J \times R^4, R)$, $r_1, r_2 \in R$ and $r_1 \leq r_2$, D^α, D^β are the standard Riemann-Liouville fractional derivatives, satisfying $0 < \alpha, \beta < 1$, $1 < \alpha + \beta < 2$, $\phi_p(t) = |t|^{p-2}t$, $p > 1$, is the p-Laplacian operator and $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

The rest of this paper is organized as follows. In section 2, we give some necessary definitions and lemmas. In section 3, the main result and proof are given. Finally, an example is presented to illustrate the main result.

2. Preliminaries

In this section, we establish some preliminary results that will be used in the next section to attain existence results for the nonlinear system (1.1)

Let $C[0, 1]$ denote the Banach space of continuous functions from $[0, 1]$ into R with the norm $\|u\|_C = \max_{t \in [0, 1]} |u(t)|$. Denote $C_{1-\alpha}[0, 1]$ by

$$C_{1-\alpha}[0, 1] = \{x \in C(0, 1] : t^{1-\alpha}x \in C[0, 1]\}.$$

Then, $C_{1-\alpha}[0, 1]$ is a Banach spaces with the norm $\|x\|_{1-\alpha} = \|t^{1-\alpha}x(t)\|_C$. It is clear that $C[0, 1] := C_0[0, 1] \subset C_{1-\alpha}[0, 1]$ with $\|x\|_{C_{1-\alpha}} \leq \|x\|_C$ for $0 < \alpha \leq 1$ and $C_{1-\alpha}[0, 1] \subset L[0, 1]$ (note $L[0, 1]$ is the space of Lebesgue integrable functions defined on $[0, 1]$). Denote $C^\alpha[0, 1]$ by

$$C^\alpha[0, 1] = \{x(t) \in C[0, 1] : (D^\alpha x)(t) \in C[0, 1] \text{ and } D^\alpha x(t)|_{t=0} = 0\}$$

Lemma 2.1: Let $0 < \beta < 1$, $\sigma \in C[0, 1]$, $M \geq 0$ and $M\Gamma(1 - \beta) < 1$, then the problem

$$\begin{cases} D^\beta u(t) + Mu(t) = \sigma(t), & 0 \leq t \leq 1, \\ u(0) = 0, \end{cases} \quad (2.1)$$

has a unique solution.

Proof. Equation (2.1) is equivalent to the following integral equation

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} (\sigma(s) - Mu(s)) ds, \quad \forall t \in J$$

Let

$$Au(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (\sigma(s) - Mu(s)) ds, \quad \forall t \in J$$

By $M \geq 0$ and $M\Gamma(1-\beta) < 1$, for any $u, v \in C[0, 1]$, we have

$$\begin{aligned} \|Au(t) - Av(t)\|_C &\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds \|u - v\|_C \\ &\leq \frac{M}{\Gamma(\beta)\beta} \|u - v\|_C \\ &< \frac{1}{\Gamma(\beta)} \cdot \frac{1}{\beta} \cdot \frac{1}{\Gamma(1-\beta)} \|u - v\|_C \\ &= \frac{\sin \Pi\beta}{\Pi\beta} \|u - v\|_C \\ &< \|u - v\|_C \end{aligned}$$

So

$$\|Au - Av\|_C < \|u - v\|_C.$$

By the Banach fixed point theorem, the operator A has a unique fixed point. That is (2.1) has a unique solution.

Lemma 2.2: Let $0 < \alpha < 1$, $h \in C_{1-\alpha}[0, 1]$, then the problem

$$\begin{cases} D^\alpha x(t) = h(t), & t \in (0, 1], \\ t^{1-\alpha} x(t)|_{t=0} = r \end{cases} \tag{2.2}$$

has a unique solution

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + rt^{\alpha-1}.$$

Proof. The conclusion is obvious, so we omit it.

Lemma 2.3: Assume that $0 < \alpha, \beta < 1$, $x(t), y(t) \in C^\alpha[0, 1]$, $\sigma_1, \sigma_2 \in C[0, 1]$, M, N be nonnegative constants, satisfying $M \geq N$ and $(M+N)\Gamma(1-\beta) < 1$, then the following fractional differential system

$$\begin{cases} D^\beta(\phi_p(D^\alpha x(t))) = \sigma_1(t) - M\phi_p(D^\alpha x(t)) - N\phi_p(D^\alpha y(t)), \\ D^\beta(\phi_p(D^\alpha y(t))) = \sigma_2(t) - M\phi_p(D^\alpha y(t)) - N\phi_p(D^\alpha x(t)) \\ D^\alpha x(t)|_{t=0} = 0, & t^{1-\alpha} x(t)|_{t=0} = r_1, \\ D^\alpha y(t)|_{t=0} = 0, & t^{1-\alpha} y(t)|_{t=0} = r_2, \end{cases} \tag{2.3}$$

has a unique solution in $C^\alpha[0, 1] \times C^\alpha[0, 1]$.

Proof. Let

$$\phi_p(D^\alpha x(t)) = \frac{u(t) + v(t)}{2} \quad \text{and} \quad \phi_p(D^\alpha y(t)) = \frac{u(t) - v(t)}{2}, \quad \forall t \in [0, 1].$$

Using (2.3), we have that

$$\begin{cases} D^\beta u(t) = \sigma_1(t) + \sigma_2(t) - (M+N)u(t) \\ u(t)|_{t=0} = \phi_p(D^\alpha x(t))|_{t=0} + \phi_p(D^\alpha y(t))|_{t=0} = 0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} D^\beta v(t) = \sigma_1(t) - \sigma_2(t) - (M - N)v(t) \\ v(t)|_{t=0} = \phi_p(D^\alpha x(t))|_{t=0} - \phi_p(D^\alpha y(t))|_{t=0} = 0, \end{cases} \quad (2.5)$$

Since, M, N are nonnegative constants, and $M \geq N$, we have

$$(M - N)\Gamma(1 - \beta) \leq (M + N)\Gamma(1 - \beta) < 1. \quad (2.6)$$

In view of $x(t), y(t) \in C^\alpha[0, 1]$, we have $D^\alpha x(t), D^\alpha y(t) \in C[0, 1]$. By (2.6) and Lemma 2.1, we know that (2.4) and (2.5) have a unique solution. In consequence, $\phi_p(D^\alpha x(t))$ and $\phi_p(D^\alpha y(t))$ are also unique. That is

$$\phi_p(D^\alpha x(t)) = \omega_1(t) \in C[0, 1], \quad \phi_p(D^\alpha y(t)) = \omega_2(t) \in C[0, 1],$$

then,

$$D^\alpha x(t) = \phi_q(\omega_1(t)), \quad D^\alpha y(t) = \phi_q(\omega_2(t)),$$

In view of the initial condition $t^{1-\alpha}x(t)|_{t=0} = r_1, t^{1-\alpha}y(t)|_{t=0} = r_2$, we obtain

$$\begin{cases} D^\alpha x(t) = \phi_q(\omega_1(t)), & t \in [0, 1] \\ D^\alpha y(t) = \phi_q(\omega_2(t)), & t \in [0, 1] \\ t^{1-\alpha}x(t)|_{t=0} = r_1, \\ t^{1-\alpha}y(t)|_{t=0} = r_2, \end{cases} \quad (2.7)$$

Let

$$x(t) = \frac{p(t) + q(t)}{2} \quad \text{and} \quad y(t) = \frac{p(t) - q(t)}{2}.$$

Using (2.7), we have

$$\begin{cases} D^\alpha p(t) = \phi_q(\omega_1(t)) + \phi_q(\omega_2(t)), & t \in (0, 1] \\ t^{1-\alpha}p(t)|_{t=0} = r_1 + r_2, \end{cases} \quad (2.8)$$

and

$$\begin{cases} D^\alpha q(t) = \phi_q(\omega_1(t)) - \phi_q(\omega_2(t)), & t \in (0, 1] \\ t^{1-\alpha}q(t)|_{t=0} = r_1 - r_2, \end{cases} \quad (2.9)$$

By Lemma 2.2, we know that both (2.8) and (2.9) have a unique solution in $C^\alpha[0, 1]$. Hence, x and y are unique too.

Lemma 2.4: Let $0 < \beta < 1$, M be nonnegative constant and $w \in C[0, 1]$ satisfies

$$\begin{cases} D^\beta w(t) + Mw(t) \geq 0, & 0 < t < 1, \\ w(0) \geq 0, \end{cases}$$

then, $w(t) \geq 0, \forall t \in [0, 1]$.

Proof. We assume that $w(t) \geq 0$ is not true. Then there exist $t^*, t_* \in (0, 1]$ such that $w(t^*) = 0, w(t_*) < 0$ and $w(t) \geq 0, \forall t \in (0, t^*), w(t) < 0, \forall t \in (t^*, t_*)$. Since $M \geq 0$, we have $D^\beta w(t) \geq 0, \forall t \in (t^*, t_*)$. This together with $D^\beta w(t) = \frac{d}{dt}I^{1-\beta}w(t)$ implies $I^{1-\beta}w(t)$ is nondecreasing on (t^*, t_*) .

Hence, for any $t \in (t^*, t_*)$, we get

$$I^{1-\beta}w(t) - I^{1-\beta}w(t^*) \geq 0.$$

On the other hand

$$\begin{aligned} I^{1-\beta}w(t) - I^{1-\beta}w(t^*) &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}w(s)ds - \frac{1}{\Gamma(1-\beta)} \int_0^{t^*} (t^*-s)^{-\beta}w(s)ds \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^{t^*} [(t-s)^{-\beta} - (t^*-s)^{-\beta}]w(s)ds + \frac{1}{\Gamma(1-\beta)} \int_{t^*}^t (t-s)^{-\beta}w(s)ds \\ &< 0, \quad \forall t \in (t^*, t_*), \end{aligned}$$

which is a contradiction. Thus the conclusion of Lemma 2.4 holds.

Lemma 2.5: If $x(t) \in C_{1-\alpha}[0, 1]$ satisfies

$$\begin{cases} D^\alpha x(t) \geq 0, & t \in (0, 1], \\ t^{1-\alpha}x(t)|_{t=0} \geq 0, \end{cases} \tag{2.10}$$

then $x(t) \geq 0$, for $t \in (0, 1]$.

Proof. By Lemma 2.2, we know that problem (2.10) has a unique solution

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds + rt^{\alpha-1}$$

Let $h(t) \geq 0$ and $r \geq 0$, then we obtain (2.10) and the conclusion of lemma 2.5.

Lemma 2.6: Let M, N be nonnegative constants, and $M \geq N$. If $u, v \in C[0, 1]$ satisfy the inequalities

$$\begin{cases} D^\beta u(t) \geq -Mu(t) + Nv(t), & t \in [0, 1] \\ D^\beta v(t) \geq -Mv(t) + Nu(t), & t \in [0, 1] \\ u(t)|_{t=0} \geq 0, \\ v(t)|_{t=0} \geq 0 \end{cases} \tag{2.11}$$

then $u(t) \geq 0, v(t) \geq 0, \forall t \in [0, 1]$.

Proof. Let $p(t) = u(t) + v(t), \forall t \in [0, 1]$. Then by (2.11), we have

$$\begin{cases} D^\beta p(t) \geq -(M-N)p(t), & t \in [0, 1] \\ p(t)|_{t=0} \geq 0 \end{cases} \tag{2.12}$$

Thus, by (2.12) and Lemma 2.4, we have that

$$p(t) \geq 0, \forall t \in [0, 1] \text{ i.e. } u(t) + v(t) \geq 0, \forall t \in [0, 1]. \tag{2.13}$$

Next, we show that $u(t) \geq 0, v(t) \geq 0, \forall t \in [0, 1]$.

Using (2.11) and (2.13), we find that

$$\begin{cases} D^\beta u(t) \geq -(M+N)u(t), & t \in [0, 1] \\ u(t)|_{t=0} \geq 0 \end{cases} \tag{2.14}$$

which, in view of (2.14) and Lemma 2.4, yield $u(t) \geq 0, \forall t \in [0, 1]$. In a similar manner, it can be shown that $v(t) \geq 0, \forall t \in [0, 1]$.

3. Main Results

In this section, we prove the existence of extremal solutions of nonlinear system (1.1). For convenience, we list the following conditions:

(H₁): There exist $x_0, y_0 \in C^\alpha[0, 1]$ and $x_0(t) \leq y_0(t)$ such that

$$\begin{cases} D^\beta(\phi_p(D^\alpha x_0(t))) \leq f(t, x_0(t), y_0(t), D^\alpha x_0(t), D^\alpha y_0(t)), & t \in [0, 1], \\ D^\alpha x_0(t)|_{t=0} = 0, & t^{1-\alpha}x(t)|_{t=0} \leq r_1, \\ D^\beta(\phi_p(D^\alpha y_0(t))) \geq g(t, y_0(t), x_0(t), D^\alpha y_0(t), D^\alpha x_0(t)), & t \in [0, 1], \\ D^\alpha y_0(t)|_{t=0} = 0, & t^{1-\alpha}y_0(t)|_{t=0} \geq r_2, \end{cases}$$

(H₂): There exist nonnegative constant M, N satisfying $M \geq N$ and $(M + N)\Gamma(1 - \beta) < 1$, such that

$$\begin{aligned} & f(t, \overline{x(t)}, \overline{y(t)}, D^\alpha \overline{x(t)}, D^\alpha \overline{y(t)}) - f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)) \\ & \leq M[\phi_p(D^\alpha x(t)) - \phi_p(D^\alpha \overline{x(t)})] + N[\phi_p(D^\alpha y(t)) - \phi_p(D^\alpha \overline{y(t)})] \\ & g(t, \overline{x(t)}, \overline{y(t)}, D^\alpha \overline{x(t)}, D^\alpha \overline{y(t)}) - g(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)) \\ & \leq M[\phi_p(D^\alpha x(t)) - \phi_p(D^\alpha \overline{x(t)})] + N[\phi_p(D^\alpha y(t)) - \phi_p(D^\alpha \overline{y(t)})] \end{aligned}$$

where $x_0(t) \leq \overline{x} \leq x \leq y_0(t), x_0(t) \leq y \leq \overline{y} \leq y_0(t)$, and

$$\begin{aligned} & f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)) - g(t, y(t), x(t), D^\alpha y(t), D^\alpha x(t)) \\ & \leq M[\phi_p(D^\alpha y(t)) - \phi_p(D^\alpha x(t))] + N[\phi_p(D^\alpha x(t)) - \phi_p(D^\alpha y(t))] \end{aligned}$$

with $x_0(t) \leq x \leq y \leq y_0(t)$.

Theorem 3.1: Suppose that conditions (H₁) and (H₂) hold. Then there is $(x^*, y^*) \in [x_0, y_0] \times [x_0, y_0]$ an extremal solution of the nonlinear problem (1.1). Moreover there exist monotone iterative sequences $\{x_n\}, \{y_n\} \subset C^\alpha$ such that $x_n \rightarrow x^*, y_n \rightarrow y^* (n \rightarrow \infty)$ uniformly on $t \in (0, 1)$, and

$$x_0 \leq x_1 \leq \dots \leq x_n \leq \dots \leq x^* \leq y^* \leq \dots \leq y_n \leq \dots \leq y_1 \leq y_0,$$

moreover, we have

$$D^\alpha x_0 \leq D^\alpha x_1 \leq \dots \leq D^\alpha x_n \leq \dots \leq D^\alpha x^* \leq D^\alpha y^* \leq \dots \leq D^\alpha y_n \leq \dots \leq D^\alpha y_1 \leq D^\alpha y_0.$$

where

$$[x_0, y_0] = \{x \in C^\alpha[0, 1] : x_0(t) \leq x(t) \leq y_0(t), t \in [0, 1]\}$$

Proof. For any $x_{n-1}, y_{n-1} \in C^\alpha[0, 1], n \geq 1$, we define

$$\begin{cases} \sigma_n^1(t) = f(t, x_{n-1}(t), y_{n-1}(t), D^\alpha x_{n-1}(t), D^\alpha y_{n-1}(t)) + M\phi_p(D^\alpha x_{n-1}(t)) + N\phi_p(D^\alpha y_{n-1}(t)), \\ \sigma_n^2(t) = g(t, y_{n-1}(t), x_{n-1}(t), D^\alpha y_{n-1}(t), D^\alpha x_{n-1}(t)) + M\phi_p(D^\alpha y_{n-1}(t)) + N\phi_p(D^\alpha x_{n-1}(t)), \end{cases}$$

and consider (2.3) as follows

$$\begin{cases} D^\beta(\phi_p(D^\alpha x_n(t))) = \sigma_n^1(t) - M\phi_p(D^\alpha x_n(t)) - N\phi_p(D^\alpha y_n(t)) & t \in (0, 1], \\ D^\beta(\phi_p(D^\alpha y_n(t))) = \sigma_n^2(t) - M\phi_p(D^\alpha y_n(t)) - N\phi_p(D^\alpha x_n(t)) & t \in (0, 1], \\ D^\alpha x_n(t)|_{t=0} = 0, & t^{1-\alpha}x_n(t)|_{t=0} = r_1, \\ D^\alpha y_n(t)|_{t=0} = 0, & t^{1-\alpha}y_n(t)|_{t=0} = r_2. \end{cases} \quad (3.1)$$

In view of Lemma 2.3, the problem (3.1) has a unique solution in $C^\alpha[0, 1] \times C^\alpha[0, 1]$.

Now, we show that $\{x_n(t)\}, \{y_n(t)\}$ satisfy the relation

$$x_{n-1} \leq x_n \leq y_n \leq y_{n-1}, \text{ and } D^\alpha x_{n-1} \leq D^\alpha x_n \leq D^\alpha y_n \leq D^\alpha y_{n-1}, \quad n = 1, 2, \dots \quad (3.2)$$

Let $u(t) = \phi_p(D^\alpha x_1(t)) - \phi_p(D^\alpha x_0(t)), v(t) = \phi_p(D^\alpha y_0(t)) - \phi_p(D^\alpha y_1(t))$.

Thus, by condition (3.1) and (H_1) , we have

$$\begin{cases} D^\beta u(t) \geq -Mu(t) + Nv(t), \\ D^\beta v(t) \geq -Mv(t) + Nu(t), \\ u(t)|_{t=0} = \phi_p(D^\alpha x_1(t))|_{t=0} - \phi_p(D^\alpha x_0(t))|_{t=0} = 0, \\ v(t)|_{t=0} = \phi_p(D^\alpha y_0(t))|_{t=0} - \phi_p(D^\alpha y_1(t))|_{t=0} = 0. \end{cases}$$

Thus, in view of Lemma 2.6, we have that $\phi_p(D^\alpha x_1(t)) \geq \phi_p(D^\alpha x_0(t)), \phi_p(D^\alpha y_0(t)) \geq \phi_p(D^\alpha y_1(t)), \forall t \in [0, 1]$. Since $\Phi_p(x)$ is nondecreasing, we have $D^\alpha x_1(t) \geq D^\alpha x_0(t), D^\alpha y_0(t) \geq D^\alpha y_1(t), \forall t \in [0, 1]$.

Let $\epsilon(t) = x_1(t) - x_0(t), \theta(t) = y_0(t) - y_1(t)$. It follows from (3.1) and (H_1) , we have

$$\begin{cases} D^\alpha \epsilon(t) \geq 0, & t \in [0, 1], \\ t^{1-\alpha} \epsilon(t)|_{t=0} \geq 0, \end{cases} \quad (3.3)$$

and

$$\begin{cases} D^\alpha \theta(t) \geq 0, & t \in [0, 1], \\ t^{1-\alpha} \theta(t)|_{t=0} \geq 0, \end{cases} \quad (3.4)$$

By Lemma 2.5, we have $x_1(t) \geq x_0(t), y_0(t) \geq y_1(t), \forall t \in [0, 1]$.

Now we put $w(t) = \phi_p(D^\alpha y_1(t)) - \phi_p(D^\alpha x_1(t))$. Applying (3.1) and (H_1) , we obtain

$$\begin{aligned} D^\beta w(t) &= D^\beta(\phi_p(D^\alpha y_1(t))) - D^\beta(\phi_p(D^\alpha x_1(t))) \\ &= g(t, y_0(t), x_0(t), D^\alpha y_0(t), D^\alpha x_0(t)) + M\phi_p(D^\alpha y_0(t)) + N\phi_p(D^\alpha x_0(t)) - M\phi_p(D^\alpha y_1(t)) \\ &\quad - N\phi_p(D^\alpha x_1(t)) - f(t, x_0(t), y_0(t), D^\alpha x_0(t), D^\alpha y_0(t)) - M\phi_p(D^\alpha x_0(t)) - N\phi_p(D^\alpha y_0(t)) \\ &\quad + M\phi_p(D^\alpha x_1(t)) + N\phi_p(D^\alpha y_1(t)) \\ &\geq -M[\phi_p(D^\alpha y_0(t)) - \phi_p(D^\alpha x_0(t))] - N[\phi_p(D^\alpha x_0(t)) - \phi_p(D^\alpha y_0(t))] + M\phi_p(D^\alpha y_0(t)) \\ &\quad + N\phi_p(D^\alpha x_0(t)) - M\phi_p(D^\alpha y_1(t)) - N\phi_p(D^\alpha x_1(t)) - M\phi_p(D^\alpha x_0(t)) - N\phi_p(D^\alpha y_0(t)) \\ &\quad + M\phi_p(D^\alpha x_1(t)) + N\phi_p(D^\alpha y_1(t)) \\ &= -(M - N)w(t). \end{aligned}$$

Also, $w(t)|_{t=0} = \phi_p(D^\alpha y_1(t))|_{t=0} - \phi_p(D^\alpha x_1(t))|_{t=0} = 0$. In view of Lemma 2.4, we have that $w(t) \geq 0, \forall t \in J$. Thus we have the relation $\phi_p(D^\alpha x_1(t)) \leq \phi_p(D^\alpha y_1(t))$. That is

$D^\alpha x_1(t) \leq D^\alpha y_1(t)$, since $\Phi_p(x)$ is nondecreasing. Therefore $D^\alpha x_0(t) \leq D^\alpha x_1(t) \leq D^\alpha y_1(t) \leq D^\alpha y_0(t)$, $\forall t \in J$ holds.

Let $\delta(t) = y_1(t) - x_1(t)$. It follows from (3.1) that

$$\begin{cases} D^\alpha \delta(t) = D^\alpha y_1(t) - D^\alpha x_1(t) \geq 0, \\ t^{1-\alpha} \delta(t)|_{t=0} = t^{1-\alpha} y_1(t)|_{t=0} - t^{1-\alpha} x_1(t)|_{t=0} = r_2 - r_1 \geq 0. \end{cases}$$

By Lemma 2.5, we obtain $y_1(t) \geq x_1(t)$, $\forall t \in (0, 1]$. Hence, we have the relation $x_0(t) \leq x_1(t) \leq y_1(t) \leq y_0(t)$.

Now we assume that

$$x_{k-1} \leq x_k \leq y_k \leq y_{k-1}, \text{ and } D^\alpha x_{k-1} \leq D^\alpha x_k \leq D^\alpha y_k \leq D^\alpha y_{k-1}, \text{ for some } k \geq 1,$$

we will prove that (3.2) is also true for $k + 1$. Set

$$u(t) = \phi_p(D^\alpha x_{k+1}(t)) - \phi_p(D^\alpha x_k(t)), \quad v(t) = \phi_p(D^\alpha y_k(t)) - \phi_p(D^\alpha y_{k+1}(t)), \quad w(t) = \phi_p(D^\alpha y_{k+1}(t)) - \phi_p(D^\alpha x_{k+1}(t)),$$

$$\epsilon(t) = x_{k+1}(t) - x_k(t), \quad \theta(t) = y_k(t) - y_{k+1}(t), \quad \delta(t) = y_{k+1}(t) - x_{k+1}(t)$$

By (H_2) and (3.1), we have that

$$\begin{cases} D^\beta u(t) \geq -Mu(t) + Nv(t), \\ D^\beta v(t) \geq -Mv(t) + Nu(t), \\ u(t)|_{t=0} = 0, \\ v(t)|_{t=0} = 0, \end{cases}$$

$$\begin{cases} D^\alpha \epsilon(t) \geq 0, \\ t^{1-\alpha} \epsilon(t)|_{t=0} \geq 0, \end{cases}$$

$$\begin{cases} D^\alpha \theta(t) \geq 0, \\ t^{1-\alpha} \theta(t)|_{t=0} \geq 0, \end{cases}$$

and

$$\begin{cases} D^\beta w(t) \geq -(M - N)w(t), \\ w(t)|_{t=0} = 0, \end{cases}$$

$$\begin{cases} D^\alpha \delta(t) \geq 0, \\ t^{1-\alpha} \delta(t)|_{t=0} \geq 0. \end{cases}$$

In view of Lemma 2.4, 2.5 and 2.6, we obtain

$$x_k \leq x_{k+1} \leq y_{k+1} \leq y_k \text{ and } D^\alpha x_k \leq D^\alpha x_{k+1} \leq D^\alpha y_{k+1} \leq D^\alpha y_k, \quad \forall t \in [0, 1]$$

.

From the above, by induction, it is not difficult to prove that

$$x_0 \leq x_1 \leq \dots \leq x_n \leq \dots \leq y_n \leq \dots \leq y_1 \leq y_0.$$

and

$$D^\alpha x_0 \leq D^\alpha x_1 \leq \dots \leq D^\alpha x_n \leq \dots \leq D^\alpha y_n \leq \dots \leq D^\alpha y_1 \leq D^\alpha y_0.$$

Applying the standard arguments, it is easy to show $\{x_n\}$ and $\{y_n\}$ are uniformly bounded and equi-continuous in $[x_0, y_0]$. By Arzela-Ascoli theorem, we have

$$\lim_{n \rightarrow \infty} x_n(t) = x^*(t), \quad \lim_{n \rightarrow \infty} y_n(t) = y^*(t), \quad \forall t \in [0, 1]$$

and

$$\lim_{n \rightarrow \infty} D^\alpha x_n(t) = D^\alpha x^*(t), \quad \lim_{n \rightarrow \infty} D^\alpha y_n(t) = D^\alpha y^*(t), \quad \forall t \in [0, 1]$$

and the limit function x^* and y^* satisfy (1.1). Moreover, $x^*, y^* \in [x_0, y_0]$. Taking the limits $n \rightarrow \infty$ in (3.1), we find that (x^*, y^*) is a solution of problem (1.1) in $[x_0, y_0] \times [x_0, y_0]$.

Finally, we show that (x^*, y^*) is an extremal solution of the system (1.1). Assume that $(x, y) \in [x_0, y_0] \times [x_0, y_0]$ is any solution for the problem (1.1), that is

$$\begin{cases} D^\beta(\phi_p(D^\alpha x(t))) = f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)), & t \in [0, 1], \\ D^\alpha x(t)|_{t=0} = 0, & t^{1-\alpha}x(t)|_{t=0} = r_1, \\ D^\beta(\phi_p(D^\alpha y(t))) = g(t, y(t), x(t), D^\alpha y(t), D^\alpha x(t)), & t \in [0, 1], \\ D^\alpha y(t)|_{t=0} = 0, & t^{1-\alpha}y(t)|_{t=0} = r_2, \end{cases} \quad (3.5)$$

Applying (3.1), (3.5), (H_2) , Lemma 2.5 and 2.6, we have

$$x_n \leq x, \quad y \leq y_n, \quad D^\alpha x_n \leq D^\alpha x, \quad D^\alpha y \leq D^\alpha y_n, \quad n = 1, 2, \dots \quad (3.6)$$

Taking the limit $n \rightarrow \infty$ in (3.6), we have $x^* \leq x, y \leq y^*$. The proof is complete.

Example: Consider the following problem

$$\begin{cases} D^{\frac{2}{3}}(\phi_4(D^{\frac{1}{2}}x(t))) = \frac{1}{6\Gamma(1-\frac{2}{3})}x^{\frac{1}{3}}(t)[D^{\frac{1}{2}}x(t)]^{\frac{1}{3}} - y^3(t)\left[D^{\frac{1}{2}}y(t) - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\right]^3, & t \in (0, 1), \\ D^{\frac{2}{3}}(\phi_4(D^{\frac{1}{2}}y(t))) = \frac{1}{6\Gamma(1-\frac{2}{3})}y^{\frac{1}{3}}(t)[D^{\frac{1}{2}}y(t)]^{\frac{1}{3}} - x^3(t)\left[D^{\frac{1}{2}}x(t) - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\right]^3, & t \in (0, 1), \\ D^{\frac{1}{2}}x(t)|_{t=0} = 0, & t^{1-\frac{1}{2}}x(t)|_{t=0} = 0, \\ D^{\frac{1}{2}}y(t)|_{t=0} = 0, & t^{1-\frac{1}{2}}y(t)|_{t=0} = 0, \end{cases} \quad (3.7)$$

where $\alpha = \frac{1}{2}, \beta = \frac{2}{3}, p = 4$ and

$$\begin{cases} f(t, x(t), y(t), D^{\frac{1}{2}}x(t), D^{\frac{1}{2}}y(t)) = \frac{1}{6\Gamma(1-\frac{2}{3})}x^{\frac{1}{3}}[D^{\frac{1}{2}}x]^{\frac{1}{3}} - y^3\left[D^{\frac{1}{2}}y - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\right]^3 \\ g(t, y(t), x(t), D^{\frac{1}{2}}y(t), D^{\frac{1}{2}}x(t)) = \frac{1}{6\Gamma(1-\frac{2}{3})}y^{\frac{1}{3}}[D^{\frac{1}{2}}y]^{\frac{1}{3}} - x^3\left[D^{\frac{1}{2}}x - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\right]^3 \end{cases}$$

Take $x_0(t) = \frac{1}{2}t^{\frac{3}{2}}, y(t) = 2t^{\frac{3}{4}}$, then $D^{\frac{1}{2}}x_0(t) = \frac{1}{2}\Gamma(\frac{5}{2})t, D^{\frac{1}{2}}y_0(t) = \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}$. It is not difficult to show that (H_1) holds.

Since the function $\sqrt[3]{x} + x^3$ is monotone increasing for $x \in R$, we obtain

$$\begin{aligned} & f(t, \overline{x(t)}, \overline{y(t)}, D^{\frac{1}{2}}\overline{x(t)}, D^{\frac{1}{2}}\overline{y(t)}) - f(t, x(t), y(t), D^{\frac{1}{2}}x(t), D^{\frac{1}{2}}y(t)) \\ = & \frac{1}{6\Gamma(1-\frac{2}{3})}(\overline{x(t)})^{\frac{1}{3}}[D^{\frac{1}{2}}\overline{x(t)}]^{\frac{1}{3}} - (\overline{y(t)})^3\left[D^{\frac{1}{2}}\overline{y(t)} - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\right]^3 - \frac{1}{6\Gamma(1-\frac{2}{3})}x^{\frac{1}{3}}(t)[D^{\frac{1}{2}}x(t)]^{\frac{1}{3}} \\ & + y^3(t)\left[D^{\frac{1}{2}}y(t) - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\right]^3, \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{6\Gamma(1-\frac{2}{3})} \sqrt[3]{2} [(D^{\frac{1}{2}}\overline{x(t)})^{\frac{1}{3}} - (D^{\frac{1}{2}}x(t))^{\frac{1}{3}}] \\ &\leq \frac{1}{6\Gamma(1-\frac{2}{3})} \sqrt[3]{2} [(D^{\frac{1}{2}}x(t))^3 - (D^{\frac{1}{2}}\overline{x(t)})^3] \\ &= \frac{1}{6\Gamma(1-\frac{2}{3})} \sqrt[3]{2} [\Phi_4(D^{\frac{1}{2}}x(t)) - \Phi_4(D^{\frac{1}{2}}\overline{x(t)})] \end{aligned}$$

where $x_0(t) \leq \overline{x(t)} \leq x(t) \leq y_0(t)$, and $x_0(t) \leq y(t) \leq \overline{y(t)} \leq y_0(t)$.

Note $M = \frac{\sqrt[3]{2}}{6\Gamma(1-\frac{2}{3})}$, $N = 0$ and

$$(M + N)\Gamma(1 - \frac{2}{3}) = \frac{\sqrt[3]{2}}{6} < 1,$$

thus the condition (H_2) holds. Therefore, there exist monotone iterative sequence $\{x_n\}$ and $\{y_n\}$, which converge uniformly to solutions of fractional problem (3.7) in $[x_0, y_0]$ by Theorem 3.1.

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The Growth and Zeros of Linear Differential Equations with Entire Coefficients of $[p, q] - \varphi(r)$ Order ^{*†}

Sheng Gui Liu¹, Jin Tu^{2,*}, Hong Zhang³

¹ School of Mechanics and Civil Engineering, China University of Mining and Technology(Beijing), Beijing, 100083, China

² College of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, China

³ School of the Tourism and Urban Management, Jiangxi University of Finance and Economics, Nanchang 330032, China

Abstract

In this paper, the authors investigate the growth and zeros of the solutions and the coefficients of higher order linear differential equations with entire coefficients of $[p, q] - \varphi$ order, where p, q are positive integers and satisfy $p \geq q \geq 1$. The theorems that we obtain extend and improve many previous results.

Key words: linear differential equations; entire functions; $[p, q] - \varphi$ order;

AMS Subject Classification(2010): 30D35, 34M10

1. Introduction and Results

In this paper, we shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions and the theory of complex linear differential equations (e.g. [9,14]). The theory of complex linear differential equations has been developed since 1960s. Many authors have investigated the complex linear differential equations

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \dots + A_0(z)f(z) = 0 \tag{1.1}$$

and

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \dots + A_0(z)f(z) = F(z) \tag{1.2}$$

and achieved many valuable results when the coefficients $A_0(z), \dots, A_{k-1}(z), F(z) (k \geq 2)$ in (1.1) and (1.2) are entire functions of finite order or finite iterate order (e.g. [1-2, 4-7, 13, 14]). In 2010, J. Tu and his co-authors investigated the complex oscillation properties of solutions of (1.1) and (1.2) when the coefficients in (1.1) or (1.2) are entire functions of $[p, q]$ -order (see [21]). In the following, we introduce some notations about $[p, q]$ -order, where p, q are positive integers

*Corresponding author E-mail adress:tujin2008@sina.com

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and satisfy $p \geq q \geq 1$. For $r \in (0, +\infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$, $i \in \mathbb{N}$ and for all sufficiently large r , we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r)$, $i \in \mathbb{N}$. Especially, we have $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Secondly, we denote the linear measure and the logarithmic measure of a set $E \subset (1, +\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E \frac{dt}{t}$.

Definition 1.1. ([12,15,16]) If $f(z)$ is a meromorphic function, the $[p, q]$ -order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q r}, \tag{1.3}$$

especially if $f(z)$ is an entire function, the $[p, q]$ -order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q r} = \overline{\lim}_{r \rightarrow \infty} \frac{\log_{p+1} M(r, f)}{\log_q r}. \tag{1.4}$$

Definition 1.2. ([15,16]) The $[p, q]$ -exponent of convergence of the (distinct) zero-sequence of $f(z)$ are respectively defined by

$$\lambda_{[p,q]}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log_p n(r, \frac{1}{f})}{\log_q r} = \overline{\lim}_{r \rightarrow \infty} \frac{\log_p N(r, \frac{1}{f})}{\log_q r}, \tag{1.5}$$

$$\bar{\lambda}_{[p,q]}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log_p \bar{n}(r, \frac{1}{f})}{\log_q r} = \overline{\lim}_{r \rightarrow \infty} \frac{\log_p \bar{N}(r, \frac{1}{f})}{\log_q r}. \tag{1.6}$$

In recent years, many authors investigated the equations (1.1) and (1.2) with entire coefficients or meromorphic coefficients of $[p, q]$ -order (e.g. [3, 11, 15, 16]) and obtain the following results.

Theorem A. ([15]) Let $A_j(z)$ ($j = 0, 1, \dots, k - 1$) be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j) | j \neq 0\} < \sigma_{[p,q]}(A_0) < \infty$. Then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

Theorem B. ([15]) Let $F(z) \not\equiv 0, A_j(z)$ ($j = 0, 1, \dots, k - 1$) be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j), \sigma_{[p+1,q]}(F) | j = 1, \dots, k - 1\} < \sigma_{[p,q]}(A_0)$. Then every solution $f(z)$ of (1.2) satisfies

$$\bar{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0),$$

with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \sigma_{[p,q]}(A_0)$.

Theorem C. ([15]) Let $F(z) \not\equiv 0, A_j(z)$ ($j = 0, \dots, k - 1$) be entire functions, and let $f(z)$ be a solution of (1.2) satisfying $\max\{\sigma_{[p,q]}(A_j), \sigma_{[p,q]}(F) | j = 0, 1, \dots, k - 1\} < \sigma_{[p,q]}(f)$. Then $\bar{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

On the basic of Definitions 1.1 and 1.2, some researchers introduce the notations of $[p, q]$ - $\varphi(r)$ order of entire functions or analytic functions in [17, 18, 19], where p, q are positive integers and satisfy $p \geq q \geq 1$.

Definition 1.3. ([17,18]) Let $\varphi : [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded continuous function, the $[p, q] - \varphi(r)$ order and $[p, q] - \varphi(r)$ lower order of an entire function $f(z)$ are respectively defined by

$$\sigma_{[p,q]}(f, \varphi) = \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)}; \quad \mu_{[p,q]}(f, \varphi) = \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)}. \tag{1.7}$$

Similar with Definition 1.3, we can also define $[p, q] - \varphi(r)$ exponent of convergence of (distinct) zero-sequence of an entire function $f(z)$.

Definition 1.4. ([17]) The $[p, q] - \varphi(r)$ exponent of convergence of (distinct) zero-sequence of an entire function $f(z)$ are respectively defined by

$$\lambda_{[p,q]}(f, \varphi) = \lim_{r \rightarrow \infty} \frac{\log_p n(r, \frac{1}{f})}{\log_q \varphi(r)}; \quad \bar{\lambda}_{[p,q]}(f, \varphi) = \lim_{r \rightarrow \infty} \frac{\log_p \bar{n}(r, \frac{1}{f})}{\log_q \varphi(r)}. \tag{1.8}$$

Remark 1.5. If $\varphi(r) = r$, the Definitions 1.1-1.2 are special cases of Definitions 1.3-1.4.

In order to get similar results with Theorems A – C by replacing $[p, q]$ -order with $[p, q] - \varphi(r)$ order, we suppose that $\varphi(r)$ has the following properties without special instructions:

Proposition 1.6. Suppose that $\varphi(r) : [0, +\infty) \rightarrow (0, +\infty)$ is a non-decreasing unbounded continuous function and satisfies (i) $\lim_{r \rightarrow \infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$, (ii) $\lim_{r \rightarrow \infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$.

Proposition 1.7. ([17]) If $\varphi(r)$ satisfies the above two conditions (i) – (ii) in Proposition 1.6: (i) then for any entire function $f(z)$, we have

$$\begin{aligned} \sigma_{[p,q]}(f, \varphi) &= \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} = \lim_{r \rightarrow \infty} \frac{\log_{p+1} M(r, f)}{\log_q \varphi(r)}; \\ \mu_{[p,q]}(f, \varphi) &= \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} = \lim_{r \rightarrow \infty} \frac{\log_{p+1} M(r, f)}{\log_q \varphi(r)}. \end{aligned}$$

(ii) then for any meromorphic function $f(z)$, we have

$$\begin{aligned} \lambda_{[p,q]}(f, \varphi) &= \lim_{r \rightarrow \infty} \frac{\log_p n(r, \frac{1}{f})}{\log_q \varphi(r)} = \lim_{r \rightarrow \infty} \frac{\log_p N(r, \frac{1}{f})}{\log_q \varphi(r)}; \\ \bar{\lambda}_{[p,q]}(f, \varphi) &= \lim_{r \rightarrow \infty} \frac{\log_p \bar{n}(r, \frac{1}{f})}{\log_q \varphi(r)} = \lim_{r \rightarrow \infty} \frac{\log_p \bar{N}(r, \frac{1}{f})}{\log_q \varphi(r)}. \end{aligned}$$

In this paper, we investigate the growth and zeros of solutions of (1.1) and (1.2) with entire coefficients of $[p, q] - \varphi(r)$ order and obtain the following results.

Theorem 1.8. Let $A_j(z)$ ($j = 0, 1, \dots, k-1$) be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j, \varphi) | j = 1, 2, \dots, k-1\} < \sigma_{[p,q]}(A_0, \varphi) < \infty$. Then every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi)$.

Theorem 1.9. Let $F(z) \not\equiv 0$, $A_j(z)$ ($j = 0, \dots, k-1$) be entire functions, and let $f(z)$ be a solution of (1.2) satisfying $\max\{\sigma_{[p,q]}(A_j, \varphi), \sigma_{[p,q]}(F, \varphi) | j = 0, 1, \dots, k-1\} < \sigma_{[p,q]}(f, \varphi)$. Then $\bar{\lambda}_{[p,q]}(f, \varphi) = \lambda_{[p,q]}(f, \varphi) = \sigma_{[p,q]}(f, \varphi)$.

Theorem 1.10. Let $F(z) \not\equiv 0$, $A_j(z)$ ($j = 0, 1, \dots, k-1$) be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j, \varphi), \sigma_{[p+1,q]}(F, \varphi) | j = 1, \dots, k-1\} < \sigma_{[p,q]}(A_0, \varphi)$. Then every solution $f(z)$ of (1.2) satisfies

$$\bar{\lambda}_{[p+1,q]}(f, \varphi) = \lambda_{[p+1,q]}(f, \varphi) = \sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi),$$

with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0, \varphi) < \sigma_{[p,q]}(A_0, \varphi)$.

Remark 1.11. The above Theorems 1.8-1.10 generalize and extend Theorems A-C and some previous results.

2. Preliminary Lemmas

Lemma 2.1. ([10, 14]) Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z| = r$ outside a set E_1 of r of finite logarithmic measure, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v_f(r)}{z}\right)^j (1 + o(1)) \quad (j \in \mathbb{N}),$$

where $v_f(r)$ is the central index of $f(z)$, $E_1 \subset (1, +\infty)$ is a set of r of finite logarithmic measure or finite linear measure in this paper, not necessarily the same at each occurrence.

Lemma 2.2. ([7, 14]) Let $g : [0, +\infty) \rightarrow R$ and $h : [0, +\infty) \rightarrow R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $E_1 \subset [1, +\infty)$ of finite logarithmic measure or finite linear measure. Then for any $d > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(dr)$ for all $r > r_0$.

Lemma 2.3. ([17]) Let $f(z)$ be an entire function satisfying $\sigma_{[p,q]}(f, \varphi) = \sigma_1$ and $\mu_{[p,q]}(f, \varphi) = \mu_1$. Then

$$\lim_{r \rightarrow \infty} \frac{\log_p v_f(r)}{\log_q \varphi(r)} = \sigma_1, \quad \lim_{r \rightarrow \infty} \frac{\log_p v_f(r)}{\log_q \varphi(r)} = \mu_1.$$

Lemma 2.4. Let $f(z)$ be an entire function of $[p, q] - \varphi(r)$ order satisfying $\sigma_{[p,q]}(f, \varphi) = \sigma_2$, where $\varphi(r)$ only satisfies $\lim_{r \rightarrow \infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$. Then there exists a set $E_2 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have

$$\lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} = \sigma_2 \quad (r \in E_2).$$

Proof. By Definition 1.3, there exists an increasing sequence $\{r_n\}_{n=1}^\infty$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{n \rightarrow \infty} \frac{\log_p T(r_n, f)}{\log_q \varphi(r_n)} = \sigma_{[p,q]}(f, \varphi) = \sigma_2,$$

there exists an $n_1 (\in \mathbb{N})$ such that for $n \geq n_1$ and for any $r \in E_2 = \cup_{n=n_1}^\infty [r_n, (1 + \frac{1}{n})r_n]$, we have

$$\frac{\log_p T(r_n, f)}{\log_q \varphi((1 + \frac{1}{n})r_n)} \leq \frac{\log_p T(r, f)}{\log_q \varphi(r)}. \tag{2.1}$$

By (2.1), for all $r \in E_2$, we have

$$\lim_{n \rightarrow \infty} \frac{\log_p T(r_n, f)}{\log_q \varphi(r_n)} \cdot \lim_{n \rightarrow \infty} \frac{\log_q \varphi(r_n)}{\log_q \varphi((1 + \frac{1}{n})r_n)} \leq \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)}. \tag{2.2}$$

By (2.12) and $\lim_{r \rightarrow \infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ ($\alpha > 1$), for all $r \in E_2$, we have

$$\lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} \geq \sigma_2. \tag{2.3}$$

On the other hand, by Definition 1.3, for all $r \in E_2$, we have

$$\lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} \leq \sigma_2. \tag{2.4}$$

By (2.3) and (2.4), for any $r \in E_2$, we have

$$\lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} = \sigma_2.$$

where $m_l E_2 = \sum_{n=n_1}^\infty \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^\infty \log(1 + \frac{1}{n}) = \infty$.

By Lemma 2.4, it is easy to obtain the following Lemma 2.5.

Lemma 2.5. Let $f_1(z), f_2(z)$ be entire functions of $[p, q] - \varphi(r)$ order satisfying $\sigma_{[p,q]}(f_1, \varphi) > \sigma_{[p,q]}(f_2)$. Then there exists a set $E_3 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_3$, we have

$$\lim_{r \rightarrow \infty} \frac{T(r, f_2)}{T(r, f_1)} = 0 \quad (r \in E_3).$$

Lemma 2.6. ([8]) Let $f(z)$ be a transcendental meromorphic function, and let $\beta > 1$ be a given constant, for any given $\varepsilon > 0$, there exist a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure

and a constant $B > 0$ that depends only on β and (i, j) (i, j are integers with $0 \leq i < j$) such that for all $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq B \left[\frac{T(\beta r, f)}{r} (\log^\beta r) \log T(\beta r, f) \right]^{j-i}.$$

3. Proofs of Theorems 1.8 - 1.10

Proof of Theorem 1.8. We divide the proof into two parts.

(i) Set $\sigma_{[p,q]}(A_0, \varphi) = \sigma_3$, first, we prove that every solution of (1.1) satisfies $\sigma_{[p+1,q]}(f, \varphi) \leq \sigma_3$. It is easy to know that equation (1.1) has no polynomial solutions under the assumptions. If $f(z)$ is a transcendental solution of (1.1), by (1.1), we get

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |A_{k-1}| \left| \frac{f^{(k-1)}(z)}{f(z)} \right| + \dots + |A_s| \left| \frac{f^{(s)}(z)}{f(z)} \right| + \dots + |A_0|. \tag{3.1}$$

Since $\max\{\sigma_{[p,q]}(A_j, \varphi) | j = 0, 1, \dots, k-1\} \leq \sigma_3$, for any given $\varepsilon > 0$ and for sufficiently large r , we have

$$|A_j(z)| \leq \exp_{p+1}\{(\sigma_3 + \varepsilon) \log_q \varphi(r)\} \quad (j = 0, 1, \dots, k-1). \tag{3.2}$$

By Lemma 2.1, there exists a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v_f(r)}{z} \right)^j (1 + o(1)) \quad (j = 1, \dots, k-1). \tag{3.3}$$

By (3.1)-(3.3), for all z satisfying $|z| = r \notin [0, 1] \cup E_1$ and $|f(z)| = M(r, f)$, we get

$$\left(\frac{v_f(r)}{r} \right)^k (1 + o(1)) \leq k \exp_{p+1}\{(\sigma_3 + \varepsilon) \log_q \varphi(r)\} \left(\frac{v_f(r)}{r} \right)^{k-1} (1 + o(1)), \tag{3.4}$$

by (3.4) and Lemma 2.2, there exists some α_1 ($1 < \alpha_1 < \alpha$) and $r \geq r_0$, we have

$$v_f(r) \leq k\alpha_1 r \exp_{p+1}\{(\sigma_3 + \varepsilon) \log_q \varphi(\alpha_1 r)\}. \tag{3.5}$$

By Lemma 2.3 and the Proposition 1.6, we have $\sigma_{[p+1,q]}(f, \varphi) \leq \sigma_3$.

(ii) On the other hand, if $f \not\equiv 0$, (1.1) can be written

$$-A_0 = \frac{f^{(k)}(z)}{f(z)} + \dots + A_j \frac{f^{(j)}(z)}{f(z)} + \dots + A_1 \frac{f'(z)}{f(z)}. \tag{3.6}$$

By (3.6), we get

$$m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_j) + \sum_{j=1}^k m\left(r, \frac{f^{(k)}}{f}\right) + \log k. \tag{3.7}$$

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Since $\max\{\sigma_{[p,q]}(A_j, \varphi) | j = 1, 2, \dots, k - 1\} < \sigma_3$ and by Lemma 2.5, there exists a set $E_2 \subset (1, +\infty)$ with infinite logarithmic measure such that for all z satisfying $|z| = r \in E_2$, we have

$$\lim_{r \rightarrow \infty} \frac{\log_p m(r, A_0)}{\log_q \varphi(r)} = \sigma_3, \quad \frac{m(r, A_j)}{m(r, A_0)} \rightarrow 0 \quad (r \in E_2, j = 1, \dots, k - 1). \tag{3.8}$$

By the lemma of logarithmic derivative, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O\{\log r T(r, f)\} \quad (r \notin E_1). \tag{3.9}$$

By (3.7)-(3.9), for all sufficiently large $r \in E_2 \setminus E_1$, we have

$$\frac{1}{2}m(r, A_0) \leq O\{\log r T(r, f)\}.$$

Hence by Proposition 1.6, we have $\sigma_{[p+1,q]}(f, \varphi) \geq \sigma_3$. Therefore, every solution $f(z) \neq 0$ of (1.1) satisfies $\sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi)$.

Proof of Theorem 1.9. Proof. If $f(z) \neq 0$, by (1.2), we get

$$\frac{1}{f} = \frac{1}{F} \left(\frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \dots + A_0 \right), \tag{3.10}$$

it is easy to see that if $f(z)$ has a zero at z_0 of order α ($\alpha > k$), and A_0, \dots, A_{k-1} are analytic at z_0 , then F must have a zero at z_0 of order $\alpha - k$, hence

$$n\left(r, \frac{1}{f}\right) \leq k\bar{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right), \tag{3.11}$$

and

$$N\left(r, \frac{1}{f}\right) \leq k\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{3.12}$$

By the lemma of logarithmic derivative and (3.10), we have

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r, A_j) + O(\log T(r, f) + \log r) \quad (r \notin E_1). \tag{3.13}$$

By (3.12), (3.13), we get

$$T(r, f) \leq k\bar{N}\left(r, \frac{1}{f}\right) + T(r, F) + \sum_{j=0}^{k-1} T(r, A_j) + O\{\log(rT(r, f))\} \quad (r \notin E_1). \tag{3.14}$$

Since $\max\{\sigma_{[p,q]}(F, \varphi), \sigma_{[p,q]}(A_j, \varphi) | j = 0, 1, \dots, k-1\} < \sigma_{[p,q]}(f, \varphi)$, by Lemma 2.5, there exists a set $E_3 \subset (1, +\infty)$ having infinite logarithmic measure such that

$$\max \left\{ \frac{T(r, F)}{T(r, f)}, \frac{T(r, A_j)}{T(r, f)} \right\} \rightarrow 0 \quad (r \in E_3, j = 0, \dots, k-1). \tag{3.15}$$

Since for all sufficiently large r , we have

$$\log T(r, f) = o\{T(r, f)\}. \tag{3.16}$$

By (3.14)-(3.16), for all $|z| = r \in E_3 \setminus E_1$, we have

$$(1 - o(1)) T(r, f) \leq O \left\{ \bar{N} \left(r, \frac{1}{f} \right) \right\} + O\{\log r\}. \tag{3.17}$$

By Definition 1.4 and Proposition 1.7 and (3.17), we get

$$\sigma_{[p,q]}(f, \varphi) \leq \bar{\lambda}_{[p,q]}(f, \varphi). \tag{3.18}$$

Since $\sigma_{[p,q]}(f, \varphi) \geq \lambda_{[p,q]}(f, \varphi) \geq \bar{\lambda}_{[p,q]}(f, \varphi)$, and by (3.18), we have

$$\bar{\lambda}_{[p,q]}(f, \varphi) = \lambda_{[p,q]}(f, \varphi) = \sigma_{[p,q]}(f, \varphi).$$

Proof of Theorem 1.10. We assume that f is a solution of (1.2). By the elementary theory of differential equations, all the solutions of (1.2) are entire functions and have the form

$$f = f^* + C_1 f_1 + C_2 f_2 + \dots + C_k f_k,$$

where C_1, \dots, C_k are complex constants, $\{f_1, \dots, f_k\}$ is a solution base of (1.1), f^* is a solution of (1.2) and has the form

$$f^* = D_1 f_1 + D_2 f_2 + \dots + D_k f_k, \tag{3.19}$$

where D_1, \dots, D_k are certain entire functions satisfying

$$D'_j = F \cdot G_j(f_1, \dots, f_k) \cdot W(f_1, \dots, f_k)^{-1} \quad (j = 1, \dots, k), \tag{3.20}$$

where $G_j(f_1, \dots, f_k)$ are differential polynomials in f_1, \dots, f_k and their derivatives with constant coefficients, and $W(f_1, \dots, f_k)$ is the Wronskian of f_1, \dots, f_k . By Theorem 1.8, we have $\sigma_{[p+1,q]}(f_j, \varphi) = \sigma_{[p,q]}(A_0, \varphi) (j = 1, 2, \dots, k)$, then by (3.19) and (3.20), we get

$$\sigma_{[p+1,q]}(f, \varphi) \leq \max\{\sigma_{[p+1,q]}(f_j, \varphi), \sigma_{[p+1,q]}(F, \varphi) | j = 1, \dots, k\} \leq \sigma_{[p,q]}(A_0, \varphi).$$

We affirm that (1.2) can only possess at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0, \varphi) < \sigma_{[p,q]}(A_0, \varphi)$. In fact, if f_* is another solution satisfying $\sigma_{[p+1,q]}(f_*, \varphi) < \sigma_{[p,q]}(A_0, \varphi)$, then $\sigma_{[p+1,q]}(f_0 - f_*, \varphi) < \sigma_{[p,q]}(A_0, \varphi)$. But $f_0 - f_*$ is a solution of (1.1), this contradicts Theorem 1.8. Then $\sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi)$ holds for all solutions of (1.2) with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0, \varphi) < \sigma_{[p,q]}(A_0, \varphi)$. By Theorem 1.9, we get that

$$\bar{\lambda}_{[p+1,q]}(f, \varphi) = \lambda_{[p+1,q]}(f, \varphi) = \sigma_{[p+1,q]}(f, \varphi)$$

holds for all solutions satisfying $\sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi)$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0, \varphi) < \sigma_{[p,q]}(A_0, \varphi)$.

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Some k -fractional integrals inequalities through generalized $\lambda_{\phi m}$ -MT-preinvexity

Chunyan Luo¹ Tingsong Du^{1,2*} Muhammad Adil Khan³
 Artion Kashuri⁴ Yanjun Shen⁵

¹Department of Mathematics, College of Science, China Three Gorges University,
 Yichang 443002, China

²Three Gorges Mathematical Research Center, China Three Gorges University,
 Yichang 443002, China

E-mail: luochunyanctgu@gmail.com tingsongdu@ctgu.edu.cn

³Department of Mathematics, University of Peshawar, Pakistan

E-mail: adilswati@gmail.com

⁴Department of Mathematics, Faculty of Technical Science, University “Ismail Qemali”,
 Vlora, Albania E-mail: artionkashuri@gmail.com

⁵Hubei Provincial Collaborative Innovation Center for New Energy Microgrid,
 China Three Gorges University, Yichang 443002, China E-mail: shenyj@ctgu.edu.cn

* Corresponding author Tingsong Du

Abstract

The authors introduce the concept of the generalized $\lambda_{\phi m}$ -MT-preinvex functions and discover a new k -fractional integral identity concerning twice differentiable preinvex mappings defined on (ϕ, m) -invex set. By using this identity, we establish the right-sided new Hermite-Hadamard type inequalities for the generalized $\lambda_{\phi m}$ -MT-preinvex mappings via k -fractional integrals. The new k -fractional integral inequalities are also applied to some special means.

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1 Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping on the interval I of real numbers and $u, v \in I$ with $u < v$. Then the following well-know Hermite-Hadamard inequality holds

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x)dx \leq \frac{f(u)+f(v)}{2}. \tag{1.1}$$

This inequality is one of the famous results for convex functions.

Many researchers generalized and extended the inequalities (1.1) involving a variety of convex functions one can see [8, 9, 12, 15, 20–22, 40, 41] and the references mentioned in these papers.

In 2013, Sarikaya et al. [32] considered the following Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals.

Theorem 1.1 *Let $f : [u, v] \rightarrow \mathbb{R}$ be a positive function along with $0 \leq u < v$ and let $f \in L^1[u, v]$. Suppose f is a convex function on $[u, v]$, then the subsequent inequalities for fractional integrals hold:*

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u^+}^\alpha f(v) + J_{v^-}^\alpha f(u)] \leq \frac{f(u) + f(v)}{2}, \tag{1.2}$$

where the symbols $J_{u^+}^\alpha f$ and $J_{v^-}^\alpha f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$ defined by

$$J_{u^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-t)^{\alpha-1} f(t) dt, \quad u < x$$

and

$$J_{v^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} f(t) dt, \quad x < v.$$

Here, $\Gamma(\alpha)$ is the gamma function and its definition is $\Gamma(\alpha) = \int_0^\infty e^{-\mu} \mu^{\alpha-1} d\mu$.

Due to the wide applications of Riemann-Liouville fractional Hermite-Hadamard type inequalities in mathematical analysis, many researchers extended Hermite-Hadamard inequality for different classes of convex functions. For example, see for convex mappings [7, 10, 16, 17, 29], for m -convex mappings [37] and (s, m) -convex mappings [3], for h -preinvex mappings [13], for harmonically convex mappings [18], for preinvex mappings [25, 31] and the references mentioned in these papers.

Also in [4], Anastassiou presented a complete theory with respect to fractional differentiation inequalities.

In 2012, Mubeen and Habibullah [24] introduced a new fractional integral that generalizes the Riemann-Liouville fractional integrals.

Definition 1.1 ([24]) *Let $f \in L^1[a, b]$, then k -Riemann-Liouville fractional integrals ${}_k J_{a^+}^\mu f(x)$ and ${}_k J_{b^-}^\mu f(x)$ of order $\mu > 0$ are defined by*

$${}_k J_{a^+}^\mu f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t) dt, \quad (0 \leq a < x < b)$$

and

$${}_k J_{b^-}^\mu f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t) dt, \quad (0 \leq a < x < b),$$

respectively, where $k > 0$ and $\Gamma_k(\mu)$ is the k -gamma function given as $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$. Note that $\Gamma_k(\mu+k) = \mu\Gamma_k(\mu)$ and ${}_k J_{a^+}^0 f(x) = {}_k J_{b^-}^0 f(x) = f(x)$.

The notion of k -Riemann-Liouville fractional integral is an significant extension of Riemann-Liouville fractional integrals. It is stressed that for $k \neq 1$ the properties of k -Riemann-Liouville fractional integrals are quite dissimilar from those of general Riemann-Liouville fractional integrals. For this, the k -Riemann-Liouville fractional integrals have aroused the interest of many researchers. Properties and integral inequalities concerning this operator can refer to [1, 2, 6, 33, 34, 38] and the references mentioned in these papers.

Let us evoke some basic definitions as follows.

Definition 1.2 ([5]) A set $K \subseteq \mathbb{R}^n$ is said to be invex set respecting the mapping $\eta : K \times K \rightarrow \mathbb{R}^n$ if $x + t\eta(y, x) \in K$ for any $x, y \in K$ and $t \in [0, 1]$.

Definition 1.3 ([39]) A function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η , if

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y), \quad \forall x, y \in K, t \in [0, 1].$$

Definition 1.4 ([27]) Let $x \in K \subseteq \mathbb{R}^n$ and let $\phi : K \rightarrow \mathbb{R}$ be a continuous function. Then the set K is said to be ϕ -convex at x respecting ϕ , if

$$x + \lambda e^{i\phi}(y - x) \in K, \quad \forall x, y \in K, \lambda \in [0, 1].$$

Definition 1.5 ([26]) A set $K \subseteq \mathbb{R}^n$ is called ϕ -invex at x with respect to $\phi(\cdot)$, if there a continuous function $\phi(\cdot) : K \rightarrow \mathbb{R}$ and a bifunction $\eta(\cdot, \cdot) : K \times K \rightarrow \mathbb{R}^n$, such that

$$x + te^{i\phi}\eta(y, x) \in K, \quad \forall x, y \in K, t \in [0, 1].$$

Definition 1.6 ([11]) A set $K \subseteq \mathbb{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Definition 1.7 ([27]) The function f on the ϕ -convex set K is said to be ϕ -convex with respect to ϕ , if

$$f(x + \lambda e^{i\phi}(y - x)) \leq (1 - \lambda)f(x) + \lambda f(y), \quad \forall x, y \in K, \lambda \in [0, 1]. \tag{1.3}$$

Definition 1.8 ([42]) The function f defined on the ϕ -invex set $K \subseteq \mathbb{R}^n$ is said to be ϕ -MT-preinvex, if it is nonnegative and for $\forall x, y \in K$ and $t \in (0, 1)$ satisfies the following inequality

$$f(x + te^{i\phi}\eta(y, x)) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}}f(y). \tag{1.4}$$

Definition 1.9 ([28]) A function: $I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be m -MT-convex, if f is positive and for $\forall x, y \in I$, and $t \in (0, 1)$, with $m \in [0, 1]$, satisfies the following inequality

$$f(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.5}$$

Definition 1.10 ([14]) A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be λ -MT-convex function, if f is positive and $\forall x, y \in I$, $\lambda \in (0, \frac{1}{2}]$ and $t \in (0, 1)$, satisfies the following inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(y). \tag{1.6}$$

Clearly, when choosing $m = 1$ and $\lambda = \frac{1}{2}$ in Definition 1.9 and Definition 1.10, respectively, the function f reduces to MT-convex function in [35]. For some significant integral inequalities in association with MT-convex functions, one can see [19, 23, 30, 36] and the references therein.

The main purpose of this paper is to introduce the class of generalized $\lambda_{\phi m}$ -MT-preinvex functions on (ϕ, m) -invex and to prove a k -fractional integral identity. By using this identity, we establish the right-sided new Hadamard-type inequalities for the generalized $\lambda_{\phi m}$ -MT-preinvex functions via k -Riemann-Liouville fractional integrals. These inequalities can be viewed as generalization of recent results that appeared in Refs. [30] and [42].

2 New definitions and a lemma

As one can see, the definitions of the ϕ -invex and m -invex have similar configurations. This observation leads us to generalized these concepts. Firstly, the so-called ‘ (ϕ, m) -invex’ may be introduced as follows.

Definition 2.1 A set $K_{\phi m} \subseteq \mathbb{R}^n$ is said to be (ϕ, m) -invex with respect to a continuous function $\phi(\cdot) : K_{\phi m} \rightarrow \mathbb{R}$ and the mapping $\eta : K_{\phi m} \times K_{\phi m} \times (0, 1] \rightarrow \mathbb{R}^n$, for some fixed $m \in (0, 1]$, if $mx + te^{i\phi}\eta(y, x, m) \in K_{\phi m}$ holds for any $x, y \in K_{\phi m}$ and $t \in (0, 1)$.

Let us note that:

- if $\phi = 0$, then we get the definition of an m -invex set,
- if the mapping $\eta(y, x, m)$ with $m = 1$ reduces to $\eta(y, x)$, then we obtain the definition of a ϕ -invex set,
- if $\phi = 0$ and $\eta(y, x, m) = y - mx$ with $m = 1$, then we obtain the definition of a convex set.

Now we define the concept of generalized $\lambda_{\phi m}$ -MT-preinvex functions.

Definition 2.2 Let $K_{\phi m} \subseteq \mathbb{R}$ is a (ϕ, m) -invex set with respect to η and ϕ . A function $f : K_{\phi m} \rightarrow \mathbb{R}_0$ is said to be generalized $\lambda_{\phi m}$ -MT-preinvex, according to η and ϕ , and $\forall x, y \in K_{\phi m}$, $t \in (0, 1)$ and $\lambda \in (0, \frac{1}{2}]$, along with some fixed $m \in (0, 1]$ satisfies the coming inequality

$$f(mx + te^{i\phi}\eta(y, x, m)) \leq \frac{m(1 - \lambda)\sqrt{1 - t}}{2\lambda\sqrt{t}}f(x) + \frac{\sqrt{t}}{2\sqrt{1 - t}}f(y). \tag{2.1}$$

Let us note that:

- if the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$, then we obtain the definition of λ_{ϕ} -MT-preinvex function,
- if the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$ and $\lambda = \frac{1}{2}$, then we obtain the definition of ϕ -MT-preinvex function,
- if $\phi = 0$, the mapping $\eta(y, x, m) = y - mx$, and $\lambda = \frac{1}{2}$, then we obtain the definition of m -MT-convex function,
- if $\phi = 0$ and the mapping $\eta(y, x, m) = y - mx$ with $m = 1$, then we obtain the definition of λ -MT-convex function,
- if $\phi = 0$, the mapping $\eta(y, x, m) = y - mx$ with $m = 1$, and $\lambda = \frac{1}{2}$, then we obtain the definition of MT-convex function.

Before presenting our main results, we prove the following lemma.

Lemma 2.1 Let $K_{\phi m} \subseteq \mathbb{R}$ be a (ϕ, m) -invex subset respecting $\phi(\cdot)$ and $\eta : K_{\phi m} \times K_{\phi m} \times (0, 1] \rightarrow \mathbb{R}$, $a, b \in K_{\phi m}$ with $\eta(b, a, m) > 0$ and some fixed $m \in (0, 1]$. Suppose that $f : K_{\phi m} \rightarrow \mathbb{R}$ is a twice differentiable mapping such that $f'' \in L[ma, ma + e^{i\phi}\eta(b, a, m)]$, we have the following identity via k -fractional integral with $k, \alpha > 0$ holds:

$$R_f(\alpha, k; \phi, \eta, m, a, b) = \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \int_0^1 \frac{1 - t^{\frac{\alpha}{k}+1} - (1 - t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} f''(ma + te^{i\phi}\eta(b, a, m))dt, \tag{2.2}$$

where

$$R_f(\alpha, k; \phi, \eta, m, a, b) := \frac{f(ma) + f(ma + e^{i\phi}\eta(b, a, m))}{2} - \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a, m))^{\frac{\alpha}{k}}} \times \left[{}_k J_{ma+}^{\alpha} f(ma + e^{i\phi}\eta(b, a, m)) + {}_k J_{(ma + e^{i\phi}\eta(b, a, m))}^{\alpha} f(ma) \right].$$

Proof. Set

$$I^* = \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \int_0^1 \frac{1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} f''(ma + te^{i\phi}\eta(b, a, m)) dt.$$

Since $a, b \in K_{\phi m}$ and $K_{\phi m}$ is a (ϕ, m) -invex subset respecting ϕ and η , for $\forall t \in (0, 1)$, we have $ma + te^{i\phi}\eta(b, a, m) \in K_{\phi m}$. Integrating by part gives, we have

$$\begin{aligned} I^* &= \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \left[\frac{1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1}}{(\frac{\alpha}{k} + 1)e^{i\phi}\eta(b, a, m)} f'(ma + te^{i\phi}\eta(b, a, m)) \Big|_0^1 \right. \\ &\quad \left. - \int_0^1 \frac{-(\frac{\alpha}{k} + 1)t^{\frac{\alpha}{k}} + (\frac{\alpha}{k} + 1)(1-t)^{\frac{\alpha}{k}}}{(\frac{\alpha}{k} + 1)e^{i\phi}\eta(b, a, m)} f'(ma + te^{i\phi}\eta(b, a, m)) dt \right] \\ &= \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \left[\frac{t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}}{(e^{i\phi}\eta(b, a, m))^2} f(ma + te^{i\phi}\eta(b, a, m)) \Big|_0^1 \right. \\ &\quad \left. - \int_0^1 \frac{\frac{\alpha}{k}t^{\frac{\alpha}{k}-1} + \frac{\alpha}{k}(1-t)^{\frac{\alpha}{k}-1}}{(e^{i\phi}\eta(b, a, m))^2} f(ma + te^{i\phi}\eta(b, a, m)) dt \right] \\ &= \frac{f(ma) + f(ma + te^{i\phi}\eta(b, a, m))}{2} - \frac{\alpha}{2k} \left[\int_0^1 (t^{\frac{\alpha}{k}-1} + (1-t)^{\frac{\alpha}{k}-1}) f(ma + te^{i\phi}\eta(b, a, m)) dt \right]. \end{aligned}$$

Using the reduction formula $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$ ($\alpha > 0$), we have

$$\frac{\alpha}{2k} \int_0^1 t^{\frac{\alpha}{k}-1} f(ma + te^{i\phi}\eta(b, a, m)) dt = \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a, m))^{\frac{\alpha}{k}}} J_{(ma+e^{i\phi}\eta(b, a, m))^-}^\alpha f(ma)$$

and

$$\frac{\alpha}{2k} \int_0^1 (1-t)^{\frac{\alpha}{k}-1} f(ma + te^{i\phi}\eta(b, a, m)) dt = \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a, m))^{\frac{\alpha}{k}}} J_{ma^+}^\alpha f(ma + e^{i\phi}\eta(b, a, m)).$$

Thus, we obtain conclusion (2.2).

Remark 2.1 *If we put $k = 1$ in Lemma 2.1, then we have:*

- (a) *for the mapping $\eta(b, a, m)$ with $m = 1$ reduces to $\eta(b, a)$, we obtain Lemma 3.1 in [14],*
- (b) *for $\alpha = 1 = m$ with the mapping $\eta(b, a, m)$ reduces to $\eta(b, a)$, we obtain Lemma 2.3 in [42],*
- (c) *for $\phi = 0, \alpha = 1 = m$ with the mapping $\eta(b, a, m) = b - ma$, we obtain Lemma 1.3 in [37].*

3 Main results

Using Lemma 2.1, we now state the following theorem.

Theorem 3.1 *Let $A_{\phi m} \subseteq \mathbb{R}_0$ be an open (ϕ, m) -invex subset respecting $\phi(\cdot)$ and $\eta : A_{\phi m} \times A_{\phi m} \times (0, 1] \rightarrow \mathbb{R}_0$, $a, b \in A_{\phi m}$ with $\eta(b, a, m) > 0$, $\lambda \in (0, \frac{1}{2}]$ and some fixed $m \in (0, 1]$. If $f : A_{\phi m} \rightarrow \mathbb{R}$ is a twice differentiable mapping such that $f'' \in L[ma, ma + e^{i\phi}\eta(b, a, m)]$ and $|f''|^q$ for $q \geq 1$ is generalized $\lambda_{\phi m}$ -MT-preinvex on $A_{\phi m}$ and $x \in [ma, ma + e^{i\phi}\eta(b, a, m)]$, then we have the following inequality for k -fractional integrals with $k, \alpha > 0$*

$$\begin{aligned} & \left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \\ & \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(q(\frac{\alpha}{k} + 1) + \frac{1}{2})}{2\Gamma(q(\frac{\alpha}{k} + 1) + 1)} \right]^{\frac{1}{q}} \left\{ \frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

Proof. Using Lemma 2.1 and the power-mean integral inequality, we obtain

$$\begin{aligned}
 & \left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \\
 & \leq \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \int_0^1 \left| \frac{1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} \right| \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right| dt \\
 & \leq \frac{(e^{i\phi}\eta(b, a, m))^2}{2(\frac{\alpha}{k} + 1)} \left(\int_0^1 1 dt \right)^{1-\frac{1}{q}} \\
 & \quad \times \left\{ \int_0^1 \left(1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right)^q \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right|^q dt \right\}^{\frac{1}{q}} \\
 & \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left\{ \int_0^1 \left(1 - t^{q(\frac{\alpha}{k}+1)} - (1-t)^{q(\frac{\alpha}{k}+1)} \right) \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right|^q dt \right\}^{\frac{1}{q}}.
 \end{aligned}$$

To prove the third inequality above, we use the following inequality

$$\left(1 - (1-t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1} \right)^q \leq 1 - (1-t)^{q(\frac{\alpha}{k}+1)} - t^{q(\frac{\alpha}{k}+1)}, \tag{3.2}$$

for any $t \in (0, 1)$, which follows from

$$(A - B)^q \leq A^q - B^q,$$

for any $A > B \geq 0$ and $q \geq 1$.

Since $|f''|^q$ is generalized $\lambda_{\phi m}$ - MT -preinvex on $A_{\phi m}$, it follows that

$$\begin{aligned}
 & \int_0^1 \left(1 - t^{q(\frac{\alpha}{k}+1)} - (1-t)^{q(\frac{\alpha}{k}+1)} \right) \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right|^q dt \\
 & \leq \int_0^1 \left(1 - t^{q(\frac{\alpha}{k}+1)} - (1-t)^{q(\frac{\alpha}{k}+1)} \right) \left\{ \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}} |f''(b)|^q \right\} dt \\
 & = \frac{m(1-\lambda)}{\lambda} \left\{ \frac{\pi}{4} - \frac{1}{2}\beta\left(q\left(\frac{\alpha}{k} + 1\right) + \frac{1}{2}, \frac{3}{2}\right) - \frac{1}{2}\beta\left(\frac{1}{2}, q\left(\frac{\alpha}{k} + 1\right) + \frac{3}{2}\right) \right\} |f''(a)|^q \\
 & \quad + \left\{ \frac{\pi}{4} - \frac{1}{2}\beta\left(q\left(\frac{\alpha}{k} + 1\right) + \frac{3}{2}, \frac{1}{2}\right) - \frac{1}{2}\beta\left(\frac{3}{2}, q\left(\frac{\alpha}{k} + 1\right) + \frac{1}{2}\right) \right\} |f''(b)|^q \\
 & = \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma\left(q\left(\frac{\alpha}{k} + 1\right) + \frac{1}{2}\right)}{2\Gamma\left(q\left(\frac{\alpha}{k} + 1\right) + 1\right)} \right] \left\{ \frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right\}.
 \end{aligned}$$

Here, we utilize the following fact that

$$\begin{aligned}
 \int_0^1 \frac{\sqrt{1-t}}{2\sqrt{t}} dt &= \int_0^1 \frac{\sqrt{t}}{2\sqrt{1-t}} dt = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{4}, \\
 \int_0^1 t^{q(\frac{\alpha}{k}+1)+\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt &= \beta\left(q\left(\frac{\alpha}{k} + 1\right) + \frac{3}{2}, \frac{1}{2}\right)
 \end{aligned}$$

and

$$\int_0^1 (1-t)^{q(\frac{\alpha}{k}+1)-\frac{1}{2}} t^{\frac{1}{2}} dt = \beta\left(\frac{3}{2}, q\left(\frac{\alpha}{k} + 1\right) + \frac{1}{2}\right)$$

where the beta function

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x, y > 0.$$

Hence, the proof is completed.

We now discuss some special cases of Theorem 3.1.

Corollary 3.1 *In Theorem 3.1, if $q = 1$, then we have*

$$\left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{2\Gamma(\frac{\alpha}{k} + 2)} \right] \left\{ \frac{m(1-\lambda)}{\lambda} |f''(a)| + |f''(b)| \right\}.$$

Corollary 3.2 *In Theorem 3.1, if we take $\lambda = \frac{1}{2}$, $q = 1$ and the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then we have the following inequality for ϕ -MT-preinvex functions*

$$\left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a))^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^\alpha f(a + e^{i\phi}\eta(b, a)) + {}_k J_{(a+e^{i\phi}\eta(b, a))^-}^\alpha f(a) \right] \right| \leq \frac{k(e^{i\phi}\eta(b, a))^2}{2(\alpha + k)} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{2\Gamma(\frac{\alpha}{k} + 2)} \right] \left\{ |f''(a)| + |f''(b)| \right\}.$$

Remark 3.1 *In Corollary 3.2, if we put $\phi = 0$ and $\eta(b, a) = b - a$, then we have the succeeding inequality for MT-preinvex functions*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2k(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^\alpha f(b) + {}_k J_{b^-}^\alpha f(a) \right] \right| \leq \frac{k(b-a)^2}{2(\alpha + k)} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{2\Gamma(\frac{\alpha}{k} + 2)} \right] \left\{ |f''(a)| + |f''(b)| \right\}.$$

Especially if we take $k = 1$ and $\alpha = 1$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\pi(b-a)^2}{64} \left\{ |f''(a)| + |f''(b)| \right\}.$$

Corollary 3.3 *In Theorem 3.1, if $|f''(x)| \leq M$, $\lambda = \frac{1}{2}$ and $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then we have the forthcoming inequality for ϕ -MT-preinvex functions*

$$\left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a))^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^\alpha f(a + e^{i\phi}\eta(b, a)) + {}_k J_{(a+e^{i\phi}\eta(b, a))^-}^\alpha f(a) \right] \right| \leq \frac{kM(e^{i\phi}\eta(b, a))^2}{2(\alpha + k)} \left[\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(q(\frac{\alpha}{k} + 1) + \frac{1}{2})}{\Gamma(q(\frac{\alpha}{k} + 1) + 1)} \right]^{\frac{1}{q}}.$$

Especially if we take $\alpha = 1$, $q = 1$ and $k = 1$, we get

$$\left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{1}{e^{i\phi}\eta(b, a)} \int_a^{a+e^{i\phi}\eta(b, a)} f(x) dx \right| \leq \frac{M\pi(e^{i\phi}\eta(b, a))^2}{32}, \quad (3.3)$$

which is the result given in [42], Theorem 2.5. Obviously, if we choose $\phi = 0$ and $\eta(b, a) = b - a$ in (3.3), then we obtain the result given in [30], Theorem 2.1.

Now, we are ready to prove our second theorem.

Theorem 3.2 *Suppose that all the assumptions of Theorem 3.1 are satisfied, then we have the following inequality*

$$\begin{aligned} & \left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \\ & \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left(\frac{\alpha}{\alpha + 2k} \right)^{1-\frac{1}{q}} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{2\Gamma(\frac{\alpha}{k} + 2)} \right]^{\frac{1}{q}} \left[\frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}. \end{aligned} \tag{3.4}$$

Proof. Using Lemma 2.1 and the Hölder’s integral inequality for $q \geq 1$, we get

$$\begin{aligned} & \left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \\ & \leq \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \int_0^1 \left| \frac{1-t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} \right| \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right| dt \\ & \leq \frac{(e^{i\phi}\eta(b, a, m))^2}{2(\frac{\alpha}{k} + 1)} \left\{ \int_0^1 \left(1-t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right) dt \right\}^{1-\frac{1}{q}} \\ & \quad \times \left\{ \int_0^1 \left(1-t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right) \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right|^q dt \right\}^{\frac{1}{q}} \\ & = \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left(\frac{\alpha}{\alpha + 2k} \right)^{1-\frac{1}{q}} \left\{ \int_0^1 \left(1-t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right) \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right|^q dt \right\}^{\frac{1}{q}}. \end{aligned}$$

By the generalized $\lambda_{\phi m}$ -MT-preinvexity of $|f''|^q$ on $A_{\phi m}$ for $q \geq 1$, we have

$$\begin{aligned} & \int_0^1 \left(1-t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right) \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right|^q dt \\ & \leq \int_0^1 \left(1-t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right) \left(\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}} |f''(b)|^q \right) dt \\ & = \frac{m(1-\lambda)}{\lambda} \left[\frac{\pi}{4} - \frac{1}{2}\beta\left(\frac{\alpha}{k} + \frac{3}{2}, \frac{3}{2}\right) - \frac{1}{2}\beta\left(\frac{1}{2}, \frac{\alpha}{k} + \frac{5}{2}\right) \right] |f''(a)|^q \\ & \quad + \left[\frac{\pi}{4} - \frac{1}{2}\beta\left(\frac{\alpha}{k} + \frac{5}{2}, \frac{1}{2}\right) - \frac{1}{2}\beta\left(\frac{3}{2}, \frac{\alpha}{k} + \frac{3}{2}\right) \right] |f''(b)|^q \\ & = \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{2\Gamma(\frac{\alpha}{k} + 2)} \right] \left[\frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right]. \end{aligned}$$

Hence, the proof is completed.

Let us discuss some special cases of Theorem 3.2.

Corollary 3.4 *In Theorem 3.2, if the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then we obtain the following inequality for λ_{ϕ} -MT-preinvex functions*

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a))^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(a + e^{i\phi}\eta(b, a)) + {}_k J_{(a+e^{i\phi}\eta(b, a))^-}^{\alpha} f(a) \right] \right| \\ & \leq \frac{k(e^{i\phi}\eta(b, a))^2}{2(\alpha + k)} \left(\frac{\alpha}{\alpha + 2k} \right)^{1-\frac{1}{q}} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{2\Gamma(\frac{\alpha}{k} + 2)} \right]^{\frac{1}{q}} \left[\frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 3.5 *In Theorem 3.2, if $\phi = 0$, $\lambda = \frac{1}{2}$ and $\eta(b, a, m) = b - ma$ with $m = 1$, then we have the following inequality for MT-convex functions*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2k(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\ & \leq \frac{k(b-a)^2}{2(\alpha+k)} \left(\frac{\alpha}{\alpha+2k} \right)^{1-\frac{1}{q}} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{2\Gamma(\frac{\alpha}{k} + 2)} \right]^{\frac{1}{q}} \left[|f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 3.6 *In Theorem 3.2, if $|f''(x)| \leq M$, $\lambda = \frac{1}{2}$ and $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then we have the following inequality for ϕ -MT-preinvex functions*

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a))^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(a + e^{i\phi}\eta(b, a)) + {}_k J_{(a+e^{i\phi}\eta(b, a))^-}^{\alpha} f(a) \right] \right| \\ & \leq \frac{kM(e^{i\phi}\eta(b, a))^2}{2(\alpha+k)} \left(\frac{\alpha}{\alpha+2k} \right)^{1-\frac{1}{q}} \left[\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k} + \frac{3}{2})}{\Gamma(\frac{\alpha}{k} + 2)} \right]^{\frac{1}{q}}. \end{aligned}$$

Especially if we take $\alpha = 1 = k$, we get

$$\left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{1}{e^{i\phi}\eta(b, a)} \int_a^{a+e^{i\phi}\eta(b, a)} f(x)dx \right| \leq \frac{M(e^{i\phi}\eta(b, a))^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} \left(\frac{\pi}{16} \right)^{\frac{1}{q}}, \tag{3.5}$$

which is the result given in [42], Theorem 2.15. Clearly, if we put $\phi = 0$ and $\eta(b, a) = b - a$ in (3.5), we obtain the result given in [30], Theorem 2.4.

A different approach leads to the following results.

Theorem 3.3 *Let $A_{\phi m} \subseteq \mathbb{R}_0$ be an open (ϕ, m) -invex subset respecting $\phi(\cdot)$ and $\eta : A_{\phi m} \times A_{\phi m} \times (0, 1) \rightarrow \mathbb{R}_0$, $a, b \in A_{\phi m}$ with $\eta(b, a, m) > 0$, and let $f : A_{\phi m} \rightarrow \mathbb{R}$ be a twice differentiable mapping such that $f'' \in L[ma, ma + e^{i\phi}\eta(b, a, m)]$. If $|f''|^q$ is generalized $\lambda_{\phi m}$ -MT-preinvex on $A_{\phi m}$, $\lambda \in (0, \frac{1}{2}]$, $m \in (0, 1]$, $q = \frac{p}{p-1}$, $q \neq p > 1$ and $x \in [ma, ma + e^{i\phi}\eta(b, a, m)]$, then we have the following inequality for k -fractional integrals with $k, \alpha > 0$*

$$\left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha+k)} \left(\frac{p(\alpha+k) - k}{p(\alpha+k) + k} \right)^{\frac{1}{p}} \left[\frac{\pi}{4} \left(\frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right) \right]^{\frac{1}{q}}. \tag{3.6}$$

Proof. Using Lemma 2.1 and Hölder’s integral inequality leads to

$$\begin{aligned} & \left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \\ & \leq \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \int_0^1 \left| \frac{1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} \right| \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right| dt \\ & \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha+k)} \left\{ \int_0^1 \left(1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^1 \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right|^q dt \right\}^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left\{ \int_0^1 \left(1 - t^{p(\frac{\alpha}{k}+1)} - (1-t)^{p(\frac{\alpha}{k}+1)} \right) dt \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^1 \left(\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}} |f''(b)|^q \right) dt \right\}^{\frac{1}{q}} \\ &= \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left(\frac{p(\alpha + k) - k}{p(\alpha + k) + k} \right)^{\frac{1}{p}} \left[\frac{\pi}{4} \left(\frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right) \right]^{\frac{1}{q}}. \end{aligned}$$

To prove the third inequality above, we use the same inequality (3.2) as Theorem 3.1, the generalized $\lambda_{\phi m}$ -MT-preinvexity of $|f''|^q$ on $A_{\phi m}$ for $q > 1$, and the following fact

$$\int_0^1 \left(1 - (1-t)^{p(\frac{\alpha}{k}+1)} - t^{p(\frac{\alpha}{k}+1)} \right) dt = \frac{p(\alpha + k) - k}{p(\alpha + k) + k}.$$

This ends the proof of Theorem 3.3.

Let us point out some special cases of Theorem 3.3.

Corollary 3.7 *In Theorem 3.3, if the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$ and $\lambda = \frac{1}{2}$, then we have the following inequality for ϕ -MT-preinvex functions*

$$\begin{aligned} &\left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a))^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(a + e^{i\phi}\eta(b, a)) + {}_k J_{(a + e^{i\phi}\eta(b, a))^-}^{\alpha} f(a) \right] \right| \\ &\leq \frac{k(e^{i\phi}\eta(b, a))^2}{2(\alpha + k)} \left(\frac{p(\alpha + k) - k}{p(\alpha + k) + k} \right)^{\frac{1}{p}} \left[\frac{\pi}{4} \left(|f''(a)|^q + |f''(b)|^q \right) \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 3.8 *In Theorem 3.3, if we put $\phi = 0$ and $\eta(b, a, m) = b - ma$ with $m = 1$, then we have the following inequality for λ -MT-convex functions*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2k(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\ &\leq \frac{k(b-a)^2}{2(\alpha + k)} \left(\frac{p(\alpha + k) - k}{p(\alpha + k) + k} \right)^{\frac{1}{p}} \left[\frac{\pi}{4} \left(\frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right) \right]^{\frac{1}{q}}. \end{aligned}$$

Especially if we take $k = 1$ and $\lambda = \frac{1}{2}$, we have

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \right| \\ &\leq \frac{(b-a)^2}{2(\alpha + 1)} \left(\frac{p(\alpha + 1) - 1}{p(\alpha + 1) + 1} \right)^{\frac{1}{p}} \left[\frac{\pi}{4} \left(|f''(a)|^q + |f''(b)|^q \right) \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 3.9 *In Theorem 3.3, if $|f''(x)| \leq M$, $\phi = 0$, $\lambda = \frac{1}{2}$ and the mapping $\eta(b, a, m) = b - ma$ with $m = 1$, then we have the following inequality for MT-convex functions*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2k(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \leq \frac{kM(b-a)^2}{2(\alpha + k)} \left(\frac{\pi}{2} \right)^{\frac{1}{q}} \left(\frac{p(\alpha + k) - k}{p(\alpha + k) + k} \right)^{\frac{1}{p}}.$$

Especially if we take $k = 1$ and $\alpha = 1$, then we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{M(b-a)^2}{4} \left(\frac{\pi}{2}\right)^{\frac{1}{q}} \left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}.$$

Finally, we are in a position to present the following result.

Theorem 3.4 *Suppose that the assumptions of Theorem 3.3 are satisfied, then we have the following inequality*

$$\begin{aligned} & \left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \\ & \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left[\frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k} \right]^{\frac{q-1}{q}} \\ & \quad \times \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(p(\frac{\alpha}{k} + 1) + \frac{1}{2})}{2\Gamma(p(\frac{\alpha}{k} + 1) + 1)} \right]^{\frac{1}{q}} \left[\frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}. \end{aligned} \tag{3.7}$$

Proof. Using Lemma 2.1 and Hölder’s inequality, we have

$$\begin{aligned} & \left| R_f(\alpha, k; \phi, \eta, m, a, b) \right| \\ & \leq \frac{(e^{i\phi}\eta(b, a, m))^2}{2} \int_0^1 \left| \frac{1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} \right| \left| f''(ma + te^{i\phi}\eta(b, a, m)) \right| dt \\ & \leq \frac{(e^{i\phi}\eta(b, a, m))^2}{2(\frac{\alpha}{k} + 1)} \left[\int_0^1 \left(1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right)^{\frac{q-p}{q-1}} dt \right]^{\frac{q-1}{q}} \\ & \quad \times \left[\int_0^1 \left(1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1} \right)^p |f''(ma + te^{i\phi}\eta(b, a, m))|^q dt \right]^{\frac{1}{q}} \\ & \leq \frac{k(e^{i\phi}\eta(b, a, m))^2}{2(\alpha + k)} \left[\int_0^1 \left(1 - t^{\frac{(\alpha+1)(q-p)}{q-1}} - (1-t)^{\frac{(\alpha+1)(q-p)}{q-1}} \right) dt \right]^{\frac{q-1}{q}} \\ & \quad \times \left[\int_0^1 \left(1 - t^{p(\frac{\alpha}{k}+1)} - (1-t)^{p(\frac{\alpha}{k}+1)} \right) |f''(ma + te^{i\phi}\eta(b, a, m))|^q dt \right]^{\frac{1}{q}}. \end{aligned} \tag{3.8}$$

By the generalize $\lambda_{\phi m}$ -MT-preinvexity of $|f''|^q$ on $A_{\phi m}$ for $q > 1$, we have

$$\begin{aligned} & \int_0^1 \left(1 - t^{p(\frac{\alpha}{k}+1)} - (1-t)^{p(\frac{\alpha}{k}+1)} \right) |f''(ma + te^{i\phi}\eta(b, a, m))|^q dt \\ & \leq \int_0^1 \left(1 - t^{p(\frac{\alpha}{k}+1)} - (1-t)^{p(\frac{\alpha}{k}+1)} \right) \left(\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}} |f''(b)|^q \right) dt \\ & = \frac{m(1-\lambda)}{\lambda} \left[\frac{\pi}{4} - \frac{1}{2}\beta\left(p\left(\frac{\alpha}{k} + 1\right) + \frac{1}{2}, \frac{3}{2}\right) - \frac{1}{2}\beta\left(\frac{1}{2}, p\left(\frac{\alpha}{k} + 1\right) + \frac{3}{2}\right) \right] |f''(a)|^q \\ & \quad + \left[\frac{\pi}{4} - \frac{1}{2}\beta\left(p\left(\frac{\alpha}{k} + 1\right) + \frac{3}{2}, \frac{1}{2}\right) - \frac{1}{2}\beta\left(\frac{3}{2}, p\left(\frac{\alpha}{k} + 1\right) + \frac{1}{2}\right) \right] |f''(b)|^q \\ & = \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(p(\frac{\alpha}{k} + 1) + \frac{1}{2})}{2\Gamma(p(\frac{\alpha}{k} + 1) + 1)} \right] \left[\frac{m(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right]. \end{aligned} \tag{3.9}$$

Also

$$\int_0^1 \left(1 - t^{\frac{(\frac{\alpha}{k}+1)(q-p)}{q-1}} - (1-t)^{\frac{(\frac{\alpha}{k}+1)(q-p)}{q-1}} \right) dt = \frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k}. \tag{3.10}$$

Utilizing (3.9) and (3.10) in (3.8), we deduce the inequality (3.7). This completes the proof of Theorem 3.4 as well.

We next discuss some special cases of Theorem 3.4.

Corollary 3.10 *In Theorem 3.4, if the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then we obtain the following inequality for λ_ϕ -MT-preinvex functions*

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\phi}\eta(b, a))}{2} - \frac{\Gamma_k(\alpha + k)}{2k(e^{i\phi}\eta(b, a))^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^\alpha f(a + e^{i\phi}\eta(b, a)) + {}_k J_{(a+e^{i\phi}\eta(b, a))^-}^\alpha f(a) \right] \right| \\ & \leq \frac{k(e^{i\phi}\eta(b, a))^2}{2(\alpha + k)} \left[\frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k} \right]^{\frac{q-1}{q}} \\ & \quad \times \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(p(\frac{\alpha}{k} + 1) + \frac{1}{2})}{2\Gamma(p(\frac{\alpha}{k} + 1) + 1)} \right]^{\frac{1}{q}} \left[\frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 3.11 *In Theorem 3.4, if we put $\phi = 0$ and $\eta(b, a, m) = b - ma$ with $m = 1$, then we obtain the following inequality for λ -MT-convex functions*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2k(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^\alpha f(b) + {}_k J_{b^-}^\alpha f(a) \right] \right| \\ & \leq \frac{k(b-a)^2}{2(\alpha + k)} \left[\frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k} \right]^{\frac{q-1}{q}} \\ & \quad \times \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(p(\frac{\alpha}{k} + 1) + \frac{1}{2})}{2\Gamma(p(\frac{\alpha}{k} + 1) + 1)} \right]^{\frac{1}{q}} \left[\frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Especially if we take $k = 1$ and $\lambda = \frac{1}{2}$, then we have the following inequality for MT-convex functions

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \\ & \leq \frac{(b-a)^2}{2(\alpha + 1)} \left[\frac{(q-p)\alpha - p + 1}{(q-p)\alpha + 2q - p - 1} \right]^{\frac{q-1}{q}} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(p(\alpha + 1) + \frac{1}{2})}{2\Gamma(p(\alpha + 1) + 1)} \right]^{\frac{1}{q}} \left[|f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 3.12 *In Theorem 3.4, if $|f''(x)| \leq M$, $\phi = 0$, $\lambda = \frac{1}{2}$ and the mapping $\eta(b, a, m) = b - ma$ with $m = 1$, then we have the following inequality for MT-convex functions*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2k(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^\alpha f(b) + {}_k J_{b^-}^\alpha f(a) \right] \right| \\ & \leq \frac{kM(b-a)^2}{2(\alpha + k)} \left[\frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k} \right]^{\frac{q-1}{q}} \left[\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(p(\frac{\alpha}{k} + 1) + \frac{1}{2})}{\Gamma(p(\frac{\alpha}{k} + 1) + 1)} \right]^{\frac{1}{q}}. \end{aligned}$$

4 Applications to special means

We begin this section by considering some particular means for two positive real numbers a, b and for this purpose we recall the following well-known means:

$$\text{Arithmetic mean: } A := A(a, b) = \frac{a+b}{2},$$

$$\text{Geometric mean: } G := G(a, b) = \sqrt{ab},$$

$$\text{Harmonic mean: } H := H(a, b) = \frac{2ab}{a+b},$$

$$\text{Power mean: } P_r := P_r(a, b) = \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}, r \geq 1,$$

$$\text{Identric mean: } I(a, b) = \begin{cases} \frac{1}{e} \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a \neq b, \\ a, & a = b, \end{cases}$$

$$\text{Logarithmic mean: } L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases}$$

and

$$\text{Generalized mean: } L_p := L_p(a, b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq 0, -1, \text{ and } a \neq b, \\ L(a, b), & p = -1 \text{ and } a \neq b, \\ I(a, b), & p = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

Clearly, L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have $H \leq G \leq L \leq I \leq A$.

Let $0 < a < b$, $\lambda \in (0, \frac{1}{2}]$ and let $M := M(a, b) : [a + \eta(b, a)] \times [a, a + \eta(b, a)] \rightarrow \mathbb{R}_+$, which is one of the above mentioned means, one can obtain various inequalities for these means.

Now, if $\eta(b, a, m)$ with $m=1$ degenerates into $\eta(b, a)$ and $\eta(b, a) := M(b, a)$, for $\phi = 0$ in (3.1), (3.4), (3.6) and (3.7), we have the following interesting inequalities concerning the above means

$$\left| R_f(\alpha, k; 0, \eta, 1, a, b) \right| \leq \frac{kM^2}{2(\alpha + k)} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma\left(q\left(\frac{\alpha}{k} + 1\right) + \frac{1}{2}\right)}{2\Gamma\left(q\left(\frac{\alpha}{k} + 1\right) + 1\right)} \right]^{\frac{1}{q}} \left\{ \frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right\}^{\frac{1}{q}}, \tag{4.1}$$

$$\left| R_f(\alpha, k; 0, \eta, 1, a, b) \right| \leq \frac{kM^2}{2(\alpha + k)} \left(\frac{\alpha}{\alpha + 2k}\right)^{1-\frac{1}{q}} \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma\left(\frac{\alpha}{k} + \frac{3}{2}\right)}{2\Gamma\left(\frac{\alpha}{k} + 2\right)} \right]^{\frac{1}{q}} \left\{ \frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right\}^{\frac{1}{q}}, \tag{4.2}$$

$$\left| R_f(\alpha, k; 0, \eta, 1, a, b) \right| \leq \frac{kM^2}{2(\alpha + k)} \left(\frac{p(\alpha + k) - k}{p(\alpha + k) + k}\right)^{\frac{1}{p}} \left\{ \frac{\pi}{4} \left(\frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q\right) \right\}^{\frac{1}{q}} \tag{4.3}$$

and

$$\begin{aligned}
 |R_f(\alpha, k; 0, \eta, 1, a, b)| &\leq \frac{kM^2}{2(\alpha + k)} \left\{ \frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k} \right\}^{\frac{q-1}{q}} \\
 &\times \left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(p(\frac{\alpha}{k} + 1) + \frac{1}{2})}{2\Gamma(p(\frac{\alpha}{k} + 1) + 1)} \right]^{\frac{1}{q}} \left\{ \frac{(1-\lambda)}{\lambda} |f''(a)|^q + |f''(b)|^q \right\}^{\frac{1}{q}}, \tag{4.4}
 \end{aligned}$$

where

$$\begin{aligned}
 |R_f(\alpha, k; 0, \eta, 1, a, b)| &= \frac{f(a) + f(a + M(a, b))}{2} - \frac{\Gamma_k(\alpha + k)}{2kM^{\frac{\alpha}{k}}(a, b)} \\
 &\times [{}_k J_{a^+}^\alpha f(a + M(b, a)) + {}_k J_{(a+M(b,a))^-}^\alpha f(a)].
 \end{aligned}$$

Letting $M = A, G, H, P_r, I, L, L_p$ in (4.1), (4.2), (4.3) and (4.4), we also get the required inequalities, and the more details are left to the reader to explore.

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Some generalizations of operator inequalities for positive linear map

Chaojun Yang and Fangyan Lu*

Abstract:

We generalize some inequalities for positive unital linear map as follows: Let A, B be positive operators on a Hilbert space with $0 < m \leq A \leq m' < M' \leq B \leq M$. Then for every positive unital linear map Φ , $\mu \in [0, 1]$ and $p > 0$,

$$\Phi^p(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp_{\mu}B^{-1})) \leq \alpha^p\Phi^p(A\sharp_{\mu}B)$$

and

$$\Phi^p(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp_{\mu}B^{-1})) \leq \alpha^p(\Phi(A)\sharp_{\mu}\Phi(B))^p$$

where $r = \min\{\mu, 1 - \mu\}$, $h' = \frac{M'}{m'}$ and $\alpha = \max\left\{\frac{(M+m)^2}{4MmK(\sqrt{h'}, 2)^R}, \frac{(M+m)^2}{4^{\frac{2}{p}}MmK(\sqrt{h'}, 2)^R}\right\}$. Furthermore, we give a squaring reversed Karcher mean inequality involving positive linear map.

1. Introduction

Through this paper, let m, m', M, M' be scalars. Other capital letters denote general elements of the C^* -algebra $B(\mathcal{H})$ of all bounded linear operators on a complex separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The Kantorovich constant is defined by $K(h, 2) = \frac{(h+1)^2}{4h}$ for $h > 0$. We write $A \geq 0$ ($A > 0$) to mean the self-adjoint operator A is positive (strictly positive). The partial order $A \leq B$ is defined as $B - A \geq 0$.

For each $\mu \in [0, 1]$, the weighted arithmetic mean ∇_{μ} and weight geometric mean \sharp_{μ} for strictly positive operator A and B are defined below:

$$A\nabla_{\mu}B = (1 - \mu)A + \mu B \quad \text{and} \quad A\sharp_{\mu}B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\mu}A^{\frac{1}{2}}.$$

When $\mu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

A linear map $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is called positive (strictly positive) if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$.

The arithmetic-geometric mean for positive operator A and B states

$$\frac{A+B}{2} \geq A\sharp B.$$

In [8], Lin give a reversed arithmetic-geometric mean inequality involving a positive linear map

$$\Phi\left(\frac{A+B}{2}\right) \leq \frac{(M+m)^2}{4Mm}\Phi(A\sharp B) \tag{1}$$

where $0 < m \leq A, B \leq M$ and Φ is a positive unital linear map.

* Corresponding author.

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It is well known that t^α is operator monotone function on $[0, \infty)$ if and only if $\alpha \in [0, 1]$. Since t^2 is not an operator monotone function, we can not obtain $A^2 \geq B^2$ directly by $A \geq B \geq 0$. However Lin [8] gave a p -th powering ($0 < p \leq 2$) of inequality (1), namely the inequality

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{p}{2}} Mm}\right)^p \Phi^p(A\sharp B) \tag{2}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{p}{2}} Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p \tag{3}$$

where $0 < m \leq A, B \leq M$ and Φ is a positive unital linear map.

In [6], the authors extend (2) and (3) to $p > 2$, which states

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{p}{2}} Mm}\right)^p \Phi^p(A\sharp B) \tag{4}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{p}{2}} Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p \tag{5}$$

where $0 < m \leq A, B \leq M$ and Φ is a positive unital linear map.

Recently the author in [4] gives inequalities that generalize the inequalities (2) to (5) and state as follows

$$\Phi^p(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \alpha^p \Phi^p(A\sharp_\mu B) \tag{6}$$

and

$$\Phi^p(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \alpha^p (\Phi(A)\sharp_\mu\Phi(B))^p \tag{7}$$

where $0 < m \leq A \leq m' < M' \leq B \leq M$, Φ be a positive unital linear map on $B(\mathcal{H})$, $\mu \in [0, 1]$, $p > 0$, $r = \min\{\mu, 1 - \mu\}$ and $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{p}{2}} Mm}\right\}$.

The ω -weighted Karcher mean $\Lambda(\omega; A_1, \dots, A_n)$ (or $\Lambda(\omega; \mathbb{A})$) of $A_1, \dots, A_n > 0$ is defined to be the unique positive definite solution of equation

$$\sum_{i=1}^n \omega_i \log(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}}) = 0,$$

where $\omega = (w_1, \dots, w_n)$ is a probability vector. Next we cite some basic properties of the Karcher mean as follows, for more details about Karcher mean, see[7].

Proposition 1.1. [7] The Karcher mean satisfies the following properties:

- (i) (consistency with scalars) $\Lambda(\omega; \mathbb{A}) = A_1^{\omega_1} \cdots A_n^{\omega_n}$ if the A_i is commute.
- (ii) (self duality) $\Lambda(\omega; A_1^{-1}, \dots, A_n^{-1})^{-1} = \Lambda(\omega; A_1, \dots, A_n)$.
- (iii) (AGH weighted mean inequalities) $(\sum_{i=1}^n \omega_i A_i^{-1})^{-1} \leq \Lambda(\omega; A_1, \dots, A_n) \leq \sum_{i=1}^n \omega_i A_i$.
- (iv) $\Phi(\Lambda(\omega; \mathbb{A})) \leq \Lambda(\omega; \Phi(\mathbb{A}))$ for any positive unital linear map Φ .
- (v) (monotonicity) If $B_i \leq A_i$ for all $1 \leq i \leq n$, then $\Lambda(\omega; \mathbb{B}) \leq \Lambda(\omega; \mathbb{A})$.

As mentioned in the abstract, we shall give refinements of inequalities (6) and (7), along with presenting a reversed Karcher mean inequality related to (iv) in Proposition 1.1 and a squaring version thereafter.

2. Main Results

Lemma 2.1. (Choi inequality.) [5] Let Φ be a unital positive linear map, then
 (i) when $A > 0$ and $-1 \leq p \leq 0$, then $\Phi(A)^p \leq \Phi(A^p)$, in particular, $\Phi(A)^{-1} \leq \Phi(A^{-1})$;
 (ii) when $A \geq 0$ and $0 \leq p \leq 1$, then $\Phi(A)^p \geq \Phi(A^p)$;
 (iii) when $A \geq 0$ and $1 \leq p \leq 2$, then $\Phi(A)^p \leq \Phi(A^p)$.

Lemma 2.2. [1] Let Φ be a unital positive linear map and A, B be positive operators. Then for $\alpha \in [0, 1]$

$$\Phi(A\sharp_{\alpha}B) \leq \Phi(A)\sharp_{\alpha}\Phi(B).$$

Lemma 2.3. [3] Let $A, B \geq 0$. Then the following norm inequality holds:

$$\|AB\| \leq \frac{1}{4}\|A + B\|^2.$$

Lemma 2.4. [2] Let $A, B \geq 0$. Then for $1 \leq r < +\infty$,

$$\|A^r + B^r\| \leq \|(A + B)^r\|.$$

Lemma 2.5. [5] (L-H inequality) If $0 \leq \alpha \leq 1$, $A, B \geq 0$ and $A \geq B$, then $A^{\alpha} \geq B^{\alpha}$.

Lemma 2.6. [9] For two operators $A, B \geq 0$ and $1 < h \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h'$ or $0 < h' \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h < 1$, we have

$$A\nabla_{\mu}B - 2r(A\nabla B - A\sharp B) \geq K(\sqrt{h}, 2)^R A\sharp_{\mu}B$$

for all $\mu \in [0, 1]$, where $r = \min\{\mu, 1 - \mu\}$ and $R = \min\{2r, 1 - 2r\}$.

Lemma 2.7. Let $0 < m \leq A \leq m' < M' \leq B \leq M$, then

$$A^{-1}\nabla_{\mu}B^{-1} - 2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) \geq K(\sqrt{h'}, 2)^R A^{-1}\sharp_{\mu}B^{-1}$$

for all $\mu \in [0, 1]$, where $r = \min\{\mu, 1 - \mu\}$, $h' = \frac{M'}{m'}$ and $R = \min\{2r, 1 - 2r\}$.

Proof. Since $0 < m \leq A \leq m' < M' \leq B \leq M$, we have $0 < \frac{m}{M} \leq (A^{-1})^{-\frac{1}{2}}(B^{-1})(A^{-1})^{-\frac{1}{2}} \leq \frac{m'}{M'} < 1$. Thus by Lemma 2.6 we have

$$A^{-1}\nabla_{\mu}B^{-1} - 2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) \geq K(\sqrt{h'}, 2)^R A^{-1}\sharp_{\mu}B^{-1}$$

where $K(\sqrt{h'}, 2) = K(\sqrt{\frac{1}{h'}}, 2)$.

Theorem 2.8. Let $0 < m \leq A \leq m' < M' \leq B \leq M$, Φ be a positive unital linear map on $B(\mathcal{H})$, $\mu \in [0, 1]$ and $p > 0$, we have

$$\Phi^p(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \alpha^p \Phi^p(A\sharp_{\mu}B) \tag{8}$$

and

$$\Phi^p(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \alpha^p (\Phi(A)\sharp_{\mu}\Phi(B))^p \tag{9}$$

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where $r = \min \{ \mu, 1 - \mu \}$, $h' = \frac{M'}{m'}$ and $\alpha = \max \left\{ \frac{(M+m)^2}{4MmK(\sqrt{h'}, 2)^R}, \frac{(M+m)^2}{4^{\frac{2}{p}} MmK(\sqrt{h'}, 2)^R} \right\}$.

Proof. By $m \leq A \leq m' < M' \leq B \leq M$ we have

$$A + MmA^{-1} \leq M + m \quad \text{and} \quad B + MmB^{-1} \leq M + m.$$

Thus we have

$$(1 - \mu)A + (1 - \mu)MmA^{-1} \leq (1 - \mu)(M + m) \quad \text{and} \quad \mu B + \mu MmB^{-1} \leq \mu(M + m). \quad (10)$$

By (10) we obtain

$$A\nabla_{\mu}B + MmA^{-1}\nabla_{\mu}B^{-1} \leq M + m. \quad (11)$$

By Lemma 2.7 we have

$$A^{-1}\nabla_{\mu}B^{-1} - 2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) \geq K(\sqrt{h'}, 2)^R A^{-1}\sharp_{\mu}B^{-1} \quad (12)$$

Compute

$$\begin{aligned} & \| \Phi(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))K(\sqrt{h'}, 2)^R Mm\Phi^{-1}(A\sharp_{\mu}B) \| \\ & \leq \frac{1}{4} \| \Phi(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R Mm\Phi^{-1}(A\sharp_{\mu}B) \|^2 \\ & \leq \frac{1}{4} \| \Phi(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R \Phi((A\sharp_{\mu}B)^{-1}) \|^2 \\ & = \frac{1}{4} \| \Phi(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R \Phi(A^{-1}\sharp_{\mu}B^{-1}) \|^2 \\ & = \frac{1}{4} \| \Phi(A\nabla_{\mu}B) + Mm\Phi(2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) + K(\sqrt{h'}, 2)^R(A^{-1}\sharp_{\mu}B^{-1})) \|^2 \\ & \leq \frac{1}{4} \| \Phi(A\nabla_{\mu}B) + Mm\Phi(A^{-1}\nabla_{\mu}B^{-1}) \|^2, \\ & \leq \frac{1}{4} (M + m)^2 \end{aligned}$$

where the first inequality is derived by Lemma 2.3, the second one is derived by Lemma 2.1, the third one is derived by (12) and the last one is derived by (11).

Therefore

$$\| \Phi(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))\Phi^{-1}(A\sharp_{\mu}B) \| \leq \frac{(M+m)^2}{4MmK(\sqrt{h'}, 2)^R}.$$

Hence

$$\Phi^2(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \left(\frac{(M+m)^2}{4MmK(\sqrt{h'}, 2)^R} \right)^2 \Phi^2(A\sharp_{\mu}B).$$

If $0 < p \leq 2$, then $0 < \frac{p}{2} \leq 1$. Therefore by the L-H inequality we get

$$\Phi^p(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \left(\frac{(M+m)^2}{4MmK(\sqrt{h'}, 2)^R} \right)^p \Phi^p(A\sharp_{\mu}B).$$

Now we obtain inequality (8) for $0 < p \leq 2$.

Next we prove (9) for $0 < p \leq 2$. Through

$$\begin{aligned}
 & \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))K(\sqrt{h'}, 2)^R Mm(\Phi(A)\sharp_\mu\Phi(B))^{-1} \| \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R Mm(\Phi(A)\sharp_\mu\Phi(B))^{-1} \|^2 \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R \Phi(A\sharp_\mu B)^{-1} \|^2 \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R \Phi(A^{-1}\sharp_\mu B^{-1}) \|^2 \\
 & = \frac{1}{4} \| \Phi(A\nabla_\mu B) + Mm\Phi(2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) + K(\sqrt{h'}, 2)^R(A^{-1}\sharp_\mu B^{-1})) \|^2 \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B) + Mm\Phi(A^{-1}\nabla_\mu B^{-1}) \|^2, \\
 & \leq \frac{1}{4}(M + m)^2
 \end{aligned}$$

where the second inequality is obtained by Lemma 2.2. Hence we get inequality (9) for $0 < p \leq 2$.

Next, put $p > 2$. We can obtain

$$\begin{aligned}
 & (K(\sqrt{h'}, 2)^R Mm)^{\frac{p}{2}} \| \Phi^{\frac{p}{2}}(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))\Phi^{-\frac{p}{2}}(A\sharp_\mu B) \| \\
 & = \| \Phi^{\frac{p}{2}}(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))(K(\sqrt{h'}, 2)^R Mm)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\mu B) \| \\
 & \leq \frac{1}{4} \| \Phi^{\frac{p}{2}}(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + (K(\sqrt{h'}, 2)^R Mm)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\mu B) \|^2 \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R Mm\Phi^{-1}(A\sharp_\mu B) \|^p \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R Mm\Phi(A^{-1}\sharp_\mu B^{-1}) \|^p \\
 & = \frac{1}{4} \| \Phi(A\nabla_\mu B) + Mm\Phi(2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) + K(\sqrt{h'}, 2)^R(A^{-1}\sharp_\mu B^{-1})) \|^p \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B) + Mm\Phi(A^{-1}\nabla_\mu B^{-1}) \|^p, \\
 & \leq \frac{1}{4}(M + m)^p
 \end{aligned}$$

where the second inequality is obtained by Lemma 2.4.

Therefore, we get inequality (8) for $p > 2$. Likewise, we have

$$\begin{aligned}
 & (K(\sqrt{h'}, 2)^R Mm)^{\frac{p}{2}} \| \Phi^{\frac{p}{2}}(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))(\Phi(A)\sharp_\mu\Phi(B))^{-\frac{p}{2}} \| \\
 & = \| \Phi^{\frac{p}{2}}(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))(K(\sqrt{h'}, 2)^R Mm)^{\frac{p}{2}}(\Phi(A)\sharp_\mu\Phi(B))^{-\frac{p}{2}} \| \\
 & \leq \frac{1}{4} \| \Phi^{\frac{p}{2}}(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + (K(\sqrt{h'}, 2)^R Mm)^{\frac{p}{2}}(\Phi(A)\sharp_\mu\Phi(B))^{-\frac{p}{2}} \|^2 \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R Mm(\Phi(A)\sharp_\mu\Phi(B))^{-1} \|^p \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R Mm\Phi(A\sharp_\mu B)^{-1} \|^p \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R Mm\Phi((A\sharp_\mu B)^{-1}) \|^p \\
 & = \frac{1}{4} \| \Phi(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + K(\sqrt{h'}, 2)^R \Phi(A^{-1}\sharp_\mu B^{-1}) \|^p \\
 & = \frac{1}{4} \| \Phi(A\nabla_\mu B) + Mm\Phi(2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) + K(\sqrt{h'}, 2)^R(A^{-1}\sharp_\mu B^{-1})) \|^p \\
 & \leq \frac{1}{4} \| \Phi(A\nabla_\mu B) + Mm\Phi(A^{-1}\nabla_\mu B^{-1}) \|^p, \\
 & \leq \frac{1}{4}(M + m)^p.
 \end{aligned}$$

Remark 2.9. Since $\frac{(M+m)^2}{4MmK(\sqrt{h'}, 2)^R} \leq \frac{(M+m)^2}{4Mm}$ and $\frac{(M+m)^2}{4^{\frac{p}{2}}MmK(\sqrt{h'}, 2)^R} \leq \frac{(M+m)^2}{4^{\frac{p}{2}}Mm}$, so under a stronger condition as Theorem 2.8, we see (8) and (9) are refinements of (6) and (7), respectively.

Corollary 2.10. Let $0 < m \leq A \leq m' < M' \leq B \leq M$, $\mu \in [0, 1]$ and $p > 0$, we have

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$$(A\nabla_{\mu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\#B^{-1}))^p \leq \alpha^p(A\#_{\mu}B)^p$$

where $r = \min\{\mu, 1 - \mu\}$, $h' = \frac{M'}{m'}$ and $\alpha = \max\left\{\frac{(M+m)^2}{4MmK(\sqrt{h'}, 2)^R}, \frac{(M+m)^2}{4^{\frac{2}{p}}MmK(\sqrt{h'}, 2)^R}\right\}$.

Proof. Put $\Phi(A) = A$ for all $A \in B(\mathcal{H})$ in Theorem 2.3, we then get the desired result.

Theorem 2.11. Let Φ be a strictly unital positive linear map, $0 < m \leq A_i \leq M$ for $i = 1, \dots, n$, $\omega = (\omega_1, \dots, \omega_n)$ be a probability vector, $t \in [-1, 0)$. Then we have

$$\Lambda(\omega; \Phi(\mathbb{A})) \leq \frac{(m+M)^2}{4mM} \Phi(\Lambda(\omega; \mathbb{A})). \tag{13}$$

Proof. By Proposition 1.1 and $0 < m \leq A_i \leq M$ we have

$$\sum_{i=1}^n \omega_i A_i + Mm \left(\sum_{i=1}^n \omega_i A_i^{-1}\right) \leq M + m.$$

First we show

$$\left(\sum_{i=1}^n \omega_i A_i\right)^2 \leq \left(\frac{(m+M)^2}{4mM}\right)^2 \left(\sum_{i=1}^n \omega_i A_i^{-1}\right)^{-2}.$$

This inequality equals to

$$\left\| \sum_{i=1}^n \omega_i A_i \sum_{i=1}^n \omega_i A_i^{-1} \right\| \leq \frac{(M+m)^2}{4Mm}.$$

Note that

$$\begin{aligned} & \left\| \left(\sum_{i=1}^n \omega_i A_i\right) Mm \left(\sum_{i=1}^n \omega_i A_i^{-1}\right) \right\| \\ & \leq \frac{1}{4} \left\| \sum_{i=1}^n \omega_i A_i + Mm \left(\sum_{i=1}^n \omega_i A_i^{-1}\right) \right\|^2 \\ & \leq \frac{1}{4} (M+m)^2. \end{aligned}$$

Use Lemma 2.5 we get

$$\sum_{i=1}^n \omega_i A_i \leq \frac{(m+M)^2}{4mM} \left(\sum_{i=1}^n \omega_i A_i^{-1}\right)^{-1}. \tag{14}$$

Thus by Proposition 1.1 and (14) we get

$$\Lambda(\omega; \Phi(\mathbb{A})) \leq \sum_{i=1}^n \omega_i \Phi(A_i) = \Phi\left(\sum_{i=1}^n \omega_i A_i\right) \leq \frac{(m+M)^2}{4mM} \Phi\left(\left(\sum_{i=1}^n \omega_i A_i^{-1}\right)^{-1}\right) \leq \frac{(m+M)^2}{4mM} \Phi(\Lambda(\omega; \mathbb{A})).$$

Remark 2.12. Since $\Phi(\Lambda(\omega; \mathbb{A})) \leq \Lambda(\omega; \Phi(\mathbb{A}))$ for any positive unital linear map, we get a reversed version of this inequality by Theorem 2.11.

Next we give a squaring version of inequality (13).

Theorem 2.13. Suppose all the assumptions of Theorem 2.11 be satisfied. Then

$$(\Lambda(\omega; \Phi(\mathbb{A})))^2 \leq \psi \Phi^2(\Lambda(\omega; \mathbb{A}))$$

where $\psi = \begin{cases} \frac{K(\frac{M}{m}, 2)^2(M+m)^2}{4Mm} & \text{for } m \leq t_0 \\ \frac{K(\frac{M}{m}, 2)(M+m)-M}{m} & \text{for } m \geq t_0 \end{cases}$, $t_0 = \frac{2Mm}{K(\frac{M}{m}, 2)(M+m)}$ and $K(\frac{M}{m}, 2) = \frac{(M+m)^2}{4Mm}$.

Proof. According to the assumption one can see that

$$m \leq \Phi(\Lambda(\omega; \mathbb{A})) \leq M \tag{15}$$

and

$$m \leq \Lambda(\omega; \Phi(\mathbb{A})) \leq M \tag{16}$$

inequality (15) implies

$$\Phi^2(\Lambda(\omega; \mathbb{A})) \leq (M + m)\Phi(\Lambda(\omega; \mathbb{A})) - Mm,$$

and inequality (16) give us

$$\Lambda^2(\omega; \Phi(\mathbb{A})) \leq (M + m)\Lambda(\omega; \Phi(\mathbb{A})) - Mm.$$

Hence

$$\begin{aligned} & \Phi^{-1}(\Lambda(\omega; \mathbb{A}))\Lambda^2(\omega; \Phi(\mathbb{A}))\Phi^{-1}(\Lambda(\omega; \mathbb{A})) \\ & \leq \Phi^{-1}(\Lambda(\omega; \mathbb{A}))((M + m)\Lambda(\omega; \Phi(\mathbb{A})) - Mm)\Phi^{-1}(\Lambda(\omega; \mathbb{A})) \\ & \leq (K(\frac{M}{m}, 2)(M + m)\Phi(\Lambda(\omega; \mathbb{A})) - Mm)\Phi^{-2}(\Lambda(\omega; \mathbb{A})) \end{aligned} \tag{17}$$

where the second inequality is derived by Theorem 2.11.

Consider the real function $f(t)$ on $(0, \infty)$ defined as

$$f(t) = \frac{K(\frac{M}{m}, 2)(M+m)t - Mm}{t^2}.$$

As a matter of fact, the inequality (17) implies that

$$\Phi^{-1}(\Lambda(\omega; \mathbb{A}))\Lambda^2(\omega; \Phi(\mathbb{A}))\Phi^{-1}(\Lambda(\omega; \mathbb{A})) \leq \max_{m \leq t \leq M} f(t).$$

Notice that

$$f(m) \geq f(M)$$

and

$$f'(t) = \frac{2Mm - K(\frac{M}{m}, 2)(M+m)t}{t^3}.$$

The function has an maximum point on

$$t_0 = \frac{2Mm}{K(\frac{M}{m}, 2)(M+m)}$$

with the maximum value

$$f(t_0) = \frac{K(\frac{M}{m}, 2)^2(M+m)^2}{4Mm}.$$

Whence

$$\max_{m \leq t \leq M} f(t) = \begin{cases} f(t_0) & \text{for } m \leq t_0 \\ f(m) & \text{for } m \geq t_0. \end{cases}$$

Notice that

$$f(m) = \frac{K(\frac{M}{m}, 2)(M+m) - M}{m}.$$

This completes the proof.

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Chaojun Yang

Department of Mathematics, Soochow University, Suzhou 215006, P. R. China

E-mail address: cjyangmath@163.com

Fangyan Lu

Department of Mathematics, Soochow University, Suzhou 215006, P. R. China

E-mail address: fylu@suda.edu.cn

Locally and globally small Riemann sums and Henstock integral of fuzzy-number-valued functions in E^n

Muawya Elsheikh Hamid^{a,b*}, Luoshan Xu ^a

^a School of Mathematical Science, Yangzhou University, Yangzhou 225002, P.R. China

^b School of Management, Ahfad University for Women, Omdurman, Sudan

Abstract: In this paper, the notions of locally and globally small Riemann sums modifications with respect to a fuzzy-number-valued functions in E^n are introduced and studied. The basic properties and characterizations are presented. In particular, it is proved that a fuzzy-number-valued functions in E^n is Henstock (H) integrable on $[a, b]$ if and only if it has ($LSRS$), and also it is proved that a fuzzy-number-valued functions in E^n is Henstock (H) integrable on $[a, b]$ if and only if it has ($GSRS$).

Keywords: Fuzzy-number-valued functions in E^n ; Henstock integral (H); Locally small Riemann sums ($LSRS$); Globally small Riemann sums ($GSRS$).

1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [12], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on.

The locally and globally small Riemann sums have been introduced by many authors from different points of views including [2, 3, 5, 6, 8, 9]. In 1986, Schurle characterized the Lebesgue integral in ($LSRS$) (locally small Riemann sums) property [8]. The ($LSRS$) property has been used to characterized the Perron (P) integral on $[a, b]$ [9]. By considering the equivalency between the (P) integral and the Henstock-Kurzweil (HK) integral, the ($LSRS$) property has been used to characterized the (HK) integral on $[a, b]$ [6].

The ($LSRS$) property brought a research to have global characterization on the Riemann sums of an (HK) integrable function on $[a, b]$. This research has been done by considering the following fact: Every (HK) integrable function on $[a, b]$ is measurable, however, there is no guarantee the boundedness of the function. A measurable function f is (HK) integrable on $[a, b]$ depends on it behaves on the set of x in which $|f(x)|$ is large, i.e. $|f(x)| \geq N$ for some N . This fact has been characterized in ($GSRS$) (globally small Riemann sums) property [6]. The ($GSRS$) property involves one characteristic of the primitive of an (HK) integrable function. That is the primitive of the (HK) integral on $[a, b]$ is ACG^* (generalized strongly absolutely continuous) on $[a, b]$. This is not a simple concept. In 2015, Indrati [5] introduced a countably Lipschitz condition of a function which is simpler than the ACG^* , and proved that the (HK) integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the (HK) integral in the ($GSRS$) property, it showed that the relationship between ($GSRS$) property and countably Lipschitz condition of an (HK) integrable function on $[a, b]$.

In 2018, Hamid et al. [2] investigated locally and globally small Riemann sums for fuzzy-number-valued functions and proved two main theorems: (1) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ if and only if $\tilde{f}(x)$ has ($LSRS$). (2) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ if and only if $\tilde{f}(x)$ has ($GSRS$).

*Corresponding author. Tel.: +8613218977118. E-mail address: muawya.ebrahim@gmail.com, mowia-84@hotmail.com (M.E. Hamid).

In this paper, we generalize locally and globally small Riemann sums from fuzzy-valued functions to n -dimensional fuzzy-numbers by means of support function. The notions of locally small Riemann sums for n -dimensional fuzzy-number-valued functions are presented and discussed. Finally, we provide a characterizations of globally small Riemann sums in n -dimensional fuzzy-number-valued functions.

The rest of this paper is organized as follows, in Section 2 we shall review the relevant concepts and properties of fuzzy-number-valued functions in E^n and the definition of Henstock integrals for fuzzy-number-valued functions in E^n . Section 3 is devoted to discussing the support function characterizations of locally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n . In section 4 we shall investigate the support function characterizations of globally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n . The last section provides the Conclusions.

2 Preliminaries

In this paper the close interval $[a, b]$ denotes a compact interval on \mathbb{R} . The set of intervals-point $\{([a_1, b_1], \xi_1), ([a_2, b_2], \xi_2), \dots, ([a_k, b_k], \xi_k)\}$ is called a division of $[a, b]$ that is $\xi_1, \xi_2, \dots, \xi_k \in [a, b]$, intervals $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$ are non-intersect and $\bigcup_{i=1}^k [a_i, b_i] = [a, b]$. Marking the division of $[a, b]$ as $P = \{([a_1, b_1], \xi_1), ([a_2, b_2], \xi_2), \dots, ([a_k, b_k], \xi_k)\}$, shortening as $P = \{[u, v]; \xi\}$ [7].

Definition 2.1 [4, 6] Let $\delta : [a, b] \rightarrow \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a δ -fine division, if the following conditions are satisfied:

- (1) $a = x_0 < x_1 < x_2 < \dots < x_n = b$;
- (2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) (i = 1, 2, \dots, n)$.

For brevity, we write $P = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in P and ξ is the associated point of $[u, v]$.

Definition 2.2 [11] E^n is said to be a fuzzy number space if $E^n = \{u : R^n \rightarrow [0, 1] : u \text{ satisfies (1)-(4) below}\}$:

- (1) u is normal, i.e., there exists a $x_0 \in R^n$ such that $u(x_0) = 1$;
- (2) u is a convex fuzzy set, i.e., $u(rx + (1 - r)y) \geq \min(u(x), u(y))$, $x, y \in R^n$, $r \in [0, 1]$;
- (3) u is upper semi-continuous;
- (4) $[u]^0 = \{x \in R^n : u(x) > 0\}$ is compact, for $0 < r \leq 1$, denote $[u]^r = \{x : x \in R^n \text{ and } u(x) \geq r\}$, $[u]^0 = \bigcup_{r \in (0, 1]} [u]^r$.

Form (1)-(4), it follows that for any $u \in E^n$ and $r \in [0, 1]$ the r -level set $[u]^r$ is a compact convex set. For any $u, v \in E^n$

$$D(u, v) = \sup_{r \in [0, 1]} d([u]^r, [v]^r), \quad (2.1)$$

where d is Hausdorff metric. It is well known that (E^n, d) is an metric space [11]. The norm of fuzzy number $u \in E^n$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{\alpha \in [u]^0} |\alpha|, \quad (2.2)$$

where the $\|\cdot\|$ is norm on E^n , $\tilde{0}$ is fuzzy number on E^n and $\tilde{0} = \chi_{\{0\}}$.

Definition 2.3 [11] For $A \in P_k(R^n)$, $x \in S^{n-1}$, define the support function of A as $\sigma(x, A) = \sup_{y \in A} \langle y, x \rangle$, where S^{n-1} is the unit sphere of R^n , i.e., $S^{n-1} = \{x \in R^n : \|x\| = 1\}$, $\langle \cdot, \cdot \rangle$ is the inner product in R^n .

Definition 2.4 [10] A fuzzy-number-valued function $\tilde{f} : [a, b] \rightarrow E^n$ is said to be Henstock integrable to $\tilde{A} \in E^n$ if for every $\varepsilon > 0$, there is a function $\delta(t) > 0$ such that for any δ -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$D\left(\sum \tilde{f}(\xi)(v - u), \tilde{A}\right) < \varepsilon, \quad (2.3)$$

where the sum \sum is understood to be over P and we write $(FH) \int_a^b \tilde{f}(t) dt = \tilde{A}$, and $\tilde{f}(t) \in FH[a, b]$.

Lemma 2.1 [11] If $u, v \in E^n$, $k \in R$, for any $r \in [0, 1]$, we have

$$[u + v]^r = [u]^r + [v]^r, [ku]^r = k[u]^r. \quad (2.4)$$

Lemma 2.2 [11] Suppose $u \in E^n$, then

(1) $u^*(r, x + y) \leq u^*(r, x) + u^*(r, y)$,

(2) if $u, v \in E^n$, $r \in [0, 1]$, then

$$d([u]^r, [v]^r) = \sup_{x \in S^{n-1}} |u^*(r, x) - v^*(r, x)|, \quad (2.5)$$

(3) $(u + v)^*(r, x) = u^*(r, x) + v^*(r, x)$,

(4) $(ku)^*(r, x) = ku^*(r, x)$, $k \geq 0$.

Lemma 2.3 [1, 11] Given $u, v \in E^n$ the distance $D : E^n \times E^n \rightarrow [0, +\infty)$ between u and v is defined by the equation

$$D(u, v) = \sup_{r \in [0, 1]} d([u]^r, [v]^r),$$
 then

- (1) (E^n, D) is a complete metric space,
- (2) $D(u + w, v + w) = D(u, v)$,
- (3) $D(u + v, w + e) \leq D(u, w) + D(v, e)$,
- (4) $D(ku, kv) = |k|D(u, v)$, $k \in R$,
- (5) $D(u + v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0})$,
- (6) $D(u + v, w) \leq D(u, w) + D(v, \tilde{0})$.

Where $u, v, w, e, \tilde{0} \in E^n$, $\tilde{0} = \mathcal{X}_{\{0\}}$.

Lemma 2.4 [1] If $\tilde{f} : [a, b] \rightarrow E^n$, then the following statements are equivalent:

(1) \tilde{f} is (FH) integrable.

(2) $f^*(\xi)(r, x)$ is (RH) integrable for any $r \in [0, 1]$ uniformly, i.e., for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ which is independent of $r \in [0, 1]$, such that for any δ -fine division $P = \{[u, v]; \xi\}$ and $r \in [0, 1]$ we have

$$|\sum f^*(\xi)(r, x)(v - u) - A^*(r, x)| < \varepsilon. \quad (2.6)$$

3 Support function characterizations of locally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n

In this section, we define the locally small Riemann sums for fuzzy-number-valued functions in n -dimensional and investigate their properties. We start with the following definition.

Definition 3.1 A fuzzy-number-valued function $\tilde{f} : [a, b] \rightarrow E^n$ is said to be have locally small Riemann sums or $(LSRS)$ if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a, b]$, we have

$$|| \sum \tilde{f}(\xi)(v - u) ||_{E^n} < \varepsilon, \quad (3.1)$$

whenever $P = \{[u, v]; \xi\}$ is a δ -fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and Σ sums over P . (Where $C = [y, z]$).

The following Theorem 3.1 shows that \tilde{f} has $(LSRS)$ is equal to the type of its support functions.

Theorem 3.1 Let $\tilde{f} : [a, b] \rightarrow E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r, x)$ of \tilde{f} has locally small Riemann sums or $(LSRS)$ if and only if for every $\varepsilon > 0$, there is a $\delta(\xi) > 0$ such that for every $t \in [a, b]$, we have

$$|\sum f^*(\xi)(r, x)(v - u)| < \varepsilon, \quad (3.2)$$

uniformly for any $r \in [0, 1]$ and $x \in S^{n-1}$, whenever $P = \{[u, v]; \xi\}$ is a δ -fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and Σ sums over P .

Proof Let $\tilde{0} \in E^n$ denote the (FH) integral of \tilde{f} on $[a, b]$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$D(\sum \tilde{f}(\xi)(v - u), \tilde{0}) < \varepsilon. \quad (3.3)$$

That is

$$\sup_{r \in [0, 1]} d([\sum \tilde{f}(\xi)(v - u)]^r, [\tilde{0}]^r) < \varepsilon. \quad (3.4)$$

By Lemma 2.2 we have

$$\sup_{r \in [0, 1]} \sup_{x \in S^{n-1}} |(\sum \tilde{f}(\xi)(v - u))^*(r, x) - \sigma(x, 0)| < \varepsilon. \quad (3.5)$$

Furthermore, by $\sigma(x, A) = \sup_{y \in A} \langle y, x \rangle$, we have

$$\sup_{r \in [0, 1]} \sup_{x \in S^{n-1}} |\sum f^*(\xi)(r, x)(v - u) - \sigma(x, 0)| < \varepsilon. \quad (3.6)$$

Hence, for any $r \in [0, 1]$, $x \in S^{n-1}$ and for any δ -fine division P we have

$$|\sum f^*(\xi)(r, x)(v - u)| < \varepsilon. \quad (3.7)$$

Where $\sigma(x, 0) = 0$.

This completes the proof. \square

Lemma 3.1 (Henstock Lemma). Let $\tilde{f} : [a, b] \rightarrow E^n$ be a fuzzy-number-valued function and Henstock integrable to \tilde{A} . Then, the support-function-wise $f^*(\xi)(r, x)$ of \tilde{f} on $[a, b]$ is Henstock integrable to $A^*(r, x)$ uniformly for any $r \in [0, 1]$, $x \in S^{n-1}$ and $\tilde{A} \in E^n$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, for δ -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$ and for any $x \in S^{n-1}$, we have

$$|\sum f^*(\xi)(r, x)(v - u) - A^*(r, x)| < \varepsilon. \quad (3.8)$$

Furthermore, for any sum of parts \sum_1 from \sum we have

$$|\sum_1 f^*(\xi)(r, x)(v - u) - A^*(r, x)| < \varepsilon. \quad (3.9)$$

Proof Let $\tilde{A} \in E^n$ denote the (FH) integral of \tilde{f} on $[a, b]$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$D(\sum \tilde{f}(\xi)(v - u), \tilde{A}) < \varepsilon. \quad (3.10)$$

That is

$$\sup_{r \in [0, 1]} d([\sum \tilde{f}(\xi)(v - u)]^r, [\tilde{A}]^r) < \varepsilon. \quad (3.11)$$

By Lemma 2.2 we have

$$\sup_{r \in [0, 1]} \sup_{x \in S^{n-1}} |(\sum \tilde{f}(\xi)(v - u))^*(r, x) - A^*(r, x)| < \varepsilon. \quad (3.12)$$

Furthermore, by $A^*(r, x) = \sup_{y \in [A]^r} \langle y, x \rangle$, we have

$$\sup_{r \in [0, 1]} \sup_{x \in S^{n-1}} |\sum f^*(\xi)(r, x)(v - u) - A^*(r, x)| < \varepsilon. \quad (3.13)$$

Hence, for any $r \in [0, 1]$, $x \in S^{n-1}$ and for any δ -fine division P we have

$$|\sum f^*(\xi)(r, x)(v - u) - A^*(r, x)| < \varepsilon.$$

For proof

$$|\sum_1 f^*(\xi)(r, x)(v - u) - A^*(r, x)| < \varepsilon, \quad (3.14)$$

the proof is similar to the Theorem 3.7 in [6].

This completes the proof. \square

Hamid et al. [2] showed that if a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ then $\tilde{f}(x)$ has LSRS. In next Theorem, we prove the above result to n -dimensional fuzzy-number-valued functions, which is an extension of the above result of Muawya et al. [2].

Theorem 3.2 Let $\tilde{f} : [a, b] \rightarrow E^n$ be a fuzzy-number-valued function. If \tilde{f} is Henstock integrable to $\tilde{F}([a, b])$, then \tilde{f} has *LSRS*.

Proof Since \tilde{f} is Henstock integrable to $\tilde{F}([a, b])$, by Theorem 3.1 the support-function-wise $f^*(\xi)(r, x)$ of \tilde{f} on $[a, b]$ is Henstock integrable to $F^*([a, b])(r, x)$ uniformly for any $r \in [0, 1]$, $x \in S^{n-1}$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, for δ -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$ and for any $x \in S^{n-1}$, we have

$$\left| \sum f^*(\xi)(r, x)(v - u) - F^*([a, b])(r, x) \right| < \frac{\varepsilon}{2}. \quad (3.15)$$

For each $t \in [a, b]$, there is a closed interval $C = [y, z] \subset (t - \delta(t), t + \delta(t))$ such that

$$|F^*([y, z])(r, x)| < \frac{\varepsilon}{2}. \quad (3.16)$$

According to Henstock lemma, for each $t \in [a, b]$ and δ -fine division $P = \{[u, v]; \xi\}$ of $C \subset (t - \delta(t), t + \delta(t))$, we have

$$\begin{aligned} \left| \sum f^*(\xi)(r, x)(v - u) \right| &\leq \left| \sum f^*(\xi)(r, x)(v - u) - F^*([a, b])(r, x) \right| + |F^*([y, z])(r, x)| \\ &< \varepsilon. \end{aligned}$$

Applies Theorem 3.1 again \tilde{f} has *LSRS*.

This completes the proof. □

Lemma 3.2 Let $\tilde{f} : [a, b] \rightarrow E^n$ be a fuzzy-number-valued function. If \tilde{f} is *(FH)* integrable with the \tilde{F} as primitive then for each number $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, such that for any $[u, v] \subset [a, b]$ with $v - u < \delta(\xi)$, we have

$$\|\tilde{F}([u, v])\|_{E^n} = \|(FH) \int_{[u, v]} \tilde{f} dx\|_{E^n} < \varepsilon. \quad (3.17)$$

Proof The continuity follows from Lemma 3.1 and the following inequality:

$$\begin{aligned} \|\tilde{F}(t) - \tilde{F}(\xi)\|_{E^n} &\leq \|\tilde{F}(t) - \tilde{F}(\xi) - \tilde{f}(\xi)(t - \xi)\|_{E^n} + \|\tilde{f}(\xi)(t - \xi)\|_{E^n} \\ &< \varepsilon. \end{aligned}$$

We only need set $\delta(\xi) < \frac{\varepsilon}{2(\|\tilde{f}(\xi)\|_{E^n} + 1)}$.

This completes the proof. □

Theorem 3.3 Let a fuzzy-number-valued function $\tilde{f} : [a, b] \rightarrow E^n$ has *LSRS*, then \tilde{f} is *(FH)* integrable on $[a, b]$.

Proof Given any $\varepsilon > 0$ and $P = \{([a, b], \xi)\} = \{([a_1, b_1], \xi_1), ([a_2, b_2], \xi_2), \dots, ([a_n, b_n], \xi_n)\}$ is a δ -fine partition of $[a, b]$. For each $i(i = 1, 2, \dots, n)$ there is a positive function δ_i with $P_i = \{([u_i, v_i], \xi_i)\}$ is a δ_i -fine partition of $[a_i, b_i]$. Since \tilde{f} has *LSRS* on $[a_i, b_i]$, then we have

$$\left\| \sum_{P_i} \tilde{f}(\xi)(v - u) \right\|_{E^n} < \frac{\varepsilon}{2n}. \quad (3.18)$$

Taken $\eta = \max\{\delta(\xi), \xi \in [a, b]\}$, according to the Lemma 3.2 we have

$$\|\tilde{F}([a_i, b_i])\|_{E^n} = \|(FH) \int_{[a_i, b_i]} \tilde{f} dx\|_{E^n} < \frac{\varepsilon}{2n}. \quad (3.19)$$

Therefore, for any δ_i -fine partition $P_i = \{([u_i, v_i], \xi_i)\}$ of $[a_i, b_i]$, we have

$$\begin{aligned} D\left(\sum_{P_i} \tilde{f}(\xi)(v - u), \tilde{F}([a_i, b_i])\right) &\leq \left\| \sum_{P_i} \tilde{f}(\xi)(v - u) \right\|_{E^n} + \|\tilde{F}([a_i, b_i])\|_{E^n} \\ &< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n}, \end{aligned}$$

for each i .

Subsequently taken $\delta^*(\xi) = \min\{\delta(\xi), \delta_i(\xi)\}$, then $P = \bigcup_{i=1}^n P_i$ denote δ^* -fine partition of $[a, b]$.

Therefore we have

$$\begin{aligned} D\left(\sum_P \tilde{f}(\xi)(v - u), \tilde{F}([a, b])\right) &= \sum_{i=1}^n D\left(\sum_{P_i} \tilde{f}(\xi)(v - u), \tilde{F}([a_i, b_i])\right) \\ &< n \cdot \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

Then \tilde{f} is *FH* integral on $[a, b]$.

This completes the proof. □

4 Support function characterizations of globally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n

The main purpose in this part is to introduce the concept of globally small Riemann sums for fuzzy-number-valued functions in n -dimensional and discuss their properties. We begin with the following definition.

Definition 4.1 A fuzzy-number-valued function $\tilde{f} : [a, b] \rightarrow E^n$ is said to be have globally small Riemann sums or (*GSRS*) if for every $\varepsilon > 0$ there exists a positive integer N such that for every $n \geq N$ there is a $\delta_n(\xi) > 0$ and for every δ_n -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$\left\| \sum_{\|\tilde{f}(\xi)\|_{E^n} > n} \tilde{f}(\xi)(v - u) \right\|_{E^n} < \varepsilon, \quad (4.1)$$

where the \sum is taken over P and for which $\|\tilde{f}(\xi)\|_{E^n} > n$.

The following Theorem 4.1 shows that \tilde{f} has (*GSRS*) is equal to the type of its support functions.

Theorem 4.1 Let $\tilde{f} : [a, b] \rightarrow E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r, x)$ of \tilde{f} has globally small Riemann sums or (*GSRS*) if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geq N$ there is a $\delta_n(\xi) > 0$ and for every δ_n -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$\left| \sum_{|f^*(\xi)(r, x)| > n} f^*(\xi)(r, x)(v - u) \right| < \varepsilon, \quad (4.2)$$

uniformly for any $r \in [0, 1]$ and $x \in S^{n-1}$, where the \sum is taken over P and for which $|f^*(\xi)(r, x)| > n$.

Proof First, we can prove the following statements are equivalent:

- (1) $\|\tilde{f}(\xi)\|_{E^n} > n$.
- (2) $|f^*(\xi)(r, x)| > n$.

In fact

$$\begin{aligned} \|\tilde{f}(\xi)\|_{E^n} > n &= \sup_{r \in [0, 1]} d([\tilde{f}(\xi)]^r, [\tilde{0}]^r) \\ &= \sup_{r \in [0, 1]} \sup_{x \in S^{n-1}} |f^*(\xi)(r, x)|. \end{aligned}$$

Second, let $\tilde{0} \in E^n$ denote the (*FH*) integral of \tilde{f} on $[a, b]$. Given $\varepsilon > 0$ there exists a positive integer N such that for every $n \geq N$ there is a $\delta_n(\xi) > 0$ and for every δ_n -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$D\left(\sum_{\|\tilde{f}(\xi)\|_{E^n} > n} \tilde{f}(\xi)(v - u), \tilde{0}\right) < \varepsilon. \quad (4.3)$$

That is

$$\sup_{r \in [0, 1]} d\left(\sum_{\|\tilde{f}_r(\xi)\|_{E^n} > n} \tilde{f}(\xi)(v - u)^r, [\tilde{0}]^r\right) < \varepsilon. \quad (4.4)$$

By Lemma 2.2 we have

$$\sup_{r \in [0, 1]} \sup_{x \in S^{n-1}} \left| \sum_{|f^*(\xi)(r, x)| > n} f(\xi)(v - u)^*(r, x) - \sigma(x, 0) \right| < \varepsilon. \quad (4.5)$$

Furthermore, by $\sigma(x, A) = \sup_{y \in A} \langle y, x \rangle$, we have

$$\sup_{r \in [0, 1]} \sup_{x \in S^{n-1}} \left| \sum_{|f^*(\xi)(r, x)| > n} f^*(\xi)(r, x)(v - u) - \sigma(x, 0) \right| < \varepsilon. \quad (4.6)$$

Hence, for any $r \in [0, 1]$, $x \in S^{n-1}$ and for any δ -fine division P we have

$$\left| \sum_{|f^*(\xi)(r, x)| > n} f^*(\xi)(r, x)(v - u) \right| < \varepsilon. \quad (4.7)$$

Where $\sigma(x, 0) = 0$.

This completes the proof. □

Hamid et al. [2] investigated that a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ if and only if $\tilde{f}(x)$ has *GSRS*. In next Theorem 4.3, we extend this result to n -dimensional fuzzy-number-valued functions. To prove this result, we need to prove the following Theorem.

Theorem 4.2 Let $\tilde{f} : [a, b] \rightarrow E^n$ be a fuzzy-number-valued function. If \tilde{f} has *GSRS* then \tilde{f} is Henstock integrable on $[a, b]$.

Proof Because \tilde{f} has *GSRS*, then by Theorem 4.1 for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geq N$ there is a $\delta_n(\xi) > 0$ and for every δ_n -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$\left| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)(v-u) \right| < \varepsilon. \tag{4.8}$$

uniformly for any $r \in [0, 1]$ and $x \in S^{n-1}$, where the \sum is taken over P and for which $|f^*(\xi)(r,x)| > n$.

For each two δ -fine divisions $P_1 = \{[u_1, v_1]; \xi_1\}$, $P_2 = \{[u_2, v_2]; \xi_2\}$ of $[a, b]$, we have

$$\begin{aligned} & \left| \sum f^*(\xi_1)(r,x)(v_1-u_1) - \sum f^*(\xi_2)(r,x)(v_2-u_2) \right| \\ & \leq \left| \sum f^*(\xi_1)(r,x)(v_1-u_1) \right| + \left| \sum f^*(\xi_2)(r,x)(v_2-u_2) \right| \\ & \leq \left| \sum_{|f^*(\xi_1)(r,x)| > n} f^*(\xi_1)(r,x)(v_1-u_1) \right| + \left| \sum_{|f^*(\xi_1)(r,x)| \leq n} f^*(\xi_1)(r,x)(v_1-u_1) \right| \\ & + \left| \sum_{|f^*(\xi_2)(r,x)| > n} f^*(\xi_2)(r,x)(v_2-u_2) \right| + \left| \sum_{|f^*(\xi_2)(r,x)| \leq n} f^*(\xi_2)(r,x)(v_2-u_2) \right| \\ & < 4\varepsilon. \end{aligned}$$

According to the properties of Cauchy, \tilde{f} is Henstock integrable on $[a, b]$.

This completes the proof. □

Theorem 4.3 Given a fuzzy-number-valued function $\tilde{f} : [a, b] \rightarrow E^n$, for each $r \in [0, 1]$ and $x \in S^{n-1}$ defined the support function $f_n^*(\xi)(r, x)$ of \tilde{f}_n by the formula:

$$f_n^*(\xi)(r, x) = \begin{cases} f^*(\xi)(r, x), \xi \in [a, b] & \text{if } |f^*(\xi)(r, x)| \leq n, \\ 0, & \text{others.} \end{cases}$$

A fuzzy-number-valued function \tilde{f} is Henstock integrable if and only if \tilde{f} has *GSRS* and $\tilde{F}_n([a, b]) \rightarrow \tilde{F}([a, b])$ as $n \rightarrow \infty$. (Where $\tilde{F}([a, b])$ and $\tilde{F}_n([a, b])$ the integral of \tilde{f} and \tilde{f}_n respectively).

Proof First we shall prove the necessity. Because a fuzzy-number-valued function \tilde{f} is Henstock integrable on $[a, b]$ uniformly for any $r \in [0, 1]$ and $x \in S^{n-1}$, i.e., for every $\varepsilon > 0$ there is a positive function δ^* , for δ^* -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$\left| \sum f^*(\xi)(r, x)(v-u) - F^*([a, b])(r, x) \right| < \frac{\varepsilon}{3}. \tag{4.9}$$

For each $n \in \mathbb{N}$, there is a positive function δ_n , for δ_n -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$\left| \sum f_n^*(\xi)(r, x)(v-u) - F_n^*([a, b])(r, x) \right| < \frac{\varepsilon}{3}, \tag{4.10}$$

for each $r \in [0, 1]$ and $x \in S^{n-1}$.

Because $\{F_n^*([a, b])(r, x)\}$ converge to $F^*([a, b])(r, x)$ of $[a, b]$ then there is a positive number N so if $n \geq N$ we have

$$|F_n^*([a, b])(r, x) - F^*([a, b])(r, x)| < \frac{\varepsilon}{3}. \tag{4.11}$$

For $n \geq N$, defined a positive function δ on $[a, b]$ by the formula:

$$\delta(\xi) = \min\{\delta^*(\xi), \delta_n(\xi)\}. \tag{4.12}$$

Therefore, for each δ -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$\begin{aligned} & \left| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)(v-u) \right| \\ &= \left| \sum f^*(\xi)(r,x)(v-u) - \sum f_n^*(\xi)(r,x)(v-u) \right| \\ &\leq \left| \sum f^*(\xi)(r,x)(v-u) - F^*([a,b])(r,x) \right| + \left| F_n^*([a,b])(r,x) - F^*([a,b])(r,x) \right| \\ &+ \left| F^*([a,b])(r,x) - \sum f_n^*(\xi)(r,x)(v-u) \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then \tilde{f} has *GSRS*.

Second we shall prove the sufficiency. Because \tilde{f} has *GSRS*, then by Theorem 4.1 for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geq N$ there is a $\delta_n(\xi) > 0$ and for every δ_n -fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$\left| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)(v-u) \right| < \varepsilon, \tag{4.13}$$

uniformly for any $r \in [0, 1]$ and $x \in S^{n-1}$, where the \sum is taken over P and for which $|f^*(\xi)(r,x)| > n$.

Note that \tilde{f}_n is Henstock integrable on $[a, b]$ for all n . Choose N so that whenever $n, m \geq N$ we have

$$\left| F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x) \right| < \varepsilon. \tag{4.14}$$

Then for $n, m \geq N$ and a suitably chosen δ -fine division $P = \{[u, v]; \xi\}$, we have

$$\begin{aligned} & \left| F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x) \right| \\ &\leq \left| F_n^*([a,b])(r,x) - \sum_{|f^*(\xi)(r,x)| \leq n} f^*(\xi)(r,x)(v-u) \right| + \left| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)(v-u) \right| \\ &+ \left| \sum_{|f^*(\xi)(r,x)| \leq m} f^*(\xi)(r,x)(v-u) - F_m^*([a,b])(r,x) \right| + \left| \sum_{|f^*(\xi)(r,x)| > m} f^*(\xi)(r,x)(v-u) \right| \\ &< 4\varepsilon. \end{aligned}$$

That is, $\{F_n^*([a,b])(r,x)\}$ converge to $F^*([a,b])(r,x)$, as $n \rightarrow \infty$. Again, for suitably chosen N and $\delta(\xi)$ and for every δ -fine division $P = \{[u, v]; \xi\}$, we have

$$\begin{aligned} & \left| \sum f^*(\xi)(r,x)(v-u) - F^*([a,b])(r,x) \right| \\ &\leq \left| \sum f^*(\xi)(r,x)(v-u) - F_N^*([a,b])(r,x) \right| + \left| F_N^*([a,b])(r,x) - F^*([a,b])(r,x) \right| \\ &\leq \left| \sum_{|f^*(\xi)(r,x)| \leq N} f^*(\xi)(r,x)(v-u) - F_N^*([a,b])(r,x) \right| + \left| \sum_{|f^*(\xi)(r,x)| > N} f^*(\xi)(r,x)(v-u) \right| \\ &+ \left| F_N^*([a,b])(r,x) - F^*([a,b])(r,x) \right| \\ &< 3\varepsilon. \end{aligned}$$

That is, \tilde{f} is Henstock integrable on $[a, b]$.

This completes the proof. □

5 conclusions

This paper introduces, first of all, the generalization of locally and globally small Riemann sums from fuzzy-valued functions to n -dimensional fuzzy-numbers by means of support function. In addition, the concept of locally small Riemann sums for n -dimensional fuzzy-number-valued functions is presented and discussed. Finally, an important result of this paper is a characterizations of globally small Riemann sums for n -dimensional fuzzy-number-valued functions.

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On systems of fractional Langevin equations of Riemann-Liouville type with generalized nonlocal fractional integral boundary conditions

Chatthai Thaiprayoon^{a,*}, Sotiris K. Ntouyas^{b,c} and Jessada Tariboon^{d,e}

^a Department of Mathematics, Faculty of Science, Burapha University, Chonburi, 20131, Thailand
E-mail: chatthai@buu.ac.th

^b Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail: sntouyas@uoi.gr

^c Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

^d Nonlinear Dynamic Analysis Research Center,
Department of Mathematics, Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok,
Bangkok 10800, Thailand

^e Centre of Excellence in Mathematics,
CHE, Sri Ayutthaya Rd., Bangkok 10400, Thailand
E-mail: jessada.t@sci.kmutnb.ac.th

Abstract

By applying Krasnoselskii's and O'Regan's fixed point theorems, in this paper, we study the existence of solutions for a coupled system consisting from Langevin fractional differential equations of Riemann-Liouville type subject to the generalized nonlocal integral boundary conditions. Examples illustrating our results are also presented.

Key words and phrases: Fractional differential equations, Krasnoselskii's fixed point theorem, O'Regan's fixed point theorem, generalized fractional integral.

AMS (MOS) Subject Classifications: 26A33; 34A08.

1 Introduction

In this paper we concentrate on the study of existence of solutions for a coupled system of Langevin fractional differential equations of Riemann-Liouville type subject to the generalized nonlocal integral boundary conditions of the form

$$\left\{ \begin{array}{l} D^{p_1} (D^{p_2} + \lambda_1) x(t) = f(t, x(t), y(t)), \quad 0 < t < T, \\ D^{q_1} (D^{q_2} + \lambda_2) y(t) = g(t, x(t), y(t)), \quad 0 < t < T, \\ x(0) = 0, \quad x(\eta) = \sum_{i=1}^n \alpha_i {}^{\mu_i} I^{\gamma_i} x(\xi_i), \\ y(0) = 0, \quad y(\kappa) = \sum_{j=1}^m \beta_j {}^{\delta_j} I^{\phi_j} y(\zeta_j), \end{array} \right. \quad (1)$$

where D^χ is the Riemann-Liouville fractional derivative of order $\chi \in \{p_1, p_2, q_1, q_2\}$, ${}^{\mu_i} I^{\gamma_i}$, ${}^{\delta_j} I^{\phi_j}$ are the Katugampola fractional integrals of orders $\gamma_i, \phi_j > 0$, respectively, $\xi_i, \zeta_j \in (0, T)$ and $\alpha_i, \beta_j \in \mathbb{R}$ for

*Corresponding author

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all $i = 1, 2, \dots, n, j = 1, 2, \dots, m, f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and λ_1, λ_2 are given constants.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, control theory, biology, economics, etc. A comprehensive study of fractional calculus and its applications is introduced in several books (see [1]-[3]). Initial and boundary value problems of nonlinear fractional differential equations and inclusions have been addressed by several researchers. For some recent results on fractional differential equations we refer in a series of papers ([4]-[12]).

In fractional calculus, the fractional derivatives are defined via fractional integrals. There are several known forms of the fractional integrals which have been studied extensively for their applications. Two of the most known fractional integrals are the Riemann-Liouville and the Hadamard fractional integral. A new fractional integral, called *generalized Riemann-Liouville fractional integral*, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, was introduced in [13]. The corresponding fractional derivatives were introduced in [14]. This integral is now known as "Katugampola fractional integral" see for example [15, pp 15, 123]. For some recent work with this new operator, for example, see [16]-[17] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [18]. For some new developments on the fractional Langevin equation in physics, see, for example, [19]-[23]. For recent results on Langevin equations with different kinds of boundary conditions we refer to [24]-[28] and the references therein.

Recently in [16], we have studied the existence and the uniqueness of solutions of a class of boundary value problems for fractional Langevin equations of Riemann-Liouville type with generalized nonlocal integral boundary conditions. Here we extend the results of [16], to a coupled system of Langevin fractional differential equations of Riemann-Liouville type subject to the generalized nonlocal integral boundary conditions. Usually in the literature the Banach's contraction mapping principle is used to prove the existence and the uniqueness of solutions, and the existence of solutions is proved via Leray-Schauder alternative. Here we apply Krasnoselskii's and O'Regan's fixed point theorems. To the best of our knowledge this is the first paper using Krasnoselskii's and O'Regan's fixed point theorems to prove the existence of solutions for coupled systems.

The paper is organized as follows: In Section 2 we will present some useful preliminaries and some auxiliary lemmas. In Section 3, we establish the main existence results by using Krasnoselskii's and O'Regan's fixed point theorems. Examples illustrating our results are presented in the final Section 4.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later.

Definition 2.1 [2] *The Riemann-Liouville fractional integral of order $p > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined by*

$$J^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the gamma function defined by $\Gamma(p) = \int_0^\infty e^{-s} s^{p-1} ds$.

Definition 2.2 [2] *The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined by*

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-p-1} f(s) ds, \quad n-1 \leq p < n,$$

where $n = [p]+1$, $[p]$ denotes the integer part of a real number p , provided the right-hand side is point-wise defined on $(0, \infty)$.

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Lemma 2.3 [2] Let $p > 0$ and $x \in C(0, T) \cap L(0, T)$. Then the fractional differential equation $D^p x(t) = 0$ has a unique solution $x(t) = \sum_{i=1}^n c_i t^{p-i}$, and the following formula holds: $J^p D^p x(t) = x(t) + \sum_{i=1}^n c_i t^{p-i}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, and $n - 1 \leq p < n$.

Lemma 2.4 ([2], page 71) Let $\alpha > 0$, $\beta > 0$ and $a \geq 0$. Then the following properties hold:

$$J^\alpha(x - a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)}(t - a)^{\beta+\alpha-1}$$

Definition 2.5 [14] The generalized (Katugampola) fractional integral of order $q > 0$ and $\rho > 0$, of a function f , for all $0 < t < \infty$, is defined as

$${}^\rho I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-q}} ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.6 [16] Let constants $\rho, q > 0$ and $p > 0$. Then the following formula holds

$${}^\rho I^q t^p = \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)} \frac{t^{p+\rho q}}{\rho^q}. \tag{2}$$

For convenience to prove our results, we set constants

$$\Omega_1 = \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \eta^{p_1+p_2-1}, \tag{3}$$

$$\Omega_2 = \sum_{i=1}^n \frac{\alpha_i \Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{\Gamma\left(\frac{p_1+p_2+\mu_i-1}{\mu_i}\right)}{\Gamma\left(\frac{p_1+p_2+\mu_i \gamma_i+\mu_i-1}{\mu_i}\right)} \frac{\xi_i^{p_1+p_2+\mu_i \gamma_i-1}}{\mu_i^{\gamma_i}}, \tag{4}$$

$$\Omega = \Omega_2 - \Omega_1 \neq 0, \tag{5}$$

and

$$\Psi_1 = \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \kappa^{q_1+q_2-1}, \tag{6}$$

$$\Psi_2 = \sum_{j=1}^m \frac{\beta_j \Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{\Gamma\left(\frac{q_1+q_2+\delta_j-1}{\delta_j}\right)}{\Gamma\left(\frac{q_1+q_2+\delta_j \phi_j+\delta_j-1}{\delta_j}\right)} \frac{\zeta_j^{q_1+q_2+\delta_j \phi_j-1}}{\delta_j^{\phi_j}}, \tag{7}$$

$$\Psi = \Psi_2 - \Psi_1 \neq 0. \tag{8}$$

Lemma 2.7 Let $\Omega, \Psi \neq 0$, $0 < p_1, p_2, q_1, q_2 \leq 1$, $\mu_i, \gamma_i > 0$, $\delta_j, \phi_j > 0$, $\eta, \kappa, \xi_i, \zeta_j \in (0, T)$, $\alpha_i, \beta_j \in \mathbb{R}$ for all $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ and $h, g \in C([0, T], \mathbb{R})$. Then the problem

$$D^{p_1}(D^{p_2} + \lambda_1)x(t) = h(t), \quad 0 < t < T, \tag{9}$$

$$D^{q_1}(D^{q_2} + \lambda_2)y(t) = g(t), \quad 0 < t < T, \tag{10}$$

$$x(0) = 0, \quad x(\eta) = \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} x(\xi_i), \tag{11}$$

$$y(0) = 0, \quad y(\kappa) = \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} y(\zeta_j), \tag{12}$$

has a unique solution given by

$$x(t) = \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[J^{p_1+p_2} h(\eta) - \lambda_1 J^{p_2} x(\eta) \right]$$

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$$- \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} (J^{p_1+p_2} h(s) - \lambda_1 J^{p_2} x(s)) (\xi_i) \Big] + J^{p_1+p_2} h(t) - \lambda_1 J^{p_2} x(t),$$

and

$$y(t) = \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \left[J^{q_1+q_2} g(\kappa) - \lambda_2 J^{q_2} y(\kappa) - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} (J^{q_1+q_2} g(s) - \lambda_2 J^{q_2} y(s)) (\zeta_j) \right] + J^{q_1+q_2} g(t) - \lambda_2 J^{q_2} y(t).$$

Proof. Applying Lemma 2.3 to the equations (9) and (10), we obtain

$$(D^{p_2} + \lambda_1)x(t) = J^{p_1} h(t) + c_1 t^{p_1-1}, \quad \text{and} \quad (D^{q_2} + \lambda_2)y(t) = J^{q_1} g(t) + d_1 t^{q_1-1},$$

which give

$$x(t) = J^{p_1+p_2} h(t) - \lambda_1 J^{p_2} x(t) + c_1 \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} t^{p_1+p_2-1} + c_2 t^{p_2-1},$$

$$y(t) = J^{q_1+q_2} g(t) - \lambda_2 J^{q_2} y(t) + d_1 \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} t^{q_1+q_2-1} + d_2 t^{q_2-1},$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. It is easy to see that the conditions $x(0) = 0, y(0) = 0$ imply that $c_2 = 0, d_2 = 0$. Thus

$$x(t) = J^{p_1+p_2} h(t) - \lambda_1 J^{p_2} x(t) + c_1 \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} t^{p_1+p_2-1}, \tag{13}$$

$$y(t) = J^{q_1+q_2} g(t) - \lambda_2 J^{q_2} y(t) + d_1 \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} t^{q_1+q_2-1}. \tag{14}$$

Taking the generalized fractional integral of order $\mu_i > 0, \gamma_i > 0$, to (13) and $\phi_j > 0, \delta_j > 0$ to (14), we have

$$\mu_i I^{\gamma_i} x(t) = \mu_i I^{\gamma_i} (J^{p_1+p_2} h(s) - \lambda_1 J^{p_2} x(s)) (t) + c_1 \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{\Gamma(\frac{p_1+p_2+\mu_i-1}{\mu_i})}{\Gamma(\frac{p_1+p_2+\mu_i\gamma_i+\mu_i-1}{\mu_i})} \frac{t^{p_1+p_2+\mu_i\gamma_i-1}}{\mu_i^{\gamma_i}}, \tag{15}$$

and

$$\delta_j I^{\phi_j} y(t) = \delta_j I^{\phi_j} (J^{q_1+q_2} g(s) - \lambda_2 J^{q_2} y(s)) (t) + d_1 \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{\Gamma(\frac{q_1+q_2+\delta_j-1}{\delta_j})}{\Gamma(\frac{q_1+q_2+\delta_j\phi_j+\delta_j-1}{\delta_j})} \frac{t^{q_1+q_2+\delta_j\phi_j-1}}{\delta_j^{\phi_j}}. \tag{16}$$

Using the second condition of (11), (12) to (15), (16) respectively, we get

$$J^{p_1+p_2} h(\eta) - \lambda_1 J^{p_2} x(\eta) + c_1 \Omega_1 = \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} (J^{p_1+p_2} h(s) - \lambda_1 J^{p_2} x(s)) (\xi_i) + c_1 \Omega_2,$$

and

$$J^{q_1+q_2} g(\kappa) - \lambda_2 J^{q_2} y(\kappa) + d_1 \Psi_1 = \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} (J^{q_1+q_2} g(s) - \lambda_2 J^{q_2} y(s)) (\zeta_j) + d_1 \Psi_2.$$

Solving the above equations for finding constants c_1, d_1 , we obtain

$$c_1 = \frac{1}{\Omega} \left[J^{p_1+p_2} h(\eta) - \lambda_1 J^{p_2} x(\eta) - \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} (J^{p_1+p_2} h(s) - \lambda_1 J^{p_2} x(s)) (\xi_i) \right],$$

and

$$d_1 = \frac{1}{\Psi} \left[J^{q_1+q_2} g(\kappa) - \lambda_2 J^{q_2} y(\kappa) - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} (J^{q_1+q_2} g(s) - \lambda_2 J^{q_2} y(s)) (\zeta_j) \right].$$

Substituting the constants c_1, d_1 into (13), (14), we obtain (13) and (13). The proof is completed. \square

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3 Main results

Let $\mathcal{C} = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, T]$ to \mathbb{R} . Let us introduce the space $X = \{x(t) | x(t) \in C([0, T])\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0, T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t) | y(t) \in C([0, T])\}$ be endowed with the norm $\|y\| = \sup\{|y(t)|, t \in [0, T]\}$. Obviously the product space $(X \times Y, \|(x, y)\|)$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$.

Throughout this paper, for convenience, we use the following expressions

$$J^z h(s, x(s), y(s))(\tau) = \frac{1}{\Gamma(z)} \int_0^\tau (\tau - s)^{z-1} f(s, x(s), y(s)) ds,$$

and

$${}^\rho I^z h(s, x(s), y(s))(\tau) = \frac{\rho^{1-z}}{\Gamma(z)} \int_0^\tau \frac{s^{\rho-1} f(s, x(s), y(s))}{(\tau^\rho - s^\rho)^{1-z}} ds,$$

where $\rho, z > 0$ and $\tau \in [0, T]$.

In view of Lemma 2.7, we define an operator $\mathcal{F} : X \times Y \rightarrow X \times Y$ by

$$\mathcal{F}(x, y)(t) = \begin{pmatrix} \mathcal{P}(x, y)(t) \\ \mathcal{Q}(x, y)(t) \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} \mathcal{P}(x, y)(t) = & \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[J^{p_1+p_2} f(s, x(s), y(s))(\eta) - \lambda_1 J^{p_2} x(s)(\eta) \right. \\ & \left. - \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} \left(J^{p_1+p_2} f(s, x(s), y(s))(\tau) - \lambda_1 J^{p_2} x(s)(\tau) \right) (\xi_i) \right] \\ & + J^{p_1+p_2} f(s, x(s), y(s))(t) - \lambda_1 J^{p_2} x(s)(t), \end{aligned} \tag{18}$$

and

$$\begin{aligned} \mathcal{Q}(x, y)(t) = & \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \left[J^{q_1+q_2} g(s, x(s), y(s))(\kappa) - \lambda_2 J^{q_2} y(s)(\kappa) \right. \\ & \left. - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} \left(J^{q_1+q_2} g(s, x(s), y(s))(s) - \lambda_2 J^{q_2} x(s) \right) (\zeta_j) \right] \\ & + J^{q_1+q_2} g(t) - \lambda_2 J^{q_2} y(t). \end{aligned} \tag{19}$$

To simplify the notations, we use in the following constants

$$\begin{aligned} \Phi(a) = & \frac{T^{a+p_2}}{\Gamma(1 + a + p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \left(\frac{\eta^{a+p_2}}{\Gamma(1 + a + p_2)} \right. \\ & \left. + \sum_{i=1}^n |\alpha_i| \left[\frac{1}{\Gamma(1 + a + p_2)} \frac{\xi_i^{a+p_2+\mu_i\gamma_i}}{\mu_i^{\gamma_i}} \frac{\Gamma\left(\frac{a+p_2+\mu_i}{\mu_i}\right)}{\Gamma\left(\frac{a+p_2+\mu_i\gamma_i+\mu_i}{\mu_i}\right)} \right] \right), \end{aligned} \tag{20}$$

and

$$\begin{aligned} \Lambda(b) = & \frac{T^{b+q_2}}{\Gamma(1 + b + q_2)} + \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{T^{q_1+q_2-1}}{|\Psi|} \left(\frac{\kappa^{b+q_2}}{\Gamma(1 + b + q_2)} \right. \\ & \left. + \sum_{j=1}^m |\beta_j| \left[\frac{1}{\Gamma(1 + b + q_2)} \frac{\zeta_j^{b+q_2+\delta_j\phi_j}}{\delta_j^{\phi_j}} \frac{\Gamma\left(\frac{b+q_2+\delta_j}{\delta_j}\right)}{\Gamma\left(\frac{b+q_2+\delta_j\phi_j+\delta_j}{\delta_j}\right)} \right] \right), \end{aligned} \tag{21}$$

where $a \in \{p_1, 0\}$ and $b \in \{q_1, 0\}$.

3.1 Existence result via Krasnoselskii’s fixed point theorem

The next result is based on the following fixed point theorem.

Lemma 3.1 (Krasnoselskii’s fixed point theorem) [29]. *Let M be a closed, bounded, convex and nonempty subset of a Banach space X . Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.2 *Suppose that the following conditions hold:*

$$(H_1) \quad |f(t, u, v)| \leq \psi(t), \quad \forall (t, u, v) \in [0, T] \times \mathbb{R}^2, \quad \text{and } \psi \in C([0, T], \mathbb{R}^+);$$

$$(H_2) \quad |g(t, u, v)| \leq \omega(t), \quad \forall (t, u, v) \in [0, T] \times \mathbb{R}^2, \quad \text{and } \omega \in C([0, T], \mathbb{R}^+);$$

If

$$\Upsilon = \max\{|\lambda_1|\Phi(0), |\lambda_2|\Lambda(0)\} < 1, \tag{22}$$

where $\Phi(0)$ and $\Lambda(0)$ are defined by (20) and (21) with $a = b = 0$, respectively. Then the problem (1) has at least one solution on $[0, T]$.

Proof. To prove our result, we set $\sup_{t \in [0, T]} |\psi(t)| = \|\psi\|$, $\sup_{t \in [0, T]} |\omega(t)| = \|\omega\|$ and choose

$$R \geq \frac{\|\psi\|\Phi(p_1) + \|\omega\|\Lambda(q_1)}{1 - \Upsilon}, \tag{23}$$

where $\Phi(p_1)$ and $\Lambda(q_1)$ are defined by (20) and (21) with $a = p_1$ and $b = q_1$, respectively. Let $B_R = \{(x, y) \in X \times Y : \|(x, y)\| \leq R\}$. We define four operators by

$$\begin{aligned} \mathcal{P}_1(x, y)(t) &= J^{p_1+p_2} f(s, x(s), y(s))(t) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[J^{p_1+p_2} f(s, x(s), y(s))(\eta) \right. \\ &\quad \left. - \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} \left(J^{p_1+p_2} f(s, x(s), y(s))(\tau) \right) (\xi_i) \right], \end{aligned}$$

$$\begin{aligned} \mathcal{P}_2(x)(t) &= -\lambda_1 J^{p_2} x(s)(t) - \lambda_1 \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[J^{p_2} x(s)(\eta) \right. \\ &\quad \left. - \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} \left(J^{p_2} x(s)(\tau) \right) (\xi_i) \right], \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_1(x, y)(t) &= J^{q_1+q_2} g(s, x(s), y(s))(t) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \left[J^{q_1+q_2} g(s, x(s), y(s))(\kappa) \right. \\ &\quad \left. - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} \left(J^{q_1+q_2} g(s, x(s), y(s))(s) \right) (\zeta_j) \right], \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_2(y)(t) &= -\lambda_2 J^{q_2} y(s)(t) - \lambda_2 \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \left[J^{q_2} y(s)(\kappa) \right. \\ &\quad \left. - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} \left(J^{q_2} y(s)(\tau) \right) (\zeta_j) \right], \end{aligned}$$

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and

$$\mathcal{F}_1(x, y)(t) = \begin{pmatrix} \mathcal{P}_1(x, y)(t) \\ \mathcal{Q}_1(x, y)(t) \end{pmatrix}, \quad \mathcal{F}_2(x, y)(t) = \begin{pmatrix} \mathcal{P}_2(x)(t) \\ \mathcal{Q}_2(y)(t) \end{pmatrix}. \tag{24}$$

Observe that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ and $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. For any $(x_1, y_1), (x_2, y_2) \in B_R$ we have

$$\begin{aligned} & |\mathcal{P}_1(x_1, y_1)(t) + \mathcal{P}_2(x_2)(t)| \\ = & \left| J^{p_1+p_2} f(s, x_1(s), y_1(s))(t) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[J^{p_1+p_2} f(s, x_1(s), y_1(s))(\eta) \right. \right. \\ & \left. \left. - \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} \left(J^{p_1+p_2} f(s, x_1(s), y_1(s))(\tau) \right) (\xi_i) \right] - \lambda_1 J^{p_2} x_2(s)(t) \right. \\ & \left. - \lambda_1 \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[J^{p_2} x_2(s)(\eta) - \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} \left(J^{p_2} x_2(s)(\tau) \right) (\xi_i) \right] \right| \\ \leq & \|\psi\| \left(\left| J^{p_1+p_2}(T) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \left[J^{p_1+p_2}(\eta) + \sum_{i=1}^n |\alpha_i| \mu_i I^{\gamma_i} \left(J^{p_1+p_2}(\tau) \right) (\xi_i) \right] \right| \right) \\ & + |\lambda_1| \|x_2\| \left(\left| J^{p_2}(T) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \left[J^{p_2}(\eta) + \sum_{i=1}^n |\alpha_i| \mu_i I^{\gamma_i} \left(J^{p_2}(\tau) \right) (\xi_i) \right] \right| \right) \\ \leq & \|\psi\| \Phi(p_1) + |\lambda_1| \|x_2\| \Phi(0). \end{aligned}$$

In a similar way, we get

$$\begin{aligned} & |\mathcal{Q}_1(x_1, y_1)(t) + \mathcal{Q}_2(y_2)(t)| \\ = & \left| J^{q_1+q_2} g(s, x_1(s), y_1(s))(t) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \left[J^{q_1+q_2} g(s, x_1(s), y_1(s))(\kappa) \right. \right. \\ & \left. \left. - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} \left(J^{q_1+q_2} g(s, x_1(s), y_1(s))(\tau) \right) (\zeta_j) \right] - \lambda_2 J^{q_2} y_2(s)(t) \right. \\ & \left. - \lambda_2 \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \left[J^{q_2} y_2(s)(\kappa) - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} \left(J^{q_2} y_2(s)(\tau) \right) (\zeta_j) \right] \right| \\ \leq & \|\omega\| \left(\left| J^{q_1+q_2}(T) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{T^{q_1+q_2-1}}{|\Psi|} \left[J^{q_1+q_2}(\kappa) + \sum_{j=1}^m |\beta_j| \delta_j I^{\phi_j} \left(J^{q_1+q_2}(\tau) \right) (\zeta_j) \right] \right| \right) \\ & + |\lambda_2| \|y_2\| \left(\left| J^{q_2}(T) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{T^{q_1+q_2-1}}{|\Psi|} \left[J^{q_2}(\kappa) + \sum_{j=1}^m |\beta_j| \delta_j I^{\phi_j} \left(J^{q_2}(\tau) \right) (\zeta_j) \right] \right| \right) \\ \leq & \|\omega\| \Lambda(q_1) + |\lambda_2| \|y_2\| \Lambda(0), \end{aligned}$$

which imply that $\|\mathcal{F}_1(x, y) + \mathcal{F}_2(x, y)\| \leq R$. This shows that $\mathcal{F}_1(x, y) + \mathcal{F}_2(x, y) \in B_R$.

For $(x_1, y_1), (x_2, y_2) \in X \times Y$ and for each $t \in [0, T]$ we have

$$\|\mathcal{P}_2(x_1) - \mathcal{P}_2(x_2)\| \leq |\lambda_1| \Phi(0) \|x_1 - x_2\|,$$

and

$$\|\mathcal{Q}_2(y_1) - \mathcal{Q}_2(y_2)\| \leq |\lambda_2| \Lambda(0) \|y_1 - y_2\|.$$

Thus

$$\|\mathcal{F}_2(x_1, y_1) - \mathcal{F}_2(x_2, y_2)\| \leq \Upsilon \|x_1 - x_2\| + \Upsilon \|y_1 - y_2\| = \Upsilon \|(x_1 - x_2, y_1 - y_2)\|,$$

which implies that \mathcal{F}_2 is a contraction mapping by (22). The continuity of f implies that the operator \mathcal{F}_1 is continuous. Also, \mathcal{F}_1 is uniformly bounded on B_R as

$$\|\mathcal{P}_1(x, y)\| \leq \|\psi\| \Phi(p_1), \quad \text{and} \quad \|\mathcal{Q}_1(x, y)\| \leq \|\omega\| \Lambda(q_1).$$

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Thus

$$\|\mathcal{F}_1(x, y)\| \leq \|\psi\|\Phi(p_1) + \|\omega\|\Lambda(q_1).$$

Next we will prove the compactness of the operator \mathcal{F}_1 . Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned} & |\mathcal{P}_1(x, y)(t_2) - \mathcal{P}_1(x, y)(t_1)| \\ & \leq \left| J^{p_1+p_2} f(s, x(s), y(s))(t_2) - J^{p_1+p_2} f(s, x(s), y(s))(t_1) \right. \\ & \quad \left. + \frac{\Gamma(p_1)(t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1})}{\Omega\Gamma(p_1+p_2)} \left[J^{p_1+p_2} f(s, x(s), y(s))(\eta) \right. \right. \\ & \quad \left. \left. - \sum_{i=1}^n \alpha_i \mu_i I^{\gamma_i} \left(J^{p_1+p_2} f(s, x(s), y(s))(\tau) \right) (\xi_i) \right] \right| \\ & \leq \|\psi\| \frac{(t_2^{p_1+p_2} - t_1^{p_1+p_2}) + 2(t_2 - t_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \|\psi\| \frac{\Gamma(p_1)(t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1})}{|\Omega|\Gamma(p_1+p_2)} J^{p_1+p_2}(\eta) \\ & \quad + \sum_{i=1}^n |\alpha_i| \mu_i I^{\gamma_i} \left(J^{p_1+p_2}(\tau) \right) (\xi_i) \end{aligned}$$

and

$$\begin{aligned} & |\mathcal{Q}_1(x, y)(t_2) - \mathcal{Q}_1(x, y)(t_1)| \\ & \leq \left| J^{q_1+q_2} g(s, x(s), y(s))(t_2) - J^{q_1+q_2} g(s, x(s), y(s))(t_1) \right. \\ & \quad \left. + \frac{\Gamma(q_1)(t_2^{q_1+q_2-1} - t_1^{q_1+q_2-1})}{\Psi\Gamma(q_1+q_2)} \left[J^{q_1+q_2} g(s, x(s), y(s))(\kappa) \right. \right. \\ & \quad \left. \left. - \sum_{j=1}^m \beta_j \delta_j I^{\phi_j} \left(J^{q_1+q_2} g(s, x(s), y(s))(\tau) \right) (\zeta_j) \right] \right| \\ & \leq \|\omega\| \frac{(t_2^{q_1+q_2} - t_1^{q_1+q_2}) + 2(t_2 - t_1)^{q_1+q_2}}{\Gamma(q_1+q_2+1)} + \|\omega\| \frac{\Gamma(q_1)(t_2^{q_1+q_2-1} - t_1^{q_1+q_2-1})}{|\Psi|\Gamma(q_1+q_2)} J^{q_1+q_2}(\kappa) \\ & \quad + \sum_{j=1}^m |\beta_j| \delta_j I^{\phi_j} \left(J^{q_1+q_2}(\tau) \right) (\zeta_j), \end{aligned}$$

which is independent of (x, y) and tends to zero as $t_2 - t_1 \rightarrow 0$. Thus, \mathcal{F}_1 is equicontinuous. So \mathcal{F}_1 is relatively compact on B_R . Hence, by the Arzelá-Ascoli theorem, \mathcal{F}_1 is compact on B_R . Thus all the assumptions of Lemma 3.1 are satisfied. So the conclusion of Lemma 3.1 implies that the problem (1) has at least one solution on $[0, T]$. This completes the proof. \square

3.2 Existence result via O'Regan's fixed point theorem

Our next existence result relies on a fixed point theorem due to O'Regan in [30].

Lemma 3.3 *Denote by U an open set in a closed, convex set C of a Banach space E . Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \rightarrow C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \rightarrow E$ is continuous and completely continuous and $F_2 : \bar{U} \rightarrow E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) F has a fixed point $u \in \bar{U}$; or

(C2) there exist a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where \bar{U} and ∂U , respectively, represent the closure and boundary of U .

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In the sequel, we will use Lemma 3.3 by taking C to be E . For more details of such fixed point theorems, we refer a paper [31] by Petryshyn. Let

$$K_r = \{(x, y) \in X \times Y : \|(x, y)\| \leq R\}.$$

Theorem 3.4 *Let $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Suppose that (22) holds. In addition we assume that:*

(H₃) *there exist a nonnegative function $z_1 \in C([0, T], \mathbb{R})$ and nondecreasing functions $\psi_1, \psi_2 : [0, \infty) \rightarrow [0, \infty)$ such that*

$$|f(t, u, v)| \leq z_1(t)[\psi_1(\|u\|) + \psi_2(\|v\|)] \text{ for all } (t, u, v) \in [0, T] \times \mathbb{R}^2;$$

(H₄) *there exist a nonnegative function $z_2 \in C([0, T], \mathbb{R})$ and nondecreasing functions $\omega_1, \omega_2 : [0, \infty) \rightarrow [0, \infty)$ such that*

$$|g(t, u, v)| \leq z_2(t)[\omega_1(\|u\|) + \omega_2(\|v\|)] \text{ for all } (t, u, v) \in [0, T] \times \mathbb{R}^2;$$

(H₅) $\sup_{r \in (0, \infty)} \frac{r}{\|z_1\|[\psi_1(r) + \psi_2(r)]\Phi(p_1) + \|z_2\|[\omega_1(r) + \omega_2(r)]\Lambda(q_1)} > \frac{1}{1 - \Upsilon}$, where $\Phi(p_1), \Lambda(q_1)$ and Υ are defined in (20), (21) and (22) respectively.

Then the problem (1) has at least one solution on $[0, T]$.

Proof. Consider the operator $\mathcal{F} : X \times Y \rightarrow X \times Y$ as that defined in (18). We decompose \mathcal{F} into a sum of two operators

$$\mathcal{F}(x, y)(t) = \mathcal{F}_1(x, y)(t) + \mathcal{F}_2(x, y)(t)$$

where $\mathcal{F}_1(x, y), \mathcal{F}_2(x, y)$ defined in (24). From (H₅) there exists a number $r_0 > 0$ such that

$$\frac{r_0}{\|z_1\|[\psi_1(r_0) + \psi_2(r_0)]\Phi(p_1) + \|z_2\|[\omega_1(r_0) + \omega_2(r_0)]\Lambda(q_1)} > \frac{1}{1 - \Upsilon}. \tag{25}$$

We shall prove that the operators \mathcal{F}_1 and \mathcal{F}_2 satisfy all the conditions of Lemma 3.3.

Step 1. *The set $\mathcal{F}(K_{r_0})$ is bounded.* We first show that $\mathcal{F}_1(K_{r_0})$ is bounded. For any $(x, y) \in \bar{K}_{r_0}$ we have

$$\|\mathcal{P}_1(x, y)\| \leq \|z_1\|[\psi_1(r_0) + \psi_2(r_0)]\Phi(p_1),$$

and

$$\|\mathcal{Q}_1(x, y)\| \leq \|z_2\|[\omega_1(r_0) + \omega_2(r_0)]\Lambda(q_1).$$

Thus

$$\|\mathcal{F}_1(x, y)\| \leq \|z_1\|[\psi_1(r_0) + \psi_2(r_0)]\Phi(p_1) + \|z_2\|[\omega_1(r_0) + \omega_2(r_0)]\Lambda(q_1).$$

This proves that $\mathcal{F}_1(\bar{K}_{r_0})$ is uniformly bounded. In a similar way we have

$$\|\mathcal{P}_2(x)\| \leq |\lambda_1|\Phi(0)\|x\|, \text{ and } \|\mathcal{Q}_2(y)\| \leq |\lambda_2|\Lambda(0)\|y\|,$$

and thus

$$\|\mathcal{F}_2(x, y)\| \leq \Upsilon r_0.$$

Step 2. *The operator \mathcal{F}_1 is continuous and completely continuous.*

By Step 1, $\mathcal{F}_1(\bar{K}_{r_0})$ is uniformly bounded. In addition for any $t_1, t_2 \in [0, T]$, we have:

$$\begin{aligned} & |\mathcal{P}_1(x, y)(t_2) - \mathcal{P}_1(x, y)(t_1)| \\ & \leq \|z_1\|[\psi_1(r_0) + \psi_2(r_0)] \left[\frac{1}{\Gamma(p_1 + p_2 + 1)} \left(t_2^{p_1 + p_2} - t_1^{p_1 + p_2} + 2(t_2 - t_1)^{p_1 + p_2} \right) \right. \\ & \quad \left. + \frac{\Gamma(p_1)(t_2^{p_1 + p_2 - 1} - t_1^{p_1 + p_2 - 1})}{|\Omega|\Gamma(p_1 + p_2)} J^{p_1 + p_2}(\eta) + \sum_{i=1}^n |\alpha_i|^{\mu_i} I^{\gamma_i} \left(J^{p_1 + p_2}(\tau) \right) (\xi_i) \right], \end{aligned}$$

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and

$$\begin{aligned} & |\mathcal{Q}_1(x, y)(t_2) - \mathcal{Q}_1(x, y)(t_1)| \\ \leq & \|z_2\|[\omega_1(r_0) + \omega_2(r_0)] \left[\frac{1}{\Gamma(q_1 + q_2 + 1)} \left(t_2^{q_1+q_2} - t_1^{q_1+q_2} + 2(t_2 - t_1)^{q_1+q_2} \right) \right. \\ & \left. + \frac{\Gamma(q_1)(t_2^{q_1+q_2-1} - t_1^{q_1+q_2-1})}{|\Psi|\Gamma(q_1 + q_2)} J^{q_1+q_2}(\kappa) + \sum_{j=1}^m |\beta_j| \delta_j I^{\phi_j} \left(J^{q_1+q_2}(\tau) \right) (\zeta_j) \right], \end{aligned}$$

which are independent of (x, y) and tends to zero as $t_2 - t_1 \rightarrow 0$. Thus, \mathcal{F}_1 is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{F}_1(\bar{K}_{r_0})$ is a relatively compact set. Now, let $(x_n, y_n) \subset \bar{K}_{r_0}$ with $\|(x_n, y_n) - (x, y)\| \rightarrow 0$. Then the limit $\|(x_n, y_n)(t) - (x, y)(t)\| \rightarrow 0$ is uniformly valid on $[0, T]$. From the uniform continuity of $f(t, x, y)$ and $g(t, x, y)$ on the compact set $[0, T] \times [-r_0, r_0] \times [-r_0, r_0]$, it follows that $\|f(t, x_n(t), y_n(t)) - f(t, x(t), y(t))\| \rightarrow 0$ and $\|g(t, x_n(t), y_n(t)) - g(t, x(t), y(t))\| \rightarrow 0$ are uniformly valid on $[0, T]$. Hence $\|\mathcal{F}_1(x_n, y_n) - \mathcal{F}_1(x, y)\| \rightarrow 0$ as $n \rightarrow \infty$ which proves the continuity of \mathcal{F}_1 . Therefore the operator \mathcal{F}_1 is continuous and completely continuous

Step 3. *The operator \mathcal{F}_2 is contractive.* This was proved in Theorem 3.2.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 3.3 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\theta \in (0, 1)$ and $(x, y) \in \partial \bar{K}_{r_0}$ such that $(x, y) = \theta \mathcal{F}(x, y)$. So, we have $\|(x, y)\| = r_0$ and

$$\|x\| \leq \|z_1\|[\psi_1(r_0) + \psi_2(r_0)]\Phi(p_1) + |\lambda_1|\Phi(0)\|x\|,$$

and

$$\|y\| \leq \|z_2\|[\omega_1(r_0) + \omega_2(r_0)]\Lambda(q_1) + |\lambda_2|\Lambda(0)\|y\|,$$

from which we get

$$\|x\| + \|y\| \leq \|z_1\|[\psi_1(r_0) + \psi_2(r_0)]\Phi(p_1) + \|z_2\|[\omega_1(r_0) + \omega_2(r_0)]\Lambda(q_1) + \Upsilon r_0,$$

or

$$\frac{r_0}{\|z_1\|[\psi_1(r_0) + \psi_2(r_0)]\Phi(p_1) + \|z_2\|[\omega_1(r_0) + \omega_2(r_0)]\Lambda(q_1)} \leq \frac{1}{1 - \Upsilon},$$

which contradicts to (25). Consequently, we have proved that the operators \mathcal{F}_1 and \mathcal{F}_2 satisfy all the conditions in Lemma 3.3. Hence, the operator \mathcal{F} has at least one fixed point $(x, y) \in \bar{K}_{r_0}$, which is the solution of the he problem (1). The proof is completed. \square

Theorem 3.5 *Let $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Suppose that (22) holds. In addition we assume that:*

(H₆) *there exist a nonnegative function $z_1 \in C([0, T], \mathbb{R})$ and a nondecreasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ such that*

$$|f(t, u, v)| \leq z_1(t)\psi(\|u\| + \|v\|) \text{ for all } (t, u, v) \in [0, T] \times \mathbb{R}^2;$$

(H₇) *there exist a nonnegative function $z_2 \in C([0, T], \mathbb{R})$ and a nondecreasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ such that*

$$|g(t, u, v)| \leq z_2(t)\omega(\|u\| + \|v\|) \text{ for all } (t, u, v) \in [0, T] \times \mathbb{R}^2;$$

(H₈) $\sup_{r \in (0, \infty)} \frac{r}{\|z_1\|\psi(r)\Phi(q_1) + \|z_2\|\omega(r)\Lambda(q_1)} > \frac{1}{1 - \Upsilon}$, *where $\Phi(p_1), \Lambda(q_1)$ and Υ are defined in (20), (21) and (22) respectively.*

Then the he problem (1) has at least one solution on $[0, T]$.

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Proof. The proof is similar to that of Theorem 3.4 and it is omitted. □

To establish some special cases, we set constants

$$R_1 = \sum_{i=1}^n \frac{\alpha_i \Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{\Gamma(p_1 + p_2)}{\Gamma(p_1 + p_2 + \gamma_i)} \xi_i^{p_1 + p_2 + \gamma_i - 1}, \quad R = R_1 - \Omega_1 \neq 0,$$

and

$$L_1 = \sum_{j=1}^m \frac{\beta_j \Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{\Gamma(q_1 + q_2)}{\Gamma(q_1 + q_2 + \phi_j)} \zeta_j^{q_1 + q_2 + \phi_j - 1}, \quad L = L_1 - \Psi_1 \neq 0,$$

$$\begin{aligned} \chi(a) = & \frac{T^{a+p_2}}{\Gamma(1+a+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|R|} \left(\frac{\eta^{a+p_2}}{\Gamma(1+a+p_2)} \right. \\ & \left. + \sum_{i=1}^n |\alpha_i| \left[\frac{\xi_i^{a+p_2+\gamma_i}}{\Gamma(1+a+p_2)} \frac{\Gamma(a+p_2+1)}{\Gamma(a+p_2+\gamma_i+1)} \right] \right), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \Theta(b) = & \frac{T^{b+q_2}}{\Gamma(1+b+q_2)} + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{T^{q_1+q_2-1}}{|L|} \left(\frac{\kappa^{b+q_2}}{\Gamma(1+b+q_2)} \right. \\ & \left. + \sum_{j=1}^m |\beta_j| \left[\frac{\zeta_j^{b+q_2+\phi_j}}{\Gamma(1+b+q_2)} \frac{\Gamma(b+q_2+1)}{\Gamma(b+q_2+\phi_j+1)} \right] \right), \end{aligned} \tag{27}$$

where $a = \{p_1, 0\}$ and $b = \{q_1, 0\}$

By setting $\mu_i = 1$ and $\delta_j = 1$, we have a boundary value problem with nonlocal Riemann-Liouville fractional integral conditions

$$\begin{cases} D^{p_1} (D^{p_2} + \lambda_1) x(t) = f(t, x(t), y(t)), & 0 < t < T, \\ D^{q_1} (D^{q_2} + \lambda_2) y(t) = g(t, x(t), y(t)), & 0 < t < T, \\ x(0) = 0, & x(\eta) = \sum_{i=1}^n \alpha_i J^{\gamma_i} x(\xi_i), \\ y(0) = 0, & y(\kappa) = \sum_{j=1}^m \beta_j J^{\phi_j} y(\zeta_j). \end{cases} \tag{28}$$

Using the above constants, we have the following corollaries.

Corollary 3.6 Suppose that (H1) and (H2) holds. If

$$M = \max\{|\lambda_1| \chi(0), |\lambda_2| \Theta(0)\} < 1, \tag{29}$$

then the problem (28) has at least one solution on $[0, T]$.

Corollary 3.7 Let $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Suppose that (29), (H3) and (H4) holds. In addition we assume that:

$$(H_9) \quad \sup_{r \in (0, \infty)} \frac{r}{\|z_1\|[\psi_1(r) + \psi_2(r)]\chi(p_1) + \|z_2\|[\omega_1(r) + \omega_2(r)]\Theta(q_1)} > \frac{1}{1-M}, \text{ where } \chi(p_1), \Theta(q_1) \text{ and } M \text{ are defined in (26), (27) and (29) respectively.}$$

Then the he problem (28) has at least one solution on $[0, T]$.

Corollary 3.8 Let $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Suppose that (29), (H6) and (H7) holds. In addition we assume that:

$$(H_{10}) \quad \sup_{r \in (0, \infty)} \frac{r}{\|z_1\|\psi(r)\chi(q_1) + \|z_2\|\omega(r)\Theta(q_1)} > \frac{1}{1-M}, \text{ where } \chi(p_1), \Theta(q_1) \text{ and } M \text{ are defined in (26), (27) and (29) respectively.}$$

Then the he problem (28) has at least one solution on $[0, T]$.

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4 Examples

In this section we present examples to illustrate our results.

Example 4.1 Consider the following system of fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions

$$\begin{cases} D^{1/2} (D^{3/5} + 0.2) x(t) = \frac{t \sin 3t \arctan x(t)}{t + 1} \frac{2 \cos t \sin y(t)}{3|x(t)| + 2} + \frac{2 \cos t \sin y(t)}{3t^2 + 2|y(t)| + 3}, & 0 < t < 1, \\ D^{2/5} (D^{4/5} + 0.25) y(t) = \frac{3t^2}{4t + 3} \frac{3x(t)}{5|x(t)| + 1} + \frac{2y(t) + 3}{3|y(t)| + 4}, & 0 < t < 1, \\ x(0) = 0, & x(0.6) = 0.2 I^{7/10} x(0.3) + 0.3 I^{3/5} x(0.6), \\ y(0) = 0, & y(0.2) = 0.2 I^{4/5} y(0.3) + 0.3 I^{2/5} y(0.7) + 0.4 I^{9/10} y(0.9), \end{cases} \tag{30}$$

Here $p_1 = 1/2, p_2 = 3/5, q_1 = 2/5, q_2 = 4/5, \lambda_1 = 0.2, \lambda_2 = 0.25, \eta = 0.6, \kappa = 0.2, \alpha_1 = 0.2, \alpha_2 = 0.3, \beta_1 = 0.2, \beta_2 = 0.3, \beta_3 = 0.4, \mu_1 = 1/2, \mu_2 = 2/5, \gamma_1 = 7/10, \gamma_2 = 3/5, \delta_1 = 3/10, \delta_2 = 3/5, \delta_3 = 2/5, \phi_1 = 4/5, \phi_2 = 2/5, \phi_3 = 9/10, \xi_1 = 0.3, \xi_2 = 0.6, \zeta_1 = 0.3, \zeta_2 = 0.7, \zeta_3 = 0.9, T = 1, f(t, x, y) = (t \sin 3t \arctan x(t))/((t + 1)(3|x(t)| + 2)) + (2 \cos t \sin y(t))/((3t^2 + 2)(2|y(t)| + 3))$ and $g(t, x, y) = (9t^2 x(t))/((4t + 3)(5|x(t)| + 1)) + (2y(t) + 3)/(3|y(t)| + 4)$. Since $f(t, x, y) \leq (t \sin 3t)/(3t + 3) + (2 \cos t)/(6t^2 + 4), g(t, x, y) \leq (9t^2)/(20t + 15) + (2/3)$ and by using the Maple program, we can find

$$\begin{aligned} \Phi(0) &= \frac{T^{p_2}}{\Gamma(1 + p_2)} \\ &+ \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{T^{p_1 + p_2 - 1}}{|\Omega|} \left(\frac{\eta^{p_2}}{\Gamma(1 + p_2)} + \sum_{i=1}^2 |\alpha_i| \left[\frac{1}{\Gamma(1 + p_2)} \frac{\xi_i^{p_2 + \mu_i \gamma_i}}{\mu_i^{\gamma_i}} \frac{\Gamma\left(\frac{p_2 + \mu_i}{\mu_i}\right)}{\Gamma\left(\frac{p_2 + \mu_i \gamma_i + \mu_i}{\mu_i}\right)} \right] \right) \\ &\approx 4.318646369, \end{aligned}$$

and

$$\begin{aligned} \Lambda(0) &= \frac{T^{q_2}}{\Gamma(1 + q_2)} \\ &+ \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{T^{q_1 + q_2 - 1}}{|\Psi|} \left(\frac{\kappa^{q_2}}{\Gamma(1 + q_2)} + \sum_{j=1}^3 |\beta_j| \left[\frac{1}{\Gamma(1 + q_2)} \frac{\zeta_j^{q_2 + \delta_j \phi_j}}{\delta_j^{\phi_j}} \frac{\Gamma\left(\frac{q_2 + \delta_j}{\delta_j}\right)}{\Gamma\left(\frac{q_2 + \delta_j \phi_j + \delta_j}{\delta_j}\right)} \right] \right) \\ &\approx 3.234126953. \end{aligned}$$

Thus $\Upsilon \approx 0.8637292738 < 1$. Hence, by Theorem 3.2, the system (30) has at least one solution on $[0, 1]$.

Example 4.2 Consider the following system of fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions

$$\begin{cases} D^{3/10} (D^{4/5} + 0.25) x(t) = \frac{t}{15} \left(\frac{|x|^2 + 2|x|}{|x| + 4} + \frac{|y|^2 + 2|y| + 2}{3|y| + 4} \right), & 0 < t < 1, \\ D^{2/5} (D^{9/10} + 0.2) y(t) = \frac{t}{5} \left(\frac{|x|^2 + |x| + 1}{2|x| + 5} + \frac{|y|^2 + 1}{|y| + 5} \right), & 0 < t < 1, \\ x(0) = 0, & x(0.1) = 1.5 I^{7/10} x(0.6) + 2 I^{3/10} x(0.8) + 2.5 I^{3/10} x(0.9), \\ y(0) = 0, & y(0.8) = 3 I^{4/5} y(0.7) + 2.5 I^{9/10} y(0.8), \end{cases} \tag{31}$$

Here $p_1 = 3/10, p_2 = 4/5, q_1 = 2/5, q_2 = 9/10, \lambda_1 = 0.25, \lambda_2 = 0.2, \eta = 0.1, \kappa = 0.8, \alpha_1 = 1.5, \alpha_2 = 2, \alpha_3 = 2.5, \beta_1 = 3, \beta_2 = 2.5, \mu_1 = 7/10, \mu_2 = 3/10, \mu_3 = 3/5, \gamma_1 = 1/2, \gamma_2 = 1/5, \gamma_3 = 3/10, \delta_1 = 7/10, \delta_2 = 3/10, \phi_1 = 4/5, \phi_2 = 9/10, \xi_1 = 0.6, \xi_2 = 0.8, \xi_3 = 0.9, \zeta_1 = 0.7, \zeta_2 = 0.8, T = 1, f(t, x, y) = (t/15)[(|x|^2 + 2|x|)/(|x| + 4) + ((|y|^2 + 2|y| + 2)/(3|y| + 4))]$ and $g(t, x, y) = (t/5)[(|x|^2 + |x| + 1)/(2|x| + 5) + ((|y|^2 + 1)/(|y| + 5))]$. By using the Maple program, we can find

$$\Phi(0) = \frac{T^{p_2}}{\Gamma(1 + p_2)}$$

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$$\begin{aligned}
 & + \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \left(\frac{\eta^{p_2}}{\Gamma(1 + p_2)} + \sum_{i=1}^3 |\alpha_i| \left[\frac{1}{\Gamma(1 + p_2)} \frac{\xi_i^{p_2+\mu_i\gamma_i}}{\mu_i^{\gamma_i}} \frac{\Gamma\left(\frac{p_2+\mu_i}{\mu_i}\right)}{\Gamma\left(\frac{p_2+\mu_i\gamma_i+\mu_i}{\mu_i}\right)} \right] \right) \\
 & \approx 1.892763483,
 \end{aligned}$$

and

$$\begin{aligned}
 \Lambda(0) & = \frac{T^{q_2}}{\Gamma(1 + q_2)} \\
 & + \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{T^{q_1+q_2-1}}{|\Psi|} \left(\frac{\kappa^{q_2}}{\Gamma(1 + q_2)} + \sum_{j=1}^2 |\beta_j| \left[\frac{1}{\Gamma(1 + q_2)} \frac{\zeta_i^{q_2+\delta_j\phi_j}}{\delta_j^{\phi_j}} \frac{\Gamma\left(\frac{q_2+\delta_j}{\delta_j}\right)}{\Gamma\left(\frac{q_2+\delta_j\phi_j+\delta_j}{\delta_j}\right)} \right] \right) \\
 & \approx 1.824427804.
 \end{aligned}$$

Thus $\Upsilon \approx 0.4731908708 < 1$. Since $|f(t, x, y)| \leq (t/15)[(|x|^2 + 2|x|)/4 + (|y|^2 + 2|y| + 2)/4]$, $|g(t, x, y)| \leq (t/5)[(|x|^2 + |x| + 1)/5 + (|y|^2 + 1)/5]$, we choose $z_1(t) = t/15$, $\psi_1(x) = (|x|^2 + 2|x|)/4$, $\psi_2(y) = (|y|^2 + 2|y| + 2)/4$, $z_2(t) = t/5$, $\omega_1(x) = (|x|^2 + |x| + 1)/5$, $\omega_2(y) = (|y|^2 + 1)/5$. We can show that

$$\begin{aligned}
 & \sup_{r \in (0, \infty)} \frac{r}{\|z_1\|[\psi_1(r) + \psi_2(r)]\Phi(p_1) + \|z_2\|[\omega_1(r) + \omega_2(r)]\Lambda(q_1)} \\
 & \approx 2.080080186 > 1.898220711 \approx \frac{1}{1 - \Upsilon}.
 \end{aligned}$$

Hence, by Theorem 3.4, the system (31) has at least one solution on $[0, 1]$.

Example 4.3 Consider the following system of fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions

$$\begin{cases}
 D^{4/5} \left(D^{9/10} + 0.3 \right) x(t) = \frac{t}{5} \left(\frac{2(|x + y|)^3 + 2|x| + |y|}{3|x| + 4} \right), & 0 < t < \frac{2}{3}, \\
 D^{3/10} \left(D^{9/10} + 0.35 \right) y(t) = \frac{t}{3} \left(\frac{(|x + y|)^2 + 1}{|x| + 2|y| + 3} \right), & 0 < t < \frac{2}{3}, \\
 x(0) = 0, & x(0.6) = 0.4 \cdot {}^{2/5}I^{7/10}x(0.2) + 0.4 \cdot {}^{4/5}I^{2/5}x(0.6), \\
 y(0) = 0, & y(0.3) = 0.8 \cdot {}^{4/5}I^{4/5}y(0.2) + 0.7 \cdot {}^{1/5}I^{9/10}y(0.5) + 0.8 \cdot {}^{7/10}I^{7/10}y(0.6),
 \end{cases} \tag{32}$$

Here $p_1 = 4/5$, $p_2 = 9/10$, $q_1 = 3/10$, $q_2 = 9/10$, $\lambda_1 = 0.3$, $\lambda_2 = 0.35$, $\eta = 0.6$, $\kappa = 0.3$, $\alpha_1 = 0.4$, $\alpha_2 = 0.4$, $\beta_1 = 0.8$, $\beta_2 = 0.7$, $\beta_3 = 0.8$, $\mu_1 = 2/5$, $\mu_2 = 4/5$, $\gamma_1 = 7/10$, $\gamma_2 = 2/5$, $\delta_1 = 4/5$, $\delta_2 = 1/5$, $\delta_3 = 7/10$, $\phi_1 = 4/5$, $\phi_2 = 9/10$, $\phi_3 = 7/10$, $\xi_1 = 0.2$, $\xi_2 = 0.6$, $\zeta_1 = 0.2$, $\zeta_2 = 0.5$, $\zeta_3 = 0.6$, $T = \frac{2}{3}$, $f(t, x, y) = (t/5) [(2(|x + y|)^3 + 2|x| + |y|)/(3|x| + 4)]$ and $g(t, x, y) = (t/3) [(|x + y|^2 + 1)/(|x| + 2|y| + 3)]$. By using the Maple program, we can find

$$\begin{aligned}
 \Phi(0) & = \frac{T^{p_2}}{\Gamma(1 + p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \left(\frac{\eta^{p_2}}{\Gamma(1 + p_2)} \right. \\
 & \left. + \sum_{i=1}^2 |\alpha_i| \left[\frac{1}{\Gamma(1 + p_2)} \frac{\xi_i^{p_2+\mu_i\gamma_i}}{\mu_i^{\gamma_i}} \frac{\Gamma\left(\frac{p_2+\mu_i}{\mu_i}\right)}{\Gamma\left(\frac{p_2+\mu_i\gamma_i+\mu_i}{\mu_i}\right)} \right] \right) \\
 & \approx 2.401980728,
 \end{aligned} \tag{33}$$

and

$$\begin{aligned}
 \Lambda(0) & = \frac{T^{q_2}}{\Gamma(1 + q_2)} + \frac{\Gamma(q_1)}{\Gamma(q_1 + q_2)} \frac{T^{q_1+q_2-1}}{|\Psi|} \left(\frac{\kappa^{q_2}}{\Gamma(1 + q_2)} \right. \\
 & \left. + \sum_{j=1}^3 |\beta_j| \left[\frac{1}{\Gamma(1 + q_2)} \frac{\zeta_i^{q_2+\delta_j\phi_j}}{\delta_j^{\phi_j}} \frac{\Gamma\left(\frac{q_2+\delta_j}{\delta_j}\right)}{\Gamma\left(\frac{q_2+\delta_j\phi_j+\delta_j}{\delta_j}\right)} \right] \right)
 \end{aligned}$$

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$$\approx 1.427481620. \quad (34)$$

Thus $\Upsilon \approx 0.7205942184 < 1$.

Since $|f(t, x, y)| \leq (t/5) [((|x + y|)^3 + |x| + |y|)/2]$, $|g(t, x, y)| \leq (t/3) [((|x + y|)^2 + 1)/3]$, we choose $z_1(t) = t/10$, $\psi(x + y) = |x + y|^3 + |x| + |y|$, $z_2(t) = t/9$, $\omega(x + y) = (|x + y|)^2 + 1$. We can show that

$$\begin{aligned} & \sup_{r \in (0, \infty)} \frac{r}{\|z_1\|[\psi_1(r) + \psi_2(r)]\Phi(p_1) + \|z_2\|[\omega_1(r) + \omega_2(r)]\Lambda(q_1)} \\ & \approx 3.980031158 > 3.579024007 \approx \frac{1}{1 - \Upsilon}. \end{aligned}$$

Hence, by Theorem 3.5, the system (32) has at least one solution on $[0, \frac{2}{3}]$.

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**SUBORDINATION RESULTS FOR CERTAIN CLASS OF
ANALYTIC FUNCTIONS ASSOCIATED WITH
MITTAG-LEFFLER FUNCTION**

MANSOUR F. YASSEN

ABSTRACT. In this paper, we introduce a new class of analytic functions associated with Mittag-Leffler function in the open unit disk. Several properties of functions belonging to this class are derived.

1. INTRODUCTION

Let \mathbf{U} be the open unit disc $\mathbf{U} = \{z : |z| < 1\}$. Also, Let $\mathcal{A}(p)$ the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_n z^{n+p-1}, \tag{1.1}$$

which are analytic in \mathbf{U} , where $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. Also $f_i(z) \in \mathcal{A}(p), (i = 1, 2)$ defined by

$$f_i(z) = z^p + \sum_{n=2}^{\infty} a_{n,i} z^{n+p-1}, \quad (i = 1, 2) \tag{1.2}$$

the convolution (or Hadamard product) of $f_1(z)$ and $f_2(z)$ given by:

$$(f_1 * f_2)(z) := (f_2 * f_1)(z) := z^p + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^{n+p-1}. \tag{1.3}$$

The Mittag-Leffler function ([11],[12]) is defined by:

$$E_{\alpha}(z) = 1 + \frac{z}{\alpha!} + \frac{z^2}{(2\alpha)!} + \frac{z^3}{(3\alpha)!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}; Re(\alpha) > 0). \tag{1.4}$$

Some interesting properties and general of Mittag-Leffler function can be found *e.g.* in [2], [3], [4], [5], [6], [9], [13], [14], [15], [16], [18], [21], [22] and [23]. The function $E_{\alpha,\beta}^{\eta,k}(z) (z \in \mathbb{C})$ introduced by Srivastava and Tomovski [20] in the form:

$$E_{\alpha,\beta}^{\eta,k}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{nk} z^n}{\Gamma(\alpha n + \beta) n!}, \quad (\alpha, \beta, \eta \in \mathbb{C}; Re(\alpha) > \max\{0, Re(k)-1\}; Re(k) > 0), \tag{1.5}$$

where

$$(\eta)_n = \frac{\Gamma(\eta + n)}{\Gamma(\eta)} = \begin{cases} 1, & n = 0, \\ \eta(\eta + 1)(\eta + 2) \dots (\eta + n - 1), & n \in \mathbb{N}. \end{cases} \tag{1.6}$$

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The function $E_{\alpha,\beta}^{\eta,k}(z)$ proved by Srivastava and Tomovski [20] is an entire function in the complex z -plane. Attiya [1] defined the function $Q_{\alpha,\beta}^{\eta,k}(z)$ by

$$Q_{\alpha,\beta}^{\eta,k}(z) = \frac{\Gamma(\alpha + \beta)}{(\eta)_k} \left(E_{\alpha,\beta}^{\eta,k}(z) - \frac{1}{\Gamma(\beta)} \right), \quad (z \in \mathbf{U}), \quad (1.7)$$

very recently, Attiya [1] introduce the operator

$$\mathcal{H}_{\alpha,\beta}^{\eta,k}(f(z)) : \mathcal{A}(1) \rightarrow \mathcal{A}(1),$$

defined, in terms of convolution by

$$\begin{aligned} \mathcal{H}_{\alpha,\beta}^{\eta,k}(f(z)) &= Q_{\alpha,\beta}^{\eta,k}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\eta + nk)\Gamma(\alpha + \beta)}{\Gamma(\eta + k)\Gamma(n\alpha + \beta)} a_n z^n \quad (z \in \mathbf{U}). \end{aligned} \quad (1.8)$$

Analogous to $\mathcal{H}_{\alpha,\beta}^{\eta,k}(f(z))$, we introduce the operator $\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z))$ as follows

$$\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) : \mathcal{A}(p) \rightarrow \mathcal{A}(p), \quad (1.9)$$

where

$$\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) = Q_{\alpha,\beta,p}^{\eta,k}(z) * f(z), \quad (z \in \mathbf{U}). \quad (1.10)$$

and

$$Q_{\alpha,\beta,p}^{\eta,k}(z) = \frac{z^{p-1}\Gamma(\alpha + \beta)}{(\eta)_k} \left(E_{\alpha,\beta}^{\eta,k}(z) - \frac{1}{\Gamma(\beta)} \right), \quad (z \in \mathbf{U}), \quad (1.11)$$

from equations (1.9), (1.10) and (1.11) we not that

$$\begin{aligned} \mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) &= Q_{\alpha,\beta,p}^{\eta,k}(z) * f(z) \\ &= z^p + \sum_{n=2}^{\infty} \frac{\Gamma(\eta + nk)\Gamma(\alpha + \beta)}{\Gamma(\eta + k)\Gamma(n\alpha + \beta)} a_n z^{n+p-1} \quad (z \in \mathbf{U}), \end{aligned} \quad (1.12)$$

when $p = 1$, the operator $\mathcal{H}_{\alpha,\beta,1}^{\eta,k}(f(z))$ is the Attiya operator $\mathcal{H}_{\alpha,\beta}^{\eta,k}(f(z))$ [1].

A function $f(z) \in \mathcal{A}(1)$ is said to be in the class $\mathcal{S}^*(\sigma)$ [7] and [19] or (star-like of order σ in \mathbf{U}) if:

$$\mathcal{S}^*(\sigma) := \left\{ f(z) : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \sigma, 0 \leq \sigma < 1, z \in \mathbf{U} \right\}. \quad (1.13)$$

A function $f(z) \in \mathcal{A}(1)$ is said to be in the class $\mathfrak{R}(\sigma)$ [7] and [17] or (pre-starlike of order σ in \mathbf{U}) if:

$$\mathfrak{R}(\sigma) := \left\{ f(z) : \frac{z}{(1-z)^{2(1-\sigma)}} * f(z) \in \mathcal{S}^*(\sigma), \sigma < 1, z \in \mathbf{U} \right\}. \quad (1.14)$$

The function $g(z)$ is called subordinate to $G(z)$, if there exist a Schwarz function $h(z)$, analytic in \mathbf{U} , with $h(0) = 0$ and $|h(z)| \leq 1$, such that $g(z) = G(h(z))$ for all $z \in \mathbf{U}$.

This subordination is denoted by $g(z) \prec G(z)$. If the function $G(z)$ is univalent in \mathbf{U} , then $g(z) \prec G(z)$ if and only if $g(0) = G(0)$ and $g(\mathbf{U}) \subset G(\mathbf{U})$.

Let \mathfrak{T} be the class of function $w(z)$ with $w(0) = 1$, which are analytic and univalent in \mathbf{U} .

Definition 1. Let $f(z) \in \mathcal{A}(p)$, then $f(z)$ is said to be in the class $\mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$ if it satisfies the following condition

$$\frac{(1-\delta)}{p} z^{-p+1} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta}{p(p-1)} z^{-p+2} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' \prec w(z), \quad (1.15)$$

where $\delta \in \mathbb{C}$, $p \in \mathbb{N} \setminus \{1\}$ and $w(z) \in \mathfrak{T}$.

The main object of our paper is to investigate and introduce some subordination results of the class $\mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$.

2. SOME LEMMAS

In our paper, we use the following lemmas:

Lemma 1.1 [10]. Let $G(z)$ be analytic function in \mathbf{U} and $w(z)$ be analytic and convex univalent in \mathbf{U} with $G(0) = w(0)$, if

$$G(z) + \frac{1}{\vartheta} z G'(z) \prec w(z), \quad (2.1)$$

where $Re(\vartheta) \geq 0$ and $\vartheta \neq 0$, then $G(z) \prec w(z)$.

Lemma 1.2 [17]. Let $\sigma < 1$, $f(z) \in \mathcal{S}^*(\sigma)$, and $G(z) \in \mathfrak{R}(\sigma)$, then, for analytic function $F(z)$ in \mathbf{U} ,

$$\frac{G * (fF)}{G * f}(\mathbf{U}) \subset \overline{co}(F(\mathbf{U})), \quad (2.2)$$

where $\overline{co}(F(\mathbf{U}))$ denote the closed convex hull of $F(\mathbf{U})$.

Lemma 1.3 [8]. Let $G(z) = 1 + \sum_{n=k}^{\infty} d_n z^n, (k \in \mathbb{N})$ be analytic function and convex univalent function in \mathbf{U} . If $Re\{G(z)\} > 0, (z \in \mathbf{U})$, then

$$Re\{G(z)\} \geq \frac{1 - |z|^k}{1 + |z|^k} \quad (k \in \mathbb{N}; z \in \mathbf{U}). \quad (2.3)$$

3. PROPERTIES OF THE CLASS $\mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$

In this section, we let $p \in \mathbb{N}$ and $p > 1$.

Theorem 3.1. *let $0 \leq \delta_1 < \delta_2$. Then $\mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta_2; w) \subset \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta_1; w)$.*

Proof. Let $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta_2; w)$ and $0 \leq \delta_1 < \delta_2$. Suppose that

$$\phi(z) = \frac{z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'. \quad (3.1)$$

Therefore, the function $\phi(z)$ in the above equation is analytic in \mathbf{U} with $\phi(0) = 1$. Differentiating the both sides of the above equation w.r.t. z , we have

$$\phi'(z) = \frac{(1-p)z^{-p}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' . \quad (3.2)$$

By using Equation (1.15), we have

$$\begin{aligned} \frac{(1-\delta_2)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta_2 z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' &= \phi(z) + \frac{\delta_2 z \phi'(z)}{(p-1)} \\ &\prec w(z). \end{aligned} \quad (3.3)$$

Using Lemma 1.1, we have

$$\phi(z) \prec w(z). \tag{3.4}$$

Since $0 \leq \delta_1/\delta_2 < 1$ and $w(z)$ is univalent in \mathbf{U} , using equations (3.1) and (3.4), we given that

$$\begin{aligned} & \frac{(1 - \delta_1)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta_1 z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' = \left(1 - \frac{\delta_1}{\delta_2} \right) \phi(z) \\ & + \frac{\delta_1}{\delta_2} \left(\frac{(1 - \delta_2)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta_2 z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' \right) \prec w(z). \end{aligned} \tag{3.5}$$

Therefore $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta_1; w)$, and the proof of Theorem 1.1 is completed. ■

Theorem 3.2. *Let $\delta > 0, \rho > 0$, and $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; \rho w + 1 - \rho)$. If $\rho \leq \rho_0$, where*

$$\rho_0 = \frac{1}{2} \left(1 - \frac{(p-1)}{\delta} \int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t} dt \right)^{-1}. \tag{3.6}$$

Then $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(0; w)$. The bound ρ_0 is sharp in the case $w(z) = 1/(1-z)$.

Proof. Let $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; \rho w + 1 - \rho)$ with $\delta > 0, \rho > 0$. Suppose that

$$\phi(z) = \frac{z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'. \tag{3.7}$$

Then we have

$$\begin{aligned} \phi(z) + \frac{\delta z \phi'(z)}{(p-1)} &= \frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' \\ &\prec \rho w(z) + 1 - \rho. \end{aligned} \tag{3.8}$$

Using Lemma 1.1, we have

$$\phi(z) \prec \frac{\rho(p-1)z^{-(p-1)/\delta}}{\delta} \int_0^z u^{((p-1)/\delta)} w(u) du + 1 - \rho = (w * \varphi)(z), \tag{3.9}$$

where

$$\varphi(z) = \frac{\rho(p-1)z^{-(p-1)/\delta}}{\delta} \int_0^z \frac{u^{((p-1)/\delta)-1}}{1-u} du + 1 - \rho. \tag{3.10}$$

If $0 < \rho \leq \rho_0$ where $\rho_0 (> 1)$ is given by (3.6), then it follows from (3.10) that

$$\begin{aligned} \operatorname{Re} \{ \varphi(z) \} &= \frac{\rho(p-1)}{\delta} \int_0^1 t^{((p-1)/\delta)-1} \operatorname{Re} \left\{ \frac{1}{1-tz} \right\} dt + 1 - \rho \\ &> \frac{\rho(p-1)}{\delta} \int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t} dt + 1 - \rho \\ &\geq \frac{1}{2} \quad (z \in \mathbf{U}). \end{aligned} \tag{3.11}$$

Using the Herglotz representation for $\varphi(z)$. Also, from Equations (3.7) and (3.9) we obtain

$$\frac{z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' \prec (w * \varphi)(z) \prec w(z), \tag{3.12}$$

since $w(z)$ is convex univalent in \mathbf{U} . Therefore $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(0; w)$. If $w(z) = 1/(1-z)$ and $f(z) \in \mathcal{A}(p)$ defined by:

$$\frac{z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' = \frac{\rho(p-1)z^{-(p-1)/\delta}}{\delta} \int_0^z \frac{u^{((p-1)/\delta)-1}}{1-u} du + 1 - \rho, \quad (3.13)$$

we can see that

$$\frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' = \rho w(z) + 1 - \rho. \quad (3.14)$$

Thus $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; \rho h + 1 - \rho)$. Also, for $\rho > \rho_0$, we have (at $z \rightarrow -1$)

$$Re \left\{ \frac{z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' \right\} \rightarrow \frac{\rho(p-1)}{\delta} \int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t} dt + 1 - \rho < \frac{1}{2}, \quad (3.15)$$

which obtains $f(z) \notin \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(0; w)$. Therefore, the value ρ_0 cannot be increased when $w(z) = 1/(1-z)$. ■

Theorem 3.3. Let $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$, $\phi(z) \in \mathcal{A}(p)$, and

$$Re \{ z^{-p} \phi(z) \} > \frac{1}{2} \quad (z \in \mathbf{U}). \quad (3.16)$$

Then $(f * \phi)(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$.

Proof. For $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$ and $\phi(z) \in \mathcal{A}(p)$, we have

$$\begin{aligned} & \frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}((f * \phi)(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}((f * \phi)(z)) \right)'' \\ &= \frac{(1-\delta)}{p} (z^{-p} \phi(z)) * \left(z^{-p+1} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' \right) \\ & \quad + \frac{\delta}{p(p-1)} (z^{-p} \phi(z)) * \left(z^{-p+2} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' \right) \\ &= (z^{-p} \phi(z)) * \varphi(z), \end{aligned} \quad (3.17)$$

where

$$\varphi(z) = \frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' . \quad (3.18)$$

Using (3.16), the function $z^{-p} \phi(z)$ has the Herglotz Representation

$$z^{-p} \phi(z) = \int_{|y|=1} \frac{d\mu(y)}{(1-yz)} \quad (z \in \mathbf{U}), \quad (3.19)$$

where $\mu(y)$ is a probability measure defined on the circle $|y| = 1$ and

$$\int_{|y|=1} d\mu(y) = 1.$$

Since $w(z)$ is convex univalent in \mathbf{U} , it follows from (3.17) to (3.19) that

$$\begin{aligned} & \frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}((f * \phi)(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}((f * \phi)(z)) \right)'' \\ &= \int_{|y|=1} \varphi(yz) d\mu(y) \prec w(z). \end{aligned} \quad (3.20)$$

This shows that $(f * \phi)(z) \in \mathfrak{F}_{\alpha, \beta, p}^{\eta, k}(\delta; w)$ and the theorem is proved. ■

Theorem 3.4. Let $f(z) \in \mathfrak{F}_{\alpha, \beta, p}^{\eta, k}(\delta; w)$, $\phi(z) \in \mathcal{A}(p)$, and

$$z^{-p+1}\phi(z) \in \Re(\sigma) \quad (\sigma < 1, z \in \mathbf{U}). \tag{3.21}$$

Then $(f * \phi)(z) \in \mathfrak{F}_{\alpha, \beta, p}^{\eta, k}(\delta; w)$.

Proof. For $f(z) \in \mathfrak{F}_{\alpha, \beta, p}^{\eta, k}(\delta; w)$ and $\phi(z) \in \mathcal{A}(p)$ from (3.17), we have

$$\begin{aligned} & \frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha, \beta, p}^{\eta, k}((f * \phi)(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha, \beta, p}^{\eta, k}((f * \phi)(z)) \right)'' \\ &= \frac{(z^{-p+1}\phi(z)) * (z\varphi(z))}{(z^{-p+1}\phi(z)) * z} \quad (z \in \mathbf{U}), \end{aligned} \tag{3.22}$$

where $\varphi(z)$ is defined as in (3.18). Since $w(z)$ is convex univalent in \mathbf{U} ,

$$\varphi(z) \prec w(z), \quad z^{-p+1}\phi(z) \in \Re(\sigma) \quad \text{and} \quad z \in \mathcal{S}^*(\sigma), \quad (\sigma < 1).$$

It follows from (3.22) and Lemma 1.2 the desired result. ■

Theorem 3.5. Let $\delta \geq 0$ and

$$f_i(z) = z^p + \sum_{n=2}^{\infty} a_{n,i} z^{n+p-1} \in \mathfrak{F}_{\alpha, \beta, p}^{\eta, k}(\delta; w), \quad (i = 1, 2) \tag{3.23}$$

where

$$w_i(z) = \gamma_i + (1 - \gamma_i) \frac{1+z}{1-z} \quad \text{and} \quad \gamma_i < 1, (i = 1, 2). \tag{3.24}$$

If $f(z) \in \mathcal{A}(p)$ is defined by

$$\mathcal{H}_{\alpha, \beta, p}^{\eta, k}(f(z)) = \int_0^z \left(\mathcal{H}_{\alpha, \beta, p}^{\eta, k}(f_1(u)) \right)' * \left(\mathcal{H}_{\alpha, \beta, p}^{\eta, k}(f_2(u)) \right)' du. \tag{3.25}$$

Then $f(z) \in \mathfrak{F}_{\alpha, \beta, p}^{\eta, k}(\delta; w)$, where

$$w(z) = \gamma + (1 - \gamma) \frac{1+z}{1-z}, \tag{3.26}$$

where γ is given by

$$\gamma = \begin{cases} p - 4p(1 - \gamma_1)(1 - \gamma_2) \left(1 - \frac{p-1}{\delta} \int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t} dt \right); & (\delta > 0) \\ p - 2p(1 - \gamma_1)(1 - \gamma_2); & (\delta = 0), \end{cases} \tag{3.27}$$

the value of γ is the best possible.

Proof. For the case when $\delta > 0$. by putting

$$G_i(z) = \frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha, \beta, p}^{\eta, k}(f_i(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha, \beta, p}^{\eta, k}(f_i(z)) \right)'' \quad (i = 1, 2), \tag{3.28}$$

for $f_i(z)$, $(i = 1, 2)$ given by (3.23), we find that

$$G_i(z) = 1 + \sum_{n=2}^{\infty} b_{n,i} z^{n-1} \prec \gamma_i + (1 - \gamma_i) \frac{1+z}{1-z} \quad (i = 1, 2),$$

and

$$\left(\mathcal{H}_{\alpha, \beta, p}^{\eta, k}(f_i(z)) \right)' = \frac{p(p-1)z^{-(p-1)(1-\delta)/\delta}}{\delta} \int_0^z u^{((p-1)/\delta)-1} G_i(u) du \quad (i = 1, 2).$$

Now, if $f(z) \in \mathcal{A}(p)$ is defined by (3.25), we find from (3.30) that

$$\begin{aligned} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z))\right)' &= \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f_1(z))\right)' * \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f_2(z))\right)' \\ &= \left(\frac{p(p-1)z^{p-1}}{\delta} \int_0^1 t^{((p-1)/\delta)-1} G_1(tz) dt\right) \\ &\quad * \left(\frac{p(p-1)z^{p-1}}{\delta} \int_1^z t^{((p-1)/\delta)-1} G_2(tz) dt\right) \\ &= \left(\frac{p(p-1)z^{p-1}}{\delta} \int_1^z t^{((p-1)/\delta)-1} G(tz) dt\right) \end{aligned} \tag{3.29}$$

where

$$G(z) = \frac{p(p-1)}{\delta} \int_0^1 u^{((p-1)/\delta)-1} (G_1 * G_2)(tz) dt. \tag{3.30}$$

Also, by using (3.29) and the Herglotz theorem, we see that

$$Re \left\{ \left(\frac{G_1(z) - \gamma_1}{1 - \gamma_1} \right) * \left(\frac{1}{2} + \frac{G_2(z) - \gamma_2}{2(1 - \gamma_2)} \right) \right\} > 0 \quad (z \in \mathbf{U}), \tag{3.31}$$

which gives

$$Re \{(G_1 * G_2)(z)\} > \gamma_0 = 1 - 2(1 - \gamma_1)(1 - \gamma_2) \quad (z \in \mathbf{U}). \tag{3.32}$$

According to Lemma 1.3, we have

$$Re \{(G_1 * G_2)(z)\} \geq \gamma_0 + (1 - \gamma_0) \left(\frac{1 - |z|}{1 + |z|} \right) \quad (z \in \mathbf{U}). \tag{3.33}$$

Now it follows from (3.31) to (3.35) that

$$\begin{aligned} Re \left\{ \frac{(1 - \delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z))\right)'' \right\} &= Re\{G(z)\} \\ &= \frac{p(p-1)}{\delta} \int_0^1 t^{((p-1)/\delta)-1} Re\{(G_1 * G_2)(tz)\} dt \\ &\geq \frac{p(p-1)}{\delta} \int_0^1 t^{((p-1)/\delta)-1} \left(\beta_0 + (1 - \beta_0) \frac{1 - t|z|}{1 + t|z|} \right) dt \\ &> p\gamma_0 + \frac{p(p-1)(1 - \gamma_0)}{\delta} \int_0^1 t^{((p-1)/\delta)-1} \frac{1 - t}{1 + t} dt \\ &= p - 4p(1 - \gamma_1)(1 - \gamma_2) \left(1 - \frac{p-1}{\delta} \int_0^1 \frac{t^{((p-1)/\delta)-1}}{1 + t} dt \right) \\ &= \gamma \quad (z \in \mathbf{U}). \end{aligned} \tag{3.34}$$

which proves that $f(z) \in \mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$ for the function $w(z)$ given by (3.26). In order to show that the bound γ is Sharp, we take the functions $f_i(z) \in \mathcal{A}(p)$ ($i = 1, 2$) defined by

$$\begin{aligned} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f_i(z))\right)' &= \frac{p(p-1)z^{-(p-1)(1-\delta)/\delta}}{\delta} \\ &\quad \times \int_0^z u^{((p-1)/\delta)-1} \left(\gamma_i + (1 - \gamma_i) \frac{1 + u}{1 - u} \right) du, \end{aligned} \tag{3.35}$$

for $i = 1, 2$ and, we have

$$\begin{aligned} G_i(z) &= \frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f_i(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f_i(z)) \right)'' \\ &= \gamma_i + (1-\gamma_i) \frac{1+z}{1-z} \quad (i = 1, 2), \end{aligned} \tag{3.36}$$

and

$$(G_1 * G_2)(z) = 1 + 4(1-\gamma_1)(1-\gamma_2) \frac{z}{1-z}. \tag{3.37}$$

Hence, for $f(z) \in \mathcal{A}(p)$ given by (3.25), we obtain

$$\begin{aligned} &\frac{(1-\delta)z^{-p+1}}{p} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)' + \frac{\delta z^{-p+2}}{p(p-1)} \left(\mathcal{H}_{\alpha,\beta,p}^{\eta,k}(f(z)) \right)'' \\ &= \frac{p(p-1)}{\delta} \int_0^1 t^{((p-1)/\delta)-1} \left(1 + 4(1-\gamma_1)(1-\gamma_2) \frac{tz}{1-tz} \right) dt \\ &\qquad\qquad\qquad \longrightarrow \gamma \quad (as \quad z \longrightarrow -1). \end{aligned} \tag{3.38}$$

The proof is simple in the case of $\delta = 0$, therefore, we omit the details involved. ■

Conclusions

we introduced the class $\mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$ of analytic functions associated with Mittag-Leffler function. Conclusion property of the class $\mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$ is obtained, sufficient condition of the class $\mathfrak{T}_{\alpha,\beta,p}^{\eta,k}(\delta; w)$ is also derived. Furthermore, several properties of functions belonging to this class are derived.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, DAMIETTA UNIVERSITY, NEW DAMIETTA 34517, EGYPT.

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND HUMANITIES IN AL-AFLAJ, PRINCE SATTAM BIN ABDULAZIZ UNIVERSITY, KINGDOM OF SAUDI ARABIA

E-mail address: mansouraliegh@yahoo.com

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