Volume 28, Number 5 October 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

ON HARMONIC MULTIVALENT FUNCTIONS DEFINED BY A NEW DERIVATIVE OPERATOR

ADRIANA CĂTAŞ[1]*, ROXANA ŞENDRUŢIU[2]

ABSTRACT. In the present paper, we define and investigate a new class of multivalent harmonic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, under certain conditions involving a new generalized differential operator. Coefficient inequalities, distortion bounds and a covering result are also obtained.

Keywords: differential operator, harmonic function, coefficient bounds.
2000 Mathematical Subject Classification: 30C45.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simple connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. (See [4] for more details.)

Denote by $S_{\mathcal{H}}(p, n), (p, n \in \mathbb{N} = \{1, 2, \ldots\})$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disc U for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_{\mathcal{H}}(p, n)$ we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p+n-1}^{\infty} b_k z^k, \quad |b_{p+n-1}| < 1. \tag{1.1}$$

Let $\tilde{S}_{\mathcal{H}}(p, n, m), (p, n \in \mathbb{N}, m \in \mathbb{N}_0 \cup \{0\})$ denote the family of functions $f_m = h + \bar{g}_m$ that are harmonic in D with the normalization

$$h(z) = z^p - \sum_{k=p+n}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^m \sum_{k=p+n-1}^{\infty} |b_k| z^k, \quad |b_{p+n-1}| < 1. \tag{1.2}$$

2. Coefficient bounds for the new classes $AL_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ and $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$

We propose for the beginning a new generalized differential operator as follows.

Definition 2.1. Let $H(U)$ denote the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and let $\mathcal{A}(p)$ be the subclass of the functions belonging

1

to $H(U)$ of the form $h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k$. For $m \in \mathbb{N}_0$, $\lambda \geq 0$, $\delta \in \mathbb{N}_0$, $l \geq 0$ we define the generalized differential operator $I^m_{\lambda,\delta}(p,l)$ on $\mathcal{A}(p)$

$$I^m_{\lambda,\delta}(p,l)h(z) = (p+l)^m z^p + \sum_{k=p+n}^{\infty} [p+\lambda(k-p)+l]^m C(\delta,k) a_k z^k, \tag{2.1}$$

$$C(\delta,k) = \begin{pmatrix} k+\delta-1 \\ \delta \end{pmatrix} = \frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(\delta+1)}. \tag{2.2}$$

Remark 2.2. When $\lambda = 1$, $p = 1$, $l = 0$, $\delta = 0$ we get Sălăgean differential operator [10]; $p = 1$, $m = 0$ gives Ruscheweyh operator [9]; $p = 1$, $l = 0$, $\delta = 0$ implies Al-Oboudi differential operator of order m (see [1]); $\lambda = 1$, $p = 1$, $l = 0$ operator (2.1) reduces to Al-Shaqsi and Darus differential operator [2] and when $p = 1$, $l = 0$ we reobtain the operator introduced by Darus and Ibrahim in [5].

Definition 2.3. Let $f \in S_{\mathcal{H}}(p,n)$, $p \in \mathbb{N}$. Using the operator (2.1) for $f = h + \bar{g}$ given by (1.1) we define the differential operator of f as

$$I^m_{\lambda,\delta}(p,l)f(z) = I^m_{\lambda,\delta}(p,l)h(z) + (-1)^m \overline{I^m_{\lambda,\delta}(p,l)g(z)} \tag{2.3}$$

$$I^m_{\lambda,\delta}(p,l)h(z) = (p+l)^m z^p + \sum_{k=p+n}^{\infty} [p+\lambda(k-p)+l]^m C(\delta,k) a_k z^k \tag{2.4}$$

$$I^m_{\lambda,\delta}(p,l)g(z) = \sum_{k=p+n-1}^{\infty} [p+\lambda(k-p)+l]^m C(\delta,k) b_k z^k. \tag{2.5}$$

Remark 2.4. When $\lambda = 1$, $l = 0$, $\delta = 0$ the operator (2.3) reduces to the operator introduced earlier in [7] by Jahangiri et al.

Definition 2.5. A function $f \in S_{\mathcal{H}}(p,n)$ belongs to the class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if

$$\frac{1}{p+l}\text{Re}\left\{\frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^m_{\lambda,\delta}(p,l)f(z)}\right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{2.6}$$

where $I^m_{\lambda,\delta}f$ is defined by (2.3), for $m \in \mathbb{N}_0$. Finally, we define the subclass

$$\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \equiv AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \cap \tilde{S}_{\mathcal{H}}(p,n,m). \tag{2.7}$$

Remark 2.6. The class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ includes a variety of well-known subclasses of $S_{\mathcal{H}}(p,n)$. For example, letting $n = 1$ we get $AL_{\mathcal{H}}(1,1,0,\alpha,1,0) \equiv HK(\alpha)$ in [6], for $n = 1$, $AL_{\mathcal{H}}(1,m-1,0,\alpha,1,0) \equiv S_H(t,u,\alpha)$ in [11], $AL_{\mathcal{H}}(p,n+p,0,\alpha,1,0) \equiv SH_p(n,\alpha)$ in [8] and $n = 1$, $AL_{\mathcal{H}}(1,m,\delta,\alpha,1,0) \equiv M_{\mathcal{H}}(m,\delta,\alpha)$ in [3].

Theorem 2.7. *Let $f = h + \bar{g}$ be given by (1.1). If*

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k| + \tag{2.8}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\le 1,$$

with $\lambda n \ge \alpha(p+l)$, *where*

$$d_{p,k}(m,\lambda,l)=[p+\lambda(k-p)+l]^m \tag{2.9}$$

then $f\in AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Proof. Using the fact that $\frac{1}{p+l}\mathrm{Re}\, w\ge\alpha$ if and only if $|(p+l)-(p+l)\alpha+w|\ge |(p+l)+(p+l)\alpha-w|$, it is sufficient to show that

$$\begin{aligned}&|(p+l)(1-\alpha)I^m_{\lambda,\delta}(p,l)f(z)+I^{m+1}_{\lambda,\delta}(p,l)f(z)|-\\ &-|(p+l)(1+\alpha)I^m_{\lambda,\delta}(p,l)f(z)-I^{m+1}_{\lambda,\delta}(p,l)f(z)|\ge 0.\end{aligned}\tag{2.10}$$

Substituting $I^m_{\lambda,\delta}(p,l)f(z)$ and $I^{m+1}_{\lambda,\delta}(p,l)f(z)$ in (2.10) yields by (2.8)

$$\begin{aligned}&|(p+l)(1-\alpha)I^m_{\lambda,\delta}(p,l)f(z)+I^{m+1}_{\lambda,\delta}(p,l)f(z)|-\\ &-|(p+l)(1+\alpha)I^m_{\lambda,\delta}(p,l)f(z)-I^{m+1}_{\lambda,\delta}(p,l)f(z)|>\\ &>2(p+l)^{m+1}(1-\alpha)\left\{1-\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|-\right.\\ &\left.-\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\right\}.\end{aligned}$$

The last expression is nonnegative by (2.8) and therefore the proof is complete. □

Remark 2.8. The harmonic function

$$\begin{aligned}f(z)=z^p&+\sum_{k=p+n}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}x_k z^k+\\ &+\sum_{k=p+n-1}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\overline{y_k z^k},\end{aligned}\tag{2.11}$$

where $\sum_{k=p+n}^{\infty}|x_k|+\sum_{k=p+n-1}^{\infty}|y_k|=1$, $0\le\alpha<1$, $m\in\mathbb{N}_0$, $\lambda n\ge\alpha(p+l)$, $\lambda\ge 0$ and $d_{p,k}(m,\lambda,l)$ is given in (2.9), show that the coefficient bound expressed by (2.8) is sharp.

Theorem 2.9. *Let* $f_m=h+\bar{g_m}$ *be given by (1.2). Then* $f_m\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ *if and only if*

$$\begin{aligned}&\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+\\ &+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\le 1,\end{aligned}\tag{2.12}$$

where $\lambda n\ge\alpha(p+l)$, $0\le\alpha<1$, $m\in\mathbb{N}_0$, $\lambda\ge 0$ *and* $d_{p,k}(m,\lambda,l)$ *is given in (2.9).*

Proof. Since $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \subset AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, we only need to prove the "only if" part of the theorem. For this part we consider that $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. Then

$$\operatorname{Re}\left\{\frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^{m}_{\lambda,\delta}(p,l)f(z)} - \alpha(p+l)\right\} =$$

$$\operatorname{Re}\left\{\frac{(p+l)^{m+1}(1-\alpha)z^p - \sum_{k=p+n}^{\infty}[(p+l)(1-\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)a_k z^k}{(p+l)^m z^p - \sum_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)a_k z^k + (-1)^{2m}\sum_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)\overline{b_k z^k}}\right.$$

$$\left. - \frac{(-1)^{2m}\sum_{k=p+n-1}^{\infty}[(p+l)(1+\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)\overline{b_k z^k}}{(p+l)^m z^p - \sum_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)a_k z^k + (-1)^{2m}\sum_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)\overline{b_k z^k}}\right\} \geq 0,$$

where $\xi(m,\lambda,l;\delta,k) = d_{p,k}(m,\lambda,l)C(\delta,k)$.
The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, we must have

$$\frac{(p+l)^{m+1}(1-\alpha) - \sum_{k=p+n}^{\infty}[(p+l)(1-\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)|a_k|r^{k-p}}{(p+l)^m - \sum_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)|a_k|r^{k-p} + \sum_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)|b_k|r^{k-p}} - \tag{2.13}$$

$$\frac{\sum_{k=p+n-1}^{\infty}[(p+l)(1+\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)|b_k|r^{k-p}}{(p+l)^m - \sum_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)|a_k|r^{k-p} + \sum_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)|b_k|r^{k-p}} \geq 0.$$

If the condition (2.12) does not hold, then the numerator in (2.13) is negative for r sufficiently close to 1. Hence there exists a $z_0 = r_o$ in $(0,1)$ for which the quotient in (2.13) is negative. This contradicts the required condition for $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ and so the proof is complete. □

3. Distortion bounds

The following theorem gives the distortion bounds for functions in $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ which yields a covering result for this class.

Theorem 3.1. *Let* $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$*, with* $0 \le \alpha < 1$*,* $\lambda n \ge \alpha(p+l)$*,* $m \in \mathbb{N}_0$*,* $\lambda \ge 0$*. Then for* $|z| = r < 1$ *one obtains*

$$|f(z)| \le (1+|b_{p+n-1}|r^{n-1})r^p + \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot \tag{3.1}$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}$$

and

$$|f(z)| \ge (1-|b_{p+n-1}|r^{n-1})r^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}.$$

Proof. We only prove the left-hand inequality. The proof for the right-hand inequality is similar and will be omitted.

Let $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. Taking the absolute value of f we have

$$|f(z)| = |z^p - \sum_{k=p+n}^{\infty} a_k z^k + (-1)^m \sum_{k=p+n-1}^{\infty} b_k \bar{z}^k| \ge (1-|b_{p+n-1}|r^{n-1})r^p - \sum_{k=p+n}^{\infty}(|a_k|+|b_k|)r^{p+n}$$

$$= (1-|b_{p+n-1}|r^{n-1})r^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda n]d_{p,p+n}(m,\lambda,l)C(\delta,p+n)}{(p+l)^{m+1}(1-\alpha)}(|a_k|+|b_k|)r^{p+n} \ge$$

$$(1-|b_{p+n-1}|r^{n-1})r^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}.$$

The bounds given in Theorem 3.1 for the functions f of the form (1.2) also hold for the functions of the form (1.1) if the coefficient condition (2.8) is satisfied. The upper bound given for $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is sharp and the equality occurs for the function

$$f(z) = z + |b_{p+n-1}|\bar{z}^p + \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p},$$

where $|b_{p+n-1}| \le \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}$ □

The following covering result follows from the left-hand inequality in Theorem 3.1.

Corollary 3.2. *If the function* $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, *then*

$$\left\{ w\colon |w| < \frac{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)-(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} - \right.$$

$$\left. -\frac{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)-E^{\alpha}_{p,\delta}(m,\lambda,l)}{[p(1-\alpha)+\lambda n+l]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)}\cdot|b_{p+n-1}| \right\} \subset f(U)$$

where $E^{\alpha}_{p,\delta}(m,\lambda,l) = [(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)$.

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Inter. J. of Math. and Mathematical Sci., **27**(2004), 1429-1436.

[2] K. Al-Shaqsi, M. Darus, *An operator defined by convolution involving polylogarithms functions*, Journal of Math. and Statistics, **4**(1)(2008), 46-50.

[3] K. Al-Shaqsi, M. Darus, *On Harmonic Functions Defined by Derivative Operator*, Journal of Inequalities and Applications, vol. 2008, Article ID 263413, doi: 10.1155/2008/263413.

[4] J. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn, Ser. A I. Math. **9**(1984), 3-25.

[5] M. Darus, R. W. Ibrahim, *On new classes of univalent harmonic functions defined by generalized differential operator*, Acta Universitatis Apulensis, **18**(2009), 61-69.

[6] J. M. Jahangiri, *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklowdowska Sect. A, **52**(1998), 57-66.

[7] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya *Sălăgean type harmonic univalent functions* South. J. Pure Appl. Math., **2**(2002), 77-82.

[8] Om P. Ahuja and J. M. Jahangiri, *Multivalent harmonic starlike functions with missing coefficients*, Math. Sci. Res. J., **7**(9)(2003), 347-352.

[9] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[10] Gr. Şt. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, Heidelberg and New York, 1013(1983), 362-372.

[11] Sibel Yalcin, *A new class of Sălăgean-type harmonic univalent functions* Appl. Math. Letters, **18**(2005), 191-198.

[1] Department of Mathematics and Computer Sciences, University of Oradea,
Str. Universităţii, No.1, 410087 Oradea, Romania
* *Corresponding author: acatas@gmail.com*

[2] Faculty of Environmental Protection, University of Oradea,
Str. B-dul Gen. Magheru, No.26, 410048 Oradea, Romania
E-mail address: roxana.sendrutiu@gmail.com

GOOD AND SPECIAL WEAKLY PICARD OPERATORS PROPERTIES FOR A CLASS OF DISCRETE LINEAR OPERATORS

LOREDANA-FLORENTINA IAMBOR, ADRIANA CĂTAŞ

Abstract. Based on the results of the weakly Picard operators theory, in this paper we study the good and special convergence of the iterates of a general class of positive linear operators of discrete type introduced by O.Agratini and I.A. Rus ([1]).

2010 AMS Mathematics Subject Classification: 47H10, 41A36.

Keywords and phrases: linear positive operators, weakly Picard operators, good and special Picard operators.

1. Introduction and Preliminaries

The study of the convergence of the sequence of successive approximations is realized in metric spaces. That is, for (X, d) metric space and $A : X \to X$ an operator, for any $x \in X$ can be considered the sequence:

$$(A^m(x))_{m\in\mathbb{N}}, \ x \in X \tag{1}$$

where $A^0 = 1_X$ and $A^m = A^{m-1} \circ A$ for $m \in \mathbb{N}^*$.

Investigating the properties of sequence (1), L. d'Apuzzo introduced in 1976 (see [3]) the good and special convergence, giving necessary and sufficient conditions for this kind of convergence (see [2]). In paper [3], she considers the good and special convergence of type M, as a particular case, in which the sequence $(d(A^m(x), A^\infty(x)))_{m\in\mathbb{N}}$, (respectively, $\left(d\left(A^m(x), A^{m-1}(x)\right)\right)_{m\in\mathbb{N}}$) is strictly decreasing for any x. I.A. Rus introduced, in paper (see [8]), the good and special weakly Picard operators .

In what follow, let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we will use the following notations:
$P(X) := \{Y \subset X | \, Y \neq \emptyset\}$;
$F_A := \{x \in X | \, A(x) = x\}$ - the fixed point set of A;
$I(A) := \{Y \in P(X) | \, A(Y) \subset Y\}$ - the family of the nonempty invariant subsets of A.

Definition 1. *(I.A. Rus - [6], [7], [8]) Let (X, d) be a metric space.*
1) An operator $A : X \to X$ is weakly Picard operator (briefly WPO) if the sequence of successive approximations $(A^m(x_0))_{m\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.
2) If the operator $A : X \to X$ is WPO and $F_A = \{x^*\}$, then by definition the operator A is Picard operator (briefly PO).
3) If the operator $A : X \to X$ is WPO, then can be considered the operator A^∞ defined by $A^\infty : X \to X, \ A^\infty(x) := \lim_{m\to\infty} A^m(x)$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The basic result in the WPO's theory is the following:

Theorem 1. *(Characterization theorem - [6], [7], [8]) An operator $A : X \to X$ is WPO if and only if there exits a partition of X, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, such that:*

(a) $X_\lambda \in I(A)$, $\forall \lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is PO, $\forall\lambda \in \Lambda$.

Definition 2. *Let (X, d) be a metric space and $A : X \to X$ a WPO.*
1) $A : X \to X$ is good WPO, if the series $\sum\limits_{m=1}^{\infty} d\left(A^{m-1}(x), A^m(x)\right)$ converges, for all $x \in X$ (see [8]). In the case that the sequence $\left(d\left(A^{m-1}(x), A^m(x)\right)\right)_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, the operator A is good WPO of type M (see [3]).
2) $A : X \to X$ is special WPO, if the series $\sum\limits_{m=1}^{\infty} d\left(A^m(x), A^\infty(x)\right)$ converges, for all $x \in X$(see [8]). When the sequence $\left(d\left(A^m(x), A^\infty(x)\right)\right)_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, A is special WPO of type M (see [3]).

In 2015, S. Mureşan and L.F. Iambor obtained the following result regarding to good and special weakly Picard operators.

Theorem 2. *([5]) Let (X, d) be a metric space and $A : X \to X$ a WPO. If A is special WPO then A is good WPO.*

In the paper [4], A.Bica and L.F. Galea(Iambor) introduced the notions of uniform good and special weakly Picard operators like this:

Definition 3. *(A.Bica, L.F. Galea - [4]) Let (X, d) be a metric space and $F \subset \{A|\, A : X \to X\}$ a family of operators on X. We say that F is a family of uniform special (good) WPO's if for any $A \in F$, A is special (good) WPO and there exist the functionals $\varphi : X \to \mathbb{R}_+$ and $\psi, \psi' : F \to \mathbb{R}_+$ such that φ is continuous and*

$$\sum_{m=1}^{\infty} d\left(A^m(x), A^\infty(x)\right) \leqslant \psi(A)\cdot\varphi(x),\ \forall\ x \in X,\ \forall\ A \in F$$

$$\textit{(respectively, } \sum_{m=1}^{\infty} d\left(A^m(x), A^{m-1}(x)\right) \leqslant \psi'(A)\cdot\varphi(x),\ \forall\ x \in X,\ \forall\ A \in F).$$

In what follow, we present the general class of linear positive operators of discrete type and some properties of these operators investigated by O. Agratini and I.A. Rus in [1].

At first they construct an approximation process of discrete type acting on the space $C([a, b])$ endowed with the Chebyshev norm $\|.\|$.

For each integer $n \geqslant 1$ they consider the following:

(i) A net on $[a, b]$ named Δ_n is fixed $(a = x_{n,0} < x_{n,1} < ... < x_{n,n} = b)$.

(ii) A system $(\psi_{n,k})_{k=\overline{0,n}}$ is given, where every $\psi_{n,k}$ belongs to $C([a, b])$.

They assume that it is a blending system with a certain connection with Δ_n, more precisely the following conditions hold:

$$\psi_{n,k} \geqslant 0,\ \left(k = \overline{0, n}\right),\ \sum_{k=0}^{n}\psi_{n,k} = e_0,\ \sum_{k=0}^{n} x_{n,k}\psi_{n,k} = e_1$$

Definition 4. *(O.Agratini, I.A. Rus - [1]) The operators $L_n : C([a, b]) \to C([a, b])$ defined by*

$$L_n(f)(x) = \sum_{k=0}^{n}\psi_{n,k}(x)\, f(x_{n,k})$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

are called ***the operators of discrete type.***

The operators of discrete type L_n, have the following properties:
1) L_n, $n \in \mathbb{N}$ are positive linear operators;
2) $L_n(e_0) = e_0$ and $L_n(e_1) = e_1$.

Theorem 3. *(O.Agratini, I.A. Rus - [1]) Let L_n, $n \in \mathbb{N}$, such that $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$. Let us denote $u_n := \min_{x\in[a,b]} [\Phi_{n,0}(x) + \Phi_{n,n}(x)]$.*

If the $u_n > 0$ the iterates sequence $(L_n^m)_{m\geqslant 1}$ verifies

$$\lim_{m\to\infty} (L_n^m f)(x) = f(a) + \frac{f(a)-f(b)}{b-a}(x-a), \ f \in C([a,b])$$

uniformly on $[a,b]$.

Theorem 4. *(O.Agratini, I.A. Rus - [1]) Let L_n, $n \in \mathbb{N}$, such that $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$. Then the operator L_n is weakly Picard operator for every $n \in \mathbb{N}$ and*

$$L_n^\infty(f) = c_1(f)e_1 + c_2(f) \overset{not}{=} f^*(x), \ f \in C([a,b])$$

where $c_1(f) = \frac{f(b)-f(a)}{b-a}$ and $c_2(f) = \frac{bf(a)-af(b)}{b-a}$.

The convergence exists on the space $(C[a,b], \|.\|_\infty)$.

In the application of Characterization theorem of weakly Picard operator, it was considerate the partition of $C([a,b])$:

$$C([a,b]) := \bigcup_{\alpha,\beta\in\mathbb{R}} X_{\alpha,\beta}$$

where $X_{\alpha,\beta} = \{f \in C([a,b]) : f(a) = \alpha, \ f(b) = \beta\}, \ \alpha, \beta \in \mathbb{R}$.

Proposition 5. *(O. Agratini, I.A. Rus - [1]) The operators of discrete type satisfied the following contraction property relative to above partition:*

$$\|L_n(f) - L_n(g)\|_\infty \leqslant (1-u_n)\|f-g\|_\infty, \ \forall \ f,g \in X_{\alpha,\beta}, \ \alpha,\beta \in \mathbb{R} \tag{2}$$

where $u_n = \min_{x\in[a,b]} [\Phi_{n,0}(x) - \Phi_{n,n}(x)]$, $u_n > 0$.

2. Main results

In this section, we will investigate some properties of the iterates of discrete type of operators in sense of good and special convergence.

Theorem 6. *The operators of discrete type $L_n, \ n \in \mathbb{N}$ are special WPO and good WPO of type M on $C([a,b])$.*

From Theorem 3, we have that $L_n, \ n \in \mathbb{N}$ is weakly Picard operator.

Let $f \in C([a,b])$. Then $f \in X_{f(a),f(b)}$ and according to (1) we infer that L_n is contraction on $X_{f(a),f(b)}$. So, the operator $L_n, \ n \in \mathbb{N}$ is special WPO of type M on $X_{f(a),f(b)}$. Finally, we get that $L_n, \ n \in \mathbb{N}$ is special WPO of type M on $C([a,b])$.

From Theorem 2, any special WPO is good WPO. Then we have that $L_n, \ n \in \mathbb{N}$ is good WPO of type M on $C([a,b])$.

Theorem 7. *The family of the operators of discrete type $\{L_n : n \in \mathbb{N}^*\}$ is family of uniform special and good WPO's on $C[a,b]$.*

Proof. Using the inequality (1), we obtain the estimation:
$\left|L_n^1(f)(x) - L_n^\infty(f)(x)\right| = \left|L_n^1(f)(x) - L_n^1(L_n^\infty(f))(x)\right| \leqslant$
$\leqslant (1-u_n)|f(x) - L_n^\infty(f)(x)| = (1-u_n)|f(x) - c_1(f)e_1 - c_2(f)| \leqslant$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

$\leqslant (1-u_n)\cdot C,\ \forall\ x\in[a,b],$
where $C = diam\,(\operatorname{Im} f) + 2\max\{|f(a)|,|f(b)|\}$, with
$diam\,(\operatorname{Im} f) = \max\{|f(x)-f(y)| : x,y\in[a,b]\}$.

The constant C was obtained using the following technique:

- If $x=a$ then:
  $|f(x)-L_n^\infty(f)(x)| \leqslant \left|f(x)-\frac{f(b)-f(a)}{b-a}\cdot a-\frac{bf(a)-af(b)}{b-a}\right| =$
  $= \left|f(x)-\frac{f(a)(b-a)}{b-a}\right| = |f(x)-f(a)| \leqslant diam\,(\operatorname{Im} f)$
- If $x=b$ then:
  $|f(x)-L_n^\infty(f)(x)| \leqslant \left|f(x)-\frac{f(b)-f(a)}{b-a}\cdot b-\frac{bf(a)-af(b)}{b-a}\right| =$
  $= \left|f(x)-\frac{f(b)(b-a)}{b-a}\right| = |f(x)-f(b)| \leqslant diam\,(\operatorname{Im} f)$
- If $x\in[a,b]$ then:
  $|f(x)-L_n^\infty(f)(x)| \leqslant \left|f(x)-\frac{f(b)-f(a)}{b-a}\cdot x-\frac{bf(a)-af(b)}{b-a}\right| =$
  $= \left|f(x)-\frac{x-a}{b-a}\cdot f(b)+\frac{b-x}{b-a}\cdot f(a)\right| \leqslant$
  $\leqslant |f(x)-f(b)| + |f(b)|\cdot\left|1-\frac{x-a}{b-a}\right| + \left|\frac{b-x}{b-a}\right|\cdot|f(a)| \leqslant$
  $\leqslant diam\,(\operatorname{Im} f) + 2\max\{|f(a)|,|f(b)|\}.$

By induction, for $m\in\mathbb{N}^*$, we have:
$|L_n^m(f)(x)-L_n^\infty(f)(x)| = \left|L_n^1\left(L_n^{m-1}(f)\right)(x)-L_n^1\left(L_n^\infty(f)\right)(x)\right| \leqslant$
$\leqslant (1-u_n)^m\cdot C,\ \forall\ x\in[a,b].$

Then, $\sum\limits_{m=1}^{\infty}|L_n^m(f)(x)-L_n^\infty(f)(x)| \leqslant$
$\leqslant \lim\limits_{m\to\infty} C\left[(1-u_n)+(1-u_n)^2+...+(1-u_n)^m\right] =$
$= \lim\limits_{m\to\infty}\left[C(1-u_n)\cdot\frac{1-(1-u_n)^m}{u_n}\right] \leqslant C\cdot\frac{1-u_n}{u_n}\quad(2)$

On the other hand, we have:
$\left|L_n^1(f)(x)-L_n^0(f)(x)\right| = \left|\sum\limits_{k=0}^{n}\psi_{n,k}(x)f(x_{n,k})-f(x)\right| =$
$= \left|\sum\limits_{k=0}^{n}\psi_{n,k}(x)\left[f(x_{n,k})-f(x)\right]\right| \leqslant C'\sum\limits_{k=0}^{n}\psi_{n,k}(x) = C'e_0,\ \forall\ x\leqslant[a,b]$, where
$C' = diam\,(\operatorname{Im} f) = \max\{|f(x)-f(y)| : x,y\in[a,b]\}.$

By induction, for $m\in\mathbb{N}$, we have:
$\left|L_n^m(f)(x)-L_n^{m-1}(f)(x)\right| = \left|L_n^1\left(L_n^{m-1}(f)\right)(x)-L_n^1\left(L_n^{m-2}(f)\right)(x)\right| \leqslant$
$\leqslant (1-u_n)^{m-1}\cdot C'e_0,\ \forall\ x\in[a,b]$

Then, $\sum\limits_{m=1}^{\infty}\left|L_n^m(f)(x)-L_n^{m-1}(f)(x)\right| \leqslant$
$\leqslant \lim\limits_{m\to\infty} C'e_0\left[1+(1-u_n)+(1-u_n)^2+...+(1-u_n)^{m-1}\right] \leqslant$
$\leqslant C'e_0\frac{1}{1-u_n},\ \forall\ f\in C[a,b]\quad(3).$

Now, the property of uniform and special WPO follows from the estimations (2) and (3). For instance, for the property of uniform special WPO we have:

$$\varphi : C[a,b]\to\mathbb{R}_+,\ \varphi(f) = diam\,(\operatorname{Im} f)+2\max\{|f(a)|,|f(b)|\}\text{ and}$$

$$\psi : \{L_n : n\in\mathbb{N}^*\}\to\mathbb{R}_+,\ \psi(L_n) = \frac{1-u_n}{u_n},\ \forall\ n\in\mathbb{N}^*$$

and for the property of uniform good WPO we have:

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

$$\varphi' : C[a,b] \to \mathbb{R}_+,\ \varphi'(f) = diam(\operatorname{Im} f) \text{ and}$$
$$\psi' : \{L_n : n \in \mathbb{N}^*\} \to \mathbb{R}_+,\ \psi'(L_n) = \frac{1}{1-u_n} e_0,\ \forall\, n \in \mathbb{N}^*.$$

It is easy to prove that $\varphi, \varphi' : C[a,b] \to \mathbb{R}_+$, $\varphi(f) = diam(\operatorname{Im} f) + 2\max\{|f(a)|, |f(b)|\}$ and $\varphi'(f) = diam(Imf)$ are seminorms on $C[a,b]$ and
$\varphi(f-g) \leqslant 2\|f-g\|_C + 2\|f\|_C$, $\varphi'(f-g) \leqslant 2\|f-g\|_C$
since $|\varphi(f) - \varphi(g)| \leqslant \varphi(f-g)$, $\forall\, f, g \in C[a,b]$ and
$|\varphi'(f) - \varphi'(g)| \leqslant \varphi'(f-g)$, $\forall\, f, g \in C[a,b]$
we infer the φ, φ' are continuous. □

References

[1] O. Agratini, I.A. Rus *Iterates of a class of discrete linear operators via contraction principle*, Comment. Math. Univ. Carolinae, 44(3), 2003, 555-563.

[2] L. D'Apuzzo, *On the convergence of the method of succesive approximation in metric spaces*, Ann. Istit. Univ. Navale Napoli, 45/46(1976/1977)(in Italian).

[3] L. D'Apuzzo, *On the notion of good and special convergence of the method of succesive approximations*, Ann. Istit. Univ. Navale Napoli, 45/46(1976/1977), 123-138, (in Italian).

[4] A. Bica, L.F. Galea *On Picard iterative properties of the Bernstein operators and an application to fuzzy numbers*, Comm. in Math. Analysis, 5(1), 2008, 8-19.

[5] S. Mureşan, L.F. Iambor, *On good and special weakly Picard operators*, Analele Universitatii Oradea, Fasc. Matematica, Tom XXII, Issue No.1, pp.5-10, 2015.

[6] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, 2001, (in Romanian).

[7] I. A. Rus, *Weakly Picard Operators and Applications*, Seminar on Fixed Point Theory Cluj-Napoca, Volume2, 41-58, 2001.

[8] I. A. Rus, *Picard operators and applications*, Sci. Math.Japon., 58 (1), pp.191-219, 2003.

Department of Mathematics and Computer Science, University of Oradea, Str. Universităţii No.1, 410087, Oradea, Romania,
E-mail address: iambor.loredana@gmail.com, acatas@gmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

General Iyengar type Inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present general Iyengar type inequalities with respect to L_p norms, with $1 \leq p \leq \infty$. The method is based on the generalized Taylor's formula.

2010 Mathematics Subject Classification: 26D10, 26D15.
Key Words and Phrases: Iyengar inequality, Taylor formula.

1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [2].

Theorem 1 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \quad (1)$$

2 Main Results

We present the following Iyengar type inequalities:

Theorem 2 *Let $n \in \mathbb{N}$, $f \in AC^n([a,b])$ (i.e. $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We assume that $f^{(n)} \in L_\infty([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left[(t-a)^{n+1} + (b-t)^{n+1} \right], \quad (2)$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$*, the right hand side of (2) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{3}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$*, for all* $k = 0, 1, ..., n-1$*, we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{4}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$*, it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right], \tag{5}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$*,* $k = 1, ..., n-1$*, from (5) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) \left[j f(a) + (N-j) f(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right], \tag{6}$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$*,*

vi) when $N = 2$ *and* $j = 1$*, (6) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{7}$$

vii) when $n = 1$ *(without any boundary conditions), we get from (7) that*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \|f'\|_{\infty,[a,b]} \frac{(b-a)^2}{4}, \tag{8}$$

a similar to Iyengar inequality (1).

2

Proof. Here $n \in \mathbb{N}$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. We assumed that

$$\left\| f^{(n)} \right\|_{\infty,[a,b]} := \left\| f^{(n)} \right\|_{L_\infty([a,b])} < +\infty.$$

By [1], we have the following generalized Taylor's formulae:

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt \tag{9}$$

and

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k = \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t)\, dt, \tag{10}$$

$\forall\, x \in [a,b]$.

Then we get

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!} (x-a)^n, \tag{11}$$

$\forall\, x \in [a,b]$,

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| = \frac{1}{(n-1)!} \left| \int_b^x (x-t)^{n-1} f^{(n)}(t)\, dt \right| =$$

$$\frac{1}{(n-1)!} \left| \int_x^b (t-x)^{n-1} f^{(n)}(t)\, dt \right| \leq \frac{1}{(n-1)!} \int_x^b (t-x)^{n-1} \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!} (b-x)^n,$$

that is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!} (b-x)^n, \tag{12}$$

$\forall\, x \in [a,b]$.

We call

$$\delta := \frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!}. \tag{13}$$

So we have

$$-\delta (x-a)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \leq \delta (x-a)^n \tag{14}$$

3

and

$$-\delta\left(b-x\right)^{n}\leq f\left(x\right)-\sum_{k=0}^{n-1}\frac{f^{(k)}\left(b\right)}{k!}\left(x-b\right)^{k}\leq\delta\left(b-x\right)^{n}, \tag{15}$$

$\forall\ x\in\left[a,b\right].$

Therefore it holds

$$\sum_{k=0}^{n-1}\frac{f^{(k)}\left(a\right)}{k!}\left(x-a\right)^{k}-\delta\left(x-a\right)^{n}\leq f\left(x\right)\leq\sum_{k=0}^{n-1}\frac{f^{(k)}\left(a\right)}{k!}\left(x-a\right)^{k}+\delta\left(x-a\right)^{n} \tag{16}$$

and

$$\sum_{k=0}^{n-1}\frac{f^{(k)}\left(b\right)}{k!}\left(x-b\right)^{k}-\delta\left(b-x\right)^{n}\leq f\left(x\right)\leq\sum_{k=0}^{n-1}\frac{f^{(k)}\left(b\right)}{k!}\left(x-b\right)^{k}+\delta\left(b-x\right)^{n}, \tag{17}$$

$\forall\ x\in\left[a,b\right].$

Let any $t\in\left[a,b\right]$, then

$$\sum_{k=0}^{n-1}\frac{f^{(k)}\left(a\right)}{(k+1)!}\left(t-a\right)^{k+1}-\frac{\delta}{(n+1)}\left(t-a\right)^{n+1}\leq\int_{a}^{t}f\left(x\right)dx\leq$$

$$\sum_{k=0}^{n-1}\frac{f^{(k)}\left(a\right)}{(k+1)!}\left(t-a\right)^{k+1}+\frac{\delta}{(n+1)}\left(t-a\right)^{n+1}, \tag{18}$$

and

$$-\sum_{k=0}^{n-1}\frac{f^{(k)}\left(b\right)}{(k+1)!}\left(t-b\right)^{k+1}-\frac{\delta}{(n+1)}\left(b-t\right)^{n+1}\leq\int_{t}^{b}f\left(x\right)dx\leq \tag{19}$$

$$-\sum_{k=0}^{n-1}\frac{f^{(k)}\left(b\right)}{(k+1)!}\left(t-b\right)^{k+1}+\frac{\delta}{(n+1)}\left(b-t\right)^{n+1}.$$

Adding (18) and (19), we obtain:

$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}\left(a\right)\left(t-a\right)^{k+1}-f^{(k)}\left(b\right)\left(t-b\right)^{k+1}\right]\right\}-$$

$$\frac{\delta}{(n+1)}\left[\left(t-a\right)^{n+1}+\left(b-t\right)^{n+1}\right]\leq\int_{a}^{b}f\left(x\right)dx\leq$$

$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}\left(a\right)\left(t-a\right)^{k+1}-f^{(k)}\left(b\right)\left(t-b\right)^{k+1}\right]\right\}+$$

$$\frac{\delta}{(n+1)}\left[\left(t-a\right)^{n+1}+\left(b-t\right)^{n+1}\right], \tag{20}$$

$\forall\ t\in\left[a,b\right].$

4

Consequently we derive:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\delta}{(n+1)}\left[(t-a)^{n+1}+(b-t)^{n+1}\right], \tag{21}$$

$\forall\ t\in[a,b]$.

Let us consider

$$g(t):=(t-a)^{n+1}+(b-t)^{n+1},\ \forall\ t\in[a,b]. \tag{22}$$

Hence

$$g'(t)=(n+1)\left[(t-a)^n-(b-t)^n\right]=0,$$

giving $(t-a)^n=(b-t)^n$ and $t-a=b-t$, that is $t=\frac{a+b}{2}$ the only critical number here.

We have $g(a)=g(b)=(b-a)^{n+1}$, and $g\left(\frac{a+b}{2}\right)=\frac{(b-a)^{n+1}}{2^n}$, which is the minimum of g over $[a,b]$.

Consequently the right hand side of (21) is minimized when $t=\frac{a+b}{2}$, with value $\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n}$. Assuming $f^{(k)}(a)=f^{(k)}(b)=0$, for $k=0,1,...,n-1$, then we obtain that

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n}, \tag{23}$$

which is a sharp inequality.

When $t=\frac{a+b}{2}$, then (21) becomes

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n}. \tag{24}$$

Next let $N\in\mathbb{N}$, $j=0,1,2,...,N$ and $t_j=a+j\left(\frac{b-a}{N}\right)$, that is $t_0=a$, $t_1=a+\frac{b-a}{N},...,t_N=b$.

Hence it holds

$$t_j-a=j\left(\frac{b-a}{N}\right),\quad (b-t_j)=(N-j)\left(\frac{b-a}{N}\right),\quad j=0,1,2,...,N. \tag{25}$$

We notice that

$$(t_j-a)^{n+1}+(b-t_j)^{n+1}=\left(\frac{b-a}{N}\right)^{n+1}\left[j^{n+1}+(N-j)^{n+1}\right], \tag{26}$$

$j = 0, 1, 2, ..., N,$

and $(k = 0, 1, ..., n-1)$

$$\left[f^{(k)}(a)(t_j - a)^{k+1} + (-1)^k f^{(k)}(b)(b - t_j)^{k+1}\right] =$$

$$\left[f^{(k)}(a) j^{k+1} \left(\frac{b-a}{N}\right)^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1} \left(\frac{b-a}{N}\right)^{k+1}\right] = \tag{27}$$

$$\left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1}\right],$$

$j = 0, 1, 2, ..., N.$

By (21) we get

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1}\right]\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \left(\frac{b-a}{N}\right)^{n+1} \left[j^{n+1} + (N-j)^{n+1}\right], \tag{28}$$

$j = 0, 1, 2, ..., N.$

If $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, then (28) becomes

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)[jf(a) + (N-j) f(b)]\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \left(\frac{b-a}{N}\right)^{n+1} \left[j^{n+1} + (N-j)^{n+1}\right], \tag{29}$$

for $j = 0, 1, 2, ..., N$.

When $N = 2$ and $j = 1$, then (29) becomes

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)[f(a) + f(b)]\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \left(\frac{b-a}{2}\right)^{n+1} 2 = \frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}. \tag{30}$$

And, if $n = 1$ (without any boundary conditions), we get from (30) that

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \leq \|f'\|_{\infty,[a,b]} \frac{(b-a)^2}{4}, \tag{31}$$

which a similar inequality to Iyengar inequality (1). ■

We give

Theorem 3 *Let $f \in AC^n([a,b])$, $n \in \mathbb{N}$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq \tag{32}$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \left[(t-a)^n + (b-t)^n \right],$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (32) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}}, \tag{33}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0,1,...,n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}}, \tag{34}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,...,$ $N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \left(\frac{b-a}{N} \right)^n \left[j^n + (N-j)^n \right], \tag{35}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1,...,n-1$, from (35) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) \left[j f(a) + (N-j) f(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \left(\frac{b-a}{N} \right)^n \left[j^n + (N-j)^n \right], \tag{36}$$

for $j = 0,1,2,...,N \in \mathbb{N}$,

vi) when $N = 2$ and $j = 1$, (36) turns to

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}}, \tag{37}$$

vii) when $n=1$ (without any boundary conditions), we get from (37) that

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq\|f'\|_{L_1([a,b])}(b-a). \tag{38}$$

Proof. Here $n\in\mathbb{N}$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Hence $f^{(n)}$ exists almost everywhere and $f^{(n)}\in L_1([a,b])$. By (9) we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right|=\frac{1}{(n-1)!}\left|\int_a^x (x-t)^{n-1}f^{(n)}(t)\,dt\right|\leq$$

$$\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\left|f^{(n)}(t)\right|dt\leq\frac{(x-a)^{n-1}}{(n-1)!}\int_a^b\left|f^{(n)}(t)\right|dt \tag{39}$$

$$=\frac{(x-a)^{n-1}}{(n-1)!}\left\|f^{(n)}\right\|_{L_1([a,b])}.$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right|\leq\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{(n-1)!}(x-a)^{n-1}, \tag{40}$$

$\forall\ x\in[a,b]$.

By (10) we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right|=\frac{1}{(n-1)!}\left|\int_b^x (x-t)^{n-1}f^{(n)}(t)\,dt\right|=$$

$$\frac{1}{(n-1)!}\left|\int_x^b (t-x)^{n-1}f^{(n)}(t)\,dt\right|\leq\frac{1}{(n-1)!}\int_x^b (t-x)^{n-1}\left|f^{(n)}(t)\right|dt\leq$$

$$\frac{(b-x)^{n-1}}{(n-1)!}\int_a^b\left|f^{(n)}(t)\right|dt=\frac{(b-x)^{n-1}}{(n-1)!}\left\|f^{(n)}\right\|_{L_1([a,b])}. \tag{41}$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right|\leq\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{(n-1)!}(b-x)^{n-1}, \tag{42}$$

$\forall\ x\in[a,b]$.

Set

$$\rho:=\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{(n-1)!}.$$

8

Hence

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \rho (x-a)^{n-1}, \tag{43}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \rho (b-x)^{n-1}, \tag{44}$$

$\forall\, x \in [a,b]$.

As in the proof of Theorem 2 we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq \frac{\rho}{n} \left[(t-a)^n + (b-t)^n \right], \tag{45}$$

$\forall\, t \in [a,b]$.

The rest of the proof is similar to the proof of Theorem 2. ■

We continue with

Theorem 4 *Let* $f \in AC^n([a,b])$, $n \in \mathbb{N}$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $f^{(n)} \in L_q([a,b])$. *Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \left[(t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}} \right], \tag{46}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, *the right hand side of (46) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \tag{47}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, *for all* $k = 0,1,...,n-1$, *we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \tag{48}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p} \right) (p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \quad (49)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from (49) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p} \right) (p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \quad (50)$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

vi) when $N = 2$ *and* $j = 1$, *(50) turns to*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p} \right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \quad (51)$$

vii) when $n = 1$ *(without any boundary conditions), we get from (51) that*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f'\|_{L_q([a,b])}}{\left(1 + \frac{1}{p} \right)} \frac{(b-a)^{1+\frac{1}{p}}}{2^{\frac{1}{p}}}. \quad (52)$$

Proof. Here $f^{(n)} \in L_q([a,b])$, where $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$. By (9) we get

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| = \frac{1}{(n-1)!} \left| \int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt \right| \leq$$

$$\frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{1}{(n-1)!} \left(\int_a^x (x-t)^{p(n-1)}\,dt \right)^{\frac{1}{p}} \left(\int_a^x \left| f^{(n)}(t) \right|^q dt \right)^{\frac{1}{q}} \leq$$

10

$$\frac{(x-a)^{\frac{p(n-1)+1}{p}}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}\left\|f^{(n)}\right\|_{L_q([a,b])}. \tag{53}$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right|\leq\frac{\left\|f^{(n)}\right\|_{L_q([a,b])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}(x-a)^{n-\frac{1}{q}}, \tag{54}$$

$\forall\ x\in[a,b]$.

By (10) we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right|=\frac{1}{(n-1)!}\left|\int_x^b(t-x)^{n-1}f^{(n)}(t)\,dt\right|\leq$$

$$\frac{1}{(n-1)!}\int_x^b(t-x)^{n-1}\left|f^{(n)}(t)\right|dt\leq$$

$$\frac{1}{(n-1)!}\left(\int_x^b(t-x)^{p(n-1)}dt\right)^{\frac{1}{p}}\left(\int_x^b\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}\leq$$

$$\frac{(b-x)^{\frac{p(n-1)+1}{p}}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}\left\|f^{(n)}\right\|_{L_q([a,b])}. \tag{55}$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right|\leq\frac{\left\|f^{(n)}\right\|_{L_q([a,b])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}(b-x)^{n-\frac{1}{q}}, \tag{56}$$

$\forall\ x\in[a,b]$.

Set

$$\gamma:=\frac{\left\|f^{(n)}\right\|_{L_q([a,b])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}, \tag{57}$$

and

$$m:=n-\frac{1}{q}>0. \tag{58}$$

So, we can write

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right|\leq\gamma(x-a)^m, \tag{59}$$

and

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right|\leq\gamma(b-x)^m, \tag{60}$$

$\forall\ x\in[a,b]$.

As in the proof of Theorem 2 we obtain:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\gamma}{(m+1)} \left[(t-a)^{m+1} + (b-t)^{m+1} \right] = \tag{61}$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}} \left(n + \frac{1}{p} \right)} \left[(t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}} \right], \tag{62}$$

$\forall\ t \in [a,b]$.

The rest of the proof is similar to the proof of Theorem 2. ■

References

[1] G.A. Anastassiou, S.S. Dragomir, *On some estimates of the remainder in Taylor's formula*, J. Math. Anal. Appl., 263 (2001), 246-263.

[2] K.S.K. Iyengar, *Note on an inequality*, Math. Student, 6 (1938), 75-76.

A NOTE ON THE APPROXIMATE SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION

F. Faizullah[1], R. Ullah[2], Jihen Majdoubi[3], I. Tlili[4], I. Khan[5]Ghaus ur Rahman

[1]Department of Mathematics, Swansea University, Singleton Park SA2 8PP UK
[2]Department of Mathematics, Women University, Swabi, Pakistan.
[3]Department of computer science college of science and humanities at Alghat Majmaah University, P.O. Box 66 Majmaah 11952, Kingdom of Saudi Arabia
[4]Department of Mechanical and Industrial Engineering, College of Engineering, Majmaah University, P.O. Box 66 Majmaah 11952, Saudi Arabia
[5]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Department of Mathematics & Statistics, University of Swat, Khyber Pakhtunkhwa, Pakistan

Abstract. By using the Caratheodory approximation method, the current article presents the analysis of exact and approximate solutions for stochastic differential equations (SDEs) in the framework of G-Brownian motion. In view of the non-linear growth and non-Lipschitz conditions, the boundedness of the Caratheodory approximate solutions $Y^q(t)$, $q \geq 1$ in the space $M_G^2([t_0,T];\mathbb{R}^n)$ has been determined. Estimate for the difference between the exact solution $Y(t)$ and the Caratheodory approximate solutions $Y^q(t)$ has been derived.

Keywords: G-Brownian motion, non-linear growth and non-Lipschitz conditions, Caratheodory approximation procedure, bounded solutions, stochastic differential equations
MSC: 60H20, 60H10, 60H35, 62L20.

1. Introduction

Stochastic differential equations (SDEs) are employed by several and diverse scientific disciplines such as chemistry, statistical physics, biology and engineering. In finance and economics, they are utilized to find out the risk measures and stochastic volatility problems. SDEs describe heavy traffic behavior of communication networks and control systems [16]. Mathematics use the concept of SDEs to incorporate random fluctuations in the model when one investigates the evolution of the number of cells in an organism infected by a virus. The weather and climate can be modeled by these equations. The clarification of fluid through porous structures and water catchment can be modeled by SDEs [17]. They are used to describe the motion of wildlife [4]. SDEs play an important role to study the animal's swarm, such as schooling of fish, flocking of birds or herding of mammals, to find resource of food in noisy and obstacle environment [30]. In physics, SDEs are used to study and model the effect of random variations on distinct physical processes. A large literature is available on the applications of SDEs in numerous fields of engineering such as computer engineering [16, 22], mechanical engineering [26, 28, 29], random vibrations [3, 24], stability theory [25] and wave processes [27]. In general, one can not find the explicit solutions for non-linear SDEs, so we have to present and study the analysis for the solutions of these equations. Moreover, the developments of computational techniques are very important for solving several demanding problems, for instance to find the optimal construction of a design and to determine input data from fundamental principles. Therefore it is valuable to know computational accuracy, which leads us to convergence results and estimates for the difference between exact and approximate solutions.

Corresponding author email (I. Khan): ilyaskhan@tdt.edu.vn

The aim of the current article is to investigate estimates for the difference between exact and approximate solutions for SDEs driven by G-Brownian motion with Caratheodory approximation procedure. In view of growth and Lipschitz conditions, the existence-uniqueness results for G-SDEs was studied by Peng [20, 21] and Gao [15]. Later, Bai and Lin [1] established the existence theory for G-SDEs with integral Lipschitz coefficients. Subject to some discontinuous coefficients, the said theory was generalized by Faizullah [11]. Let $0 \leq t_0 \leq t \leq T < \infty$. Consider the following SDE in the framework of G-Brownian motion

$$\begin{aligned} dY(t) =& \kappa(t, Y(t))dt + \lambda(t, Y(t))d\langle W, W\rangle(t) \\ &+ \mu(t, Y(t))dW(t), \end{aligned} \tag{1.1}$$

with initial value $Y(t_0) \in \mathbb{R}^n$. The given coefficients $g(.,x), h(.,x)$ and $w(.,x)$ belong to space $M_G^2([t_0,T];\mathbb{R}^n)$, for all $x \in \mathbb{R}^n$. SDE (1.1) in the integral form is expressed as the following

$$\begin{aligned} Y(t) = Y(t_0) &+ \int_{t_0}^t \kappa(s, Y(s))ds + \int_{t_0}^t \lambda(s, Y(s))d\langle W, W\rangle(s) \\ &+ \int_{t_0}^t \mu(s, Y(s))dW(s), \end{aligned} \tag{1.2}$$

on $t \in [t_0, T]$. Its solution is a process $Y \in M_G^2([t_0,T];\mathbb{R}^n)$ and satisfying SDE (1.2). The rest of the current paper contains three sections. Building on the previous notions of G-expectation, section 2 presents the fundamental definitions and results of G-Browinian motion, sub-expectation, Grownwall's inequality, Doobs martingale inequality, G-Itô's integral and Hölder's inequality etc. Section 3 reveals the Caratheodory approximate solutions procedure for SDEs driven by G-Brownian motion. This section give an important result, which shows that the Caratheodory approximate solutions are bounded. Section 4 derives estimates for the difference between approximate and exact solutions to SDEs driven by G-Brownian motion.

2. Preliminaries

We present some basic results and notions required for the subsequent sections of the current article. We don't give detailed literature on basic notions of G-expectation, so readers are suggested to consult the more depth oriented papers [9, 13, 18, 20, 21]. Let Ω be a given basic non-empty set. Assume $\mathcal{H}$ be a space of linear real functions defined on Ω so that (i) $1 \in \mathcal{H}$ (ii) for every $n \geq 1$, $Y_1, Y_2, ..., Y_n \in \mathcal{H}$ and $\varphi \in C_{b.Lip}(\mathbb{R}^n)$ it satisfies $\varphi(Y_1, Y_2, ..., Y_n) \in \mathcal{H}$ i.e., subject to Lipschitz bounded functions, $\mathcal{H}$ is stable. Then $(\Omega, \mathcal{H}, E)$ is a sub-expectation space, where E is a sub-expectation defined as follows.

Definition 2.1. A functional $E : \mathcal{H} \to \mathbb{R}$ satisfying the below four features is known as a sub-expectation. Let $X, Y \in \mathcal{H}$, then

(**1**) Monotonicity: $E(X) \leq E(Y)$ if $X \leq Y$.
(**2**) Constant preservation: $E(M_1) = M_1$, for all $M_1 \in \mathbb{R}$.
(**3**) Positive homogeneity: $E(N_1 Y) = N_1 E(Y)$, for all $N_1 \in \mathbb{R}^+$.
(**4**) Sub-additivity: $E(X) + E(Y) \geq E(X + Y)$.

Moreover, let Ω be the space of all $\mathbb{R}^n$-valued continuous paths $(w_t)_{t\geq 0}$ starting from zero. In addition, assume that subject to the below distance, Ω is a metric space

$$\rho(w^1, w^2) = \sum_{i=1}^{\infty} \frac{1}{2^i} (\max_{t\in[0,k]} |w_t^1 - w_t^2| \wedge 1).$$

Fix $T \geq 0$ and set

$$L_{ip}^0(\Omega_T) = \{\phi(W_{t_1}, W_{t_2}, ..., W_{t_m}) : m \geq 1, t_1, t_2, ..., t_m \in [0, T], \phi \in C_{b.Lip}(\mathbb{R}^{m\times n}))\},$$

where W is the canonical process, $L_{ip}^0(\Omega_t) \subseteq L_{ip}^0(\Omega_T)$ for $t \leq T$ and $L_{ip}^0(\Omega) = \cup_{n=1}^{\infty} L_{ip}^0(\Omega_n)$. The completion of $L_{ip}^0(\Omega)$ under the Banach norm $E[|.|^p]^{\frac{1}{p}}$, $p \geq 1$ is denoted by $L_G^p(\Omega)$, where $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$ for $0 \leq t \leq T < \infty$. Generated by the canonical process $\{W(t)\}_{t\geq 0}$, the filtration is represented as $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$, $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. Suppose $\pi_T = \{t_0, t_1, ..., t_N\}$, $0 \leq t_0 \leq t_1 \leq ... \leq t_N \leq \infty$ be a division of $[0, T]$. For $p \geq 1$, $M_G^{p,0}(0, T)$ denotes a set of the processes given by

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t), \tag{2.1}$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N-1$. Furthermore, the completion of $M_G^{p,0}(0, T)$ with the below given norm is indicated by $M_G^p(0, T)$, $p \geq 1$

$$\|\eta\| = \{\int_0^T E[|\eta_s|^p] ds\}^{1/p}.$$

Definition 2.2. An n-dimensional stochastic process $\{W(t)\}_{t\geq 0}$ is called a G-Brownian motion if

(**1**) $W(0) = 0$.
(**2**) For any $t, m \geq 0$, $W_{t+m} - W_t$ is G-normally distributed and independent from $W_{t_1}, W_{t_2}, ...W_{t_n}$, for $n \in N$ and $0 \leq t_1 \leq t_2 \leq, ..., \leq t_n \leq t$,

Definition 2.3. Let $\eta_t \in M_G^{2,0}(0, T)$ having the form (2.1). Then the G-quadratic variation process $\{\langle W\rangle_t\}_{t\geq 0}$ and G-Itô's integral $I(\eta)$ are respectively defined by

$$\langle W\rangle_t = W_t^2 - 2\int_0^t W_s dW(s),$$
$$I(\eta) = \int_0^T \eta_s dW(s) = \sum_{i=0}^{N-1} \xi_i (W_{t_{i+1}} - W_{t_i}).$$

The following two lemmas can be found in the book [19]. They are known as Hölder's and Gronwall's inequalities respectively, .

Lemma 2.4. *Assume $m, n > 1$ such that $\frac{1}{m} + \frac{1}{n} = 1$ and $\beta \in L^2$ then $\alpha\beta \in L^1$ and*

$$\int_a^b \alpha\beta \leq \left(\int_a^b |\alpha|^m\right)^{\frac{1}{m}} \left(\int_a^b |\beta|^n\right)^{\frac{1}{n}}.$$

3

Lemma 2.5. *Let $\alpha(t) \geq 0$ and $\beta(t)$ be continuous real functions defined on $[a, b]$. If for all $t \in [a, b]$,*

$$\beta(t) \leq K + \int_a^b \alpha(s)\beta(s)ds,$$

where $K \geq 0$, then

$$\beta(t) \leq Ke^{\int_a^t \alpha(s)ds},$$

for all $t \in [a, b]$.

The following lemma, known as Doob's martingale inequality, is borrowed from [15].

Lemma 2.6. *Assume $[c, d]$ be a bounded interval of $\mathbb{R}_+$. Consider an $\mathbb{R}^n$ valued G-martingale $\{X(t) : t \geq 0\}$. Then we have*

$$E(\sup_{c \leq t \leq d} |Y(t)|^p) \leq (\frac{p}{p-1})^p E(|Y(d)|^P),$$

where $p > 1$ and $Y(t) \in L_G^p(\Omega, \mathbb{R}^d)$. In particular, if $p = 2$ then $E(\sup_{c \leq t \leq d} |Y(t)|^2) \leq 4E(|Y(d)|^2)$.

3. Caratheodory approximate solutions

We now present the Caratheodory approximation procedure for equation (1.2). Let $q \geq 1$ be any positive integer. For $t \in [t_0 - 1, t_0]$, we set $Y^q(t) = Y_0$ and for $t \in [t_0, T]$,

$$\begin{aligned} Y^q(t) = Y_0 &+ \int_{t_0}^t \kappa(s, Y^q(s - \frac{1}{q}))ds + \int_{t_0}^t \lambda(s, Y^q(s - \frac{1}{q}))d\langle W, W\rangle(s) \\ &+ \int_{t_0}^t \mu(s, Y^q(s - \frac{1}{q}))dW(s). \end{aligned} \tag{3.1}$$

The approximate solutions $Z^q(.)$ can be determined step-by-step on the intervals $[t_0, t_0 + \frac{1}{q}]$, $(t_0 + \frac{1}{q}, t_0 + \frac{2}{q}]$ and son on with the following procedure. For $t \in [t_0, t_0 + \frac{1}{q}]$, we have

$$\begin{aligned} Y^q(t) = Y_0 &+ \int_{t_0}^t \kappa(s, Y_0)ds + \int_{t_0}^t \lambda(s, Y_0)d\langle W, W\rangle(s) \\ &+ \int_{t_0}^t \mu(s, Y_0)dW(s), \end{aligned}$$

and for $t \in (t_0 + \frac{1}{q}, t_0 + \frac{2}{q}]$,

$$\begin{aligned} Y^q(t) = Y^q(t_0 + \frac{1}{q}) &+ \int_{t_0+\frac{1}{q}}^t \kappa(s, Y^q(s - \frac{1}{q}))ds + \int_{t_0+\frac{1}{q}}^t \lambda(s, Y^q(s - \frac{1}{q}))d\langle W, W\rangle(s) \\ &+ \int_{t_0+\frac{1}{q}}^t \mu(s, Y^q(s - \frac{1}{q}))dW(s), \end{aligned}$$

etc. All through this article, we assume two conditions, described as follows. Let M be a positive constant. For any $t \in [t_0, T]$ and $\kappa(t, 0), \lambda(t, 0), \mu(t, 0) \in L^2$,

$$|\kappa(t, 0)|^2 + |\lambda(t, 0)|^2 + |\mu(t, 0)|^2 \leq M, \tag{3.2}$$

which is weakened linear growth condition. Let $t \in [t_0, T]$. For every $u, v \in \mathbb{R}^n$, there exists a concave non-decreasing function $\Psi(.) : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(0) = 0$ and for $s > 0$, $\Psi(s) > 0$ such that

$$|\kappa(t,u) - \kappa(t,v)|^2 + |\lambda(t,u) - \lambda(t,v)|^2 + |\mu(t,u) - \mu(t,v)|^2 \le \Psi(|u-v|^2), \tag{3.3}$$

where $\int_{0+} \frac{ds}{\Psi(s)} = \infty$ and for all $s \ge 0$, $C, D > 0$, $\Psi(s) \le C + Ds$. Assumption (3.3) is a non-uniform Lipschitz condition. Subject to conditions (3.2) and (3.3), we assume that problem (1.1) has a unique solution $Y(t) \in M_G^2([t_0, T]; \mathbb{R}^n)$ [1].

Lemma 3.1. *Let assumptions* (3.2) *and* (3.2) *are satisfied. For every* $q \ge 1$ *and any* $T > 0$,

$$\sup_{t_0 \le t \le T} E(|Y^q(t)|^2) \le N_1, \tag{3.4}$$

where $N_1 = H_1 e^{H_2(T-t_0)}$, $H_1 = 4E|Y_0|^2 + 8T(T+2)(2M+C)$, $H_2 = 8(T+2)D$ *and* M, C, D *are arbitrary positive constants.*

Proof. In view of the inequality $|\sum_{i=1}^4 c_i|^2 \le 7\sum_{i=1}^4 |c_i|^2$, from (3.1) we derive

$$|Y^q(t)|^2 \le 4|Y_0|^2 + 4\Big|\int_{t_0}^t \kappa(s, Y^q(s-\frac{1}{q}))ds\Big|^2 + 4\Big|\int_{t_0}^t \lambda(s, Y^q(s-\frac{1}{q}))d\langle W,W\rangle(s)\Big|^2$$
$$+ 4\Big|\int_{t_0}^t \mu(s, Y^q(s-\frac{1}{q}))dW(s)\Big|^2.$$

Apply G-subexpectation on both sides. Then by virtue of the Doob's martingale, Holder's and BDG [5] inequalities we have

$$\begin{aligned}
E(\sup_{t_0 \le s \le t} |Y^q(s)|^2) &\le 4E(|Y_0|^2) + 4T\int_{t_0}^t E|\kappa(s, Y^q(s-\frac{1}{q}))|^2 ds + 4T\int_{t_0}^t E|\lambda(s, Y^q(s-\frac{1}{q}))|^2 ds \\
&+ 16\int_{t_0}^t E|\mu(s, Y^q(s-\frac{1}{q}))|^2 ds \\
&\le 4E(|Y_0|^2) + 8T\int_{t_0}^t E[|\kappa(s, Y^q(s-\frac{1}{q})) - \kappa(s,0)|^2 + |\kappa(s,0)|^2]ds \\
&+ 8T\int_{t_0}^t E[|\lambda(s, Y^q(s-\frac{1}{q})) - \lambda(s,0)|^2 + |\lambda(s,0)|^2]ds \\
&+ 32\int_{t_0}^t E[|\mu(s, Y^q(s-\frac{1}{q})) - \mu(s,0)|^2 + |\mu(s,0)|^2]ds.
\end{aligned}$$

Using (3.2) and (3.3), we derive

$$\begin{aligned}
E(\sup_{t_0 \le s \le t} |Y^q(s)|^2) &\le 4E(|Y_0|^2) + 16T^2M + 32TM + 8(T+2)\int_{t_0}^t E[\Psi(|Y^q(s-\frac{1}{q})|^2)]ds \\
&\le 4E(|Y_0|^2) + 16T^2M + 32TM + 8(T+2)\int_{t_0}^t [C + DE|Y^q(s-\frac{1}{q})|^2]ds \\
&\le 4E(|Y_0|^2) + 16T^2M + 32TM + 8T(T+2)C + 8(T+2)D\int_{t_0}^t E[\sup_{t_0 \le r \le s} |Y^q(r)|^2]ds
\end{aligned}$$

By virtue of the Grownwall's inequality, we derive

$$E(\sup_{t_0\le s\le t}|Y^q(s)|^2)\le H_1e^{H_2(t-t_0)},$$

where $H_1=4E|Y_0|^2+8T(T+2)(2M+C)$ and $H_2=8(T+2)D$. Consequently, supposing $t=T$, we obtain

$$E(\sup_{t_0\le s\le T}|Y^q(s)|^2)\le H_1e^{H_2(T-t_0)}=N_1.$$

The proof stands completed. □

In a similar way as lemma 3.1, we can prove the following result.

Lemma 3.2. *Subject to the growth condition* (3.2)*, for any* $T>0$,

$$\sup_{t_0\le t\le T}E(|Y(t)|^2)\le N_1, \tag{3.5}$$

where N_1 *is a positive constant.*

4. Estimates for the difference between exact and Caratheodory approximate solutions

We first give an important result. Then in view of weakened growth and non-uniform Lipschitz conditions, we derive an estimate for the difference between the approximate and exact solutions to problem (1.1).

Lemma 4.1. *Let* $0\le r<t\le T$*. Suppose that the assumptions of lemma 3.1 are satisfied. For all* $q\ge 1$

$$E[|Z^q(t)-Z^q(u)|^2]\le G_1(t-u), \tag{4.1}$$

where $G_1=12(T+2)(M+C+DN_1)$*,* M*,* C*,* D *and* N_1 *are positive constants.*

Proof. In view of the fundamental inequality $|\sum_{i=1}^3 c_i|^2\le 7\sum_{i=1}^3|c_i|^2$, for any $q\ge 1$ and $0\le r<t\le T$, from (3.1) we derive

$$|Y^q(t)-Y^q(u)|^2\le 3|\int_u^t\kappa(s,Y^q(s-\frac{1}{q}))ds|^2+3|\int_u^t\lambda(s,Y^q(s-\frac{1}{q}))d\langle W,W\rangle(s)|^2$$
$$+3|\int_u^t\mu(s,Y^q(s-\frac{1}{q}))dW(s)|^2.$$

Apply G-subexpectation on both sides. Then by virtue of the Doob's martingale, Holder's and BDG [5] inequalities we have

$$
\begin{aligned}
|Y^q(t)-Y^q(r)|^2 \le\; & 3T\int_u^t E|\kappa(s,Y^q(s-\frac{1}{q}))|^2ds + 3T\int_{t_0}^t E|\lambda(s,Y^q(s-\frac{1}{q}))|^2ds \\
& + 12\int_u^t |\mu(s,Y^q(s-\frac{1}{q}))|^2ds \\
& \le 6T\int_u^t E[|\kappa(s,Y^q(s-\frac{1}{q}))-\kappa(s,0)|^2+|\kappa(s,0)|^2]ds \\
& + 6T\int_u^t E[|\lambda(s,Y^q(s-\frac{1}{q}))-\lambda(s,0)|^2+|\lambda(s,0)|^2]ds \\
& + 24\int_u^t E[|\mu(s,Y^q(s-\frac{1}{q}))-\mu(s,0)|^2+|\mu(s,0)|^2]ds.
\end{aligned}
$$

Using (3.2) (3.3), we derive

$$
\begin{aligned}
|Y^q(t)-Y^q(u)|^2 \le\; & 6TM(t-u)+6TM(t-u)+24M(t-u)+12(T+2)\int_u^t E[\Psi(|Y^q(s-\frac{1}{q})|^2)]ds \\
& \le 12TM(t-u)+24M(t-u)+12C(T+2)(t-u)+12D(T+2)\int_u^t E[|Y^q(s-\frac{1}{q})|^2]ds \\
& \le 12TM(t-u)+24M(t-u)+12C(T+2)(t-u) \\
& + 12D(T+2)\int_u^t E[\sup_{t_0\le r\le s}|Y^q(r)|^2]ds
\end{aligned}
$$

In view of lemma 3.1, we have

$$
\begin{aligned}
|Y^q(t)-Y^q(u)|^2 \le\; & 12TM(t-u)+24M(t-u)+12C(T+2)(t-u) \\
& + 12D(T+2)N_1(t-u)
\end{aligned}
$$

Consequently,

$$|Y^q(t)-Y^q(u)|^2 \le G_1(t-u),$$

where $G_1 = 12(T+2)(M+C+DN_1)$. The proof is complete. □

Next lemma can be proved by using similar arguments as used in lemma 4.1.

Lemma 4.2. *Let $0 \le r < t \le T$. Subject to conditions* (3.2) *and* (3.3),

$$E[|Z(t)-Z(u)|^2] \le G_1(t-u),$$

where G_1 is a positive constant.

Theorem 4.3. *Assume* (3.2) *and* (3.3) *holds. Then*

$$E(\sup_{t_0\le s\le T}|Y(s)-Y^q(s)|^2) \le 6T(T+2)[C+\frac{2D}{q}]e^{12(T+2)D(T-t_0)},$$

where C and D are positive constants.

7

Proof. By using the inequality $|\sum_{i=1}^{3} c_i|^2 \leq 7\sum_{i=1}^{3}|c_i|^2$, from (1.2) and (3.1) we obtain

$$|Y(t)-Y^q(t)|^2 \leq 3|\int_{t_0}^{t}[\kappa(s,Y(s))-\kappa(s,Y^q(s-\frac{1}{q}))]ds|^2+3|\int_{t_0}^{t}[\lambda(s,Y(s))-\lambda(s,Y^q(s-\frac{1}{q}))]d\langle W,W\rangle(s)|^2$$
$$+3|\int_{t_0}^{t}[\mu(s,Y(s))-\mu(s,Y^q(s-\frac{1}{q}))]dW(s)|^2.$$

Apply G-subexpectation on both sides. Then by virtue of the Doob's martingale, Holder's and BDG [5] inequalities we derive

$$E(\sup_{t_0\leq s\leq t}|Y(s)-Y^q(s)|^2) \leq 3T\int_{t_0}^{t}E[|\kappa(s,Y(s))-\kappa(s,Y^q(s-\frac{1}{q}))|^2]ds$$
$$+3T\int_{t_0}^{t}E[|\lambda(s,Y(s))-\lambda(s,Y^q(s-\frac{1}{q}))|^2]ds$$
$$+12\int_{t_0}^{t}E[|\mu(s,Y(s))-\mu(s,Y^q(s-\frac{1}{q}))|^2]ds.$$

Using the non-uniform Lipschitz condition we get

$$E(\sup_{t_0\leq s\leq t}|Y(s)-Y^q(s)|^2) \leq 6(T+2)\int_{t_0}^{t}E[\Psi(|Y(s)-Y^q(s-\frac{1}{q})|^2)]ds$$
$$\leq 6T(T+2)C+6(T+2)D\int_{t_0}^{t}E[|Y(s)-Y^q(s-\frac{1}{q})|^2]ds$$
$$=6T(T+2)C+6(T+2)D\int_{t_0}^{t}E[|Y(s)-Y^q(s)+Y^q(s)-Y^q(s-\frac{1}{q})|^2]ds$$
$$\leq 6T(T+2)C+12(T+2)D\int_{t_0}^{t}E[|Y(s)-Y^q(s)|^2]ds$$
$$+12(T+2)D\int_{t_0}^{t}E[|Y^q(s)-Y^q(s-\frac{1}{q})|^2]ds$$

Utilizing lemma 4.1, we determine

$$E(\sup_{t_0\leq s\leq t}|Y(s)-Y^q(s)|^2) \leq 6T(T+2)C+12(T+2)D\int_{t_0}^{t}E(\sup_{t_0\leq r\leq s}|Y(r)-Y^q(r)|^2)ds+12T(T+2)D\frac{1}{q}$$

Finally, the Grownwall's inequality gives

$$E(\sup_{t_0\leq s\leq t}|Y(s)-Y^q(s)|^2) \leq [6T(T+2)C+12T(T+2)D\frac{1}{q}]e^{12(T+2)D(t-t_0)}.$$

Consequently, by letting $t=T$, we get

$$E(\sup_{t_0\leq s\leq T}|Y(s)-Y^q(s)|^2) \leq 6T(T+2)[C+\frac{2D}{q}]e^{12(T+2)D(T-t_0)}.$$

The proof stands completed. □

5. Conclusion

This paper opens several new research directions with arising the following open problems. What will be the estimates for the difference between exact and Caratheodory approximate solutions to G-SFDEs under non-linear growth and non-Lipschitz conditions? How can one solve the stated problem for G-NSFDEs? Can one gives estimates for the difference between exact and Caratheodory approximate solutions to backward stochastic differential equations in the framework of G-Brownian motion? Under what conditions, can we develop the mentioned theory for stochastic pantograph equations [2, 12, 31, 32]? We hope the current paper will play an essential role to establish a foundation for the concepts briefly discussed.

6. Acknowledgement

This work is supported by the Commonwealth Scholarship Commission in the United Kingdom with project number PKRF-2017-429.

References

[1] X. Bai, Y. Lin, On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with Integral-Lipschitz coefficients, Acta Mathematicae Applicatae Sinica, English Series, 3(30) (2014) 589-610.

[2] C. T.H. Baker and E. Buckwar, Continuous Θ-methods for the stochastic pantograph equation, Electronic Transactions on Numerical Analysis, 11 (2000) 131-151.

[3] V.V Bolotin, Random vibrations of elastic systems, Martinus Nijhoff,The Hague (1984).

[4] D. R. Brillinger, H. K. Preisler, A. A. Ager, J. G. Kie, B. S. Stewart, Employing stochastic differential equations to model wildlife motion, Bull. Braz. Math. Soc., 33(3), (2002) 385-408.

[5] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths, Potential Anal., 34 (2010) 139–161.

[6] F. Faizullah, Existence and uniqueness of solutions to SFDEs driven by G-Brownian motion with non-Lipschitz conditions, Journal of Computational Analysis and Applications, 2(23) (2017) 344-354.

[7] F. Faizullah, On the pth moment estimates of solutions to stochastic functional differential equations in the G-framework, SpringerPlus, 5(872) (2016) 1-11.

[8] F. Faizullah, A note on p-th moment estimates for stochastic functional differential equations in the framework of G-Brownian motion, Iranian Journal of Science and Technology, Transaction A: Science, 3(40) (2016) 1-8.

[9] F. Faizullah, Existence results and moment estimates for NSFDEs driven by G-Brownian motion, Journal of Computational and Theoretical Nanoscience, 7(13) (2016) 1-8.

[10] F. Faizullah, Existence of solutions for G-SFDEs with Cauchy-Maruyama Approximation Scheme, Abstract and Applied Analysis, http://dx.doi.org/10.1155/2014/809431, (2014) 1–8.

[11] F. Faizullah, Existence of solutions for stochastic differential equations under G-Brownian motion with discontinuous coefficients, Zeitschrift fr Naturforschung A., 67A (2012) 692–698.

[12] Z. Fan, M. Liu and W. Cao, Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations, J. Math. Anal. Appl., 325 (2007) 11421159.

[13] M. Hu, S. Peng, Extended conditional G-expectations and related stopping times, arXiv:1309.3829v1[math.PR] 16 Sep 2013.

[14] M. Hu, S. Ji, S. Peng, Y. Song, Backward stochastic differential equations driven by G-Brownian motion , Stochastic Processes and their Applications, 124 (2014) 759784.

[15] F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and thier Applications, 2 (2009) 3356–3382.

[16] J.H. Gubner, Probability and random processes for electrical and computer engineering, Cambridge University Press, 2006.

[17] lyas Khan, F. Ali, N. A. Shah, Interaction of magnetic field with heat and mass transfer in free convection flow of a Walters-B fluid, The European Physical Journal Plus, 131(77) (2016) 1-15.

[18] X. Li, S. Peng, Stopping times and related Ito's calculus with G-Brownian motion, Stochastic Processes and thier Applications, 121 (2011) 1492–1508.

[19] X. Mao, *Stochastic Differential Equations and their Applications*, Horwood Publishing Chichester, 1997.

[20] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito's type, The abel symposium 2005, Abel symposia 2, edit. benth et. al., Springer-vertag., (2006) 541-567.

[21] S. Peng, Multi-dimentional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and thier Applications, 12 (2008) 2223–2253.

[22] S. Primak, V. Kontorovitch, V. Lyandres, Stochastic methods and their applications to communications: Stochastic differential equations approach, John Wiley & Sons Ltd, Chichester England, 2004.

[23] Y. Ren, Q. Bi, R. Sakthivel, Stochastic functional differential equations with infinite delay driven by G-Brownian motion, Mathematical Methods in the Applied Sciences, 36(13) (2013) 1746–1759.

[24] J.B Roberts, P.D. Spanos, Random vibration and statistical linearization, Dover, New York (2003).

[25] B. Skalmierski, A. Tylikowski, Stability of dynamical systems, Polish Scientific Editors, Warsaw (1973).

[26] B. Skalmierski, A. Tylikowski, Stochastic processes in dynamics, Polish Scientific editors, Warsaw (1982).

[27] K. Sobezyk, Stochastic waive propagation, Polish Scientific editors, Warsaw (1984).

[28] K. Sobezyk, Stochastic differential equations with applications to Physics and Engineering, Kluwer Academic, Dordrecht (1991).

[29] K. Sobezyk, Jr. B.F. Spencer, Random Fatigue: Frome data to theory Academic Press, Boston (1992).

[30] T. V. Ton, N. T. H. Linh, A.Yagi, A stochastic differential equations model for foraging swarms, arXiv:1509.00063v1 [math.PR] 28 Aug 2015.

[31] Y. Xiao, M. Song and M. Liu, Convergence and stability of semi-implicit Euler method with variable stepsize for a linear stochastic pantograph differential equation, International Journal of Numerical Analysis and Modeling, 2(8) (2011) 214225.

[32] Y. Xiao, H.Y. Zhang, A note on convergence of semi-implicit Euler methods for stochastic pantograph equations, Computers and Mathematics with Applications, 59 (2010) 1419-1424.

Behavior of a system of higher-order difference equations

M. A. El-Moneam* A. Q. Khan† E. S. Aly‡ M. A. Aiyashi§

Abstract

We study the local stability about equilibria, periodicity nature of positive solutions and existence of unbounded solutions of higher-order system of rational difference equations. The results presented here are considerably extended and improve some existing results in the literature. Finally theoretical results are verified numerically.

Keywords: difference equations; local stability; periodicity; unbounded solutions

AMS subject classifications: 39A10, 40A05

1 Introduction

In [1], Bajo and Liz have investigated the global behavior of the difference equation: $x_{n+1} = \frac{x_{n-1}}{a+bx_{n-1}x_n}$, where $a,\ b,\ x_0,\ x_{-1} \in \mathbb{R}_+^2$. Aloqeili [2] has investigated the stability and semi-cycle analysis of the difference equation: $x_{n+1} = \frac{x_{n-1}}{a-x_{n-1}x_n}$, $n = 0, 1, \cdots$, where $a,\ x_0,\ x_{-1} \in \mathbb{R}_+^2$. For systemic study of difference equations and systems of difference equations, we refer the reader [3–7] and references cited therein. Motivated by the above studies, our aim in this paper is to investigate the local stability about equilibria, periodicity nature of the positive solutions and existence of unbounded solutions of the following higher-order system of difference equations:

$$x_{n+1} = \frac{\alpha_1 x_{n-k}}{\beta_1 - \gamma_1 \prod_{i=0}^{k} y_{n-i}},\ y_{n+1} = \frac{\alpha_2 y_{n-k}}{\beta_2 - \gamma_2 \prod_{i=0}^{k} x_{n-i}},\ n = 0, 1, \cdots, \tag{1}$$

where $\alpha_i,\ \beta_i,\ \gamma_i$ for $i = 1, 2$ and $x_{-j},\ y_{-j}$ for $j = 0, 1, \cdots, k$ are belong to $\mathbb{R}_+^2$. The rest of the paper is organized as follows: Existence of equilibria and local stability are studied in Section 2. Section 3 deals with the study of periodicity nature and existence of unbounded solutions of system (1). In Section 4, numerical simulations are presented to verify theocratical discussion. A brief conclusion is given in last Section.

*Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: maliahmedibrahim@jazanu.edu.sa

†Department of Mathematics, University of Azad Jammu and Kashmir, Muzaffarabad 13100, Pakistan, e-mail: abdulqadeerkhan1@gmail.com

‡Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: elkhateeb@jazanu.edu.sa

§Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: maiyashi@jazanu.edu.sa

2 Existence of equilibria and local stability

In this section, we will study the existence of equilibria and local stability of system (1). The results about the existence of equilibria are summarized into following Lemma:

Lemma 1. *System (1) has two equilibria in the interior of* $\mathbb{R}_+^2$*. More precisely*

(i) $\forall$ *parametric values, system (1) has a unique boundary equilibrium point* $O(0,0)$*;*

(ii) If $\alpha_1 < \beta_1$ *and* $\alpha_2 < \beta_2$*, then* $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ *is the unique positive equilibrium point of system (1).*

Hereafter we will study the local stability of system (1) about boundary equilibrium $(0,0)$ and the unique positive equilibrium point $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ of system (1).

Lemma 2. *For local dynamics about* $O(0,0)$ *and* $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$*, the following statements hold:*

(i) For equilibrium $O(0,0)$*, the following holds:*

(i.1) $O(0,0)$ *is locally asymptotically stable if* $\alpha_1 < \beta_1$ *and* $\alpha_2 < \beta_2$*;*

(i.2) $O(0,0)$ *is a unstable if* $\alpha_1 > \beta_1$ *or* $\alpha_2 < \beta_2$*.*

(ii) $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ *is unstable.*

Proof. $(i.1)$ The linearized system of (1) about $(0,0)$ becomes: $X_{n+1} = J_{(0,0)}X_n$ where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}, \quad J_{(0,0)} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha_1}{\beta_1} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_2}{\beta_2} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

. The characteristic equation of $J_(0,0)$ about $(0,0)$ is

$$\lambda^{2k+2} - \left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}\right)\lambda^{k+1} + \frac{\alpha_1\alpha_2}{\beta_1\beta_2} = 0. \tag{2}$$

If $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$ then all roots of (2) lie inside unit disk. So $O(0,0)$ of system (1) is locally asymptotically stable.

$(i.2)$ It is easy to show that if $\alpha_1 > \beta_1$ or $\alpha_2 > \beta_2$ then $O(0,0)$ is unstable.

(ii). The linearized system of (1) about $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ becomes: $X_{n+1} = J_A X_n$ where

$$J_A = E_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} & \dots & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & \dots & \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1,\ \lambda_2,\ \dots,\ \lambda_{2k+2}$ denote the $2k+2$ eigenvalues of matrix E. Let $D = diag(d_1,\ d_2,\ \dots,\ d_{2k+2})$ be a diagonal matrix, where $d_1 = d_{k+2} = 1$, $d_i = d_{k+1+i} = 1 - i\epsilon,\ i = 2,\ 3,\ \cdots,\ k+1$ for $0 < \epsilon < 1$. Clearly, D is invertible. In computing DED^{-1}, we obtain that

$$DED^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & d_1d_{k+1}^{-1} & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+2}^{-1}}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+3}^{-1}}{\alpha_1} & \dots & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+1}^{-1}}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+2}^{-1}}{\alpha_1} \\ d_2d_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{k+1}d_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_1^{-1}}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_2^{-1}}{\alpha_2} & \dots & \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_k^{-1}}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_{k+1}^{-1}}{\alpha_2} & 0 & 0 & \dots & 0 & d_{k+2}d_{2k+2}^{-1} \\ 0 & 0 & \dots & 0 & 0 & d_{k+3}d_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & d_{2k+2}d_{2k+1}^{-1} & 0 \end{pmatrix}. \tag{3}$$

From $d_1 > d_2 > \cdots > d_{k+1} > 0$ and $d_{k+2} > d_{k+3} > \cdots > d_{2k+2} > 0$ it implies that $d_2d_1^{-1} < 1,\ d_3d_2^{-1} < 1,\ \cdots,\ d_{k+1}d_k^{-1} < 1$ and $d_{k+3}d_{k+2}^{-1} < 1,\ d_{k+4}d_{k+3}^{-1} < 1,\ \cdots,\ d_{2k+2}d_{2k+1}^{-1} < 1$. Furthermore,

$$\begin{aligned} d_1d_{k+1}^{-1} + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+2}^{-1}}{\alpha_1} + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+3}^{-1}}{\alpha_1} + \cdots + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+1}^{-1}}{\alpha_1} + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+2}^{-1}}{\alpha_1} &= \\ \frac{1}{1-(k+1)\epsilon} + \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1}\left(1 + \frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-k\epsilon} + \frac{1}{1-(k+1)\epsilon}\right) > 1. \end{aligned}$$

Also

$$\begin{aligned} \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_1^{-1}}{\alpha_2} + \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_2^{-1}}{\alpha_2} + \cdots + \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_k^{-1}}{\alpha_2} + \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_{k+1}^{-1}}{\alpha_2} + d_{k+2}d_{2k+2}^{-1} &= \\ \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2}\left(1 + \frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-k\epsilon} + \frac{1}{1-(k+1)\epsilon}\right) + \frac{1}{1-(k+1)\epsilon} > 1. \end{aligned}$$

It is well-known fact that E has the same eigenvalues as DED^{-1}. Hence, we obtain

$$\begin{aligned}\max_{1\leq m\leq 2k+2}|\lambda_m| \leq \|DED^{-1}\|_\infty &= \max\{d_2d_1^{-1},\cdots,d_{k+1}d_k^{-1},d_{k+3}d_{k+2}^{-1},\cdots,d_{2k+2}d_{2k+1}^{-1},\frac{1}{1-(k+1)\epsilon}\\ &+\frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1}\left(1+\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-k\epsilon}+\frac{1}{1-(k+1)\epsilon}\right),\frac{1}{1-(k+1)\epsilon}\\ &+\frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2}\left(1+\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-k\epsilon}+\frac{1}{1-(k+1)\epsilon}\right)\} > 1.\end{aligned}$$

This implies that $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}},\left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ of system (1) is unstable. □

3 Periodicity nature and existence of unbounded solutions

In this section, we will study the periodicity nature and existence of unbounded solutions of system (1). Let us denote $a_1=\gamma_1y_{-k}y_{1-k}\cdots y_0,\ a_2=\gamma_2x_{-k}x_{1-k}\cdots x_0$ to study the periodicity nature of positive solution of system (1).

Theorem 1. *If $a_1=\beta_1-\alpha_1$ and $a_2=\beta_2-\alpha_2$, then system (1) has prime period-(k+1) solutions.*

Proof. From system (1) and $a_1=\beta_1-\alpha_1,\ a_2=\beta_2-\alpha_2$, we have

$$x_1=\frac{\alpha_1x_{-k}}{\beta_1-\gamma_1\prod_{i=0}^{k}y_{-i}}=\frac{\alpha_1x_{-k}}{\beta_1-a_1}=x_{-k},\ y_1=\frac{\alpha_2y_{-k}}{\beta_2-\gamma_2\prod_{i=0}^{k}x_{-i}}=\frac{\alpha_2y_{-k}}{\beta_2-a_2}=y_{-k}.$$

$$x_2=\frac{\alpha_1x_{1-k}}{\beta_1-\gamma_1\prod_{i=0}^{k}y_{1-i}}=\frac{\alpha_1x_{1-k}}{\beta_1-\gamma_1y_1y_0y_{-1}\cdots y_{1-k}}=\frac{\alpha_1x_{1-k}}{\beta_1-\gamma_1y_0y_{-1}\cdots y_{1-k}y_{-k}}=\frac{\alpha_1x_{1-k}}{\beta_1-a_1}=x_{1-k},$$

$$y_2=\frac{\alpha_2y_{1-k}}{\beta_2-\gamma_2\prod_{i=0}^{k}x_{1-i}}=\frac{\alpha_2y_{1-k}}{\beta_2-\gamma_2x_1x_0x_{-1}\cdots x_{1-k}}=\frac{\alpha_2y_{1-k}}{\beta_2-\gamma_2x_0x_{-1}\cdots x_{1-k}x_{-k}}=\frac{\alpha_2y_{1-k}}{\beta_2-a_2}=y_{1-k}.$$

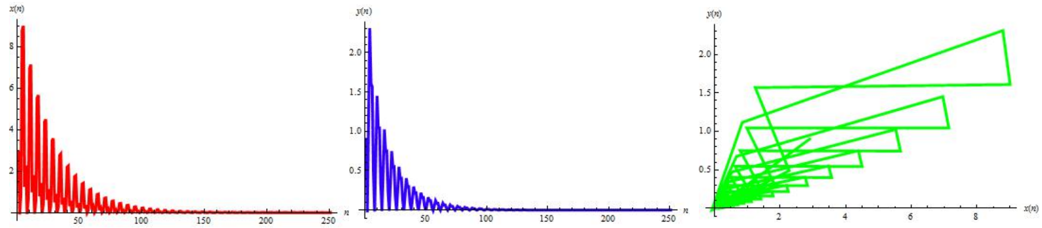
By induction, one has

$$x_{k+2}=\frac{\alpha_1x_1}{\beta_1-\gamma_1\prod_{i=0}^{k}y_{k+1-i}}=\frac{\alpha_1x_1}{\beta_1-\gamma_1y_{k+1}y_ky_{k-1}\cdots y_1}=\frac{\alpha_1x_1}{\beta_1-\gamma_1y_0y_{-1}\cdots y_{1-k}y_{-k}}=\frac{\alpha_1x_1}{\beta_1-a_1}=x_1,$$

$$y_{k+2}=\frac{\alpha_2y_1}{\beta_2-\gamma_2\prod_{i=0}^{k}x_{k+1-i}}=\frac{\alpha_2y_1}{\beta_2-\gamma_2x_{k+1}x_kx_{k-1}\cdots x_1}=\frac{\alpha_2y_1}{\beta_2-\gamma_2x_0x_{-1}\cdots x_{1-k}x_{-k}}=\frac{\alpha_2y_1}{\beta_2-a_2}=y_1.$$

□

Theorem 2. *Assume that $\beta_1<\alpha_1,\beta_2<\alpha_2$. Then, every positive solution $\{(x_n,y_n)\}$ of system (1) tends to ∞ as $n\to\infty$.*



(a) (b) (c)

Figure 1: Plots for system (5)

Proof. From system (1), it follows that

$$x_{n+1} = \frac{\alpha_1 x_{n-k}}{\beta_1 - \gamma_1 \prod_{i=0}^{k} y_{n-i}} \geq \frac{\alpha_1 x_{n-k}}{\beta_1} > x_{n-k}, \quad y_{n+1} = \frac{\alpha_2 y_{n-k}}{\beta_2 - \gamma_2 \prod_{i=0}^{k} x_{n-i}} \geq \frac{\alpha_2 y_{n-k}}{\beta_2} > y_{n-k}. \tag{4}$$

From first equation of (4), we have $x_{(k+1)n+1} > x_{(k+1)n-k}$, and $x_{(k+1)n+(k+2)} > x_{(k+1)n+1}$. Hence, the subsequences $\{x_{(k+1)n+1}\}, \cdots, \{x_{(k+1)n+(k+1)}\}$ are increasing, *i.e.*, the sequence $\{x_n\}$ is increasing. So, $x_n \to \infty$ as $n \to \infty$. Similarly, from second equation of (4) one gets: $y_{(k+1)n+1} > y_{(k+1)n-k}$ and $y_{(k+1)n+(k+2)} > y_{(k+1)n+1}$. Hence, the subsequences $\{y_{(k+1)n+1}\}, \cdots, \{y_{(k+1)n+(k+1)}\}$ are increasing, *i.e.*, the sequence $\{y_n\}$ is increasing. So, $y_n \to \infty$ as $n \to \infty$. □

4 Numerical simulations

In this section we will present numerical simulations to verify theoretical results.

Example 1. *If* $\alpha_1 = 50, \beta_1 = 63, \gamma_1 = 4, \alpha_2 = 90, \beta_2 = 122, \gamma_2 = 2$ *then system (1) with* $x_{-5} = 3.9, x_{-4} = 1.5, x_{-3} = 12.4, x_{-2} = 11.9, x_{-1} = 1.6, x_0 = 2.9, y_{-5} = 2.6, y_{-4} = 3.8, y_{-3} = 5.8, y_{-2} = 3.5, y_{-1} = 3.1, y_0 = 0.9$ *can be written as:*

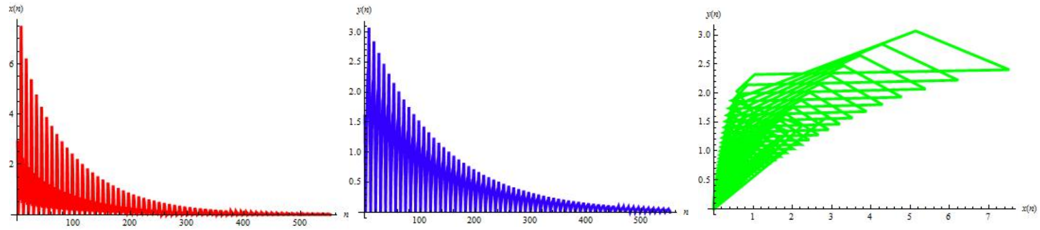
$$x_{n+1} = \frac{50 x_{n-5}}{63 - 4 y_n y_{n-1} y_{n-2} y_{n-3} y_{n-4} y_{n-5}}, \quad y_{n+1} = \frac{90 y_{n-5}}{122 - 2 x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4} x_{n-5}}. \tag{5}$$

Moreover, in Fig. 1 the plot of x_n *is shown in Fig. 1a, the plot of* y_n *is shown in Fig. 1b and global attractor of system (5) is shown in Fig. 1c.*

Example 2. *If* $\alpha_1 = 15.5, \beta_1 = 17, \gamma_1 = 27, \alpha_2 = 11.2, \beta_2 = 12, \gamma_2 = 23$*, then system (1) with* $x_{-8} = 1.9, x_{-7} = 1.7, x_{-6} = 2.5, x_{-5} = 0.9, x_{-4} = 1.5, x_{-3} = 10.4, x_{-2} = 6.9, x_{-1} = 0.6, x_0 = 2.9, y_{-8} = 2.8, y_{-7} = 1.6, y_{-6} = 1.8, y_{-5} = 2.6, y_{-4} = 2.8, y_{-3} = 2.8, y_{-2} = 3.5, y_{-1} = 2.1, y_0 = 1.6$ *can be written as*

$$\begin{aligned} x_{n+1} &= \frac{15.5 x_{n-8}}{17 - 27 y_n y_{n-1} y_{n-2} y_{n-3} y_{n-4} y_{n-5} y_{n-6} y_{n-7} y_{n-8}}, \\ y_{n+1} &= \frac{11.2 y_{n-8}}{12 - 23 x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4} x_{n-5} x_{n-6} x_{n-7} x_{n-8}}. \end{aligned} \tag{6}$$

Moreover, in Fig. 2 the plot of x_n *is shown in Fig. 2a, the plot of* y_n *is shown in Fig. 2b and global attractor of system (6) is shown in Fig. 2c.*



(a) (b) (c)

Figure 2: Plots for system (6)

5 Conclusion and future work

This work is related to the qualitative behavior of a system of higher-order rational difference equations. We have proved that under some restrictions to parameters, system (1) has a boundary equilibrium $O(0,0)$ and the unique positive equilibrium point $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ in the closed first quadrant $\mathbb{R}_+^2$. We have analyzed the local stability about equilibria, periodicity nature of positive solutions and existence of unbounded solutions of system (1). Finally, theoretical results are verified numerically. Besides the local properties, the global stability of under consideration system (1), which is our further aim to study.

Acknowledgements
A. Q. Khan research is supported by the Higher Education Commission(HEC) of Pakistan.

References

[1] I. Bajo, E. Liz, Global behaviour of a second-order nonlinear difference equation, *Journal of Difference Equations and Applications*, 17(10)(2011):1471-1486.

[2] M. Aloqeili, Dynamics of a rational difference equation, *Applied Mathematics and Computation*, 176(2)(2006):768-774.

[3] E. A. Grove, G. Ladas, Periodicities in nonlinear difference equations, Chapman and Hall/CRC Press, Boca Raton, (2004).

[4] H. Sedaghat, Nonlinear difference equations:theory with applications to social science models, Kluwer Academic Publishers, Dordrecht, (2003).

[5] V. L. Kocic, G. Ladas, Global behavior of nonlinear difference equations of higher-order with applications, Kluwer Academic Publishers, Dordrecht, (1993).

[6] E. Camouzis, G. Ladas, Dynamics of third-order rational difference equations:with open problems and conjectures, Chapman and Hall/HRC, Boca Raton, (2007).

[7] V. L. Kocic, G. Ladas, Global attractivity in a second order nonlinear difference equations, *Journal of Mathematical Analysis and Applications*, 180(1993):144-150.

ON APPROXIMATING THE GENERALIZED EULER-MASCHERONI CONSTANT*

TI-REN HUANG[1], BO-WEN HAN[2], XIAO-YAN MA[2], AND YU-MING CHU[3,**]

ABSTRACT. In the article, we provide several sharp bounds for the the generalized Euler-Mascheroni constant, which are the generalizations of the previously results on the Euler-Mascheroni constant.

1. INTRODUCTION

It is well known that the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \tag{1.1}$$

is convergent towards the Euler-Mascheroni constant

$$\gamma = 0.57721566490115328\cdots. \tag{1.2}$$

The Euler-Mascheroni constant has been involved in a variety of mathematical formulas and results [1-6], many special functions are closely related to the Euler-Mascheroni constant [7-63]. Recently, the bounds for $\gamma_n - \gamma$ have attracted the attention of many researchers.

Alzer [64] proved that the double inequality

$$\frac{1}{2n+1} \le \gamma_n - \gamma \le \frac{1}{2n}$$

holds for $n \ge 1$.

In [65], Tóth proved that the two-sided inequality

$$\frac{1}{2n+\frac{2}{5}} < \gamma_n - \gamma \le \frac{1}{2n+\frac{1}{3}} \tag{1.3}$$

takes place for $n \ge 1$.

Chen [66] proved that $\alpha = (2\gamma-1)/(1-\gamma)$ and $\beta = 1/3$ are the best possible constants such that the double inequality

$$\frac{1}{2n+\alpha} \le \gamma_n - \gamma < \frac{1}{2n+\beta} \tag{1.4}$$

holds for $n \ge 1$.

2010 *Mathematics Subject Classification.* Primary: 11Y60; Secondary: 40A05, 33B15.

Key words and phrases. Euler-Mascheroni constant, gamma function, psi function, asymptotic expansion.

*The research was supported by the Natural Science Foundation of China (Grants Nos. 61673169, 11301127, 11701176, 11626101, 11401531, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant no. Y201635325), the Natural Science Foundation of Zhejiang Province (Grant No. LQ17A010010) and the Science Foundation of Zhejiang Sci-Tech University (Grant No. 14062093-Y).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

In [67], Qiu and Vuorinen proved that the double inequality

$$\frac{1}{2n} - \frac{\lambda}{n^2} < \gamma_n - \gamma \leq \frac{1}{2n} - \frac{\mu}{n^2} \tag{1.5}$$

holds for $n \geq 1$ if and only if $\lambda \geq 1/12$ and $\mu \leq \gamma - 1/2$.

Let $a > 0$. Then the generalized Euler-Mascheroni constant $\gamma(a)$ is defined by

$$\gamma(a) = \lim_{n\to\infty} \left(\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log \frac{a+n-1}{a} \right), \tag{1.6}$$

which was introduced by Knopp [68]. We clearly see that $\gamma(1) = \gamma$. Recently, the generalized Euler-Mascheroni constant $\gamma(a)$ has been the subject of intensive research [69-71].

In [70], Sîntămărian introduced the sequences

$$x_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log \frac{a+n}{a}, \tag{1.7}$$

$$y_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log \frac{a+n-1}{a}, \tag{1.8}$$

and proved that the double inequalities

$$\frac{1}{2(n+a)} \leq \gamma(a) - x_n \leq \frac{1}{2(n+a-1)}, \tag{1.9}$$

$$\frac{1}{2(n+a)} \leq y_n - \gamma(a) \leq \frac{1}{2(n+a-1)} \tag{1.10}$$

hold for $n \geq 1$.

In [71], Berinde and Mortici established Theorems 1.1 and 1.2 as follows.

Theorem 1.1. *The double inequalities*

$$\frac{1}{2(n+a) - \frac{1}{4}} < \gamma(a) - x_n < \frac{1}{2(n+a) - \frac{1}{3}}, \tag{1.11}$$

$$\frac{1}{2(n+a) - \frac{4}{3}} < y_n - \gamma(a) < \frac{1}{2(n+a) - \frac{5}{3}} \tag{1.12}$$

hold for $a > 0$ and $n \geq 2$.

Theorem 1.2. (a) *The inequality*

$$\frac{1}{2(n+a) - \frac{1}{3} + \frac{1}{18n}} \leq \gamma(a) - x_n \tag{1.13}$$

holds for $a \geq 13/30$ and any integer $n \geq 1$.

(b) *The inequality*

$$\frac{1}{2(n+a) - \frac{5}{3} + \frac{1}{18n}} \leq y_n - \gamma(a) \tag{1.14}$$

holds for $a \geq 17/30$ and $n \geq 1$.

The main purpose of this article is to generalize inequalities (1.4) and (1.5) to the generalized Euler-Mascheroni constant $\gamma(a)$. Our main results are the following Theorems 1.3 and 1.4.

Theorem 1.3. *Let $a > 0$, $n \geq 1$. Then one has*

(1) *the double inequality*

$$\frac{1}{2(n+a)-\alpha_1} \leq \gamma(a) - x_n < \frac{1}{2(n+a)-\beta_1} \tag{1.15}$$

holds with the best possible constants

$$\alpha_1 = 2(1+a) - \frac{1}{\psi(1+a)-\log(1+a)}, \quad \beta_1 = \frac{1}{3}; \tag{1.16}$$

(2) *the two-sided inequality*

$$\frac{1}{2(n+a)-\alpha_2} \leq y_n - \gamma(a) < \frac{1}{2(n+a)-\beta_2} \tag{1.17}$$

is valid with the best possible constants

$$\alpha_2 = 2(1-d), \qquad \beta_2 = \frac{5}{3}, \tag{1.18}$$

where

$$d = \max\{\tilde{f}_2(a), \tilde{f}_2(1+a), \tilde{f}_2(2+a)\}, \quad \tilde{f}_2(x) = \frac{1}{2(\psi(x+1)-\log(x))} - x.$$

Theorem 1.4. *Let $a > 0$, $n \geq 1$. Then the double inequalities*

$$\frac{1}{2(n+a)} + \frac{\alpha_3}{(n+a)^2} \leq \gamma(a) - x_n < \frac{1}{2(n+a)} + \frac{\beta_3}{(n+a)^2}, \tag{1.19}$$

$$\frac{1}{2(n+a-1)} + \frac{\alpha_4}{(n+a-1)^2} < y_n - \gamma(a) \leq \frac{1}{2(n+a-1)} + \frac{\beta_4}{(n+a-1)^2} \tag{1.20}$$

hold with the best possible constants

$$\alpha_3 = (1+a)^2[\log(1+a) - \psi(1+a)] - \frac{1+a}{2}, \quad \beta_3 = \frac{1}{12}, \tag{1.21}$$

$$\alpha_4 = -\frac{1}{12}, \quad \beta_4 = a^2[\psi(a) - \log(a)] + \frac{a}{2}. \tag{1.22}$$

2. Lemmas

In order to prove our main results, we need the following formulas and lemmas.

For $x > 0$, the classical gamma function $\Gamma(x)$ and psi function $\psi(x)$ [72-84] are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively.

The psi function $\psi(x)$ has the following recurrence and asymptotic formulas [85]

$$\psi(n+x) = \frac{1}{(n-1)+x} + \frac{1}{(n-2)+x} + \cdots + \frac{1}{2+x} + \frac{1}{1+x} + \frac{1}{x} + \psi(x), \tag{2.1}$$

$$\psi(x) \sim \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty) \tag{2.2}$$

According to (2.1) and the definitions of x_n and y_n given in (1.7) and (1.8), we clearly see that x_n and y_n can be rewritten as

$$x_n = \psi(n+a) - \psi(a) - \log\frac{n+a}{a}, \tag{2.3}$$

$$y_n = \psi(n+a) - \psi(a) - \log\frac{n+a-1}{a}. \tag{2.4}$$

It follows from (1.6) and (2.2) that

$$\gamma(a) = \lim_{n\to\infty} y_n \tag{2.5}$$

$$= \lim_{n\to\infty} (\psi(n+a) - \log(n+a-1) + \log(a) - \psi(a)) = \log(a) - \psi(a).$$

Therefore,

$$\gamma(a) - x_n = \log(n+a) - \psi(n+a), \tag{2.6}$$

$$y_n - \gamma(a) = \psi(n+a) - \log(n+a-1). \tag{2.7}$$

Lemma 2.1. *The function*

$$f_1(x) = \frac{1}{\log(x) - \psi(x)} - 2x \tag{2.8}$$

is strictly decreasing from $(1,\infty)$ *onto* $(-1/3, 1/\gamma - 2)$.

The function

$$f_2(x) = \frac{1}{\psi(x+1) - \log(x)} - 2x \tag{2.9}$$

is strictly decreasing from $[2,\infty)$ *onto* $(1/3, f_2(2)]$.

Proof. Differentiating $f_1(x)$ gives

$$(\log(x) - \psi(x))^2 f_1'(x) = \psi'(x) - \frac{1}{x} - 2(\log(x) - \psi(x))^2.$$

It follows from the inequalities

$$\psi'(x) - \frac{1}{x} < \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7},$$

$$\log(x) - \psi(x) > \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4}$$

given in [86] that

$$(\log(x) - \psi(x))^2 f_1'(x) < \frac{1}{50400x^8} F_1(x), \tag{2.10}$$

where

$$F_1(x) = -207 - 3840(x-1) - 6580(x-1)^2 - 3640(x-1)^3 - 700(x-1)^4 < 0 \tag{2.11}$$

for $x \in (1,\infty)$.

Therefore, the monotonicity of $f_1(x)$ follows easily from (2.10) and (2.11).

Clearly, $f_1(1) = 1/\gamma - 2$. The limiting value $\lim_{x\to\infty} f_1(x) = -1/3$ follows from the asymptotic formula (2.2).

Differentiating $f_2(x)$ leads to

$$2\left(\psi(x+1) - \log(x)\right)^2 f_2'(x) = \frac{1}{x} + \frac{1}{x^2} - \psi'(x) - 2(\psi(x) + \frac{1}{x} - \log(x))^2.$$

It follows from the inequalities

$$\frac{1}{x}+\frac{1}{x^2}-\psi'(x)<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5},$$
$$\psi(x)+\frac{1}{x}-\log(x)>\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}$$

for $x>0$ given in [86] that

$$2\left(\psi(x+1)-\ln(x)\right)^2 f_2'(x)<-\frac{F_2(x)}{3175200x^{12}},$$

where

$$\begin{aligned}F_2(x)&=3217636+17887632(x-2)+39443124(x-2)^2\\&+47009928(x-2)^3+33797841(x-2)^4+15180480(x-2)^5\\&+4189500(x-2)^6+652680(x-2)^7+44100(x-2)^8>0\end{aligned}$$

for $x\geq 2$.

Therefore, $f_2(x)$ is a strictly decreasing function on $[2,\infty)$. The limit $\lim_{x\to\infty} f_2(x)=1/6$ follows from the asymptotic formula (2.2). □

Remark 1. *Qi et. al. [87] proved that the function $f_2(x)$ defined by (2.9) is strictly decreasing on $(12/5,\infty)$.*

The following Lemma 2.2 can be found in [88, 89].

Lemma 2.2. *The function*

$$f_3(x)=x^2(\psi(x)-\log(x))+\frac{x}{2} \tag{2.12}$$

is strictly decreasing from $(0,\infty)$ onto $(-1/12,0)$ and completely monotonic on $(0,\infty)$.

3. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. From (2.6) we clearly see that inequality (1.15) can be rewritten as

$$-\beta<\frac{1}{\log(n+a)-\psi(n+a)}-2(n+a)<-\alpha.$$

It follows from Lemma 2.1 that the sequence

$$f_1(n+a)=\frac{1}{\log(n+a)-\psi(n+a)}-2(n+a)$$

is strictly decreasing, which leads to the conclusion that

$$-\frac{1}{3}=\lim_{n\to\infty} f_1(n)<f_1(n)\leq f_1(1)=\frac{1}{\log(1+a)-\psi(1+a)}-2(1+a).$$

Therefore,

$$\alpha_1=2(1+a)-\frac{1}{\psi(1+a)-\log(1+a)},\quad \beta_1=\frac{1}{3}$$

are the best possible constants such that inequality (1.15) holds.

From (2.7) we clearly see that inequality (1.17) is equivalent to

$$1-\frac{\beta}{2}<\frac{1}{2(\psi(n+a)-\log(n+a-1))}-(n+a-1)\leq 1-\frac{\alpha}{2}.$$

It follows from Lemma 2.1 that the sequence

$$\tilde{f}_2(n+a-1) = \frac{1}{2(\psi(n+a) - \log(n+a-1))} - (n+a-1)$$

is strictly decreasing for $n \geq 2$, which leads to the conclusion that

$$\frac{1}{6} = \lim_{n\to\infty} \tilde{f}_2(n) < \tilde{f}_2(n) \leq \max\left\{\tilde{f}_2(a), \tilde{f}_2(1+a), \tilde{f}_2(2+a)\right\} = d.$$

Therefore,

$$\alpha_2 = 2(1-d), \qquad \beta_2 = \frac{5}{3} \tag{3.1}$$

are the best possible constants such that inequality (1.17) holds.

Proof of Theorem 1.4. From (2.6) and (2.7) we know that inequalities (1.19) and (1.20) can be rewritten as

$$\alpha_3 \leq (n+a)^2 \left(\log(n+a) - \psi(n+a)\right) - \frac{(n+a)}{2} < \beta_3,$$

$$\alpha_4 < (n+a-1)^2 \left(\psi(n+a-1) - \log(n+a-1)\right) + \frac{(n+a-1)}{2} \leq \beta_4,$$

respectively.

It follows from Lemma 2.2 that the sequence

$$\tilde{f}_3(n+a-1) = (n+a-1)^2 \left(\psi(n+a-1) - \log(n+a-1)\right) + \frac{(n+a-1)}{2}$$

is strictly decreasing for $n \in \mathbb{N}$.

Note that

$$\lim_{n\to\infty} f_3(n) = -\frac{1}{12}.$$

Therefore,

$$\alpha_3 = (1+a)^2[\log(1+a) - \psi(1+a)] - \frac{1+a}{2}, \quad \beta_3 = \frac{1}{12},$$

$$\alpha_4 = -\frac{1}{12}, \qquad \beta_4 = (a)^2[\psi(a) - \log(a)] + \frac{a}{2}$$

are the best possible constants such that inequalities (1.19) and (1.20) hold.

Remark 2. (1) *Let $a = 1$. Then Theorem 1.3(2) leads to inequality (1.4) with the best possible constants $\alpha = (2\gamma - 1)/(1-\gamma)$ and $\beta = 1/3$.*

(2) *Let $a = 1$. Then inequality (1.20) becomes inequality (1.5) with the best possible constants $\alpha = 1/12$ and $\beta = \gamma - 1/2$.*

(3) *From Theorem 1.3 we know that both the upper bounds $1/[2(n+a) - 1/3]$ for $\gamma(a) - x_n$ and $1/[2(n+a) - 5/3]$ for $y_n - \gamma(a)$ given in (1.11) and (1.12) are sharp for any $a > 0$.*

References

[1] P. A. Panzone, Formulas for the Euler-Mascheroni constant, Rev. Un. Mat. Argentina, 2009, **50**(1), 161–164.

[2] J. Choi and H. M. Srivastava, Integral representations for the Euler-Mascheroni constant γ, Integral Transforms Spec. Funct., 2010, **21**(9-10), 675–690.

[3] T. Burić and N. Elezović, Approximants of the Euler-Mascheroni constant and harmonic numbers, Appl. Math. Comput., 2013, **222**, 604–611.

[4] P. A. Panzone, An approximation formula for Euler-Mascheroni constant, Rev. Un. Mat. Argentina, 2016, **57**(1), 9–22.

[5] T.-R. Huang, B.-W. Han, X.-Y. Ma and Y.-M. Chu, Optimal bounds for the generalized Euler-Mascheroni constant, J. Inequal. Appl., 2018, **2018**, Article 118, 9 pages.

[6] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.

[7] J. C. Sampedro, On a new constant related to Euler's constant, Ramanujan J., 2018, **46**(1), 77–89.

[8] N. Kurokawa and Y. Taguchi, A p-analogue of Euler's constant and congruence zeta functions, Proc. Japan Acad Ser. A Math. Sci., 2018, **94**(2), 13–16.

[9] O. Furdni, Multiple fractional part integrals and Euler's constant, Miskolc Math. Notes, 2016, **17**(1), 255–266.

[10] A. W. Addison, A series representation for Euler's constant, Amer. Math. Monthly, 1967, **74**, 823–824.

[11] W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang and F. Qi, Convexity of the generalized sine function and the generalized hyperbolic sine function, J. Approx. Theory, 2013, **174**, 1–9.

[12] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.

[13] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Monotonicity, convexity, and inequalities invoving the Agard distortion function, Abstr. Appl. Anal., 2011, **2011**, Article ID 671765, 8 pages.

[14] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.

[15] Y.-M. Chu and T.-H. Zhao, Concavity of the error functions with respect to Hölder means, Math. Inequal. Appl., 2016, **19**(2), 589–595.

[16] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.

[17] Y.-M. Chu and X.-M. Zhang, Multiplicative concavity of the integral of multiplicatively concave functions, J. Inequal. Appl., 2010, **2010**, Article ID 845930, 8 pages.

[18] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, J. Inequal. Appl., 2010, **2010**, Article ID 507560, 11 pages.

[19] Y.-M. Chu, G.-D. Wang and X.-H. Zhang, Schur convexity and Hadamard's inequality, Math. Inequal. Appl., 2010, **13**(4), 725–731.

[20] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 2018, **21**(2), 469–479.

[21] Zh.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, Math. Inequal. Appl., 2017, **20**(4), 1107–1120.

[22] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.

[23] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.

[24] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.

[25] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.

[26] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.

[27] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformaation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.

[28] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.

[29] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.

[30] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.

[31] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.

[32] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726.

[33] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.

[34] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.

[35] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.

[36] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.

[37] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.

[38] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.

[39] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.

[40] Zh.-H. Yang and Y.-M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 2017, **20**(3), 729–735.

[41] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, Monotonicity rule for the quotient of two function and its applications, J. Inequal. Appl., 2017, **2017**, Article 106, 13 pages.

[42] H.-Z. Xu, Y.-M. Chu and W.-M. Qian, Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means, J. Inequal. Appl., 2018, **2018**, Article 127, 13 pages.

[43] W.-M. Gong, Y.-Q. Song, M.-K. Wang and Y.-M. Chu, A sharp double inequality between Seiffert, arithmetic, and geometric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 684834, 7 pages.

[44] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.

[45] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.

[46] W.-F. Xia, Y.-M. Chu and G.-D. Wang, The optimal upper and lower power mean bounds for a convex combiantion of the arithmetic and logarithmic means, Abstr. Appl. Anal., 2010, **2010**, Article ID 604804, 9 pages.

[47] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.

[48] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.

[49] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu and G.-D. Wang, Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean, J. Math. Inequal., 2011, **5**(3), 301–306.

[50] Y.-M. Chu and W.-F. Xia, Two double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.

[51] Y.-M. Chu and B.-Y. Long, Best possible inequalities between generalized logarithmic mean and classical means, Abstr. Appl. Anal., 2010, **2010**, Article ID 303286, 13 pages.
[52] W.-M. Qian and Y.-M. Chu, Best possible bounds for Yang mean using generalized logarithmic mean, Math. Probl. Eng., 2016, **2016**, Article ID 8901258, 7 pages.
[53] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., 2012, **6**(4), 567–577.
[54] Y.-M. Chu, M.-K. Wang and G.-D. Wang, The optimal generalized logarithmic mean bounds for Seiffert's mean, Acta Math. Sci., 2012, **32B**(4), 1619–1626.
[55] Y.-M. Chu, Y.-M. Li, W.-F. Xia and X.-H. Zhang, Best possible inequalities for the harmonic mean of error function, J. Inequal. Appl., 2014, **2014**, Article 525, 9 pages.
[56] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A best-possible double inequality between Seiffert and harmonic means, J. Inequal. Appl., 2011, **2011**, Article 94, 7 pages.
[57] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., 2011, **2011**, Article ID 657935, 7 pages.
[58] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geoemtric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[59] B.-Y. Long and Y.-M. Chu, Optimal power mean bounds for the weighted geometric mean of classical means, J. Inequal. Appl., 2010, **2010**, Article ID 905697, 6 pages.
[60] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.
[61] Y.-M. Chu, B.-Y. Long and B.-Y. Liu, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[62] M.-K. Wang, Y.-M. Chu and Y.-F. Qiu, Some comparison inequalities for generalized Muirhead and identric means, J. Inequal. Appl., 2010, **2010**, Article ID 295620, 10 pages.
[63] W.-M. Qian, X.-H. Zhang and Y.-M. Chu, Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.
[64] H. Alzer, Inequalities for the gamma and polygamma functions, Abh. Math. Sem. Univ. Hamburg, 1998, **68**, 363–372.
[65] L. Tóth, Problem E3432, Amer. Math. Monthly, 1991, **98**(3), 264–264.
[66] C.-P. Chen, Inequalities for the Euler-Mascheroni constant, Appl. Math. Lett., 2010, **23**(2), 161–164.
[67] S.-L. Qiu and M. Vuorinen, Some properties of the gamma and psi functions, with applications, Math. Comput., 2005, **74**(250), 723–742.
[68] K. Knopp, Theory and Applications of Infinite Series, Dover Publications, New York, 1990.
[69] C. Mortici, Improved convergence towards generalized Euler-Mascheroni constant, Appl. Math. Comput., 2010, **215**(9), 3443-3448.
[70] A. Sîntămărian, A generalization of Euler's constant, Numer. Algorithms, 2007, **46**(2), 141–151.
[71] V. Berinde and C. Mortici, New sharp estimates of the generalized Euler-Mascheroni constant, Math. Inequal. Appl., 2013, **16**(1), 279–288.
[72] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[73] T.-H. Zhao and Y.-M. Chu, A class of logarithmically completely monotonic functions associated with gamma function, J. Inequal. Appl., 2010, **2010**, Article ID 392431, 11 pages.
[74] T.-H. Zhao, Y.-M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal., 2011, **2011**, Article ID 896483, 13 pages.
[75] M. Adil Khan, A. Iqbal, M. Suleman and Y.-M. Chu, Hermite-Hadamard typ inequalities for fractional integrals via Green's function, J. Inequal. Appl., 2018, **2018**, Article 161, 15 pages.
[76] M. Adil Khan, Y.-M. Chu, A. Kashuri, R. Liko and G. Ali, Conformable fractional integrals version of Hermite-Hadamard inequalities and their applications, J. Funct. Spaces, 2018, **2018**, Article ID 6928130, 9 pages.
[77] M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, 2018, **2018**, Article ID 5357463, 12 pages.
[78] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, 2018, **8**, Article ID 075101, 18 pages, DOI: 10.1063/1.5031954.

[79] M. Adil Khan, Z. M. Al-sahwi and Y.-M. Chu, New estimations for Shannon and Zipf-Mandelbrot entropies, Entropy, 2018, **20**, Article 608, 10 pages, DOI: 10.3390/e2008608.
[80] M. Adil Khan, Y.-M. Chu, A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions, J. Comput. Anal. Appl., 2019, **26**(8), 1487–1503.
[81] M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 2017, **15**, 1414–1430.
[82] Y.-M. Chu, M. Adil khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl, 2017, **2017**, Article 93, 12 pages.
[83] Y.-Q. Song, M. Adil Khan, S. Zaheer Ullah and Y.-M. Chu, Integral inequalities involving strongly convex functions, J. Funct. Spaces, 2018, **2018**, Article ID 6595921, 8 pages.
[84] M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, J. Inequal. Appl., 2018, **2018**, Article 70, 14 pages.
[85] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U. S. Government Printing Office, Washington, 1964.
[86] C.-P. Chen and F. Qi, The best lower and upper bounds of harmonic sequence, RGMIA Res. Rep. Coll., 2003, **6**(2), Article 14, 5 pages.
[87] F. Qi, R.-Q. Cui, C.-P. Chen and B.-N. Guo, Some completely monotonic functions involving polygamma functions and an applications, J. Math. Anal. Appl., 2005, **310**(1), 303–308.
[88] S.-L. Qiu and M. Vuorinen, Some properties of the gamma and psi functions, with applications, Math. Comput., 2005, **74**(250), 723–742.
[89] C.-P. Chen, F. Qi and H. M. Srivastava, Some properties of functions related to the gamma and psi functions, Integral Transforms Spec. Funct., 2010, **21**(1-2), 153–166.

Ti-Ren Huang, College of Science, Hunan City University, Yiyang 413000, China
E-mail address: huangtiren@163.com

Bo-Wen Han, Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China
E-mail address: 2648440793@qq.com

Xiao-Yan Ma, Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China
E-mail address: mxymath@126.com

Yu-Ming Chu (Corresponding author), Department of Mathematics, Huzhou University, Huzhou 313000, China
E-mail address: chuyuming2005@126.com

General study on Volterra integral equations of the second kind in space with weight function

M. E. Nasr [1,2] and M. F. Jabbar[1]

[1]Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

[2]Department of Mathematics, Collage of Science and Arts -Al Qurayyat, Jouf University, Kingdom of Saudi Arabia

mohamed.naser@fsc.bu.edu.eg[1], and masar.lefta@gmail.com[1]

Abstract

This paper is devoted to present a new and simple algorithm to prove that the function $\varphi_n(x)$ is a good approximation to the solution $\varphi(x)$ for Volterra integral equations (VIEs) of the second kind in the space $L^2_{p(x)}[0, 2\pi]$ with weight function $p(x)$. This approximation is discussed in details with help of the Valleé-Poussin's and Fèjer's, operators. Special attention is given to study the convergence analysis and estimation of an upper bound for the error of the approximated solution.

Key-Words: Volterra integral equations; Valleé-Poussin's and Fèjer's operators; Convergence analysis;

1. Introduction

In this paper, we present the approximate solution for Volterra integral equations (VIEs) of the second kind in the space $L^2_{p(x)}[0, 2\pi]$ with weight function $p(x) \geq 1$ where $p(x)$ is a summable function on $[0, 2\pi]$

$$\varphi(x) = f(x) + \lambda \int_0^x k(x,y)\varphi(y)dy, \qquad 0 \leq x, y \leq 2\pi, \tag{1}$$

where the functions $f(x)$, $k(x,y)$ belong to $L^2_{p(x)}[0, 2\pi]$ and are 2π-periodic functions, $\frac{1}{\lambda}$ is a regular value of the kernel $k(x,y)$ and the kernel $k(x,y)$ satisfies the following conditions

1. $\{\int_0^x p(y)|k(x,y)|^2 dy\}^{\frac{1}{2}} = \chi(x) \in L^2_{p(x)}[0, 2\pi]$;

2. $|\lambda| \|k(x,y)\|_{L^2_p} < 1$,

where

$$\|k(x,y)\|_{L^2_p} = \|k(x,y)\|_{L^2_{p(x)}[0,2\pi]} = \left[\int_0^{2\pi}\int_0^x p(x)p(y)|k(x,y)|^2 dydx\right]^{\frac{1}{2}}.$$

The simplicity of finding a solution for Fredholm integral equations (FIEs) of the second kind with degenerate kernel naturally leads one to think of replacing the given equation (1) by FIE with degenerate kernel, see [1, 2, 8, 9]. The solution of the new equation is taken as an approximate solution of the original equation. The study employs Dzyadyk's method which is based on the linear polynomial operator ([3]-[5]).

Eq.(1) can be written in the new form

$$\varphi(x) = f(x) + \lambda \int_0^x \tilde{k}(x,y)\varphi(y)dy, \tag{2}$$

where

$$\tilde{k}(x,y) = e(x,y)k(x,y), \qquad e(x,y) = \begin{cases} 1, & \text{for} \quad y \le x, \\ 0, & \text{for} \quad y > x. \end{cases} \tag{3}$$

From (3), it is found that the kernel $\tilde{k}(x,y)$ in (2) satisfies the following conditions (A^*)

1. $\{\int_0^{2\pi} p(y)|\tilde{k}(x,y)|^2 dy\}^{\frac{1}{2}} = \rho(x) \in L^2_{p(x)}[0,2\pi]$;

2. $|\lambda| \|\tilde{k}(x,y)\|_{L^2_p} < 1$,

where

$$\|\tilde{k}(x,y)\|_{L^2_p[0,2\pi]} = \left[\int_0^{2\pi}\int_0^{2\pi} p(x)p(y)|\tilde{k}(x,y)|^2 dydx\right]^{\frac{1}{2}}.$$

Now, instead of Eqs.(1) and (2), let us solve the following equations

$$\varphi_n(x) = U_n(f;x) + \lambda \int_0^{2\pi} U_n[\tilde{k}(.,y);x]\varphi_n(y)dy, \qquad 0 \le x, y \le 2\pi, \tag{4}$$

The notation $U_n[\tilde{k}(.,y);x]$ will mean that the operator U_n acts on $\tilde{k}(x,y)$ as a function of x and at the same time, the variable y plays the role of the parameter.

Now, since the functions $U_n(f;x)$ and $U_n[\tilde{k}(.,y);x]$ are both trigonometric polynomials of order n with respect to x, the solution $\varphi_n(x)$ of the Eq.(4) will also be trigonometric polynomial of order n in x. It is well known that the problem of determination of the solution of Fredholm integral equation of the second kind with degenerate kernel is reduced to the solution of corresponding system of algebraic equations [11]. In this study, it will be proved that the function $\varphi_n(x)$ is a good approximation to the solution $\varphi(x)$ of Eq.(1) on the space $L^2_{p(x)}[0,2\pi]$. This approximation is discussed in details for Valleé-Poussin's and Fèjer's operators.

2. Preliminaries

Starting from the known linear polynomial operators $U_n(g;x)$ which are good approximation to the function $g(x)$ in the space $L^2_{p(x)}$, and have the form:

$$U_n(g;x) = \frac{1}{\pi}\int_0^{2\pi} g(t)U_n(x-t)dt = \frac{1}{\pi}\int_0^{2\pi} g(x-t)U_n(t)dt, \tag{5}$$

where

$$U_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos(kx), \tag{6}$$

$\lambda_k^{(n)}$ are constants which define the method of approximation.

Theorem 1. *[6]*

For $k(x,y)$ belongs to $L^2_p[0,2\pi]$, such that $|\lambda|\|\tilde{k}(x,y)\|_{L^2_p} < 1$, and $f(x)$ belongs to $L^2_{p(x)}$, then the integral equation

$$\varphi(x) = f(x) + \lambda \int_0^{2\pi} k(x,y)\varphi(y)dy,$$

has an unique solution $\varphi(x)$ in $L^2_{p(x)}[0,2\pi]$.

Now, with the help of the following theorem we will find the condition by which the equation (4) has an unique solution.

Theorem 2. *[6]*

If A and B are two bounded linear operators in Banach space E, while A has an inverse and $\|B\|_E\|A^{-1}\|_E < 1$, then the operator $(A+B)$ has also an inverse and

$$\|(A+B)^{-1}\|_E \leq \|A^{-1}\|_E(1-\|B\|_E\|A^{-1}\|_E)^{-1}.$$

To find this condition, we write both of Eqs.(2) and (4) in the operator form

$$(I-\lambda\tilde{K})\varphi = f, \qquad (I-\lambda U_n(\tilde{K}))\varphi_n = f_n,$$

where

$$\tilde{K}\varphi = \int_{-\pi}^{\pi} \tilde{k}(x,y)\varphi(y)dy, \qquad U_n(\tilde{K})\varphi_n = \int_{-\pi}^{\pi} U_n[\tilde{k}(.,y);x]\varphi_n(y)dy.$$

It is obvious that $I-\lambda\tilde{K} = A, \quad \lambda(\tilde{K}-U_n(\tilde{K})) = B$, are two bounded linear operators in the space $L^2_{p(x)}$.

It is well-known that the operator $I-\lambda\tilde{K}$ has an inverse for each λ such that $\frac{1}{\lambda}$ is a regular value of $\tilde{K}$ [6]. So Eq.(2) has an unique solution and we can write

$$\varphi = (I+\lambda R)f = f + \lambda Rf,$$

where $(I-\lambda\tilde{K})^{-1} = (I+\lambda R)$ and R is the resolvent of the operator $\tilde{K}$. From theorem 2 if $|\lambda|\|(I-\lambda\tilde{K})^{-1}\|_E\|\tilde{K}-U_n(\tilde{K})\|_E < 1$, then $(I-\lambda U_n(\tilde{K}))$ has also an inverse, thereby Eq.(4) has an unique solution and can be written in the form

$$\varphi_n = (I+\lambda R_n)f_n = f_n + \lambda R_n f_n,$$

where $(I-\lambda U_n(\tilde{K}))^{-1} = I+\lambda R_n$ and R_n is the resolvent of the operator $U_n(\tilde{K})$.

Now, we return to the functional representation of resolvents $R(x,y;\lambda); R_n(x,y;\lambda)$ and equations (2) and (4). Knowing the resolvent $R(x,y;\lambda)$, we at once obtain the solution of the original equation (2) with an arbitrary right hand side $f(x)$ in the following form

$$\varphi(x) = f(x) + \lambda\int_0^{2\pi} R(x,y;\lambda)f(y)dy.$$

Also, the solution of Eq.(4) can be represented through the resolvent as follows

$$\varphi_n(x) = f_n(x) + \lambda\int_0^{2\pi} R_n(x,y;\lambda)f_n(y)dy.$$

Theorem 3.

For any kernel $k(x,y) \in L_p^2[0,2\pi]$, if the linear polynomial operator U_n of order n is defined in $L^2_{p(x)}$ and if the function $f(x) \in L^2_{p(x)}$, then

$$U_n\left[\int_a^b k(.,y)f(y)dy;x\right] = \int_a^b U_n[k(.,y);x]f(y)dy.$$

The proof of this theorem is very similar to the proof of a theorem in [4].

3. Auxiliary definitions and theorems

Definition 1.

The averaged-modulus of continuity of the kernel $k(x,y) \in L_p^2[0,2\pi]$ is defined as follows

$$w_{L_p^2}(k;t) = w_{L_p^2}(t) = \frac{1}{2\pi}\sup_{|s|\le t}\left[\int_0^{2\pi}\int_0^{x-s} p(x)p(y)[k(x-s,y)-k(x,y)]^2dxdy\right]^{\frac{1}{2}}. \tag{7}$$

Lemma 1.

The function $w_{L_p^2}(t)$ has the following properties:

1. $w_{L_p^2}(t) \to 0$ for $t \to 0$;
2. $w_{L_p^2}(t)$ is positive and monotonic increasing;
3. $w_{L_p^2}(t_1 +_2) \le w_{L_p^2}(t_1) + w_{L_p^2}(t_2)$;
4. $w_{L_p^2}(t)$ is continuous;
5. for any positive real number η, the following inequality holds $w_{L_p^2}(\eta t) \le (1+\eta)w_{L_p^2}(t)$.

Also, by the averaged-modulus of continuity with respect to x and y of a function $\tilde{k}(x,y) = e(x,y)k(x,y)$ defined in $[0,2\pi]$, we mean the following function $\Omega_{L_p^2}(t)$

$$\Omega_{L_p^2}(k;t) = \Omega_{L_p^2}(t) = \frac{1}{2\pi}\sup_{|s|\le t}\left[\int_0^{2\pi}\int_0^{2\pi} p(x)p(y)[k(x,y)[e(x-s,y)-e(x,y)]]^2dxdy\right]^{\frac{1}{2}}. \tag{8}$$

It is evident that the function $\Omega_{L_p^2}(t)$ satisfies the above properties of the modulus of continuity (1-5).

Definition 2.

The value of the following norm

$$\delta_n(\tilde{k}) = \delta(\tilde{k};U_n) = \|U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)\|_{L_p^2} = \left[\int_0^{2\pi}\int_0^{2\pi} p(x)p(y)[U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)]^2dxdy\right]^{\frac{1}{2}}, \tag{9}$$

will play an important role in estimating the error arising from replacement of Eq.(1) by Eq.(4). The following theorem provides an estimate of $\delta(\tilde{k},U_n)$.

Theorem 4.

For any kernel $\tilde{k}(x,y) \in L_p^2[0,2\pi]$, and for any linear polynomial operator $U_n(g;x)$, we always have the following inequality

$$\delta_n(\tilde{k}) \leq 2\left[w_{L_p^2}(\frac{1}{n}) + \Omega_{L_p^2}(\frac{1}{n})\right]\int_{-\pi}^{\pi}[n|t|+1]|U_n(t)|dt. \tag{10}$$

Proof. Using Minkowski inequality and equalities (5) and (7), we obtain

$$\begin{aligned}
\delta_n(\tilde{k}) &= \|U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)\|_{L_p^2} = \\
&= \frac{1}{\pi}\left\|\int_{-\pi}^{\pi}[\tilde{k}(x-t,y) - \tilde{k}(x,y)]U_n(t)dt\right\|_{L_p^2} \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)\left[\int_{-\pi}^{\pi}U_n(t)(\tilde{k}(x-t,y) - \tilde{k}(x,y))\right]^2 dydx\right]^{\frac{1}{2}} \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[\tilde{k}(x-t,y) - \tilde{k}(x,y)]^2 dydx\right]^{\frac{1}{2}} dt \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[e(x-t,y)k(x-t,y) - e(x,y)k(x,y)]^2 dydx\right]^{\frac{1}{2}} dt \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)k(x,y)[e(x-t,y) - e(x,y)]^2 dydx\right]^{\frac{1}{2}} \\
&\quad + \left[\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)e(x-t,y)[k(x-t,y) - k(x,y)]^2 dydx\right]^{\frac{1}{2}} dt\right] \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)k(x,y)[e(x-t,y) - e(x,y)]^2 dydx\right]^{\frac{1}{2}} \\
&\quad + \left[\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)e(x-t,y)[k(x-t,y) - k(x,y)]^2 dydx\right]^{\frac{1}{2}} dt\right] \\
&\leq 2\int_{-\pi}^{\pi}|U_n|[w_{L_p^2}(t) + \Omega_{L_p^2}(t)]dt \leq \\
&\leq 2\left[w_{L_p^2}(\frac{1}{n}) + \Omega_{L_p^2}(\frac{1}{n})\right]\int_{-\pi}^{\pi}[n|t|+1]|U_n(t)|dt.
\end{aligned}$$

□

Definition 3.

We define the error of approximation of $\tilde{k}(x,y)$ as follows

$$\begin{aligned}
E^*_{n,m}(\tilde{k})_{L_p^2} &= \|\tilde{k}(x,y) - T^*_{n,m}(x,y)\|_{L_p^2} \\
&= \inf_{T_{n,m}(x,y)}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)[\tilde{k}(x,y) - T_{n,m}(x,y)]^2 dxdy\right]^{\frac{1}{2}}, \\
E^*_{n,\infty}(\tilde{k})_{L_p^2} &= \|\tilde{k}(x,y) - T^*_{n,\infty}(x,y)\|_{L_p^2} \\
&= \inf_{T_{n,\infty}(x,y)}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)[\tilde{k}(x,y) - T_{n,\infty}(x,y)]^2 dxdy\right]^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned} E^*_{\infty,m}(\tilde{k})_{L^2_p} &= \|\tilde{k}(x,y) - T^*_{\infty,m}(x,y)\|_{L^2_p} \\ &= \inf_{T_{\infty,m}(x,y)} \left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} p(x)p(y)[\tilde{k}(x,y) - T_{\infty,m}(x,y)]^2 dxdy\right]^{\frac{1}{2}}, \end{aligned}$$

where $T^*_{n,m}(x,y)$ denotes the trigonometric polynomial in x of order n and in y of order m of best approximation of $\tilde{k}(x,y)$ in the metric $L^2_p[0,2\pi]$, $T^*_{n,\infty}(x,y)$ denotes the trigonometric polynomial in x of order n of best approximation of $\tilde{k}(x,y)$ in the metric $L^2_p[0,2\pi]$, $T^*_{\infty,m}(x,y)$ denotes the trigonometric polynomial in y of order m of best approximation of $\tilde{k}(x,y)$ in the metric $L^2_p[0,2\pi]$. The estimates of how rapidly the quantities $E^*_{n,m}(\tilde{k})_{L^2_p}$, $E^*_{n,\infty}(\tilde{k})_{L^2_p}$ and $E^*_{\infty,m}(\tilde{k})_{L^2_p}$ tend to zero as $n \to \infty, m \to \infty$ are given in [10], where

$$E^*_{n,m}(\tilde{k})_{L^2_p} \to 0, \qquad n,m \to \infty,$$

$$E^*_{n,m}(\tilde{k})_{L^2_p} \geq E^*_{n,\infty}(\tilde{k})_{L^2_p}, \qquad E^*_{n,m}(\tilde{k})_{L^2_p} \geq E^*_{\infty,m}(\tilde{k})_{L^2_p}$$

then

$$E^*_{n,\infty}(\tilde{k})_{L^2_p} \to 0, \quad as \quad n \to \infty, \tag{11}$$

$$E^*_{\infty,m}(\tilde{k})_{L^2_p} \to 0, \quad as \quad m \to \infty. \tag{12}$$

Now, we will mention the bounds of the norm (9) for various linear polynomial operators U_n as the following cases:

Case 1: Valleè-Poussin's method [5]:

$U_n = V_n$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |V_n(t)|dt \leq \frac{1}{3} + \frac{2\sqrt{3}}{\pi}, \tag{13}$$

from Eq.(10) and definition 3, we get

$$E^*_{n,\infty}(\tilde{k})_{L^2_p} \leq 12\pi\left[w_{L^2_p}(\frac{1}{n}) + \Omega_{L^2_p}(\frac{1}{n})\right]. \tag{14}$$

By using the inequality (13) and considering that the method of Valleè-Poussin's V_n leaves trigonometric polynomial of order n invariant, then

$$\begin{aligned} \delta_n(\tilde{k};V_n) &= \|\tilde{k}(x,y) - V_n(\tilde{k}(.,y);x)\|_{L^2_p} \\ &= \|\tilde{k}(x,y) - T^*_{n,\infty}(x,y) - V_n[\tilde{k}(.,y) - T^*_{n,\infty}(.,y);x]\|_{L^2_p} \\ &\leq E^*_{n,\infty}(\tilde{k})_{L^2_p} + \frac{1}{\pi}\int_{-\pi}^{\pi}|V_n(t)|\Big[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} p(x)p(y)[\tilde{k}(x-t,y) - T^*_{n,\infty}(x-t,y)]^2 dydx\Big]^{\frac{1}{2}} dt \\ &\leq \left[1 + \frac{1}{\pi}\int_{-\pi}^{\pi}|V_n(t)|dt\right] E^*_{n,\infty}(\tilde{k})_{L^2_p} \simeq 2.436 E^*_{n,\infty}(\tilde{k})_{L^2_p}, \end{aligned} \tag{15}$$

and from (14) we get

$$\delta_n(\tilde{k};V_n) \leq 29.232\pi[w_{L^2_p}(\frac{1}{n}) + \Omega_{L^2_p}(\frac{1}{n})]. \tag{16}$$

Case 2: Féjer's method [5]:

$U_n = F_n$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |F_n(t)|dt = 1, \tag{17}$$

$$\int_{-\pi}^{\pi}(1+n|t|)|F_n(t)|dt < 6(1+\ln n), \quad \forall n \geq 3. \tag{18}$$

We let $n' = \frac{\sqrt{n}}{2}, a_i(y), b_i(y), a_i^*(y)$ and $b_i^*(y)$ denote the corresponding coefficients of Fourier series in the variable x of the functions $\tilde{k}(x,y)$ and $V_{n'}[\tilde{k}(.,y);x]$. Then,

$$\begin{aligned}
&\|V_{n'}(\tilde{k}(.,y);x) - F_n[V_{n'}(\tilde{k}(.,y);x)]\|_{L_p^2} \\
&= \left\| \sum_{i=1}^{2n'} \frac{i}{n}[a_i^*(y)\cos ix + b_i^*(y)\sin ix] \right\|_{L_p^2} \\
&\leq \left\| \left[\sum_{i=1}^{2n'} \left(\frac{i}{n}\right)^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^{2n'} [a_i^*(y)\cos ix + b_i^*(y)\sin ix]^2 \right]^{\frac{1}{2}} \right\|_{L_p^2} \\
&\leq \left[\sum_{i=1}^{2n'} \left(\frac{i}{n}\right)^2 \right]^{\frac{1}{2}} \left\| \left[\sum_{i=1}^{2n'} [a_i^{*2}(y) + b_i^{*2}(y)] \right]^{\frac{1}{2}} \right\|_{L_p^2} \\
&\leq \frac{1}{\sqrt{\pi} n} \left[\sum_{i=1}^{2n'} i^2 \right]^{\frac{1}{2}} \|\tilde{k}(x,y)\|_{L_p^2} \leq \frac{1}{\sqrt{\pi}\, n}(2n')^{\frac{3}{2}} \|\tilde{k}(x,y)\|_{L_p^2} \\
&\leq \frac{1}{\sqrt{\pi}\, n^{\frac{1}{4}}} \|\tilde{k}(x,y)\|_{L_p^2}.
\end{aligned}$$

Thereby

$$\begin{aligned}
\delta(\tilde{k};F_n) &= \|\tilde{k}(x,y) - F_n(\tilde{k}(.,y);x)\|_{L_p^2} \\
&= \|\tilde{k}(x,y) - V_{n'}(\tilde{k}(.,y);x) + V_{n'}(\tilde{k}(.,y);x) - F_n(V_{n'}(\tilde{k}(.,y);x)) + F_n(V_{n'} - \tilde{k});x)\|_{L_p^2} \\
&\leq \|\tilde{k}(x,y) - V_{n'}(\tilde{k}(.,y);x)\|_{L_p^2} + \|F_n(V_{n'} - \tilde{k});x)\|_{L_p^2} + \|V_{n'}(\tilde{k}(.,y);x) - F_n(V_{n'}(\tilde{k}(.,y);x))\|_{L_p^2} \\
&\leq \left(1 + \frac{1}{\pi}\int_{-\pi}^{\pi} |F_n(t)|dt\right)(2.5)E_{n',\infty}^*(\tilde{k})_{L_p^2} + \frac{1}{\sqrt{\pi}\, n^{\frac{1}{4}}}\|\tilde{k}(x,y)\|_{L_p^2},
\end{aligned} \tag{19}$$

from Eqs.(17) and (19), we get

$$\delta(\tilde{k};F_n) \leq 5E_{n',\infty}^*(\tilde{k})_{L_p^2} + \frac{1}{\sqrt{\pi}\, n^{\frac{1}{4}}}\|\tilde{k}(x,y)\|_{L_p^2}. \tag{20}$$

Also, from Eqs.(18) and (10), we have

$$\delta(\tilde{k};F_n) \leq 12(1+\ln n)[w_{L_p^2}(\frac{1}{n}) + \Omega_{L_p^2}(\frac{1}{n})]. \tag{21}$$

Now from (16), (20) and (21) it is clear that $\delta_n(\tilde{k}) \to 0$ as $n \to \infty$ for Valleé-Poussin's and Féjer's methods for every periodic function $\tilde{k}(x,y) \in L_p^2[0,2\pi]$, $w_{L_p^2}(\frac{1}{n}) = o(1/\ln n)$ and $\Omega_{L_p^2}(\frac{1}{n}) = o(1/\ln n)$.

Definition 4.

The following quantities will play an important role in estimating the error of our approximation

$$\xi(\tilde{k};U_n;\varphi)=\xi_n=\left\|\int_0^{2\pi}\tilde{k}(x,y)[\varphi(y)-U_n(\varphi;y)]dy\right\|_{L_p^2}, \tag{22}$$

$$\gamma_m=\gamma_m(U_n;\varphi)=\sum_{i=1}^{m}|1-\lambda_i^{(n)}|E_{i-1}(\varphi)_{L_p^2}, \tag{23}$$

where

$$E_n(\varphi)_{L_p^2}=\inf_{T_n}\|\varphi(x)-T_n(x)\|_{L_p^2},$$

$T_n(x)$ is a trigonometric polynomial of order n in x, $m\le n$.

Theorem 5.

For any kernel $\tilde{k}(x,y)\in L_p^2[0,2\pi]$ and for linear polynomial operator $U_n(g;x)$ the following inequality holds

$$\begin{aligned}\xi_n(\tilde{k})=\xi_n(\tilde{k};U_n;\varphi)&=\left\|\int_0^{2\pi}\tilde{k}(x,y)[\varphi(y)-U_n(\varphi;y)]dy\right\|_{L_p^2}\\ &\le E^*_{\infty,m}(\tilde{k})_{L_p^2}\|\varphi(y)-U_n(\varphi;y)\|_{L_p^2}+\sqrt{\frac{2}{\pi}}\gamma_m(U_n;\varphi)\left[\int_0^{2\pi}p(x)dx\right]^{\frac{1}{2}}\left[\|\tilde{k}(x,y)\|_{L_p^2}+E^*_{\infty,m}(\tilde{k})_{L_p^2}\right],\end{aligned} \tag{24}$$

for any positive integer $m\le n$.

Proof. For any function $\varphi(x)\in L_p^2$ with Fourier coefficients c_i and d_i in view of Bunyakovskii inequality and $p(x)\ge 1$, we obtain

$$\begin{aligned}|c_i\cos ix+d_i\sin ix|&=\inf_{T_{i-1}(t)}\frac{1}{\pi}\left|\int_{-\pi}^{\pi}[\varphi(t)-T_{i-1}(t)]\cos(i(x-t))dt\right|\\ &\le\frac{1}{\pi}\inf_{T_{i-1}(t)}\left[\int_{-\pi}^{\pi}p(t)[\varphi(t)-T_{i-1}(t)dt]^2\right]^{\frac{1}{2}}.\left[\int_{-\pi}^{\pi}\frac{[\cos(i(x-t))]^2}{p(t)}dt\right]^{\frac{1}{2}}\\ &\le\sqrt{\frac{2}{\pi}}\inf_{T_{i-1}(t)}\left[\int_{-\pi}^{\pi}p(t)|\varphi(t)-T_{i-1}(t)|^2dt\right]^{\frac{1}{2}}\\ &\le\sqrt{\frac{2}{\pi}}E^*_{i-1}(\varphi)_{L_p^2},\end{aligned}$$

therefore

$$\|c_i\cos ix+d_i\sin ix\|_{L_p^2}\le\sqrt{\frac{2}{\pi}}E^*_{i-1}(\varphi)_{L_p^2}\left(\int_{-\pi}^{\pi}p(x)dx\right)^{\frac{1}{2}}.$$

Letting

$$T_{\infty,m}(x,y)=\sum_{i=0}^{m}a_i(x)\cos iy+b_i(x)\sin iy,$$

$$E^*_{\infty,m}(\tilde{k})_{L^2_p} = \inf_{a_i,b_i} \left\| \tilde{k}(x,y) - \sum_{i=0}^{m} a_i(x)\cos iy + b_i(x)\sin iy \right\|_{L^2_p},$$

and taking into consideration (23) and using Bunyakovskii inequality, we obtain

$$\begin{aligned}
\xi_n &= \xi_n(\tilde{k};U_n;\varphi) = \left\| \int_0^{2\pi} \tilde{k}(x,y)[\varphi(y) - U_n(\varphi;y)]dy \right\|_{L^2_p} \\
&= \left[\int_0^{2\pi} p(x) \left[\int_0^{2\pi} \tilde{k}(x,y)[\varphi(y) - U_n(\varphi;y)]dy \right]^2 dx \right]^{\frac{1}{2}} \\
&\leq \Big[\int_0^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \Big[\int_0^{2\pi} |\tilde{k}(x,y) - T_{\infty,m}(x,y)||\varphi(y) - U_n(\varphi;y)|dy \\
&+ \left| \int_0^{2\pi} (\tilde{k}(x,y) + T_{\infty,m}(x,y) - \tilde{k}(x,y))(\varphi(y) - U_n(\varphi;y))dy \right| \Big]^2 dx \Big]^{\frac{1}{2}} \\
&\leq \left[\int_0^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \left[\int_0^{2\pi} |\tilde{k}(x,y) - T_{\infty,m}(x,y)||\varphi(y) - U_n(\varphi;y)|dy \right]^2 dx \right]^{\frac{1}{2}} \\
&+ \Big[\int_0^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \Big[\int_0^{2\pi} (\tilde{k}(x,y) + T_{\infty,m}(x,y) - \tilde{k}(x,y)). \\
&\left(\sum_{i=1}^{m} (1 - \lambda_i^{(n)})(c_i\cos iy + d_i\sin iy) \right) dy \Big]^2 dx \Big]^{\frac{1}{2}} \\
&\leq E^*_{\infty,m}(\tilde{k})_{L^2_p} \|\varphi(y) - U_n(\varphi;y)\|_{L^2_p} \\
&+ \sqrt{\frac{2}{\pi}} \gamma_m(U_n;\varphi) \left[\int_0^{2\pi} p(x)dx \right]^{\frac{1}{2}} \left[\|\tilde{k}(x,y)\|_{L^2_p} + E^*_{\infty,m}(\tilde{k})_{L^2_p} \right].
\end{aligned}$$

□

4. The approximate solution and its error bounds

The following theorem shows that for sufficiently good linear methods $U_n(g;x)$, the difference between the polynomials $\varphi_n(x)$ and the original solution $\varphi(x)$ is sufficiently small.

Theorem 6.

If the kernel $\tilde{k}(x,y)$ in Eq.(2) satisfies the assumptions (A^*), all functions appearing in (2) are $2\pi-$periodic in x and y, then any linear polynomial operator $U_n(g;x)$, if $|\lambda|R\delta(\tilde{k};U_n) < 1$ and if Eq.(1) is replaced by Eq.(4), the following inequality holds

$$\|\varphi(x) - \varphi_n(x)\|_{L^2_p} \leq (1 + \alpha_n(\tilde{k}))\|\varphi(x) - U_n(\varphi;x)\|_{L^2_p}, \tag{25}$$

in which

$$\alpha_n(\tilde{k}) = |\lambda|R \left[\delta(\tilde{k};U_n) + \frac{\xi(\tilde{k};U_n;\varphi)}{\|\varphi(x) - U_n(\varphi;x)\|_{L^2_p}} \right] / [1 - |\lambda|R\delta(\tilde{k};U_n)], \tag{26}$$

where $\delta(\tilde{k};U_n)$ and $\xi(\tilde{k};U_n;\varphi)$ are defined in (9) and (22), respectively, and $R = 1 + |\lambda|\|R(x,y)\|_{L^2_p}$, where $R(x,y)$ denotes the resolvent of the kernel $\tilde{k}(x,y)$.

Proof. Using theorem 3, and Eq.(2), we represent the solution $\varphi_n(x)$ of Eq.(4) in the form

$$\begin{aligned}\varphi_n(x) &= U_n(f;x) + \lambda U_n\left[\int_0^{2\pi}\tilde{k}(.,y)\varphi_n(y)dy;x\right]\\ &= U_n(f;x) + \lambda U_n\left[\int_0^{2\pi}\tilde{k}(.,y)[\varphi_n(y)-\varphi(y)]dy + \int_0^{2\pi}\tilde{k}(.,y)\varphi_n(y)dy;x\right]\\ &= \lambda\int_0^{2\pi}U_n[\tilde{k}(.,y);x][\varphi_n(y)-\varphi(y)]dy + U_n\left[f(.)+\lambda\int_0^{2\pi}\tilde{k}(.,y)\varphi(y)dy;x\right]\\ &= \lambda\int_0^{2\pi}U_n[\tilde{k}(.,y);x][\varphi_n(y)-\varphi(y)]dy + U_n(\varphi;x),\end{aligned} \tag{27}$$

it follows that

$$\varphi_n(x) - U_n(\varphi;x) = \lambda\int_0^{2\pi}\tilde{k}(x,y)[\varphi_n(y)-U_n(\varphi;y)]dy + g_n(x), \tag{28}$$

where

$$g_n(x) = \lambda\int_0^{2\pi}[U_n(\tilde{k}(.,y);x)-\tilde{k}(x,y)][\varphi_n(y)-\varphi(y)]dy + \lambda\int_{-\pi}^{\pi}\tilde{k}(x,y)[U_n(\varphi;y)-\varphi(y)]dy.$$

Thus, by Eqs.(9), (10) and (22) we get the estimate

$$\begin{aligned}\|g_n(x)\|_{L_p^2} &\le |\lambda|\left\|\int_0^{2\pi}[U_n(\tilde{k}(.,y);x)-\tilde{k}(x,y)][\varphi_n(y)-\varphi(y)]dy\right\|_{L_p^2}\\ &+ |\lambda|\left\|\int_0^{2\pi}\tilde{k}(x,y)[U_n(\varphi;y)-\varphi(y)]dy\right\|_{L_p^2}\\ &\le |\lambda|\delta(\tilde{k};U_n)\left[\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2} + \|U_n(\varphi;x)-\varphi(x)\|_{L_p^2}\right] + |\lambda|\xi(\tilde{k};U_n;\varphi).\end{aligned} \tag{29}$$

In view of $|\lambda|\|\tilde{k}(x,y)\|_{L_p^2} < 1$, Eq.(28) has an unique solution given by

$$\varphi_n(x) - U_n(\varphi;x) = g_n(x) + \lambda\int_0^{2\pi}R(x,y)g_n(y)dy.$$

Therefore

$$\begin{aligned}\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2} &\le \|g_n(x)\|_{L_p^2}\left[1+|\lambda|\|R(x,y)\|_{L_p^2}|\right] = R\|g_n(x)\|_{L_p^2}\\ &\le R|\lambda|\left[\delta(\tilde{k};U_n)[\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2} + \|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2}] + \xi(\tilde{k};U_n;\varphi)\right].\end{aligned}$$

Taking into consideration $|\lambda|R\delta(\tilde{k};U_n) < 1$, we obtain

$$\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2} \le \frac{|\lambda|R[\delta(\tilde{k};U_n)\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2} + \xi(\tilde{k};U_n;\varphi)]}{1-|\lambda|R\delta(\tilde{k};U_n)}$$

Therefore

$$\begin{aligned}&\|\varphi(x)-\varphi_n(x)\|_{L_p^2} \le \|\varphi(x)-U_n(\varphi;x)\|_{L_p^2} + \|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2}\\ &\le \|\varphi(x)-U_n(\varphi;x)\|_{L_p^2} + \frac{|\lambda|R[\delta(\tilde{k};U_n)\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2} + \xi(\tilde{k};U_n;\varphi)]}{1-|\lambda|R\delta(\tilde{k};U_n)}\\ &\le (1+\alpha_n(\tilde{k}))\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2},\end{aligned} \tag{30}$$

where α_n is given by (26). Thus, the inequality (25) is proved. □

5. The results

It is well-known that in [7], one cannot achieve an error less than the corresponding to the best approximation. The error estimate in (25) with rate of convergence $\alpha_n(\tilde{k})$, means that, the rate of convergence of $\varphi_n(x)$ to $\varphi(x)$ is comparable with the rate of convergence of the best approximate, which means that the error estimate (25) is optimal. Applying theorem 6, and also the corresponding results from section 3, we obtain the following results:

In the case of the application of Valleé-Poussin's method:

From [10] and (25) we obtain

$$\|\varphi(x)-\varphi_n(x)\|_{L_p^2} \leq (1+\alpha_n(\tilde{k}))(\frac{4}{3}+\frac{2\sqrt{3}}{\pi})E_n^*(\varphi)_{L_p^2} \leq (1+\alpha_n(\tilde{k}))(2.5)E_n^*(\varphi)_{L_p^2},$$

where by (15) we have

$$\alpha_n(\tilde{k}) \leq |\lambda| R \frac{2.5E_{n,\infty}^*(\tilde{k})_{L_p^2}+E_{\infty,m}^*(\tilde{k})_{L_p^2}}{1-\lambda R(2.5)E_{n,\infty}^*(\tilde{k})_{L_p^2}},$$

then $\alpha_n(\tilde{k}) \to 0$ as $n \to \infty$ for all $\varphi(x) \in L_{p(x)}^2$, $\tilde{k}(x,y) \in L_{p(x)}^2[0,2\pi]$.

In the case of the application of Féjer's method:

The quantity $\alpha_n(\tilde{k})$ in the relation (25) will not tend to zero for any solution $\varphi(x)$, but will tend to zero only under the condition that "the solution $\varphi(x)$ belongs to some subclasses of integrable functions". Restricting ourselves to the Holder classes $W^{(r)}H^{\beta}(L_p^2)$ where r is a non-negative integer and $0<\beta\leq 1$, we obtain the following case:

In order that $\alpha_n(\tilde{k}) \to 0$ as $n \to \infty$ considering (20), (21) and [10], it is sufficient that the following conditioned be satisfied

$$\varphi(x) \in W^{(0)}H^{\beta}(L_p^2), \quad \text{i.e. } r=0, \quad 0<\beta\leq 1, \quad w(\frac{1}{n})_{L_p^2}=o(1/\ln n), \quad \Omega(\frac{1}{n})_{L_p^2}=o(1/\ln n).$$

6. Conclusion and remarks

In this article, we presented the approximate solutions of the Volterra integral equations of the second kind in the space $L_{p(x)}^2[0,2\pi]$ with weight function $p(x)$ with the help of the Valleé-Poussin's and Fèjer's operators. In the same time, we proved that the function $\varphi_n(x)$ is a good approximation to the exact solution $\varphi(x)$ for the Volterra integral equations. From the obtained approximate solutions using ADM, we can conclude that the proposed approach is easy to implement and computationally very attractive. A good agreement between the theoretical study with the obtained approximate solutions have been obtained.

References

[1] M. A. Abdou, M. E. Nasr and M. A. Abdel-Aty, Study of the Normality and Continuity for the Mixed Integral Equations with Phase-Lag Term, Inter. J. of Math. Analysis, **11** (2017), 787–799. https://doi.org/10.12988/ijma.2017.7798

[2] M. A. Abdou, M. E. Nasr and M. A. Abdel-Aty, A study of normality and continuity for mixed integral equations, J. of Fixed Point Theory Appl., **20**(1) (2018). https://doi.org/10.1007/s11784-018-0490-0

[3] V. K. Dzyadyk, V. T. Gavrilyuk and O. I. Stepanets, On the best approximations of Holder class functions by Rogozinski polynomials, Dokl. Akad. Nauk Ukr. SSR. Ser. A, **3**, 1969.

[4] V. K. Dzyadyk, On the approximations of linear methods to the approximations by polynomials of functions which are solution of Fredholm integral equation of the second kind I and II, Urain. Matem. Zh. **22**, 1970.

[5] V. K. Dzyadyk, Approximation Methods for Solutions of Differential and Integral Equations, The Netherlands, VSP, 1995.

[6] A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis, Dover Pubns, 1999.

[7] P. P. Korovkin, Linear Operators and Approximation Theory, Hindustan (India), 1960 (Translated from the Russian).

[8] M. E. Nasr and M. F. Jabbar, An Approximate Solution for Volterra Integral Equations of the Second Kind in Space with Weight Function, Inter. J. of Math. Analysis, 11 (2017), 849–861.

[9] M. E. Nasr and M. A. Abdel-Aty, Analytical discussion for the mixed integral equations, J. of Fixed Point Theory Appl., 20(3) (2018). https://doi.org/10.1007/s11784-018-0589-3

[10] A. F. Timan, Theory of Approximations of Functions of a Real Variable, Dover, New York, 1994.

[11] F. G. Tricomi, Integral Equations, Dover, New York, 1985.

A Modified SSDP Method for Nonlinear Semidefinie Programming*

Jianling Li† Chunting Lu Hui Zhang
College of Mathematics and Information Science, Guangxi University,
Nanning, Guangxi, 530004, China

Abstract In this paper, we investigate nonlinear semidefinite programming and propose a modified sequential semidefinite programming (SSDP for short) algorithm without a penalty function or a filter. At each iteration, the search direction is yielded by solving a linear semidefinite programming subproblem and a quadratic semidefinite programming subproblem. The nonmonotone line search ensures that the objective function or constraint violation function is sufficiently reduced. Under some appropriate conditions, the global convergence of the proposed algorithm is shown. Some preliminary numerical results are reported.

Key words nonlinear semidefinite programming; sequential semidefinite programming; nonmonotone line search; global convergence

1 Introduction

Consider the following nonlinear semidefinite programming (NLSDP):

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & G(x) \preceq 0, \end{array} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is assumed to be a smooth and real value function, $G : \mathbb{R}^n \to \mathbb{S}^m$ is a smooth and matrix value function. $\mathbb{S}^m$ represents the set of all real symmetric matrices. The symbol $A \preceq B$ means that $A - B$ is a negative semidefinite matrix.

Nonlinear semidefinite programming has many real-world applications, such as engineering design, optimal structure design, optimal robust control and robust feedback control design (see [1]-[4]). In recent years, the investigation of NLSDP has attracted much attention. The main solution methods for NLSDP are augmented Lagrange method [5]-[10], interior point method [11]-[15], SSDP method [16]-[21]. In this paper, our focus is on SSDP method. Correa and Ramirez in [16] proposed an SSDP algorithm. At each iteration, the search direction is generated by solving a traditional quadratic semidefinite programming (QSDP for short) subproblem. A subdifferentiable penalty function is used as a merit function to design line search. Under some conditions, the algorithm is globally convergent. However, it is not easy for the choice of an appropriate penalty parameter. Gomez in [17] proposed a filter-type SSDP algorithm for nonlinear semidefinite programming problem. For each iteration point, by solving a trust-region type QSDP subproblem

*Project supported by the National Natural Science Foundation (No. 11561005), the National Science Foundation of Guangxi (No. 2016GXNSFAA380248
† Corresponding author. E-mail: jianlingli@126.com

to get search direction. When objective function value or the constraint violation function is improved, the trial point is accepted by filter. Chen in [21] proposed a trust region SSDP method without a penalty function or a filter. The search direction is obtained by solving trust region QSDP subproblem. Whether the trial point is accepted or not depends on the decline of the objective function or constraint violation function.

In all above SSDP algorithms, the traditional QSDP subproblem, which generated the search direction, may be incompatible. Motivated by the idea of modified SQP methods for nonlinear programming, in this paper, we proposed a modified SSDP algorithm for NLSDP (1.1). At each iteration, the search direction is yielded by solving a linear semidefinite programming (LSDP for short) subproblem and a modified QSDP subproblem. Nonmonotone line search technique is used to determine step size.

The paper is organized as follows. In the next section, the algorithm is described in detail. The global convergence is shown in Section 3. Some preliminary numerical results are reported in Section 4 and some concluding remarks are given in the final section.

2 Description of Algorithm

In this section, we first restate some concepts and notations about nonlinear semidefinite programming, and then describe the proposed algorithm.

Let $G(x):\ \mathbb{R}^n \to \mathbb{S}^m$ be a matrix value function, we use the notation

$$DG(x) = \left(\frac{\partial G(x)}{\partial x_1}, \cdots, \frac{\partial G(x)}{\partial x_n}\right)^{\mathrm{T}} \tag{2.1}$$

for its differential operator evaluated at x. For any $d = (d_1, \cdots, n) \in \mathbb{R}^n$, $DG(x)d$ is defined by

$$DG(x)d = \sum_{i=1}^{n} d_i \frac{\partial G(x)}{\partial x_i}. \tag{2.2}$$

The adjoint operator $DG(x)^*$ of the linear operator $DG(x)$ satisfies

$$DG(x)^*Y = \left(< \frac{\partial G(x)}{\partial x_1}, Y >, < \frac{\partial G(x)}{\partial x_2}, Y >, \cdots, < \frac{\partial G(x)}{\partial x_n}, Y >\right)^{\mathrm{T}}, \quad \forall\, Y \in \mathbb{S}^m. \tag{2.3}$$

where $< A, B >$ means the inner product of the matrix A and B.

Definition 2.1 [16] Let $\widetilde{x} \in \mathbb{R}^n$ be a feasible point of NLSDP (1.1), if there exists $\widetilde{Y} \in \mathbb{S}^m$ satisfying the following KKT conditions

$$\nabla_x L(\tilde{x}, \tilde{Y}) = \nabla f(\tilde{x}) + DG(\tilde{x})^* \tilde{Y} = 0, \tag{2.4}$$

$$\tilde{Y} \succeq 0, \qquad < G(\tilde{x}), \tilde{Y} >= 0, \tag{2.5}$$

where $L : \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R}$ is the Lagrangian function of NLSDP (1.1), that is,

$$L(x, \lambda, Y) = f(x) + < Y, G(x) >,$$

then $\widetilde{x}$ is called a KKT point of NLSDP (1.1), the matrix $\widetilde{Y}$ is called a Lagrangian multiplier associated with $\widetilde{x}$.

Let $x^k \in \mathbb{R}^n$ be the current iterate point. In order to generate search directions, we borrow the ideas in [22] and construct the following linear semidefinite programming (LSDP (x^k) for short):

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & G(x^k) + DG(x^k)d \preceq zI_m, \\ & z \geq 0, \end{aligned} \tag{2.6}$$

where I_m is the m order identity. Obviously, the feasible set of LSDP(x^k)(2.6) is not empty, so there exists an optimal solution of (2.6). Let $(\hat{d^k}^{\mathrm{T}},\ z_k)^{\mathrm{T}}$ be an optimal solution of (2.6), then we construct a quadratic semidefinite programming (QSDP (x^k, H_k) for short) as follows:

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \ & \nabla f(x^k)^{\mathrm{T}} d + \tfrac{1}{2} d^{\mathrm{T}} H_k d \\ \text{s.t.} \quad & G(x^k) + DG(x^k)d \preceq z_k I_m. \end{aligned} \tag{2.7}$$

If H_k is a symmetric positive definite matrix, then the solution of QSDP(x^k, H_k) (2.7) is unique.

To measure the degree of feasibility at the iterate point, we define the degree of constraint violation as follows:

$$h(x) = (\lambda_1(G(x)))_+, \tag{2.8}$$

where $\lambda_1(\cdot)$ is the largest eigenvalue of a matrix, $(\alpha)_+ = \max\{0, \alpha\}$. Obviously, $h(x) = 0$ is equivalent with that x is a feasible point of NLSDP (1.1).

Let d^k be the solution of QSDP(x^k, H_k) (2.7). Similar to the idea of filter method, we hope that the search direction d^k can improve the feasibility of the iterate point or the value of the objective function. In other words, if d^k satisfies

$$\nabla f(x^k)^{\mathrm{T}} d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k, \tag{2.9}$$

and t satisfies

$$f(x^k + td^k) \leq \max_{0\leq j\leq m(k)} \{f(x^{k-j})\} - t\alpha (d^k)^{\mathrm{T}} H_k d^k, \tag{2.10}$$

$$h(x^k + td^k) \leq \beta \max_{0\leq j\leq m(k)} \{h(x^{k-j})\}, \tag{2.11}$$

where $\alpha \in (0, \frac{1}{2})$, $m(0) = 0$, $m(k) = \min\{m(k-1)+1,\ M\}$, M is a positive integer, then the corresponding trial step $x^k + td^k$ is accepted.

If d^k does not satisfy (2.9), that is,

$$\nabla f(x_k)^{\mathrm{T}} d^k > -\tfrac{1}{2}(d^k)^{\mathrm{T}} H_k d^k, \tag{2.12}$$

then let $t = 1$. If the following inequality

$$h(x^k + d^k) \leq \beta \max_{0\leq j\leq m(k)} \{h(x^{k-j})\} \tag{2.13}$$

hold, then the corresponding trial step $x^k + d^k$ is accepted.

Based on the above strategy, we now present the new algorithm in detail.

Algorithm A

S0. Given $x^0 \in \mathbb{R}^n,\ H_0 = I_m,\ \alpha \in (0, \frac{1}{2}),\ \sigma \in (0,1),\ \beta \in (\frac{1}{2}, 1),\ m(0) = 0,\ apositiveinteger\ M$. Let $k := 0$.
S1. Solve LSDP(x^k) (2.6) to get a solution $(\hat{d^k}^{\mathrm{T}}, z_k)^{\mathrm{T}}$. If $\hat{d^k} = 0$ and $z_k \neq 0$, stop.
S2. Solve QSDP (x^k, H_k) (2.7) to get the solution d^k. If $d^k = 0$, stop.
S3. If d^k satisfies (2.9), then let t_k be the first number in the sequence of $\{1, \sigma, \sigma^2, \cdots\}$ satisfying the following inequality

$$f(x^k + td^k) \leq \max_{0 \leq j \leq m(k)} \{f(x^{k-j})\} - t\alpha(d^k)^{\mathrm{T}} H_k d^k, \tag{2.14}$$

and go to S4; otherwise, let $t_k = 1$ and go to S4.
S4. Let $x^{k+1} = x^k + t_k d^k$. If the following inequality

$$h(x^{k+1}) \leq \beta \max_{0 \leq j \leq m(k)} \{h(x^{k-j})\}, \tag{2.15}$$

holds, then set $m(k+1) = \min\{m(k) + 1,\ M\}$. Update H_k such that H_{k+1} is a positive definite matrix. Let $k = k+1$ and go to S1; otherwise, go into the restoration phase to obtain a new point x^{k+1}. Let $k = k+1$ and go to S1.

Remark. In the restoration phase, our aim is to decrease the value of $h(x)$. The restoration algorithm is similar to the one given by Long et al. [23].

3 Global Convergence

In this section, we first show that Algorithm A is well-defined, and then show the global convergence. To this end, the following assumptions are necessary.

***A* 1** The iterate $\{x^k\}$ remains in a closed, bounded subset $\mathcal{X}$.

***A* 2** The objective function $f(x)$ and the constraint function $G(x)$ are twice continuously differentiable in $\mathbb{R}^n$.

***A* 3** There exist two constants $0 < a \leq b$ such that $a\|d\|^2 \leq d^{\mathrm{T}} H_k d \leq b\|d\|^2$ for any $d \in \mathbb{R}^n$.

In what follows, we analyze the feasibility of Algorithm A. To this end, it is necessary to extend the definition of infeasible stationary point for nonlinear programming [24] to nonlinear semidefinite programming.

Definition 3.1 Let $\widetilde{x} \in \mathbb{R}^n$ be an infeasible point of $NLSDP$ (1.1), if

$$\min_{d \in \mathbb{R}^n} \max\{\lambda_1(G(\widetilde{x}) + DG(\widetilde{x})d), 0\} = \max\{\lambda_1(G(\widetilde{x})), 0\} = h(\widetilde{x}), \tag{3.1}$$

then $\widetilde{x}$ is called an infeasible sationary of $NLSDP$ (1.1).

Lemma 3.1 Supposed that the assumptions A1-A3 hold, if Algorithm A terminates at x^k, then x^k is either an infeasible stationary point or a KKT point of $NLSDP$ (1.1).

Proof. The proof is divided into two cases.

Case A. If Algorithm A terminates in S1, then $\hat{d}^k = 0$ and $z_k \neq 0$. We know from $LSDP(x^k)$ (2.6) that $z_k = h(x^k)$, so $h(x^k) \neq 0$, which implies x^k is an infeasible point of $NLSDP$ (1.1).

In the following, we prove that x^k is an infeasible stationary point of $NLSDP$ (1.1), namely, x^k satisfies:

$$\min_{d\in\mathbb{R}^n} \max\{\lambda_1(G(x^k)+DG(x^k)d),0\} = \max\{\lambda_1(G(x^k)),0\} = h(x^k).$$

By contradiction, suppose that the conclusion is not true. So there exists $d^{k,0} \in \mathbb{R}^n$ such that

$$\widehat{z} := \max\{\lambda_1(G(x^k)+DG(x^k)d^{k,0}),0\} < h(x^k). \tag{3.2}$$

Clearly, $(d^{k,0^{\mathrm{T}}}, \hat{z})^{\mathrm{T}}$ is a feasible solution of $LSDP(x^k)$ (2.6). Note that z_k is a solution of $LSDP(x^k)$ (2.6), so we obtain

$$z_k \leq \widehat{z} < h(x^k), \tag{3.3}$$

this contradicts $z_k = h(x^k)$. Therefore, x^k is an infeasible stationary point of $NLSDP$ (1.1).

Case B. If Algorithm A terminates in S2, then the solution d^k of $QSD(x^k, H_k)$ (2.7) is zero, i.e., $d^k = 0$. Further, $d^k = 0$ satisfies KKT condition of $QSDP(x^k, H_k)$ (2.7), that is to say, there exists $Y_k \in \mathbb{S}^m$, such that

$$\nabla f(x^k) + DG(x^k)^* Y_k = 0, \tag{3.4}$$

$$G(x^k) \preceq z_k I_m, \tag{3.5}$$

$$Y_k \succeq 0, \qquad < G(x^k) - z_k I_m, Y_k >= 0. \tag{3.6}$$

In what follows, we prove that $z_k = 0$. By contradiction, supposed that $z_k \neq 0$, obviously, $(0^{\mathrm{T}}, z_k)^{\mathrm{T}}$ is a solution of $LSDP(x^k)$ (2.6) from (3.5). Therefore, x^k is an infeasible point of $NLSDP$ (1.1). Since Algorithm A does not stop in S1, $z_k < h(x^k)$.

On the other hand, it follows from (3.5) that

$$\lambda_1(G(x^k)) \leq z_k.$$

In view of $z_k > 0$, we obtain $h(x^k) = \max\{\lambda_1(G(x^k)),0\} \leq z_k$. This contradict $z_k < h(x^k)$. Therefore, $z_k = 0$.

Substituting $z_k = 0$ into (3.5), and conbining with (3.4) and (3.6), we know that x^k is a KKT point of $NLSDP$ (1.1). □

Lemma 3.2 If d^k satisfies the inequality (2.9), then the line search (2.14) is performed.

Proof. It is sufficient to show that there exsits $t \in (0,1)$ such that (2.14) hold.

In view of $\nabla f(x^k)^{\mathrm{T}}d^k \le -\frac{1}{2}(d^k)^{\mathrm{T}}H_kd^k$, so in combination with the positive definiteness of H_k, we know that there exists $d^k \ne 0$ such that $\nabla f(x^k)^{\mathrm{T}}d^k < 0$. For convinence, denote

$$f(x^{l(k)}) = \max_{0\le j\le m(k)}\{f(x^{k-j})\}. \tag{3.7}$$

By contradiction, if the conclusion is not true, then for all $t \in (0,1)$, we have

$$f(x^k+td^k) - f(x^{l(k)}) > -t\alpha(d^k)^{\mathrm{T}}H_kd^k \ge 2t\alpha\nabla f(x^k)^{\mathrm{T}}d^k. \tag{3.8}$$

From (3.7), it is obvious that $f(x^{l(k)}) \ge f(x^k)$, so combining with (3.8), we have

$$f(x^k+td^k) - f(x^k) \ge f(x^k+td^k) - f(x^{l(k)}) > 2t\alpha\nabla f(x^k)^{\mathrm{T}}d^k, \tag{3.9}$$

equivalently,

$$\tfrac{[f(x^k+td^k)-f(x^k)]}{t} > 2\alpha\nabla f(x^k)^{\mathrm{T}}d^k. \tag{3.10}$$

Let $t \to 0^+$, taking the limit for the both sides, it follows that

$$\nabla f(x^k)^{\mathrm{T}}d^k \ge 2\alpha\nabla f(x^k)^{\mathrm{T}}d^k.$$

This implies $\alpha \in [\frac{1}{2},\propto)$ due to $\nabla f(x^k)^{\mathrm{T}}d^k < 0$,. This contradicts $\alpha \in (0,\frac{1}{2})$. Hence, the desired result holds. $\square$

Lemma 3.3 Supposed that the assumptions A1-A3 hold, then there exists $\bar{t} > 0$ such that $t_k \ge \bar{t}$ for k sufficiently large,.

Proof. According to Algorithm A, without loss of generality, suppose that the search direction d^k satisfies (2.10), that is,

$$\nabla f(x^k)^{\mathrm{T}}d^k \le -\frac{1}{2}(d^k)^{\mathrm{T}}H_kd^k.$$

By Taylor expansion, (3.7) and the assumptions A1-A3, we have

$$\begin{aligned}
&f(x^k+t_kd^k) - f(x^{l(k)}) + t_k\alpha(d^k)^{\mathrm{T}}H_kd^k\\
&= f(x^k) + t_k\nabla f(x^k)^{\mathrm{T}}d^k + \tfrac{1}{2}t_k^2(d^k)^{\mathrm{T}}\nabla^2 f(y^k)d^k - f(x^{l(k)}) + t_k\alpha(d^k)^{\mathrm{T}}H_kd^k\\
&\le f(x^k) + t_k\nabla f(x^k)^{\mathrm{T}}d^k + \tfrac{1}{2}t_k^2(d^k)^{\mathrm{T}}\nabla^2 f(y^k)d^k - f(x^k) + t_k\alpha(d^k)^{\mathrm{T}}H_kd^k\\
&= t_k\nabla f(x^k)^{\mathrm{T}}d^k + \tfrac{1}{2}t_k^2(d^k)^{\mathrm{T}}\nabla^2 f(y^k)d^k + t_k\alpha(d^k)^{\mathrm{T}}H_kd^k\\
&\le -\tfrac{1}{2}t_k(d^k)^{\mathrm{T}}H_kd^k + \tfrac{1}{2}t_k^2(d^k)^{\mathrm{T}}\nabla^2 f(y^k)d^k + t_k\alpha(d^k)^{\mathrm{T}}H_kd^k\\
&\le -at_k(\tfrac{1}{2}-\alpha)\|d^k\|^2 + \tfrac{1}{2}t_k^2M\|d^k\|^2,
\end{aligned} \tag{3.11}$$

where y^k is between x^k and $x^k+t_kd^k$, M is a positive integer such that $\|\nabla^2 f(x)\| \le M$.

Let $\bar{t} = \frac{a(1-2\alpha)}{M} > 0$, so (2.10) holds for $t_k \ge \bar{t}$ and $\alpha \in (0,\frac{1}{2})$. $\square$

Lemma 3.4 Supposed that the assumptions A1-A3 hold, $\{x^k\}$ is an infinite sequence generated by Algorithm A, then $\lim\limits_{k\to\infty} h(x^k) = 0$.

Proof. Since $m(k+1) \leq m(k)+1$, we have

$$h(x^{l(k+1)}) = \max_{0\leq j\leq m(k+1)}\{h(x^{k+1-j})\} \leq \max_{0\leq j\leq m(k)+1}\{h(x^{k+1-j})\} = \max\{h(x^{k+1}),\ h(x^{l(k)})\} = h(x^{l(k)}),$$

this implies that the sequence $\{h(x^{l(k)})\}$ is not increasing for k. Combining with $h(x^{l(k)}) \geq 0$, we conclude that $\{h(x^{l(k)})\}$ is convergent.

By Algorithm A, we have

$$h(x^{k+1}) \leq \beta \max_{0\leq j\leq m(k)}\{h(x^{k-j})\} = \beta h(x^{l(k)}). \tag{3.12}$$

Replace k by $l(k)-1$. we obtain

$$h(x^{l(k)}) \leq \beta h(x^{l(l(k)-1)}), \tag{3.13}$$

which together with $\beta \in (\frac{1}{2}, 1)$ gives $\lim\limits_{k\to\infty} h(x^{l(k)}) = 0$. Further, by (3.12), we can conclude that $\lim\limits_{k\to\infty} h(x^k) = 0$. □

Theorem 3.1 Supposed that the assumptions A1-A3 hold, $\{x^k\}$ is an infinite sequence generated by Algorithm A, d^k is the solution of $QSDP(x^k, H_k)$ (2.7). If the multiplier corresponding to d^k is uniform bounded, then there exists $\widetilde{\mathcal{K}} \subseteq \{1, 2, \cdots\}$ such that $\lim\limits_{k\in\widetilde{\mathcal{K}}} d^k = 0$.

Proof. By the assumption A1, we know that $\{x^k\}$ is bounded, so there exists an infinite index set $\mathcal{K} \subseteq \{1, 2, \cdots\}$, such that $\{x^k\}_{\mathcal{K}}$ is convergent. Let $\lim\limits_{k\in\mathcal{K}} x^k = x^*$.

We consider the following two cases:

Case 1. The index set $\mathcal{K}_0 = \{k \in \mathcal{K} \mid \nabla f(x^k)^{\mathrm{T}} d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k\}$ is infinite.

By (2.14), we obtain

$$f(x^{k+1}) = f(x^k + t_k d^k) \leq f(x^{l(k)}) - t_k \alpha (d^k)^{\mathrm{T}} H_k d^k \leq f(x^{l(k)}),\quad \forall\ k \in \mathcal{K}_0. \tag{3.14}$$

Since $m(k+1) \leq m(k)+1$, we obtain

$$f(x^{l(k+1)}) \leq \max_{0\leq j\leq m(k)+1}\{f(x^{k+1-j})\} = \max\{f(x^{k+1}), f(x^{l(k)})\} = f(x^{l(k)}). \tag{3.15}$$

This implies that the sequence $\{f(x^{l(k)})\}$ is not increasing. Combining with the boundedness of $\{f(x^{l(k)})\}$, it follows that $\{f(x^{l(k)})\}_{\mathcal{K}_0}$ is convergent.

For $\{l(k)-1,\ \ k \in \mathcal{K}_0\}$, we obtain

$$f(x^{l(k)}) \leq f(x^{l(l(k)-1)}) - t_{l(k)-1}\alpha(d^{l(k)-1})^{\mathrm{T}} H_{l(k)-1} d^{l(k)-1}. \tag{3.16}$$

Since $\{f(x^{l(k)})\}$ is convergent, we have

$$\lim_{\mathcal{K}_0} t_{l(k)-1}\alpha(d^{l(k)-1})^{\mathrm{T}} H_{l(k)-1} d^{l(k)-1} = 0,$$

By Lemma 3.3, we know that there exists $\bar{t} > 0$ such that $t_{l(k)-1} \geq \bar{t} > 0$, so by the assumption A3, we obtain

$$\lim_{\mathcal{K}_0} d^{l(k)-1} = 0. \tag{3.17}$$

The uniform continuity of $f(x)$ implies that

$$\lim_{\mathcal{K}_0} f(x^{l(k)-1}) = \lim_{\mathcal{K}_0} f(x^{l(k)}). \tag{3.18}$$

Let $\hat{l}(k) = l(k+M+2)$, it is not difficult to prove by induction that for any given $j \geq 1$,

$$\lim_{\mathcal{K}_0} \|d^{\hat{l}(k)-j}\| = 0, \tag{3.19}$$

$$\lim_{\mathcal{K}_0} f(x^{\hat{l}(k)-j}) = \lim_{\mathcal{K}_0} f(x^{l(k)}). \tag{3.20}$$

For any $k \in \mathcal{K}_0$, we obtain $x^{k+1} = x^{\hat{l}(k)} - \sum_{j=1}^{\hat{l}(k)-k-1} t_{\hat{l}(k)-j} d^{\hat{l}(k)-j}$. Note that $\hat{l}(k) - k - 1 \leq M + 1$ and (3.19), we get $\lim_{\mathcal{K}_0} \|x^{k+1} - x^{\hat{l}(k)}\| = 0$. So it follows from the convergence of $\{f(x^{l(k)})\}$ and the uniform continuity of $f(x)$ that

$$\lim_{\mathcal{K}_0} f(x^{k+1}) = \lim_{\mathcal{K}_0} f(x^{l(k)}).$$

So let $k \ (\in \mathcal{K}_0) \to \infty$, taking the limit in (3.14), we have

$$\lim_{\mathcal{K}_0} t_k \alpha (d^k)^{\mathrm{T}} H_k d^k = 0. \tag{3.21}$$

Similar to the proof of (3.17), we obtain $\lim_{\mathcal{K}_0} d^k = 0$. Hence, let $\widetilde{\mathcal{K}} = \mathcal{K}_0$ and the conclusion follows.

Case 2. The index set $\mathcal{K}_0 = \{k \in \mathcal{K} \mid \nabla f(x^k)^{\mathrm{T}} d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k\}$ is finite, which implies that $\mathcal{K}_1 = \{k \in \mathcal{K} \mid \nabla f(x^k)^{\mathrm{T}} d^k > -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k\}$ is infinite.

By contradiction, supposed that the conclusion is not true, then $\lim_{\mathcal{K}_1} d^k \neq 0$. So there exist $\mathcal{K}_2 \subseteq \mathcal{K}_1$ and a constant $\varepsilon > 0$, such that $\|d^k\| > \varepsilon$ for $k \in \mathcal{K}_2$.

Since d^k is the solution of $QSDP(x^k, H_k)$ (2.7), by KKT condition of $QSD(x^k, H_k)$ (2.7) , it follows that there exists a positive semidefinite matrix Y_k such that

$$\nabla f(x^k) + H_k d^k + DG(x^k)^* Y_k = 0, \tag{3.22}$$

$$\mathrm{Tr}((G(x^k) + DG(x^k) d^k - z_k I_m) Y_k) = 0, \tag{3.23}$$

According to the assumption of Theorem 3.1, there exists $\widetilde{M} > 0$ such that $\|Y_k\|_F \leq \widetilde{M}$.

By Lemma 3.4, we know $\lim_{k \to \infty} h(x^k) = 0$, hence there exists $k_0 > 0$, such that

$$h(x^k) \leq \frac{1}{2\widetilde{M}m} a\varepsilon^2, \quad \text{for } k \ (\in \mathcal{K}_2) > k_0, \tag{3.24}$$

combining with $\|d^k\| > \varepsilon$ and the assumption A3, we obtain

$$h(x^k) \leq \frac{1}{2\tilde{M}m} (d^k)^{\mathrm{T}} H_k d^k. \tag{3.25}$$

It follows from (2.2) that

$$\mathrm{Tr}(DG(x^k) d^k Y_k) = \mathrm{Tr}((\sum_{i=1}^{n} d_i^k \frac{\partial G(x^k)}{\partial x_i}) Y_k) = \sum_{i=1}^{n} \mathrm{Tr}(\frac{\partial G(x^k)}{\partial x_i} Y_k) d_i^k = \sum_{i=1}^{n} < \frac{\partial G(x^k)}{\partial x_i},\ Y_k > d_i^k. \tag{3.26}$$

It follows from (3.23) that

$$\mathrm{Tr}((DG(x^k)d^k)Y_k) = \mathrm{Tr}((G(x^k) - z_k I_m)Y_k), \tag{3.27}$$

so (3.26) and (3.27) give rise to

$$\sum_{i=1}^{n} < \frac{\partial G(x^k)}{\partial x_i},\ Y_k > d_i^k = \mathrm{Tr}((G(x^k) - z_k I_m)Y_k). \tag{3.28}$$

By (3.22) and (3.28), we have

$$\begin{aligned} \nabla f(x^k)^{\mathrm{T}} d^k &= -(d^k)^{\mathrm{T}} H_k d^k - (DG(x^k)^* Y_k)^{\mathrm{T}} d^k \\ &= -(d^k)^{\mathrm{T}} H_k d^k - \sum_{i=1}^{n} < \frac{\partial G(x^k)}{\partial x_i}, Y_k > d_i^k \\ &= -(d^k)^{\mathrm{T}} H_k d^k + \mathrm{Tr}((G(x^k) - z_k I_m)Y_k). \end{aligned} \tag{3.29}$$

By *Neumann* Inequality, we obtain

$$\begin{aligned} \mathrm{Tr}((G(x^k) - z_k I_m)Y_k) &\le \sum_{i=1}^{m} \lambda_i (G(x^k) - z_k I_m)\lambda_i(Y_k) \\ &\le \sum_{i=1}^{m} \lambda_i (G(x^k) - z_k I_m)\|Y_k\|_F \\ &\le \sum_{i=1}^{m} \lambda_i (G(x^k) - z_k I_m)\widetilde{M} \\ &\le \sum_{i=1}^{m} \lambda_i (G(x^k))\widetilde{M}, \end{aligned} \tag{3.30}$$

the last inequality above is due to $z_k \ge 0$. According to the definition (2.8) of $h(x^k)$ and (3.30), we obtain

$$\mathrm{Tr}((G(x^k) - z_k I_m)Y_k) \le \widetilde{M} m h(x^k) \le \frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k. \tag{3.31}$$

Substituting (3.31) into (3.29), it follows that

$$\nabla f(x^k)^{\mathrm{T}} d^k \le -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k,$$

which contradicts the definition of $\mathcal{K}_1$. Hence, the conclusion is true. □

Theorem 3.2 Supposed that $\{x^k\}$ is an infinite sequence generated by Algorithm A, and the assumptions in Theorem 3.1 hold, then any accumulation point of $\{x^k\}$ is a KKT point of *NLSDP* (1.1).

Proof. Supposed that x^* is an accumulation point of $\{x^k\}$, then there exists $\mathcal{K} \subseteq \{1, 2, \cdots\}$, such that $\lim_{k \in \mathcal{K}} x^k = x^*$. In view of the assumption A3, without loss of generality, we suppose that $\lim_{k \in \mathcal{K}} H_k = H_*$.

By Lemma 3.4, we have $\lim_{k\in\mathcal{K}} h(x^k) = h(x^*) = 0$, which means that x^* is a feasible point of $NLSDP$ (1.1).

By Theorem 3.1, there exists $\widetilde{\mathcal{K}} \subseteq \{1,2,\cdots\}$ such that $\lim_{\widetilde{\mathcal{K}}} d^k = d^* = 0$. By the proof of Theorem 3.1, we know that $\widetilde{\mathcal{K}} \subseteq \mathcal{K}$.

According to KKT conditions of $QSDP$ (2.7), we obtain

$$\begin{aligned} &\nabla f(x^k) + H_k d^k + DG(x^k)^* Y_k = 0, \\ &Y_k \succeq 0, \quad \mathrm{Tr}((G(x^k) + DG(x^k)d^k - z_k I_m)Y_k) = 0. \end{aligned}$$

Let $k(\in \widetilde{\mathcal{K}}) \to \infty$, taking the limit, we obtain

$$\begin{aligned} &\nabla f(x^*) + DG(x^*)^* Y_* = 0, \\ &Y_* \succeq 0, \quad < G(x^*), Y_* >= 0. \end{aligned}$$

This implies that x^* is a KKT point of $NLSDP$ (1.1). □

4 Numerical experiments

In this section, preliminary numerical experiments of Algorithm A is implemented. Algorithm A was coded by Matlab (2014a) and run on the computer with Windows 7 (64 bite), Intel(R) Core(TM) i7-4790 CPU @ 3.60GHz 3.60GHz, RAM: 4.00GB.

In the numerical experiments, the parameters are chosen as follows: $\alpha = 0.25$, $\beta = 0.85$, $\sigma = 0.5$, $M = 3$. And the termination criteria of Algorithm A is: $\| d^k \| \leq 10^{-4}$.

The test problem is chosen from [11].

Problem 1. Nearest Correlation Matrix (NCM) Problem:

$$\begin{array}{ll} \min & f(X) = \frac{1}{2}\|X - C\|_F^2 \\ \text{s.t} & X \preceq \epsilon I, \\ & X_{ii} = 1, i = 1, 2, ..., m, \end{array} \tag{4.1}$$

where $C \in \mathbb{S}^m$ is a given matrix, $X \in \mathbb{S}^m$, ϵ is a scalar.

In the implementation, $\epsilon = 10^{-3}$, C is generated randomly, which diagonal elements are 1. We test ten times for every fixed dimensionality.

We compare Algorithm A with the ones in [11] (denoted by Algo. YYH) and [14] (denoted by Algo. YYY).

The numerical results are listed in Table 1. The meaning of the notations in Table 1 are described as follows:

n : the dimensionality of independent variable;
m : the dimensionality of $\mathcal{G}(x)$;
$\mathrm{A}-Iter$: the average number of evaluation of iterations.

Table 1. Numerical results of NCM

n	m	x^0	Algorithm	A-$Iter$
			Algorithm A	15
10	5	$(0.5,...,0.5)^{\mathrm{T}}$	Algo. YYY	8
			Algo. YYH	9
			Algorithm A	15
45	10	$(0.5,...,0.5)^{\mathrm{T}}$	Algo. YYY	8
			Algo. YYH	10
			Algorithm A	17
105	15	$(0.5,...,0.5)^{\mathrm{T}}$	Algo. YYY	10
			Algo. YYH	11
			Algorithm A	17
190	20	$(0.5,...,0.5)^{\mathrm{T}}$	Algo. YYY	11
			Algo. YYH	12

5 Concluding remarks

In this paper, we have presented a new SSDP algorithm for nonlinear semidefinite programming. Two subproblems, which are constructed skillfully, are solved to generate the search directions. The nonmonotone line search ensures that the objective function or constraint violation function is sufficiently reduced. The global convergence of the proposed algorithm is shown under some mild conditions. The preliminary numerical results show that the proposed algorithm is effective.

References

[1] Konno, H., Kawadai, N. and Wu, D.: Estimation of failure probability using semi-definite logitmodel. Computational Management Science, 1: 59-73 (2003)

[2] Apkarian, P., Noll, D and Tuan, D.: Fixed-order HI control design via partially argmented Lagrangian method. International Journal of Control, 11: 1137-1148 (2003)

[3] Kanno, Y. and Takewaki, I.: Sequential semidefinite program for maximum robustness design of structures under load uncertainty. Journal of Optimization Theroy and Applications, 130: 265-287 (2006)

[4] Freund, R. W., Jarre, F. and Vogelbusch, C. H.: Nonlinear semidefinite programming: sensitivity, convergence and an application in passive reduce-order modeling. Mathematical Programming, 109: 581-611 (2007)

[5] Sun, J., Zhang, L. W. and Wu, Y.: Properties of the augmented Lagrangian in nonlinear semidefinite optimization. Journal of Optimization Theory and Applications, 129: 437-456 (2006)

[6] Noll, D.: Local convergence of an augmented Lagrangian method for matrix inequality constrained programming. Optimization Methods and Software, 22: 777 – 802 (2007)

[7] Sun, D. F., Sun, J. and Zhang, L. W.: The rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming. Mathwmatical Programming, 114: 349-391 (2008)

[8] Luo, H. Z., Wu, H. X. and Chen, G. T.: On the convergence of augmented Lagrangian methods for nonlinear semidefinite programming. Journal of Global Optimization, 54: 599-618 (2012)

[9] Wu, H. X., Luo, H. Z. and Ding, X. D.: Global convergence of modified augmented Lagrangian methods for nonlinear semidefinite programming[J]. Computational Optimization and Applications, 56: 531-558 (2013)

[10] Wu, H. X., Luo, H. Z. and Yang, J. F.: Nonlinear separation approach for the augmented Lagrangianin nonlinear semidenite programming. Journal of Global Optimization, 59(4): 695-727 (2014)

[11] Yamashita, H., Yabe, H. and Harada, K.: A primal – dual interior point method for nonlinear semidefinite programming. Mathematical Programming, 135: 89-121 (2012)

[12] Yamashita, H. and Yabe, H.: Local and superlinear convergence of a primal-dual interior point method for nonlinear semidefinite programming. Mathematical Programming, 132: 1-30 (2012)

[13] Aroztegui, M., Herskovits, J. and Roche, J. R.: A feasible direction interior point algorithm for nonlinear semidefinite programming. Structral and Multidisciplinary Optimization, 50(6): 1019-1035 (2014)

[14] Yamakawa, Y., Yamashita, N. and Yabe, H.: A differentiable merit function for the shifted perturbed Karush-Kuhn-Tucker conditions of the nonlinear semidefinite programming. Pacific Journal of Optimization, 11: 557-579 ((2015))

[15] Li, J. L., Yang, Z. P. and Jian, J. B.: A globally convergent QP-free algorithm for nonlinear semidefinite programming. Journal of Inequalities and Applications, 145: 1-21 (2017),

[16] Correa, R. and Ramirez, H.: A global algorithm for nonlinear semidefinite programming. SIAM Journal on Optimization, 15: 303-318 (2004)

[17] Gomez, W. and Ramirez, H.: A filter algorithm for nonlinear semidefinite programming. Computation and Applied Mathematics, 29: 297-32 (2010)

[18] Zhu, Z. B. and Zhu, H. L.: A filter method for nonlinear semidefinite programming with global convergence. Acta Mathematica Sinica (English Series), 30: 1810-1826 (2014)

[19] Li, C. J. and Sun, W. Y.: On filter-successive linearization methods for nonlinear semidefinite programming. Science in China (Series A), Mathematics, 52: 2341-2361 (2009)

[20] Zhao, Q.: Global convergence of new method for semidefinite programming. South East Asian Journal of Mathematics and Mathematical Sciences, 3: 189-198 (2013)

[21] Chen, Z. W. and Miao, S. C.: A penalty-free method with trust region for nonlinear semidefinite programming. Asia-Pacific Journal of Operational Reasearch, 32(1): 1-24 (2015)

[22] Zhang, J. L. and Zhang, X. S.: A robust SQP method for optimization with inequality constraints. Journal of Computational Mathematics, 21(2): 247-256 (2003)

[23] Long, J., Ma, C. F. and Nie, P. Y.: A new filter method for solving nonlinear complementarity problems. Application Mathematics and Computation, 185: 705-718 (2007)

[24] Liu, X. W. and Yuan, Y. X.: A robust algorithm for optimization with general equality and inequality constraints. SIAM Journal on Optimization, 22(2): 517-534 (2000)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Approximation by Sublinear and Max-product Operators using Convexity

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we consider quantitatively using convexity the approximation of a function by general positive sublinear operators with applications to Max-product operators. These are of Bernstein type, of Favard-Szász-Mirakjan type, of Baskakov type, of Meyer-Köning and Zeller type, of sampling type, of Lagrange interpolation type and of Hermite-Fejér interpolation type. Our results are both: under the presence of smoothness and without any smoothness assumption on the function to be approximated which fulfills a convexity property.

2010 AMS Mathematics Subject Classification: 41A17, 41A25, 41A36.
Keywords and Phrases: positive sublinear operators, Max-product operators, modulus of continuity, convexity.

1 Background

We make

Remark 1 *Let* $f \in C([a,b])$, $x_0 \in (a,b)$, $0 < h \leq \min(x_0 - a, b - x_0)$, *and* $|f(t) - f(x_0)|$ *is convex in* $t \in [a,b]$.

By Lemma 8.1.1, p. 243 of [1] we have that

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f,h)}{h} |t - x_0|, \quad \forall\ t \in [a,b], \tag{1}$$

where

$$\omega_1(f,h) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq h}} |f(x) - f(y)|, \tag{2}$$

the first modulus of continuity of f.

1

We also make

Remark 2 *Let $f \in C^n([a,b])$, $n \in \mathbb{N}$, $x_0 \in (a,b)$, $0 < h \leq \min(x_0 - a, b - x_0)$, and $\left|f^{(n)}(t) - f^{(n)}(x_0)\right|$ is convex in $t \in [a,b]$. We have that*

$$f(t) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k + I_t, \tag{3}$$

where

$$I_t = \int_{x_0}^{t} \left(\int_{x_0}^{t_1} \ldots \left(\int_{x_0}^{t_{n-1}} \left(f^{(n)}(t_n) - f^{(n)}(x_0) \right) dt_n \right) \ldots \right) dt_1. \tag{4}$$

Assuming $f^{(k)}(x_0) = 0$, $k = 1, ..., n$, we get

$$f(t) - f(x_0) = I_t. \tag{5}$$

By Lemma 8.1.1, p. 243 of [1] we have

$$\left|f^{(n)}(t) - f^{(n)}(x_0)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h}|t - x_0|, \quad \forall\, t \in [a,b]. \tag{6}$$

Furthermore it holds

$$|I_t| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h}\frac{|t - x_0|^{n+1}}{(n+1)!}, \quad \forall\, t \in [a,b]. \tag{7}$$

Hence we derive that

$$|f(t) - f(x_0)| \overset{(5)}{\leq} \frac{\omega_1\left(f^{(n)}, h\right)}{h}\frac{|t - x_0|^{n+1}}{(n+1)!}, \quad \forall\, t \in [a,b]. \tag{8}$$

We have proved the following results:

Theorem 3 *Let $f \in C([a,b])$, $x \in (a,b)$, $0 < h \leq \min(x - a, b - x)$, and $|f(\cdot) - f(x)|$ is convex over $[a,b]$. Then*

$$|f(\cdot) - f(x)| \leq \frac{\omega_1(f, h)}{h}|\cdot - x|, \quad \text{over } [a,b]. \tag{9}$$

Theorem 4 *Let $f \in C^n([a,b])$, $n \in \mathbb{N}$, $x \in (a,b)$, $0 < h \leq \min(x - a, b - x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[a,b]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$|f(\cdot) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h}\frac{|\cdot - x|^{n+1}}{(n+1)!}, \quad \text{over } [a,b]. \tag{10}$$

We give

Definition 5 *Call* $C_+([a,b]) := \{f : [a,b] \to \mathbb{R}_+ \text{ and continuous}\}$. *Let* L_N *from* $C_+([a,b])$ *into* $C_+([a,b])$ *be a sequence of operators satisfying the following properties (see also [6], p. 17):*

(i) (positive homogeneous)

$$L_N(\alpha f) = \alpha L_N(f), \ \forall \ \alpha \geq 0, \ \forall \ f \in C_+([a,b]), \tag{11}$$

(ii) (Monotonicity)
if $f, g \in C_+([a,b])$ *satisfy* $f \leq g$, *then*

$$L_N(f) \leq L_N(g), \ \ \forall \ N \in \mathbb{N}, \tag{12}$$

(iii) (Subadditivity)

$$L_N(f+g) \leq L_N(f) + L_N(g), \ \ \forall \ f, g \in C_+([a,b]). \tag{13}$$

We call L_N *positive sublinear operators.*

We make

Remark 6 *As in [6], p. 17, we get that for* $f, g \in C_+([a,b])$

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f-g|)(x), \ \ \forall \ x \in [a,b]. \tag{14}$$

From now on we assume that $L_N(1) = 1$, $\forall \ N \in \mathbb{N}$. *Hence it holds*

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x), \ \ \forall \ x \in [a,b], \ \forall \ N \in \mathbb{N}, \tag{15}$$

see also [6], p. 17.

We obtain the following results:

Theorem 7 *Let* $f \in C_+([a,b])$, $x \in (a,b)$, $0 < h \leq \min(x-a, b-x)$, *and* $|f(\cdot) - f(x)|$ *is a convex function over* $[a,b]$. *Let* $\{L_N\}_{N\in\mathbb{N}}$ *positive sublinear operators from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall \ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1(f,h)}{h} L_N(|\cdot - x|)(x), \ \ \forall \ N \in \mathbb{N}. \tag{16}$$

Proof. By (9) and (15). ■

Theorem 8 *Let* $f \in C^n([a,b], \mathbb{R}_+)$, $n \in \mathbb{N}$, $x \in (a,b)$, $0 < h \leq \min(x-a, b-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[a,b]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Let* $\{L_N\}_{N\in\mathbb{N}}$ *positive sublinear operators from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall \ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} L_N\left(|\cdot - x|^{n+1}\right)(x), \ \ \forall \ N \in \mathbb{N}. \tag{17}$$

3

Proof. By (10) and (15). ■

We continue with

Theorem 9 *Let* $f \in C_+([a,b])$, $x \in (a,b)$, $0 < L_N(|\cdot - x|)(x) \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N}$, *and* $|f(\cdot) - f(x)|$ *is a convex function over* $[a,b]$. *Here* L_N *are positive sublinear operators from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \omega_1(f, L_N(|\cdot - x|)(x)), \quad \forall\ N \in \mathbb{N}. \tag{18}$$

If $L_N(|\cdot - x|)(x) \to 0$, *then* $L_N(f)(x) \to f(x)$, *as* $N \to +\infty$.

Proof. By (16). ■

Theorem 10 *Let* $f \in C^n([a,b], \mathbb{R}_+)$, $n \in \mathbb{N}$, $x \in (a,b)$, $0 < L_N\left(|\cdot - x|^{n+1}\right)(x) \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N}$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[a,b]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Here* $\{L_N\}_{N\in\mathbb{N}}$ *are positive sublinear operators from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, L_N\left(|\cdot - x|^{n+1}\right)(x)\right)}{(n+1)!}, \quad \forall\ N \in \mathbb{N}. \tag{19}$$

If $L_N\left(|\cdot - x|^{n+1}\right)(x) \to 0$, *then* $L_N(f)(x) \to f(x)$, *as* $N \to +\infty$.

Proof. By (17). ■

Next we combine both Theorems 7, 8:

Theorem 11 *Let* $f \in C^n([a,b], \mathbb{R}_+)$, $n \in \mathbb{Z}_+$, $x \in (a,b)$, $0 < h \leq \min(x-a, b-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[a,b]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Let* $\{L_N\}_{N\in\mathbb{N}}$ *positive sublinear operators from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} L_N\left(|\cdot - x|^{n+1}\right)(x), \quad \forall\ N \in \mathbb{N};\ n \in \mathbb{Z}_+. \tag{20}$$

The initial conditions $f^{(k)}(x) = 0$, $k = 1, ..., n$, *are void when* $n = 0$.

In this article we study under convexity quantitatively the approximation properties of Max-product operators to the unit. These are special cases of positive sublinear operators. We present also results regarding the convergence to the unit of general positive sublinear operators under convexity. Special emphasis is given to our study about approximation under the presence of smoothness. Our work is inspired from [6].

Under our convexity conditions the derived convergence inequalities are elegant and compact with very small constants.

2 Main Results

Here we apply Theorem 11 to Max-product operators.

We make

Remark 12 *We start with the Max-product Bernstein operators ([6], p. 10)*

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad \forall\ N \in \mathbb{N}, \tag{21}$$

$p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-1}$, $x \in [0,1]$, $\bigvee$ *stands for maximum, and* $f \in C_+([0,1]) = \{f : [0,1] \to \mathbb{R}_+ \text{ is continuous}\}$, *where* $\mathbb{R}_+ := [0,\infty)$.

Clearly $B_N^{(M)}$ *is a positive sublinear operators from* $C_+([0,1])$ *into itself with* $B_N^{(M)}(1) = 1$.

By [6], p. 31, we have

$$B_N^{(M)}(|\cdot - x|)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1], \forall\ N \in \mathbb{N}. \tag{22}$$

And by [2] we get:

$$B_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1], m, N \in \mathbb{N}. \tag{23}$$

Denote by

$$C_+^n([0,1]) = \{f : [0,1] \to \mathbb{R}_+, n\text{-times continuously differentiable}\}, \quad n \in \mathbb{Z}_+.$$

We present

Theorem 13 *Let* $f \in C_+^n([0,1])$, $n \in \mathbb{Z}_+$, $x \in (0,1)$ *and* $N^* \in \mathbb{N} : 0 < \frac{1}{\sqrt{N^*+1}} \leq \min(x, 1-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[0,1]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{6\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \quad \forall\ N \in \mathbb{N} : N \geq N^*. \tag{24}$$

It holds $\lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} B_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(23)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} \frac{6}{\sqrt{N+1}} =$$

(setting $h := \frac{1}{\sqrt{N+1}}$)

$$\frac{6\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \tag{25}$$

proving the claim. ■

We make

Remark 14 *Here we focus on the truncated Favard-Szász-Mirakjan operators*

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{26}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, *see also [6], p. 11.*

By [6], p. 178-179 we have

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, N \in \mathbb{N}. \tag{27}$$

And by [2] we get

$$T_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, m, N \in \mathbb{N}. \tag{28}$$

The operators $T_N^{(M)}$ *are positive sublinear from* $C_+([0,1])$ *into itself with* $T_N^{(M)}(1) = 1$, $\forall\, N \in \mathbb{N}$.

We give

Theorem 15 *Let* $f \in C_+^n([0,1])$, $n \in \mathbb{Z}_+$, $x \in (0,1)$ *and* $N^* \in \mathbb{N} : 0 < \frac{1}{\sqrt{N^*}} \leq \min(x, 1-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[0,1]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \leq \frac{3\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N}}\right)}{(n+1)!}, \quad \forall\, N \in \mathbb{N} : N \geq N^*. \tag{29}$$

It holds $\lim_{N \to +\infty} T_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} T_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(28)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} \frac{3}{\sqrt{N}} =$$

(setting $h := \frac{1}{\sqrt{N}}$)

$$\frac{3\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N}}\right)}{(n+1)!}, \tag{30}$$

proving the claim. ■

We make

Remark 16 *Next we study the truncated Max-product Baskakov operators (see [6], p. 11)*

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \ f \in C_+([0,1]), \ N \in \mathbb{N}, \tag{31}$$

where

$$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}.$$

From [6], pp. 217-218, we get ($x \in [0,1]$)

$$\left(U_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{12}{\sqrt{N+1}}, \quad N \geq 2, \ N \in \mathbb{N}. \tag{32}$$

And as in [2], we obtain ($m \in \mathbb{N}$)

$$\left(U_N^{(M)}(|\cdot - x|^m)\right)(x) \leq \frac{12}{\sqrt{N+1}}, \quad N \geq 2, \ N \in \mathbb{N}, \ \forall \ x \in [0,1]. \tag{33}$$

Also it holds $U_N^{(M)}(1)(x) = 1$, and $U_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself, $\forall \ N \in \mathbb{N}$.

We give

Theorem 17 *Let $f \in C_+^n([0,1])$, $n \in \mathbb{Z}_+$, $x \in (0,1)$ and $N^* \in \mathbb{N} - \{1\} : 0 < \frac{1}{\sqrt{N^*+1}} \leq \min(x, 1-x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[0,1]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{12\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \quad \forall \ N \in \mathbb{N} : N \geq N^*. \tag{34}$$

It holds $\lim_{N \to +\infty} U_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} U_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(33)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)},h\right)}{h\,(n+1)!}\frac{12}{\sqrt{N+1}}=$$

(setting $h:=\frac{1}{\sqrt{N+1}}$)

$$\frac{12\omega_1\left(f^{(n)},\frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \tag{35}$$

proving the claim. ■

We make

Remark 18 *Here we study Max-product Meyer-Köning and Zeller operators (see [6], p. 11) defined by*

$$Z_N^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{\infty}s_{N,k}(x)\,f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty}s_{N,k}(x)},\quad \forall\ N\in\mathbb{N},\ f\in C_+([0,1]), \tag{36}$$

$s_{N,k}(x)=\binom{N+k}{k}x^k,\ x\in[0,1].$

By [6], p. 253, we get that

$$Z_N^{(M)}(|\cdot-x|)(x)\leq\frac{8\left(1+\sqrt{5}\right)}{3}\frac{\sqrt{x}\,(1-x)}{\sqrt{N}},\ \ \forall\ x\in[0,1],\ N\geq 4. \tag{37}$$

And by [2], we derive that

$$Z_N^{(M)}(|\cdot-x|^m)(x)\leq\frac{8\left(1+\sqrt{5}\right)}{3}\frac{\sqrt{x}\,(1-x)}{\sqrt{N}}, \tag{38}$$

$\forall\ x\in[0,1],\ N\geq 4,\ \forall\ m\in\mathbb{N}.$

The ceiling $\left\lceil\frac{8\left(1+\sqrt{5}\right)}{3}\right\rceil=9$, *and using a basic calculus technique (see [4]) we get that* $g(x):=(1-x)\sqrt{x}$ *has an absolute maximum over* $(0,1]:g\left(\frac{1}{3}\right)=\frac{2}{3\sqrt{3}}$. *That is* $(1-x)\sqrt{x}\leq\frac{2}{3\sqrt{3}},\ \forall\ x\in[0,1]$.

Consequently it holds

$$Z_N^{(M)}(|\cdot-x|^m)(x)\leq\frac{6}{\sqrt{3}\sqrt{N}}, \tag{39}$$

$\forall\ x\in[0,1],\ \forall\ N\in\mathbb{N}:N\geq 4,\ \forall\ m\in\mathbb{N}.$

Also it holds $Z_N^{(M)}(1)=1$, *and* $Z_N^{(M)}$ *are positive sublinear operators from* $C_+([0,1])$ *into itself,* $\forall\ N\in\mathbb{N}$.

We give

Theorem 19 *Let* $f \in C_{+}^{n}([0,1])$, $n \in \mathbb{Z}_{+}$, $x \in (0,1)$ *and* $N^{*} \in \mathbb{N} : N^{*} \geq 4$ *with* $0 < \frac{1}{\sqrt{N^{*}}} \leq \min(x, 1-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[0,1]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Then*

$$\left|Z_{N}^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{6}{\sqrt{3}(n+1)!}\right) \omega_{1}\left(f^{(n)}, \frac{1}{\sqrt{N}}\right), \quad \forall\, N \in \mathbb{N} : N \geq N^{*}. \tag{40}$$

It holds $\lim_{N \to +\infty} Z_{N}^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|Z_{N}^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_{1}\left(f^{(n)}, h\right)}{h(n+1)!} Z_{N}^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(39)}{\leq}$$

$$\frac{\omega_{1}\left(f^{(n)}, h\right)}{h(n+1)!} \frac{6}{\sqrt{3}\sqrt{N}} =$$

(setting $h := \frac{1}{\sqrt{N}}$)

$$\left(\frac{6}{\sqrt{3}(n+1)!}\right) \omega_{1}\left(f^{(n)}, \frac{1}{\sqrt{N}}\right), \tag{41}$$

proving the claim. ■

We make

Remark 20 *Here we mention the Max-product truncated sampling operators (see [6], p. 13) defined by*

$$W_{N}^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \quad x \in [0,\pi], \tag{42}$$

$f : [0,\pi] \to \mathbb{R}_{+}$, *continuous,*

and

$$K_{N}^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin^{2}(Nx-k\pi)}{(Nx-k\pi)^{2}} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^{2}(Nx-k\pi)}{(Nx-k\pi)^{2}}}, \quad x \in [0,\pi], \tag{43}$$

$f : [0,\pi] \to \mathbb{R}_{+}$, *continuous.*

By convention we take $\frac{\sin(0)}{0} = 1$, *which implies for every* $x = \frac{k\pi}{N}$, $k \in \{0, 1, ..., N\}$ *that we have* $\frac{\sin(Nx-k\pi)}{Nx-k\pi} = 1$.

We define the Max-product truncated combined sampling operators (see also [5])

$$M_{N}^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \rho_{N,k}(x) f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \rho_{N,k}(x)}, \quad x \in [0,\pi], \tag{44}$$

$f \in C_+([0,\pi])$, where

$$M_N^{(M)}(f)(x) := \begin{cases} W_N^{(M)}(f)(x), \text{ if } \rho_{N,k}(x) := \frac{\sin(Nx-k\pi)}{Nx-k\pi}, \\ K_N^{(M)}(f)(x), \text{ if } \rho_{N,k}(x) := \left(\frac{\sin(Nx-k\pi)}{Nx-k\pi}\right)^2. \end{cases} \tag{45}$$

By [6], p. 346 and p. 352 we get

$$\left(M_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{\pi}{2N}, \tag{46}$$

and by [3] ($m \in \mathbb{N}$) we have

$$\left(M_N^{(M)}(|\cdot - x|^m)\right)(x) \leq \frac{\pi^m}{2N}, \ \forall\, x \in [0,\pi], \forall\, N \in \mathbb{N}. \tag{47}$$

Also it holds $M_N^{(M)}(1) = 1$, and $M_N^{(M)}$ are positive sublinear operators from $C_+([0,\pi])$ into itself, $\forall\, N \in \mathbb{N}$.

We give

Theorem 21 *Let $f \in C^n([0,\pi], \mathbb{R}_+)$, $n \in \mathbb{Z}_+$, $x \in (0,\pi)$ and $N^* \in \mathbb{N} : 0 < \frac{1}{N^*} \leq \min(x, \pi - x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[0,\pi]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$\left|M_N^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{\pi^{n+1}}{2(n+1)!}\right)\omega_1\left(f^{(n)}, \frac{1}{N}\right), \tag{48}$$

$\forall\, N \in \mathbb{N} : N \geq N^*$; $n \in \mathbb{Z}_+$.

It holds $\lim_{N \to +\infty} M_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we have:

$$\left|M_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} M_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(47)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!}\frac{\pi^{n+1}}{2N} =$$

(setting $h := \frac{1}{N}$)

$$\left(\frac{\pi^{n+1}}{2(n+1)!}\right)\omega_1\left(f^{(n)}, \frac{1}{N}\right), \tag{49}$$

proving the claim. ■

We make

Remark 22 *The Chebyshev knots of first kind* $x_{N,k} := \cos\left(\frac{2(N-k)+1}{2(N+1)}\pi\right) \in (-1,1)$, $k \in \{0,1,...,N\}$, $-1 < x_{N,0} < x_{N,1} < ... < x_{N,N} < 1$, *are the roots of the first kind Chebyshev polynomial* $T_{N+1}(x) := \cos((N+1)\arccos x)$, $x \in [-1,1]$.

Define ($x \in [-1,1]$)

$$h_{N,k}(x) := (1 - x \cdot x_{N,k}) \left(\frac{T_{N+1}(x)}{(N+1)(x - x_{N,k})} \right)^2, \tag{50}$$

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see p. 12 of [6]) are defined by

$$H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall\ N \in \mathbb{N}, \tag{51}$$

for $f \in C_+([-1,1])$, $\forall\ x \in [-1,1]$.

By [6], p. 287, we have

$$H_{2N+1}^{(M)}(|\cdot - x|)(x) \leq \frac{2\pi}{N+1}, \quad \forall\ x \in [-1,1], \forall\ N \in \mathbb{N}. \tag{52}$$

And by [3], we get that

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^m \pi}{N+1}, \quad \forall\ x \in [-1,1], \forall\ m, N \in \mathbb{N}. \tag{53}$$

Notice $H_{2N+1}^{(M)}(1) = 1$, *and* $H_{2N+1}^{(M)}$ *maps* $C_+([-1,1])$ *into itself, and it is a positive sublinear operator. Furthermore it holds* $\bigvee_{k=0}^{N} h_{N,k}(x) > 0$, $\forall\ x \in [-1,1]$. *We also have* $h_{N,k}(x_{N,k}) = 1$, *and* $h_{N,k}(x_{N,j}) = 0$, *if* $k \neq j$, *and* $H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, *for all* $j \in \{0,1,...,N\}$, *see [6], p. 282.*

We give

Theorem 23 *Let* $f \in C^n([-1,1], \mathbb{R}_+)$, $n \in \mathbb{Z}_+$, $x \in (-1,1)$ *and let* $N^* \in \mathbb{N}$: $0 < \frac{1}{N^*+1} \leq \min(x+1, 1-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[-1,1]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1,...,n$. *Then*

$$\left|H_{2N+1}^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{2^{n+1}\pi}{(n+1)!}\right) \omega_1\left(f^{(n)}, \frac{1}{N+1}\right), \tag{54}$$

$\forall\ N \geq N^*$, $N \in \mathbb{N}$; $n \in \mathbb{Z}_+$.

It holds $\lim_{N \to +\infty} H_{2N+1}^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|H_{2N+1}^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} H_{2N+1}^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(53)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h\,(n+1)!}\left(\frac{2^{n+1}\pi}{N+1}\right) =$$

(setting $h := \frac{1}{N+1}$)

$$\left(\frac{2^{n+1}\pi}{(n+1)!}\right)\omega_1\left(f^{(n)}, \frac{1}{N+1}\right), \tag{55}$$

proving the claim. ■

We make

Remark 24 *Let $f \in C_+([-1,1])$. Let the Chebyshev knots of second kind $x_{N,k} = \cos\left(\left(\frac{N-k}{N-1}\right)\pi\right) \in [-1,1]$, $k = 1, ..., N$, $N \in \mathbb{N} - \{1\}$, which are the roots of $\omega_N(x) = \sin(N-1)t\sin t$, $x = \cos t \in [-1,1]$. Notice that $x_{N,1} = -1$ and $x_{N,N} = 1$.*

Define

$$l_{N,k}(x) := \frac{(-1)^{k-1}\omega_N(x)}{(1+\delta_{k,1}+\delta_{k,N})(N-1)(x-x_{N,k})}, \tag{56}$$

$N \geq 2$, $k = 1, ..., N$, and $\omega_N(x) = \prod_{k=1}^{N}(x - x_{N,k})$ and $\delta_{i,j}$ denotes the Kronecher's symbol, that is $\delta_{i,j} = 1$, if $i = j$, and $\delta_{i,j} = 0$, if $i \neq j$.

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ± 1, are defined by ([6], p. 12)

$$L_N^{(M)}(f)(x) = \frac{\bigvee_{k=1}^{N} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k=1}^{N} l_{N,k}(x)}, \quad x \in [-1,1]. \tag{57}$$

By [6], pp. 297-298 and [3], we get that

$$L_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^{m+1}\pi^2}{3(N-1)}, \tag{58}$$

$\forall\, x \in (-1,1)$ and $\forall\, m \in \mathbb{N}$; $\forall\, N \in \mathbb{N}$, $N \geq 4$.

We see that $L_N^{(M)}(f)(x) \geq 0$ is well defined and continuous for any $x \in [-1,1]$. Following [6], p. 289, because $\sum_{k=1}^{N} l_{N,k}(x) = 1$, $\forall\, x \in [-1,1]$, for any x there exists $k \in \{1, ..., N\} : l_{N,k}(x) > 0$, hence $\bigvee_{k=1}^{N} l_{N,k}(x) > 0$. We have that $l_{N,k}(x_{N,k}) = 1$, and $l_{N,k}(x_{N,j}) = 0$, if $k \neq j$. Furthermore it holds $L_N^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, all $j \in \{1, ..., N\}$, and $L_N^{(M)}(1) = 1$.

By [6], pp. 289-290, $L_N^{(M)}$ are positive sublinear operators.

Finally we present

Theorem 25 *Let $f \in C^n([-1,1], \mathbb{R}_+)$, $n \in \mathbb{Z}_+$, $x \in (-1,1)$ and let $N^* \in \mathbb{N}$: $N^* \geq 4$, with $0 < \frac{1}{N^*-1} \leq \min(x+1, 1-x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[-1,1]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$\left|L_N^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{2^{n+2}\pi^2}{3(n+1)!}\right)\omega_1\left(f^{(n)}, \frac{1}{N-1}\right), \tag{59}$$

$\forall\ N \in \mathbb{N} : N \geq N^* \geq 4$; $n \in \mathbb{Z}_+$.

It holds $\lim_{N \to +\infty} L_N^{(M)}(f)(x) = f(x)$.

Proof. Using (20) we get:

$$\left| L_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} L_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(58)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} \left(\frac{2^{n+2}\pi^2}{3(N-1)} \right) =$$

(setting $h := \frac{1}{N-1}$)

$$\left(\frac{2^{n+2}\pi^2}{3(n+1)!} \right) \omega_1\left(f^{(n)}, \frac{1}{N-1}\right), \tag{60}$$

proving the claim. ∎

References

[1] G. Anastassiou, *Moments in probability and approximation theory*, Pitman Research Notes in Mathematics Series, Longman Group UK, New York, NY, 1993.

[2] G. Anastassiou, *Approximation by Sublinear Operators*, Acta Mathematica Universitatis Comenianae, 87(2018)(2), 237-250.

[3] G. Anastassiou, *Approximation by Max-Product Operators*, Fasc. Math. 60(2018), 5-28.

[4] G. Anastassiou, *Approximation of Fuzzy numbers by Max-product operators*, Transylvanian Journal of Mathematics and Mechanics, 9(2017)(2),117-123.

[5] G. Anastassiou, *Approximations by Multivariate Sublinear and Max-product Operators under Convexity*, Demonstratio Mathematica, 51(2018), 85-105.

[6] B. Bede, L. Coroianu, S. Gal, *Approximation by Max-Product type Operators*, Springer, Heidelberg, New York, 2016.

Symmetric identities for Carlitz's generalized twisted q-Bernoulli numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : In this paper, we study the symmetry for the Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$. We obtain some interesting identities of the power sums and the Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

Key words : Symmetric properties, power sums, Bernoulli numbers and polynomials, Carlitz's generalized twisted q-Bernoulli numbers and polynomials, p-adic invariant integral on $\mathbb{Z}_p$.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Bernoulli polynomials, q-Bernoulli polynomials, the second kind Bernoulli polynomials, Euler polynomials, tangent polynomials, and Bell polynomials were studied by many authors(see [1, 3, 4, 5, 6, 7, 8, 9, 10]). Recently, Y. He obtained several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [1]). D. Kim *et al.*[3] studied some identities of symmetry for generalized Carlitz's q-Bernoulli numbers and polynomials by using the p-adic integrals on $\mathbb{Z}_p$ in p-adic field. The purpose of this paper is to obtain some interesting identities of the power sums and Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

Let p be a fixed prime number. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-4])} .$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

For $g \in UD(\mathbb{Z}_p)$ the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x), \text{ (cf. [2, 3, 4])} . \tag{1.1}$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. It is easy to see that

$$\int_X g(x)d\mu_q(x) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x). \tag{1.2}$$

Let $T_p = \cup_{N\ge 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$(cf. [6, 10]).

2. Symmetric identities for Carlitz's generalized twisted q-Bernoulli numbers and polynomials

Mathematicians investigated interesting properties of symmetry for special polynomials using p-adic invariant integral on $\mathbb{Z}_p$(see [1, 3, 4, 5]). If we take $\chi^0 = 1$, then [5] is the special case of this paper. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $(d, p) = 1$. For $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$, the twisted q-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$ are defined by

$$\beta_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y) q^y [x+y]_q^n d\mu_1(y).$$

We introduce the Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ attached to χ. The Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ attached to χ are defined by

$$\beta_{n,\chi,q,\zeta}(x) = \int_X \chi(y)\phi_\zeta(y) q^y [x+y]_q^n d\mu_1(y).$$

When $x = 0$, $\beta_{n,\chi,q,\zeta} = \beta_{n,\chi,q,\zeta}(0)$ is called the n-th Carlitz's generalized twisted q-Bernoulli numbers $\beta_{n,\chi,q,\zeta}$. We note that

$$\sum_{n=0}^{\infty} \beta_{n,\chi,q,\zeta} \frac{t^n}{n!} = \int_X \chi(y)\zeta^y q^y e^{[x+y]_q t} d\mu_1(x).$$

Let w_1 and w_2 be natural numbers. Then, by (1.1) and (1.2), we obtain

$$\begin{aligned}
&\frac{1}{w_1}\int_X \chi(y)\zeta^{w_1 y} q^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q t} d\mu_1(y) \\
&= \lim_{N\to\infty} \frac{1}{w_1}\frac{1}{dw_2 p^N} \sum_{y=0}^{dw_2 p^N - 1} \chi(y)\zeta^{w_1 y} q^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q t} q^{w_1 y} \\
&= \lim_{N\to\infty} \frac{1}{dw_1 w_2 p^N} \sum_{i=0}^{dw_2 - 1} \chi(i) q^{w_1 i}\zeta^{w_1 i} \sum_{y=0}^{p^N - 1} \zeta^{dw_1 w_2 y} q^{dw_1 w_2 y} e^{[w_1 w_2 x + w_2 j + w_1 i + dw_1 w_2 y]_q t}.
\end{aligned} \tag{2.1}$$

From (2.1), we can derive the following equation (2.2):

$$\begin{aligned}
&\frac{1}{w_1}\sum_{j=0}^{dw_1 - 1} \chi(j)\zeta^{w_2 j} q^{w_2 j} \int_X \chi(y)\zeta^{w_1 y} q^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q t} d\mu_1(y) \\
&= \lim_{N\to\infty} \frac{1}{dw_1 w_2 p^N} \sum_{j=0}^{dw_1 - 1}\sum_{i=0}^{dw_2 - 1}\sum_{y=0}^{p^N - 1} \chi(i)\chi(j)\zeta^{w_2 j}\zeta^{w_1 i} q^{w_2 j} q^{w_1 i} \\
&\qquad \times e^{[w_1 w_2 x + w_2 j + w_1 i + dw_1 w_2 y))]_q t} \zeta^{dw_1 w_2 y} q^{dw_1 w_2 y}.
\end{aligned} \tag{2.2}$$

By the same method as (2.2), we obtain

$$\begin{aligned}
&\frac{1}{w_2}\sum_{j=0}^{dw_2 - 1} \chi(j)\zeta^{w_1 j} q^{w_1 j} \int_X \chi(y)\zeta^{w_2 y} q^{w_2 y} e^{[w_1 w_2 x + w_1 j + w_2 y]_q t} d\mu_1(y) \\
&= \lim_{N\to\infty} \frac{1}{dw_1 w_2 p^N} \sum_{j=0}^{dw_2 - 1}\sum_{i=0}^{dw_1 - 1}\sum_{y=0}^{p^N - 1} \chi(i)\chi(j)\zeta^{w_1 i}\zeta^{w_2 i} q^{w_1 j} q^{w_2 i} \\
&\qquad \times e^{[w_1 w_2 x + w_1 j + w_2 i + dw_1 w_2 y]_q t} \zeta^{dw_1 w_2 y} q^{dw_1 w_2 y}.
\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we have the following theorem.

Theorem 1. For $w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{1}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{w_2 j}\int_X \chi(y)\zeta^{w_1 y}q^{w_1 y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_1(y)\\&=\frac{1}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{w_1 j}\int_X \chi(y)\zeta^{w_2 y}q^{w_2 y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_1(y).\end{aligned} \tag{2.4}$$

By substituting Taylor series of e^{xt} into (2.4) and after calculations, we obtain the following corollary.

Corollary 2. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\begin{aligned}&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{w_2 j}\int_X \chi(y)\zeta^{w_1 y}q^{w_1 y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)\\&=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{w_1 j}\int_X \chi(y)\zeta^{w_2 y}q^{w_2 y}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_1(y).\end{aligned}$$

By Corollary 2, we have the following theorem.

Theorem 3. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\begin{aligned}&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{w_2 j}\beta_{n,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{w_1 j}\beta_{n,\chi,q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right).\end{aligned}$$

By (2.5), we can derive the following equation:

$$\begin{aligned}&\int_X \chi(y)\zeta^{w_1 y}q^{w_1 y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)\\&=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[j]_{q^{w_2}}^i q^{w_2(n-i)j}\int_X \chi(y)\zeta^{w_1 y}q^{w_1 y}\left[w_2x+y\right]_{q^{w_1}}^{n-i}d\mu_1(y)\\&=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x\right).\end{aligned} \tag{2.5}$$

Again, by (2.5), and Theorem 3, we have

$$\begin{aligned}&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{w_2 j}\int_X \chi(y)\zeta^{w_1 y}q^{w_1 y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)\\&=\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{w_2 j}\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x\right)\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x\right)\sum_{j=0}^{dw_1-1}\zeta^{w_2 j}q^{w_2(n-i+1)j}[j]_{q^{w_2}}^i\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x\right)S_{n,i}(dw_1,\zeta^{w_2},q^{w_2}|\chi),\end{aligned} \tag{2.6}$$

where

$$S_{n,i}(w_1,\zeta,q|\chi)=\sum_{j=0}^{w_1-1}\chi(j)\zeta^j q^{(n-i+h)j}[j]_q^i.$$

By the same method as (2.6), we get

$$\begin{aligned}&\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1j}q^{w_1j}\int_X\chi(y)\zeta^{w_2y}q^{w_2y}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_1(y)\\&=\sum_{i=0}^n\binom{n}{i}\frac{[w_1]_q^i[w_2]_q^{n-i}}{w_2}\beta_{n-i,\chi,q^{w_2},\zeta^{w_2}}(w_1x)\,S_{n,i}(dw_2,\zeta^{w_1},q^{w_1}|\chi).\end{aligned}\tag{2.7}$$

Therefore, by (2.6) and (2.7), we have the following theorem.

Theorem 4. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\begin{aligned}&\sum_{i=0}^n\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2x)\,S_{n,i}(dw_1,\zeta^{w_2},q^{w_2}|\chi)\\&=\sum_{i=0}^n\binom{n}{i}\frac{[w_1]_q^i[w_2]_q^{n-i}}{w_2}\beta_{n-i,\chi,q^{w_2},\zeta^{w_2}}(w_1x)\,S_{n,i}(dw_2,\zeta^{w_1},q^{w_1}|\chi).\end{aligned}$$

Remark 5. Let $w_1, w_2 \in \mathbb{N}, n \geq 0$, and χ be the trivial character. Then we have

$$\begin{aligned}&\sum_{i=0}^n\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,q^{w_1},\zeta^{w_1}}(w_2x)\,S_{n,i}(w_1|\zeta^{w_2},q^{w_2})\\&=\sum_{i=0}^n\binom{n}{i}\frac{[w_1]_q^i[w_2]_q^{n-i}}{w_2}\beta_{n-i,q^{w_2},\zeta^{w_2}}(w_1x)\,S_{n,i}(w_2|\zeta^{w_1}q^{w_1}).\end{aligned}$$

REFERENCES

1. Yuan He, *Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials*, Advances in Difference Equations, 246 (2013), 10 pages.
2. T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., 9 (2002), 288-299.
3. D. Kim, T. Kim, Dmitry V. Dolgy, J.-J. Seo, *A note on identities of symmetry for generalized Carlitz's q-Bernoulli polynomials,* Int. Journal of Math. Analysis, 8 (2014), 907-914.
4. B. A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials,* J. Nonlinear Math. Phys., 12 (2005), 412-422.
5. C. S. Ryoo, *Symmetric identities for Carlitz's twisted q-Bernoulli numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$*, Global Journal of Pure and Applied Mathematics, 11 (2015), pp. 2413-2417.
6. C. S. Ryoo, *On the symmetric properties for the generalized twisted q-tangent numbers and polynomials*, Advanced Studies in Theoretical Physics, 9 (2015), pp. 63-69.
7. C. S. Ryoo, (2011). *On the alternating sums of powers of consecutive odd integers*, *Journal of Computational Analysis and Applications*, 13 (2011), pp. 1019-1024.
8. C. S. Ryoo, *Calculating zeros of q-extension of the second kind Bernoulli polynomials*, *Journal of Computational Analysis and Applications*, 15 (2013), pp. 248-254.
9. C. S. Ryoo, *Differential equations associated with generalized Bell polynomials and their zeros*, *Open Mathematics*, 14(2016), pp. 807-815.
10. Y. Simsek, *Theorem on twisted L-function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math. 12 (2006), pp. 237-246.

An efficient optimal algorithm for high frequency in wavelet based image reconstruction

Jingjing Liu[*], Guoxi Ni[†]
LCP, Institute of Applied Physics and Computational Mathematics,Beijing,China

Abstract

Wavelet algorithms for high-resolution image reconstruction has been shown effectively, it relies on the decomposition of low/high frequency, and hard/soft thresholding arguments are often used to denoise for high frequency. In this paper, instead of using this kind of thresholding arguments,, we apply the gradient based shrinkage thresholding optimization for high-frequency, in this way, we can keep the useful information in the original signal as much as possible, coupling the shrinkage thresholding optimization with the wavelet algorithm, we get an efficient reconstruction algorithm. Numerical results show we obtain a higher resolution,better peak signal-to-noise ratios and lower relative errors.

Key words: Wavelet; high-resolution; image reconstruction; shrinkage thresholding; high frequency.

1 Introduction

Increasing the resolution is important and necessary for many applications, lots of studies have been done on the high-resolution image reconstruction [13, 14, 18, 20, 21, 22, 23, 24, 27].

Among the methods in image processing, wavelet method is a well developed technology [6, 9, 10, 12, 26]. In this method, global patterns are represented by densely distributed coefficients obtained from low-pass filtering, while local features are represented by coefficients obtained from high-pass filtering. This makes it easy for us to distinct between smooth and sharp image components. In this way wavelet frames can effectively separate smooth image components and non smooth ones, and the wavelet-based procedure is essentially to approximate iteratively the densely distributed coefficients folded by the given low-pass filter. To overcome the incompatibility of symmetry and exact reconstruction, bi-orthogonal wavelet system is thus proposed in image processing, see [1, 8, 25].

[*]E-mail: liujingjing0618@126.com
[†]E-mail: gxni@iapcm.ac.cn

The relatively complex hard/soft thresholding methods [11] are often used to denoise for high frequency information, but some useful information will lose in the processing because of its cut off action. Preserving useful high frequency part while removing noise is the main goal in image denoising, some techniques developed in the past years has shown their advantage than the hard- and soft-thresholding in the wavelet field, for example, the wavelet packet method,it is based on the further decomposition of wavelet coefficients by packets, and this leads to an essentially translation invariant wavelet packet system.

To get an efficient algorithm while keep useful information in high frequency as much as possible, we consider the optimization strategy instead of hard/soft thresholding method for the high frequency components, this strategy is based on the classic variation technology, and has been previously used in image reconstruction, because of its computational complex, a fast iterative shrinkage-thresholding algorithms are proposed in [2, 4], this kind of method , which can be viewed as an extension of the classical gradient algorithm, is attractive due to its simplicity, it is adequate for solving large-scale problems even with dense matrix data in image reconstruction,to improve the convergence rate, a more fast iterative shrinkage-thresholding algorithm with a significantly better global convergence rate is introduced in [3, 28], this algorithm improves the convergence rate from $O(1/k)$ to $O(1/k^2)$, it relies on computing the next iteration based on the values not only in the previous one, but also in two previously computed steps.

In this paper, we are intent to improve the wavelet algorithm in image reconstruction. We begin with the bi-orthogonal wavelet systems, and obtain the decomposition formula, which represent a perfect reconstruction equation for the symbols of the low-pass and the high-pass filters,theoretical analysis shows that the noise is contains in high-frequency part, and the hard/soft thresholding argument will inevitably delete some useful information, instead of using this kind of thresholding argument for high-frequency components of the original image, we take advantage of shrinkage thresholding algorithm for the optimization of high-frequency, it has been proved that it has notable effect in image denoising, to get the algorithm more efficient, we apply some accelerating iteration argument in shrinkage thresholding algorithm.

The outline of the paper is as follows. the algorithms are derived in section 2. Numerical examples are given in section 3 to illustrate the effectiveness of the algorithms. Some concluding remarks are given in section 4.

2 Reconstruction algorithm

In this section, we construct a shrinkage thresholding optimization coupling with the wavelet based algorithm for high resolution image reconstruction.

2.1 Iterative scheme

Refer to [5], we obtain that using the periodic boundary condition and ordering the discretized values of f and g in a row-by-row fashion, we obtain $M1M2 \times M1M2$ linear system of the form:

$$Lf = g \tag{1}$$

where f is original image, g low-resolution image, $L = L^x \otimes L^y$ is the blurring matrix which is made up from each sensor, and L^x, L^y have circulant structure as follow:

$$L^x = \frac{1}{L} \cdot circulant(a),$$

where

$$a = [1, \cdots, 1, \frac{1}{2}, 0, \cdots, 0, \frac{1}{2}, 1, \cdots, 1]^t,$$

where $circulant(a)$ represents circulant matrix, and the first $L/2$ entries in a are equal to 1, the last $L/2-1$ entries are equal to 1. The matrix L^y can be define similarly, these matrix are circulant matrices, then we get that the matrix L is a block-circulant-circulant-block (BCCB)matrix [17].

By the biorthogonal wavelet theory [7, 19], the symbols of the refinement masks and wavelet masks satisfy the following equation

$$\overline{\hat{a}^d}\hat{a} + \sum_{\nu \in Z_K^2 \setminus \{(0,0)\}} \overline{\hat{b}_\nu^d}\hat{b}_\nu^d = 1 \tag{2}$$

where K is sensor size.

The equation (2) is not only for the reconstruction of function but also for image reconstruction, the matrix representation of the perfect reconstruction from biorthogonal system can be written as

$$L^d L + \sum_{\nu \in Z_K^2 \setminus \{(0,0)\}} M_\nu^d M_\nu = I, \tag{3}$$

here denote by L, L^d, M_ν^d, M_ν the matrices generated by the symbols of the refinement and wavelet masks $\hat{a}, \overline{\hat{a}^d}, \hat{b_\nu}, \overline{\hat{b_\nu^d}}$, respectively.

Since $g = Lf$ is just the observed high-resolution image, and the other $M_\nu f, \nu \neq (0,0)$, represent the high-frequency components of f, from equation (3) we obtain an iterative algorithm

$$f_{n+1} = L^d g + (\sum_{\nu \in Z_K^2 \setminus \{(0,0)\}} M_\nu^d M_\nu) f_n \tag{4}$$

In the usual denoising procedure, the high frequency components are often penalized by a factor, this smoothes the original signals,so a nonlinear denoising scheme can be built into equation (4), and thus obtain an iterative algorithm

$$f_{n+1} = L^d g + \sum_{\nu \in Z_2^2 \setminus \{(0,0)\}} M_\nu^d T(M_\nu f_n). \tag{5}$$

where T is a denoising operator, a hard/soft thresholding wavelet denoising algorithm is presented in [7], in which a further decomposition by the translation invariant wavelet packets is used, this can remedy the smoothing effect on the original signals in some sense, some useful information in high frequency is still lost, this motivates us to consider an efficient optimal method to keep information as much as possible.

2.2 Shrinkage thresholding optimization for high frequency

Let $b = M_\nu f_n$,we consider the following formulation:

$$x^* = \min_x F(x), \quad F(x) = ||Ax - b||^2, \tag{6}$$

where $A = L^d L$,and L is the blurring matrix in the last section, the norm $||\cdot||$ is the inner product, and x is the vector we are looking for, this is the classical least square problem. The optimization problem (6) can be cast as a second order cone programming problem and thus could be solved via interior point methods.

Usually this problem is not only in large scale but also involves dense matrix data, which often precludes the use and potential advantage of sophisticated interior point methods. This motivated a simpler gradient-based algorithms for solving it, the gradient algorithm generates a sequence $\{x_k\}$ via

$$x_k = x_{k-1} - t_k \nabla F(x_{k-1}),$$

where $t_k > 0$ is a suitable stepsize. It can be viewed as an approximal regularization of the linearized function F at x_{k-1}, and can be written equivalently as

$$x_k = \min_x \{F(x_{k-1}) + (x - x_{k-1}, \nabla F(x_{k-1})) + \frac{1}{2t_k}||x - x_{k-1}||^2\}.$$

After ignoring constant terms, this can be rewritten as

$$x_k = \min_x \{\frac{1}{2t_k}||x - (x_{k-1} - t_k \nabla F(x_{k-1}))||^2\},$$

the computation of x_k reduces to solving a one-dimensional minimization problem for each of its components, which produces

$$x_k = \mathcal{T}_{\lambda t_k}(x_{k-1} - t_k \nabla F(x_{k-1})),$$

considering the expression of $F(x)$ in (6), we get:

$$x_k = \mathcal{T}_{\lambda t_k}(x_{k-1} - 2t_k A^T(Ax_{k-1} - b)), \tag{7}$$

where $\mathcal{T}_\alpha : R^2 \to R^2$ is the thresholding operator defined by

$$\mathcal{T}_\alpha(x_i) = (|x_i| - \alpha)_+ sgn(x_i). \tag{8}$$

This algorithm (7) is a kind of iterative shrinkage thresholding algorithm similar as that in [15].

It has been proved that for large-scale problems this first order methods are only practical option, and the sequence x_k converges quite slowly to its solution, that is

$$F(x_k) - F(x^*) = O(1/k),$$

namely, it shares a sublinear global rate of convergence.

To improve the efficiency of the iterative shrinkage thresholding algorithm (ISTA) with better global rate of convergence, Beck etc.[3] consider an improved fast iteration, that is the x_k in (7) is not dependent on the previous point x_{k-1}, but rather on the point y_k which is a linear combination of the previous two point $\{x_{k-1}, x_{k-2}\}$, with this modification, they get a fast ISTA, and the convergence rate is

$$F(x_k) - F(x^*) = O(1/k^2).$$

In this way, we get x^* from b according to the above iterative shrinkage thresholding algorithm.

2.3 Summary of Algorithms

For convenience,we call our shrinkage thresholding algorithm in wavelet based reconstruction algorithm as STWL,to compare with the hard/soft thresholding wavelet reconstructed algorithm (abbr. TWL) in [7]. Now we embed the Shrinkage thresholding optimization algorithm into the iteration scheme (5), denote the previous two iterations as $\{f_{n-1}, f_{n-2}\}$, then our algorithm for the model equation (5) can be summarized as following:

(1) Choose an initial approximation f_0 (e.g., $f_0 = L^d g$);
(2) Iterate until convergence:
Outer circulation:

$$f_{n+1} = L^d g + \sum_{\nu \in Z_2^2 \setminus \{(0,0)\}} M_\nu^d \tilde{T}(M_\nu f_n). \tag{9}$$

Begin inner loop:

To get optimal high frequency part $\tilde{T}(M_\nu f_n)$. Let $b_\nu = M_\nu f_n$, $y_{1,\nu} = M_\nu f_n, h_{0,\nu} = 1, t_1 = 1$, then a fast iteration for (7) is

$$h_{k,\nu} = \mathcal{T}_{\lambda t_k}(y_{k,\nu} - 2t_k A^T (Ay_{k,\nu} - b_\nu)),$$

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2},$$

$$y_{k+1,\nu} = h_{k,\nu} + (\frac{t_k - 1}{t_{k+1}})(h_{k,\nu} - h_{k-1,\nu}),$$

where $\mathcal{T}_{\lambda t_k}$ is defined as in (8), λ is estimated by the method given in [11] which uses the median of the absolute value of the entries in the vector $M_\nu f_n$.

End inner loop.

Return the optimized results $y^*_{n,\nu}$ for high frequency $\tilde{T}(M_\nu f_n)$, that is

$$f_{n+1} = L^d g + \sum_{\nu \in Z_2^2 \backslash \{(0,0)\}} M_\nu^d y^*_{n,\nu}.$$

End outer circulation.

Remark: When the operator $\tilde{T}$ in (9) is realized by soft/hard thresholding operator T,then this reduces to the soft/hard thresholding wavelet reconstruction algorithm as that in [7].

3 Experimental results

In this part, we present the efficiency and accuracy of our shrinkage thresholding wavelet based reconstruction algorithm(abbr. STWL),and compare with the hard/soft thresholding wavelet reconstructed method (abbr. TWL). As usual,we evaluate the methods using the peak signal-to-noise ratio (PSNR),relative error (RE) and cpu time cost they are defined by

$$RE = \frac{\|f - f_c\|_2}{\|f\|_2},$$

and

$$PSNR = 10 \log_{10} \frac{\|f\|_2^2}{\|f - f_c\|_2^2}$$

for 1D signals, while as

$$PSNR = 10 \log_{10} \frac{255^2 NM}{\|f - f_c\|_2^2}$$

for 2D images, respectively, with the size of the signals (images) is $N \times M$. Where f is original image, and f_c is restored image.

In our tests, $N = 1$ for 1D signals while $N = M$ for 2D images. Here we take 2×2 and 4×4 sensor arrays in 2D.

3.1 1D denoisy signal recovery

We take the original signal data from the WaveLab toolbox at *http://statweb. stanford.edu/ wavelab/* developed by Donoho's research group. Fig. 1(a) shows the original signal f. Fig. 1(b) depicts the contaminated signal with white noise at signal-to-noise ratio ($SNR = 25$), here we use matlab function *awgn* to add noise to the original signal. The results of denoising by the above two algorithms with periodic conditions are shown in Fig. 1(c) and (d), respectively. From the data results of experiments in table 1, it shows that our algorithm STWL has a better performance, and has a better time efficiency than TWL.

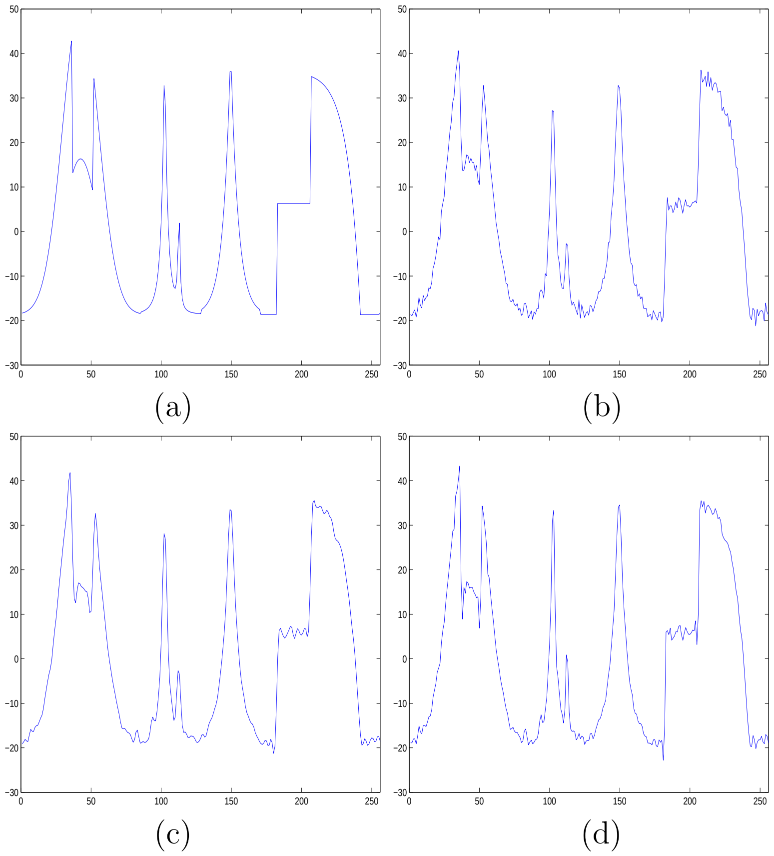


Fig. 1: (a) Original signal; (b) Contaminated by white noise at $SNR = 25$; (c) Reconstructed signal from the algorithm TWL; (d) Reconstructed signal from our algorithm $STWL$.

Table 1: Corresponding PSNR, RE and timecost values using algorithm TWL in [7] and our algorithm STWL.

Algorithms	$SNR = 25$			$SNR = 30$		
	PSNR	RE	timecost	PSNR	RE	timecost
TWL	47.1560	0.0946	0.7956	48.0624	0.0904	0.9984
$STWL$	53.3382	0.0695	0.6396	57.5931	0.0562	0.7332

3.2 High-resolution image reconstruction

In this section, we use the classical "boat", the "Lena" and the "cameraman" images with size of 256×256 as the original images for our tests, and consider 2×2 sensor arrays and 4×4 sensor arrays respectively, and the Gaussian white noise is added to these original images.

7

In the algorithm TWL, the thresholding value λ is chosen to be $\sigma_{n,\nu}\sqrt{2\log(M_1M_2)}$, where the variance $\sigma_{n,\nu}$ is estimated by the median of the absolute value of the entries in the vector $M_\nu f_n$ named high frequency term. In our algorithm STWL, we have decomposed the image data into low and high frequency parts, and denoise the high frequency information by shrinkage thresholding method, the Lipschitz constant is computable in the examples since the eigenvalues of the matrix A^TA can be easily calculated using the two-dimensional cosine transform [17]. For simplicity, we will only consider the matrices for the periodic case.

3.2.1 2×2 sensor array

For 2×2 sensor arrays, the corresponding refinement mask m is the piecewise linear spine,

$$m(-1)=\frac{1}{4}, m(0)=\frac{1}{2}, m(1)=\frac{1}{4},$$

and $m(\alpha)=0$ for all other α. The nonzero terms of the dual mask of m used in this paper are

$$m^d(-2)=-\frac{1}{8}, m^d(-1)=\frac{1}{4}, m^d(0)=\frac{3}{4}, m^d(1)=\frac{1}{4}, m^d(2)=-\frac{1}{8}.$$

The dual pair of the wavelet masks are $r_\alpha := (-1)^\alpha m^d(1-\alpha)$ and $r^d(\alpha) := (-1)^\alpha m(1-\alpha)$, see [12] for details.

The tensor product dual pair of the refinement symbols are given by $\hat{a}(\omega)=\hat{m}(\omega_1)\hat{m}(\omega_2)$, $\hat{a}^d(\omega)=\hat{m}^d(\omega_1)\hat{m}^d(\omega_2)$, and the corresponding wavelet symbols are $\hat{b}_{(0,1)}(\omega)=\hat{m}(\omega_1)\hat{r}(\omega_2)$, $\hat{b}^d_{(0,1)}(\omega)=\hat{m}^d(\omega_1)\hat{r}^d(\omega_2)$, $\hat{b}_{(1,0)}(\omega)=\hat{r}(\omega_1)\hat{m}(\omega_2)$, $\hat{b}^d_{(1,0)}(\omega)=\hat{r}^d(\omega_1)\hat{m}^d(\omega_2)$, $\hat{b}_{(1,1)}(\omega)=\hat{r}(\omega_1)\hat{r}(\omega_2)$, $\hat{b}^d_{(1,1)}(\omega)=\hat{r}^d(\omega_1)\hat{r}^d(\omega_2)$, where $\omega=(\omega_1,\omega_2)$.

Although we give here only the details of the refinable functions and their corresponding wavelets with dilation $2I$, the whole theory can be carried over to the general isotropic integer dilation matrices.

The wavelet matrices are formed by the tensor product,and we consider

$$Z_2^2=\{(0,0),(0,1),(1,0),(1,1)\}.$$

In particular, we have

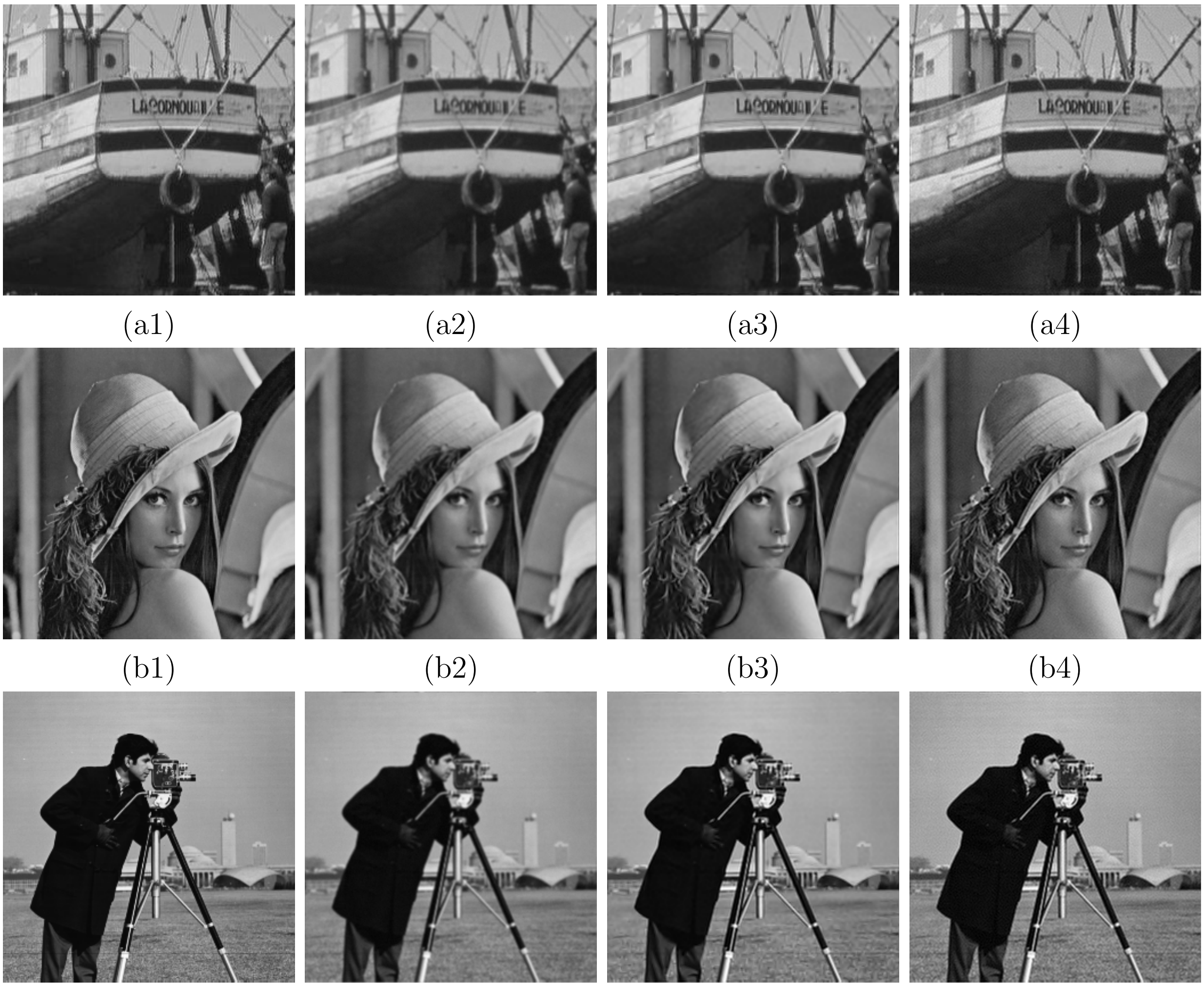
$$L=L_2\otimes L_2,\quad M_{(0,1)}=L_2\otimes M_2,\ M_{(1,0)}=M_2\otimes L_2,\quad M_{(1,1)}=M_2\otimes M_2,$$
$$L^d=L_2^d\otimes L_2^d,\quad M^d_{(0,1)}=L_2^d\otimes M_2^d,\ M^d_{(1,0)}=M_2^d\otimes L_2^d,\quad M^d_{(1,1)}=M_2^d\otimes M_2^d.$$

where

$$L_2=circulant(\frac{1}{2},\frac{1}{4},0,\cdots,0,\frac{1}{4}),\ L_2^d=circulant(\frac{3}{4},\frac{1}{4},-\frac{1}{8},0,\cdots,0,-\frac{1}{8},\frac{1}{4})$$

$$M_2=circulant(\frac{1}{4},-\frac{3}{4},\frac{1}{4},\frac{1}{8},0,\cdots,0,\frac{1}{8}),\ M_2^d=circulant(\frac{1}{4},0,\cdots,0,\frac{1}{4},-\frac{1}{2}).$$

Fig.2 demonstrate the reconstructed high-resolution image for the "boat" , the "Lena" and the "cameraman" images respectively, in these figures, (a1)-(c1) are the original images, (a2)-(c2) are with noise $PSNR = 40dB$, (a3)-(c3) are the denoisy images with the algorithm TWL, and (a4)-(c4) are obtained by our algorithm STWL. Table 2 gives the PSNR, RE, and the cputime of the reconstructed images for different levels of Gaussian noise, our algorithm shows less RE, less cputime and better PSNR, we can conclude that our algorithm STWL is better than the original algorithm TWL in [7].



(a1) (a2) (a3) (a4)

(b1) (b2) (b3) (b4)

(c1) (c2) (c3) (c4)

Fig. 2: (a1)-(c1) the original images; (a2)-(c2) the noisy images; (a3)-(c3) the reconstructed image by algorithm TWL; (a4)-(c4) the reconstructed images by our algorithm STWL.

Table 2: Comparison of PSNR, RE and cputime values using algorithm TWL and STWL for 2×2 sensor arrays with different kinds noise level.

Image	Evaluation	TWL		$STWL$	
		$SNR = 35$	$SNR = 40$	$SNR = 35$	$SNR = 40$
boat	PSNR	81.9938	82.2142	82.4965	82.4724
	RE	0.0166	0.0164	0.0162	0.0162
	timecost	10.3585	9.9529	5.9280	5.8344
Lena	PSNR	84.0785	84.1040	84.6254	85.0012
	RE	0.0149	0.0149	0.0145	0.0143
	timecost	11.3881	11.7157	5.9436	5.5380
cameraman	PSNR	85.8451	86.1629	86.1727	86.8056
	RE	0.0137	0.0135	0.0135	0.0130
	timecost	11.2165	9.7033	5.9124	5.6472

3.2.2 4×4 sensor array

In this case, we give the refinable and wavelet masks with dilation $4I$ that used to generate the matrices for 4×4 sensor arrays.

For 4×4 sensor arrays, the corresponding mask is

$$m(\alpha) = \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \alpha = -2, \cdots, 2,$$

with $m(\alpha) = 0$ for all other α. The nonzero terms of a dual refinement mask of m is

$$m^d(\alpha) = -\frac{1}{16}, \frac{1}{8}, \frac{5}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{8}, -\frac{1}{16}, \alpha = -3, \cdots, 3.$$

The nonzero terms of the corresponding wavelet masks are

$$r_1(\alpha) = -\frac{1}{8}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{8}, \alpha = -2, \cdots, 2,$$

$$r_2(\alpha) = -\frac{1}{16}, -\frac{1}{8}, \frac{5}{16}, -\frac{1}{4}, \frac{5}{16}, -\frac{1}{8}, -\frac{1}{16}, \alpha = -2, \cdots, 4,$$

$$r_3(\alpha) = \frac{1}{16}, \frac{1}{8}, -\frac{7}{16}, 0, \frac{7}{16}, -\frac{1}{8}, -\frac{1}{16}, \alpha = -2, \cdots, 4.$$

The dual wavelet masks are

$$r_1^d(\alpha) = (-1)^{1-\alpha} r_3(1-\alpha), r_2^d(\alpha) = (-1)^{1-\alpha} m(1-\alpha), r_3^d(\alpha) = (-1)^{1-\alpha} r_1(1-\alpha)$$

The observed high-resolution image g is generated by applying the bivariate lowpass filter on the true image f,again, we consider periodic boundary condition. The matrices

$$L, L^d, M_\nu, M_\nu^d, \nu \in Z_4^2 \setminus \{(0,0)\}$$

can be generated by the corresponding filters.

Fig.3 shows the reconstructed high-resolution image for the "boat" , the "Lena" and the "cameraman" images, (a1)-(c1) are blurred with noise $PSNR = 40dB$, (a2)-(c2) are obtained from the algorithm TWL, and (a1)-(c1) are obtained from our algorithm STWL. From Table 3, we can also find that our algorithm shows less RE, less cputime and better PSNR, since the problem is more difficult than the 2×2 sensor case, we need more cputime consuming, we can see that the performance of our algorithm STWL is much better than the original algorithm TWL.

Table 3: Comparison of PSNR, RE and cputime values using algorithm TWL and STWL for 4×4 sensor arrays with different kinds noise level.

Image	Evaluation	TWL		$STWL$	
		$SNR = 30$	$SNR = 40$	$SNR = 30$	$SNR = 40$
boat	PSNR	67.3135	67.4297	68.3053	68.5968
	RE	0.0345	0.0343	0.0329	0.0324
	timecost	19.5157	20.4673	15.9277	15.8809
Lena	PSNR	69.9279	69.9769	71.3131	71.4538
	RE	0.0303	0.0302	0.0283	0.0281
	timecost	20.9041	20.0773	16.1773	15.9745
cameraman	PSNR	72.9596	73.2392	73.6651	73.9353
	RE	0.0260	0.0257	0.0251	0.0248
	timecost	19.8589	20.9041	14.8981	15.6313

4 Conclusions

In this paper, we constructed a shrinkage thresholding algorithm in wavelet based image reconstruction, instead of using the hard/soft thresholding algorithm we apply the iterative shrinkage thresholding algorithm for the optimization for high frequency. Our new algorithm works effectively both in one-dimensional and two-dimensional situations, numerical tests show that this algorithm gives higher resolution, larger signal-to noise ratios,lower relative errors and less cputime.

Fig. 3: (a1)-(c1) is the noisy images,(a2)-(c2) is reconstructed from the algorithm TWL; (a3)-(c3) is reconstructed from our algorithm STWL.

Acknowledgments

This work is supported by China Natural Science Fund (No.91130020) and Defense Industrial Technology Development Program (No.B1520133015).

References

[1] Chenglong Bao, Bin Dong, Likun Hou, Zuowei Shen,Image restoration by minimizing zero norm of wavelet frame coefficients, Inverse Problems, 32(11), (2016),115004.

[2] A. Beck, M. Teboulle, Fast gradient-based algorithms for constrained total variation image denoising and deblurring problems, IEEE Trans. Image Process. 18, 2419-2434(2009).

[3] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM Journal on Imaging Sciences. 2(1), 183-202(2009).

[4] J. Bioucas-Dias, M. Figueiredo, A new TWIST: Two-step iterative shrinkage/thresholding algorithms for image restoration, IEEE Trans. Image Process.,16, 2992-3004(2007).

[5] N. Bose, K. Boo, High-resolution image reconstruction with multisensors, Int. J. Imag. Syst. Technol. 9, 294-304(1998).

[6] A. Chambolle, R. A. DeVore, N. Y. Lee, and B. J. Lucier, Nonlinear wavelet image processing: Variational problems, compression, and noise removal through wavelet shrinkage, IEEE Trans. Image Proces., 7, 319-335 (1998).

[7] R.H. Chan, T. Chan, L. Shen, Z. Shen, Wavelet algorithms for high-resolution image reconstruction, SIAM J. Sci. Comput. 24(4), 1408-1432(2003).

[8] R.H. Chan, T. Chan, L. Shen, Z. Shen, Wavelet deblurring algorithms for spatially varying blur from high-resolution image reconstruction, Linear Algebra Appl. 366, 139-155(2003).

[9] R. H. Chan, S. D. Riemenschneider, L. Shen, Z. Shen, Tight frame: an efficient way for high-resolution image reconstruction, Applied and Computational Harmonic Analysis. 17, 91-115(2004).

[10] T. Chan and J. Shen, Variational image inpainting, Commun. Pure Appl. Math, vol. 58, 579-619 (2005).

[11] D. Donoho, De-noising by soft-thresholding, IEEE Trans, Inform. Theory, 41, 613-627(1995).

[12] I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conf. Ser. in Appl. Math., 61. SIAM, Philadelphia, 1992.

[13] M. Elad, A. Feuer, Restoration of a single superresolution image from several blurred, noisy and undersampled measured images, IEEE Trans. Image Process. 6, 1646-1658(1997).

[14] M. Elad, A. Feuer, Superresolution restoration of an image sequence: adaptive filtering approach, IEEE Trans. Image Process. 8(3), 387-395(1999).

[15] M. A. T. Figueiredo and R. D. Nowak, An EM algorithm for wavelet-based image restoration, IEEE Trans. Image Process., 12, 906-916(2003).

[16] T. M. Stadtmiller, J. C. Gillete, R. C. Hardie, Aliasing reduction in staring infrared images utilizing subpixel techniques, Aerospace and Electronics Conference, 2(11), 874-880(1995).

[17] P. C. Hansen, J. G. Nagy and D. P. O'Leary, Deblurring Images: Matrices, Spectra and Filtering, SIAM, Philadelphia, 2006.

[18] T. Huang, R. Tsay, Multiple frame image restoration and registration, in: T. S. Huanf(Ed.), Advances in Computer Vision and Image Processing, vol. 1, 317-339(1984).

[19] H. Ji, S. Riemenschneider, and Z. Shen, Multivariate compactly supported fundamental refinable functions, dual and biorthogonal wavelets, Stud. Appl. Math.,102, 862-874(1999).

[20] M. Ng, R. Chan, T. Chan, A. Yip, Cosine transform preconditioners for high resolution image reconstruction, Linear Algebra Appl. 316, 89-104(2000).

[21] M. Ng, R. Chan, W. Tang, A fast algorithm for deeblurring models with Neumann boundary conditions, SIAM J.Sci. Comput. 21, 851-866(2000).

[22] M. K. Ng, N, Bose, Analysis of displacement errors in high-resolution image reconstruction with multisensors, IEEE Trans. Circuits Systems I Fund. Theory Appl. 49(6), 806-813(2002).

[23] A. Patti, M. Sezan, A. Tekalp, Superresolution video reconstruction with arbitrary sampling lattices and nonzero aperture time, IEEE Trans. Image Process. 6, 1064-1076(1997).

[24] R. Schultz, R. Stevenson, Extraction of high-resolution frames from video sequences, IEEE Trans. Image Process. 5, 996-1011(1996).

[25] Z. Shen, Extension of matrices with Laurent polynomial entries, in proceedings of the 15th IMACS World Congress 1997 on Scientific Computation, Modelling and Applied Mathematics, A. Syclow, ed., 57-61(1997).

[26] L. Shen, Q. Sun, Bi-orthogonal wavelet system for high-resolution image reconstruction, IEEE Trans. Signal Process. 52(7),1997-2011(2003).

[27] P. Vandewalle, J. Kovacevic, and M. Vetterli, Reproducible research in signal processing What, why, and how, IEEE Signal Process., 26, 37-47(2009).

[28] M. Zulfiquar, A. Bhotto, M. O. Ahmad, M. N. S. Swamy, An improved fast iterative shrinkage thresholding algorithm for image deblurring, SIAM Journal on Imaging Sciences. 8(3),1640-1657(2015).

Negative Domain Local fractional Inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

This research is about inequalities in a local fractional environment over a negative domain. The author presents the following types of analytic local fractional inequalities: Opial, Hilbert-Pachpatte, comparison of means, Poincare and Sobolev. The results are with respect to uniform and L_p norms, involving left and right Riemann-Liouville fractional derivatives.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.
Keywords and phrases: Local fractional derivative, Riemann-Liouville fractional derivative, Opial inequality, Hilbert Pachpatte, Poincare inequality, fractional inequalities.

1 Introduction

Many sources motivate us to write this work. The first one comes next. It is the famous Opial inequality ([13]):

$$\int_0^a |y'(x)\, y(x)|\, dx \leq \frac{a}{2} \int_0^a |y'(x)|^2\, dx, \tag{1}$$

where $y(x)$ is absolutely continuous function and $y(0) = 0$. The above inequality is proved sharp.

The well known Ostrowski ([14]) inequality also motivates this work and has as follows:

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty, \tag{2}$$

where $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality.

Next $D_{*a}^{\rho} f$ indicates the left Caputo fractional derivative of order $\rho > 0$, anchored at $a \in \mathbb{R}$, see [10], p. 50.

1

The author in [7], pp. 82-83, proved the following left Caputo fractional Landau inequality: Let $0 < \nu \leq 1$, $f \in AC^2([0,b])$ (i.e. $f' \in AC([0,b])$, absolutely continuous functions), $\forall\ b > 0$. Suppose $\|f\|_{\infty,\mathbb{R}_+} < +\infty$, $D_{*0}^{\nu+1} f \in L_\infty(\mathbb{R}_+)$, and

$$\left\|D_{*a}^{\nu+1} f\right\|_{\infty,[a,+\infty)} \leq \left\|D_{*0}^{\nu+1} f\right\|_{\infty,\mathbb{R}_+}, \ \ \forall\ a \geq 0. \tag{3}$$

Then

$$\|f'\|_{\infty,\mathbb{R}_+} \leq (\nu+1)\left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} (\Gamma(\nu+2))^{-\frac{1}{\nu+1}} \left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu}{\nu+1}} \left(\left\|D_{*0}^{\nu+1} f\right\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{\nu+1}}, \tag{4}$$

that is $\|f'\|_{\infty,\mathbb{R}_+}$ is finite.

The last inequality is another inspiration.

The author's monographs [2], [3], [4], [5], [6], [8], motivate and support largely this work too. See also [1].

Under the point of view of local fractional differentiation the author examines the broad area of analytic inequalities and produces a variety of well-known inequalities in a local fractional setting over a negative domain to all possible directions.

2 Background

We mention

Definition 1 *([11]) Let $x, x' \in [a,b]$, $f \in C([a,b])$. The Riemann-Liouville (R-L) fractional derivative of a function f of order q $(0 < q < 1)$ is defined as*

$$D_x^q f(x') = \begin{cases} D_{x+}^q f(x'), & x' > x, \\ D_{x-}^q f(x'), & x' < x \end{cases} =$$

$$\frac{1}{\Gamma(1-q)} \begin{cases} \frac{d}{dx'} \int_x^{x'} (x'-t)^{-q} f(t)\, dt, & x' > x, \\ -\frac{d}{dx'} \int_{x'}^{x} (t-x')^{-q} f(t)\, dt, & x' < x, \end{cases} \tag{5}$$

the left and right R-L fractional derivatives, respectively.

We need

Definition 2 *([11], [12]) The local fractional derivative of order q $(0 < q < 1)$ of a function $f \in C([a,b])$ is defined as*

$$D^q f(x) = \lim_{x' \to x} D_x^q (f(x') - f(x)). \tag{6}$$

More generally we define

2

Definition 3 *([9]) Let $N \in \mathbb{Z}_+$, $0 < q < 1$, the local fractional derivative of order $(N+q)$ of a function $f \in C^N([a,b])$ is defined by*

$$D^{N+q} f(x) = \lim_{x' \to x} D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \right). \tag{7}$$

If $N = 0$, then Definition 3 collapses to Definition 2.

We need

Definition 4 *(related to Definition 3) Let $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. Set*

$$F(x, x'-x; q, N) := D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \right). \tag{8}$$

Let $x' - x := t$, then $x' = x + t$, and

$$F(x, t; q, N) = D_x^q \left(f(x+t) - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} t^n \right). \tag{9}$$

We make

Remark 5 *Here $x', x \in [a,b]$, and $a \le x + t \le b$, equivalently $a - x \le t \le b - x$. From $a \le x \le b$, we get $a - x \le 0 \le b - x$. We assume here that $F(x, \cdot; q, N) \in C^1([a-x, b-x])$. Clearly, then it holds*

$$D^{N+q} f(x) = F(x, 0; q, N), \tag{10}$$

and $D^{N+q} f(x)$ exists in $\mathbb{R}$.

We would need:

Theorem 6 *([9]) Let $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. Here $x, x' \in [a,b]$, and $F(x, \cdot; q, N) \in C^1([a-x, b-x])$. Then*

$$f(x') = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n + \frac{D^{N+q} f(x)}{\Gamma(q+1)} |x'-x|^q + \tag{11}$$

$$\frac{1}{\Gamma(q+1)} \int_0^{x'-x} \frac{dF(x,t;q,N)}{dt} |(x'-x) - t|^q \, dt.$$

Corollary 7 *(to Theorem 6, $N = 0$) Let $f \in C([a,b])$, $x, x' \in [a,b]$, and $F(x, \cdot; q, 0) \in C^1([a-x, b-x])$. Then*

$$f(x') = f(x) + \frac{D^q f(x)}{\Gamma(q+1)} |x'-x|^q + \tag{12}$$

$$\frac{1}{\Gamma(q+1)} \int_0^{x'-x} \frac{dF(x,t;q,0)}{dt} |(x'-x) - t|^q \, dt.$$

We make

Remark 8 *Let $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. Here $x, x' \in [a,b] : x' < x$, and $F(x, \cdot; q, N) \in C^1([a-x, b-x])$, $0 < q < 1$. By Theorem 6 we get*

$$f(x') = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!}(x'-x)^n + \frac{D^{N+q}f(x)}{\Gamma(q+1)}(x-x')^q -$$

$$\frac{1}{\Gamma(q+1)} \int_{x'-x}^{0} \frac{dF(x,t;q,N)}{dt}(t-x'+x)^q \, dt. \tag{13}$$

Clearly then we get:

Let $f \in C^N([a,0])$, $a < 0$, $N \in \mathbb{Z}_+$, $F(0,\cdot;q,N) \in C^1([a,0])$, $0 < q < 1$. Then, for any $x \in [a,0]$, we derive

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n + \frac{D^{N+q}f(0)}{\Gamma(q+1)}(-x)^q -$$

$$\frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0,t;q,N)}{dt}(t-x)^q \, dt. \tag{14}$$

In this article we will use a lot (14).

Remark 9 *Let $f \in C^N([a,0])$, $N \in \mathbb{Z}_+$, $a < 0$, $x \in [a,0]$; $F(0,\cdot;q,N) \in C^1([a,0])$, $0 < q < 1$. Then, by (14), we have*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n + \frac{D^{N+q}f(0)}{\Gamma(q+1)}(-x)^q \tag{15}$$

$$-\frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0,t;q,N)}{dt}(t-x)^q \, dt.$$

Assume that $f^{(n)}(0) = 0$, $n = 0, 1, ..., N$, and $D^{N+q}f(0) = 0$ $(= F(0,0;q,N) = D_0^q f(0))$.

Then

$$-f(x) = \frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0,t;q,N)}{dt}(t-x)^q \, dt, \tag{16}$$

$\forall\, x \in [a,0]$.

Here it is

$$F(0,t;q,N) = D_0^q(f(t)) \in C^1([a,0]),$$

where D_0^q is the right Riemann-Liouville fractional derivative.

Let $a \le x \le w \le 0$, then

$$-f(w) = \frac{1}{\Gamma(q+1)} \int_{w}^{0} \frac{dF(0,t;q,N)}{dt}(t-w)^q \, dt. \tag{17}$$

4

Consider $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. *Then*

$$|f(w)| = \frac{1}{\Gamma(q+1)} \int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right| (t-w)^q \, dt \leq$$

$$\frac{1}{\Gamma(q+1)} \left(\int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} \left(\int_w^0 (t-w)^{qp_1} \, dt \right)^{\frac{1}{p_1}} =$$

$$\frac{1}{\Gamma(q+1)} \frac{(-w)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}} \left(\int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} =$$

$$\frac{1}{\Gamma(q+1)} \frac{(-w)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}} (z(w))^{\frac{1}{q_1}}, \tag{18}$$

where

$$z(w) := \int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt, \tag{19}$$

all $a \leq x \leq w \leq 0$, *and* $z(0) = 0$.

From

$$-z(w) = \int_0^w \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt,$$

we get

$$-z'(w) = (-z(w))' = \left| \frac{dF(0,w;q,N)}{dw} \right|^{q_1}, \tag{20}$$

and

$$\left| \frac{dF(0,w;q,N)}{dw} \right| = (-z'(w))^{\frac{1}{q_1}}. \tag{21}$$

Therefore we obtain

$$|f(w)| \left| \frac{dF(0,w;q,N)}{dw} \right| \leq \tag{22}$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} (-w)^{\frac{qp_1+1}{p_1}} (z(w))^{\frac{1}{q_1}} (-z'(w))^{\frac{1}{q_1}}.$$

Hence it holds

$$\int_x^0 |f(w)| \left| \frac{dF(0,w;q,N)}{dw} \right| dw \leq \tag{23}$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \int_x^0 (-w)^{\frac{qp_1+1}{p_1}} (z(w)(-z'(w)))^{\frac{1}{q_1}} dw \leq$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left(\int_x^0 (-w)^{qp_1+1} dw \right)^{\frac{1}{p_1}} \left(\int_x^0 z(w)(-z'(w)) \, dw \right)^{\frac{1}{q_1}} =$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left(\frac{(-x)^{qp_1+2}}{qp_1+2} \right)^{\frac{1}{p_1}} \left(-\frac{z^2(w)}{2} \Big|_x^0 \right)^{\frac{1}{q_1}} = \tag{24}$$

5

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}(qp_1+2)^{\frac{1}{p_1}}}(-x)^{\frac{qp_1+2}{p_1}}\frac{(z(x))^{\frac{2}{q_1}}}{2^{\frac{1}{q_1}}}.$$

We have proved that

$$\int_x^0 |f(w)|\left|\frac{dF(0,w;q,N)}{dw}\right|dw \leq$$

$$\frac{(-x)^{q+\frac{2}{p_1}}}{2^{\frac{1}{q_1}}\Gamma(q+1)\left[(qp_1+1)(qp_1+2)\right]^{\frac{1}{p_1}}}\left(\int_x^0\left|\frac{dF(0,w;q,N)}{dw}\right|^{q_1}dw\right)^{\frac{2}{q_1}}. \quad (25)$$

We have established the following negative domain L_p-Opial type local right fractional inequality:

Theorem 10 *Let $p_1, q_1 > 1 : \frac{1}{p_1}+\frac{1}{q_1}=1$; $f \in C^N([a,0])$, $N \in \mathbb{Z}_+$, $a<0$, $x \in [a,0]$; $F(0,\cdot;q,N) \in C^1([a,0])$, $0<q<1$. Assume that $f^{(n)}(0)=0$, $n=0,1,...,N$, and $D^{N+q}f(0)=0$ $(= F(0,0;q,N)=D_0^q f(0))$. [Here it is $F(0,t;q,N)=D_0^q(f(t)) \in C^1([a,0])$, where D_0^q is the right Riemann-Liouville fractional derivative]. Then*

$$\int_x^0 |f(t)|\left|\frac{dF(0,t;q,N)}{dt}\right|dt \leq$$

$$\frac{(-x)^{q+\frac{2}{p_1}}}{2^{\frac{1}{q_1}}\Gamma(q+1)\left[(qp_1+1)(qp_1+2)\right]^{\frac{1}{p_1}}}\left(\int_x^0\left|\frac{dF(0,t;q,N)}{dt}\right|^{q_1}dt\right)^{\frac{2}{q_1}}, \quad (26)$$

$\Leftrightarrow$

it holds

$$\int_x^0 |f(t)|\left|\frac{dD_0^q(f(t))}{dt}\right|dt \leq$$

$$\frac{(-x)^{q+\frac{2}{p_1}}}{2^{\frac{1}{q_1}}\Gamma(q+1)\left[(qp_1+1)(qp_1+2)\right]^{\frac{1}{p_1}}}\left(\int_x^0\left|\frac{dD_0^q(f(t))}{dt}\right|^{q_1}dt\right)^{\frac{2}{q_1}}, \quad (27)$$

$\forall\ x \in [a,0]$.

The case $p_1=q_1=2$ follows:

Corollary 11 *All as in Theorem 10, with $p_1=q_1=2$. Then*

$$\int_x^0 |f(t)|\left|\frac{dF(0,t;q,N)}{dt}\right|dt \leq \quad (28)$$

$$\frac{(-x)^{q+1}}{2\Gamma(q+1)\sqrt{(q+1)(2q+1)}}\left(\int_x^0\left|\frac{dF(0,t;q,N)}{dt}\right|^2dt\right),$$

$\Leftrightarrow$

6

it holds

$$\int_x^0 |f(t)| \left| \frac{dD_0^q (f(t))}{dt} \right| dt \leq \tag{29}$$

$$\frac{(-x)^{q+1}}{2\Gamma(q+1)\sqrt{(q+1)(2q+1)}} \left(\int_x^0 \left(\frac{dD_0^q (f(t))}{dt} \right)^2 dt \right),$$

$\forall\ x \in [a, 0]$.

We make

Remark 12 *Let f_1, f_2 according to the assumptions of Theorem 10. Then*

$$-f_1(x_1) = \frac{1}{\Gamma(q+1)} \int_{x_1}^0 \frac{dF_1(0, t_1; q, N)}{dt_1} (t_1 - x_1)^q\, dt_1, \tag{30}$$

$\forall\ x_1 \in [a_1, 0]$, $a_1 < 0$;

$$-f_2(x_2) = \frac{1}{\Gamma(q+1)} \int_{x_2}^0 \frac{dF_2(0, t_2; q, N)}{dt_2} (t_2 - x_2)^q\, dt_2, \tag{31}$$

$\forall\ x_2 \in [a_2, 0]$, $a_2 < 0$.

Here it is

$$F_i(0, t_i; q, N) = D_0^q (f_i(t_i)) \in C^1([a_i, 0]), \quad i = 1, 2;$$

where D_0^q is the right Riemann-Liouville fractional derivative.

Consider $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$.

Hence

$$|f_i(x_i)| \leq \frac{1}{\Gamma(q+1)} \int_{x_i}^0 \left| \frac{dF_i(0, t_i; q, N)}{dt_i} \right| (t_i - x_i)^q\, dt_i, \tag{32}$$

$i = 1, 2$; $\forall\ x_i \in [a_i, 0]$.

We get by Hölder's inequality:

$$|f_1(x_1)| \leq$$

$$\frac{1}{\Gamma(q+1)} \left(\int_{x_1}^0 (t_1 - x_1)^{qp_1} dt_1 \right)^{\frac{1}{p_1}} \left(\int_{x_1}^0 \left| \frac{dF_1(0, t_1; q, N)}{dt_1} \right|^{q_1} dt_1 \right)^{\frac{1}{q_1}} \leq \tag{33}$$

$$\frac{1}{\Gamma(q+1)} \frac{(-x_1)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}} \left\| \frac{dF_1(0, t_1; q, N)}{dt_1} \right\|_{q_1, [a_1, 0]},$$

$\forall\ x_1 \in [a_1, 0]$.

Similarly, we obtain

$$|f_2(x_2)| \leq \frac{1}{\Gamma(q+1)} \frac{(-x_2)^{\frac{qq_1+1}{q_1}}}{(qq_1+1)^{\frac{1}{q_1}}} \left\| \frac{dF_2(0, t_2; q, N)}{dt_2} \right\|_{p_1, [a_2, 0]}, \tag{34}$$

$\forall\ x_2 \in [a_2, 0]$.

Therefore we have

$$|f_1(x_1)||f_2(x_2)| \leq \frac{1}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{35}$$

$$(-x_1)^{\frac{qp_1+1}{p_1}} (-x_2)^{\frac{qq_1+1}{q_1}} \left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]} \leq$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p_1}} b^{\frac{1}{q_1}} \leq \frac{a}{p_1} + \frac{b}{q_1}$*)*

$$\frac{1}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]$$

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]}, \tag{36}$$

$\forall\ x_i \in [a_i, 0]$, $i = 1, 2$.

So far we have established

$$\frac{|f_1(x_1)||f_2(x_2)|}{\left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]} \leq \frac{1}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{37}$$

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]},$$

$\forall\ x_i \in [a_i, 0]$, $i = 1, 2$.

The denominator of left hand side of (37) can be zero only when $x_1 = 0$ *and* $x_2 = 0$. *By integrating (37) over* $[a_1, 0] \times [a_2, 0]$ *we get*

$$\int_{a_1}^{0} \int_{a_2}^{0} \frac{|f_1(x_1)||f_2(x_2)|\, dx_1 dx_2}{\left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]} \leq \frac{a_1 a_2}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{38}$$

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]}.$$

We have proved the following negative domains local right fractional Hilbert-Pachpatte inequality:

Theorem 13 *Let* $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$*;* $i = 1, 2$ *for* $f_i \in C^N([a_i, 0])$*,* $N \in \mathbb{Z}_+$*,* $a_i < 0$*;* $F_i(0, \cdot; q, N) \in C^1([a_i, 0])$*,* $0 < q < 1$*. Assume that* $f_i^{(n)}(0) = 0$*,* $n = 0, 1, ..., N$*, and* $D^{N+q} f_i(0) = 0$ *,* $i = 1, 2$ *(i.e.* $F_i(0, 0; q, N) = D_0^q f_i(0) = 0$*). [Here it is* $F_i(0, t_i; q, N) = D_0^q(f_i(t_i)) \in C^1([a_i, 0])$*, where* D_0^q *is the right Riemann-Liouville fractional derivative]. Then*

$$\int_{a_1}^{0} \int_{a_2}^{0} \frac{|f_1(x_1)||f_2(x_2)|\, dx_1 dx_2}{\left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]} \leq \frac{a_1 a_2}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{39}$$

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]},$$

$\Leftrightarrow$

it holds

$$\int_{a_1}^{0}\int_{a_2}^{0} \frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{(-x)^{qp_1+1}}{p_1}+\frac{(-x_2)^{qq_1+1}}{q_1}\right]} \leq \frac{a_1a_2}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{40}$$

$$\left\| \frac{dD_0^q(f_1(t_1))}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dD_0^q(f_2(t_2))}{dt_2} \right\|_{p_1,[a_2,0]}.$$

We make

Remark 14 *Let $f \in C^N([a,0])$, $a<0$, $N \in \mathbb{Z}_+$, $F(0,\cdot;q,N) \in C^1([a,0])$, $0<q<1$. Then for any $x \in [a,0]$, we have*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n + \frac{D^{N+q}f(0)}{\Gamma(q+1)}(-x)^q \tag{41}$$

$$-\frac{1}{\Gamma(q+1)}\int_x^0 \frac{dF(0,t;q,N)}{dt}(t-x)^q\,dt.$$

Assume that $f^{(n)}(0)=0$, $n=0,1,...,N$. Here $D^{N+q}f(0)=F(0,0;q,N)=D_0^qf(0)$, where D_0^q is the right Riemann-Liouville fractional derivative.

So far we have

$$f(x) = \frac{D^{N+q}f(0)}{\Gamma(q+1)}(-x)^q + R(x), \tag{42}$$

where

$$R(x) := -\frac{1}{\Gamma(q+1)}\int_x^0 \frac{dF(0,t;q,N)}{dt}(t-x)^q\,dt. \tag{43}$$

We also assume that $D_0^qf \in C^1([a,0])$.

We can rewrite

$$R(x) = -\frac{1}{\Gamma(q+1)}\int_x^0 \left(\frac{d}{dt}D_0^qf(t)\right)(t-x)^q\,dt. \tag{44}$$

We notice that

$$|R(x)| \leq \frac{1}{\Gamma(q+1)}\int_x^0 \left|\frac{d}{dt}D_0^qf(t)\right|(t-x)^q\,dt \leq$$

$$\frac{1}{\Gamma(q+1)}\left\|\frac{d}{dt}D_0^qf(t)\right\|_{\infty,[a,0]} \frac{(-x)^{q+1}}{q+1}.$$

That is

$$|R(x)| \leq \frac{(-x)^{q+1}}{\Gamma(q+2)} \left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]}, \tag{45}$$

$\forall \; x \in [a,0]$.

Hence, it holds

$$\int_a^0 f(x)\,dx = \frac{D^{N+q} f(0)}{\Gamma(q+1)} \int_a^0 (-x)^q\,dx + \int_a^0 R(x)\,dx = \tag{46}$$

$$\frac{D^{N+q} f(0)}{\Gamma(q+1)} \frac{(-a)^{q+1}}{q+1} + \int_a^0 R(x)\,dx = \frac{D^{N+q} f(0)}{\Gamma(q+2)} (-a)^{q+1} + \int_a^0 R(x)\,dx.$$

Therefore, we get

$$\int_a^0 f(x)\,dx - \frac{D^{N+q} f(0)}{\Gamma(q+2)} (-a)^{q+1} = \int_a^0 R(x)\,dx. \tag{47}$$

Consequently, we derive

$$\left| \int_a^0 f(x)\,dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^{q+1} \right| \leq \int_a^0 |R(x)|\,dx \leq \tag{48}$$

$$\frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]}}{\Gamma(q+2)} \int_a^0 (-x)^{q+1}\,dx = \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]}}{\Gamma(q+2)} \frac{(-a)^{q+2}}{q+2}$$

$$= \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]} (-a)^{q+2}}{\Gamma(q+3)}. \tag{49}$$

We have proved the following negative domain local right fractional comparison of means results:

Theorem 15 *Let $f \in C^N([a,0])$, $a<0$, $N \in \mathbb{Z}_+$, $D_0^q f \in C^1([a,0])$, $0<q<1$. Assume $f^{(n)}(0) = 0$, $n = 0,1,...,N$. Then*

$$\left| \int_a^0 f(x)\,dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^{q+1} \right| \leq \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]} (-a)^{q+2}}{\Gamma(q+3)}, \tag{50}$$

$\Leftrightarrow$

$$\left| \frac{1}{(-a)} \int_a^0 f(x)\,dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^q \right| \leq \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]} (-a)^{q+1}}{\Gamma(q+3)}. \tag{51}$$

We make

Remark 16 *All as in Theorem 10. Then*

$$-f(x) = \frac{1}{\Gamma(q+1)} \int_x^0 \frac{dF(0,t;q,N)}{dt} (t-x)^q\,dt, \tag{52}$$

10

$\forall\ x \in [a, 0]$.

Let $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. *Thus*

$$|f(x)| \leq \frac{1}{\Gamma(q+1)} \int_x^0 \left| \frac{dF(0,t;q,N)}{dt} \right| (t-x)^q \, dt \leq$$

$$\frac{1}{\Gamma(q+1)} \left(\int_x^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} \left(\int_x^0 (t-x)^{qp_1} \, dt \right)^{\frac{1}{p_1}} \leq$$

$$\frac{1}{\Gamma(q+1)} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]} \frac{(-x)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}}. \tag{53}$$

That is

$$|f(x)| \leq \frac{(-x)^{\frac{qp_1+1}{p_1}}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}, \tag{54}$$

$\forall\ x \in [a, 0]$.

Therefore

$$|f(x)|^{q_1} \leq \frac{(-x)^{q_1(q+1)-1}}{(\Gamma(q+1))^{q_1} (qp_1+1)^{\frac{q_1}{p_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}^{q_1}. \tag{55}$$

Consequently, it holds

$$\int_a^0 |f(x)|^{q_1} \, dx \leq$$

$$\frac{(-a)^{q_1(q+1)}}{\left[\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}\right]^{q_1} q_1(q+1)} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}^{q_1}. \tag{56}$$

That is

$$\|f\|_{q_1,[a,0]} \leq \frac{(-a)^{(q+1)}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}} (q_1(q+1))^{\frac{1}{q_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}. \tag{57}$$

We have proved the following negative domain local right fractional Poincare inequality:

Theorem 17 *All as in Theorem 10. Then*

$$\|f\|_{q_1,[a,0]} \leq \frac{(-a)^{(q+1)}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}} (q_1(q+1))^{\frac{1}{q_1}}} \left\| (D_0^q(f))' \right\|_{q_1,[a,0]}. \tag{58}$$

We make

Remark 18 *All as in Theorem 10, plus $r > 0$. By (54) we have*

$$|f(x)| \leq \frac{(-x)^{\frac{qp_1+1}{p_1}}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}, \tag{59}$$

$\forall\, x \in [a,0]$.

Hence it holds

$$|f(x)|^r \leq \frac{(-x)^{r\left(q+\frac{1}{p_1}\right)}}{\left[\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}\right]^r} \left\| \left(D_0^q(f)\right)' \right\|_{q_1,[a,0]}^r. \tag{60}$$

Consequently, we get

$$\int_a^0 |f(x)|^r dx \leq \frac{(-a)^{r\left(q+\frac{1}{p_1}\right)+1}}{\left[\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}\right]^r \left[r\left(q+\frac{1}{p_1}\right)+1\right]} \left\| \left(D_0^q(f)\right)' \right\|_{q_1,[a,0]}^r. \tag{61}$$

We have proved the following negative domain local ritgh fractional Sobolev type inequality:

Theorem 19 *All as in Theorem 10, plus $r > 0$. Then*

$$\|f\|_{r,[a,0]} \leq \frac{(-a)^{q+\frac{1}{p_1}+\frac{1}{r}}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}} \left[r\left(q+\frac{1}{p_1}\right)+1\right]^{\frac{1}{r}}} \left\| \left(D_0^q(f)\right)' \right\|_{q_1,[a,0]}. \tag{62}$$

References

[1] F.B. Adda, J. Cresson, *Fractional differentiation equations and the Schrödinger equation*, Applied Math. & Computation, 161 (2005), 323-345.

[2] G.A. Anastassiou, *Quantitative Approximations*, CRC press, Boca Raton, London, New York, 2001.

[3] G.A. Anastassiou, *Fractional differentiation inequalities*, Springer, Heidelberg, New York, 2009.

[4] G.A. Anastassiou, *Probabilistic Inequalities*, World Scientific, Singapore, New York, 2010.

[5] G.A. Anastassiou, *Advanced Inequalities*, World Scientific, Singapore, New York, 2010.

[6] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, 2011.

[7] G.A. Anastassiou, *Advances on Fractional Inequalities*, Springer, Heidelberg, New York, 2011.

[8] G.A. Anastassiou, *Intelligent Comparisons: Analytic Inequalities*, Springer, Heidelberg, New York, 2016.

[9] G.A. Anastassiou, *Local Fractional Taylor Formula*, J. of Computational Analysis and Applications, accepted, 2018.

[10] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Heidelberg, New York, 2010.

[11] K.M. Kolwankar, *Local fractional calculus: a Review*, arXiv: 1307:0739v1 [nlin.CD] 2 Jul. 2013.

[12] K.M. Kolwankar and A.D. Gangal, *Local fractional calculus: a calculus for fractal space-time*, Fractals: theory and applications in engineering, 171-181, London, New York, Springer, 1999.

[13] Z. Opial, *Sur une inégalité*, Ann. Polon. Math. 8 (1960), 29-32.

[14] A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Approximate controllability for semilinear integro-differential control equations in Hilbert spaces

Yong Han Kang[1], Jin-Mun Jeong[2,*] and Ah-ran Park[3]

[1]Institute of Liberal Education, Catholic University of Daegu
Gyeongsan, 712-702, South Korea
E-mail:yonghann@cu.ac.kr

[2,3]Department of Applied Mathematics, Pukyong National University
Busan 48513, South Korea
E-mail: *jmjeong@pknu.ac.kr(Corresponding author), alanida@naver.com

Abstract

This paper deals with the approximate controllability for a class of semilinear integro-differential functional control equations, which is provided under general sufficient conditions on the system operator, controller and nonlinear terms. Our used tool is applying results similar to Fredholm alternative for nonlinear operators under restrictive assumptions. Finally, a simple example to which our main result can be applied is given.

Keywords: approximate controllability, semilinear control equations, integro-differential control equations, controller, Fredholm alternative.

AMS Classification: Primary 93B05, 35F25

1 Introduction

In this paper, we deal with the approximate controllability for semilinear integro-differential functional control equations in the form

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \quad 0 < t \le T, \\ x(0) &= x_0 \end{cases} \tag{1.1}$$

Email: alanida@naver.com(A. park), *jmjeong@pknu.ac.kr(J. jeong), nabifly@hanmail.net(E. Y. Ju)
This work was supported by a Research Grant of Pukyong National University(2021 Year).

in a Hilbert space H, where k belongs to $L^2(0,T)(T>0)$ and g is a nonlinear mapping as detailed in Section 2. The principal operator A generates an analytic semigroup $(S(t))_{t\geq 0}$ and B is a bounded linear operator from another Hilbert space U to H.

The controllability problem is a question of whether is possible to steer a dynamic system from an initial state to an arbitrary final state using the set of admissible controls. Naito [13] was the first to deal with the range condition argument of controller in order to obtain the approximate controllability of a semilinear control system. In [3, 9, 17, 18], they have studied continuously about controllability of semilinear systems dominated by linear parts(in case $g \equiv 0$) by assuming that $S(t)$ is compact operator for each $t>0$ as matters connected with [13]. Another approach used to obtain sufficient conditions for approximate solvability of nonlinear equations is a fixed point theorem combined with technique of operator transformations by configuring the resolvent as seen in [2]

The controllability for various nonlinear equations has been studied by many authors, for example, see [5, 6, 12] for local controllability of neutral functional differential systems with unbounded delay, [10, 14] for neutral evolution integrodifferential systems with state dependent delay.

Sukavanam and Tomar [15] studied the approximate controllability for the general retarded initial value problem by assuming that the Lipschitz constant of the nonlinear term is less then 1, and Wang [17] for general retarded semilinear equations assuming the growth condition of the nonlinear term and the compactness of the semigroup.

In this paper, authors want to use a different method than the previous one. Our used tool is the theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda T(x) - F(x) = y$ in dependence on the real number λ, where T and F are nonlinear operators defined a Banach space X with values in a Banach space Y. In order to obtain the approximate controllability for a class of semilinear integro-differential functional control equations, it is necessary to suppose that T acts as the identity operator while F related to the nonlinear term of (1.1) is completely continuous

In Section 2, we introduce regularity properties for (1.1). Since we apply the Fredholm theory in the proof of the main theorem, we assume some compactness of the embedding between intermediate spaces. Then by virtue of Aubin [1], we can show that the solution mapping of a control space to the terminal state space is completely continuous. Based on Section 2, it is shown the sufficient conditions on the controller and nonlinear terms for approximate controllability for (1.1) by using the Fredholm theory. Finally, a simple example to which our main result can be applied is given.

2 Semilinear functional equations

Let V and H be complex Hilbert spaces forming a Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

by identifying the antidual of H with H. Therefore, for the brevity, we may regard that $||u||_* \le |u| \le ||u||$ for all $u \in V$, where the notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^*, respectively as usual. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0.$$

Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u,v), \quad u, \ v \in V.$$

Then A is a bounded linear operator from V to V^*. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. For the sake of simplicity we assume that $c_1 = 0$ and hence the closed half plane $\{\lambda : \text{Re}\,\lambda \ge 0\}$ is contained in the resolvent set of A. It is known that A generates an analytic semigroup $S(t)$ in both H and V^*. As seen in Lemma 3.6.2 of [16], there exists a constant $M > 0$ such that

$$|S(t)x| \le M|x| \quad \text{and} \quad ||S(t)x||_* \le M||x||_*, \tag{2.1}$$

The following initial value problem for the abstract linear parabolic equation

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + k(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{2.2}$$

By virtue of Theorem 3.3 of [4](or Theorem 3.1 of [9]), we have the following result on the corresponding linear equation (2.2).

Proposition 2.1. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) For $x_0 \in V$ and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (2.2) belonging to

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$$

and satisfying

$$||x||_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.3}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then, there exists a unique solution x of (2.2) belonging to

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.4}$$

where C_1 is a constant depending on T.

By virtue of Proposition 2.1, we have the following lemma.

Lemma 2.1. *Suppose that* $k \in L^2(0,T;H)$ *and* $x(t) = \int_0^t S(t-s)k(s)ds$ *for* $0 \le t \le T$. *Then there exists a constant* C_2 *such that*

$$||x||_{L^2(0,T;H)} \le C_2 T ||k||_{L^2(0,T;H)}, \tag{2.5}$$

and

$$||x||_{L^2(0,T;V)} \le C_2 \sqrt{T} ||k||_{L^2(0,T;H)}. \tag{2.6}$$

Consider the following initial value problem for the abstract semilinear parabolic equation

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \\ x(0) &= x_0. \end{cases} \tag{2.7}$$

Let U be a Hilbert space and the controller operator B be a bounded linear operator from U to H.

Let $g : \mathbb{R}^+ \times V \times U \to H$ be a nonlinear mapping satisfying the following:

Assumption (F).

(i) For any $x \in V$, $u \in U$ the mapping $g(\cdot, x, u)$ is strongly measurable;

(ii) There exist positive constants L_0, L_1, L_2 such that

(a) $u \mapsto g(t,x,u)$ is an odd mapping $(g(\cdot,x,-u) = -g(\cdot,x,u))$;

(b) for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $u, \hat{u} \in U$,

$$|g(t,x,u) - g(t,\hat{x},\hat{u})| \le L_1||x-\hat{x}|| + L_2||u-\hat{u}||_U,$$
$$|g(t,0,0)| \le L_0.$$

For $x \in L^2(0,T;V)$, we set

$$f(t,x,u) = \int_0^t k(t-s)g(s,x(s),u(s))ds$$

where k belongs to $L^2(0,T)$.

Lemma 2.2. *Let Assumption (F) be satisfied. Assume that* $x \in L^2(0,T;V)$ *for any* $T > 0$. *Then* $f(\cdot,x,u) \in L^2(0,T;H)$ *and*

$$\begin{aligned} ||f(\cdot,x,u))||_{L^2(0,T;H)} &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} \\ &+ ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}). \end{aligned} \tag{2.8}$$

Moreover if $x,\ \hat{x} \in L^2(0,T;V)$, *then*

$$\begin{aligned} &||f(\cdot,x,u) - f(\cdot,\hat{x},\hat{u})||_{L^2(0,T;H)} \\ &\le ||k||_{L^2(0,T)}\sqrt{T}(L_1||x-\hat{x}||_{L^2(0,T;V)} + L_2||u-\hat{u}||_{L^2(0,T;U)}). \end{aligned} \tag{2.9}$$

4

The proof is easily from Assumption (F), and using the Hölder inequality.

By virtue of Theorem 2.1 of [8], we have the following result on (2.7).

Proposition 2.2. *Let Assumption (F) be satisfied. Then there exists a unique solution x of (2.7) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $x_0 \in H$. Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_3(|x_0| + ||u||_{L^2(0,T;U)}). \tag{2.10}$$

Corollary 2.1. *Assume that the embedding $D(A) \subset V$ is completely continuous. Let Assumption (F) be satisfied, and x_u be the solution of equation (2.7) associated with $u \in L^2(0,T;U)$. Then the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$.*

Proof. If u is bounded in $L^2(0,T;U)$, then so is x_u in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ by (2.8). Since $D(A)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is completely continuous in view of Theorem 2 of [1], the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$. □

3 Approximate controllability

Throughout this section, we assume that $D(A)$ is compactly embedded in V. Let $x(T;f,u)$ be a state value of the system (2.7) at time T corresponding to the nonlinear term f and the control u. We define the reachable sets for the system (2.7) as follows:

$$R_T(f) = \{x(T;f,u) : u \in L^2(0,T;U)\},$$
$$R_T(0) = \{x(T;0,u) : u \in L^2(0,T;U)\}.$$

Definition 3.1. *The system (2.7) is said to be approximately controllable in the time interval $[0,T]$ if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T;f,u)$ of (2.7) satisfies $|x(T;f,u) - x_1| < \epsilon$, that is, if $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H, then the system (2.9) is called approximately controllable at time T.*

Let us introduce the theory of the degree for completely continuous perturbations of the identity operator, which is the infinite dimensional version of Borsuk's theorem. Let $0 \in D$ be a bounded open set in a Banach space X, $\overline{D}$ its closure and ∂D its boundary. The number $d[I-T;D,0]$ is the degree of the mapping $I-T$ with respect to the set D and the point 0 (see Fučik et al. [7] or Lloid [11]).

Theorem 3.1. *(Borsuk's theorem) Let D be a bounded open symmetric set in a Banach space X, $0 \in D$. Suppose that $T : \overline{D} \to X$ be odd completely continuous operator satisfying $T(x) \neq x$ for $x \in \partial D$. Then $d[I - T; D, 0]$ is odd integer. That is, there exists at least one point $x_0 \in D$ such that $(I - T)(x_0) = 0$.*

Definition 3.2. *Let T be a mapping defined by on a Banach space X with value in a real Banach space Y. The mapping T is said to be a (K, L, α)-homeomorphism of X onto Y if*

(i) T is a homeomorphism of X onto Y;

(ii) there exist real numbers $K > 0$, $L > 0$, and $\alpha > 0$ such that

$$L||x||_X^\alpha \leq ||T(x)||_Y \leq K||x||_X^\alpha, \quad \forall x \in X.$$

Lemma 3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ a continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty.$$

Proof. Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\} \subset X$ such that

$$||\lambda T(x_n) - F(x_n)||_Y \leq M \tag{3.1}$$

as $x_n \to \infty$. From (3.1) it follows that

$$\frac{\lambda T(x_n)}{||x_n||_X^\alpha} - \frac{F(x_n)}{||x_n||_X^\alpha} \to 0.$$

Hence, we have

$$\limsup_{n \to \infty} \frac{|\lambda| ||T(x_n)||_Y}{||x_n||_X^\alpha} = N,$$

and so, $|\lambda|K \geq N \geq |\lambda|L$. It is a contradiction with $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}]$. □

Proposition 3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator. Suppose that for $\lambda \neq 0$,*

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty. \tag{3.2}$$

Then $\lambda T - F$ maps X onto Y.

Proof. We follow the proof Theorem 1.1 in Chapter II of Fučik et al. [7]. Suppose that there exists $y \in Y$ such that $\lambda T(x) = y$. Then from (3.2) it follows that $FT^{-1} : Y \to Y$ is an odd completely continuous operator and

$$\lim_{||y||_Y \to \infty} ||y - FT^{-1}(\frac{y}{\lambda})||_Y = \infty.$$

Let $y_0 \in Y$. There exists $r > 0$ such that

$$||y - FT^{-1}(\frac{y}{\lambda})||_Y > ||y_0||_Y \geq 0$$

for each $y \in Y$ satisfying $||y||_Y = r$. Let $Y_r = \{y \in Y : ||y||_Y < r\}$ be a open ball. Then by view of Theorem 3.1, we have $d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0]$ is an odd number. For each $y \in Y$ satisfying $||y||_Y = r$ and $t \in [0, 1]$, there is

$$||y - FT^{-1}(\frac{y}{\lambda}) - ty_0||_Y \geq ||y - FT^{-1}(\frac{y}{\lambda})||_Y - ||y_0||_Y > 0$$

and hence, by the homotopic property of degree, we have

$$d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, y_0] = d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0] \neq 0.$$

Hence, by the existence theory of the Leray-Schauder degree, there exists a $y_1 \in Y_r$ such that

$$y_1 - FT^{-1}(\frac{y_1}{\lambda}) = y_0.$$

We can choose $x_0 \in X$ satisfying $\lambda T(x_0) = y_1$, and so, $\lambda T(x_0) - F(x_0) = y_0$. Thus, it implies that $\lambda T - F$ is a mapping of X onto Y. □

Combining Lemma 3.1. and Proposition 3,1, we have the following results.

Corollary 3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then $\lambda T - F$ maps X onto Y. Therefore, if $N = 0$, then for all $\lambda \neq 0$ the operator $\lambda T - F$ maps X onto Y.

First we consider the approximate controllability of the system (2.7) in case where the controller B is the identity operator on H under Assumption (F) on the nonlinear operator f in Section 2. Hence, noting that $H = U$, we consider the linear system given by

$$\begin{cases} \frac{d}{dt}y(t) &= Ay(t) + u(t), \\ y(0) &= x_0, \end{cases} \tag{3.3}$$

and the following semilinear control system

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + v(t), \\ x(0) &= x_0. \end{cases} \tag{3.4}$$

Theorem 3.2. *Assume that*

$$\limsup_{||u||\to\infty} \frac{||f(\cdot, x_u, u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} < 1. \tag{3.5}$$

Under the Assumption (F) we have

$$R_T(0) \subset R_T(f).$$

Therefore, if the linear system (3.3) with $f = 0$ is approximately controllable, then so is the semilinear system (3.4).

Proof. Let $x(t)$ be solution of (3.4) corresponding to a control u. First, we show that there exist a $v \in L^2(0, T; H)$ such that

$$\begin{cases} v(t) &= u(t) - f(t, x(t), v(t)), \quad 0 < t \le T, \\ v(0) &= u(0). \end{cases}$$

Let us define an operator $F : L^2(0, T; H) \to L^2(0, T; H)$ as

$$Fv = -f(\cdot, x_v, v).$$

Then by Corollary 2.1, F is a compact mapping from $L^2(0, T; H)$ to itself, and we have

$$\lim_{||v||\to\infty} ||\lambda I(v) - F(v)||_{L^2(0,T;H)} = \infty,$$

where the identity operator I on $L^2(0, T; H)$ is an odd $(1, 1, 1)$-homeomorphism. Thus, from (3.5) and Corollary 3.1, if $\lambda \ge 1$ then $\lambda I - F$ maps $L^2(0, T; H)$ onto itself. Hence, we have showed that there exists a $v \in L^2(0, T; H)$ such that $v(t) = u(t) - f(t, y(t), v(t))$. Let y and x be solutions of (3.3) and (3.4) corresponding to controls u and v, respectively. Then, equation (3.4) is rewritten as

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + v(t), \ 0 < t \le T \\ &= Ax(t) + f(t, x(t), v(t)) + u(t) - f(t, y(t), v(t)) \\ &= Ax(t) + u(t) \end{aligned}$$

with $x(0) = x_0$, which means

$$\begin{aligned} x(t) =& S(t)x_0 + \int_0^t S(t-s)\{f(s,x(s),v(s)) + v(s)\}ds \\ =& S(t)x_0 + \int_0^t S(t-s)u(s)ds = y(t), \end{aligned}$$

where y be solution of (3.3) corresponding to a control u. Therefore, we have proved that $R_T(0) \subset R_T(f)$. □

Corollary 3.2. *Let us assume that*

$$||k||_{L^2(0,T)}\sqrt{T}\big(L_1C_3 + L_2\big) < 1,$$

where C_3 is the constant in Proposition 2.2. Under the Assumption (F), we have

$$R_T(0) \subset R_T(f)$$

in case where $B \equiv I$.

Proof. By Lemma 2.2 and Proposition 2.2, we have

$$\begin{aligned} &||Fu||_{L^2(0,T;H)} = ||f(\cdot, x_u, u)||_{L^2(0,T;H)} \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}) \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}\big\{L_1C_3\big(|x_0| + ||u||_{L^2(0,T;U)}\big) + L_2||u||_{L^2(0,T;U)}\big\}. \end{aligned}$$

Hence, we have

$$\limsup_{||u||\to\infty} \frac{||F(u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;U)}} \le ||k||_{L^2(0,T)}\sqrt{T}\big(L_1C_3 + L_2\big).$$

Thus, from Theorem 3.2, it follows that if $\lambda \ge 1$ then $\lambda I - F$ maps $L^2(0,T;H)$ onto itself, and so, by the same argument as in the proof of theorem it holds that $R_T(0) \subset R_T(f)$. □

From now on, we consider the initial value problem for the semilinear parabolic equation (2.7). Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H.

Assumption (B) There exists a constant $\beta > 0$ such that $\mathrm{R}(f) \subset \mathrm{R}(B)$ and

$$||Bu|| \ge \beta||u||, \quad \forall u \in L^2(0,T;U).$$

Consider the linear system given by

$$\begin{cases} \frac{d}{dt}y(t) &= Ay(t) + Bu(t), \\ y(0) &= x_0. \end{cases} \tag{3.6}$$

Theorem 3.3. *Under the Assumptions (3.5), (B) and (F), we have*

$$R_T(0) \subset R_T(f).$$

Therefore, if the linear system (3.6) with $f = 0$ is approximately controllable, then so is the semilinear system (2.7).

Proof. Let y be a solution of the linear system (3.6) with $f = 0$ corresponding to a control u, and let x be a solutions of the semilinear system (3.4) corresponding to a control v. Set $v(t) = u(t) - B^{-1}f(t, x(t), v(t))$. Then, system (2.9) is rewritten as

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + Bv(t), \ 0 < t \leq T \\ &= Ax(t) + f(t, x(t), v(t)) + Bu(t) - f(t, x(t), v(t))\end{aligned}$$

with $x(0) = x_0$. Hence, we have

$$\begin{aligned}x(t) =&S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s), v(s)) + v(s))\}ds \\ =&S(t)x_0 + \int_0^t S(t-s)u(s))ds = y(t).\end{aligned}$$

Thus, we obtain that $R_T(0) \subset R_T(f)$. □

References

[1] Aubin, J. P., Un thèoréme de compacité, C. R. AQcad. Sci.,, 256 (1963), 5042–5044.

[2] K. Balachandran and J. P. Dauer, Controllability of nonlinear systems in Banach spaces; a survey, J. optim. Theory Appl., 115 (2002), 7–28.

[3] J. P. Dauer and N. I. Mahmudov, *Approximate controllability of semilinear functional equations in Hilbert spaces*, J. Math. Anal., 273 (2002), 310–327.

[4] G. Di Blasio, K. Kunisch and E. Sinestrari, *L^2-regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl., 102 (1984), 38–57.

[5] X. Fu,*Controllability of neutral functional differential systems in abstract space*, Appl. Math. Comput., 141 (2003), 281–296.

[6] X. Fu, J. Lu and Y. You, *Approximate controllability of a semilinear neutral evolution systems with delay*, Inter. J. Control, 87 (2014), 665-681.

[7] Fučik, S., Nečas, J., Souček, J., & Souček, V.(1973), *lecture Notes in Mathematics 346*, Springer-verlag, Belin-Heidelberg-NewYork.

[8] J. M. Jeong and H. Kim, *Controllability for semilinear functional integrodifferential equations*, Bull. Korean Math. Soc. 46(3) (2009), 463–475.

[9] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynamics and Control Systems, 5 (1999), no. 3, 329–346.

[10] Y. C. Kwun, S. H. Park, D. K. Park and S. J. Park, *Controllability of semilinear neutral functional differential evolution equations with nonlocal conditions*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., 15 (2008), 245-??57.

[11] N. G. Lloid, Degree Theory, Cambridge Univ. Press. 1978.

[12] F. Z. Mokkedem and X. Fu, *Approximate controllability of a semi-linear neutral evolution systems with infinite delay* Internat. J. Robust Nonlinear Control, 27, (2017), 1122-1146.

[13] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim., 25 (1987), 715–722.

[14] B. Radhakrishnan and K. Balachandran, *Controllability of neutral evolution integrodifferential systems with state dependent delay*, J. Optim. Theory Appl., 153 (2012), 85-97.

[15] N. Sukavanam and N. K. Tomar, *Approximate controllability of semilinear delay control system*, Nonlinear Func. Anal.Appl., 12 (2007), 53–59.

[16] H. Tanabe, *Equations of Evolution*, Pitman-London, 1979.

[17] L. Wang, *Approximate controllability for integrodifferential equations and multiple delays*, J. Optim. Theory Appl., 143 (2009), 185–206.

[18] H. X. Zhou, Approximate controllability for a class of semilinear abstract equations, SIAM J. Control Optim., 21 (1983), 551–565.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Convergence theorems and approximating endpoints for multivalued Suzuki mappings in hyperbolic spaces

Preeyalak Chuadchawna[1], Ali Farajzadeh[2] and Anchalee Kaewcharoen[3]*
[2]Department of Mathematics, Razi University, Kermanshah, 67149, Iran
E-mail: farajzadehali@gmail.com
[1,3] Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
[3] Center of Excellence in Nonlinear Analysis and Optimization, Faculty of Science
Naresuan University Phitsanulok 65000, Thailand
E-mails: Chuadchawna@hotmail.com; anchaleeka@nu.ac.th

November 13, 2018

Abstract

The objective of this paper is to determine a modified SP-iteration process for multi-valued mappings and to establish the convergence theorems for sequences generated by modified SP-iteration processes involving multi-valued Suzuki mappings converging to endpoints in uniformly convex hyperbolic spaces. The numerical example for supporting our main result is also presented.

Keywords: modified SP-iteration; Δ-convergence theorem; strong convergence theorem; endpoint; hyperbolic space.

MSC: Primary 47H10; Seconday 54H25.

1 Introduction

The distance from u in a metric space (X, d) to a nonempty subset E of X is defined by

$$\operatorname{dist}(u, E) := \inf\{d(u, v) : v \in E\}.$$

It is denoted by $K(E)$ the family of nonempty compact subsets of E. The Hausdorff distance on $K(E)$ is defined by

$$H(U, V) := \max\{\sup_{u\in U} \operatorname{dist}(u, V), \sup_{v\in V} \operatorname{dist}(v, U)\} \quad \text{for all } U, V \in K(E).$$

*Corresponding author.

1

For an element x in E, if $x \in T(x)$, then x is said to be a fixed point of T. Moreover, if $\{x\} = T(x)$, then x is said to be an endpoint of T. It denote by $Fix(T)$ the set of all fixed points of T and by $End(T)$ the set of all endpoints of T. We can see that for every a multi-valued mapping T, $End(T) \subset Fix(T)$ and whenever t is a single-valued mapping, $End(T) = Fix(T)$.

The notion of endpoints for multi-valued mappings is significant notion which put between the notion of fixed points for single-valued mappings and the notion of fixed points for multi-valued mappings.

Aubin and Siegel [3] were first studied the existence of endpoints for special kind of contractive mappings on complete metric spaces. The endpoint results for several types of contractive mappings have been quickly developed and many of papers have showed (see, e.g.,[9],[18],[20],[21]).

On the other hand, Panyanak [15] presented the existence of endpoints for multi-valued nonexpansive mappings in uniformly convex Banach spaces. Next, Kudtha and Panyanak [13] proved the existence of endpoints for Suzuki mappings in uniformly convex hyperbolic spaces.

Recently, Panyanak [16] established the convergence theorems to an endpoint for modified Ishikawa iteration of multi-vaued nonexpansive mappings in uniformly convex Banach spaces.

Motivated and inspired by above mention, we prove the convergence results to an endpoint for modified SP-iteration of multi-valued Suzuki mappings in uniformly convex hyperbolic spaces. The numerical example for supporting our main result is also presented.

2 Preliminaries

For this paper, we work in the setting of a hyperbolic space which is defined by Kohlenbach [12].

Definition 2.1 A hyperbolic space [12] is a metric space (X, d) together with a mapping $W : X^2 \times [0,1] \to X$ satisfying the following statements:

(W1) $d(u, W(x, y, \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y)$;

(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;

(W3) $W(x, y, \alpha) = W(y, x, (1-\alpha))$;

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$,

for all $x, y, u, z, w \in X$ and $\alpha, \beta \in [0,1]$.

If $x, y \in X$ and $\alpha \in [0,1]$, then we use the notion $(1-\alpha)x \oplus \alpha y$ for $W(x, y, \alpha)$. A hyperbolic space (X, d, W) is said to be *uniformly convex* [14] if for any $r > 0$ and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$ such that for all $u, x, y \in X$, we have

$$d(W(x, y, \frac{1}{2}), u) \leq (1-\delta)r,$$

provided $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is well known as a modulus of uniformly convexity of

X. We call η monotone if it decreases with r (for a fixed ε), i.e., for any given $\varepsilon > 0$ and for any $r_2 \geq r_1 > 0$, we have $\eta(r_2, \varepsilon) \leq \eta(r_1, \varepsilon)$.

A nonempty subset E of a hyperbolic space X is *convex* if $W(x, y, \alpha) \in E$ for any $x, y \in E$ and $\alpha \in [0, 1]$.

Obviously, uniformly convex Banach spaces are uniformly convex hyperbolic spaces, CAT(0) spaces are also uniformly convex hyperbolic spaces, [14].

Definition 2.2 [7] A multi-valued mapping $T : E \to CB(E)$ is called to be a Suzuki mapping if

$$\frac{1}{2}\text{dist}(x, T(x)) \leq d(x, y) \text{ implies } H(T(x), T(y)) \leq d(x, y) \qquad (1)$$

for all $x, y \in X$.

Definition 2.3 [1] A multi-valued mapping $T : E \to CB(E)$ is said to satisfy condition (E_μ) provided that

$$\text{dist}(x, T(y)) \leq \mu\text{dist}(x, T(x)) + d(x, y), \text{ for all } x, y \in E.$$

We say that T satisfies condition (E) whenever T satisfies condition (E_μ) for some $\mu \geq 1$.

Lemma 2.4 *[6] If E is a nonempty closed convex subset of X and $T : E \to CB(E)$ is a multi-valued Suzuki mapping, then T satisfies the condition (E_3).*

We need the following definition of convergence in hyperbolic spaces [5] which is called Δ-convergence.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. Define a function $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \text{ for all } x \in X.$$

The asymptotic radius of a bounded sequence $\{x_n\}$ with respect to a nonempty subset K of X is defined and denoted by

$$r_K(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a nonempty subset K of X is defined and denoted by

$$AC_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}), \text{ for all } y \in K\}.$$

Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-lim of $\{x_n\}$.

The sequence $\{x_n\}$ is called to be *regular* relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_j}\})$ for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. It is known that every bounded sequence in a Banach space has a regular subsequence (see [8]). The proof is metric in nature and carries over to the present setting without change.

Lemma 2.5 *[4] Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η and E is a closed convex subset of X if $\{x_n\}$ is a bounded sequence in E, then the asymptotic center of $\{x_n\}$ is in E.*

Lemma 2.6 *[10] Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq c$, $\limsup_{n\to\infty} d(y_n, x) \leq c$, $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c$ for some $c \geq 0$, then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.7 *[11] Every bounded sequence in a complete CAT(0)(and hence hyperbolic) space has a Δ-convergent subsequence.*

Lemma 2.8 *[7] If $\{x_n\}$ is a bounded sequence in complete uniformly convex hyperbolic space (X, d, W) with $A(\{x_n\}) = \{p\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

Definition 2.9 [8] Let E be a nonempty subset of a metric space (X, d) and $x \in X$. The *radius* of E relative to x is defined by

$$r_x(E) := \sup\{d(x, y) : y \in E\}.$$

The *diameter* of E is defined by

$$\text{diam}(E) := \sup\{d(x, y) : x, y \in E\}.$$

Definition 2.10 [2] Let $T : E \to CB(E)$ be a multi-valued mapping. A sequence $\{x_n\}$ in E is called an *approximate fixed point sequence* (resp. *an approximate endpoint sequence*) for T if $\lim_{n\to\infty} \text{dist}(x_n, T(x_n)) = 0$ (resp. $\lim_{n\to\infty} r_{x_n}(T(x_n)) = 0$). A mapping T is said to have *the approximate fixed point property* (resp. *the approximate endpoint property*) if it has an approximate fixed point sequence (resp. an approximate endpoint sequence) in E.

Lemma 2.11 *[15] Let E be a nonempty subset of X, $\{x_n\}$ be a sequence in E and $T : E \to K(E)$ be a multi-valued mapping. Then $r_{x_n}(T(x_n)) \to 0$ if and only if dist$(x_n, T(x_n)) \to 0$ and diam$(T(x_n)) \to 0$.*

Lemma 2.12 *[13] Let E be a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Next, we also need the following definitions that will be used in the next section.

A sequence $\{x_n\}$ in E is said to be Fejér monotone with respect to E if

$$d(x_{n+1}, q) \leq d(x_n, q) \text{ for all } q \in E \text{ and } n \in \mathbb{N}.$$

Definition 2.13 [13] Let E be a nonempty subset of a hyperbolic space X. A mapping $T : E \to K(E)$ is said to satisfy condition (J) if there exists a nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$, $h(r) > 0$ for $r \in (0,\infty)$ such that

$$r_x(T(x)) \geq h(\mathrm{dist}(x, End(T))) \text{ for all } x \in E.$$

The mapping T is called *semicompact* if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} r_{x_n}(T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $q \in E$ such that $\lim_{n\to\infty} x_{n_j} = q$.

3 Main results

For this part, we start by introducing the notion of the modified SP-iteration process for multi-valued mappings. Notice that it is an improvement of the one so called the SP-iteration process given in Phuengrattana and Suantai [17]. They [17] also showed that SP-iteration process is a generalized version and the sequence generated by the SP-iteration process converges faster than Ishikawa for the class of nondecreasing and continuous functions.

Let X be a hyperbolic space and E be a nonempty convex subset of X, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$ and $T : E \to K(E)$ be a multi-valued mapping. The sequence generated by the modified SP-iteration is defined by $z_1 \in E$,

$$\begin{cases} y_n = W(u_n, z_n, \gamma_n) \\ w_n = W(v_n, y_n, \beta_n) \\ z_{n+1} = W(x_n, w_n, \alpha_n), \end{cases} \tag{2}$$

where $u_n \in T(z_n)$ such that $d(z_n, u_n) = r_{z_n}(T(z_n))$, $v_n \in T(y_n)$ such that $d(v_n, y_n) = r_{y_n}(T(y_n))$ and $x_n \in T(w_n)$ such that $d(x_n, w_n) = r_{w_n}(T(w_n))$.

We need the following important Lemmas that will be used in the sequel.

Lemma 3.1 *Let E be a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping. If $\{z_n\}$ is a sequence in E, then the following holds:*

$$z_n \xrightarrow{\triangle} z,\ dist(z_n, T(z_n)) \to 0 \text{ and } diam(T(z_n)) \to 0 \text{ imply } z \in End(T).$$

Proof. From Lemma 2.5, we obtain that $z \in E$. For each $n \in \mathbb{N}$, we can choose $w_n \in T(z_n)$ such that $d(z_n, w_n) = \mathrm{dist}(z_n, T(z_n))$. By passing throught a subsequence, we may assume that $\{z_n\}$ is regular relative to E. Let $A(E, \{z_n\}) = \{z\}$ and $r = r(E, \{z_n\})$. By similar way in the proof of Lemma 2.12, we obtain that $z \in End(T)$. ■

5

Lemma 3.2 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). Then $\lim_{n\to\infty} d(z_n, q)$ exists for each $q \in End(T)$.*

Proof. Let T be a multi-valued Suzuki mapping and $q \in End(T)$. Therefore,

$$\frac{1}{2}\text{dist}(q, T(q)) = 0 \leq d(q, y_n), \tag{3}$$

$$\frac{1}{2}\text{dist}(q, T(q)) = 0 \leq d(q, w_n), \tag{4}$$

and

$$\frac{1}{2}\text{dist}(q, T(q)) = 0 \leq d(q, z_n), \tag{5}$$

for all $n \in \mathbb{N}$. This implies that

$$H(T(q), T(y_n)) \leq d(q, y_n), \tag{6}$$

$$H(T(q), T(w_n)) \leq d(q, w_n), \tag{7}$$

and

$$H(T(q), T(z_n)) \leq d(q, z_n). \tag{8}$$

Using (2) and (8), we obtain that

$$\begin{aligned} d(y_n, q) &= d(W(u_n, z_n, \gamma_n), q) \\ &\leq (1-\gamma_n)d(u_n, q) + \gamma_n d(z_n, q) \\ &= (1-\gamma_n)\text{dist}(u_n, T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)H(T(z_n), T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)d(z_n, q) + \gamma_n d(z_n, q) \\ &\leq d(z_n, q). \end{aligned} \tag{9}$$

Next, using (2), (6) and (9)

$$\begin{aligned} d(w_n, q) &= d(W(v_n, y_n, \beta_n), q) \\ &\leq (1-\beta_n)d(v_n, q) + \beta_n d(y_n, q) \\ &= (1-\beta_n)\text{dist}(v_n, T(q)) + \beta_n d(y_n, q) \\ &\leq (1-\beta_n)H(T(y_n), T(q)) + \beta_n d(y_n, q) \\ &\leq (1-\beta_n)d(y_n, q) + \beta_n d(y_n, q) \\ &\leq d(y_n, q) \leq d(z_n, q). \end{aligned} \tag{10}$$

Again, using (2), (7) and (10)

$$\begin{aligned} d(z_{n+1}, q) &= d(W(x_n, w_n, \alpha_n), q) \\ &\leq (1-\alpha_n)d(x_n, q) + \alpha_n d(w_n, q) \\ &= (1-\alpha_n)\mathrm{dist}(x_n, T(q)) + \alpha_n d(w_n, q) \\ &\leq (1-\alpha_n)H(T(w_n), T(q)) + \alpha_n d(w_n, q) \\ &\leq (1-\alpha_n)d(w_n, q) + \alpha_n d(w_n, q) \\ &\leq d(w_n, q) \\ &\leq d(z_n, q). \end{aligned} \tag{11}$$

This shows that sequence $\{d(z_n, q)\}$ is decreasing and bounded below. Thus $\lim_{n\to\infty} d(z_n, q)$ exists for each $q \in End(T)$.

■

Next, we prove Δ-convergence theorem for a multi-valued mapping in hyperbolic spaces.

Theorem 3.3 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). Then $\{z_n\}$ Δ-converges to an endpoint of T.*

Proof. First we will prove that $r_{z_n}(T(z_n)) \to 0$. Let $q \in End(T)$. Since T is a multi-valued Suzuki mapping and

$$\frac{1}{2}\mathrm{dist}(q, T(q)) = 0 \leq d(q, z_n)$$

for all $n \in \mathbb{N}$, then

$$H(T(q), T(z_n)) \leq d(q, z_n).$$

From Lemma 3.2, we know that for each $q \in End(T)$, $\lim_{n\to\infty} d(z_n, q)$ exists. Let $\lim_{n\to\infty} d(z_n, q) = t \geq 0$. If $t = 0$, then

$$\begin{aligned} d(z_n, u_n) &\leq d(z_n, q) + d(q, u_n) \\ &= d(z_n, q) + \mathrm{dist}(T(q), u_n) \\ &\leq d(z_n, q) + H(T(q), T(z_n)) \\ &\leq d(z_n, q) + d(z_n, q). \end{aligned}$$

Taking $n \to \infty$ on above inequality, we have

$$\lim_{n\to\infty} r_{z_n}(T(z_n)) = \lim_{n\to\infty} d(z_n, u_n) = 0.$$

If $t > 0$, then

$$\begin{aligned} d(y_n, q) &= d(W(u_n, z_n, \gamma_n), q) \\ &\leq (1-\gamma_n)d(u_n, q) + \gamma_n d(z_n, q) \\ &= (1-\gamma_n)\text{dist}(u_n, T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)H(T(z_n), T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)d(z_n, q) + \gamma_n d(z_n, q) \\ &\leq d(z_n, q). \end{aligned}$$

Letting limsup as $n \to \infty$ on the both sides of above inequality, we have

$$\limsup_{n\to\infty} d(y_n, q) \leq \limsup_{n\to\infty} d(z_n, q) \leq t. \tag{12}$$

From (11), we have $d(z_{n+1}, q) \leq d(w_n, q)$.
Then we obtain that

$$t \leq \liminf_{n\to\infty} d(z_{n+1}, q) \leq \liminf_{n\to\infty} d(w_n, q). \tag{13}$$

From the proof in (10), we have $d(w_n, q) \leq d(y_n, q)$.
Taking liminf as $n \to \infty$ on above inequality and using (13),

$$t \leq \liminf_{n\to\infty} d(y_n, q). \tag{14}$$

Combine (12) and (14), we obtain that

$$\lim_{n\to\infty} d(W(u_n, z_n, \gamma_n), q) = \lim_{n\to\infty} d(y_n, q) = t. \tag{15}$$

Since

$$\begin{aligned} d(u_n, q) &= dist(u_n, T(q)) \\ &\leq H(T(z_n), T(q)) \leq d(z_n, q), \end{aligned}$$

this implies that

$$\limsup_{n\to\infty} d(u_n, q) \leq t. \tag{16}$$

By (15), (16), $\lim_{n\to\infty} d(z_n, q) = t$ together with Lemma 2.6, we have

$$\lim_{n\to\infty} d(u_n, z_n) = 0. \tag{17}$$

From the condition of the modified SP-iteration, so

$$\lim_{n\to\infty} r_{z_n}(T(z_n)) = \lim_{n\to\infty} d(u_n, z_n) = 0. \tag{18}$$

Hence by the both cases we can conclude that $r_{z_n}(T(z_n)) \to 0$. It follows from Lemma 2.11, we have $\text{dist}(z_n, T(z_n)) \to 0$ and $\text{diam}(T(z_n)) \to 0$.

To show that $\{z_n\}$ Δ-converges to an endpoint of T. Now we prove that $W_\omega(z_n) := \cup_{\{s_n\}\subset\{z_n\}} AC(E, \{s_n\}) \subset End(T)$ and $W_\omega(z_n)$ consists of exactly one point. Let $s \in W_\omega(z_n)$. Therefore there exists a subsequence $\{s_n\}$ of $\{z_n\}$ such that $AC(E, \{s_n\}) = \{s\}$. From Lemma 2.5 and Lemma 2.7, there exists a subsequence $\{t_n\}$ of $\{s_n\}$ such that Δ-$\lim_{n\to\infty} t_n = t \in E$. Since $\text{dist}(t_n, T(t_n)) \to 0$ and $\text{diam}(T(t_n)) \to 0$ and it follows from Lemma 3.1, we have $t \in End(T)$ and $\lim_{n\to\infty} d(z_n, t)$ exists by Lemma 3.2. Thus by Lemma 2.8 we have $s = t \in End(T)$. This shows that $W_\omega(z_n) \subset End(T)$. Next, we prove that $W_\omega(z_n)$ consists of exactly one point. Let $\{s_n\}$ be a subsequence of $\{z_n\}$ such that $AC(E, \{s_n\}) = \{s\}$ and $AC(E, \{z_n\}) = \{z\}$. Since $s \in W_\omega(z_n) \subset End(T)$ and from Lemma 3.2, we know that $\{d(z_n, s)\}$ exists. By Lemma 2.8, $z = s$. Therefore the proof is completed.
■

Next, we present the following key lemma for proving the strong convergence theorem.

Lemma 3.4 *Let E be a nonempty closed subset of a complete hyperbolic space X and $\{w_n\}$ be a Fejér monotone sequence with respect to E. Then $\{w_n\}$ converges strongly to an element of E if and only if $\lim_{n\to\infty} dist(w_n, E) = 0$.*

Proof. Assume that $\{w_n\}$ converges strongly to $q \in E$. Thus $\lim_{n\to\infty} d(w_n, q) = 0$. Because $0 \le \text{dist}(w_n, E) \le d(w_n, q)$, therefore $\lim_{n\to\infty} \text{dist}(w_n, E) = 0$.
Conversely, suppose that $\lim_{n\to\infty} \text{dist}(w_n, E) = 0$. Since $\{w_n\}$ is a Fejér monotone sequence with respect to E, we have

$$d(w_{n+1}, q) \le d(w_n, q) \text{ for all } q \in E.$$

Thus $\inf_{q\in E} d(w_{n+1}, q) \le \inf_{q\in E} d(w_n, q)$, which means that $\text{dist}(w_{n+1}, E) \le \text{dist}(w_n, E)$. Therefore $\lim_{n\to\infty} \text{dist}(w_n, E)$ exists. By hypothesis, we obtain that $\lim_{n\to\infty} \text{dist}(w_n, E) = 0$. Next, we show that $\{w_n\}$ is a Cauchy sequence in E. Let $r > 0$. Since $\lim_{n\to\infty} \text{dist}(w_n, E) = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\text{dist}(w_n, E) < \frac{r}{2} \text{ for all } n \ge n_0.$$

Inparticular, $\inf\{d(w_{n_0}, q) : q \in E\} < \frac{r}{2}$.
Therefore there exists $q_0 \in E$ such that $d(w_{n_0}, q_0) < \frac{r}{2}$. For any $n, m \ge n_0$, we have

$$\begin{aligned} d(w_{n+m}, w_n) &\le d(w_{n+m}, q_0) + d(q_0, w_n) \\ &\le d(w_{n_0}, q_0) + d(q_0, w_{n_0}) \\ &\le \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

This means that a sequence $\{w_n\}$ is a Cauchy sequence in E. Since E is a closed subset of a complete hyperbolic space X, we have E is also complete. Then $\{w_n\}$ must be convergent to a point in E. ■

9

Theorem 3.5 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). If T satisfies condition (J), then $\{z_n\}$ converges strongly to an endpoint of T.*

Proof. First, we will show that $End(T)$ is closed. Let $\{z_n\} \subseteq End(T)$ such that $z_n \to z \in E$. We will prove that $z \in End(T)$. Since T is a multi-valued Suzuki mapping, therefore T satisfies condition (E_3). Then

$$\text{dist}(z_n, Tz) \leq 3\text{dist}(z_n, T(z_n)) + d(z_n, z) \to 0 \text{ as } n \to \infty.$$

This implies that $z \in T(z)$. Next, we show that $\{z\} = T(z)$. Take any point $w \in T(z)$. Since T is a multi-valued Suzuki mapping,

$$\frac{1}{2}\text{dist}(z_n, T(z_n)) = 0 \leq d(z_n, z) \text{ implies that } H(T(z_n), T(z)) \leq d(z_n, z).$$

Since $z_n \in End(T)$, we have

$$\begin{aligned} d(w, z) &\leq d(w, z_n) + d(z_n, z) \\ &= \text{dist}(w, T(z_n)) + d(z_n, z) \\ &\leq H(T(z), T(z_n)) + d(z_n, z) \\ &\leq d(z_n, z) + d(z_n, z) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $w = z$. Because $w \in T(z)$ is arbitrary, then $T(z) = \{z\}$, so $z \in End(T)$. Thus $End(T)$ is closed. Next, as in the proof of Theorem 3.3, we have $r_{z_n}(T(z_n)) \to 0$ and it follows from T satisfies condition (J),

$$h(\text{dist}(z_n, End(T))) \leq r_{z_n}(T(z_n)) \to 0.$$

This implies that $\lim_{n\to\infty} h(\text{dist}(z_n, End(T))) = 0$. Since $h : [0, \infty) \to [0, \infty)$ is nondecreasing with $h(0) = 0$, $h(r) > 0$ for $r \in (0, \infty)$, we obtain that $\lim_{n\to\infty} \text{dist}(z_n, End(T)) = 0$. As in the proof of Lemma 3.2 implies that $\{z_n\}$ is Fejér monotone with respect to $End(T)$. By applying Lemma 3.4, we obtain the desired result. ■

Theorem 3.6 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). If T is semicompact, then $\{z_n\}$ converges strongly to an endpoint of T.*

Proof. As in the proof of Theorem 3.3, $r_{z_n}(T(z_n)) \to 0$ and T is semicompact, we may assume a subsequence $z_{n_k} \to z$ for some $z \in E$. Again, as in the proof of Theorem 3.3, we obtain that $r_{z_{n_k}}(T(z_{n_k})) \to 0$. By Lemma 2.11, we also get

$\text{dist}(z_{n_k}, T(z_{n_k})) \to 0$ as $k \to \infty$. Since T is a multi-valued Suzuki mapping, therefore T satisfies condition(E_3). Because of

$$\begin{aligned} \text{dist}(z, T(z)) &\leq d(z, z_{n_k}) + \text{dist}(z_{n_k}, T(z)) \\ &\leq d(z, z_{n_k}) + 3\text{dist}(z_{n_k}, T(z_{n_k})) + d(z_{n_k}, z) \to 0 \text{ as } k \to \infty, \end{aligned}$$

we obtain that $z \in T(z)$. Next, we show that $\{z\} = T(z)$.
Notice that $\frac{1}{2}\text{dist}(z, T(z)) = 0 \leq d(z_{n_k}, z)$ for all $k \in \mathbb{N}$. Since T is a multi-valued Suzuki mapping, we have

$$H(T(z_{n_k}, T(z))) \leq d(z_{n_k}, z).$$

We now let $u \in T(z)$ and choose $w_{n_k} \in T(z_{n_k})$ so that $d(u, w_{n_k}) = \text{dist}(u, T(z_{n_k}))$. For all $k \in \mathbb{N}$, we obtain that

$$\begin{aligned} d(z, u) &\leq d(z, z_{n_k}) + d(z_{n_k}, w_{n_k}) + d(w_{n_k}, u) \\ &\leq d(z, z_{n_k}) + r_{z_{n_k}}(T(z_{n_k})) + \text{dist}(u, T(z_{n_k})) \\ &\leq d(z, z_{n_k}) + r_{z_{n_k}}(T(z_{n_k})) + H(T(z), T(z_{n_k})) \\ &\leq d(z, z_{n_k}) + r_{z_{n_k}}(T(z_{n_k})) + d(z, z_{n_k}). \end{aligned}$$

Taking limit as $k \to \infty$, we get that $z = u$ for all $u \in T(z)$ and so $\{z\} = T(z)$. Hence $z \in End(T)$. By Lemma 3.2, $\lim_{n\to\infty} d(z_n, q)$ exists for each $q \in End(T)$, it follows that $z_n \to z$ as $n \to \infty$. This completes the proof. ■

4 Numerical example

In this section, we give an example shows that there exists a mapping which is a multi-valued Suzuki mapping but is not a nonexpansive mapping. Furthermore, we illustrate that a sequence generated by the modified SP-iteration process (2) converges to an endpoint of the multi-valued Suzuki mapping.

Example 4.1 Let $X = \mathbb{R}$ with metric defined by $d(x, y) = |x-y|$ and $E = [0, 3]$. Define $W : X^2 \times [0, 1] \to X$ by $W(x, y, \alpha) := \alpha x + (1 - \alpha)y$ for all $x, y \in X$ and $\alpha \in [0, 1]$. Then (X, d, W) is a complete uniformly hyperbolic space with a monotone modulus of uniform convexity and E is a nonempty compact convex subset of X. Let $T : E \to K(E)$ defined by

$$Tz = \begin{cases} \{0\}, & z \neq 3; \\ \{1\}, & z = 3. \end{cases}$$

By [19] showed that the mapping T is a Suzuki mapping. But T is not a nonexpansive mapping if we take $x = 2.9$ and $y = 3$. Moreover, $End(T) = \{0\}$. For initial point $z_0 = 0.1$ and $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{\sqrt{3n + 7}}$. Therefore $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$. Set stop parameter to $|z_n - 0| \leq 10^{-12}$, where 0 is an endpoint of T. By using MATLAB, we compute the sequence generated by the modified SP-iteration process (2) converging to 0 as in Table 1 and Figure 1.

11

iterate	the modified SP-iteration process
z_0	0.1
z_1	0.024068308483
z_2	0.006367770608
z_3	0.001780515413
z_4	0.000516698713
:	:
z_{20}	0.000000000008
z_{21}	0.000000000003
z_{22}	0.000000000000

Table 1: Sequences generated by SP-iteration process

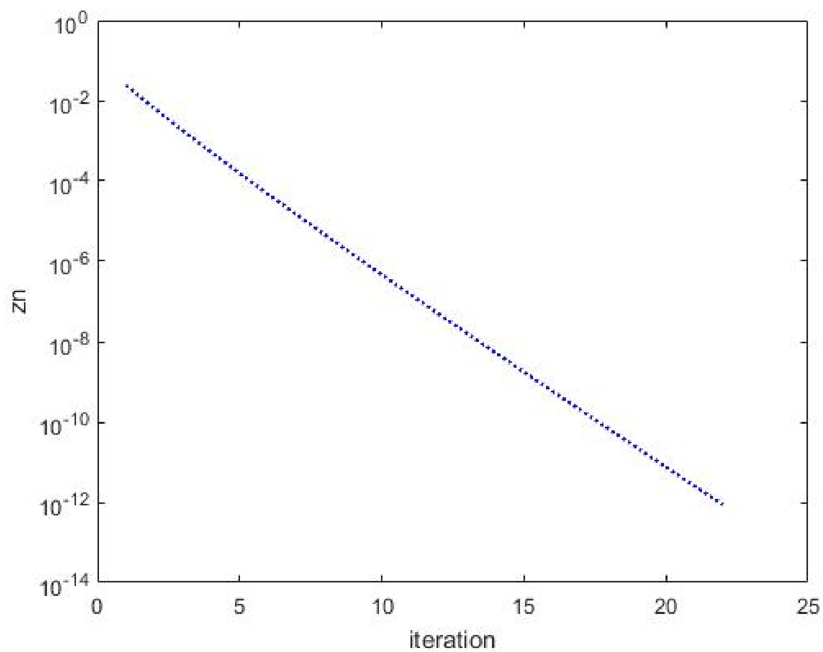


Figure 1 Convergence of iterative sequences generated by SP-iteration process

Acknowledgement

The first and the third authors would like to express their deep thanks to Naresuan University for supporting this research.

References

[1] A. Abkar, M.Eslamian, Common fixed point results in CAT(0) spaces, *Nonlinear Anal.*, 74, 1835-1840(2011).

[2] A. Amini-Harandi, Endpoints of set-valued contractions in metric spaces, *Nonlinear Anal.*, 72, 132-134(2010).

[3] J.P. Aubin, J. Siegel, Fixed points and stationary points of dissipative multi-valued maps, *Proc. Amer. Math. Soc.*, 78, 391-398(1980).

[4] S. Dhompongsa, W. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, *Nonlinear Convex Anal.*, 8,35-45(2007).

[5] S. Dhompongsa, B. Panyanak, On Δ-convergence theorem in CAT(0) spaces, *Comput. Math. Appl.*, 56(10), 2572-2579(2008).

[6] R. Espinola, P. Lorenzo, A. Nicolae, Fixed points selections and common fixed points for nonexpansive-type mappings, *Math. Anal. Appl.*, 382,503-515(2011).

[7] J. García-Falset, E. Llorens-Fuster, T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, *Math. Anal. Appl.*, 375, 185-195(2011).

[8] K. Goebel, W.A. Kirk, Topics in metric fixed point theory, *Cambridge Univ.Press, Cambridge* (1990).

[9] M.S. Kahn, K.R. Rao, Y.J. Cho, Common stationary points for set-valued mappings, *Internat. Math. Math. Sci.*, 16, 733-736(1993).

[10] A.R. Khan, H. Fukhar-ud-din, M.A.A. Kuan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, *Fixed Point Theory Appl.*, 2012, 54(2012).

[11] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear anal.*, 68, 3689-3696(2008).

[12] U. Kohlenbach, Some logical metatheorems with applications in functional analysis, *Trans. Am. Math. Soc.*, 357, 89-128(2005).

[13] A. Kudtha, B. Panyanak, Common endpoints for Suzuki mappings in uniformly convex hyperbolic spaces, *Thai.J. Math*, 159-168, A.M.M.(2017).

[14] L. Leustean, A quadratic rate of asymptotic regularity for CAT(0) spaces, *Math. Anal. Appl.*, 235, 386-399(2007).

[15] B. Panyanak, Endpoints of multivalued nonexpansive mappings in geodesic spaces, *Fixed Point Theory Appl.*, 2015, 147(2015).

[16] B. Panyanak, Approximating endpoints of multi-valued nonexpansive mappings in Banach spaces, *Fixed Point Theory Appl.*(2018).

[17] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, *Comput. Appl. Math.*, 235,3006-3014(2011).

[18] S.L. Singh, S.N. Mishra, Coincidence points, hybrid fixed and stationary points of orbitally weakly dissipative maps, *Math. Japan.*, 39,451-459(1994).

[19] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *Math. Anal. Appl.*, 340, 1088-1095(2008).

[20] Y. Yamamoto, A path following algorithm for stationary point problems, *Oper. Res. Soc. Japan*, 30,181-199(1987).

[21] Y. Yamamoto, Fixed point algorithms for stationary point problems, *Mathematical Programming*(Tokyo, 1988), *Math. Appl.* (Japanese Ser.), *SCIPRESS*, Tokyo 6, 283-307(1989).

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 5, 2020

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 5, 2020

(continued)