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Computational Mathematics

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Control problems for semilinear impulsive differential control systems

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Abstract

In this paper, we establish the approximate controllability for the semilinear impulsive differential equation in relation to the the corresponding linear control system based on the regularity for the equation under natural assumptions such as the local Lipschitz continuity of nonlinear term.

Keywords: approximate controllability, semilinear equation, ,impulsive differential equation, local lipschitz continuity, controller operator, reachable set

AMS Classification Primary 35B37; Secondary 93C20

1 Introduction

In this paper, we are concerned with the approximate controllability for the semilinear impulsive control system in Hilbert spaces:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \quad t \in (0, T], \quad t = t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Let H be identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections be continuous. Here, A is the operator associated with a sesquilinear form $a(\cdot, \cdot)$ defined on $V \times V$ satisfying Gårding's inequality:

$$(Au, v) = a(u, v), \quad u, \ v \in V$$

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where V is a Hilbert space such that $V \subset H \subset V^*$. Then $-A$ generates an analytic semigroup in both H and V^*(see [1, Theorem 3.6.1]) and so the equation (1.1) may be considered as an equation in H as well as in V^*. The nonlinear operator f from $[0, T] \times V$ to H is assumed to be locally Lipschitz continuous with respect to the second variable. Let U be a Banach space of control variables and the controller operator B be a bounded linear operator from the Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$. The impulsive condition

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m,$$

is a combination of traditional evolution systems. Let $x(t; f, u)$ be a solution of the equation (1.1) associated with a nonlinear term f and a control u. We will show the approximate controllability for the equation (1.1), namely that the reachable set $R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\}$ is a dense subset of H. This kind of equations arise naturally in biology, in physics, control engineering problem, etc.

In the first part of this paper we establish the wellposedness and regularity property for the following equation:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0, T], \quad t = t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{1.2}$$

The regularity for the semilinear heat equations has been developed as seen in Barbu [2] and [3, 4, 5, 6].

In this paper, based on the regularity for (1.2), we intend to establish the approximate controllability for (1.1). Approximate controllability for semilinear control systems can be founded in [7-15]. Similar considerations of linear and semilinear systems have been dealt with in many references, linear problems in the book [15] and Nakagiri [14], semilinear cases with the uniform bounded nonlinear term in [16], and with the uniform Lipschtz continuous nonlinear term in [3, 17, 18, 19]. However, there are few papers treating the systems with local Lipschipz continuity, we can just find a recent article Wang [20]. Among these literatures, in [17, 20], they assumed that the semigroup $S(t)$ generated by A is compact in order to guarantee the compactness of the solution mapping, and investigated the approximate controllability for the equation (1.1).

In this paper, in order to show that the main result of Naito [17] is extended to the nonlinear differential equation, we assume that the embedding $D(A) \subset V$ is compact instead of the compact property of semigroup used in [17, 21]. Then by virtue of the result in Aubin [22], we can take advantage of the fact that the solution mapping $u \in L^2(0, T; U) \mapsto x(T; f, u)$ is compact. Under natural assumptions such as the local Lipschtiz continuity of nonlinear term, we obtain the approximate controllability for the equation (1.1) when the corresponding linear system is approximately controllable.

The paper is organized as follows. In section 2, the results of general linear evolution equations besides notations and assumptions are stated. In section 3, we investigate the approximate controllability for the problem (1.1). The approach used here is similar to that developed in [1, 3] on the general semilnear evolution equations, which is an important role to extend the theory of practical nonlinear partial differential equations.

2 Regularity for semilinear impulsive systems

The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. We assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u, \ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. Then we consider the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.3}$$

where each space is dense in the next one which continuous injection. It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity, we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \text{Re}\,\lambda \ge 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0,T;X)$ is the collection of all strongly measurable square integrable functions from $(0,T)$ into X and $W^{1,2}(0,T;X)$ is the set of all absolutely continuous functions on $[0,T]$ such that their derivative belongs to $L^2(0,T;X)$. $C([0,T];X)$ will denote the set of all continuously functions from $[0,T]$ into X with the supremum norm. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0,T;D(A)) \cap W^{1,2}(0,T;H),$$
$$\mathcal{W}_1(T) = L^2(0,T;V) \cap W^{1,2}(0,T;V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0,T];V), \quad \mathcal{W}_1(T) \subset C([0,T];H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \le M_0||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \le M_0||x||_{\mathcal{W}_1(T)}. \tag{2.4}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \le M, \quad ||s(t)||_* \le M.$$

Let f be a nonlinear mapping from V into H. We need to impose the following conditions on nonlinear term f.

Assumption (F). There exists a function $L : \mathbb{R}_+ \to \mathbb{R}$ such that $L(r_1) \le L(r_2)$ for $r_1 \le r_2$ and

$$|f(t,x)| \le L(r), \quad |f(t,x) - f(t,y)| \le L(r)||x-y||$$

hold for any $t \in [0,T]$, $||x|| \le r$ and $||y|| \le r$.

Assumption (I). The functions $I_k : V \to H$ are continuous and there exist positive constants $L(I_k)$ and $\beta \in (1/3, 1]$ such that

$$|A^\beta I_k(x)| \le L(I_k)||x||, \quad |A^\beta I_k(x) - I_k(y)| \le L(I_k)||x-y||, \quad k = 1,2,\cdots,m$$

for each $x, y \in V$, and

$$||x(t_k^-)|| \le K, \quad k = 1,2,\cdots,m.$$

From now on, we establish the following results on the local solvability of the following equation;

$$\begin{cases} x'(t) + Ax(t) = f(t,x(t)) + k(t), \quad t \in (0,T], \quad t \ne t_k, \\ \ \ k = 1,2,\cdots,m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)),\ k = 1,2,\cdots,m, \\ x(0) = x_0. \end{cases} \tag{2.5}$$

Let us rewrite $(Fx)(t) = f(t,x(t))$ for each $x \in L^2(0,T;V)$. Then there is a constant, denoted again by $L(r)$, such that

$$||Fx||_{L^2(0,T;H)} \le L(r)\sqrt{T}, \quad ||Fx_1 - Fx_2||_{L^2(0,T;H)} \le L(r)||x_1 - x_2||_{L^2(0,T;V)}$$

hold for $x_1,\ x_2 \in B_r(T) = \{x \in L^2(0,T;V) : ||x||_{L^2(0,T;V)} \le r\}$. Here, we note that by using interpolation theory, we have that for any $t > 0$,

$$L^2(0,t;V) \cap W^{1,2}(0,t;V^*) \subset C([0,t];H).$$

Thus, for any $t > 0$, there exists a constant $c > 0$ such that

$$||x||_{C([0,t];H)} \le c||x||_{L^2(0,t;V)\cap W^{1,2}(0,t;V^*)}. \tag{2.6}$$

Let

$$0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m = T.$$

Then by Assumption (I) and (2.5), it is immediately seen that

$$x \in W^{1,2}(t_i, t_{i+1}; V^*), \quad i = 0, \cdots, m-1.$$

Thus by virtue of Assumption (I) and (2.6), we may consider that there exists a constant $C_3 > 0$ such that

$$\max_{0 \le t \le T} \{|x(t)| : x \text{ is a solution of (2.5)}\} \le C_3 ||x||_{L^2(0,T:V)}. \tag{2.6}$$

With the notations (2.2), (2.3), we have

$$(V, V^*)_{1/2,2} = H, \quad (D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [23]). From now on, we establish the following results on the solvability of the equation (2.5).

Theorem 2.1. *1) Let Assumption (F) be satisfied. Assume that $x_0 \in H$, $k \in L^2(0, T; V^*)$. Then, there exists a time $T_0 \in (0, T)$ such that the equation (2.5) admits a solution*

$$x \in W_1(T_0) \subset C([0, T_0]; H). \tag{2.7}$$

2) Under Assumption (F) for the nonlinear mapping f, there exists a unique solution x of (2.5) such that

$$x \in \mathcal{W}_1(T) \equiv L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H), \quad T > 0.$$

for any $x_0 \in H$, $k \in L^2(0, T; V^*)$. Moreover, there exists a constant C_1 such that

$$||x||_{\mathcal{W}_1(T)} \le C_1(1 + |x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

where C_1 is a constant depending on T.
3) Let Assumptions (F) and (I) be satisfied and $(x_0, k) \in H \times L^2(0, T; V)$. Then the solution x of the equation (2.5) belongs to $x \in \mathcal{W}_1 \equiv L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ and the mapping

$$H \times L^2(0, T; V^*) \ni (x_0, k) \mapsto x \in \mathcal{W}_1(T) \tag{2.9}$$

is continuous.

Corollary 2.1. *Suppose that $k \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.10}$$

Proof. From Theorem 2.3 of [24], it follows that there exists a $C > 0$ such that

$$||x||_{L^2(0,T;D(A))} \leq C||k||_{L^2(0,T;H)}. \tag{2.11}$$

Moreover, we have

$$||x||^2_{L^2(0,T;H)} \leq M \int_0^T t \int_0^t |k(s)|^2 ds dt \leq M \frac{T^2}{2} \int_0^T |k(s)|^2 ds. \tag{2.12}$$

Since

$$(D(A), H)_{1/2,2} = V,$$

there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||^{1/2}_{D(A)} |u|^{1/2}. \tag{2.13}$$

Thus, by (2.11), (2.12) and (2.13), if $C_2 = C_0\sqrt{CT}(M/2)^{1/4}$, then the inequality (2.10) holds. □

3 Approximate Controllability

Let U be a Banach space of control variables. Here B is a linear bounded operator from $L^2(0,T;U)$ to $L^2(0,T;H)$, which is called a controller. Consider the following nonlinear impulsive control systems.

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \quad t \in (0,T], \\ x(0) = x_0. \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \ k = 1, 2, \cdots, m. \end{cases} \tag{3.1}$$

Let $x(T; f, u)$ be a state value of the system (3.1) at time T corresponding to the nonlinear term f and the control u. Let $S(\cdot)$ be the analytic semigroup generated by $-A$. Then the solution $x(t; f, u)$ can be written as

$$x(t; f, u) = S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s, f, u)) + (Bu)(s)\}ds + \sum_{0<t_k<t} S(t-s)I_k(x(t_k^-)),$$

and in view of Theorem 2.1

$$||x(\cdot; f, u)||_{\mathcal{W}_1(T)} \leq C_1(1 + |x_0| + ||B|| ||u||_{L^2(0,T;U)}). \tag{3.2}$$

We define the reachable sets for the system (3.1) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0,T;U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0,T;U)\}.$$

Definition 3.1. *The system (3.1) is said to be approximately controllable at time T if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T;f,u)$ of (3.1) satisfies $|x(T;f,u) - x_1| < \epsilon$, that is, $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H.*

We define a linear bounded operator $\hat{S}$ from $L^2(0,T;H)$ to H by

$$\hat{S}p = \int_0^T S(T-t)p(t)dt,$$

for $p(\cdot) \in L^2(0,T;H)$.
Assumption (B) For any $\varepsilon > 0$, $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} |\hat{S}p - \hat{S}Bu| \le \varepsilon \\ ||Bu||_{L^2(0,t;H)} \le q_1||p||_{L^2(0,t;H)}, \quad 0 \le t \le T \end{cases}$$

where q is a constant independent of p.

Assumption (F1) The nonlinear operator f is a nonlinear mapping of $[0,T] \times H$ into H satisfying the following. There exists a constant $L_1 = L_1(r) > 0$ such that

$$|f(t,x) - f(t,y)| \le L_1||x-y||, \quad t \in [0,T],$$

hold for $||x|| \le r$ and $||y|| \le r$.

Assumption (H) We assume the following inequality condition:

$$max\{q,1\}\{1-M_2\}^{-1}C_2L_1\sqrt{T} < 1.$$

where C_2 is the constant in (2.10),

$$M_2 = C_2\sqrt{T}L_1 + (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3T^{3\beta/2}\sum_{0 \le t_k \le T} L(I_k).$$

Lemma 3.1. *Let u_1 and u_2 be in $L^2(0,T;U)$. Then under Assumption(B) and Assumption$(F1)$, one has that, for $0 \le t \le T$,*

$$||x(t:f,u_1) - x(t:f,u_2)||_{L_2(0,T;V)} \le \{1-M_2\}^{-1}C_2\sqrt{t}||Bu_1 - Bu_2||_{L^2(0,T;H)}. \quad (3.3)$$

Proof. Let $x_1(t) = x(t : f, u_1)$ and $x_2(t) = x(t : f, u_2)$. Then for $0 \le t \le T$,we have

$$\begin{aligned} x_1(t) - x_2(t) = &\int_0^t S(t-s)\{f(s, x_1(s)) - f(s, x_2(s))\}ds \\ &+ \int_0^t S(t-s)\{Bu_1 - Bu_2\}ds \\ &+ \sum_{0 \le t_k \le T} S(t-s)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}. \end{aligned} \tag{3.4}$$

By $Assumption(F1)$ and (2.10), we obtain

$$||\int_0^t S(t-s)\{f(s, x_1(s)) - f(s, x_2(s))\}ds||_{L^2(0,t;V)} \le C_2\sqrt{t}L_1||x_1 - x_2||_{L^2(0,t;V)}.$$

Moreover, by Lemma 2.5 of (2.11) and Theorem 3.1, we have

$$||\int_0^t S(t-s)\{Bu_1 - Bu_2\}ds||_{L^2(0,t;V)} \le C_2\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,t;H)}$$

and

$$\begin{aligned} &||\sum_{0 \le t_k \le t} S(t-s)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}||_{L^2(0,t;V)} \\ &\le (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3 t^{3\beta/2} \sum_{0 \le t_k \le t} L(I_k)||x_1(t_k^-) - x_2(t_k^-)||_{L^2(0,t;V)}. \end{aligned}$$

Thus, from (3.4) it follows that

$$\begin{aligned} &||x(t; f, u_1) - x(t; f, u_2)||_{L^2(0,T;V)} \\ &\le C_2\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,T;H)} + C_2\sqrt{T}L_1||x_1 - x_2||_{L^2(0,T;V)} \\ &+ (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3 t^{3\beta/2} \sum_{0 \le t_k \le t} L(I_k)||x_1(t_k^-) - x_2(t_k^-)||_{L^2(0,T;V)}. \end{aligned}$$

$\square$

Theorem 3.1. *Under Assumptions (B),(F1), and (H) the system(4.1) is approximately controllable on* $[0, T]$.

Proof. The reachable set for the system(4.1) is given by

$$R_T = \{x(T; f, u) : u \in L^2(0, T; U)\}.$$

We will show that $D(A) \subset \overline{R_T(f)}$, i.e., for given $\varepsilon > 0$ and $\xi_T \in D(A)$, there exists $u \in L^2(0, T; U)$ such that

$$|\xi_T - x(T; f, u)| < \varepsilon, \tag{3.5}$$

where

$$x(T;,f,u) = S(T)x_0 + \int_0^T S(T-s)\{f(s,x(s,f,u)) + (Bu)(s)\}ds + \sum_{0<t_k<T} S(T-s)I_k(x(t_k^-)). \tag{3.6}$$

As $\xi_T \in D(A)$ there exists a $p \in L^2(0,T;H)$such that

$$\hat{S}p = \xi_T - S(T)x_0,$$

for instance, take $p(s) = (\xi_T - sA\xi_T) - S(s)x_0/T$. Let $u_1 \in L^2(0,T;U)$be arbitrary fixed. Since by Assumption (B) there exists $u_2 \in L^2(0,T;U)$ such that

$$|\hat{S}(p - f(\cdot, x(\cdot; f, u_1))) - \hat{S}Bu_2| < \frac{\varepsilon}{4}, \tag{3.7}$$

it follows that

$$|\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_1)) - \hat{S}Bu_2| < \frac{\varepsilon}{4}. \tag{3.8}$$

We can also choose $w_2 \in L^2(0,T;U)$ by Assumption (B) such that

$$|\hat{S}(f(\cdot, x(\cdot; f, u_2)) - f(\cdot, x(\cdot; f, u_1))) - \hat{S}Bw_2| < \frac{\varepsilon}{8} \tag{3.9}$$

$$||Bw_2||_{L^2(0,T;H)} \leq q||f(\cdot, x(\cdot; f, u_2)) - f(\cdot, x(\cdot; f, u_1))||_{L^2(0,T;H)}.$$

Choose a constant r_1 satisfying

$$||x(\cdot; f, u_1)||_{C([0,T];H)} \leq r_1, ||x(\cdot; f, u_2)||_{C([0,T];H)} \leq r_1.$$

Therefor, in view of Lemma 3.1 and Assumption (B)

$$\begin{aligned} ||Bw_2||_{L^2(0,T;H)} &\leq q||f(s, x(s; f, u_2)) - f(s, x(s; f, u_1))||_{L^2(0,T;H)} \\ &\leq qL_1||x(t; f, u_1) - x(t; f, u_2)||_{L^2(0,T;V)} \\ &\leq q\{1 - M_2\}^{-1}C_2L_1\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,T;H)}. \end{aligned} \tag{3.10}$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$|\hat{S}(f(\cdot, x(\cdot; f, u_3)) - f(\cdot, x(\cdot; f, u_2))) - \hat{S}Bw_3| < \frac{\varepsilon}{8}$$

$$||Bw_3||_{L^2(0,T;H)} \leq q||f(\cdot, x(\cdot; f, u_3)) - f(\cdot, x(\cdot; f, u_2))||_{L^2(0,T;H)}.$$

Let r_2 be a constant satisfying $r_2 \geq r_1$ and

$$||x(\cdot; f, u + 3)||_{C([0,T];H)} \leq r_2.$$

Then, in a similar way to (3.10) we have

$$\begin{aligned}||Bw_3||_{L^2(0,T;H)} &\le q||f(s,x(s;f,u_3)) - f(s,x(s;f,u_2))||_{L^2(0,T;H)}\\ &\le qL_1||x(t;f,u_3) - x(t;f,u_2)||_{L^2(0.T;V)}\\ &\le q\{1-M_2\}^{-1}C_2L_1\sqrt{T}||Bu_2 - Bu_3||_{L^2(0,T;H)}\\ &\le (q\{1-M_2\}^{-1}C_2L_1\sqrt{T})^2||Bu_1 - Bu_2||_{L^2(0,T;H)}.\end{aligned}$$

By proceeding with this process and from

$$\begin{aligned}&||B(u_n - u_{n+1})||_{L^2(0,T;H)}\\ &\quad = ||Bw_n||_{L^2(0,T;H)} \le (q\{1-M_2\}^{-1}C_2L_1\sqrt{T})^{n-1}||B(u_2-u_1)||_{L^2(0,T;H)}.\end{aligned}$$

Here, nothing that Assumption (H) is equivalent to

$$q\{1-M_2\}^{-1}C_2L_1\sqrt{T} < 1,$$

it follows that there exists $u^* \in L^2(0,T;H)$ such that

$$\lim_{n\to\infty} Bu_n = u^* \quad \text{in} \quad L^2(0,T;H).$$

From(3.8),(3.9) it follow that

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_2)) - \hat{S}Bu_3|\\ &\quad = |\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_1)) - \hat{S}Bu_2 + \hat{S}Bw_2\\ &\quad - [\hat{S}f(\cdot,x(\cdot;f,u_2)) - \hat{S}f(\cdot,x(\cdot;f,u_1))]|\\ &\quad < (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.\end{aligned}$$

By choosing $w_n \in L^2(0,T;U)$ by Assumption (B), such that

$$|\hat{S}(f(\cdot,x(\cdot;f,u_n)) - f(\cdot,x(\cdot;f,u_{n-1}))) - \hat{S}Bw_n| < \frac{\varepsilon}{2^{n+1}}$$

putting $u_{n+1} = u_n - w_n$ we have

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_n)) - \hat{S}Bu_{n+1}|\\ &\quad < (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1,2,....\end{aligned}$$

Therefor, for $\varepsilon > 0$ there exists integer N such that

$$|\hat{S}Bu_{N+1} - \hat{S}Bu_N| < \frac{\varepsilon}{2},$$

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_N)) - \hat{S}Bu_N|\\ &\quad \le |\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_N)) - \hat{S}Bu_{N+1}| + |\hat{S}Bu_{N+1} - \hat{S}Bu_N|\\ &\quad \le (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \le \varepsilon.\end{aligned}$$

Thus, the system (3.1) is approximately controllable on $[0,T]$ as N tends to infinity. □

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Homoclinic solutions for a class of difference equations with asymptotically linear nonlinearity

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Abstract

A class of difference equations with asymptotically linear nonlinearity are considered in this paper. The existence of homoclinic solutions of the equations are obtained by using generalized saddle point theorem.

Key words: Generalized saddle point theorem; Difference equations; $(PS)_c$ sequence; Homoclinic solutions.

1 Introduction

In this paper, we consider the following difference equation

$$Lu_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n$$

is a Jacobi operator ([14]), here $\{a_n\}$ and $\{b_n\}$ are real valued T-periodic sequences, and T is a positive integer.

As in the literature, a solution $u = \{u_n\}$ of (1.1) is homoclinic solution if

$$\lim_{|n|\to\infty} u_n = 0. \tag{1.2}$$

This problem appears in the following discrete nonlinear schrödinger equation

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - f_n(\psi_n), \quad n \in \mathbb{Z}, \tag{1.3}$$

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where

$$\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$$

is the discrete one-dimension Laplacian. And the potential $V = \{v_n\}$ is real valued T-periodic sequences, i.e., $v_{n+T} = v_n$, for all $n \in \mathbb{Z}$. Moreover, we assume that the nonlinearity $f_n(u)$ is gauge invariant, i.e.,

$$f_n(e^{i\theta}u) = e^{i\theta} f_n(u), \quad \theta \in \mathbb{R}.$$

We consider special solutions of (1.3)

$$\psi_n = u_n e^{-i\omega t},$$

where $\omega \in \mathbb{R}$ is the temporal frequency and $\{u_n\}$ is a real valued sequence such that

$$\lim_{|n|\to\infty} \psi_n = 0.$$

Such solutions are called solitons. Inserting the soliton Ansatz into (1.3), then

$$-\Delta u_n + v_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{1.4}$$

and

$$\lim_{|n|\to\infty} u_n = 0 \tag{1.5}$$

holds. Therefore, in order to looking for solitons of equation (1.3), we just need to get the homoclinic solutions of equation (1.4), which is a special case of (1.1) with $a_n \equiv -1$ and $b_n = 2 + v_n$.

It is well known that the operator L is a bounded and self-adjoint operator in l^2. Its spectrum $\sigma(L)$ is a union of a finite number of closed intervals and the complement $\mathbb{R}\backslash\sigma(L)$ consists of a finite number of open intervals called spectral gaps. Two of them are semi-infinite (see [14]). In particular, $T = 1$, then finite gaps do not exist. In general, finite gaps do exist. The most interesting case of equation (1.1) is when the frequency ω belongs to a finite gap. The solitons of (1.3) with the temporal frequency ω belonging to a spectral gap, in particular to a finite gap are important. Such solitons are called gap solitons. Fix any finite spectral gap and denote it by (α, β).

Discrete nonlinear schrödinger equation (DNLS) is one of the most important inherently discrete models, It appears in a great variety of applications, such as nonlinear optics, solid state, condensed matter physics and biology (see [1–3, 5, 13] and reference therein). It also has been successfully applied to the modeling of localized pulse propagation in optical fibers and wave guides, to the study of energy relaxation in solids, to the behavior of amorphous material, to the modeling of self-trapping of vibrational energy in proteins or studies related to the denaturation of the DNA double strand ([6, 7, 18]). In the past decade, the periodic DNLS equations have been considered in the physics literature ([15]). For example, results

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on numerical simulation of gap solitons in a particular periodic DNLS equation are obtained in [4].

With the development of variational techniques, solitons of the periodic DNLS equations have become a hot topic. The existence of solitons for the periodic DNLS equations with superlinear nonlinearity (see [10, 11] and reference therein) and with saturable nonlinearity ([16, 17]) have been studied, respectively. Discrete soliton is a kind of homoclinic solutions. In this paper, we employ generalized saddle point theorem developed by Liu and Shen in [9] and obtain homoclinic solutions of equation (1.1).

The organization of this paper is as follows. In Section 2, we introduce the functional, and its critical points are solutions of the problem and remind a critical point theorem, then present the main result. The detailed proofs of the main result is given in Section 3.

2 Preliminaries and main results

Throughout this paper, we assume that

(V) $\omega \notin \sigma(L)$ and $\omega \in (\alpha, \beta)$.

(f_1) $f_n \in C(\mathbb{R}, \mathbb{R})$, $f_n(u)u \geq 0$ for all $u \in \mathbb{R}$.

(f_2) Assume that f_n is asymptotically linear at infinity, i.e.,

$$\lim_{|u|\to\infty} \frac{f_n(u)}{u} = 0. \tag{2.1}$$

(f_3) $f_n(u) = o(u)$ as $u \to 0$.

To study the homoclinic solutions, we consider the real sequence spaces

$$l^p = \left\{ u = \{u_n\}_{n\in\mathbb{Z}} : \forall\, n \in \mathbb{Z}, u_n \in \mathbb{R}, \|u\|_{l^p} = \left(\sum_{n\in\mathbb{Z}} |u_n|^p\right)^{\frac{1}{p}} < \infty \right\}. \tag{2.2}$$

Between l^p spaces the following elementary embedding holds,

$$l^q \subset l^p, \quad \|u\|_{l^p} \leq \|u\|_{l^q}, \quad 1 \leq q \leq p \leq \infty. \tag{2.3}$$

To state our results, we fix some notation. Let

$$A = L - \omega \ \text{ and } \ E = l^2(\mathbb{Z}).$$

Consider the functional J defined on E by

$$J(u) = \frac{1}{2}\ (Au, u) - \sum_{n\in\mathbb{Z}} F_n(u_n), \tag{2.4}$$

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where $(\cdot,\cdot)$ is the inner product in E, $\|\cdot\|$ is the corresponding norm in E. $F_n(u)$ is the primitive function of $f_n(u)$, *i.e.*,

$$F_n(u)=\int_0^u f_n(s)ds.$$

Standard arguments show that the functional $J\in C^1(E,\mathbb{R})$ and equation (1.1) is easily recognized as the corresponding Euler-Lagrange equation for J. Thus, critical points of J are solutions of equation (1.1).

It is easy to get the derivative of J,

$$(J'(u),v)=(Au,v)_E-\sum_{n\in\mathbb{Z}} f_n(u_n)v_n,\quad \forall\, v\in E. \tag{2.5}$$

By (V), then we have the orthogonal decomposition $E=E^+\oplus E^-$ corresponding to the spectral decomposition of A with respect to the positive and negative part of the spectrum, and

$$(Au,u)_E\geq(\beta-\omega)\|u\|_E^2,\qquad u\in E^+,$$

$$(Au,u)_E\leq(\alpha-\omega)\|u\|_E^2,\qquad u\in E^-.$$

For any $u,v\in E$, letting $u=u^++u^-$ with $u^\pm\in E^\pm$ and $v=v^++v^-$ with $v^\pm\in E^\pm$, we can define an equivalent inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on E by

$$(u,v)=(Au^+,v^+)_E-(Au^-,v^-)_E \text{ and } \|u\|=(u,u)^{\frac{1}{2}},$$

respectively. So J can be rewritten as

$$J(u)=\frac{1}{2}\|u^+\|^2-\frac{1}{2}\|u^-\|^2-\sum_{n\in\mathbb{Z}}F_n(u_n)\equiv\frac{1}{2}\|u^+\|^2-\frac{1}{2}\|u^-\|^2-I(u). \tag{2.6}$$

Note that if ω lies in a finite spectral gap, then $\dim E^-=\infty$ and the problem (1.1) and (1.2) is strongly indefinite. Now our main result can be stated as the following:

Theorem 2.1. *Suppose that conditions (V), $(f_1)-(f_3)$ are satisfied, then equation (1.1) at least has one solution.*

Let $R>0$. Set

$$M=\{u\in E^-:\|u\|\leq R\}.$$

Let $\{e_k\}$ be a total orthonormal sequence in E^-, we define a norm on E^- by

$$\|u\|_{E^-}=\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}|<u,e_k>|.$$

Let $P_\pm : E \to E^\pm$ be the orthogonal projection of E onto $E^\pm$. We denote by τ the topology on E generated by the norm

$$\|u\|_\tau = \max\left(\|P_+u\|, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} | < P_-u, e_k > |\right).$$

Remark 2.1. Note that if $u_n \xrightarrow{\tau} u$, then $P_+u_n \to P_+u$ and $P_-u_n \rightharpoonup P_-u$.

Definition 2.1. Let $J \in C^1(E)$, we say J is $\tau-$upper semicontinuous if $u_n \xrightarrow{\tau} u$ implies

$$J(u) \geq \overline{\lim_{n\to\infty}} J(u_n).$$

Definition 2.2. Let $J \in C^1(E)$, we say J' is weakly sequentially continuous, if $u_n \rightharpoonup u$ implies $J'(u_n) \to J'(u_n)$, as $n \to \infty$.

The purpose of this paper is to use the generalized saddle point theorem to solve some strongly indefinite problems with asymptotically linear nonlinearity. The following lemma is the generalized saddle point theorem taken from [9] and will play an important role in the proofs of our main results.

Lemma 2.1. *Assume that $J \in C^1(E, \mathbb{R})$ is $\tau-$upper semicontinuous and J' is weakly sequentially continuous. If*

$$b := \inf_{E^+} J > \sup_{\partial M} J, \quad d = \sup_{M} J < \infty,$$

then for some $c \in [b, d]$, there is a sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \textit{ as } n \to \infty. \tag{2.7}$$

Such a sequence is called a Palais-Smale sequence on the level c, or a $(PS)_c$ sequence.

3 Proofs of main results

Lemma 3.1. *Assume that (V) and $(f_1) - (f_3)$ are satisfied. Then J is $\tau-$upper semicontinuous, and J' is weakly sequentially continuous.*

Proof. Let $u^{(k)} \xrightarrow{\tau} u$ and $c = \overline{\lim_{k\to\infty}} J(u^{(k)})$. Then there is a subsequence, still denoted by $\{u^{(k)}\}$ such that $J(u^{(k)}) \to c$. By Remark 2.1 we have

$$u^{(k)+} \to u^+ \quad \text{and} \quad u^{(k)-} \rightharpoonup u^-, \quad \text{as} \quad k \to \infty. \tag{3.1}$$

Passing to a subsequence if necessary, we have $u_n^{(k)} \to u_n$ for all $n \in \mathbb{Z}$, as $k \to \infty$, hence, $F_n(u_n^{(k)}) \to F_n(u_n)$. Since $F_n(u^{(k)}) \geq 0$, using the Fatou lemma we have

$$I(u) = \sum_{n\in\mathbb{Z}} \lim_{k\to\infty} F_n(u_n^{(k)}) \leq \underline{\lim_{k\to\infty}} \sum_{n\in\mathbb{Z}} F_n(u_n^{(k)}) = \underline{\lim_{k\to\infty}} I(u^{(k)}). \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\begin{aligned} -J(u) &= \frac{\|u^-\|^2}{2} - \frac{\|u^+\|^2}{2} + I(u) \\ &\leq \lim_{k\to\infty}\left(\frac{\|u^{(k)-}\|^2}{2} - \frac{\|u^{(k)+}\|^2}{2} + I(u^{(k)})\right) \\ &= \lim_{k\to\infty}\left(-J(u^{(k)})\right) = -c. \end{aligned}$$

So $J(u) \geq c$ and J is $\tau-$upper semicontinuous.

Finally, we show that J' is weakly sequentially continuous. Let $u^{(k)} \rightharpoonup u$ in E, we have that $u_n^{(k)} \to u_n$ for all $n \in \mathbb{Z}$, as $k \to \infty$. and there exists $M > 0$ such that $\|u^{(k)}\| \leq M$ and $\|u\| \leq M$. By (f_3), there exists constant C_0 such that $|f_n(u)| \leq C_0|u|$ for $|u| \leq M$.

For any $v \in E$ fix $0 < N \in \mathbb{N}$ such that $\sum_{|n|>N} |v_n|^2 < \frac{\varepsilon^2}{16C_0^2M^2}$. Therefore, we have

$$\begin{aligned} |I'(u^{(k)})v - I'(u)v| &\leq |\sum_{n=-N}^{N}(f_n(u_n^{(k)}) - f_n(u_n))v_n| + |\sum_{|n|>N}(f_n(u_n^{(k)}) - f_n(u_n))v_n| \\ &\leq |\sum_{n=-N}^{N}(f_n(u_n^{(k)}) - f_n(u_n))v_n| + C_0(\|u^{(k)}\| + \|u\|)\left(\sum_{|n|>N}|v_n|^2\right)^{\frac{1}{2}} \\ &\leq |\sum_{n=-N}^{N}(f_n(u_n^{(k)}) - f_n(u_n))v_n| + \frac{\varepsilon}{2}. \end{aligned}$$

Note that $f_n(u_n^{(k)}) \to f_n(u_n)$, as $k \to \infty$, then there exists k_0 such that for $k \geq k_0$,

$$|\sum_{n=-N}^{N}(f_n(u_n^{(k)}) - f_n(u_n))v_n| < \frac{\varepsilon}{2}.$$

So $|I'(u^{(k)})v - I'(u)v| < \varepsilon$, for all $k \geq k_0$. By the definition of J', then J' is weakly sequentially continuous. $\square$

Proof of Theorem 2.1.

By (f_2) and (f_3), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f_n(u)| \leq C_\varepsilon|u|, \quad |F_n(u)| \leq C_\varepsilon|u|^2.$$

For $u \in E^+$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} F_n(u_n) \\ &\geq \frac{1}{2}\|u\|^2 - C_\varepsilon\|u\|^2 \\ &= (\frac{1}{2} - C_\varepsilon)\|u\|^2. \end{aligned}$$

So $\inf_{E^+} J > -\infty$.

For $u \in E^-$, since $F(u) \geq 0$, we have

$$\begin{aligned} J(u) &= -\frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} F_n(u_n) \\ &\leq -\frac{1}{2}\|u\|^2. \end{aligned}$$

For R large enough, we have

$$\inf_{E^+} J > \sup_{\partial M} J, \quad \sup_M J < \infty,$$

where $M = \{u \in E^- : \|u\| \leq R\}$.

By Lemma 2.1, for some $c \in \mathbb{R}$, there is a sequence $\{u^{(k)}\}$ such that

$$J(u^{(k)}) \to c \text{ and } J'(u^{(k)}) \to 0 \ as \ k \to \infty.$$

Let $\widetilde{u}^{(k)} = u^{(k)+} - u^{(k)-}$, then $\|\widetilde{u}^{(k)}\| = \|u^{(k)}\|$ and

$$\begin{aligned} \|u^{(k)}\| &= \|\widetilde{u}^{(k)}\| \geq (J'(u^{(k)}), \widetilde{u}^{(k)}) \\ &= \|u^{(k)+}\|^2 + \|u^{(k)-}\|^2 - \sum_{n\in\mathbb{Z}} f_n(u_n^{(k)})\widetilde{u}_n^{(k)} \\ &\geq \|u^{(k)}\|^2 - \sum_{n\in\mathbb{Z}} C_\varepsilon |u_n^{(k)}|(|u_n^{(k)+}| + |u_n^{(k)-}|) \\ &\geq \|u^{(k)}\|^2 - C_\varepsilon\|u^{(k)}\|\|u^{(k)+}\| - C_\varepsilon\|u^{(k)}\|\|u^{(k)-}\| \\ &= \|u^{(k)}\|^2 - C_\varepsilon\|u^{(k)}\|^2. \end{aligned}$$

It implies $\{u^{(k)}\}$ is bounded.

Next we may extract a subsequence, still denoted by $\{u^{(k)}\}$, such that $u^{(k)} \rightharpoonup u$ and $u_n^{(k)} \to u_n$ for all $n \in \mathbb{Z}$. Moreover, we have

$$(J'(u), v) = \lim_{k\to\infty} (J'(u^{(k)}), v) = 0, \ \forall v \in E,$$

so $J'(u) = 0$ and u is a homoclinic solution of (1.1). □

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APPROXIMATION OF ALMOST CAUCHY'S POINTS BY CAUCHY'S POINTS

GWANG HUI KIM AND HWAN-YONG SHIN

Abstract. In this paper, we investigate Hyers–Ulam stability of Cauchy's mean value points which is a extended and generalized version of I. R. Peter and D. Popa's theorem [10] and then, as applications, we obtain Hyers-Ulam stability results of Lagrange's mean value points which refine the result of P. Găvruţă, J. Huang and Y. Li [5].

1. Introduction

The concept of Hyers–Ulam stability was raised by S. M. Ulam [11] in 1940. We are given a group G and a metric group G' with metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$? Ulam's question was partially solved by D. H. Hyers [6] in the case of approximately additive functions and when the groups in the question are Banach spaces. Due to the question of Ulam and the answer of Hyers, the stability of functional equations is called after their names. For more information of Hyers–Ulam stability, we can refer to [1, 2].

A similar problem of Ulam's question can be formulated for the mean value points : "Assume that a function f satisfies a mean value theorem with a point η. If ξ is a point near to η, does there exists a function g near to f satisfying the same mean value theorem with the point ξ?" [10].

It seems that the first result to the previous question was given by D. H. Hyers and S. M. Ulam [7] in the case of differential expressions.

Theorem 1.1. (*D. H. Hyers, S. M. Ulam, 1954,* [7]) *Let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable in a neighborhood N of a point η. Suppose that $f^{(n)}(\eta) = 0$ and $f^{(n)}(x)$ changes sign at η. Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable in N and satisfies $|f(x) - g(x)| < \delta$ for all $x \in N$, there exists a point $\xi \in N$ such that $g^{(n)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.*

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In 2003, M. Das, T. Riedel and P. K. Sahoo [3] proved the stability problem for Flett's mean value points by using Theorem 1.1. Subsequently, some authors applied the idean from [3] to prove the Hyers–Ulam stability of various mean value points [5, 8, 9, 10]. Especially, P. Găvruţă, S.-M. Jung and Y. Li [5] proved the following stability result of Lagrange's mean value points which is a point η of a differentiable function $f : [a, b] \to \mathbb{R}$ satisfying $\frac{f(b)-f(a)}{b-a} = f'(\eta)$.

Theorem 1.2. (*P. Găvruţă, S.-M. Jung, Y. Li, 2010,* [5]) *Let a, b, η be real numbers satisfying $a < \eta < b$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function and η is the unique Lagrange's mean value point of f in an open interval (a, b) and moreover that $f''(\eta) \neq 0$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in [a, b]$, then there is a Lagrange's mean value point $\xi \in (a, b)$ of g with $|\xi - \eta| < \varepsilon$.*

Hereafter, Theorem 1.2 was generalized by I. R. Peter and D. Popa [10] by proving the stability of Cauchy's mean value points which is a point η of two differentiable functions $f, g : [a, b] \to \mathbb{R}$ satisfying

$$(f(b) - f(a))g'(\eta) - (g(b) - g(a))f'(\eta) = 0.$$

Let I be an open interval which contains the interval (a, b).

Theorem 1.3. (*I. R. Peter, D. Popa, 2013,* [10]) *Assume that $f, g : I \to \mathbb{R}$ are continuously differentiable functions, η is the unique Cauchy's mean value point of the pair (f, g) in I and f, g are twice continuously differentiable in a neighborhood of η, satisfying*

$$f''(\eta)(g(b) - g(a)) - g''(\eta)(f(b) - f(a)) \neq 0.$$

Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $f_1, g_1 : (a, b) \to \mathbb{R}$ are continuously differentiable functions with the property that $|f(x) - f_1(x)| < \delta$ and $|g(x) - g_1(x)| < \delta$ for all $x \in [a, b]$ there exists a Cauchy mean value point $\xi \in (a, b)$ of (f_1, g_1) with $|\eta - \xi| < \varepsilon$.

In this paper, we prove Hyers–Ulam stability of Cauchy's mean value points which is a extended and generalized version of Theorem 1.3 and then, as applications, we obtain the stability results of Lagrange's mean value points which refine Theorem 1.2.

2. Hyers–Ulam Stability of Cauchy's mean value points

We now present a main theorem, which is a Hyers–Ulam stability of Cauchy's mean value points for real-valued differentiable functions on $[a, b]$.

Theorem 2.1. *Let $f, g, f_1, g_1 : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Cauchy's mean value point of the pair (f, g) in the interval (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose the following control function*

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

changes sign at η. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x)-f_1(x)| < \delta$ and $|g(x) - g_1(x)| < \delta$ for all $x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (f_1, g_1) with $|\xi - \eta| < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given and $N \subseteq (a, b)$ be any neighborhood of η. Consider the auxiliary function $G_{f,g}(x) : [a, b] \to \mathbb{R}$ corresponding to (f, g) defined by

$$G_{f,g}(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$$

for all $x \in [a, b]$. Evidently $G_{f,g}(x)$ is continuous on $[a, b]$ and differentiable on $[a, b]$. Further, we have

$$G'_{f,g}(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x), \quad x \in [a, b].$$

Since η is the Cauchy's mean value point of (f, g), we get $G'_{f,g}(\eta) = 0$. Thus it follows from the assumption that there exists a neighborhood $(\eta - r, \eta + r) \subseteq N$ of η such that $G'_{f,g}(x)$ changes sign at η in $(\eta - r, \eta + r) \subseteq N$ for some $r > 0$ with $\eta - r > a$. Then it follows from Theorem 1.1 that there exists a $\overline{\delta} > 0$ such that for any differentiable function H on $[a, b]$ with $|H(x)-G_{f,g}(x)| < \overline{\delta}$ for x in $(\eta-r, \eta+r)$, there exists a point $\zeta \in (\eta-r, \eta+r)$ satisfying $H'(\zeta) = 0$ and $|\zeta - \eta| < \varepsilon$.

For a continuous function $f : [a, b] \to \mathbb{R}$ define

$$M_f := \max\{|f(x)| : x \in [a, b]\}$$

and analogously M_g. Define $G_{f_1,g_1}(x) : [a, b] \to \mathbb{R}$ be the corresponding auxiliary function defined as

$$G_{f_1,g_1}(x) = (f_1(b) - f_1(a))g_1(x) - (g_1(b) - g_1(a))f_1(x)$$

for all $x \in [a, b]$.

For some fixed $\lambda > 0$, let

$$\delta := \min\Big\{\frac{\overline{\delta}}{4M_f + 4M_g + 4\lambda}, \lambda\Big\}.$$

and let $f_1, g_1 : [a, b] \to \mathbb{R}$ be any differentiable functions satisfying $|f(x) - f_1(x)| < \delta$ and $|g(x)-g_1(x)| < \delta$ for all $x \in N\cup\{a, b\}$. Then one can easy to see that $G_{f_1,g_1}(x)$ is differentiable

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in N. And it follows that

$$\begin{aligned}|f_1(b)-f_1(a)| &\leq |f_1(b)-f(b)|+|f(b)-f(a)|+|f(a)-f_1(a)| \\ &\leq 2\lambda+2M_f.\end{aligned}$$

By the same reason we obtain that

$$|g_1(b)-g_1(a)| \leq 2\lambda+2M_g.$$

These yield that

$$\begin{aligned}|G_{f,g}(x)-G_{f_1,g_1}(x)| &= |(f(b)-f(a))g(x)-(g(b)-g(a))f(x) \\ &\quad -(f_1(b)-f_1(a))g_1(x)+(g_1(b)-g_1(a))f_1(x)| \\ &= |(f(b)-f(a))g(x)-(f_1(b)-f_1(a))g(x) \\ &\quad +(f_1(b)-f_1(a))g(x)-(f_1(b)-f_1(a))g_1(x) \\ &\quad +(g_1(b)-g_1(a))f_1(x)-(g_1(b)-g_1(a))f(x) \\ &\quad +(g_1(b)-g_1(a))f(x)-(g(b)-g(a))f(x)| \\ &\leq (|f(b)-f_1(b)|+|f(a)-f_1(a)|)|g(x)| \\ &\quad +|f_1(b)-f_1(a)|\cdot|g(x)-g_1(x)| \\ &\quad +|g_1(b)-g_1(a)|\cdot|f_1(x)-f(x)| \\ &\quad +(|g_1(b)-g(b)|+|g_1(a)-g(a)|)|f(x)| \\ &\leq (2M_g+|f_1(b)-f_1(a)|+|g_1(b)-g_1(a)|+2M_f)\delta \\ &\leq (4M_f+4M_g+4\lambda)\delta \\ &\leq \bar{\delta}\end{aligned}$$

for all $x \in (\eta-r,\eta+r) \subseteq N$. Hence, there exists a point $\xi \in (\eta-r,\eta+r)$ such that $G'_{f_1,g_1}(\xi)=0$ and $|\xi-\eta|<\varepsilon$. We note that $G'_{f_1,g_1}(\xi)=0$ implies

$$(f_1(b)-f_1(a))g_1'(\xi)-(g_1(b)-g_1(a))f_1'(\xi)=0.$$

Hence, the point ξ is a Cauchy's mean value point of (f_1,g_1) and the proof is complete. □

The following corollary is a refined result of Theorem 1.3.

Corollary 2.2. *Let $f,g,f_1,g_1:[a,b]\to\mathbb{R}$ be countinuously differentiable functions and η be a Cauchy's mean value point of the pair (f,g) in the interval (a,b) and $N\subseteq(a,b)$ be a neighborhood of η. Suppose either η is unique Cauchy's mean value point of (f,g) or f, g have second derivative at η such that*

$$[f(b)-f(a)]g''(\eta) \neq [g(b)-g(a)]f''(\eta). \tag{2.1}$$

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Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x)-f_1(x)| < \delta$ and $|g(x)-g_1(x)| < \delta$ for all $x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (f_1, g_1) with $|\xi - \eta| < \varepsilon$.

Proof. Let $G_{f,g} : [a, b] \to \mathbb{R}$ be defined as

$$G_{f,g}(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a))$$

for all $x \in [a, b]$. Suppose η is a unique Cauchy's mean value point of (f, g). Then we obtain that $G_{f,g}(a) = G_{f,g}(b)$ and $\eta \in (a, b)$ is a unique point such that $G'_{f,g}(\eta) = 0$. These yield that $G'_{f,g}(x)$ changes sign at η.

If f and g have second derivative and satisfy (2.1), we have $G''_{f,g}(\eta) \neq 0$. Thus associating this fact and $G'_{f,g}(\eta) = 0$, we get $G'_{f,g}(x)$ changes sign at η.

Rewriting the fact $G_{f,g}$ changes sign at η, we obtain

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

changes sign at η. By applying Theorem 2.1, we get the desired result. □

If we take $f_1, g_1 : [a, b] \to \mathbb{R}$ by $f_1 := h$ and $g_1 := g$ in Theorem 2.1 and Corollary 2.2, then we get the following two corollaries.

Corollary 2.3. *Let $f, g, h : [a, b] \to \mathbb{R}$ be differentiable and η be a Cauchy's mean value point of the pair (f, g) in the interval (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose the following control function*

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

changes sign at η. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - h(x)| < \delta$ for all $x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (g, h) with $|\xi - \eta| < \varepsilon$.

Corollary 2.4. *Let $f, g, h : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Cauchy's mean value point of the pair (f, g) in the interval (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose either η is a unique Cauchy's mean value point of (f, g) or f, g have second derivative at η such that*

$$(f(b) - f(a))g''(\eta) \neq (g(b) - g(a))f''(\eta).$$

Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x)-h(x)| < \delta$ for all $x \in N\cup\{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (g, h) with $|\xi - \eta| < \varepsilon$.

The following theorem is another type of Hyers-Ulam stability for Cauchy's mean value points.

Theorem 2.5. *Let a, b, ξ be real numbers satisfying $a < \xi < b$. Assume that $f, g : [a, b] \to \mathbb{R}$ are countinuously differentiable functions such that*

$$g'(x), \frac{f'(x)g''(x) - f''(x)g'(x)}{g'(x)^2} \neq 0$$

for all $x \in [a, b]$. If

$$\Big|\frac{f'(\xi)}{g'(\xi)} - \frac{f(b) - f(a))}{g(b) - g(a)}\Big| \leq \varepsilon \tag{2.2}$$

for some $\varepsilon > 0$, then there exists a Cauchy's mean value point η of (f, g) on (a, b) satisfying

$$|\xi - \eta| \leq \frac{\varepsilon}{min_{x\in[a,b]}\Big|\frac{f'(x)g''(x)-f''(x)g'(x)}{g'(x)^2}\Big|}.$$

Proof. Due to Cauchy's mean value theorem, there exists a point $\eta \in (a, b)$ such that

$$\frac{f'(\eta)}{g'(\eta)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Hence it follows from (2.2) that

$$\Big|\frac{f'(\xi)}{g'(\xi)} - \frac{f'(\eta)}{g'(\eta)}\Big| \leq \varepsilon.$$

If $\xi = \eta$ then the proof is clear. Otherwise, we assume that $a < \eta < \xi < b$. Since f and g have second derivative on $[a, b]$, by Lagrange's mean value theorem, there exists a point $\xi_0 \in (\eta, \xi)$ such that

$$\Big|(\xi - \eta)\Big(\frac{f'(\xi_0)g''(\xi_0) - f''(\xi_0)g'(\xi_0)}{g'(\xi_0)^2}\Big)\Big| = \Big|\frac{f'(\eta)}{g'(\eta)} - \frac{f'(\xi)}{g'(\xi)}\Big|.$$

Since f', f'', g', g'' are continuous on $[a, b]$, we obtain

$$|\xi - \eta| = \Big|\frac{\frac{f'(\eta)}{g'(\eta)} - \frac{f'(\xi)}{g'(\xi)}}{\frac{f'(\xi_0)g''(\xi_0)-f''(\xi_0)g'(\xi_0)}{g'(\xi_0)^2}}\Big| \leq \frac{\varepsilon}{min_{x\in[a,b]}\Big|\frac{f'(x)g''(x)-f''(x)g'(x)}{g'(x)^2}\Big|},$$

which complete the proof. □

3. Applications to Lagrange's mean value points

In this section, we obtain stability results of Lagrange's mean value points for the differentiable functions on $[a, b]$.

Corollary 3.1. *Let $f, g : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Lagrange's mean value point of f in (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose the following control function*

$$f(b) - f(a) - (b - a)f'(x)$$

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changes sign at η. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in N \cup \{a, b\}$ there exists a point $\xi \in N$ such that ξ is a Lagrange's mean value point of g with $|\xi - \eta| < \varepsilon$.

Proof. Consider the auxiliary function $G_f(x) : [a, b] \to \mathbb{R}$ corresponding to f defined by

$$G_f(x) = (f(b) - f(a))x - f(x)(b - a)$$

for all $x \in [a, b]$. Then the proof goes through the same way as that of Theorem 2.1. $\square$

Example 3.2. Let $f : [-2\pi, 2\pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos x - 1, & \text{if} \quad x \leq 0, \\ 1 - \cos x, & \text{if} \quad x > 0. \end{cases}$$

It is obvious to see that there exist three Lagrange's mean value points $-\pi, 0, \pi$ of f. Let N_i be a neighborhood of $(-1)^i\pi$ for each $i = 1, 2$. We can easily check that $f(2\pi) - f(-2\pi) - (2\pi - (-2\pi))f'(x) = -4\pi f'(x)$ changes sign at $\pm\pi$. Therefore, by Corollary 3.1, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every differentiable function g satisfying $|f(x) - g(x)| < \delta$ for all $x \in N_i \cup \{\pm 2\pi\}$ then there exists a point $\xi_i \in N_i$ such that ξ_i is a Lagrange's mean value point of g and $|\xi_i - (-1)^i\pi| < \varepsilon$. However, $f(2\pi) - f(-2\pi) - (2\pi - (-2\pi))f'(x) = -4\pi f'(x)$ does not change sign at 0, and so we cannot apply Corollary 3.1 for the function f at the Lagrange's mean value point 0.

Let $N := (-\frac{\pi}{4}, \frac{\pi}{4})$ and $\delta > 0$ be given. And let $g : [-2\pi, 2\pi] \to \mathbb{R}$ be defined by

$$g(x) := f(x) + \frac{\delta}{1024}(x^3 - 4\pi^2 x)$$

for all $x \in [-2\pi, 2\pi]$. Then

$$|f(x) - g(x)| = \frac{\delta}{1024}|x^3 - 4\pi^2 x| < \frac{\delta}{1024}((2\pi)^3 + 4\pi^2(2\pi)) < \delta$$

for all $x \in N \cup \{\pm 2\pi\}$. But, for all $x \in N$, the following inequlity holds

$$\frac{g(2\pi) - g(-2\pi)}{4\pi} - g'(x) > 0.$$

Therefore, we can conclude that there is no Lagrange's mean value point of g in N.

The following refined result of Theorem 1.2 is obtained as a corollary of Corollary 3.1.

Corollary 3.3. *Let $f, g : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Lagrange's mean value point of f in (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose either η is a unique Lagrange's mean value point of f or f has second derivative at η with $f''(\eta) \neq 0$. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all*

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$x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Lagrange's mean value point of g with $|\xi - \eta| < \varepsilon$.

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Weak Galerkin Finite Element Method for Convection-Diffusion-Reaction Problems

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Abstract

In this paper, a weak Galerkin (WG) finite element method is proposed for solving the convection-diffusion-reaction problems. The main idea of WG finite element methods is the use of weak functions and their corresponding discrete weak derivatives in standard weak form of the model problem. We show that the continuous time WG finite element method preserves the energy conservation law as well the optimal order error estimate in L^2 norm. Numerical experiment is conducted to confirm the theoretical results.

Keywords: WG finite element method, convection-diffusion-reaction equation, energy conservation law, error estimate.

1 Introduction

The convection-diffusion-reaction processes appear in many areas of science and technology. For example, fluid dynamics, heat and mass transfer hydrology and so on. In this paper, we consider the following convection-diffusion-reaction equation:

$$\begin{aligned} u_t - \nabla\cdot(\lambda\nabla u) + b\cdot\nabla u + cu &= f, \quad (x,t)\in\Omega\times(0,T], && (1.1)\\ u(x,0) &= 0, \quad x\in\Omega, && (1.2)\\ u(x,t)|_\Gamma &= g, \quad t\in(0,T], && (1.3)\end{aligned}$$

where Ω is a bounded region in R^2, with a Lipschitz continuous boundary $\Gamma = \partial\Omega$, $u_t = \frac{\partial u}{\partial t}$, and ∇u denote the gradient of function $u = u(x,t)$. Further $\lambda > 0$ is a diffusion coefficient, b is a convection coeffient and f, g are given functions.

The standard weak form of equations $(1.1) - (1.3)$ seeks $u \in L^2(0,T; H^1(\Omega))$ such that $u = g$ on $\partial\Omega\times(0,T)$ and

$$(u_t, v) + (\lambda\nabla u, \nabla v) - (bu, \nabla v) + (cu, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{1.4}$$

The WG finite element method refers to a general finite element technique for partial differential equation where the differential operators (e.g., gradient, divergence, curl, Laplacian) are approximated by weak forms. The method, first introduced by Wang and Ye [1] for solving a second order elliptic problems, is a newly developed finite element method. Since

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then, some WG finite element methods have been developed to solve other problems, such as parabolic equation [2, 3, 4], Stokes equations [5, 6], Helmholtz equation [7], Biharmonic equation [8, 9] and Navier-Stokes equations [10, 11], etc.

In general, WG finite element formulations for partial differential equation can be derived naturally by replacing usual derivatives by variational forms. The implementations of all these possible extension are based on the computation of these weak operators.

The rest of this paper is organized as follows. In section 2, we shall introduce some preliminaries and notations for Sobolev spaces. We define the weak gradient and discrete weak gradient operator and the weak finite element spaces and present semi-discrete WG finite element method for problem (1.1) − (1.3) in section 3 and section 4, respectively. In section 5, we prove the energy conservation law of the continuous time WG approximation, and in section 6 we present optimal order error estimate in L^2 norm for the WG finite element approximations. Finally, we present a numerical example to verify theory.

2 Preliminaries and notations

We use standard definitions for the Sobolev spaces $H^m(\Omega)$ and their associated inner products $(\cdot,\cdot)_{m,\Omega}$, norms $\|\cdot\|_{m,\Omega}$, and seminorms $|\cdot|_{m,\Omega}$ for $m \geq 0$ [12, 13]. For any integers $m \geq 0$ the seminorm $|\cdot|_{m,\Omega}$ is given by

$$|v|_{m,\Omega} = (\sum_{|\alpha|=m} \int_{\Omega} |\partial^{\alpha} v|^2 d\Omega)^{1/2},$$

with the usual notation

$$\alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2, \quad \partial^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}.$$

The Sobolev norm $\|\cdot\|_{n,\Omega}$, is given by

$$\|v\|_{n,\Omega} = (\sum_{j=0}^{n} |v|_{j,\Omega}^2)^{1/2}.$$

The space $H(div;\Omega)$ is defined as the set of vector-valued functions on Ω which, together with their divergence, are square integrable; i.e,

$$H(div;\Omega) = \{v : v \in [L^2(\Omega)]^2, \nabla \cdot v \in L^2(\Omega)\}.$$

The norm in $H(div;\Omega)$ is defined by

$$\|v\|_{H(div;\Omega)} = (\|v\|^2 + \|\nabla \cdot v\|^2)^{1/2}.$$

3 A weak Gradient operator and its approximation

In this section we introduce a weak gradient operator defined on a space of generalized functions. Let K be any polygonal domain with interior K^0 and boundary ∂K. A weak function on the region K refers to vector-valued function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{1/2}(\partial K)$. The first component v_0 can be understood as the value of v in interior of K, and the second component v_b is the value of v on the boundary of ∂K. Denote by $W(K)$ the space of weak function associated with K ; i.e.,

$$W(K) := \{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in H^{1/2}(\partial K)\}. \tag{3.1}$$

Definition 3.1. *For any $v \in W(K)$, the weak gradient of v is defined as a linear functional $\nabla_d v$ in the dual space of $H(div, K)$ whose action on each $q \in H(div, K)$ is given by*

$$\int_K \nabla_d v \cdot q dK = -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot \boldsymbol{n} ds, \tag{3.2}$$

where $\boldsymbol{n}$ is the outward normal direction to ∂K.

Next, we introduce a discrete weak gradient operator by defining ∇_d in a polynomial subspace of $H(div, K)$. To this end , for any non-negative integer $r \geq 0$ denote by $P_r(K)$ the set of polynomials on K with degree no more than r. Let $V(K, r) \subset [P_r(K)]^2$ be a subspace of the space of vector-valued polynomials of degree r. A discrete weak gradient operator, denoted by $\nabla_{d,r}$, is defined so that $\nabla_{d,r} v \in V(K, r)$ is the unique solution of the following equation

$$\int_K \nabla_{d,r} v \cdot q dK = -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot \mathbf{n} ds, \quad \forall q \in V(K, r). \tag{3.3}$$

4 A weak Galerkin finite element scheme

Let T_h be triangular partition of the domain Ω with mesh size h. Assume that the partition T_h is shape regular so that the routine inverse inequality holds true (see[13]). In the general spirit of Galerkin procedure, we shall design a WG method for (1.4) by following two basic principles: first replacing $H^1(\Omega)$ by a space of discrete weak functions defined on the finite element partition T_h and the boundary of triangular elements; second replacing the classical gradient operator by a discrete weak gradient operator $\nabla_{d,r}$ for weak functions on each triangle T.

For each $T \in T_h$. Denote by $P_j(T^0)$ the set of polynomials with degree no more than j and $P_\ell(\partial T)$ the set of polynomial on ∂T with degree no more than ℓ. A discrete weak function $v = \{v_0, v_b\}$ on T refers to a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P_j(T^0)$ and $v_b \in P_\ell(\partial T)$ with $j \geq 0$ and $\ell \geq 0$. Denote this space by $W(T, j, \ell)$, i.e.,

$$W(T, j, \ell) = \{v = \{v_0, v_b\} : v_0 \in P_j(T^0), v_b \in P_\ell(\partial T)\}. \tag{4.1}$$

The corresponding finite element space would be defined by patching $W(T, j, \ell)$ over all the triangles $T \in T_h$. In other words, the weak finite element space is given by

$$S_h(j, \ell) = \{v = \{v_0, v_b\} : \{v_0, v_b\}|_T \in W(T, j, \ell), \forall T \in T_h\}. \tag{4.2}$$

Denote by $S_h^0(j, \ell)$ the subspace of $S_h(j, \ell)$ with vanishing boundary values on $\partial\Omega$, i.e.,

$$S_h^0(j, \ell) = \{v = \{v_0, v_b\} \in S_h(j, \ell), v_b|_{\partial T \cap \partial\Omega} = 0, \forall T \in T_h\}. \tag{4.3}$$

To investigate the approximation properties of the discrete weak space $S_h(j, \ell)$, we define three projections in this paper. The first two are local projections defined on each triangle T:

one is $Q_h u = \{Q_0 u, Q_b u\}$, the L^2 projection of $H^1(T)$ onto $P_j(T^0) \times P_{j+1}(\partial T)$ and another is R_h, the L^2 projection of $[L^2(T)]^2$ onto $V(T,r)$. The third projection Π_h is assumed to exist and satisfy the following property: for $q \in H(div,\Omega)$ with mildly added regularity, $\Pi_h q \in H(div,\Omega)$ such that $\Pi_h q \in V(T,r)$ on each $T \in T_h$, and

$$(\nabla \cdot q, v_0)_T = (\nabla \cdot \Pi_h q, v_0)_T, \quad \forall v_0 \in P_j(T). \tag{4.4}$$

It is easy to see the following two useful identities:

$$\nabla_{d,r}(Q_h u) = R_h(\nabla u), \ \ \forall u \in H^1(T), \tag{4.5}$$

and for any $q \in H(div,\Omega)$

$$\sum_{T\in T_h} (-\nabla \cdot q, v_0)_T = \sum_{T\in T_h} (\Pi_h q, \nabla_{d,r} v)_T, \ \ \forall v = \{v_0, v_b\} \in S_h^0(j,\ell). \tag{4.6}$$

Now for any $u, v \in S_h(j,\ell)$, we introduce the following bilinear form

$$a(u,v) = (\lambda \nabla_{d,r} u, \nabla_{d,r} v) - (b u_0, \nabla_{d,r} v) + (c u_0, v_0), \tag{4.7}$$

where

$$\begin{aligned} (\lambda \nabla_{d,r} u, \nabla_{d,r} v) &= \int_\Omega \lambda \nabla_{d,r} u \cdot \nabla_{d,r} v d\Omega, \\ (b u_0, \nabla_{d,r} v) &= \int_\Omega b u_0 \cdot \nabla_{d,r} v d\Omega, \\ (c u_0, v_0) &= \int_\Omega c u_0 v_0 d\Omega. \end{aligned}$$

We pose the continuous time WG finite element method based on (3.3) and (1.4) which is to find $u_h(t) = \{u_0(\cdot,t), u_b(\cdot,t)\}$, belonging to $S_h(j,\ell))$ for $t > 0$, satisfying $u_b = Q_b g$ on $\partial\Omega$, and the following equation

$$((u_h)_t, v_0) + a(u_h, v) = (f, v_0), \qquad \forall v = \{v_0, v_b\} \in S_h^0(j,\ell), \tag{4.8}$$

where

$$a(u_h, v) = (\lambda \nabla_{d,r} u, \nabla_{d,r} v) - (b u_h, \nabla_{d,r} v) + (c u_h, v_0),$$

where, $Q_b g$ is an approximation of the boundary value in the polynomial space $P_\ell(\partial T \cap \partial\Omega)$. For simplicity, $Q_b g$ shall be taken as the standard L^2 projection for each boundary segment.

5 Energy conservation property of WG

In this section, we investigate the energy conservation property of the semi-discrete WG finite element approximation u_h. The solution u of the problem $(1.1) - (1.3)$ has the following energy preserving property on each $K \in T_h$ [2].

$$\int_{t-\Delta t}^{t+\Delta t} \int_K u_t dx dt + \int_{t-\Delta t}^{t+\Delta t} \int_{\partial K} q \cdot \mathbf{n} ds dt \ = \ \int_{t-\Delta t}^{t+\Delta t} \int_K f dx dt, \tag{5.1}$$

where $q = -\lambda \nabla u + b u$ is the flow rate of heat energy.

4

We claim that the semi-discrete WG for (1.1) $-$ (1.3) preserves the energy conservation property in (5.1). Choosing in (4.8) the test function $v = \{v_0, v_b = 0\}$ so that $v_0 = 1$ on K and $v_0 = 0$ elsewhere. We then obtain by integration over the time period $[t - \Delta t, t + \Delta t]$

$$\int_{t-\Delta t}^{t+\Delta t} \int_K u_t dx dt + \int_{t-\Delta t}^{t+\Delta t} a(u_h, v) dt \quad = \quad \int_{t-\Delta t}^{t+\Delta t} \int_K f dx dt, \tag{5.2}$$

where

$$a(u_h, v) \quad = \quad \int_K \lambda \nabla_{d,r} u_h \cdot \nabla_{d,r} v dx - \int_K b u_0 \cdot \nabla_{d,r} v dx + \int_K c u_0 dx.$$

Using the definition of operators R_h and $\nabla_{d,r}$ in (4.4), we obtain

$$\begin{aligned} \int_K \lambda \nabla_{d,r} u_h \cdot \nabla_{d,r} v dx &= \int_K R_h(\lambda \nabla_{d,r} u_h) \cdot \nabla_{d,r} v dx \\ &= -\int_K \nabla \cdot R_h(\lambda \nabla_{d,r} u_h) dx \\ &= -\int_{\partial K} R_h(\lambda \nabla_{d,r} u_h) \cdot \mathbf{n} ds, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \int_K b u_0 \cdot \nabla_{d,r} v dx &= \int_K R_h(b u_0) \cdot \nabla_{d,r} v dx \\ &= -\int_K \nabla \cdot R_h(b u_0) dx \\ &= -\int_{\partial K} R_h(b u_0) \cdot \mathbf{n} ds. \end{aligned} \tag{5.4}$$

Now substituting (5.3) and (5.4) into (5.2) yields

$$\int_{t-\Delta t}^{t+\Delta t} \int_K u_t dx dt + \int_{t-\Delta t}^{t+\Delta t} \int_{\partial K} R_h(-\lambda \nabla_{d,r} u_h + b u_0) \cdot \mathbf{n} ds \quad = \quad \int_{t-\Delta t}^{t+\Delta t} \int_K f dx dt,$$

which provides a numerical flux.

$$q_h \cdot \mathbf{n} = R_h(-\lambda \nabla_{d,r} u_h + b u_0) \cdot \mathbf{n}.$$

The numerical flux $q_h \cdot \mathbf{n}$ can be verified to be continuous across the edge of each element K through a selection of the test function $v = \{v_0, v_b\}$ so that $v_0 \equiv 0$ and v_b are arbitrary.

6 Error analysis

In this section, we derive optimal order error estimate for the semi-discrete scheme (4.8) in L^2 norm. Let us begin with proving the elliptic property of WG finite element method for equation (1.1).

Lemma 6.1. *Let $S_h(j,\ell)$ be the weak finite element space defined in* (4.2) *and $a(u_h, v)$ be the bilinear form given in* (4.8)*. There exists positive constant α satisfying*

$$a(v_h, v_h) \geq \alpha(\|\nabla_{d,r} v_h\|^2 + \|v_0\|^2),$$

for all $v_h \in S_h(j,\ell)$.

Proof. Taking $u = v$ in equation (4.8) we have

$$a(v_h, v_h) \quad = \quad (\lambda \nabla_{d,r} v, \nabla_{d,r} v) - (b v_0, \nabla_{d,r} v) + (c v_0, v_0). \tag{6.1}$$

Let $A = \|b\|_{L^\infty(\Omega)}$ and $B = \|c\|_{L^\infty(\Omega)}$ be the L^∞-norm of the coefficients b and c, respectively and using Cauchy- Schwarz inequality we have.

$$\begin{aligned} |(b v_0, \nabla_{d,r} v)| &\leq \|b\|_{L^\infty(\Omega)} \|\nabla_{d,r} v\| \|v_0\|, \\ &\leq A \|\nabla_{d,r} v\| \|v_0\| \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} |(c v_0, v_0)| &\leq \|c\|_{L^\infty(\Omega)} \|v_0\|^2 \\ &\leq B \|v_0\|^2. \end{aligned} \tag{6.3}$$

Substituting (6.2) and (6.3) into (6.1) we obtain

$$a(v_h, v_h) \quad \geq \quad |\lambda| \|\nabla_{d,r} v\|^2 + A \|\nabla_{d,r} v\| \|v_0\| - B \|v_0\|^2,$$

by using Young-inequality, we have

$$\begin{aligned} a(v_h, v_h) &\geq (|\lambda| + \frac{1}{2\epsilon}) \|\nabla_{d,r} v\|^2 + (\frac{\epsilon A^2}{2} - B) \|v_0\|^2 \\ &\geq \alpha_1 \|\nabla_{d,r} v\|^2 + \alpha_2 \|v_0\|^2 \\ &\geq \alpha(\|\nabla_{d,r} v\|^2 + \|v_0\|^2), \end{aligned}$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$, which completes the proof. ∎

Lemma 6.2. ([*2*])
For $u \in H^{1+\kappa}(\Omega)$ with $\kappa > 0$, we have

$$\|\Pi_h(\lambda \nabla u) - \lambda R_h(\nabla u)\| \quad \leq \quad C h^\kappa \|u\|_{1+\kappa}. \tag{6.4}$$

Lemma 6.3. ([*14*])
For $u \in H^{1+\kappa}(\Omega)$ with $\kappa > 0$, we have

$$\|u - \Pi_h u\| \quad \leq \quad C h^\kappa \|u\|_{1+\kappa}. \tag{6.5}$$

6.1 Continuous time WG finite element method

Our aim is to prove the following estimate in L^2 norm for the semi-discrete approximation.

Theorem 6.1. *Let $u \in H^{1+\kappa}(\Omega)$ with $\kappa > 0$ and u_h be the solutions of* (1.1) − (1.3) *and* (4.8) *respectively. Denote by $e = u_h - Q_h u$ the difference between WG approximation and the L^2 projection of the exact solution $u = u(x,t)$. Then there exists a constant C such that*

$$\|e\|^2 + \int_0^t \alpha \|e\|^2 ds \quad \leq \quad \|e(\cdot,0)\|^2 + Ch^{2\kappa} \int_0^t \|u\|^2_{1+\kappa} ds \tag{6.6}$$

Proof. Let $v = \{v_0, v_b\} \in S_h^0(j,\ell)$ be the testing function. By testing (1.1) − (1.3) against v_0, together with (4.6) we arrive at

$$\begin{aligned} (f, v_0) &= (u_t, v_0) + \sum_{T\in T_h} (-\nabla \cdot (\lambda \nabla u), v_0)_T + \sum_{T\in T_h} (\nabla \cdot (bu), v_0) + (cu, v_0) \\ &= (u_t, v_0) + (\Pi_h(\lambda\nabla u), \nabla_{d,r} v) - (\Pi_h(bu), \nabla_{d,r} v) + (cu, v_0). \end{aligned} \tag{6.7}$$

Adding and subtracting the term

$$a(Q_h u, v) \equiv (\lambda \nabla_{d,r}(Q_h u), \nabla_{d,r} v) - (b(Q_0), \nabla_{d,r} v) + (c(Q_0 u), v_0),$$

on the right hand side of the equation (6.7) and using $(Q_h u_t, v_0) = (u_t, v_0)$ we obtain

$$\begin{aligned} (f, v_0) &= (Q_h u_t, v_0) + (\Pi_h(\lambda\nabla u) - \lambda\nabla_{d,r}(Q_h u), \nabla_{d,r} v) \\ &- (\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} v) + (cu - c(Q_0 u), v_0) \\ &+ (\lambda\nabla_{d,r}(Q_h u), \nabla_{d,r} v) - (b(Q_0), \nabla_{d,r}) \\ &+ (c(Q_0 u), v_0), \end{aligned}$$

by using $R_h(\nabla u) = \nabla_{d,r}(Q_h u)$ for $u \in H^1$ and (4.8) we obtain

$$\begin{aligned} ((u_h)_t, v_0) + a(u_h, v) &= (Q_h u_t, v_0) + (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r} v) \\ &- (\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} v) + (cu - c(Q_0 u), v_0) \\ &+ a(Q_h u, v), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} ((u_h - Q_h)_t, v_0) + a(u_h - Q_h u, v) = (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r} v) \\ -(\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} v) + (cu - c(Q_0 u), v_0). \end{aligned} \tag{6.8}$$

Equation (6.8) shall be called the error equation for the WG finite element method (4.8). Substituting v in (6.8) by $e = \{u_h - Q_h u\} = \{e_0, e_b\} = \{u_0 - Q_0 u, u_b - Q_b u\}$, we have

$$\begin{aligned} (e_t, e) + a(e, e) &= (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r} e) - (\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} e) \\ &+ (cu - c(Q_0 u), e). \end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \beta\|\nabla_{d,r} e\|^2 + \alpha\|e\|^2 \quad = \quad \sum_{i=1}^{3} I^{(i)}, \tag{6.9}$$

where

$$\begin{aligned}
I^{(1)} &= (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r}e) \\
I^{(2)} &= (\Pi_h(bu) - bu_0, \nabla_{d,r}e) \\
I^{(3)} &= (cu - c(Q_0 u), e).
\end{aligned}$$

To estimate $I^{(1)}$, by Cauchy-Schwarz inequality and Young inequality, we have

$$|I^{(1)}| \leq \frac{1}{2\beta}\|\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u)\|^2 + \frac{\beta}{2}\|\nabla_{d,r}e\|^2$$

by lemma (6.2), we have

$$|I^{(1)}| \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \frac{\beta}{2}\|\nabla_{d,r}e\|^2. \tag{6.10}$$

To estimate $I^{(2)}$, by Cauchy-Schwarz inequality and Young inequality, we have

$$|I^{(2)}| \leq \frac{1}{2\beta}\|\Pi_h(bu) - bu_0\|^2 + \frac{\beta}{2}\|\nabla_{d,r}e\|^2,$$

by lemma (6.3), we have

$$|I^{(2)}| \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \frac{\beta}{2}\|\nabla_{d,r}e\|^2. \tag{6.11}$$

To estimate $I^{(3)}$, again by Cauchy-Schwarz inequality, Young inequality and lemma(6.3), we have

$$\begin{aligned}
|I^{(3)}| &\leq \frac{1}{2\alpha}\|cu - c(Q_0 u)\|^2 + \frac{\alpha}{2}\|e\|^2 \\
&\leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \frac{\alpha}{2}\|e\|^2. \qquad (6.12)
\end{aligned}$$

Substituting (6.10), (6.11), and (6.12), into (6.9) we get

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \beta\|\nabla_{d,r}e\|^2 + \alpha\|e\|^2 \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \beta\|\nabla_{d,r}e\|^2 + \frac{\alpha}{2}\|e\|^2.$$

It follows that

$$\frac{d}{dt}\|e\|^2 + \alpha\|e\|^2 \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2.$$

Thus, integrating with respect to t, we obtain

$$\|e\|^2 + \int_0^t \alpha\|e\|^2 ds \leq \|e(\cdot,0)\|^2 + Ch^{2\kappa}\int_0^t \|u\|_{1+\kappa}^2 ds,$$

which completes the proof. □

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6.2 Optimal order of error estimation in L^2

To get an optimal order of error estimate in L^2, the idea, similar to Wheeler's projection as in [14, 15], is used where an elliptic projection E_h onto the discrete weak space $S_h(j,\ell)$ is defined as the following: Find $E_h u \in S_h(j,\ell)$ such that $E_h u$ is the L^2 projection of the trace of u on the boundary $\partial\Omega$ and

$$\begin{aligned}(\lambda\nabla_{d,r}E_h u, \nabla_{d,r}w) + (b\cdot\nabla_{d,r}E_h u, w) &= (-\nabla\cdot(\lambda\nabla u), w) \\ &+ (-bu, \nabla w) \quad \forall w \in S_h^0(j,\ell). \qquad (6.13)\end{aligned}$$

In view of the weak formulation of the convection-diffusion-reaction problem.

$$\begin{aligned}-\nabla\cdot(\lambda\nabla u) + b\cdot\nabla u &= F, \quad in\ \Omega, \qquad (6.14)\\ u &= g, \quad on\ \partial\Omega, \qquad (6.15)\end{aligned}$$

this defined may be expressed by using that $E_h u$ is the WG finite element approximation of the solution of the corresponding convection-diffusion problem with exact solution u.

Lemma 6.4. (*see*[*1*])
Assume that problem (6.14) − (6.15) *has the* $H^{1+s}(\Omega)$ *regularity* ($s \in (0,1]$). *Let* $u \in H^{1+\kappa}(\Omega)$ *be the exact solution of* (6.14) − (6.15), *and* $E_h u$ *be a WG approximation of* u *defined in* (6.13). *Let* $Q_h u = \{Q_0 u, Q_b u\}$ *be the* L^2 *projection of* u *in the corresponding finite element space. Then there exists a constant* C *such that*

$$\|Q_0 u - E_h u\| \leq C(h^{\kappa+1}\|F - Q_0 F\| + h^{\kappa+s}\|u\|_{\kappa+1})$$

and

$$\|\nabla_{d,r}(Q_h u - E_h u\| \leq Ch^{\kappa}\|u\|_{\kappa+1}.$$

Theorem 6.2. *Under the assumption of Theorem* (6.1) *and the assumption that the corresponding convection-diffusion problem has the* H^{1+s} *regularity* ($s \in (0,1]$), *there exists a constant* C *such that*

$$\begin{aligned}\|u_h(t) - Q_h u(t)\| &\leq \|u_h(0) - Q_h u(0)\| + Ch^{\kappa+s}(\|\psi\|_{\kappa+1} + \int_0^t \|u_t\|_{\kappa+1} ds)\\ &+ Ch^{s+1}(\int_0^t (\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|) ds)\\ &+ Ch^{s+1}(\|f(0) - Q_0 f(0)\| + \|u_t(0) - Q_0 u_t(0)\|) \qquad (6.16)\end{aligned}$$

Proof. The error in the problem (1.1) − (1.3) is written as a sum of two terms,

$$u_h(t) - Q_h u(t) = \theta(t) + \rho(t), \qquad (6.17)$$

where

$$\theta = u_h - E_h u, \qquad \rho = E_h u - Q_h u.$$

The error bound for ρ easily by lemma (6.4) as the following [2]

$$\begin{aligned}\|\rho\| &\leq C(h^{s+1}(\|f - Q_0 f\| + \|u_t - Q_0 u_t\|)\\ &+ h^{\kappa+s}(\|\psi\|_{\kappa+1} + \int_0^t \|u_t\|_{\kappa+1} ds)). \qquad (6.18)\end{aligned}$$

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Now, to estimate θ, we note that by our definitions

$$\begin{aligned}(\theta_t, w) + a(\theta, w) &= ((u_h)_t, w) + a(u_h, w) - (E_h u_t, w) - a(E_h u_h, w)\\ &= (f, w) - (E_h u_t, w) - a(E_h u_h, w)\\ &= (f, w) + (\nabla \cdot (\lambda \nabla u), w) + (b \cdot \nabla u, w) - (cu, w) - (E_h u_t, w)\\ &= (u_t, w) - (E_h u_t, w)\\ &= (Q_h u_t, w) - (E_h u_t, w)\\ &= -(\rho_t, w),\end{aligned}$$

which is

$$(\theta_t, w) + a(\theta, w) = -(\rho_t, w), \quad \forall w \in S_h^0(j, \ell), t > 0, \tag{6.19}$$

where we have used the fact that the operator E_h commutes with time differentiation. Since $\theta \in S_h^0(j, \ell)$, we may choose $w = \theta$ in (6.19) and obtain

$$(\theta_t, \theta) + a(\theta, \theta) = -(\rho_t, \theta), \quad t > 0, \tag{6.20}$$

by using lemma (6.1) we have

$$a(\theta, \theta) \geq \alpha(\|\nabla_{d,r}\theta\|^2 + \|\theta_0\|^2) > 0.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = \|\theta\|\frac{d}{dt}\|\theta\| \leq \|\rho_t\|\|\theta\|,$$

and integrating with respect to t, we obtain

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\| ds. \tag{6.21}$$

using lemma (6.3), we have

$$\begin{aligned}|\theta(0)\| &= \|u_h(0) - E_h u(0)\|\\ &\leq \|u_h(0) - Q_h u(0)\| + \|E_h u(0) - Q_h u(0)\|\\ &\leq \|u_h(0) - Q_h u(0)\| + C(h^{s+1}(\|f(0) - Q_0 f(0)\|\\ &+ \|u_t(0) - Q_0 u_t(0)\|) + h^{\kappa+s}\|\psi\|_{\kappa+1}),\end{aligned} \tag{6.22}$$

and since

$$\begin{aligned}\|\rho_t\| &= \|E_h u_t - Q_h u_t\|\\ &\leq C(h^{s+1}(\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|)\\ &+ h^{\kappa+s}\|u_t\|_{\kappa+1}).\end{aligned} \tag{6.23}$$

Substituting (6.18) and (6.21) into (6.17), we have an optimal order of error estimate in L^2 which completes the proof. $\square$

7 Numerical result

In this section, we present some numerical results to illustrate the theoretical analysis in the previous section. We consider the following convection-diffusion-reaction problem.

$$u_t - \nabla \cdot (D\nabla u) + b \cdot \nabla u + cu \quad = \quad f, \qquad in\ \Omega \times J, \tag{7.1}$$

with homogeneous Dirichlet boundary condition and initial condition. The data for problem (7.1) taken as follows: let $D = 100$, Ω be a unit square, i.e., $\Omega = [0,1] \times [0,1]$, time interval be $J = (0,T) = (0,1)$, the absorption coefficient is $c = 1$ and the velocity vector has been taken as $b = (\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}))$, we can get the initial and boundary conditions and source term $f(x,t)$ according to the corresponding analysis solution of example. First, we partition the square domain $\Omega = (0,1) \times (0,1)$ in to $N \times N$ sub-square uniformly. Then we divide each square element into two triangles by the diagonal line with a negative slopeso that we complete the construction of the triangular mesh let $h = 1/N(N = 4, 8, 16, 32, 64)$ be mesh size for triangular meshes.

In the example, the analytical solution is chosen as

$$u = sin(\pi x)sin(\pi y)exp(-t).$$

Numerical error results and convergence rate are listed in Table 7.1 and convergence rate in Figure 1.

Table 7.1:numerical result

h	L^2-**error**	L^2-**order**
1/4	3.7148e-00	
1/8	9.4454e-01	1.97
1/16	2.3719e-01	1.99
1/32	5.9383e-02	2.00
1/64	1.4875e-02	2.00

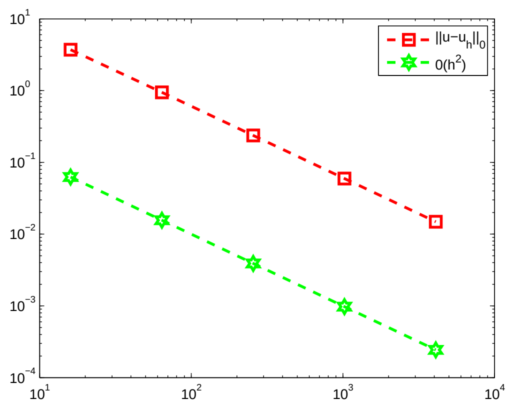

Figure 1: Convergence rate for $\kappa = 1$ and $s = 1$.

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The Generalized Moment Problem on White Noise Spaces

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Abstract

Our purpose in this paper, is to derive the main properties of the generalized moment functions defined on some types of white noise spaces. A new version of Wick product on some spaces of generalized functions is introduced. Applying the direct connection between the theory of construction for hypercomplex systems and white noise analysis, we setup a framework to construct a lot of spaces of generalized functions connected with different examples of hypercomplex systems.

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1 Introduction

In this paper, the main properties of the generalized moment functions defined on some types of white noise spaces are derived. A new version of Wick product with respect to non-Gaussian measures, the associated Hermite transform and the characterization theorem for the constructed spaces of generalized functions are introduced. Let Q denotes a locally compact basis on the space $\mathbb{R}^n$. The linear space of bounded continuous complex-valued functions $C_b(Q)$ is complete normed space with respect to the norm

$$\|f\|_\infty = \sup_{x\in Q} |f(x)|,$$

where f define on Q. We will denote by $C_b^\infty(Q)$ the space of infinitely differential bounded functions on Q, and by $\mathcal{S}(Q)$ the linear subspace of $C_b^\infty(Q)$ formed by the set of functions on Q such that $x^\alpha D^\beta f(x)$ is bounded on Q, where $\alpha, \beta \in \mathbb{Z}_+^n$. The space of continuous linear functional on $\mathcal{S}(Q)$ is called tempered distribution space and is denoted by $\mathcal{S}'(Q)$. There exist many works aims to investigate white noise spaces. Some of these works devoted to deal with the construction of spaces of test, generalized functions and operators acting in these spaces using the Wiener-Itô-Segal isomorphism and various riggings of the Fock space [2,9]. Distribution play a crucial role in the study of PDEs and quantum field theory [5,11], where quantum field are defined as operator valued distributions. The contemporary theory of generalized functions of infinitely many variables originates from the works of Berezanskyi and Samoilenko [3] and Hida [9]. In [3], the spaces of test and generalized functions were constructed as infinite tensor products of one-dimensional spaces. In [9], the classical approach to the construction of the theory of generalized functions was, in fact, used, but all functions under consideration were functions of a point of the infinite-dimensional space on which the Gaussian measure was defined; this measure played the same role as the Lebesgue measure in the classical theory of generalized functions. This paper is organized as follows: In section 2, we give the main properties of the generalized moment functions defined on the space of rabidly decreasing functions on Q. In section 3, a new way for constructing spaces of generalized functions is given. In section 4, we derive the main relations between the construction of hypercomplex system and the Theory of white noise analysis.

2 The moment problem on $\mathcal{S}(Q)$

The elements of $\mathcal{S}(Q)$ are called rabidly decreasing functions and for each $\alpha, \beta \in \mathbb{Z}_+^n$, $\mathcal{S}(Q)$ is equipped with the family of seminorms

$$\|f\|_{\alpha,\beta} = \sup_{x\in Q} |x^\alpha D^\beta f(x)|$$

In this section, we devoted to give a full description of the integral

$$\phi(x) = \int_Q \lambda(x) d\mu(\lambda), \quad \mu \in \mathcal{M}_+(Q),$$

where $\lambda : Q \to \mathbb{C}$ belongs to the linear space of bounded continuous complex-valued functions $C_b(Q)$ and the measure μ belongs to the space of positive Radon measures $\mathcal{M}_+(Q)$. Let $s = (s_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(s_0 > 0)$ be an n-sequence of real numbers. We set

$$\mathcal{L}_s(x^\alpha) = s_\alpha, \quad \alpha \in \mathbb{Z}_+^n.$$

The n-sequence $s = (s_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ is called quasi-positive definite if $\mathcal{L}_s$ is quasi-positive definite (i.e., $\mathcal{L}_s(f\bar{f}) \geq 0$ for all $f \in \mathcal{S}(Q)$). The n-sequence s is called a generalized moment sequence if there exists a Radon measure μ on Q such that $x^\alpha \in L_1(\mu)$ and $s_\alpha = \int_Q x^\alpha d\mu(x)$ for all $\alpha \in \mathbb{Z}_+^n$. When such measure exists, then it is called a representing measure of the sequence s. Let $\mathcal{F} = f_1, ..., f_m$ be a finite family in $\mathcal{S}(Q)$, and

$$Q_\mathcal{F} = \{q \in Q; f_j(q) \geq 0, \quad i = 1, ..., m\}.$$

Clearly, we have $m_j = sup_{x\in Q} f_j(x) < \infty$, setting

$$\hat{f}_j(x) = m_j^{-1} f_j(x), \quad x \in Q \quad if \quad m_j > 0,$$

and

$$\hat{f}_j = f_j, \quad m_j = 0,$$

$j = 1, ..., m$. We define $\hat{\mathcal{F}} = \{0, 1, \hat{f}_1, ..., \hat{f}_m\}$, and we will denote by $\Delta_{\mathcal{F}}$ the set of all products of the form $f_1...f_i(1-g_1)...(1-g_i)$ for functions $f_1, ..., f_i, g_1..., g_i \in \hat{\mathcal{F}}$ and integers $i, j \geq 1$.

Theorem 2.1. Let $\Pi(\mathcal{F})$ denote the convex set of all linear mappings $\mathcal{L} : \mathcal{S}(Q) \to \mathbb{R}$ such that $\mathcal{L}(1) = 1$ and $\mathcal{L}(f) \geq 0$ for all $f \in \Delta_{\mathcal{F}}$. Then we have

$$0 \leq \mathcal{L}(f) \leq 1$$

for all $\mathcal{L} \in \Pi(\mathcal{F})$ and $f \in \Delta_{\mathcal{F}}$.

Proof. Let $f_1...f_k \in \Delta_{\mathcal{F}}$, where either $f_j \in \hat{\mathcal{F}}$ or $1 - f_j \in \hat{\mathcal{F}}$ for all $j = 1, ..., k$. We have

$$f_1...f_n = (1 - f_1) + f_1(1 - f_2) + ... + f_1...f_{k-1}(1 - f_k)$$

This implies $\mathcal{L}(1 - f) \geq 0$, whence $\mathcal{L}(f) \leq 1$.

Remark. Let $\Gamma_+(Q)$ be the positive cone generated by $\Delta_{\mathcal{F}}$. From the previous proof we notice that if $f \in \Delta_{\mathcal{F}}$, then $1 - f \in \Gamma_+(Q)$. Moreover, we notice that if $f, g \in \Delta_{\mathcal{F}}$, then $(1 - f)g \in \Gamma_+(Q)$. In particular, if $\mathcal{L} : \mathcal{S}(Q) \to \mathbb{R}$ is positive on $\Delta_{\mathcal{F}}$, then $\mathcal{L}((1 - f)g) \geq 0$ for all $f, g \in \Delta_{\mathcal{F}}$. Finally, we notice that $\mathcal{L}(1) = 0$ implies $\mathcal{L} = 0$.

Lemma 2.2. Let $\mathcal{L}$ be an extreme point of the convex set $\Pi(\mathcal{F})$. Then $\mathcal{L}$ is multiplicative on $\mathcal{S}(Q)$.

Proof. Suppose $f \in \Delta_{\mathcal{F}}$ be fixed. Sufficiently, we need to prove that

$$\mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g) \quad for \quad all \quad g \in \Delta_{\mathcal{F}}$$

Let $d = \mathcal{L}(f)$. We have the following possibilities:

1. If $0 < d < 1$, we consider the linear functionals $\mathcal{L}_1(h) = d^{-1}\mathcal{L}(fh)$ and $\mathcal{L}_2(h) = (1 - d)^{-1}\mathcal{L}((1 - f)h), \quad h \in \mathcal{S}(Q)$. Clearly, $\mathcal{L}_1, \mathcal{L}_2 \in \Pi(\mathcal{F})$. since $\mathcal{L} = d\mathcal{L}_1 + (1 - d)\mathcal{L}_2$ and $\mathcal{L}$ is an extreme point of $\Pi(\mathcal{F})$, this implies $\mathcal{L} = \mathcal{L}_1$, whence $\mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g)$.

2. If $d = 0$, then the functional $\mathcal{L}_0(h) = \mathcal{L}(fh)$ is positive on $\Delta_{\mathcal{F}}$ and $\mathcal{L}_0(1) = 0$, applying the above remark implies $\mathcal{L}_0 = 0$, whence $\mathcal{L}(fg) = 0 = \mathcal{L}(f)\mathcal{L}(g)$.

3. If $d = 1$, we use the above discussion to the functional $\mathcal{L}_1(g) = \mathcal{L}((1 - f)g)$, and obtain $\mathcal{L}(fg) = \mathcal{L}(g) = \mathcal{L}(f)\mathcal{L}(g)$.

Theorem 2.3. For every linear functional $\mathcal{L} \in \Pi(\mathcal{F})$ there exists a uniquely probability measure μ on Q such that

$$\mathcal{L}(f) = \int_Q f d\mu$$

for all $f \in \mathcal{S}(Q)$.

Proof. Let $\mathcal{L}_0 \in \Pi(\mathcal{F})$ be an extreme point. Then $\mathcal{L}_0$ is multiplicative on $\mathcal{S}(Q)$, by the above lemma. Thus, for the sequence $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n$ defined by $\mathcal{L}_0(t_j) = \gamma_j$, we have $\mathcal{L}_0(f) = f(\gamma)$ for all $f \in \mathcal{S}(Q)$. But we have $0 \leq \mathcal{L}_0(f) \leq 1, \quad f \in \Delta_{\mathcal{F}}$, by Theorem 2.1, we obtain that

$$|\mathcal{L}_0(f)| = |f(\gamma)| \leq ||f||_Q = sup_{t\in Q}|f(t)|, \quad f \in \mathcal{S}(Q).$$

If $f \in \Pi(\mathcal{F})$ is of the form $\mathcal{L} = \sum_{j\in I} c_j L_j$, where $c_j \geq 0, \quad \sum_{j\in I} c_j = 1$, $\mathcal{L}_j$ an extreme point of $\Pi(\mathcal{F})$, then

$$|\mathcal{L}(f)| \leq \sum_{j\in I} c_j|\mathcal{L}_j(f)| \leq \sum_{j\in I} c_j||f||_Q = ||f||_Q, \quad f \in \mathcal{S}(Q).$$

Let $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}(\gamma_0 > 0$ be a generalized moment sequence. Then the linear form $\mathcal{L} = \gamma_0^{-1}$ is an element of $\Pi(\mathcal{F})$, and by using the result obtained from the above Theorem we have:

Corollary 2.4. Let Q is compact and $\mathcal{F} = f_0 = 1, f_1, ..., f_m$ be a finite family which generates the space $\mathcal{S}(Q)$. An n-sequence of real numbers $s = (s_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(s_0 > 0)$ is a generalized moment sequence if and only if the linear form $\mathcal{L}_s$ is nonnegative on the set $\Delta_{\mathcal{F}}$.

3 The spaces of generalized functions

This section is devoted to give the main relations between the construction of hypercomplex system and the Theory of white noise analysis. We will consider the following rigging of a Hilbert space H_0 with positive and negative spaces H_+ and H_-:

$$H_- \supseteq H_0 \supseteq H_+. \tag{3.1}$$

Let $\mathbf{I}_0^+ : H_- \longrightarrow H_+$ be the canonical isometry transferring the negative space H_- onto the positive space H_+. A biorthogonal basis $(p_n, q_n)_{n=0}^\infty$ in the space H_0 can be understood as sequences $(p_n)_{n=0}^\infty \subset H_+$ and $(q_n = \mathbf{I}_0^- p_n)_{n=0}^\infty \subset H_-$, where the first sequence is an orthogonal basis in the positive space H_+ and the second is an orthogonal basis in the negative space H_-. Hence, these systems of sequences p_n and q_n are biorthogonal:

$$(p_n, q_n)_{H_0} = \delta_{n,m} h_n, \quad h_n = \|p_n\|_{H_+}^2 = \|q_n\|_{H_-}^2, \quad n, m \in \mathbb{Z}_+, \tag{3.2}$$

for all $\varphi \in H_+$,

$$\varphi = \sum_{n=0}^{\infty} \varphi_n p_n, \quad \varphi_n = (\varphi, q_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^{\infty} |\varphi_n|^2 h_n = \|\varphi\|_{H_+}^2 < \infty, \tag{3.3}$$

for all $\xi \in H_-$,

$$\xi = \sum_{n=0}^{\infty} \xi_n q_n, \quad \xi_n = (\xi, p_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^{\infty} |\xi_n|^2 h_n = \|\xi\|_{H_-}^2 < \infty, \tag{3.4}$$

$$(\xi, \varphi)_{H_0} = \sum_{n=0}^{\infty} \xi_n \overline{\varphi_n} h_n. \tag{3.5}$$

Let $(p_n)_{n=0}^{\infty}$ be an arbitrary total sequence of vectors p_n of a Hilbert space H_0. It is easy to prove that such sequence $(h_n)_{n=0}^{\infty}$ of positive numbers h_n exists for which the set of test functions

$$H_+ = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \, p_n \mid \varphi_n \in \mathbb{C} : \|\varphi\|_{H_+}^2 = \sum_{n=0}^{\infty} |\varphi_n|^2 \, h_n < \infty \right\}, \tag{3.6}$$

with the corresponding scalar product is the positive space with respect to H_0. Note that, it is necessary to assume in addition the fulfilment of the following necessary and sufficient condition on $(p_n)_{n=0}^{\infty}$: an arbitrary sequence $(\varphi^{(i)})_{i=0}^{\infty}$ of vectors $\varphi^{(i)} \in H_+$ with finite sequences of coordinates $\varphi_n^{(i)}$ which is fundamental in H_+ and converges to 0 in H_0 must converge to 0 in H_+. This condition will always be fulfilled in our case. Similarly, for the negative space H_-, by replacing p_n by q_n, we have the set of generalized functions as follows

$$H_- = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n \, q_n \mid \xi_n \in \mathbb{C} : \|\xi\|_{H_-}^2 = \sum_{n=0}^{\infty} |\xi_n|^2 \, h_n < \infty \right\}. \tag{3.7}$$

As pointed out from [1-3], there exists a quasinuclear rigging such that, the zero space H_0 is a hypercomplex system $L_2(Q, dm(p))(p \in Q)$ and we assume that

$$\mathbf{I}_\chi^+ : H_+ \longrightarrow H_1^\chi, \qquad \mathbf{I}_\chi^- : H_- \longrightarrow H_{-1}^\chi.$$

such that

$$\langle \mathbf{I}_\chi^- \xi, \mathbf{I}_\chi^+ \varphi \rangle_{L_2(Q,dm(p))} = \langle \xi, \varphi \rangle_{H_0}, \quad \xi \in H_-, \varphi \in H_+.$$

So, we have a biunitary map $\{\mathbf{I}_\chi^-, \mathbf{I}_\chi^+\}$. This mapping transfers the rigging of the space H_0 to a rigging of the hypercomplex space $L_2(Q, dm(p))$:

$$\begin{array}{ccccc} H_- & \supseteq & H_0 & \supseteq & H_+. \\ \Big\downarrow \mathbf{I}_\chi^- & & & & \Big\downarrow \mathbf{I}_\chi^+ \\ H_{-1}^\chi & \supseteq & L_2(Q, dm(p)) & \supseteq & H_1^\chi, \end{array} \tag{3.8}$$

Hence, we consider the space H_1^χ is a positive space of the form

$$H_1^\chi = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \, \chi_n \mid \, : \|\varphi\|^2_{H_1^\chi} = \sum_{n=0}^{\infty} |\varphi_n|^2 \, (n!)^2 K^n < \infty \right\}, \tag{3.9}$$

where $p_n = \chi_n, h_n = (n!)^2 K^n, n \in \mathbb{Z}_+$, ($K > 1$ is a fixed sufficiently large number), and consists of continuous functions on Q. Similarly, for the space H_1^χ, we have

$$H_{-1}^\chi = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n \, \chi_n \mid \, : \|\xi\|^2_{H_{-1}^\chi} = \sum_{n=0}^{\infty} |\xi_n|^2 \, (n!)^2 K^n < \infty \right\}, \tag{3.10}$$

The system $(\chi_n, q_n^\chi)_{n=0}^\infty$, where $q_n^\chi = \mathbf{I}_1^- \chi_n \in H_{-1}^\chi$, is a biorthogonal basis of the space $L_2(Q, dm(p))$. It is essential to introduce the rigging of the hypercomplex space $L_2(Q, dm(p))$ by means of projective and inductive limits of Hilbert spaces which are constructed by rules of type (3.6), (3.8) and (3.9). For every $q \in \mathbb{N}$, we define the Hilbert space of type (3.6):

$$H_q^\chi = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \chi_n \in H_0 : \|\varphi\|^2_{H_q^\chi} = \sum_{n=0}^{\infty} |\varphi_n|^2 (n!)^2 K^{qn} < \infty \right\}. \tag{3.11}$$

Then, we have the rigging:

$$(\Psi^\chi)' \supseteq H_{-q}^\chi \supseteq L_2(Q, dm(p)) \supseteq H_q^\chi \supseteq \Psi^\chi, \tag{3.12}$$

$$\Psi^\chi = \operatorname{pr}\lim_{q \in \mathbb{N}} H_q^\chi = \bigcap_{q \in \mathbb{N}} H_q^\chi, \qquad (\Psi^\chi)' = \operatorname{ind}\lim_{q \in \mathbb{N}} H_{-q}^\chi = \bigcup_{q \in \mathbb{N}} H_{-q}^\chi,$$

$$H_{-q}^\chi = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n q_n^\chi : \|\xi\|^2_{H_{-q}^\chi} = \sum_{n=0}^{\infty} |\xi_n|^2 (n!)^2 K^{-qn} < \infty \right\}, \tag{3.13}$$

with the action

$$(\xi, \varphi)_{L_2(Q, dm(p))} = \sum_{n=0}^{\infty} \xi_n \overline{\varphi_n} (n!)^2 K^{qn}, \quad \varphi \in H_q^\chi, \;\; \xi \in H_{-q}^\chi.$$

To illustrate the above result, we give the following example

Example 3.1. In the classical case when $H_0 := L_2(\mathbb{R}, dx)$ with respect to the Lebesgue measure dx and ordinary convolution. Then, the generalized character $\chi(x, \lambda) = e^{\lambda x}$ ($\lambda \in \mathbb{C}$) and $\chi_n(x) = x^n$ ($x \in \mathbb{R}, \; n \in \mathbb{Z}_+$). Therefore, the space (3.11) consists of entire functions $\varphi(x)$ and $\varphi_n(x)$ are the Taylor coefficients of $\varphi(x)$. Formula (3.2) gives their representation as the Fourier coefficients using the scalar product $(\xi, \varphi)_{H_0}$, ($\xi \in H_{-1}^\chi, \; \varphi \in H_1^\chi$).

Remark. Obviously, such a generalization gives the possibility of constructing a lot of spaces of generalized functions connected with different examples of hypercomplex systems.

6

4 The generalized Wick product

In this section, we devoted to introduce a new version of Wick product with respect to non-Gaussian measures, the associated Hermite transform and the characterization theorem for the constructed spaces of generalized functions. Wick is the first one introduced the product between two functions in white noise space, so this product carry his name [13]. He was used as a tool to renormalize certain infinite quantities in quantum field theory. Later on, the Wick product was considered, in a stochastic ordinary and partial differential equations (see, e.g., [6,8,10]). Under the assumption that $\|\chi\|^2_{H_0} \leq C^n$ for some $C > 0$, we define a new Wick product, called χ-Wick product on the space H^{χ}_{-q}. Then, we give the definition of the χ-Hermite transform and apply it to establish a characterization theorem for the space H^{χ}_{-q}.

Definition 4.1. Let $\xi = \sum_{m=0}^{\infty} \xi_m q^{\chi}_m$, $\eta = \sum_{n=0}^{\infty} \eta_n q^{\chi}_n \in H^{\chi}_{-q}$ with $\xi_m, \eta_n \in \mathbb{C}$. The χ-Wick product of ξ, η, denoted by $\xi \diamond_{\chi} \eta$, is defined by the formula

$$\xi \diamond_{\chi} \eta = \sum_{m,n=0}^{\infty} \xi_m \eta_n q^{\chi}_{m+n}. \tag{4.1}$$

It is important to show that the spaces $H^{\chi}_{-q}, H^{\chi}_{q}$ are closed under χ-Wick product.

Lemma 4.2. If $\xi, \eta \in H^{\chi}_{-q}$ and $\varphi, \psi \in H^{\chi}_{q}$, we have

(i) $\xi \diamond_{\chi} \eta \in H^{\chi}_{-q}$,

(ii) $\varphi \diamond_{\chi} \psi \in H^{\chi}_{q}$.

Proof. If $\xi = \sum\limits_{m=0}^{\infty} \xi_m q^{\chi}_m$, $\eta = \sum\limits_{n=0}^{\infty} \eta_n q^{\chi}_n \in H^{\chi}_{-q}$, then for some $q_1 \in \mathbb{N}$ we have

$$\sum_{m=0}^{\infty} |\xi_m|^2 K^{-q_1 m} < \infty \text{ and } \sum_{n=0}^{\infty} |\eta_n|^2 K^{-q_1 n} < \infty. \tag{4.2}$$

We note that

$$\xi \diamond_{\chi} \eta = \sum_{m,n=0}^{\infty} \xi_m \eta_n q^{\chi}_{m+n} = \sum_{l=0}^{\infty} \left(\sum_{m+n=l}^{\infty} \xi_m \eta_n \right) q^{\chi}_l = \sum_{l=0}^{\infty} \zeta_l q^{\chi}_l, \tag{4.3}$$

where $\zeta_l = \sum\limits_{m+n=l}^{\infty} \xi_m \eta_n$. With $q = q_1 + p$ we have

$$\begin{aligned}
\sum_{l=0}^{\infty} |\zeta_l|^2 K^{-ql} &= \sum_{l=0}^{\infty} \left| \sum_{m+n=l}^{\infty} \xi_m \eta_n \right|^2 K^{-q_1 l} K^{-pl} \\
&\leq \sum_{l=0}^{\infty} \left(\sum_{m+n=l}^{\infty} |\xi_m|^2 K^{-q_1 m} \right) \left(\sum_{m+n=l}^{\infty} |\eta_n|^2 K^{-q_1 n} \right) K^{-pl} \\
&\leq \left(\sum_{l=0}^{\infty} K^{-pl} \right) \left(\sum_{m=0}^{\infty} |\xi_m|^2 K^{-q_1 m} \right) \left(\sum_{n=0}^{\infty} |\eta_n|^2 K^{-q_1 n} \right) \\
&< \infty,
\end{aligned} \tag{4.4}$$

which proves $(\boldsymbol{i})$. The proof of $(\boldsymbol{ii})$ is similar. ■

The following important algebraic properties of the χ-Wick product follow directly from Definition 4.1.

Lemma 4.3. For each $\xi, \eta, \zeta \in H_{-q}^{\chi}$, we get

$(\boldsymbol{i})$ $\xi \diamond_{\chi} \eta = \eta \diamond_{\chi} \xi$ (Commutative law),

$(\boldsymbol{ii})$ $\xi \diamond_{\chi} (\eta \diamond_{\chi} \zeta) = (\xi \diamond_{\chi} \eta) \diamond_{\chi} \zeta$ (Associative law),

$(\boldsymbol{ii})$ $\xi \diamond_{\chi} (\eta + \zeta) = \xi \diamond_{\chi} \eta + \xi \diamond_{\chi} \zeta$ (Distributive law).

Remark. According to Lemmas 4.2. and 4.3., we can conclude that the spaces H_{-q}^{χ} and H_{q}^{χ} form topological algebras with respect to the χ-Wick product.

From the above arguement, the χ-Wick product satisfies all the ordinary algebraic rules for multiplication. But, there are some problems when limit operations are involved. To treat these situations it is convenient to apply a transformation, called the χ-Hermite transform, which converts χ-Wick products into ordinary (complex) products and convergence in H_{-q}^{χ} into bounded, pointwise convergence in a certain neighborhood of 0 in $\mathbb{C}$.

Definition 4.4. Let $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}$ with $\xi_n \in \mathbb{C}$. Then, the χ-Hermite transform of ξ, denoted by $\mathcal{H}_{\chi}\xi$, is defined by

$$\mathcal{H}_{\chi}\xi(z) = \sum_{n=0}^{\infty} \xi_n z^n \in \mathbb{C} \quad \text{(when convergent)}. \tag{4.5}$$

In the following, we define for $0 < M, q < \infty$ the neighborhoods of zero in $\mathbb{C}$ which denoted it by $\mathbb{O}_{q,M}(0)$:

$$\mathbb{O}_{q,M}(0) = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} |z^n|^2 K^{qn} < M^2 \right\}. \tag{4.6}$$

It is easy to see that

$$q \leq p,\ N \leq M \Rightarrow \mathbb{O}_{q,N}(0) \subseteq \mathbb{O}_{q,M}(0). \tag{4.7}$$

Note that, if $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}, z \in \mathbb{O}_{q,M}(0)$ for some $0 < M, q < \infty$, we have the estimate

$$\begin{aligned} \sum_{n=0}^{\infty} |\xi_n||z^n| &= \sum_{n=0}^{\infty} |\xi_n||z^n| K^{-\frac{qn}{2}} K^{\frac{qn}{2}} \\ &\leq \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn} \right)^{\frac{1}{2}} \\ &< M \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn} \right)^{\frac{1}{2}} \\ &< \infty. \end{aligned} \tag{4.8}$$

The conclusion above can be stated as follows:

Proposition 4.5. If $\xi \in H^{\chi}_{-q}$, then $\mathcal{H}_{\chi}\xi$ converges for all $z \in \mathbb{O}_q(M)$ for all $q, M < \infty$.

Proposition 4.6. If $\xi, \eta \in H^{\chi}_{-q}$, then

$$\mathcal{H}_{\chi}(\xi \diamond_{\chi} \eta)(z) = \mathcal{H}_{\chi}\xi(z).\mathcal{H}_{\chi}\eta(z). \tag{4.9}$$

for all z such that $\mathcal{H}_{\chi}\xi$ and $\mathcal{H}_{\chi}\eta$ exist.

Proof. The proof is an immediate consequence of Definitions 4.1. and 4.4.

Let $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H^{\chi}_{-q}$, with $\xi_n \in \mathbb{R}$. Then, the number $\xi_0 = \mathcal{H}_{\chi}\xi(0) \in \mathbb{R}$ is called the generalized expectation of ξ and is denoted by $\mathbb{E}(\xi)$. Suppose that $V \ni z \mapsto f(z) \in \mathbb{C}$ is an analytic function, where V is a neighborhood of $\mathbb{E}(\xi)$. Assume that the Taylor series of f around $\mathbb{E}(\xi)$ has coefficients in $\mathbb{R}$. Then, the χ-Wick version $f^{\diamond_{\chi}}$ of f is defined by

$$H^{\chi}_{-q} \ni \xi \mapsto f^{\diamond_{\chi}}(\xi) = \mathcal{H}^{-1}\big(f \circ \mathcal{H}_{\chi}(\xi)\big) \in H^{\chi}_{-q}. \tag{4.10}$$

Example 4.7. If the function $f : \mathbb{C} \to \mathbb{C}$ is entire, then $f^{\diamond_{\chi}}$ is defined for all $\xi \in H^{\chi}_{-q}$. For example, the χ-Wick exponential is defined by

$$\exp^{\diamond_{\chi}}(\xi) = \sum_{j=0}^{\infty} \frac{1}{j!}\xi^{\diamond_{\chi} n}. \tag{4.11}$$

5 Concluding Remarks

The space of continuous linear functional on $\mathcal{S}(Q)$ are called tempered distributions, and is denoted by $\mathcal{S}'(Q)$. Let $L \in \mathcal{S}'(Q)$ and $\alpha \in \mathbb{Z}^d_+$. The weak derivative $D^{\alpha}L$ (or the derivative of the sense of distributions) is given by

$$(D^{\alpha}L)(f) = (-1)^{|\alpha|}L(D^{\alpha}f) \tag{5.1}$$

for $f \in (Q)$. This corresponds to $D^{\alpha}L\{g\} = L\{D^{\alpha}g\}$. Note that distribution always has a weak derivative. A function f is completely monotonic if for each $\alpha \in \mathbb{Z}^n_+$, $(-1)^{|\alpha|}D^{\alpha}f(x) \geq 0$ on $\mathbb{R}^n_+$; see [4, 7, 12] for many properties of completely monotonic functions. Bernstien's theorem asserts that f is completely monotonic if and only if $f(x) = \int_{\mathbb{R}^n} e^{-x.t}d\mu(t)$ where μ is a positive measure supported on a subset of $\mathbb{R}^n_+$. If assume that $Q = \mathbb{R}^n$. So, $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Let x^{α} be denote the product $x_1^{\alpha_1} ... x_n^{\alpha_n}$, $\mathbb{Z}^n_+$ denote the set of n-tuples $(\alpha_1, ..., \alpha_n)$ where each α_i is a non-negative integer, $|\alpha| = \sum_{i=1}^n \alpha_i$ and D^{α} denote the partial differential operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$. Then, we obtain the special case $\mathcal{S}(Q) = \mathcal{S}(\mathbb{R}^n)$ is the space of rabidly decreasing function on $\mathbb{R}^n$ (so-called Schwartz space) and its dual $\mathcal{S}'(Q) = \mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distribution on $\mathbb{R}^n$.

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QUADRATIC TYPE FUNCTIONAL INCLUSIONS ON SQUARE-SYMMETRIC GROUPOIDS AND HYERS-ULAM STABILITY

GWANG HUI KIM AND HWAN-YONG SHIN

Abstract. We consider that a set-valued map $F : X \to \mathcal{P}_0(Y)$ satisfying the functional inclusion $F(x * y)\Diamond F(x * y^{-1}) \subseteq \sigma_\Diamond(F(x)\Diamond F(y))$(or $\sigma_\Diamond(F(x)\Diamond F(y)) \subseteq \sigma_\Diamond(F(x * y)\Diamond F(x * y^{-1})))$ admits a unique selection $f : X \to Y$ satisfying the functional equation $f(x*y)\diamond f(x* y^{-1}) = \sigma_\diamond(f(x) \diamond f(y))$ in appropriate conditions, where $(X, *)$, $(Y, \diamond)$ are square-symmetric groupoids and $\Diamond$ is the extension of $\diamond$ to the collection $\mathcal{P}_0(Y)$ of all nonempty subsets of Y.

1. Introduction

Let $(X, *), (Y, \diamond)$ be groupoids with binary operations. If the binary oepration $*$ satisfies the following inequality

$$(x * y) * (x * y) = (x * x) * (y * y), \quad x, y \in X$$

then the operation $*$ is called square-symmetric. Note that the square symmetric $*$ implies that $\sigma_*(x) := x*x$ is an endomorphism. A binary operation $*$ such that σ_* is an automorphism of $(X, *)$ is called divisible and the corresponding groupoid is said to be a divisible groupoid. The triple $(Y, \diamond, d)$ is called a metric groupoid if $(Y, \diamond)$ is a groupoid, (Y, d) is a metric space and $\diamond$ is a continuous operation with respect to the topology of (Y, d). For a nonempty set Y we denote by $\mathcal{P}_0(Y)$ the collection of all nonempty subsets of Y. The diameter of a set $A \in \mathcal{P}_0(Y)$ is defined by

$$\delta(A) := \sup\{d(x, y)|x, y \in A\}.$$

The Lipschitz modulus of a function $f : X \to Y$ is the smallest real extended number L with the property

$$d(f(x), f(y)) \le Ld(x, y), \quad x, y \in Y.$$

The Lipschitz modulus of a function f is denoted by $Lipf$. A selection of a set-valued mapping $F : X \to \mathcal{P}_0$ is a single-valued map $f : X \to Y$ with the property $f(x) \in F(x)$ for all $x \in X$.

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In a linear normed space $(Y, \|\cdot\|)$ we define the following families of sets

$$\begin{aligned} c(Y) &:= \{A : A \in \mathcal{P}_0(Y), A \text{ is convex set}\} \\ ccl(Y) &:= \{A : A \in \mathcal{P}_0(Y), A \text{ is closed and convex set}\} \\ cc(Y) &:= \{A : A \in \mathcal{P}_0(Y), A \text{ is compact and convex set}\}. \end{aligned}$$

The theory of stability of functional equations had been formulated by Ulam [14]. In 1941, Hyers [3] had answered affirmatively the question of Ulam for Banach spaces and it represents the starting point of the Hyers–Ulam stability of functional equations. Let us recall the Hyers' result.

Theorem 1.1. [3] *Let X be a linear normed space, Y a Banach space and $\varepsilon > 0$. If a function $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in X \tag{1.1}$$

then there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leq \varepsilon, \quad x \in X. \tag{1.2}$$

Smajdor [13] and Gajda and Ger [2] observed an interesting connection between the stability of the Cauchy functional equation and set-valued functions satisfying $F(x+y) \subseteq F(x) + F(y)$. If $f : X \to Y$ satisfies (1.1), then the set-valued mapping $F : X \to \mathcal{P}_0$ defined by

$$F(x) = f(x) + \overline{B}(0, \varepsilon), \quad x \in X,$$

where $\overline{B}(0, \varepsilon)$ is the closed ball in Y centered at 0 and radius $\varepsilon > 0$, implies that $F(x+y) \subseteq F(x) + F(y)$ for $x, y \in X$, and the function g from relation (1.2) is an additive selection of F. Naturally Gajda and Ger [2] considered under what conditions a set-valued mapping with $F(x+y) \subseteq F(x) + F(y)$ admits an additive selection and they obtained the following theorem.

Theorem 1.2. [2] *Let $(S, +)$ be a commutative semigroup with zero element, X a Banach space over $\mathbb{R}$ and $F : S \to ccl(X)$ a set-valued mapping with convex and closed values such that $F(x+y) \subseteq F(x) + F(y)$ for $x, y \in S$ and $\sup_{x \in S} \delta(F(x)) < \infty$. Then F admits a unique additive selection.*

For the last two decades, many mathematicians have developed Theorem 1.2 [6, 9, 10, 11] and investigated various properties of functional inclusion and its connectedness of Hyers–Ulam stability of functional equations [4, 5, 7, 8, 12].

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The aim of this paper is to study some properties for set-valued mappings satisfying the following quadratic type functional inclusions

$$\sigma_{\lozenge}(F(x)\lozenge F(y)) \subseteq F(x*y)\lozenge F(x*y^{-1})$$
$$F(x*y)\lozenge F(x*y^{-1}) \subseteq \sigma_{\lozenge}(F(x)\lozenge F(y))$$

and obtain Hyers-Ulam stability of functional equation.

2. Main results

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this section, suppose that the operation $\lozenge$ satisfies the following condition : for all $\varepsilon > 0$ there exists $\eta > 0$ such that if $\delta(A), \delta(B) < \eta, A, B \in \mathcal{P}_0(Y)$, then

$$\delta(A\lozenge B) < \varepsilon$$

and we assume that X and Y have unique identity id_X and id_Y respectively.

If the operation $\diamond$ satisfies that

$$(x_1 \diamond y_1) \diamond (x_2 \diamond y_2) = (x_1 \diamond x_2) \diamond (y_1 \diamond y_2)$$

for all $x_1, x_2, y_1, y_2 \in Y$, then we say $\diamond$ is bisymmetric operation.

Lemma 2.1. [9] *If $(Y, \diamond)$ is a groupoid with a bisymmetric operation, then $\sigma_\diamond$ is increasing endomorphism of $(\mathcal{P}_0(Y), \lozenge, \subseteq)$.*

Now, we present the main theorem of this paper.

Theorem 2.2. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Y, \diamond, d)$ a complete metric bisymmetric divisible groupoid and $F : X \to \mathcal{P}_0(Y)$ with $F(id_X) = \{id_Y\}$ a set-valued mapping such that*

$$\sigma_{\lozenge}(F(x)\lozenge F(y)) \subseteq F(x*y)\lozenge F(x*y^{-1}) \tag{2.1}$$

for all $x, y \in X$. Assume that

$$\lim_{m\to\infty} \delta(F \circ \sigma_*^{-m}(x))Lip(\sigma_\diamond^{2m}) = 0, \quad \text{and} \tag{2.2}$$
$$\sigma_{\lozenge}^{2n} \circ F \circ \sigma_*^{-n}(x) \in cl(Y)$$

for all $x \in X$ and $n \in \mathbb{N}_0$. Then there exists a unique selection $f : X \to Y$ of F such that

$$\sigma_\diamond(f(x) \diamond f(y)) = f(x*y) \diamond f(x*y^{-1}) \tag{2.3}$$

for all $x, y \in X$.

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Proof. First we prove that there exists a selection of F satisfying (2.3). Consider the set valued mapping $F_n : X \to \mathcal{P}_0(Y)$ corresponding to F defined by

$$F_0 := F, \quad F_n := \sigma_{\Diamond}^{2n} \circ F \circ \sigma_*^{-n}. \tag{2.4}$$

for each $n \in \mathbb{N}$. Letting x, y by $\sigma_*^{-n-1}(x)$ in (2.14) respectively, we get

$$\sigma_{\Diamond}^{2} \circ F(\sigma_*^{-n-1}(x)) \subseteq F(\sigma_*^{-n}(x)) \tag{2.5}$$

for all $x \in X$. By composing $\sigma_{\diamond}^{2n}$ to the both sides of (2.5) and using Lemma 2.1, we obtain

$$\sigma_{\Diamond}^{2n+2} \circ F \circ \sigma_*^{-n-1}(x) \subseteq \sigma_{\Diamond}^{2n} \circ F \circ \sigma_*^{-n}(x)$$

for all $x \in X$ and $n \in \mathbb{N}_0$. This means that $\{F_n(x)\}_{n=0}^{\infty}$ is a decreasing sequence of closed subsets of the Banach space Y. Let $s, t \in F_m(x)$ for some fixed $m \in \mathbb{N}$. Denoting $\sigma_{\diamond}^{-2m}(s) = u$, $\sigma_{\diamond}^{-2m}(t) = v$, we have

$$\begin{aligned} d(s,t) &= d(\sigma_{\diamond}^{2m}(u), \sigma_{\diamond}^{2m}(v)) \leq Lip(\sigma_{\diamond}^{2m}) \cdot d(u,v) \\ &\leq Lip(\sigma_{\diamond}^{2m}) \delta(F \circ \sigma_*^{-m}(x)) \end{aligned}$$

and this implies that

$$\delta(F_m(x)) \leq Lip(\sigma_{\diamond}^{2m}) \cdot \delta(F \circ \sigma_*^{-m}(x)) \tag{2.6}$$

for all $x \in X$. Taking the limit $m \to \infty$ of (2.6), we find that

$$\lim_{m \to \infty} \delta(F_m(x))$$

for all $x \in X$. It is follows from the Cantor intersection theorem in the complete metric spaces that

$$\bigcap_{n=0}^{\infty} F_n(x) \tag{2.7}$$

is singleton $f(x)$. Since the function $f : X \to Y$ satisfies $f(x) \in F_0(x) = F(x)$ for all $x \in X$, f is a selection of F.

Putting x, y for $\sigma_*^{-n}(x)$ and $\sigma_*^{-n}(y)$, respectively in (2.14) and applying $\sigma_{\Diamond}^{2n}$ to the both sides of (2.14), we arrive at

$$\sigma_{\Diamond}(F_n(x) \Diamond F_n(y)) \subseteq F_n(x * y) \Diamond F_n(x * y^{-1}) \tag{2.8}$$

for all $x, y \in X$ and $n \in \mathbb{N}_0$. Since $\{f(x)\} = \bigcap_{n=0}^{\infty} F_n(x), x \in X$, we have $\sigma_{\diamond}(f(x) \diamond f(y)) \in \sigma_{\Diamond}(F_n(x) \Diamond F_n(y))$, for all $x, y \in X$, $n \in \mathbb{N}_0$. Therefore, in view of (2.8), we get

$$d(\sigma_{\diamond}(f(x) \diamond f(y)), f(x * y) \diamond f(x * y^{-1})) \leq \delta(F_n(x * y) \diamond F_n(x * y^{-1})) \tag{2.9}$$

for all $x, y \in X$ and $n \in \mathbb{N}_0$. Taking the limit $n \to \infty$ of (2.9), it is reduced to the equation

$$\sigma_{\diamond}(f(x) \diamond f(y)) = f(x * y) \diamond f(x * y^{-1}), \text{ for all } x, y \in X. \tag{2.10}$$

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To show the uniqueness of f, assume that $g : X \to Y$ is a selection of F such that

$$\sigma_\diamond(g(x) \diamond g(y)) = g(x * y) \diamond g(x * y^{-1}), \text{ for all } x, y \in X. \tag{2.11}$$

From (2.10) and (2.11), it follows that

$$\begin{aligned} f(x) &= \sigma_\diamond^{2n} \circ f \circ \sigma_*^{-n}(x), \\ g(x) &= \sigma_\diamond^{2n} \circ g \circ \sigma_*^{-n}(x) \end{aligned}$$

for all $x \in X, n \in \mathbb{N}$. Hence, for $x \in X$ and $n \in \mathbb{N}$, we see that

$$\begin{aligned} d(f(x), g(x)) &= d(\sigma_\diamond^{2n} \circ f \circ \sigma_*^{-n}(x), \sigma_\diamond^{2n} \circ g \circ \sigma_*^{-n}(x)) \\ &= Liq(\sigma_\diamond^{2n}) d(f \circ \sigma_*^{-n}(x), g \circ \sigma_*^{-n}(x)) \\ &\leq Liq(\sigma_\diamond^{2n}) \delta(F \circ \sigma_\circ^{-n}(x)). \end{aligned}$$

Taking $n \to \infty$, we arrive at the desired conclusion. □

Next, we are going to establish another theorem about the inclusion (2.14).

Theorem 2.3. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Y, \diamond, d)$ a metric bisymmetric divisible groupoid and A a divisible subgroupoid of $(\mathcal{P}_0(Y), \Diamond)$. Suppose that $F : X \to A$ with $F(id_X) = \{id_Y\}$ is a set-valued mapping subject to the condition* (2.14) *and satisfying*

$$\lim_{n\to\infty} \delta(F \circ \sigma_*^n(x)) Lip(\sigma_\diamond^{-2n}) = 0, \quad x \in X. \tag{2.12}$$

Then F is single-valued mapping and

$$\sigma_\Diamond(F(x) \Diamond F(y)) = F(x * y) \Diamond F(x * y^{-1}), \quad \text{for all } x, y \in X. \tag{2.13}$$

Proof. Consider the function $G_n : X \to A$ corresponding to F defined by

$$G_0 := F, \quad G_n := \sigma_\Diamond^{-2n} \circ F \circ \sigma_*^n$$

for each $n \in \mathbb{N}$. Replacing x, y by $\sigma_*^n(x)$ in (2.12) respectively, and then composing on both sides by $\sigma_\Diamond^{-2n-2}$, we have

$$\sigma_\Diamond^{-2n} \circ F \circ \sigma_*^n(x) \subseteq \sigma_\Diamond^{-2n-2} \circ F \circ \sigma_*^{n+1}(x)$$

for all $x, y \in X$. This means that $\{G_n(x)\}_{n=0}^\infty$ is an increasing sequence of $(A, \Diamond)$. By the similar argument in the proof of Theorem 2.2, we see that

$$\lim_{n\to\infty} \delta(G_n(x)) \leq \lim_{n\to\infty} \delta(F \circ \sigma_*^n(x)) Lip(\sigma_\diamond^{-2n}) = 0, \quad \text{for all } x \in X.$$

It implies that $\delta(G_n(x)) = 0$ for every $n \in \mathbb{N}_0$ and $G_n(x)$ is single-valued for all $n \in \mathbb{N}_0$. Therefore, in view of (2.14), $G_0 = F$ satisfies (2.13) and the proof is completed. □

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Corollary 2.4. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Z, \|\cdot\|)$ a Banace space over $\mathbb{R}$, $p, q \in \mathbb{R}$, $p+q \neq 0$, $p+q \neq 1$, and $F : X \to c(Z)$ with $F(id_X) = \{0_Z\}$ a set-valued mapping such that*

$$p(p+q)F(x) + q(p+q)F(y)) \subseteq pF(x*y) + qF(x*y^{-1}) \tag{2.14}$$

for all $x, y \in X$. Assume that there exists $M > 0$ such that

$$\delta(F(x)) \leq M, \quad and$$
$$F \circ \sigma_*^{-n}(x) \in cl(Y)$$

for all $x, y \in X$ and $n \in \mathbb{Z}$. Then there exists a unique selection $f : X \to Z$ of F such that

$$p(p+q)f(x) + q(p+q)f(y) = pf(x*y) + qf(x*y^{-1}), \quad x, y \in X. \tag{2.15}$$

Proof. Consider the operation $\diamond : Z \times Z \to Z$ is defined by

$$x \diamond y = px + qy, \quad x, y \in Z,$$

where $p, q \in \mathbb{R}$ are given real numbers. Then the triple $(Z, \diamond, \|\cdot\|)$ is a metric groupoid with a bisymmetric operation. For all $U, V \in \mathcal{P}_0(Z)$, the operation $\Diamond$ is naturally defined by

$$U \Diamond V = pU + qV$$

and we have $\sigma_\Diamond(U) = (p+q)U$ and in general, $\sigma_\Diamond^n(U) = (p+q)^n U$, for all $n \in \mathbb{N}$. And we get

$$Lip(\sigma_\diamond^{2n}) = |p+q|^{2n}, \quad n \in \mathbb{Z}.$$

If $|p+q| < 1$, we have

$$\sigma_\Diamond^{2n} \circ F \circ_*^{-n}(x) = (p+q)^{2n} F \circ \sigma_*^{-n}(x) \in cl(Z)$$

and

$$\delta(F \circ \sigma_*^{-n}) \leq M|p+q|^{2n}, \quad x \in X, n \in \mathbb{N}_0,$$

thus, by Theorem 2.2, there exists a unique selection of F satisfying (2.15).

If $|p+q| > 1$, we obtain

$$\delta(F \circ \sigma_*^n) Lip(\sigma_\diamond^{-2n}) \leq \frac{M}{|p+q|^{2n}}, \quad x \in X, n \in \mathbb{N}_0.$$

By using Theorem 2.3, F is single-valued mapping satisfying (2.15). We arrive at the desired conclusion. $\square$

Corollary 2.5. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Z, \|\cdot\|)$ a Banach space over $\mathbb{R}$, $p, q, \varepsilon > 0, p+q < 1$, and $z \in Z$. Assume that $f : X \to Z$ is a function satisfying*

$$\|pf(x*y) + qf(x*y^{-1}) - p(p+q)f(x) - q(p+q)f(y) - z\| \leq \varepsilon, \quad x, y \in X.$$

Then there exists a unique function $g : X \to Z$ satisfying

$$pg(x*y) + qg(x*y^{-1}) = p(p+q)g(x) + q(p+q)g(y) + z, \quad x, y \in X \tag{2.16}$$

and

$$\|f(x) - g(x)\| \leq \frac{\varepsilon}{(1-p-q)(p+q)}, \quad x \in X. \tag{2.17}$$

Proof. Consider the auxiliary set-valued mapping $G_f : X \to ccl(Z)$ corresponding to f defined by

$$G_f(x) = f(x) + \frac{1}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon) - z), \quad \text{if } x \in X - \{id_X\}$$

and $G_f(id_X) = \{0_Z\}$. Then we obtain

$$\begin{aligned}
p(p+q)G_f(x) + q(p+q)G_f(y) &= p(p+q)f(x) + \frac{p(p+q)}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon) - z) \\
&\quad + q(p+q)f(y) + \frac{q(p+q)}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon) - z) \\
&\subseteq pf(x*y) + qf(x*y^{-1}) + (\overline{B}(0,\varepsilon) - z) \\
&\quad + \frac{(p+q)^2}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon) - z) \\
&= pf(x*y) + \frac{p}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon) - z) \\
&\quad + qf(x*y^{-1}) + \frac{q}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon) - z) \\
&= pG_f(x*y) + qG_f(x*y^{-1})
\end{aligned}$$

for all $x, y \in X$. By the definition of $\delta(G_f(x))$, we have

$$\delta(G_f(x)) \leq \frac{2\varepsilon}{(1-p-q)(p+q)}$$

for all $x \in X$. Since all conditions of Corollary 2.4 are equipped, G_f has a unique selection $h : X \to Z$ such that

$$ph(x*y) + qh(x*y^{-1}) = p(p+q)h(x) + q(p+q)h(y), \quad x, y \in X.$$

Defining the function $g : X \to Z$ as

$$g(x) = h(x) + \frac{z}{(1-p-q)(p+q)}$$

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for all $x \in X$, we see that the function g satisfies (2.16) and (2.17). □

Next, we will introduce some theorems and corollaries which are obtained by the similar proofs of Theorem 2.2, 2.3 and Corollary 2.4, 2.5.

Theorem 2.6. *Let $(X,*)$ be a square-symmetric divisible groupoid, $(Y,\diamond,d)$ a complete metric bisymmetric divisible groupoid and $F: X \to \mathcal{P}_0(Y)$ with $F(id_X) = \{id_Y\}$ a set-valued mapping such that*

$$F(x*y)\Diamond F(x*y^{-1}) \subseteq \sigma_{\Diamond}(F(x)\Diamond F(y)) \tag{2.18}$$

for all $x, y \in X$. Assume that

$$\lim_{m\to\infty} \delta(F\circ\sigma_*^m(x))Lip(\sigma_{\Diamond}^{-2m}) = 0, \quad and$$
$$\sigma_{\Diamond}^{2n}\circ F\circ\sigma_*^{-n}(x) \in cl(Y)$$

for all $x \in X$ and $n \in \mathbb{N}_0$. Then there exists a unique selection $f: X \to Y$ of F such that

$$\sigma_{\diamond}(f(x)\diamond f(y)) = f(x*y)\diamond f(x*y^{-1})$$

for all $x, y \in X$.

Theorem 2.7. *Let $(X,*)$ be a square-symmetric divisible groupoid, $(Y,\diamond,d)$ a metric bisymmetric divisible groupoid and A a divisible subgroupoid of $(\mathcal{P}_0(Y),\Diamond)$. Suppose that $F: X \to A$ with $F(id_X) = \{id_Y\}$ is a set-valued mapping subject to the condition (2.18) and satisfying*

$$\lim_{n\to\infty} \delta(F\circ\sigma_*^{2n}(x))Lip(\sigma_{\Diamond}^{2n}) = 0, \quad x \in X.$$

Then F is single valued and

$$\sigma_{\Diamond}(F(x)\Diamond F(y)) = F(x*y)\Diamond F(x*y^{-1}), \quad for\ all\ x, y \in X.$$

Corollary 2.8. *Let $(X,*)$ be a square-symmetric divisible groupoid, $(Z,\|\cdot\|)$ a Banace space over $\mathbb{R}$, $p, q \in \mathbb{R}$, $p+q \neq 0$, $p+q \neq 1$, and $F: X \to c(Z)$ with $F(id_X) = \{0_Z\}$ a set-valued mapping subject to the condition (2.18). Assume that there exists $M > 0$ such that*

$$\delta(F(x)) \le M, \quad and$$
$$F\circ\sigma_*^{-n}(x) \in cl(Z)$$

for all $x, y \in X$ and $n \in \mathbb{N}_0$. Then there exists a unique selection $f: X \to Z$ of F such that

$$p(p+q)f(x) + q(p+q)f(y) = pf(x*y) + qf(x*y^{-1}), \quad x, y \in X.$$

Corollary 2.9. *Let* $(X, *)$ *be a square-symmetric divisible groupoid,* $(Z, \|\cdot\|)$ *a Banach space over* $\mathbb{R}$*,* $p, q, \varepsilon > 0, p+q > 1,$ *and* $z \in Z$*. Assume that* $f : X \to Z$ *is a function satisfying*

$$\|pf(x*y) + qf(x*y^{-1}) - p(p+q)f(x) - q(p+q)f(y) - z\| \leq \varepsilon, \quad x, y \in X.$$

Then there exists a unique function $g : X \to Z$ *satisfying*

$$pg(x*y) + qg(x*y^{-1}) = p(p+q)g(x) + q(p+q)g(y) + z, \quad x, y \in X$$

and

$$\|f(x) - g(x)\| \leq \frac{\varepsilon}{(p+q-1)(p+q)}, \quad x \in X.$$

Remark. If $(X, +, \cdot)$ is a vector space and $*$ is defined by $x * y = x + y$ and $z = 0$, $p = q = 1$, $y^{-1} = -y$ in Corollary 2.9, then it is a same result of Czerwik [1].

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Explicit identities involving r-Bell polynomials

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Abstract : In this paper, we study differential equations arising from the generating functions of the r-Bell polynomials. We give explicit identities for the r-Bell polynomials.

Key words : Differential equations, Bell polynomials, r-Bell polynomials.

2000 Mathematics Subject Classification : 05A19, 11B68, 11S40, 11S80, 11B83, 34A30, 65L99.

1 Introduction

The moments of the Poisson distribution are well-known to be connected to the combinatorics of the Bell and Stirling numbers(see [1, 4, 5]). As is well known, the Bell numbers B_n are given by the generating function

$$e^{(e^t-1)} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.1}$$

The Bell polynomials $B_n(\lambda)$ are given by the generating function

$$e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \tag{1.2}$$

The Bell polynomials $B_n(\lambda)$ satisfy the relation $B_n(\lambda) = E_\lambda[Z^n], n \in \mathbb{N}$, where Z is a Poisson random variable with parameter $\lambda > 0$.

The r-Bell polynomials $G_n(x,r)$ are defined by the exponential generating function:

$$\sum_{n=0}^{\infty} G_n(x,r) \frac{t^n}{n!} = e^{rt+x(e^t-1)}, \text{ (see [4])}, \tag{1.3}$$

where, r may be real or complex numbers. Note that $B_n(x) = G_n(x,0)$. The first few examples of r-Bell polynomials $G_n(x,r)$ are

$$\begin{aligned}
&G_0(x,r) = 1,\\
&G_1(x,r) = r + x,\\
&G_2(x,r) = r^2 + x + 2rx + x^2,\\
&G_3(x,r) = r^3 + x + 3rx + 3r^2x + 3x^2 + 3rx^2 + x^3,\\
&G_4(x,r) = r^4 + x + 4rx + 6r^2x + 4r^3x + 7x^2 + 12rx^2\\
&\qquad\qquad + 6r^2x^2 + 6x^3 + 4rx^3 + x^4,\\
&G_5(x,r) = r^5 + x + 5rx + 10r^2x + 10r^3x + 5r^4x + 15x^2 + 35rx^2\\
&\qquad\qquad + 30r^2x^2 + 10r^3x^2 + 25x^3 + 30rx^3 + 10r^2x^3 + 10x^4 + 5rx^4 + x^5.
\end{aligned}$$

From (1.2) and (1.3), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} G_n(x,r)\frac{t^n}{n!} &= e^{(e^t-1)x}e^{rt}\\ &= \left(\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} r^m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_k(x)r^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{1.4}$$

Comparing the coefficients on both sides of (1.4), we obtain

$$G_n(x,r) = \sum_{k=0}^{n}\binom{n}{k}B_k(x)r^{n-k} \quad (n \geq 0).$$

Similarly we also have

$$G_n(x+y,r) = \sum_{k=0}^{n}\binom{n}{k}G_k(x,r)B_{n-k}(y).$$

Recently, many mathematicians have have studied the differential equations arising from the generating function of special polynomials(see [2, 3, 6, 7, 8, 9]). In this paper, we study differential equations arising from the generating function of r-Bell polynomials. We give explicit identities for the r-Bell polynomials.

2 Explicit identities involving r-Bell polynomials

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials(see [7, 8, 13]). In this section, we study differential equations arising from the generating functions of r-Bell polynomials.

Let

$$F = F(t,x,r) = \sum_{n=0}^{\infty} G_n(x,r)\frac{t^n}{n!} = e^{rt+(e^t-1)x}, \quad x, r \in \mathbb{C}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned}F^{(1)} = \frac{d}{dt}F(t,x,r) = \frac{d}{dt}\left(e^{rt+(e^t-1)x}\right)\\ &= e^{rt+(e^t-1)x}(r+xe^t)\\ &= re^{rt+(e^t-1)x} + xe^{(r+1)t+(e^t-1)x}\\ &= rF(t,x,r) + xF(t,x,r+1),\end{aligned} \tag{2.2}$$

$$\begin{aligned}F^{(2)} &= \frac{d}{dt}F^{(1)} = rF^{(1)}(t,x,r) + xF^{(1)}(t,x,r+1)\\ &= r^2F(t,x,r) + x(2r+1)F(t,x,r+1) + x^2F(t,x,r+2),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}F^{(3)} &= \frac{d}{dt}F^{(2)}\\ &= r^2F^{(1)}(t,x,r) + x(2r+1)F^{(1)}(t,x,r+1) + x^2F^{(1)}(t,x,r+2)\\ &= r^3F(t,x,r) + x\left(r^2+(2r+1)(r+1)\right)F(t,x,r+1)\\ &\quad + x^2(3r+3)F(t,x,r+2) + x^3F(t,x,r+3).\end{aligned}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x,r) \\ &= \sum_{i=0}^{N} a_i(N,x,r)F(t,x,r+i), (N=0,1,2,\ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we get

$$\begin{aligned} F^{(N+1)} = \frac{dF^{(N)}}{dt} &= \sum_{i=0}^{N} a_i(N,x,r)F^{(1)}(t,x,r+i) \\ &= \sum_{i=0}^{N} a_i(N,x,r)\left\{(r+i)F(t,x,r+i) + xF(t,x,r+i+1)\right\} \\ &= \sum_{i=0}^{N} a_i(N,x,r)(r+i)F(t,x,r+i) \\ &\qquad + x\sum_{i=0}^{N} a_i(N,x,r)F(t,x,r+(i+1)) \\ &= \sum_{i=0}^{N}(r+i)a_i(N,x,r)F(t,x,r+i) \\ &\qquad + x\sum_{i=1}^{N+1} a_{i-1}(N,x,r)F(t,x,r+i). \end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x,r)F(t,x,r+i). \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$a_0(N+1,x,r) = ra_0(N,x,r), \quad a_{N+1}(N+1,x,r) = xa_N(N,x,r), \tag{2.7}$$

and

$$a_i(N+1,x,r) = (r+i)a_{i-1}(N,x,r) + xa_{i-1}(N,x,r), (1 \le i \le N). \tag{2.8}$$

In addition, by (2.4), we get

$$F(t,x,r) = F^{(0)}(t,x,r) = a_0(0,x,r)F(t,x,r). \tag{2.9}$$

By (2.9), we get

$$a_0(0,x,r) = 1. \tag{2.10}$$

It is not difficult to show that

$$\begin{aligned} &rF(t,x,r) + xF(t,x,r+1) \\ &= F^{(1)}(t,x,r) \\ &= \sum_{i=0}^{1} a_i(1,x,r)F(t,x,r+1) \\ &= a_0(1,x,r)F(t,x,r) + a_1(1,x,r)F(t,x,r+1). \end{aligned} \tag{2.11}$$

Thus, by (2.11), we also get

$$a_0(1,x,r)=r,\quad a_1(1,x,r)=x. \tag{2.12}$$

From (2.7), we note that

$$a_0(N+1,x,r)=ra_0(N,x,r)=\cdots=r^Na_0(1,x,r)=r^{N+1}, \tag{2.13}$$

and

$$a_{N+1}(N+1,x,r)=xa_N(N,x,r)=\cdots=x^Na_1(1,x,r)=x^{N+1}. \tag{2.14}$$

For $i=1,2,3$ in (2.8), we have

$$a_1(N+1,x,r)=x\sum_{k=0}^{N}(r+1)^ka_0(N-k,x,r), \tag{2.15}$$

$$a_2(N+1,x,r)=x\sum_{k=0}^{N-1}(r+2)^ka_1(N-k,x,r), \tag{2.16}$$

and

$$a_3(N+1,x,r)=x\sum_{k=0}^{N-2}(r+3)^ka_2(N-k,x,r). \tag{2.17}$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$a_i(N+1,x,r)=x\sum_{k=0}^{N-i+1}(r+i)^ka_{i-1}(N-k,x,r). \tag{2.18}$$

Here, we note that the matrix $a_i(j,x,r)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix}1 & r & r^2 & r^3 & \cdots & r^{N+1}\\ 0 & x & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & x^2 & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & x^3 & \cdots & \cdot\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & x^{N+1}\end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,x,r)$. By (2.15), (2.16), and (2.17), we get

$$a_1(N+1,x,r)=x\sum_{k_1=0}^{N}(r+1)^{k_1}a_0(N-k_1,x,r)=\sum_{k_1=0}^{N}(r+1)^{k_1}r^{N-k_1},$$

$$\begin{aligned}a_2(N+1,x,r)&=x\sum_{k_2=0}^{N-1}(r+2)^{k_2}a_1(N-k_2,x,r)\\&=x^2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-1-k_2}(r+1)^{k_1}(r+2)^{k_2}r^{N-k_2-k_1-1},\end{aligned}$$

and

$$\begin{aligned}&a_3(N+1,x,r)\\&=x\sum_{k_3=0}^{N-2}(r+3)^{k_3}a_2(N-k_3,x,r)\\&=x^3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(r+3)^{k_3}(r+2)^{k_2}(r+1)^{k_1}r^{N-k_3-k_2-k_1-2}.\end{aligned}$$

Continuing this process, we have

$$\begin{aligned} &a_i(N+1,x,r) \\ &= x^i \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} \left(\prod_{l=1}^{i}(r+l)^{k_l}\right) r^{N-i+1-\sum_{l=1}^{i} k_l}. \end{aligned} \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.1 *For $N = 0, 1, 2, \ldots$, the differential equation*

$$F^{(N)} = \sum_{i=0}^{N} a_i(N,x,r)e^{it}F(t,x,r)$$

has a solution

$$F = F(t,x,r) = e^{rt+(e^t-1)x},$$

where

$$\begin{aligned} &a_0(N,x,r) = r^N, \\ &a_N(N,x,r) = x^N, \\ &a_i(N,x,r) = x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i}(r+l)^{k_l}\right) r^{N-i-\sum_{l=1}^{i} k_l}, \\ &\qquad (1 \le i \le N). \end{aligned}$$

From (2.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x,r) = \sum_{k=0}^{\infty} G_{k+N}(x,r)\frac{t^k}{k!}. \tag{2.20}$$

From Theorem 1 and (2.20), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} G_{k+N}(x,r)\frac{t^k}{k!} = F^{(N)} &= \left(\sum_{i=0}^{N} a_i(N,x,r)e^{it}\right) F(t,x,r) \\ &= \sum_{i=0}^{N} a_i(N,x,r) \left(\sum_{l=0}^{\infty} i^l \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} G_m(x,r)\frac{t^m}{m!}\right) \\ &= \sum_{i=0}^{N} a_i(N,x,r) \left(\sum_{k=0}^{\infty}\sum_{m=0}^{k} \binom{k}{m} i^{k-m} G_m(x,r)\frac{t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N}\sum_{m=0}^{k} \binom{k}{m} i^{k-m} a_i(N,x,r)G_m(x,r)\right) \frac{t^k}{k!}. \end{aligned} \tag{2.21}$$

By comparing the coefficients on both sides of (2.21), we obtain the following theorem.

Theorem 2.2 *For $k, N = 0, 1, 2, \ldots$, we have*

$$G_{k+N}(x,r) = \sum_{i=0}^{N}\sum_{m=0}^{k} \binom{k}{m} i^{k-m} a_i(N,x,r)G_m(x,r), \tag{2.22}$$

where where

$$\begin{aligned} &a_0(N,x,r) = r^N, \quad a_N(N,x,r) = x^N, \\ &a_i(N,x,r) = x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i}(r+l)^{k_l}\right) r^{N-i-\sum_{l=1}^{i} k_l}, \\ &\qquad (1 \le i \le N). \end{aligned}$$

Let us take $k = 0$ in (2.22). Then, we have the following corollary.

Corollary 2.3 *For $N = 0, 1, 2, \ldots,$ we have*

$$G_N(x, r) = \sum_{i=0}^{N} a_i(N, x, r).$$

For $N = 0, 1, 2, \ldots,$ the functional equation $F^{(N)} = \sum_{i=0}^{N} a_i(N, x, r)e^{it}F(t, x, r)$ has a solution $F = F(t, x, r) = e^{rt+(e^t-1)x}$. Here is a plot of the surface for this solution. In Figure 1(left), we

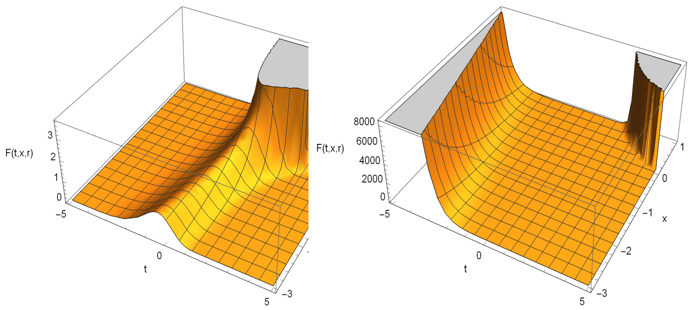

Figure 1: The surface for the solution $F(t, x, r)$

choose $-3 \le x \le 1, -5 \le t \le 5$, and $r = -2$. In Figure 1(right), we choose $-3 \le x \le 3, -5 \le t \le 5$, and $r = 2$.

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A CLASS INVOLVING DERIVATIVES OF RATIO OF THE ANALYTIC FUNCTIONS

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Abstract. The class of functions defined using linear combination of the derivatives of ratio of the normalized analytic function with the identity function is considered in this manuscript. Further, the sharp bounds on the Hankel determinants and estimates on the higher order Schwarzian derivatives for the first three consecutive derivatives are investigated.

1. Introduction

Let $\mathcal{A}$ be the family of functions f in the open unit disk $\mathbb{D}$ and satisfying the normalization conditions $f(0) = 0 = f'(0) - 1$. Let the collection $\mathcal{S} \subset \mathcal{A}$ contains univalent functions in $\mathbb{D}$. An analytic function f is subordinate to another analytic function g if there is an analytic function w with $|w(z)| \leq |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$ and we write $f \prec g$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. The classes $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions, respectively, are defined by $\operatorname{Re}(zf'(z)/f(z)) > 0$ and $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$. There are several sufficient conditions for functions to be univalent. Among them the simplest one is to verify $\operatorname{Re} f'(z) > 0$ in $z \in \mathbb{D}$. However, there are several other sufficient conditions for univalency were investigated in the recent years. Obradovič [17] proved that if $f \in \mathcal{A}$ satisfy $|f''(z)| < 1/2$, then f is convex in $\mathbb{D}$. Later, this condition was generalized by Frasin [7]. For $0 < \gamma \leq 1$, Tunseki [24] investigated the conditions on the expressions $f'(z) - (1-\gamma)f(z)/z$ and $zf''(z) - \gamma f'(z)$ for the sufficient conditions of starlikeness and convexity. Frasin [8] obtained some sufficient conditions on $f'''(z)$ for starlikeness and convexity. In particular, he proved that when the function $f \in \mathcal{A}$ with $f''(0) = 0$ satisfies $|f'''(z)| < 1$, then f is starlike in $\mathbb{D}$ and if $|f'''(z)| < 1/2$, then f is convex in $\mathbb{D}$, see [8, Cororllary 2, Cororllary 3, p. 65].

Motivated by this, in 2010, Uyanik *et al.* [25] introduced and investigated a new subclass of $\mathcal{A}$ defined using the linear combination of the derivatives of ratio of the normalized analytic function with the identity function. For $\beta_1, \beta_2 \in \mathbb{C}, \lambda > 0$ and $f \in \mathcal{A}$ he defined $\mathcal{V}(\beta_1, \beta_2, \lambda)$ as follows:

$$\left|\beta_1 z \left(\frac{f(z)}{z}\right)' + \beta_2 z^2 \left(\frac{f(z)}{z}\right)''\right| \leq \lambda.$$

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He obtained sufficient condition for normalized analytic functions to be in the class $\mathcal{V}(\beta_1,\beta_2,\lambda)$. He proved that the nth coefficient of functions in this class is bounded by $\lambda/((n-1)(|\beta_1|+(n-2)|\beta_2|))$.

It is well-known that the function in the class $\mathcal{S}$ satisfy $|a_n| \leq 2$ $(n=2,3,\cdots)$. Moreover, if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then $f \in \mathcal{S}^*$ and if $\sum_{n=2}^{\infty} n^2|a_n| \leq 1$, then $f \in \mathcal{K}$. There is another important quantity related to coefficients, called the Hankel determinant, which enable us to determine the necessary condition on coefficient functional for functions belonging to a given class of functions. For given natural numbers n, q, the Hankel determinant $H_{q,n}(f)$ of a function $f(z)=\sum_{n=0}^{\infty} a_n z^n,\ a_1=1$ is defined by means of the following determinant

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

It is easy to see that the functional $H_{2,1}(f)=a_3-a_2^2$ is the well-known Fekete-Szegö functional. However, the second Hankel determinant is given by $H_{2,2}(f):=a_2a_4-a_3^2$. Further, the third Hankel determinant is $H_{3,\,1}(f):=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2)$. The Hankel determinant $H_{q,n}(f)$ for the class $\mathcal{S}$ was investigated by Pommerenke [19] and Hayman [10]. For more details, see [4,5,11,13,19,21] and the references cited therein.

The Schwarzian derivative of a locally univalent function f, defined by

$$\mathbf{S}(f)(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian derivative is an important quantity in Univalent Function Theory. Further properties were investigated by Nehari [16]. He obtained the necessary and sufficient conditions for $f \in \mathcal{S}$. The higher order Schwarzian derivative [9, 23]), is defined by $\sigma_3(f)=\mathbf{S}(f)$ and for any integer $n \geq 4$, it is given by

$$\sigma_{n+1}(f) = (\sigma_n(f))' - (n-1)\sigma_n(f)\frac{f''}{f'}.$$

Droff and Szynal [6] studied the higher order Schwarzian derivative for convex functions. Now $\sigma_n(f)(0) =: \mathbf{S}_n$ and $\mathbf{S}_3=\sigma_3(f)(0)=6(a_3-a_2^2)$, $\mathbf{S}_4=\sigma_4(f)(0)=24(a_4-3a_2a_3+2a_2^3)$ and $\mathbf{S}_5=\sigma_5(f)(0)=24(5a_5-20a_2a_4-9a_3^2+48a_3a_2^2-24a_2^4)$. The sharp bound on $|\mathbf{S}_i|$ $(i=2,3,4)$, for $f \in \mathcal{K}$, investigated by Droff and Szynal. The generalization of their work, recently, carried out in [3] by Cho *et al.*

We shall investigate, the estimates on the Hankel determinants and the higher order Schwarzian derivatives by associating the functions of the class under consideration with the Carathéodory functions. Now we recall those results which shall be needed for investigation of our results. Let $\mathcal{P}$ denote the class of Carathéodory [1,2] functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

Let $\mathcal{B}$ be the class of analytic functions $w(z) = \sum_{n=1}^{\infty} c_n z^n$ $(z \in \mathbb{D})$ and satisfying the condition $|w(z)| < 1$ for $z \in \mathbb{D}$. The function $w \in \mathcal{B}$ and $p \in \mathcal{P}$ are related as $p(z) = (1 + w(z))/(1 - w(z))$. Consider a functional $\Psi(w) = |c_3 + \alpha c_1 c_2 + \beta c_1^3|$ for $w \in \mathcal{B}$ and $\alpha, \beta \in \mathbb{R}$.

Lemma 1.1. [20, Lemma 2, p. 128] *If $w \in \mathcal{B}$, then for any real numbers α and β the following sharp estimate $\Psi(w) \leq \Phi(\alpha, \beta)$ holds, where*

$$\Phi(\alpha,\beta) = \begin{cases} 1, & \text{if } (\alpha,\beta) \in \Omega_1 \cup \Omega_2, \\ |\beta|, & \text{if } (\alpha,\beta) \in \Omega_3 \cup \Omega_4 \cup \Omega_5 \\ \frac{2}{3}(|\alpha|+1)\left(\frac{|\alpha|+1}{3(|\alpha|+\beta+1)}\right)^{1/2}, & \text{if } (\alpha,\beta) \in \Omega_6 \cup \Omega_7. \end{cases}$$

Here the sets Ω_i's are defined by
$\Omega_1 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \leq 1/2, \;\; -1 \leq \beta \leq 1\right\}$,
$\Omega_2 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{1}{2} \leq |\alpha| \leq 2, \;\; \frac{4}{27}(|\alpha|+1)^3 - (|\alpha|+1) \leq \beta \leq 1\right\}$,
$\Omega_3 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \leq 2, \;\; \beta \geq 1\right\}$,
$\Omega_4 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : 2 \leq |\alpha| \leq 4, \;\; \beta \geq \frac{1}{12}(\alpha^2+8)\right\}$, *and*
$\Omega_5 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \geq 4, \; \beta \geq \frac{2}{3}(|\alpha|-1)\right\}$.
$\Omega_6 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{1}{2} \leq |\alpha| \leq 2, \; -\frac{2}{3}(|\alpha|+1) \leq \beta \leq \frac{4}{27}(|\alpha|+1)^3 - (|\alpha|+1)\right\}$,
$\Omega_7 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \geq 2, \; -\frac{2}{3}(|\alpha|+1) \leq \beta \leq \frac{2|\alpha|(|\alpha|+1)}{\alpha^2+2|\alpha|+4}\right\}$.

Lemma 1.2. [14,15, Libera and Zlotkiewicz] *If $p \in \mathcal{P}$ has the form given by* (1.1) *with $p_1 \geq 0$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{1.2}$$

and

$$4p_3 = p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)y \tag{1.3}$$

for some x and y such that $|x| \leq 1$ and $|y| \leq 1$.

Lemma 1.3. [22, Ravichandran and Verma] *Let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{a}$ satisfy the inequalities* $0 < \hat{\alpha} < 1$, $0 < \hat{a} < 1$ *and*

$$8a(1-a)[(\hat{\alpha}\hat{\beta} - 2\hat{\gamma})^2 + (\hat{\alpha}(\hat{a}+\hat{\alpha}) - \hat{\beta})^2] + \hat{\alpha}(1-\hat{\alpha})(\hat{\beta} - 2\hat{a}\hat{\alpha})^2 \leq 4\hat{a}\hat{\alpha}^2(1-\hat{\alpha})^2(1-\hat{a}).$$

If $p \in \mathcal{P}$ has the form given by (1.1), *then*

$$|\hat{\gamma}p_1^4 + \hat{a}p_2^2 + 2\hat{\alpha}p_1p_3 - (3/2)\hat{\beta}p_1^2p_2 - p_4| \leq 2.$$

Lemma 1.4. [18, Ohno and Sugawa] *For any real numbers a, b and c, let the quantity $Y(a,b,c)$ be given by*

$$Y(a,b,c) = \max_{z \in \overline{\mathbb{D}}} \left\{|a + bz + cz^2| + 1 - |z|^2\right\},$$

where $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. If $ac \geq 0$, then

$$Y(a,b,c) = \begin{cases} |a| + |b| + |c|, & \text{if } |b| \geq 2(1-|c|), \\ 1 + |a| + \frac{b^2}{4(1-|c|)}, & \text{if } |b| < 2(1-|c|). \end{cases}$$

Further, if $ac < 0$, *then*

$$Y(a,b,c) = \begin{cases} 1-|a|+\frac{b^2}{4(1-|c|)}, & \textit{if } -4ac(c^{-2}-1) \leq b^2 \textit{ and } |b| < 2(1-|c|), \\ 1+|a|+\frac{b^2}{4(1+|c|)}, & \textit{if } b^2 < \min\{4(1+|c|)^2, -4ac(c^{-2}-1)\}, \\ R(a,b,c), & \textit{otherwise}, \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a|+|b|-|c|, & \textit{if } |c|(|b|+4|a|) \leq |ab|, \\ -|a|+|b|+|c|, & \textit{if } |ab| \leq |c|(|b|-4|a|), \\ (|c|+|a|)\sqrt{1-\frac{b^2}{4ac}}, & \textit{otherwise.} \end{cases}$$

2. Coefficient bounds

The following theorem gives the sharp upper bound for Fekete-Szegö functional and Hankel determinant for functions in the class $\mathcal{V}(\beta_1, \beta_2, \lambda)$.

Theorem 2.1. *Let* $0 < \beta_1 < 1, 0 < \beta_2 < 1$ *and* $f \in \mathcal{V}(\beta_1, \beta_2, \lambda)$. *Then the following sharp inequalities hold:*

(1) $|a_3 - \mu a_2^2| \leq \frac{\lambda}{2\beta_1+\beta_2} \max\left\{1; \frac{2\lambda(\beta_1+\beta_2)|\mu|}{{\beta_2}^2}\right\}, \ \mu \in \mathbb{C}.$

(2) $|a_2 a_4 - {a_3}^2| \leq \frac{\lambda^2(\beta_1+\beta_2)^2}{3\beta_1(\beta_1+2\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}.$

(3) $|a_2 a_3 - a_4| \leq \begin{cases} \frac{\lambda}{3(\beta_1+2\beta_2)}, & 0 < \lambda \leq \frac{(3\sqrt{2}-2){\beta_1}^2+(3\sqrt{2}-2)\beta_1\beta_2}{3\beta_1+6\beta_2}; \\ \frac{2\beta_1(\beta_1+\beta_2)+3\lambda(\beta_1+2\beta_2)}{9\sqrt{3}\beta_1(\beta_1+\beta_2)(\beta_1+2\beta_2)}\lambda, & \lambda > \frac{(3\sqrt{2}-2){\beta_1}^2+(3\sqrt{2}-2)\beta_1\beta_2}{3\beta_1+6\beta_2}. \end{cases}$

Proof. Since $f \in \mathcal{V}(\beta_1, \beta_2, \lambda)$, it follows that there exists a Schwarz function $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{B}$ such that

$$\beta_1 z \left(\frac{f(z)}{z}\right)' + \beta_2 z^2 \left(\frac{f(z)}{z}\right)'' = \lambda w(z)). \tag{2.1}$$

In the view of interconnection $w(z) = (p(z)-1)/(p(z)+1) \in \mathbf{B}$ if and only if $p \in \mathcal{P}$ between the Schwarz function w and the Carathéodory function $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$, from (2.1), we get

$$a_2 = \frac{\lambda p_1}{2\beta_1}, \quad a_3 = \frac{\lambda(2p_2 - {p_1}^2)}{8(\beta_1+\beta_2)}, \tag{2.2}$$

and

$$a_4 = \frac{\lambda(4p_3 - 4p_1p_2 + {p_1}^3)}{24(\beta_1+2\beta_2)}, \quad a_5 = \frac{\lambda(8p_4 - 8p_1p_3 - 4{p_2}^2 + 6{p_1}^2p_2 - {p_1}^4)}{64(\beta_1+3\beta_2)}. \tag{2.3}$$

(1) From (2.2), Using the result [see [12]], for any complex number μ,

$$\left|p_2 - \mu p_1^2\right| \leq 2\max\{1; |2\mu - 1|\},$$

we have

$$\begin{aligned} |a_3-\mu{a_2}^2| &= \frac{\lambda}{4(\beta_1+\beta_2)}\left[p_2-\frac{{\beta_1}^2+2\mu\lambda(\beta_1+\beta_2)}{2{\beta_1}^2}p_1^2\right] \\ &= \frac{\lambda}{2(\beta_1+\beta_2)}\max\left\{1,\frac{2\lambda(\beta_1+\beta_2)}{{\beta_1}^2}|\mu|\right\}. \end{aligned} \tag{2.4}$$

The equality holds in case of the function f defined by (2.1) with choice of the function $w(z)=z$.

(2) Using (2.2) and (2.3), we have

$$\begin{aligned} a_2a_4-{a_3}^2 = \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} & \left[({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2){p_1}^4\right. \\ & -4({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2){p_1}^2p_2-12\beta_1(\beta_1+2\beta_2){p_2}^2 \\ & \left.+16(\beta_1+\beta_2)^2p_1p_3\right]. \end{aligned} \tag{2.5}$$

Putting equivalent expressions for p_2 and p_3 in terms of p_1 from (1.2) and (1.3) in (2.5), we have

$$\begin{aligned} a_2a_4-{a_3}^2 = \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} & \left[\{-3\beta_1(\beta_1+2\beta_2)(4-{p_1}^2)+4(\beta_1+\beta_2)^2{p_1}^2\}\right. \\ & \left.\times(4-{p_1}^2)x^2+8p_1(4-{p_1}^2)(\beta_1+\beta_2)^2(1-|x|^2)y\right]. \end{aligned} \tag{2.6}$$

Because $p\in\mathcal{P}$, and the class $\mathcal{P}$ is invariant under rotation, without loss of any generality, we can set $p_1=|p_1|=:s\in[0,2]$. Further, since $|x|\le 1$ and $|y|\le 1$ for some $x,y\in\mathbb{C}$, using this facts and the triangle inequality in (2.6) we can write

$$|a_2a_4-{a_3}^2| \;\le\; T\left[\left|-\frac{3\beta_1(\beta_1+2\beta_2)(4-s^2)+4(\beta_1+\beta_2)^2s^2}{8(\beta_1+\beta_2)^2}x^2\right|+s(1-|x|^2)\right], \tag{2.7}$$

where

$$T:=\frac{\lambda^2(4-s^2)}{24\beta_1(\beta_1+2\beta_2)}.$$

We note that for $s=p_1=0$, and $s=p_1=2$ from (2.7), we have $|a_2a_4-{a_3}^2|\le \lambda^2/4(\beta_1+\beta_2)^2$ and $|a_2a_4-{a_3}^2|=0$, respectively.

Now we assume that $s\in(0,2)$. Then, form (2.7), we obtain

$$|a_2a_4-{a_3}^2|\le\frac{\lambda^2}{24\beta_1(\beta_1+2\beta_2)}s(4-s^2)F(a,b,c), \tag{2.8}$$

where

$$F(a,b,c):=|a+bx+cx^2|+1-|x|^2,$$

with

$$a:=0,\quad b:=0 \text{ and } c:=-\frac{3\beta_1(\beta_1+2\beta_2)(4-s^2)+4(\beta_1+\beta_2)^2s^2}{8(\beta_1+\beta_2)^2s}.$$

Here it is easily seen that $ac=0$. Here we have two cases now:

(i) When $0<\beta_1<\beta_2(\sqrt{3}-1)$ and $s^*\le s<2$, we obtain $|b|\ge 2(1-|c|)$. Therefore, by Lemma 1.4, we have

$$\begin{aligned}|a_2a_4-{a_3}^2| &\le \frac{\lambda^2(\beta_1+\beta_2)^2 s(4-s^2)}{24\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)}F(a,b,c)\\ &= \frac{\lambda^2(\beta_1+\beta_2)^2 s(4-s^2)}{24\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)}\left(\frac{3\beta_1(\beta_1+2\beta_2)(4-s^2)+4(\beta_1+\beta_2)^2s^2}{8(\beta_1+\beta_2)^2 s}\right)\\ &= \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)}g(s),\end{aligned}$$

where $g:[s^*,2)\to\mathbb{R}$ is defined by

$$g(s):=3\beta_1(\beta_1+2\beta_2)(4-s^2)+4(\beta_1+\beta_2)^2(4-s^2)s^2.$$

Cleanly, g has maximum at

$$s=s_1:=\frac{2\sqrt{-{\beta_1}^2-2\beta_1\beta_2+{\beta_2}^2}}{\sqrt{{\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2}},$$

we have

$$\begin{aligned}|a_2a_4-{a_3}^2| &\le \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)}g(s_1)\\ &= \frac{\lambda^2(\beta_1+\beta_2)^2}{3\beta_1(\beta_1+2\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}.\end{aligned}$$

(ii) When $0<\beta_1\le\beta_2(\sqrt{3}-1)$ and $0<s<s^*$, $\beta_1>\beta_2(\sqrt{3}-1)>0$ and $0<s<2$, we obtain $|b|<2(1-|c|)$. Therefore, by Lemma 1.4, we have

$$\begin{aligned}|a_2a_4-{a_3}^2| &\le \frac{\lambda^2(\beta_1+\beta_2)^2 s(4-s^2)}{24\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)}F(a,b,c)\\ &= \frac{\lambda^2(4-s^2)s}{24\beta_1(\beta_1+2\beta_2)}\\ &= \frac{\lambda^2}{24\beta_1(\beta_1+2\beta_2)}h(s),\end{aligned}$$

where the function $h:(0,2)\to\mathbb{R}$ is defined by

$$h(s):=(4-s^2)s.$$

Further computation reveals that h has its maximum at $s=s_2:=2/\sqrt{3}$, and thus we have

$$|a_2a_4-{a_3}^2|\le\frac{\lambda^2}{24\beta_1(\beta_1+2\beta_2)}h(s_2)=\frac{2\sqrt{3}\lambda^2}{27\beta_1(\beta_1+2\beta_2)}.$$

Therefore, from (i) and (ii), we conclude that

$$|a_2a_4-{a_3}^2|\le\frac{\lambda^2(\beta_1+\beta_2)^2}{3\beta_1(\beta_1+2\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}.$$

The equality holds in case of the function defined in (2.1) with

$$w(z) = \frac{z(u_0 - 2z^2)}{2 - u_0 z},$$

where $u_0 = 2\sqrt{-{\beta_1}^2 - 2\beta_1\beta_2 + {\beta_2}^2}/\sqrt{{\beta_1}^2 + 2\beta_1\beta_2 + 4{\beta_2}^2}$.

(3) To find the estimates on the functional $|a_2a_3 - a_4|$, we shall express the coefficients (a_i) in terms of Schwarz's coefficients (c_i). From (2.1), we have

$$a_2 = \frac{c_1\lambda}{\beta_1}, \quad a_3 = \frac{c_2\lambda}{2(\beta_1+\beta_2)}, \quad a_4 = \frac{c_3\lambda}{3(\beta_1+2\beta_2)}, \; a_5 = \frac{c_4\lambda}{4(\beta_1+3\beta_2)}. \tag{2.9}$$

Using (2.10), we get

$$\begin{aligned} |a_2a_3 - a_4| &= \left| -\frac{\lambda^2 c_1c_2}{2\beta_1(\beta_1+\beta_2)} + \frac{\lambda c_3}{3(\beta_1+2\beta_2)} \right| \\ &= \frac{\lambda}{3(\beta_1+2\beta_2)} \left| -\frac{3\lambda(\beta_1+2\beta_2)}{2\beta_1(\beta_1+\beta_2)} c_1c_2 + c_3 \right| \\ &= \frac{\lambda}{3(\beta_1+2\beta_2)} \Phi(\mu,\nu), \end{aligned}$$

where $\Phi(\mu,\nu) := |c_3 + \mu c_1 c_2 + \nu {c_1}^3|$ with

$$\mu := -\frac{3\lambda(\beta_1+2\beta_2)}{2\beta_1(\beta_1+\beta_2)}, \text{ and } \nu := 0.$$

Assume that Ω_i's are as defined in lemma 1.1 with μ and ν as given above. We now complete the proof in the following cases.

(i) Suppose that $0 < \lambda \leq \beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$, then we see that $-1/2 \leq \mu \leq 1/2$ and $-1 \leq \nu \leq 1$. So, we conclude that $(\mu,\nu) \in \Omega_1$.

(ii) Let $\beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2) \leq \lambda \leq \beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. Then we can easily verify that $-2 \leq \mu \leq -1/2$ and $(4/27)(|\mu|+1)^3 - (|\mu|+1) \leq \nu \leq 1$ holds and we get $(\mu,\nu) \in \Omega_2$.

(iii) Let $\beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2) \leq \lambda \leq 4\beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. Now we see that $-2 \leq \mu \leq -1/2$ and $-2(|\mu|+1)/3 \leq \lambda \leq (4/27)(|\mu|+1)^3 - (|\mu|+1)$ hold for all such positive values of λ and so $(\mu,\nu) \in \Omega_6$.

(iv) Let $\lambda \geq 4\beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. Then we see that the conditions $\mu \leq -2$ and $-(2/3)(|\mu|+1) \leq \nu \leq 2|\mu|(|\mu|+1)/(\mu^2+|\mu|+4)$ hold. Therefore, $(\mu,\nu) \in \Omega_7$.

Now by using Lemma 1.1, the cases (i) and (ii), we conclude that if

$$0 < \lambda \leq \beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2),$$

then $\Phi(\mu,\nu) \leq 1$. Further, the (iii) and (iv) hold, then

$$\Phi(\mu,\nu) \leq (2\beta_1(\beta_1+\beta_2) + 3\lambda(\beta_1+2\beta_2))/3\sqrt{3}\beta_1(\beta_1+\beta_2)$$

for $\lambda \geq \beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. The result is sharp in case of the function f defined by (2.1) with choice of the Schwarz function $w(z) = z^3$ and $w(z) = z(t_1 +$

$z)/(1+t_1z)$, respectively, where

$$t_1=\left(\frac{|\mu|-1}{3(|\mu|-1-\nu)}\right)^{\frac{1}{2}}.$$

This ends the proof. ∎

Remark 2.2. In the case, when β_1 and β_2 are real numbers, from the result [25, Corollary 2.1, p.383], we conclude that

$$|a_n|\leq\frac{\lambda}{(n-1)(|\beta_1|+(n-2)|\beta_2|)}.$$

Using the above results, we deduce the following estimates on the third Hankel determinant:

Corollary 2.3. *Let $0<\beta_1<1, 0<\beta_2<1$ and $f\in\mathcal{V}(\beta_1,\beta_2;\lambda)$. Then the following holds:*

$$|H_{3,\,1}(f)|\leq\begin{cases}\tau_1\lambda^2, & 0<\lambda\leq\lambda_1;\\ \tau_2\lambda, & \lambda_1<\lambda\leq\lambda_2;\\ \tau_3, & \lambda\geq\lambda_2,\end{cases}$$

where

$$\tau_1:=\frac{\begin{array}{r}24\lambda(\beta_1+\beta_2)^2(\beta_1+2\beta_2)(\beta_1+3\beta_2)+\beta_1({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)\\ \times(5\beta_1+6\beta_2)(5\beta_1+14\beta_2)\end{array}}{2(\alpha g_3+\beta h_3)B_1^2},$$

$$\tau_2:=\frac{\begin{array}{r}6\lambda^2(\beta_1+\beta_2)(\beta_1+2\beta_2)(\beta_1+3\beta_2)+4\lambda\beta_1(\beta_1+3\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)\\ +9\beta_1(\beta_1+2\beta_2)^2({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)\end{array}}{\beta_1(\beta_1+2\beta_2)^2(\beta_1+3\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}$$

and

$$\tau_3:=\frac{\mu^2\left(1296\mu^4+3456\mu^3+2304\mu^2+1740\mu+1015\right)}{5184(12\mu+7)}.$$

Theorem 2.4. *Let $f\in\mathcal{V}(\beta_1,\beta_2;\lambda)$, then the following sharp inequalities hold:*

(1) *If $0<\beta_1<1$ and $0<\beta_2<1$, then*

$$|\mathbf{S}_3|\leq\begin{cases}\frac{3\lambda}{\beta_1+\beta_2}, & 0<\lambda\leq\frac{{\beta_1}^2}{2(\beta_1+\beta_2)};\\ \frac{6\lambda^2}{{\beta_1}^2}, & \lambda>\frac{{\beta_1}^2}{2(\beta_1+\beta_2)}.\end{cases}$$

(2) *(a) If either of the set of conditions $0<\lambda\leq\lambda^*$ or ${\lambda_1}^*\leq\lambda\leq\lambda^{**}$ and*

$$\begin{aligned}\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}[\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}^2-27{\beta_1}^2(\beta_1+\beta_2)^2]\\ \leq 324(\beta_1+2\beta_2)\lambda^2(\beta_1+\beta_2)^3\end{aligned}$$

holds, then

$$|\mathbf{S}_4|\leq\frac{24\lambda}{3(\beta_1+2\beta_2)}.$$

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(b) If either of the set of conditions ${\lambda_2}^* \leq \lambda \leq 4{\lambda_1}^*$ *and* $\frac{9}{4}\sqrt{\frac{\beta_1(\beta_1+2\beta)}{6}} \leq \beta_1 + \beta_2$ *or* $\lambda \geq 4{\lambda_1}^*$ *holds, then*

$$|\mathbf{S}_4| \leq \frac{48\lambda^4}{{\beta_1}^3}.$$

(3) *If* $0 < \beta_1 < 1, 0 < \beta_2 < 1$ *and* $0 < \lambda < 3\beta_1(\beta_1 + 2\beta_2)/8(\beta_1 + 3\beta_3)$, *then*

$$|\mathbf{S}_5| \leq \frac{720\lambda}{\beta_1 + 3\beta_2}.$$

Proof. Let $f \in \mathcal{V}(\beta_1, \beta_2; \lambda)$. Then, to find the estimates on the higher order Schwarzian derivatives, we shall express the coefficients (a_i) in terms of Schwarz's coefficients (c_i). From (2.1), we have

$$a_2 = \frac{c_1\lambda}{\beta_1}, \quad a_3 = \frac{c_2\lambda}{2(\beta_1 + \beta_2)}, \quad a_4 = \frac{c_3\lambda}{3(\beta_1 + 2\beta_2)}, \; a_5 = \frac{c_4\lambda}{4(\beta_1 + 3\beta_2)}. \tag{2.10}$$

Using first part of Theorem 2.1, we have

$$\begin{aligned} |\mathbf{S}_3| &= 6|a_3 - {a_2}^2)| \\ &\leq \frac{3\lambda}{\beta_1 + \beta_2} \max\left\{1, \frac{2(\beta_1 + \beta_2)\lambda}{{\beta_1}^2}\right\}. \end{aligned}$$

The function for the equality holds by (2.1) with the choice $w(z) = z$.

Now we consider the estimate on $|S_4|$. From (2.10), we obtain

$$\begin{aligned} \mathbf{S}_4 &= 24(a_4 - 3a_2a_3 + 2{a_3}^2) \\ &= \frac{24\lambda}{3(\beta_1 + 2\beta_2)}\left[\frac{6(\beta_1 + 2\beta_2)\lambda^2}{{\beta_1}^3}{c_1}^3 - \frac{9(\beta_1 + 2\beta_2)\lambda}{2\beta_1(\beta_1 + \beta_2)}c_1c_2 + c_3\right] \\ &= \frac{24\lambda}{3(\beta_1 + 2\beta_2)}\Upsilon(\mu, \nu) \end{aligned}$$

where $\Upsilon(\mu, \nu) := c_3 + \mu c_1 c_2 + \nu {c_1}^3$ with

$$\mu := -\frac{9(\beta_1 + 2\beta_2)\lambda}{2\beta_1(\beta_1 + \beta_2)}, \quad \text{and} \quad \nu := \frac{6(\beta_1 + 2\beta_2)\lambda^2}{{\beta_1}^3}.$$

Assume that Ω_i's are as defined in Lemma 1.1 with μ and ν as given above. We now complete the proof with the following cases.

(i) Suppose that $0 < \lambda \leq {\lambda_1}^*$. In this case, we see that $-1/2 \leq \mu \leq 1/2$ holds. Moreover, $-1 \leq \nu \leq 1$ holds if and only if $0 < \lambda \leq {\lambda_2}^*$, where ${\lambda_1}^* := \beta_1(\beta_1+\beta_2)/9(\beta_1+2\beta_2)$ and ${\lambda_2}^* := \beta_1\sqrt{\beta_1/6(\beta_1 + 2\beta_2)}$. Thus, for all $0 < \lambda \leq \min\{{\lambda_1}^*, {\lambda_2}^*\}$, we conclude that $(\mu, \nu) \in \Omega_1$.

(ii) Next suppose that ${\lambda_1}^* < \lambda \leq 4{\lambda_1}^*$, then we see that the condition $-2 \leq \mu \leq -1/2$ holds. Further, $(4/27)(\mu + 1)^3 - (\mu + 1) \leq \nu \leq 1$ holds if and only if $0 < \lambda \leq {\lambda_2}^*$

and

$$\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}[\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}^2 \\ -27{\beta_1}^2(\beta_1+\beta_2)^2] \leq 324(\beta_1+2\beta_2)\lambda^2(\beta_1+\beta_2)^3. \quad (2.11)$$

So, if ${\lambda_1}^* \leq \lambda \leq \lambda^{**} := \min\{4{\lambda_1}^*, {\lambda_2}^*\}$ and (2.11) hold, then $(\mu,\nu) \in \Omega_2$.

(iii) Let ${\lambda_2}^* \leq \lambda \leq {\lambda_3}^* := 4(\beta_1+\beta_2)/9(\beta_1+2\beta_2)$ and $\beta_1+\beta_2 \geq 9\sqrt{\beta_1(\beta_1+2\beta_2)}/4\sqrt{6}$. Then, we can easily verify that $|\mu| \leq 2$ and $\nu \geq 1$. Therefore, $(\mu,\nu) \in \Omega_5$.

(iv) Let $4{\lambda_1}^* \leq \lambda \leq 8{\lambda_1}^*$. Now we see that $-4 \leq \mu \leq -2$ and $\nu \geq (\mu^2+8)/12$ hold for all such positive values of λ and hence $(\mu,\nu) \in \Omega_6$.

(v) Let $\lambda \geq 8{\lambda_1}^*$. Then we see that $\mu \leq -4$ and $\nu \geq 2(|\mu|-1)$, so $(\mu,\nu) \in \Omega_7$.

Now by using Lemma 1.1 and the cases (i) and (ii), we conclude that if $0 < \lambda \leq \min\{{\lambda_1}^*, {\lambda_2}^*\}$ or ${\lambda_1}^* \leq \lambda \leq \lambda^{**}$ and (2.11) hold, then $\Upsilon(\mu,\nu) \leq 1$. Further, from the cases $(iii)-(v)$ and Lemma 1.1, we conclude that $\Upsilon(\mu,\nu) \leq \nu$, for ${\lambda_2}^* \leq \lambda \leq {\lambda_3}^*$ and $\beta_1+\beta_2 \geq 9\sqrt{\beta_1(\beta_1+2\beta_2)}/4\sqrt{6}$ or $\lambda \geq 4{\lambda_1}^*$. The result is sharp in case of the function f defined by (2.1) with choice of the Schwarz function $w(z)=z^3$ and $w(z)=z$, respectively. This completes the proof.

Now we find the estimate on $|\mathbf{S}_5|$. Using 2.2 and 2.3, we get

$$\begin{aligned}
\mathbf{S}_5 &= 24(5a_5-20a_2a_4-9a_3^2+48a_2^2a_3-24a_2^4) \\
&= \frac{-720\lambda}{\beta_1+3\beta_2}[\hat{\gamma}p_1^4+\hat{a}p_2^2+2\hat{\alpha}p_1p_3-(3/2)\hat{\beta}p_1^2p_2-p_4)] \\
&= \frac{-720\lambda}{\beta_1+3\beta_2}\Psi(\hat{\gamma},\hat{a},\hat{\alpha},\hat{\beta}), \qquad (2.12)
\end{aligned}$$

where $\Psi(\hat{\gamma},\hat{a},\hat{\alpha},\hat{\beta}) := \hat{\gamma}p_1^4+\hat{a}p_2^2+2\hat{\alpha}p_1p_3-(3/2)\hat{\beta}p_1^2p_2-p_4$ with the parameters $\hat{\gamma},\hat{a},\hat{\alpha}$ and $\hat{\beta}$ are given by

$$\hat{\gamma} := \frac{\beta_1+3\beta_2}{15}\left(\frac{15}{8(\beta_1+3\beta_2)}+\frac{27\lambda}{8(\beta_1+\beta_2)^2}+\frac{10\lambda}{\beta_1(\beta_1+2\beta_2)}+\frac{36\lambda^2}{{\beta_1}^2(\beta_1+\beta_2)}+\frac{36\lambda^3}{{\beta_1}^4}\right),$$

$$\hat{a} := \frac{\beta_1+3\beta_2}{6}\left(\frac{8\lambda}{\beta_1(\beta_1+2\beta_2)}+\frac{3}{\beta_1+3\beta_2}\right),$$

$$\hat{\alpha} := \frac{\beta_1+3\beta_2}{5}\left(\frac{5}{2(\beta_1+3\beta_2)}+\frac{9\lambda}{2(\beta_1+\beta_2)^2}\right)$$

and

$$\hat{\beta} := \frac{2(\beta_1+3\beta_2)}{45}\left(\frac{45}{4(\beta_1+3\beta_2)}+\frac{27\lambda}{2(\beta_1+\beta_2)^2}+\frac{20\lambda}{\beta_1(\beta_1+2\beta_2)}+\frac{72\lambda^2}{{\beta_1}^2(\beta_1+\beta_2)}\right).$$

We assume that $0<\beta_1<1$ and $0<\beta_2<1$ and $0<\lambda<3\beta_1(\beta_1+2\beta_2)/8(\beta_1+3\beta_3)$. Under these conditions, it is a simple matter to verify that $0<\hat{\alpha}<1$ and $0<\hat{a}<1$. Moreover, with these restrictions all conditions of Lemma 1.3 are fulfilled and thus, we get $|\Psi(\hat{\gamma},\hat{a},\hat{\alpha},\hat{\beta})| \leq 2$. Thus, the result follows from (2.12). ∎

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EXPLICIT FORMULAE OF CAUCHY POLYNOMIALS WITH A q PARAMETER IN TERMS OF r-WHITNEY NUMBERS

F. A. SHIHA

Abstract. The Cauchy polynomials with a q parameter were recently defined, and several arithmetical properties were studied. In this paper, we establish explicit formulae for computing the Cauchy polynomials with a q parameter in terms of r-Whitney numbers of the first kind. We also obtain several properties and combinatorial identities.

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Key Words. Cauchy numbers and polynomials, r-Whitney numbers, Stirling numbers.

1. Introduction

The Cauchy polynomials of the first kind $c_n(z)$ are defined by

$$c_n(z) = \int_0^1 (x-z)_n \, dx, \tag{1.1}$$

and the Cauchy polynomials of the second kind $\hat{c}_n(z)$ are defined by

$$\hat{c}_n(z) = \int_0^1 (-x+z)_n \, dx, \tag{1.2}$$

where $(y)_n = \prod_{i=0}^{n-1}(y-i)$ is the falling factorial with $(y)_0 = 1$. The exponential generating function of these polynomials are

$$\sum_{n=0}^{\infty} c_n(z) \frac{t^n}{n!} = \frac{t}{(1+t)^z \ln(1+t)}. \tag{1.3}$$

$$\sum_{n=0}^{\infty} \hat{c}_n(z) \frac{t^n}{n!} = \frac{t(1+t)^z}{(1+t) \ln(1+t)}. \tag{1.4}$$

(see [7, 4]). When $z = 0$, $c_n(0) = c_n$ and $\hat{c}_n(0) = \hat{c}_n$ are the Cauchy numbers of the first and second kind (see [2, 9, 12, 8]).

Recently [5] obtained a representation of the integer values of Cauchy polynomials in terms of r-Stirling numbers of the first kind $s_r(n,k)$ [3]. For all integers $n, r \geq 0$,

$$c_n(r) = \sum_{k=0}^{n} s_r(n+r, k+r) \frac{1}{k+1}, \tag{1.5}$$

$$\hat{c}_n(-r) = \sum_{k=0}^{n} (-1)^k s_r(n+r, k+r) \frac{1}{k+1}. \tag{1.6}$$

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Given variables y and m and a positive integer k, define the generalized rising and falling factorials of order k with increment m by

$$[y|m]_k = \prod_{j=0}^{k-1}(y+jm), \qquad [y|m]_0 = 1$$

$$(y|m)_k = \prod_{j=0}^{k-1}(y-jm), \qquad (y|m)_0 = 1.$$

Komatsu [6] introduced the Poly-Cauchy polynomials and numbers with a q parameter, and the Cauchy polynomials and numbers with a q parameter as special cases.

Let q be a real number with $q \neq 0$, Komatsu [6] defined the Cauchy polynomials with a q parameter of the first kind $c_n^q(z)$ by

$$c_n^q(z) = \int_0^1 (x-z|q)_n \, dx \tag{1.7}$$

and the Cauchy polynomials with a q parameter of the second kind $\hat{c}_n^q(z)$ by

$$\hat{c}_n^q(z) = \int_0^1 (-x+z|q)_n \, dx. \tag{1.8}$$

The exponential generating functions are

$$\sum_{n=0}^{\infty} c_n^q(z)\frac{t^n}{n!} = (1+qt)^{\frac{-z}{q}} \sum_{k=0}^{\infty}\left(\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}\,\frac{1}{k+1}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} \hat{c}_n^q(z)\frac{t^n}{n!} = (1+qt)^{\frac{z}{q}} \sum_{k=0}^{\infty}\left(-\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}\,\frac{1}{k+1}. \tag{1.10}$$

If $z=0$, then $c_n^q(0) = c_n^q$ and $\hat{c}_n^q(0) = \hat{c}_n^q$ are the Cauchy numbers with q parameter of the first and second kind, respectively. If $q=1$, then $c_n^1(z) = c_n(z)$ and $\hat{c}_n^1(z) = \hat{c}_n(z)$.

The r-Whitney numbers of the first and second kind were introduced by Mezö [10]. For non-negative integers n and k with $0 \le k \le n$, let $w(n,k) = w_{q,r}(n,k)$ denote the r-Whitney numbers of the first kind, which are defined by

$$q^n(x)_n = \sum_{k=0}^{n} w(n,k)\,(qx+r)^k. \tag{1.11}$$

Let $W(n,k) = W_{q,r}(n,k)$ denote the r-Whitney numbers of the second kind, which are defined by

$$(qx+r)^n = \sum_{k=0}^{n} q^k\, W(n,k)\,(x)_k. \tag{1.12}$$

Usually r is taken to be a non-negative integer and q a positive integer, but both may also be regarded as real numbers [11]. The exponential generating function of $w(n,k)$ is given by [10]

$$\sum_{n\ge k} w(n,k)\frac{t^n}{n!} = (1+qt)^{\frac{-r}{q}}\left(\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}, \tag{1.13}$$

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2. Basic results

Replace x by $\frac{x-r}{q}$ in (1.11), then the r-Whitney numbers of the first kind $w(n,k)$ are given by

$$(x-r|q)_n = \prod_{j=0}^{n-1}(x-r-jq) = \sum_{k=0}^{n} w(n,k)\, x^k, \qquad q \neq 0, \tag{2.1}$$

Using (1.7), we get the following theorem.

Theorem 1. *The Cauchy polynomials with q parameter of the first kind $c_n^q(r)$, $q \neq 0$ can be written explicitly as*

$$c_n^q(r) = \sum_{k=0}^{n} w(n,k)\, \frac{1}{k+1}. \tag{2.2}$$

The first few polynomials are
$c_0^q(r) = 1$,
$c_1^q(r) = -r + \frac{1}{2}$,
$c_2^q(r) = r^2 + (q-1)r - \frac{1}{2}q + \frac{1}{3}$,
$c_3^q(r) = -r^3 - \frac{3}{2}(2q-1)r^2 + (-2q^2+3q-1)r + q^2 - q + \frac{1}{4}$,
$c_4^q(r) = r^4 + (6q-2)r^3 + (11q^2-9q+2)r^2 + (6q^3-11q^2+6q-1)r - 3q^3 + \frac{11}{3}q^2 - \frac{3}{2}q + \frac{1}{5}$.

Remark 1. *If $r=0$, then $c_n^q(0) = c_n^q$ are the Cauchy numbers with q parameter of the first kind* [6]

$$c_n^q = \int_0^1 (x|q)_n \, dx = \sum_{k=0}^{n} q^{n-k}\, s(n,k)\, \frac{1}{k+1},$$

where $s(n,k)$ are the Stirling numbers of the first kind.

If $q=1$, we have $c_n^1(r) = c_n(r)$ and $w_{1,r}(n,k)$ are reduced to $s_r(n+r, k+r)$, and hence we obtain the explicit formula (1.5).

From (1.13), we can easily derive the exponential generating function of $c_n^q(r)$ as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n^q(r) \frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} w(n,k) \frac{1}{k+1}\, \frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} w(n,k) \frac{t^n}{n!}\, \frac{1}{k+1} \\
&= (1+qt)^{\frac{-r}{q}} \sum_{k=0}^{\infty} \left(\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}\, \frac{1}{k+1} \\
&= (1+qt)^{\frac{-r}{q}} \sum_{k=0}^{\infty} \left(\frac{\ln(1+qt)}{q}\right)^{k+1} \frac{1}{(k+1)!}\, \frac{q}{\ln(1+qt)} \\
&= \frac{q(1+qt)^{\frac{-r}{q}}}{\ln(1+qt)} \sum_{k=1}^{\infty} \left(\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!} \\
&= \frac{q(1+qt)^{\frac{-r}{q}}}{\ln(1+qt)} \left((1+qt)^{\frac{1}{q}} - 1\right).
\end{aligned}$$

When $r = 0$, we get the exponential generating function of c_n^q

$$\sum_{n=0}^{\infty} c_n^q \frac{t^n}{n!} = \frac{q}{\ln(1+qt)} \left((1+qt)^{\frac{1}{q}} - 1\right)$$

According to (2.1),

$$(-x-r|q)_n = \prod_{j=0}^{n-1}(-x-r-jq) = \sum_{k=0}^{n} w(n,k)\,(-1)^k\,x^k, \qquad q \neq 0. \tag{2.3}$$

Using (1.7), we get the following theorem.

Theorem 2. *The Cauchy polynomials with q parameter of the second kind $\hat{c}_n^q(r)$, $q \neq 0$ can be written explicitly as*

$$\hat{c}_n^q(-r) = \sum_{k=0}^{n}(-1)^k\,w(n,k)\,\frac{1}{k+1}. \tag{2.4}$$

The first few polynomials are
$\hat{c}_0^q(r) = 1$,
$\hat{c}_1^q(r) = r - \frac{1}{2}$,
$\hat{c}_2^q(r) = r^2 - (q+1)r + \frac{1}{2}q + \frac{1}{3}$,
$\hat{c}_3^q(r) = r^3 - \frac{3}{2}(2q+1)r^2 + (2q^2+3q+1)r - q^2 - q - \frac{1}{4}$,
$\hat{c}_4^q(r) = r^4 - (6q+2)r^3 + (11q^2+9q+2)r^2 - (6q^3+11q^2+6q+1)r + 3q^3 + \frac{11}{3}q^2 + \frac{3}{2}q + \frac{1}{5}$.

Remark 2. *If $r = 0$, then $\hat{c}_n^q(0) = \hat{c}_n^q$ are the Cauchy numbers with q parameter of the second kind* [6]

$$\hat{c}_n^q = \int_0^1 (-x|q)_n\,dx = \sum_{k=0}^{n} q^{n-k}\,s(n,k)\,\frac{(-1)^k}{k+1},$$

Similarly, we can obtain the exponential generating function of $\hat{c}_n^q(r)$:

$$\begin{aligned}\sum_{n=0}^{\infty} \hat{c}_n^q(r)\frac{t^n}{n!} &= (1+qt)^{\frac{r}{q}} \sum_{k=0}^{\infty}\left(-\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}\,\frac{1}{k+1} \\ &= \frac{q(1+qt)^{\frac{r}{q}}}{\ln(1+qt)}\left(1-(1+qt)^{\frac{-1}{q}}\right).\end{aligned} \tag{2.5}$$

And

$$\sum_{n=0}^{\infty} \hat{c}_n^q \frac{t^n}{n!} = \frac{q}{\ln(1+qt)}\left(1-(1+qt)^{\frac{-1}{q}}\right). \tag{2.6}$$

Replace x by $\frac{x-r}{q}$ in (1.12), then the r-Whitney numbers of the second kind $W(n,k)$ are given by

$$x^n = \sum_{k=0}^{n} W(n,k)(x-r|q)_k = \sum_{k=0}^{n} W(n,k)\prod_{j=0}^{k-1}(x-r-jq), \qquad q \neq 0. \tag{2.7}$$

Thus, the relation between $c_n^q(r)$, $\hat{c}_n^q(r)$ and $W(n,k)$ can be obtained as follows:

$$\sum_{k=0}^{n} W(n,k)\,c_k^q(r) = \int_0^1 \sum_{k=0}^{n} W(n,k)(x-r|q)_k\,dx = \int_0^1 x^n\,dx = \frac{1}{n+1} \tag{2.8}$$

(2.9)
$$\sum_{k=0}^{n} W(n,k)\,\hat{c}_k^q(-r) = \int_0^1 \sum_{k=0}^{n} W(n,k)(-x-r|q)_k\,dx = \int_0^1 (-1)^n\,x^n\,dx = \frac{(-1)^n}{n+1}$$

Cheon et al. [1] gave the following representation of $w(n,k)$ in terms of $s(n,k)$

$$w(n,k) = \sum_{i=k}^{n} \binom{n}{i} (-1)^{n-i}\,q^{i-k}\,[r|q]_{n-i}\,s(i,k).$$

Hence,

Corollary 1. *The Cauchy polynomials $c_n^q(r)$ can be computed by using $s(n,k)$ as follows:*

$$\begin{aligned} c_n^q(r) &= \sum_{k=0}^{n}\sum_{i=k}^{n} \binom{n}{i} (-1)^{n-i}\,q^{i-k}\,[r|q]_{n-i}\,s(i,k)\,\frac{1}{k+1} \\ &= \sum_{i=0}^{n}\sum_{k=0}^{i} \binom{n}{i} (-1)^{n-i}\,q^{i-k}\,[r|q]_{n-i}\,s(i,k)\,\frac{1}{k+1}. \end{aligned} \tag{2.10}$$

When $q=1$, we obtain the identity

$$c_n(r) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}\,[r|1]_{n-i}c_i. \tag{2.11}$$

The r-Whitney numbers $w_{q,r}(n,k)$ satisfy the following identity [1].

$$w_{q,r+s}(n,k) = \sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}\,[r|q]_{n-j}\,w_{q,s}(j,k), \tag{2.12}$$

hence, we obtain the following theorem.

Theorem 3. *For $n \ge 0$, we have*

$$c_n^q(r+s) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\,[r|q]_{n-j}\,c_j^q(s). \tag{2.13}$$

Proof.

$$\begin{aligned} c_n^q(r+s) &= \sum_{k=0}^{n} w_{q,r+s}(n,k)\,\frac{1}{k+1} \\ &= \sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}\,[r|q]_{n-j}\,w_{q,s}(j,k)\,\frac{1}{k+1} \\ &= \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{n-j}\binom{n}{j}\,[r|q]_{n-j}\,w_{q,s}(j,k)\,\frac{1}{k+1} \\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\,[r|q]_{n-j}\,c_j^q(s). \end{aligned}$$

□

Remark 3. *For $s = 0$, we get*

$$c_n^q(r) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [r|q]_{n-j}\, c_j^q. \tag{2.14}$$

For $q = 1$, we get

$$c_n(r+s) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [r|1]_{n-j}\, c_j(s). \tag{2.15}$$

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Global dynamics of Chikungunya virus with two routes of infection

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Abstract

In this paper, we address the stability analysis of within-host Chikungunya virus (CHIKV) infection models with antibodies. We incorporate two modes of infections, attaching a CHIKV to a host monocyte, and contacting an infected monocyte with an uninfected monocyte. The global stability analysis of the equilibria are established using Lyapunov method. The existence and global stability of the steady states are determined by the basic reproduction number $\mathcal{R}_0$. We have proven that the CHIKV-free equilibrium E_0 is globally asymptotically stable when $\mathcal{R}_0 \leq 1$, and the infected equilibrium E_1 is globally asymptotically stable when $\mathcal{R}_0 > 1$. The theoretical results are confirmed by numerical simulations.

1 Introduction

During last decades, many researchers have developed and analyzed several mathematical models human pathogens (see e.g. [1]-[16]). Chikungunya virus (CHIKV) is an alphavirus causes chikungunya fever. CHIKV is a mosquito-transmitted and is transmitted by the Aedes albopictus and Aedes agypti mosquito. Most of authors develop the mathematical models to describe the disease transmission mosquito and human populations. Recently, Wang and Liu [16] have proved a mathematical model for the within-host CHIKV dynamics as:

$$\dot{s} = \beta - \delta s - \eta s y, \tag{1}$$

$$\dot{y} = \eta s y - \epsilon y, \tag{2}$$

$$\dot{p} = \pi y - c p - r x p, \tag{3}$$

$$\dot{x} = \lambda + \rho x p - m x, \tag{4}$$

Here, s, y, p and x are the concentrations of uninfected monocytes, infected monocytes, CHIKV pathogen and antibodies, respectively. β and δ represent the birth rate and death rate constants of the uninfected monocytes, respectively. The monocytes become infected at rate ηsy, where η is the infection rate constant. Constants ϵ, c and m represent, respectively, the death rate of the infected monocytes, CHIKV and antibodies. Constant π is the generation rate of the CHIKV from actively infected monocytes. Antibodies attack the CHIKV at rate rxp. Once antigen is encountered, the antibodies expand at a constant rate λ and proliferate at rate ρxp. In a very recent work, Elaiw et al. [17], [18] have studied the global stability analysis of a class of CHIKV dynamics models. The models presented in [16]-[18] assume that the uninfected monocyte becomes infected by contacting with CHIKV(CHIKV-to-monocyte transmission). Kristin and Mork [19] reported that the CHIKV can also spread by infected-to-monocyte transmission. Viral danamics models with both cellular and viral infections have been studied in several works [20]-[24]. However, the dynamics of CHIKV with two routes of infection did not studied before.

Our aim is to propose and analyse a CHIKV dynamics model with two routes of infection. We calculate the basic reproduction number $\mathcal{R}_0$, and construct Lyapunov functions to prove the global stability of the equilbria.

2 CHIKV dynamics model

We investigate the following CHIKV dynamics model with CHIKV-to-monocyte and infected-to-monocyte with two routes of infection:

$$\dot{s} = \beta - \delta s - \eta_1 sp - \eta_2 sy, \tag{5}$$

$$\dot{y} = \eta_1 sp + \eta_2 sy - \epsilon y, \tag{6}$$

$$\dot{p} = \pi y - cp - rxp, \tag{7}$$

$$\dot{x} = \lambda + \rho xp - mx. \tag{8}$$

Here, the uninfected monocytes become infected at rate $(\eta_1 y + \eta_2 p)s$, where η_1 and η_2 are the CHIKV-monocyte and infected-monocyte incidence constants, respectively.

2.1 Nonnegativity and boundedness

Lemma 1 *There exist $M_1, M_2, M_3 > 0$, such that the following compact set is positively invariant for system (5)-(8)*

$$\Gamma_1 = \{(s, y, p, x) \in \mathbb{R}^4_{\geq 0} : 0 \leq s, y \leq M_1, 0 \leq p \leq M_2, 0 \leq x \leq M_3\}$$

Proof. We have

$$\dot{s}\mid_{s=0} = \beta > 0, \qquad \dot{y}\mid_{y=0} = \eta_1 sp \geq 0, \quad \text{for all} \ \ s, p \geq 0,$$

$$\dot{p}\mid_{p=0} = \pi y \geq 0, \quad \text{for all} \ \ y \geq 0, \quad \dot{x}\mid_{x=0} = \lambda > 0.$$

Thus $\mathbb{R}^4_{\geq 0}$ positively invariant with respect to system (5)-(8). Let us define

$$F_1(t) = s(t) + y(t),$$
$$F_2(t) = p(t) + \frac{r}{\rho}x(t).$$

Then from Eqs. (5)-(8) we get

$$\begin{aligned}\dot{F}_1(t) &= \beta - \delta s(t) - \epsilon y(t) \\ &\leq \beta - \sigma_1(s(t) + y(t)) \\ &= \beta - \sigma_1 F_1(t),\end{aligned}$$

where, $\sigma_1 = min\{\delta, \epsilon\}$. Hence $F_1(t) \leq M_1$, if $F_1(0) \leq M_1$, where $M_1 = \frac{\beta}{\sigma_1}$. It follows that $0 \leq s(t), y(t) \leq M_1$ if $0 \leq s(0) + y(0) \leq M_1$. Moreover, we have

$$\begin{aligned}\dot{F}_2(t) &= \pi y(t) - cp(t) + \frac{r}{\rho}\lambda - \frac{mr}{\rho}x(t) \\ &\leq \pi M_1 + \frac{r}{\rho}\lambda - \sigma_2\left(p(t) + \frac{r}{\rho}x(t)\right) \\ &= \pi M_1 + \frac{r}{\rho}\lambda - \sigma_2 F_2(t),\end{aligned}$$

where, $\sigma_2 = min\{c, m\}$. Hence $F_2(t) \leq M_2$, if $F_2(0) \leq M_2$, where $M_2 = \frac{\pi M_1 + \frac{r}{\rho}\lambda}{\sigma_2}$. Since $p(t)$ and $x(t)$ are all nonnegative, then $0 \leq p(t) \leq M_2$ and $x(t) \leq M_3$ if $0 \leq p(0) + \frac{r}{\rho}x(0) \leq M_2$, where $M_3 = \frac{\rho M_2}{r}$.
■

2.2 Equilbria

We define the basic reproduction number

$$\mathcal{R}_0 = \frac{(\eta_1\pi m + \eta_2 cm + \eta_2 r\lambda)\beta}{\epsilon\delta(cm + r\lambda)}.$$

Lemma 2 *(i) if $\mathcal{R}_0 \leq 1$, then there exists only one equilbrium $E_0 \in \Gamma_1$ (ii) if $\mathcal{R}_0 > 1$, then there exist two equilbria $E_0 \in \Gamma_1$ and $E_1 \in \overset{\circ}{\Gamma}_1$, where $\overset{\circ}{\Gamma}_1$ is the interior of Γ_1.*

Proof. Any equilbrium satisfying

$$\beta - \delta s - \eta_1 sp - \eta_2 sy = 0, \tag{9}$$

$$\eta_1 sp + \eta_2 sy - \epsilon y = 0, \tag{10}$$

$$\pi y - cp - rxp = 0, \tag{11}$$

$$\lambda + \rho xp - mx = 0. \tag{12}$$

By solving Eqs. (9)-(12) we we get two equilbria a CHIKV-free equilbrium $E_0 = (s_0, 0, 0, x_0)$, where $s_0 = \frac{\beta}{\delta}$ and $x_0 = \frac{\lambda}{m}$. Moreover we have

$$C_1 p^3 + C_2 p^2 + C_3 p + C_4 = 0,$$

where

$$\begin{aligned}
C_1 &= -c\pi\epsilon\eta_1\rho^2 - c^2\epsilon\eta_2\rho^2,\\
C_2 &= 2cm\pi\epsilon\eta_1\rho + 2c^2 m\epsilon\eta_2\rho + \pi r\epsilon\eta_1\lambda\rho + 2cr\epsilon\eta_2\lambda\rho - c\pi\delta\epsilon\rho^2 + \pi^2\beta\eta_1\rho^2 + c\pi\beta\eta_2\rho^2\\
C_3 &= -cm^2\pi\epsilon\eta_1 - c^2m^2\epsilon\eta_2 - m\pi r\epsilon\eta_1\lambda - 2cmr\epsilon\eta_2\lambda - r^2\epsilon\eta_2\lambda^2 + 2cm\pi\delta\epsilon\rho - 2m\pi^2\beta\eta_1\rho\\
&\quad - 2cm\pi\beta\eta_2\rho + \pi r\delta\epsilon\lambda\rho - \pi r\beta\eta_2\lambda\rho,\\
C_4 &= -cm^2\pi\delta\epsilon + m^2\pi^2\beta\eta_1 + cm^2\pi\beta\eta_2 - m\pi r\delta\epsilon\lambda + m\pi r\beta\eta_2\lambda.
\end{aligned}$$

Let define a function $X(p)$ as:

$$X(p) = C_1 p^3 + C_2 p^2 + C_3 p + C_4 = 0.$$

Then we obtain

$$\begin{aligned}
X(0) &= C_4,\\
X\left(\frac{m}{\rho}\right) &= -\frac{mr^2\epsilon\eta_2\lambda^2}{\rho} < 0.
\end{aligned}$$

The constant C_4 can be written as

$$C_4 = m\pi\delta\epsilon(cm + r\lambda)\left(\frac{(\eta_1\pi m + \eta_2 cm + \eta_2 r\lambda)\beta}{\epsilon\delta(cm + r\lambda)} - 1\right)$$

Then $C_4 > 0$ if the following condition is satisfied

$$\frac{(\eta_1\pi m + \eta_2 cm + \eta_2 r\lambda)\beta}{\epsilon\delta(cm + r\lambda)} > 1, \tag{13}$$

then there exists $p_1 \in (0, \frac{m}{\rho})$ such that $X(p_1) = 0$. Therefore, if condition (13) is satisfied, then

$$\begin{aligned}
s_1 &= \frac{\epsilon c(m - \rho p_1) + \epsilon r\lambda}{\eta_1\pi(m - \rho p_1) + \eta_2 c(m - \rho p_1) + \eta_2 r\lambda} > 0,\\
y_1 &= \frac{p_1(c(m - \rho p_1) + r\lambda)}{\pi(m - \rho p_1)} > 0, \qquad x_1 = \frac{\lambda}{m - \rho p_1} > 0.
\end{aligned}$$

Then an infected equilbrium $E_1 = (s_1, y_1, p_1, x_1)$ exists when $\mathcal{R}_0 > 1$.

Now we show that $E_0 \in \Gamma_1$ and $E_1 \in \overset{\circ}{\Gamma}_1$. Clearly, $E_0 \in \Gamma_1$. From the equilbrium conditions of E_1 we have

$$\beta = \delta s_1 + \eta_1 s_1 p_1 + \eta_2 s_1 y_1 \Rightarrow \delta s_1 + \epsilon y_1 = \beta \Rightarrow 0 < s_1 < \frac{\beta}{\delta} \leq M_1, 0 < y_1 < \frac{\beta}{\epsilon} \leq M_1.$$

Moreover, from Eqs. (11) and (12) we have

$$cp_1 = \pi y_1 + \frac{r}{\rho}\lambda - \frac{mr}{\rho}x_1 \Rightarrow cp_1 + \frac{mr}{\rho}x_1 = \pi y_1 + \frac{r}{\rho}\lambda < \pi M_1 + \frac{r}{\rho}\lambda$$
$$p_1 < \frac{\pi M_1 + \frac{r}{\rho}\lambda}{c} \leq M_2, x_1 < \frac{\rho}{r}\frac{\pi M_1 + \frac{r}{\rho}\lambda}{m} \leq \frac{\rho M_2}{r} = M_3.$$

It follows that, $E_1 \in \overset{\circ}{\Gamma}$. ■

3 Global properties

Define a function $G(z) = z - 1 - \ln z$.

Theorem 1 *If $\mathcal{R}_0 \leq 1$, then E_0 is globally asymptotically stable in Γ_1.*

Proof. Letting $\mathcal{R}_0 \leq 1$ and constructing a Lyapunov function $U_0(s, y, p, x)$ as:

$$U_0(s, y, p, x) = s_0 G\left(\frac{s}{s_0}\right) + y + \frac{\eta_1 s_0}{c + rx_0}p + \frac{r\eta_1 s_0}{\rho(c + rx_0)}x_0 G\left(\frac{x}{x_0}\right).$$

Calculating $\frac{dU_0}{dt}$ along system (5)-(8) we obtain

$$\begin{aligned}\frac{dU_0}{dt} &= \left(1 - \frac{s_0}{s}\right)\left(\beta - \delta s - \eta_1 sp - \eta_2 sy\right) + \eta_1 sp + \eta_2 sy - \epsilon y \\ &+ \frac{\eta_1 s_0}{c + rx_0}\left(\pi y - cp - rxp\right) + \frac{r\eta_1 s_0}{\rho(c + rx_0)}\left(1 - \frac{x_0}{x}\right)\left(\lambda + \rho xp - mx\right) \\ &= \left(1 - \frac{s_0}{s}\right)\left(\beta - \delta s\right) + \eta_2 s_0 y - \epsilon y + \frac{\eta_1 s_0}{c + rx_0}\pi y + \frac{r\eta_1 s_0}{\rho(c + rx_0)}\left(1 - \frac{x_0}{x}\right)\left(\lambda - mx\right)\end{aligned}$$

$$\begin{aligned}\frac{dU_0}{dt} &= -\delta\frac{(s - s_0)^2}{s} + \epsilon\left(\frac{\eta_2 s_0}{\epsilon} + \frac{\eta_1 s_0 \pi}{\epsilon(c + rx_0)} - 1\right)y - \frac{r\eta_1 s_0 m}{\rho(c + rx_0)}\frac{(x - x_0)^2}{x} \\ &= -\delta\frac{(s - s_0)^2}{s} - \frac{r\eta_1 s_0 m}{\rho(c + rx_0)}\frac{(x - x_0)^2}{x} + \epsilon(\mathcal{R}_0 - 1)y. \end{aligned} \tag{14}$$

Since $\mathcal{R}_0 \leq 1$, then for all $s, y, p, x > 0$ we have $\frac{dU_0}{dt} \leq 0$. Let $W_0 = \{(s, y, p, x) : \frac{dU_0}{dt} = 0\}$. It can be easily shown that $\frac{dU_0}{dt} = 0$ at E_0. Appling LaSalle's invariance principle, we get E_0 is globally asymptotically stable when $\mathcal{R}_0 \leq 1$. ■

Theorem 2 *If $\mathcal{R}_0 > 1$, then E_1 is globally asymptotically stable in $\overset{\circ}{\Gamma}_1$.*

Proof. Define

$$U_1(s, y, p, x) = s_1 G\left(\frac{s}{s_1}\right) + y_1 G\left(\frac{y}{y_1}\right) + \frac{\eta_1 s_1 p_1}{\pi y_1}p_1 G\left(\frac{p}{p_1}\right) + \frac{r}{\rho}\frac{\eta_1 s_1 p_1}{\pi y_1}x_1 G\left(\frac{x}{x_1}\right).$$

Calculating $\frac{dU_1}{dt}$ along the trajectories of (5)-(8) we obtain

$$\begin{aligned}\frac{dU_1}{dt} &= \left(1-\frac{s_1}{s}\right)\left(\beta-\delta s-\eta_1 sp-\eta_2 sy\right)+\left(1-\frac{y_1}{y}\right)\left(\eta_1 sp+\eta_2 sy-\epsilon y\right)\\ &+\frac{\eta_1 s_1 p_1}{\pi y_1}\left(1-\frac{p_1}{p}\right)\left(\pi y-cp-rxp\right)+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda+\rho xp-mx\right)\\ &= \left(1-\frac{s_1}{s}\right)\left(\beta-\delta s\right)+\eta_1 s_1 p+\eta_2 s_1 y-\eta_1 sp\frac{y_1}{y}-\eta_2 sy_1-\epsilon y+\epsilon y_1+\frac{\eta_1 s_1 p_1}{y_1}y-\eta_1 s_1 p_1\frac{p_1 y}{py_1}\\ &-\frac{\eta_1 s_1 p_1}{\pi y_1}cp+\frac{\eta_1 s_1 p_1}{\pi y_1}cp_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rxp_1-\frac{r\eta_1 s_1 p_1}{\pi y_1}x_1 p+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda-mx\right).\end{aligned}$$

Applying the equilbrium conditions for E_1

$$\beta=\eta_1 s_1 p_1+\eta_2 s_1 y_1+\delta s_1,\quad \epsilon y_1=\eta_1 s_1 p_1+\eta_2 s_1 y_1,\quad cp_1=\pi y_1-rx_1 p_1,\quad \lambda=mx_1-\rho x_1 p_1.$$

we get

$$\begin{aligned}\frac{dU_1}{dt} &= -\delta\frac{(s-s_1)^2}{s}+\left(1-\frac{s_1}{s}\right)\left(\eta_1 s_1 p_1+\eta_2 s_1 y_1\right)\\ &-\eta_1 s_1 p_1\frac{spy_1}{s_1 p_1 y}-\eta_2 s_1 y_1\frac{s}{s_1}+\eta_1 s_1 p_1+\eta_2 s_1 y_1-\eta_1 s_1 p_1\frac{p_1 y}{py_1}+\eta_1 s_1 p_1\\ &-2\frac{\eta_1 s_1 p_1}{\pi y_1}rx_1 p_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rxp_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rx_1 p_1\frac{x_1}{x}-\frac{r\eta_1 s_1 p_1 m}{\rho\pi y_1}\frac{(x-x_1)^2}{x}.\end{aligned}\tag{15}$$

Eq. (15) can be simplified as:

$$\begin{aligned}\frac{dU_1}{dt} &= -\delta\frac{(s-s_1)^2}{s}+\eta_2 s_1 y_1\left[2-\frac{s_1}{s}-\frac{s}{s_1}\right]+\eta_1 s_1 p_1\left[3-\frac{s_1}{s}-\frac{spy_1}{s_1 p_1 y}-\frac{p_1 y}{py_1}\right]\\ &-\frac{\eta_1 s_1 p_1}{\pi y_1}rx_1 p_1\left[2-\frac{x}{x_1}-\frac{x_1}{x}\right]-\frac{r\eta_1 s_1 p_1 m}{\rho\pi y_1}\frac{(x-x_1)^2}{x}\\ &= -\delta\frac{(s-s_1)^2}{s}-\frac{\eta_2 y_1(s-s_1)^2}{s}+\eta_1 s_1 p_1\left[3-\frac{s_1}{s}-\frac{spy_1}{s_1 p_1 y}-\frac{p_1 y}{py_1}\right]\\ &+\frac{\eta_1 s_1 p_1}{\pi y_1}rp_1\frac{(x-x_1)^2}{x}-\frac{r\eta_1 s_1 p_1 m}{\rho\pi y_1}\frac{(x-x_1)^2}{x}\\ &= -(\delta+\eta_2 y_1)\frac{(s-s_1)^2}{s}-\frac{\eta_1 s_1 p_1}{\pi y_1}\frac{r\lambda}{\rho x_1}\frac{(x-x_1)^2}{x}+\eta_1 s_1 p_1\left[3-\frac{s_1}{s}-\frac{spy_1}{s_1 p_1 y}-\frac{p_1 y}{py_1}\right].\end{aligned}$$

We use the following arithmetic mean-geometric mean inequality rule. If $a_i\geq 0$, $i=1,2,...,n$, then

$$\frac{1}{n}\sum_{i=1}^{n}a_i\geq\sqrt[n]{\prod_{i=1}^{n}a_i},\tag{16}$$

where equality holding if and only if $a_1=a_2=...=a_n$. It follows that

$$\frac{1}{3}\left(\frac{s_1}{s}+\frac{spy_1}{s_1 p_1 y}+\frac{p_1 y}{py_1}\right)\geq 1.$$

Therefore, $\frac{dU_1}{dt}\leq 0$ for all $s,y,p,x>0$ and $\frac{dU_1}{dt}=0$ if and only if $s=s_1, y=y_1, p=p_1$ and $x=x_1$. It follows that the global stability of E_1 is induced from LaSalle's invariance principle. ■

Table 1: The value of the parameters of model (5)-(8).

Parameter	Value	Parameter	Value
β	2	δ	0.1
η_1, η_2	varied	ϵ	0.5
π	4	c	0.1
r	0.5	λ	1.4
m	1	ρ	0.2

4 Numerical Simulations

We will use the values of the parameters given in Table 1. Moreover, we similate the system with three different initial values as:

IV1: $s(0) = 14.0, y(0) = 1.0, p(0) = 1.0$, and $x(0) = 1.0$,

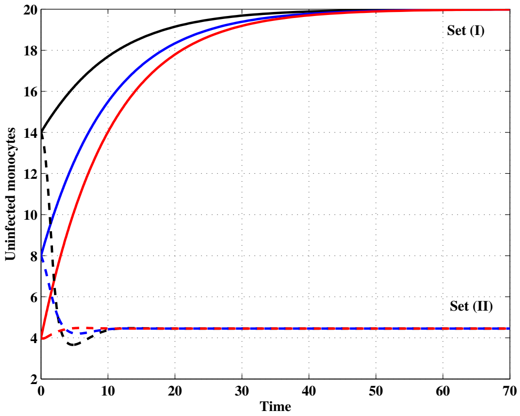
IV2: $s(0) = 8.0, y(0) = 2.0, p(0) = 3.0$, and $x(0) = 4.0$,

IV3: $s(0) = 4.0, y(0) = 3.5, p(0) = 6.0$, and $x(0) = 7.0$.

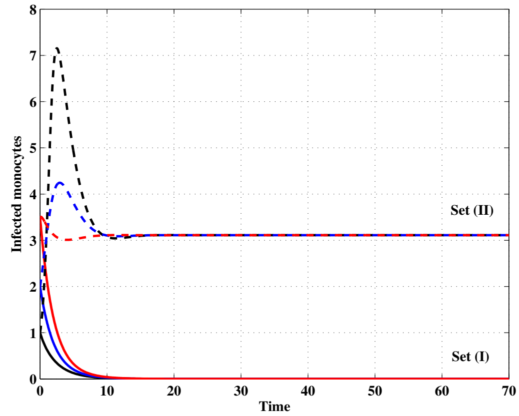
Then we consider two sets of the values of η_1 and η_2 as follows:

Set (I): We choose $\eta_1 = \eta_2 = 0.001$. The value of $\mathcal{R}_0$ is computed as $\mathcal{R}_0 = 0.2400 < 1$. Figure 1 shows that, the concentrations of the uninfected monocytes and B cells return to their values $s_0 = \frac{\beta}{\delta} = 20$ and $x_0 = \frac{\lambda}{m} = 1.4$, respectively. On the other hand, the concentrations of infected monocytes and CHIKV particles are declining and reaching zero for the initial values IV1-IV3. This shows that, E_0 is GAS which agrees with the result of Theorem 1.

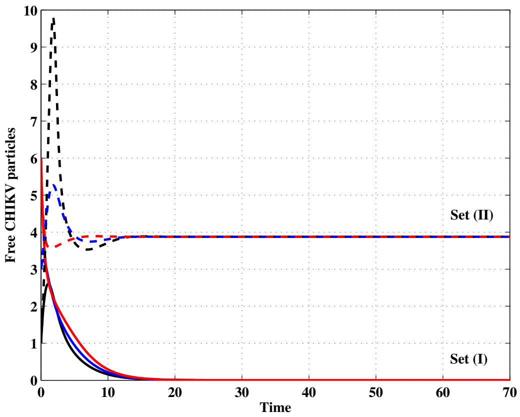
Set (II): We take $\eta_1 = \eta_2 = 0.05$. Then, we calculate $\mathcal{R}_0 = 12.0 > 1$. We comput the equilbria as $E_0(20.0, 0, 0, 1.4)$ and $E_1 = (4.45, 3.10, 3.87, 6.22)$. Figure 1 shows that when $\mathcal{R}_0 > 1$, the states of the system tend to E_1 for all the three initial values IV1-IV3. This confirms that the validity of Theorem 2.



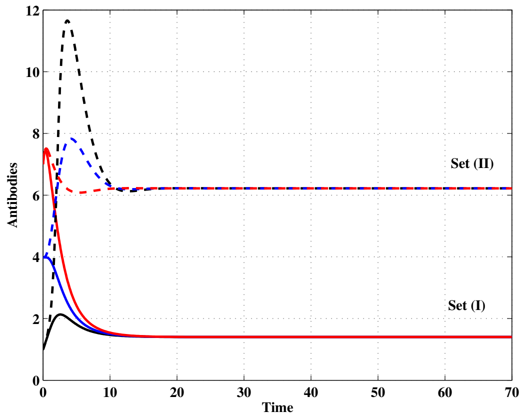
(a) Uninfected monocytes.



(b) Infected monocytes.



(c) Free CHIKV particles.



(d) Antibodies.

Figure 1: The simulation of trajectories of system (5)-(8).

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Weighted norm inequalities of θ-type Calderón-Zygmund operators and commutators on λ-central Morrey space

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Abstract: In this paper, the weighted boundedness for θ-type Calderón-Zygmund operators T_θ is established on the λ-central Morrey space. Futhermore, the weighted norm inequalities for commutators of $[b, T_\theta]$ generated by T_θ and BMO functions on the weighted λ-central Morrey space is also given.

Keywords: θ-type Calderón-Zygmund operator; weighted λ-central Morrey space; commutator

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1 Introduction and notation

The theory of Calderón-Zygmund operators has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, which has been extensively studied (see [7-10, 16-17], for instance). In 1985, Yabuta introduced certain θ-type Calderón-Zygmund operators to facilitate his study of certain classes of pseudodifferential operators (see [36]). Following the terminology of Yabuta, we recall the so-called θ-type Calderón-Zygmund operators. Let θ be a non-negative and non-decreasing function on $\mathbb{R}^+ = (0, \infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty. \tag{1.1}$$

A measurable function $K(\cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a θ-type Calderón-Zygmund kernel if it satisfies

$$|K(x, y)| \le C|x - y|^{-n} \tag{1.2}$$

and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le C\theta\left(\frac{|x - x'|}{|x - y|}\right)|x - y|^{-n}, \quad \text{as} \quad |x - y| \ge 2|x - x'|. \tag{1.3}$$

Definition 1.1[36] Let T_θ be a linear operator from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class. One can say that T_θ is a θ-type Calderón-Zygmund operator if it satisfies the following conditions:

(1) T_θ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;

(2) there is a θ-type Calderón-Zygmund kernel $K(x, y)$ such that

$$T_\theta f(x) := \int_{\mathbb{R}^n} K(x, y) f(y) \mathrm{d}y, \quad \text{as} \quad f \in C_c^\infty(\mathbb{R}^n) \quad \text{and} \quad x \notin \operatorname{supp} f. \tag{1.4}$$

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It is easy to see that the classical Calderón-Zygmund operator with standard kernel is a special case of θ-type operator T_θ as $\theta(t) = t^\delta$ with $0 < \delta \leq 1$. Given a locally integrable function b, the commutator generated by T_θ and b is defined by

$$[b, T_\theta]f(x) = b(x)T_\theta f(x) - T_\theta(b \cdot f)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]K(x, y)f(y)\mathrm{d}y. \tag{1.5}$$

Such type of operators is extensively applied in PDE with non-smooth area. Many authors concentrates on the boundedness of this operators on various function spaces, we refer the reader to see [19-20, 27, 29-30, 33-35] for its developments and applications. In [27], Quek-Yang established the boundedness of T_θ on spaces such as weighted Lebesgue spaces and weak Lebesgue spaces, weighted Hardy spaces and weak Hardy spaces. Ri-Zhang obtained the bounedness of T_θ on Hardy spaces with non-doubling measures and non-homogeneous metric measure spaces in [29-30]. Wang proved the boundedness of T_θ and $[b, T_\theta]$ on the generalized weighted Morrey spaces in [33]. Inspired by the results mentioned previously, a natural and interesting problem is to consider whether the θ-type Calderón-Zygmund operators T_θ and their commutators $[b, T_\theta]$ are bounded on λ-central Morrey space or not. The purpose of this paper is to give an surely answer.

On the other hand, the well-known Morrey spaces which introduced originally by Morrey [23] in relation to the study of partial differential equations, were widely investigated during last decades, including the study of classical operators of harmonic analysis in various generalizations of these spaces. Morrey-type spaces appeared to be quite useful in the study of the local behavior of the solutions of partial differential equations, a priori estimates and other topics. They are also widely used in applications to regularity properties of solutions to PDE including the study of Navier-Stokes equations (see [32] and references therein). The ideas of Morrey (see [23]) were further developed by Campanato in 1964 (see [11]). In 1975, Adam proved the boundedness of Riesz potential on the classical Morrey space in [1]. Later, in 1987, the boundedness of singular integrals and Hardy-Littlewood maximal functions on Morrey spaces was obtained By Chiarenza and Frasca in [13]. A more systematic study of these (and even more general) spaces, we refer the readers to see [2-3, 6, 26, 28, 31].

In [5], Beurling introduced a pair of dual Banach spaces, A^q and $B^{q'}$ with $1/q + 1/q' = 1$. After that, Feichtinger found the folling way to describe B^q as

$$\|f\|_{B^q} = \sup_{k \geq 0}(2^{-kn/q}\|f\chi_k\|_{L^q}) < \infty, \tag{1.6}$$

where χ_0 is the characteristic function of the unit ball defined by $\{x \in \mathbb{R}^n : |x| \leq 1\}$ and χ_k is the characteristic function of the annulus, that is $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$ with $k \in \mathbb{Z}^+$. By duality, the beurling algebra A^q can be written as

$$\|f\|_{A^q} = \sum_{k=0}^{\infty} 2^{kn/q'}\|f\chi_k\|_{L^q} < \infty. \tag{1.7}$$

Later, a new Hardy space HA^q related to the Beurling algebra A^q was introduced by Chen and Lau (see [12]). Denotes $B(0, R)$ be a cube centered at the origin with the side-length $R > 0$. Let

$f_{B(0,R)} = \frac{1}{|B(0,R)|}\int_{B(0,R)} f(x)\mathrm{d}x$ be the integral average of f on B. Then using duality, the dual space of HA^q can be described by $\mathrm{CBMO^q}$ with the following norm,

$$\|f\|_{\mathrm{CBMO}^q} = \sum_{R\geq 1}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|f(x)-f_{B(0,R)}|^q\mathrm{d}x\right)^{1/q} < \infty. \tag{1.8}$$

Later, Lu and Yang (see [21-22]) introduced the homogeneous new Hardy type space $\mathrm{H\dot{A}}_q$ and they proved that the dual space of $\mathrm{H\dot{A}}_q$ can be written by

$$\|f\|_{\mathrm{C\dot{B}MO}^q} = \sum_{R\geq 0}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|f(x)-f_{B(0,R)}|^q\mathrm{d}x\right)^{1/q} < \infty. \tag{1.9}$$

Obviously, the space of $\mathrm{C\dot{B}MO}^q$ is the homogeneous central bounded mean oscillation space depending on q and it can be regarded as an extention of the classical BMO since the famous John-Nirenberg inequality no longer hold in such space.

Alverez, Lakey and Guzmán-Partida introduced the λ-central bounded mean oscillation space and the λ-central Morrey space in 2000 (see [4]), respectively.

Definition 1.2[4] Let $\lambda < 1/n$ and $1 < q < \infty$. Then we say that a function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the λ-central bounded mean oscillation space $\mathrm{C\dot{B}MO}^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{C\dot{B}MO}^{q,\lambda}} = \sum_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)-f_{B(0,R)}|^q\mathrm{d}x\right)^{1/q} < \infty. \tag{1.10}$$

Definition 1.3[4] Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. Then the λ-central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ by the following norm

$$\|f\|_{\dot{B}^{q,\lambda}} = \sum_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)|^q\mathrm{d}x\right)^{1/q} < \infty. \tag{1.11}$$

It is very important to study the weighted norm inequalities for some integral operators on classical L^p spaces, one may see [14, 24-25] et al. for more details. In 2009, Komori-Furuya and Shirai (see [18]) defined the weighted Morrey space and showed the boundedness of some classical integral operators and their commutators on the weighted Morrey spaces. In this paper, we will prove the weighted boundedness of θ-type Calderón-Zygmund operator T_θ on the weighted λ-central Morrey space. Before giving the main results, we introduce the following definitions.

Definition 1.4[37] Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. Then the weighted λ-central Morrey space $\dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}(\mathbb{R}^n)$ is defined by

$$\left\{f \in \dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}(\mathbb{R}^n) : \|f\|_{\dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}} = \sum_{R>0}\left(\frac{1}{\omega_1(B(0,R))^{1+\lambda q}}\int_{B(0,R)}|f(x)|^q\omega_2(x)\mathrm{d}x\right)^{1/q} < \infty\right\}, \tag{1.12}$$

where ω_1 and ω_2 are non-negative and local integrable functions. Moreover, if $\omega_1 = \omega_2 = \omega$, we denote $\dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}(\mathbb{R}^n) = \dot{\mathrm{B}}^{q,\lambda}_{\omega}(\mathbb{R}^n)$.

Definition 1.5[37] Let $\lambda < 1/n$ and $1 < q < \infty$. Then we say that a function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the weighted λ-central bounded mean oscillation space $\mathrm{C\dot{B}MO}^{q,\lambda}_{\omega}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{C\dot{B}MO}^{q,\lambda}_{\omega}} = \sum_{R>0} \left(\frac{1}{\omega(B(0,R))^{1+\lambda q}} \int_{B(0,R)} |f(x) - f_{B,\omega}|^q \omega(x)\mathrm{d}x \right)^{1/q} < \infty, \tag{1.13}$$

where the definition of $f_{B,\omega}$ is $f_{B,\omega} = \frac{1}{\omega(B)} \int_B f(x)\omega(x)\mathrm{d}x$.

Definition 1.6[25] We say a non-negative function $\omega(x)$ belongs to the Muckenhoupt class A_p with $1 < p < \infty$ if there exist a constant $C > 1$ such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x)\mathrm{d}x \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'}\mathrm{d}x \right)^{p-1} < \infty,$$

where $1/p + 1/p' = 1$ and $[\omega]_{A_p}$ denotes the infimum of C. Moreover, we define $A_\infty = \bigcup_{1<p<\infty} A_p$.

Obviously, by the classical Hölder inequality, there is $A_p \subset A_q \subset A_\infty$ for $1 < p < q < \infty$.

Our results can be stated as follows.

Theorem 1.1 Let T_θ be defined by (1.4) with θ satisfies (1,1). Suppose that $1 < p < \infty$, $\lambda < 0$ and $\omega(x) \in A_p$, then there exists a constant $C > 0$ independent of f, such that, for any $f \in \dot{\mathrm{B}}^{p,\lambda}_{\omega}$,

$$\|T_\theta(f)\|_{\dot{\mathrm{B}}^{p,\lambda}_{\omega}} \le C\|f\|_{\dot{\mathrm{B}}^{p,\lambda}_{\omega}}.$$

Theorem 1.2 Let $[b, T_\theta]$ be defined by (1.5) with θ satisfies $\int_0^1 \frac{\theta(t)}{t}|\log t|dt < \infty$. Suppose that $1 < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $b \in \mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}$, $\omega(x) \in A_p$ and $\lambda = \lambda_1 + \lambda_2$ with $\lambda_i < 0 (i = 1, 2)$, then there exists a constant $C > 0$ independent of f, such that, for any $f \in \dot{\mathrm{B}}^{p_2,\lambda_2}_{\omega}$,

$$\|[b, T_\theta]f\|_{\dot{\mathrm{B}}^{p,\lambda}_{\omega}} \le C\|b\|_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}} \|f\|_{\dot{\mathrm{B}}^{p_2,\lambda_2}_{\omega}}.$$

Let us give some necessary notations. Throughout the paper C will denote a positive constant whose value may change at each appearance. In the following, unless otherwise stated, for any real number $p > 1$, we denote p' by $1/p + 1/p' = 1$. Moreover, we say that a weight ω satisfies the doubling condition if there exists a constant D, such that for any cube $Q \in \mathbb{R}^n$, we have $\omega(2Q) \le D\omega(Q)$. For simplicity, we denote $\omega \in \Delta_2$ if ω satisfies the doubling condition.

2 Preliminary Lemmas

Lemmas 2.1[15] If $\omega \in A_p$ for some $1 \le p < \infty$, then $\omega \in \Delta_2$. More precisely, for all $\alpha > 1$, we have

$$\omega(\alpha Q) \le \alpha^{np}[\omega]_{A_p}\omega(Q).$$

Lemmas 2.2[27] Let $1 < p < \infty$ and $\omega \in A_p$. Then, the θ-type Calderón-Zygmund operator T_θ is bounded on L^p_ω.

Lemmas 2.3[18] If $\omega \in \Delta_2$, then there exists a constant $D > 1$ such that for any cube B,

$$\omega(2B) \ge D\omega(B).$$

Lemmas 2.4[37] If $\omega \in A_p$ for some $1 \leq p < \infty$, then for any $k \in \mathbb{Z}^+$, $s < 0$ and any cube $B \in \mathbb{R}^n$,

$$\omega(2^k B)^s \leq D_1^{ks}\omega(B)^s,$$

where D_1 is a positive constant which belongs to the interval $(1, 2)$.

3 Proof of Theorems

Proof of Theorem 1.1. For a fixed cube $B = B(0, R)$, we may decompose $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Then we obain

$$\begin{aligned}\frac{1}{\omega(B)^{1+\lambda p}}\int_B |T(f)(x)|^p\omega(x)\mathrm{d}x &\leq \frac{1}{\omega(B)^{1+\lambda p}}\int_B |T(f_1)(x)|^p\omega(x)\mathrm{d}x\\ &\quad \frac{1}{\omega(B)^{1+\lambda p}}\int_B |T(f_2)(x)|^p\omega(x)\mathrm{d}x =: C(I_1 + I_2).\end{aligned}$$

From Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}I_1 &= \frac{1}{\omega(B)^{1+\lambda p}}\int_B |T(f_1)(x)|^p\omega(x)\mathrm{d}x\\ &\leq \frac{1}{\omega(B)^{1+\lambda p}}\int_{2B} |f(x)|^p\omega(x)\mathrm{d}x\\ &\leq C\|f\|^p_{\dot{B}^{p,\omega}_\omega}\frac{\omega(2B)^{1+\lambda p}}{\omega(B)^{1+\lambda p}}.\end{aligned}$$

As $1 + \lambda p \geq 0$, by using Lemma 2.1, then there exists a constant $C > 0$ independent of f such that

$$I_1 \leq C\|f\|^p_{\dot{B}^{p,\omega}_\omega}. \tag{3.1}$$

On the other hand, by using Lemma 2.4, we can also get (3.1) with an similar argument in the case of $1 + \lambda p < 0$.

Next let's estimate I_2. Noting that $x \in B$ and $y \in (2B)^c$, then there exists a constant $C > 0$ such that $|y| < C|x - y|$. Thus, we have

$$|T_\theta(f_2)| \leq \int_{\mathbb{R}^n} |K(x, y)||f(y)|\mathrm{d}y \leq C\int_{|y|>2r} 1/|y|^n|f(y)|\mathrm{d}y$$

Furthermore, by using Definition 1.6 and the Hölder's inequality, we can get

$$\begin{aligned}&\int_{|y|>2r} 1/|y|^n|f(y)|\mathrm{d}y = \sum_{j=1}^{\infty}\int_{2^jr<|y|<2^{j+1}r} 1/|y|^n|f(y)|\mathrm{d}y\\ &\leq \sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left(\int_{2^{j+1}B}|f(y)|^p\omega(y)\mathrm{d}y\right)^{\frac{1}{p}}\left(\int_{2^{j+1}B}\omega(y)^{\frac{-p'}{p}}\mathrm{d}y\right)^{\frac{1}{p'}}\\ &\leq \sum_{j=1}^{\infty}\frac{1}{2^jB}\left(\frac{1}{\omega(2^{j+1}B)^{1+\lambda p}}\int_{2^{j+1}B}|f(y)|^p\omega(y)\mathrm{d}y\right)^{\frac{1}{p}}\omega(2^{j+1}B)^{\frac{1+\lambda p}{p}}\left[\left(\int_{|2^{j+1}B|}\omega(y)^{1-p'}\mathrm{d}y\right)^{p-1}\right]^{\frac{1}{p}}\\ &\leq C\|f\|_{\dot{B}^{\lambda,q}_\omega}\sum_{j=1}^{\infty}(\omega(2^{j+1}B))^{\lambda}.\end{aligned}$$

Then, by the fact $\lambda < 0$ and Lemma 2.4, we get

$$I_2 = \frac{1}{\omega(B)^{1+\lambda p}} \int_B \int |T_\theta(f_2)(x)|^p \omega(x) \mathrm{d}x \leq C \frac{\omega(2^{j+1}B)^{\lambda p}}{\omega(B)^{\lambda p}} \|f\|^p_{\dot{\mathrm{B}}^{\lambda,q}_\omega} \leq C\|f\|^p_{\dot{\mathrm{B}}^{\lambda,q}_\omega} \qquad (3.2).$$

Combing with the estimates of I_1 and I_2, we finish the proof of **Theorem 1.1**. □

Proof of Theorem 1.2. For a fixed cube $B = B(0, R)$, we decompose $f = f_1 + f_2$ as in the proof of Theorem 1.1. Then we have

$$\begin{aligned}\frac{1}{\omega(B)^{1+\lambda p}} \int_B |[b, T_\theta]f(x)|^p \omega(x) \mathrm{d}x \leq & \frac{1}{\omega(B)^{1+\lambda p}} \int_B |[b, T_\theta]f_1(x)|^p \omega(x) \mathrm{d}x \\ & + \frac{1}{\omega(B)^{1+\lambda p}} \int_B |[b, T_\theta]f_2(x)|^p \omega(x) \mathrm{d}x =: I + II.\end{aligned}$$

To estimate I, we may split as

$$\begin{aligned}I \leq & \frac{1}{\omega(B)^{1+\lambda p}} \int_B |b(x) - b_{B,\omega}|^p |T_\theta(f_1)(x)|^p \omega(x) \mathrm{d}x \\ & + \frac{1}{\omega(B)^{1+\lambda p}} \int_B |T_\theta(f_1(b - b_{B,\omega}))(x)|^p \omega(x) \mathrm{d}x \\ =: & I_1 + I_2.\end{aligned}$$

First, we give the estimate of I_1. Noting that $p < p_2$, by the Hölder's inequality and Lemma 2.2, one has

$$\begin{aligned}I_1 = & \frac{1}{\omega(B)^{1+\lambda p}} \int_B |b(x) - b_{B,\omega}|^p \omega(x)^{\frac{p}{p_1}} |T_\theta(f_1)(x)|^p \omega(x)^{1-\frac{p}{p_1}} \mathrm{d}x \\ \leq & \frac{1}{\omega(B)^{1+\lambda p}} \left(\int_B |b(x) - b_{B,\omega}|^{p_1} \omega(x) \right)^{\frac{p}{p_1}} \left(\int_B |T_\theta(f_1)(x)|^{p_2} \omega(x) \mathrm{d}x \right)^{1-\frac{p}{p_1}} \\ \leq & C \frac{1}{\omega(B)^{1+\lambda p}} \left(\frac{1}{\omega(B)^{1+\lambda_1 p_1}} \int_B |b(x) - b_{B,\omega}|^{p_1} \omega(x) \right)^{\frac{p}{p_1}} \omega(B)^{\frac{p}{p_1}+\lambda_1 p} \left(\int_{2B} |f_1(x)|^{p_2} \omega(x) \mathrm{d}x \right)^{\frac{p}{p_2}} \\ \leq & C\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}} \|f\|^p_{\dot{B}^{p_2,\lambda_2}_\omega} \left(\frac{\omega(2B)^{\frac{1}{p_2}+\lambda_2}}{\omega(B)^{\frac{1}{p_2}+\lambda_2}} \right)^p.\end{aligned}$$

If $\frac{1}{p_2} + \lambda_2 \geq 0$, we can use Lemma 2.1 to get $\frac{\omega(2B)^{\frac{1}{p_2}+\lambda_2}}{\omega(B)^{\frac{1}{p_2}+\lambda_2}} \leq C$. Moreover, we can also use Lemma 2.4 to get the same estimate for the case of $\frac{1}{p_2} + \lambda_2 < 0$. Thus, we have

$$I_1 \leq C\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega} \|f\|^p_{\dot{\mathrm{B}}^{p_2,\lambda_2}_\omega} \qquad (3.3).$$

For I_2, by the Hölder's inequality, we can obtain

$$\begin{aligned}I_2 \leq & \frac{1}{\omega(B)^{1+\lambda p}} \int_{2B} |f(x)(b(x) - b_{B,\omega})|^p \omega(x) \mathrm{d}x \\ \leq & \frac{1}{\omega(B)^{1+\lambda p}} \left(\int_{2B} (|b(x) - b_{B,\omega}|^p \omega(x)^{\frac{p}{p_1}})^{\frac{p_1}{p}} \mathrm{d}x \right)^{\frac{p}{p_1}} \left((\int_{2B} |f(x)|^p \omega(x)^{1-\frac{p}{p_1}})^{\frac{p_1}{p_1-p}} \mathrm{d}x \right)^{1-\frac{p}{p_1}} \\ \leq & \frac{1}{\omega(B)^{1+\lambda p}} \left(\frac{1}{\omega(2B)^{1+\lambda_1 p_1}} \int_{2B} |b(x) - b_{B,\omega}|^{p_1} \omega(x) \mathrm{d}x \right)^{\frac{p}{p_1}} \omega(2B)^{\frac{p}{p_1}+\lambda_1 p} \\ & \times \left(\frac{1}{\omega(2B)^{1+\lambda_2 p_2}} \int_{2B} |f(x)|^{p_2} \omega(x) \right)^{\frac{p}{p_2}} \omega(2B)^{\frac{p}{p_2}+\lambda_2 p} \\ \leq & C\|f\|^p_{\dot{B}^{p_2,\lambda_2}_\omega} \|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega} \frac{\omega(2B)^{1+\lambda p}}{\omega(B)^{1+\lambda p}}.\end{aligned}$$

If $1+\lambda p>0$, we can use Lemma 2.1 to get $\frac{\omega(2B)^{1+\lambda p}}{\omega(B)^{1+\lambda p}}\le C$. Moreover, in the case of $1+\lambda p<0$, we can also get the above estimate by using Lemma 2.4 with a similar argument.

Combining the estimates of I_1 and I_2, we have

$$I\le C\|f\|^p_{\dot{B}^{p_2,\lambda_2}_\omega}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega}. \qquad (3.4).$$

Now we are going to give the estimate of II. First, we may give the following estimates.

$$\begin{aligned}|[b,T_\theta]f_2(x)|^p&\le C\left(\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|}{|x-y|^n}|f_2(y)|\right)^p\\&\le C\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}(|b(x)-b_{B,\omega}|+|b_{B,\omega}-b(y)|)\mathrm{d}y\right)^p\\&\le C\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y\right)^p|b(x)-b_{B,\omega}|^p\\&\quad+C\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y|b(y)-b_{B,\omega}|\mathrm{d}y\right)^p.\end{aligned}$$

Thus, we can decompose II as

$$\begin{aligned}II&=\frac{1}{\omega(B)^{1+\lambda p}}\int_B|[b,T_\theta]f_2(x)|^p\omega(x)\mathrm{d}x\\&\le\frac{1}{\omega(B)^{1+\lambda p}}\int_B\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y\right)^p|b(x)-b_{B,\omega}|^p\omega(x)\mathrm{d}x\\&+\frac{1}{\omega(B)^{1+\lambda p}}\int_B\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y|b(y)-b_{B,\omega}|\mathrm{d}y\right)^p\omega(x)\mathrm{d}x\\&=II_1+II_2.\end{aligned}$$

For II_1, by the same estimate as in the proof of Theorem 1.1, we can obtain that

$$\int_{|y|>2r}1/|y|^n|f(y)|\mathrm{d}y\le C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2},$$

which implies

$$\begin{aligned}II_1&\le\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_{B,\omega}|^p\omega(x)\mathrm{d}x\\&\le\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{1+\lambda p}}\left(\int_B|b(x)-b_{B,\omega}|^{p_1}\omega(x)\mathrm{d}x\right)^{p/p_1}\times\left(\int_B\omega(x)^{\frac{p_1-p}{p}}\frac{p_1}{p_1-p}\mathrm{d}x\right)^{1-p/p_1}\\&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega}\omega(B)^{p/p_1+\lambda_1p+1-p/p_1}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{1+\lambda p}}\\&=C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{\lambda_2 p}}\\&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega},\end{aligned}$$

where in the last inequality we use the fact $\lambda_2<0$ and Lemma 2.4.

Next, we turn to estimate II_2. Noticing that $1/p = 1/p_1 + 1/p_2$, by using the Hölder's inequality, then we have

$$\begin{aligned}
\int_{|y|>2r} \frac{|f(y)|}{|y|^n}|b(y)-b_{B,\omega}|\mathrm{d}y &= \sum_{j=1}^{\infty}\int_{2^j r<|y|<2^{j+1}r} \frac{|f(y)|}{|y|^n}|b(y)-b_{B,\omega}|\mathrm{d}y \\
&\le \sum_{j=1}^{\infty}\frac{1}{|2^jB|}\int_{2^{j+1}B\setminus 2^jB}|f(y)||b(y)-b_{B,\omega}|\mathrm{d}y \\
&\le \sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left(\int_{2^{j+1}B}|f(y)||b(y)-b_{B,\omega}|\omega(y)^{\frac{1}{p}}\omega(y)^{\frac{1}{-p}}\right)\mathrm{d}y \\
&\le \sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left(\int_{2^{j+1}B}|f(y)|^{p_2}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_2}}\left(\int_{2^{j+1}B}|b(y)-b_{B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&\quad\times\left(\int_{2^{j+1}B}\omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{1}{p'}}.
\end{aligned}$$

By the fact that $\omega(x)\in A_p$, we get

$$\begin{aligned}
\left(\int_{2^{j+1}B}\omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{1}{p'}} &= \left[\left(\int_{2^{j+1}B}\omega(y)^{1-p'}\mathrm{d}y\right)^{p-1}\right]^{\frac{1}{p}} \\
&\le C\left(\frac{|2^{j+1}B|}{\omega(2^{j+1}B)}\right)^{\frac{1}{p}}|2^{j+1}B|^{\frac{p-1}{p}} \\
&= C\frac{|2^{j+1}B|}{\omega(2^{j+1}B)^{1/p_1+1/p_2}}.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
\int_{|y|>2r}\frac{|f(y)|}{|y|^n}|b(y)-b_{B,\omega}|\mathrm{d}y &\le C\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\omega(2^{j+1}B)^{1/p_2+\lambda_2} \\
&\quad\times\left(\int_{2^{j+1}B}|b(y)-b_{B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}}\times\frac{|2^{j+1}B|}{\omega(2^{j+1}B)^{1/p_1+1/p_2}} \\
&\le C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-1/p_1}\left(\int_{2^{j+1}B}|b(y)-b_{B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&\le C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-1/p_1}\left(\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&\quad+C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-1/p_1}\left(\int_{2^{j+1}B}|b_{B,\omega}-b_{2^{j+1}B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&=: C(II_{21}+II_{22}).
\end{aligned}$$

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For II_{21}, by the definition of $C\dot{B}MO_{\omega}^{p,\lambda}(\mathbb{R}^n)$, the fact $\lambda<0$ and Lemma 2.4, we have

$$\begin{aligned}\frac{1}{\omega(B)^{1+p\lambda}}\int_B II_{21}^p\cdot\omega(x)\mathrm{d}x &\le \frac{\|f\|^p_{\dot{B}_{\omega}^{\lambda_2,q_2}}\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}}{\omega(B)^{1+\lambda p}}\int_B\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda p}\mathrm{d}x\\ &\le C\|f\|^p_{\dot{B}_{\omega}^{\lambda_2,q_2}}\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda p}}{\omega(B)^{\lambda p}}\\ &\le C\|f\|^p_{\dot{B}_{\omega}^{\lambda_2,q_2}}\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}\end{aligned}$$

Next, we will give the estimate of II_2. First, we have the following inequality

$$|b_{B,\omega}-b_{2^{j+1}B,\omega}|\le\sum_{k=0}^{j}|b_{2^{k+1}B,\omega}-b_{2^kB,\omega}|.$$

Then for any $0<k\le j$, we obtain

$$\begin{aligned}|b_{2^{k+1}B,\omega}-b_{2^kB,\omega}| &\le \frac{1}{\omega(2^kB)}\int_{2^kB}|b_{2^{k+1}B,\omega}-b(y)|\omega(y)\mathrm{d}y\\ &\le \frac{1}{\omega(2^kB)}\left(\int_{2^kB}|b_{2^{k+1}B,\omega}-b(y)|^{p_1}\omega(y)\mathrm{d}y\right)^{1/p_1}\left(\int_{2^kB}(\omega(y)^{1-\frac{1}{p_1}})^{\frac{p_1}{p_1-1}}\mathrm{d}y\right)^{1-1/p_1}\\ &\le C\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}\omega(2^kB)^{\lambda_1}.\end{aligned}$$

Using the Lemma 2.4 and the fact $\lambda_1<0$, we get

$$\sum_{k=0}^{j}|b_{2^{k+1}B,\omega}-b_{2^kB,\omega}|\le C\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}\omega(B)^{\lambda_1}D_1^{(j+1)\lambda_1},$$

where D_1 is a positive constant and belongs to the interval $(1,2)$.

Thus, using Lemma 2.4 again, we obtain

$$\begin{aligned}\frac{1}{\omega(B)^{1+p\lambda}}\int II_{22}^p\cdot\omega(x)\mathrm{d}x &\le C\|f\|^p_{\dot{B}_{\omega}^{\lambda_2,q_2}}\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}\frac{\left(\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-\frac{1}{p_1}+\frac{1}{p_1}}D_1^{(j+1)\lambda}\omega(B)^{\lambda_1}\right)^p}{\omega(B)^{p\lambda}}\\ &\le C\|f\|^p_{\dot{B}_{\omega}^{\lambda_2,q_2}}\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}\sum_{j=1}^{\infty}D_1^{(j+1)\lambda_2p}D_1^{(j+1)\lambda_1p}\\ &\le C\|f\|^p_{\dot{B}_{\omega}^{\lambda_2,q_2}}\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}\sum_{j=1}^{\infty}D_1^{(j+1)\lambda p}\\ &\le C\|f\|^p_{\dot{B}_{\omega}^{\lambda_2,q_2}}\|b\|^p_{C\dot{B}MO_{\omega}^{p_1,\lambda_1}}.\end{aligned}$$

Combining the estimates of I, II, II_1, II_2, II_{21} and II_{22}, we finishe the proof of **Theorem 1.2**. □

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Stability of latent CHIKV infection model with CHIKV-to-monocyte and infected-to-monocyte transmissions

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Abstract

We investigate the global stability of within-host Chikungunya virus (CHIKV) infection model with CHIKV-to-monocyte and infected-to-monocyte transmissions. We take into account the antibody immune response. The model incorporates both latently infected monocytes which do not generate the CHIKV, and actively infected monocytes. The global stability analysis of the equilibria are established using Lyapunov method. The theoretical results are confirmed by numerical simulations. The effect of latently infection has been discussed.

1 Introduction

Chikungunya virus (CHIKV) is an alphavirus causes chikungunya fever. CHIKV is a mosquito-transmitted and is transmitted by the Aedes albopictus and Aedes agypti mosquito. Mathematical models have been constructed to describe the CHIKV transmission in mosquito and human populations [1]-[7]). Modeling and analysis of within-host CHIKV dynamics have first studied in [8]. The model presented in [8] has negelected the latently infected monocytes. Therefore, Elaiw et al. [9] have modified the model by considering five compartments, uninfected monocytes (s), latently infected monocytes (w), actively infected monocytes (y), CHIKV pathogen (p) and antibodies (x). The model is given as:

$$\dot{s} = \beta - \delta s - \frac{\eta sp}{1+\theta p}, \tag{1}$$

$$\dot{w} = (1-n)\frac{\eta sp}{1+\theta p} - (b+d)w, \tag{2}$$

$$\dot{y} = n\frac{\eta sp}{1+\theta p} + dw - \epsilon y, \tag{3}$$

$$\dot{p} = \pi y - cp - rxp, \tag{4}$$

$$\dot{x} = \lambda + \rho xp - mx, \tag{5}$$

where, β and δ represent the generation and death rate constants of the uninfected monocytes, respectively. The uninfected monocytes become infected at rate ηsp, where η is the infection rate constant. θ is the saturation constant. Constants b, ϵ, c and m represent, respectively, the death rate of the latently infected monocytes, actively infected monocytes, CHIKV and antibodies. We assume that a fraction $(1-n)$ of the CHIKV-contacted monocytes becomes latently infected monocytes and the remaining n becomes actively infected monocytes, where $0 < n < 1$. The latently infected monocytes are transmitted to actively infected monocytes at rate bw. Constant π is the generation rate of the CHIKV from actively infected monocytes. Antibodies attack the CHIKV at rate rxp. Once antigen is encountered, the antibodies expand at a constant rate λ and proliferate at rate ρxp. Latently infected cells have been considered in viral infection models in several papers (see e.g. [11]-[15]).

Model (1)-(5) assumes that the uninfected monocyte becomes infected by contacting with CHIKV(CHIKV-to-monocyte transmission). Kristin and Mork [16] reported that the CHIKV can also spread by infected-to-monocyte transmission. Cellular and viral infections have been considered in several viral infection models [17]-[20]. However, the dynamics of CHIKV with CHIKV-to-monocyte and infected-to-monocyte transmissions did not studied before.

The aim of the present paper is to construct and analyze a CHIKV dynamics model with both CHIKV-to-monocyte and infected-to-monocyte transmissions. The model incorporates two types of infected monocytes, latently infected monocytes which do not generate the CHIKV, and actively infected monocytes. We use Lyapunov method to prove the global stability of the proposed model .

2 Presentation of the model and mathematical problem

We propose a CHIKV model as:

$$\dot{s} = \beta - \delta s - \eta_1 sp - \eta_2 sy, \tag{6}$$

$$\dot{w} = (1-n)(\eta_1 sp + \eta_2 sy) - (b+d)w, \tag{7}$$

$$\dot{y} = n(\eta_1 sp + \eta_2 sy) + dw - \epsilon y, \tag{8}$$

$$\dot{p} = \pi y - cp - rxp, \tag{9}$$

$$\dot{x} = \lambda + \rho xp - mx, \tag{10}$$

Here, the uninfected monocytes become infected at rate $(\eta_1 y + \eta_2 p)s$, where η_1 and η_2 are constants.

2.1 Basic properties

Lemma 1 *There exist $M_1, M_2, M_3 > 0$, such that the following compact set is positively invariant for system (6)-(10)*

$$\Gamma_2 = \{(s,w,y,p,x) \in \mathbb{R}^5_{\geq 0} : 0 \leq s,w,y \leq M_1^L, 0 \leq p \leq M_2^L, 0 \leq x \leq M_3^L\}$$

Proof. We have

$$\begin{aligned}
&\dot{s}\mid_{s=0} = \beta > 0,\\
&\dot{w}\mid_{w=0} = (1-n)(\eta_1 sp + \eta_2 sy) \geq 0, \quad \text{for all} \quad s,p \geq 0,\\
&\dot{y}\mid_{y=0} = n(\eta_1 sp) + dw \geq 0, \quad \text{for all} \quad s,p,w \geq 0,\\
&\dot{p}\mid_{p=0} = \pi y \geq 0, \quad \text{for all} \quad y \geq 0,\\
&\dot{x}\mid_{x=0} = \lambda > 0.
\end{aligned}$$

Then, $\mathbb{R}^5_{\geq 0}$ is positively invariant for system (6)-(10). We let

$$\begin{aligned}
H_1(t) &= s(t) + w(t) + y(t),\\
H_2(t) &= p(t) + \frac{r}{\rho} x(t),
\end{aligned}$$

then

$$\begin{aligned}
\dot{H}_1(t) &= \beta - \delta s(t) - bw(t) - \epsilon y(t)\\
&\leq \beta - \sigma_1^L (s(t) + w(t) + y(t))\\
&= \beta - \sigma_1^L H_1(t),
\end{aligned}$$

where, $\sigma_1^L = min\{\delta, b, \epsilon\}$. Hence $H_1(t) \leq M_1^L$, if $H_1(0) \leq M_1^L$, where $M_1^L = \frac{\beta}{\sigma_1^L}$. Hence , $0 \leq s(t), w(t), y(t) \leq M_1^L$ if $0 \leq s(0) + w(0) + y(0) \leq M_1^L$. Moreover, we have

$$\begin{aligned}
\dot{H}_2(t) &= \pi y(t) - cp(t) + \frac{r}{\rho}\lambda - \frac{mr}{\rho}x(t) \\
&\leq \pi M_1^L + \frac{r}{\rho}\lambda - \sigma_2\left(p(t) + \frac{r}{\rho}x(t)\right) \\
&= \pi M_1^L + \frac{r}{\rho}\lambda - \sigma_2 H_2(t),
\end{aligned}$$

where, σ_2 is defined before. Hence $H_2(t) \leq M_2^L$, if $H_2(0) \leq M_2^L$, where $M_2^L = \frac{\pi M_1^L + \frac{r}{\rho}\lambda}{\sigma_2}$. Thus, $0 \leq p(t) \leq M_2^L$ and $x(t) \leq M_3^L$ if $0 \leq p(0) + \frac{r}{\rho}x(0) \leq M_2^L$, where $M_3^L = \frac{\rho M_2^L}{r}$. ■

2.2 Equilibria

We define the basic reproduction number as:

$$\mathcal{R}_0^L = \frac{\beta(d + bn)(\eta_1\pi m + \eta_2 cm + \eta_2 r\lambda)}{\epsilon\delta(cm + r\lambda)(b + d)}.$$

Lemma 2 *(i) if $\mathcal{R}_0^L \leq 1$, then there exists only one equilibrium E_0, (ii) if $\mathcal{R}_0^L > 1$, then there exist two equilibria E_0 and E_1.*

Proof. The equilibria of system (6)-(10) satisfying

$$\beta - \delta s - \eta_1 sp - \eta_2 sy = 0 \tag{11}$$

$$(1 - n)(\eta_1 sp + \eta_2 sy) - (b + d)w = 0, \tag{12}$$

$$n(\eta_1 sp + \eta_2 sy) + dw - \epsilon y = 0, \tag{13}$$

$$\pi y - cp - rxp = 0, \tag{14}$$

$$\lambda + \rho xp - mx = 0. \tag{15}$$

Solving Eqs. (11)-(15) there exists a CHIKV-free equilibrium $E_0 = (s_0, 0, 0, 0, x_0)$, where $s_0 = \frac{\beta}{\delta}$ and $x_0 = \frac{\lambda}{m}$. From Eqs. (11)-(15) we have

$$s = \frac{\pi\beta}{\pi(\delta + p\eta_1) + p(c + rx)\eta_2}, \tag{16}$$

$$w = \frac{(1 - n)p\beta(\pi\eta_1 + (c + rx)\eta_2)}{(b + d)(\pi(\delta + p\eta_1) + p(c + rx)\eta_2)}, \tag{17}$$

$$y = \frac{p(c + rx)}{\pi}, \tag{18}$$

$$x = \frac{\lambda}{m - \rho p}. \tag{19}$$

Substituting from Eqs. (16)-(19) into (13) we get

$$D_1 p^3 + D_2 p^2 + D_3 p + D_4 = 0,$$

where

$$\begin{aligned}
D_1 &= -c(b+d)\epsilon(\pi\eta_1 + c\eta_2)\rho^2,\\
D_2 &= 2bcm\pi\epsilon\eta_1\rho + 2cdm\pi\epsilon\eta_1\rho + 2bc^2m\epsilon\eta_2\rho + 2c^2dm\epsilon\eta_2\rho + b\pi r\epsilon\eta_1\lambda\rho + d\pi r\epsilon\eta_1\lambda\rho + 2bcr\epsilon\eta_2\lambda\rho\\
&\quad + 2cdr\epsilon\eta_2\lambda\rho - bc\pi\delta\epsilon\rho^2 - cd\pi\delta\epsilon\rho^2 + d\pi^2\beta\eta_1\rho^2 + bn\pi^2\beta\eta_1\rho^2 + cd\pi\beta\eta_2\rho^2 + bcn\pi\beta\eta_2\rho^2,\\
D_3 &= -bcm^2\pi\epsilon\eta_1 - cdm^2\pi\epsilon\eta_1 - bc^2m^2\epsilon\eta_2 - c^2dm^2\epsilon\eta_2 - bm\pi r\epsilon\eta_1\lambda - dm\pi r\epsilon\eta_1\lambda - 2bcmr\epsilon\eta_2\lambda\\
&\quad - 2cdmr\epsilon\eta_2\lambda - br^2\epsilon\eta_2\lambda^2 - dr^2\epsilon\eta_2\lambda^2 + 2bcm\pi\delta\epsilon\rho + 2cdm\pi\delta\epsilon\rho - 2dm\pi^2\beta\eta_1\rho - 2bmn\pi^2\beta\eta_1\rho\\
&\quad - 2cdm\pi\beta\eta_2\rho - 2bcmn\pi\beta\eta_2\rho + b\pi r\delta\epsilon\lambda\rho + d\pi r\delta\epsilon\lambda\rho - d\pi r\beta\eta_2\lambda\rho - bn\pi r\beta\eta_2\lambda\rho,\\
D_4 &= -bcm^2\pi\delta\epsilon - cdm^2\pi\delta\epsilon + dm^2\pi^2\beta\eta_1 + bm^2n\pi^2\beta\eta_1 + cdm^2\pi\beta\eta_2 + bcm^2n\pi\beta\eta_2 - bm\pi r\delta\epsilon\lambda\\
&\quad - dm\pi r\delta\epsilon\lambda + dm\pi r\beta\eta_2\lambda + bmn\pi r\beta\eta_2\lambda.
\end{aligned}$$

Let

$$X_2(p) = D_1p^3 + D_2p^2 + D_3p + D_4 = 0.$$

Then

$$\begin{aligned}
X_2(0) &= D_4,\\
X_2\left(\frac{m}{\rho}\right) &= -\frac{(b+d)mr^2\epsilon\eta_2\lambda^2}{\rho} < 0.
\end{aligned}$$

D_4 can be written as:

$$D_4 = m\pi(b\delta\epsilon + d\delta\epsilon)(cm + r\lambda)\left(\mathcal{R}_0^L - 1\right).$$

Then $D_4 > 0$ if $\mathcal{R}_0^L > 1$. Then there exists $p_1 \in (0, \frac{m}{\rho})$ such that $X_2(p_1) = 0$.If $\mathcal{R}_0^L > 1$, then system (6)-(10) has infected equilibrium $E_1 = (s_1, y_1, p_1, x_1)$, where

$$\begin{aligned}
s_1 &= \frac{\pi\beta}{\pi(\delta + p_1\eta_1) + p_1(c + rx_1)\eta_2} > 0, & w_1 &= \frac{(1-n)p_1\beta(\pi\eta_1 + (c + rx_1)\eta_2)}{(b+d)(\pi(\delta + p_1\eta_1) + p_1(c + rx_1)\eta_2)} > 0,\\
y_1 &= \frac{p_1(c(m - \rho p_1) + r\lambda)}{\pi(m - \rho p_1)} > 0, & x_1 &= \frac{\lambda}{m - \rho p_1} > 0.
\end{aligned}$$

■

3 Global properties

Define a function $G(z) = z - 1 - \ln z$.

Theorem 1 *If $\mathcal{R}_0^L \le 1$, then E_0 is globally asymptotically stable in Γ_2.*

Proof. Let

$$V_0 = s_0 G\left(\frac{s}{s_0}\right) + \frac{d}{bn + d}w + \frac{b+d}{bn+d}y + \frac{\eta_1 s_0}{c + rx_0}p + \frac{r\eta_1 s_0}{\rho(c + rx_0)}x_0 G\left(\frac{x}{x_0}\right).$$

Calculating $\frac{dV_0}{dt}$ along system (6)-(10) we obtain

$$\begin{aligned}
\frac{dV_0}{dt} &= \left(1-\frac{s_0}{s}\right)\left(\beta-\delta s-\eta_1 sp-\eta_2 sy\right)+\frac{d}{bn+d}\Big[(1-n)(\eta_1 sp+\eta_2 sy)-(b+d)w\Big]\\
&+\frac{b+d}{bn+d}\Big[n(\eta_1 sp+\eta_2 sy)+dw-\epsilon y\Big]+\frac{\eta_1 s_0}{c+rx_0}\left(\pi y-cp-rxp\right)\\
&+\frac{r\eta_1 s_0}{\rho(c+rx_0)}\left(1-\frac{x_0}{x}\right)\left(\lambda+\rho xp-mx\right)\\
&=\left(1-\frac{s_0}{s}\right)\left(\beta-\delta s\right)+\eta_2 s_0 y-\frac{b+d}{bn+d}\epsilon y+\frac{\eta_1 s_0}{c+rx_0}\pi y+\frac{r\eta_1 s_0}{\rho(c+rx_0)}\left(1-\frac{x_0}{x}\right)\left(\lambda-mx\right)
\end{aligned}$$

$$\begin{aligned}
\frac{dV_0}{dt} &= -\delta\frac{(s-s_0)^2}{s}+\frac{\epsilon(b+d)}{bn+d}\left(\frac{\eta_2 s_0(bn+d)}{\epsilon(b+d)}+\frac{\eta_1 s_0\pi(bn+d)}{\epsilon(b+d)(c+rx_0)}-1\right)y-\frac{r\eta_1 s_0 m}{\rho(c+rx_0)}\frac{(x-x_0)^2}{x}\\
&=-\delta\frac{(s-s_0)^2}{s}-\frac{r\eta_1 s_0 m}{\rho(c+rx_0)}\frac{(x-x_0)^2}{x}+\frac{\epsilon(b+d)}{bn+d}(\mathcal{R}_0-1)y. \qquad (20)
\end{aligned}$$

Since $\mathcal{R}_0 \le 1$, then $\frac{dV_0}{dt} \le 0$ for all $s, w, y, p, x > 0$. Let $D_0 = \{(s, w, y, p, x) : \frac{dV_0}{dt} = 0\}$. One can show that $D_0 = \{E_0\}$. LaSalle's invariance principle implies that E_0 is globally asymptotically stable when $\mathcal{R}_0 \le 1$. ■

Theorem 2 *If $\mathcal{R}_0 > 1$, then E_1 is globally asymptotically stable in $\overset{\circ}{\Gamma}_2$.*

Proof. Let

$$\begin{aligned}
V_1(s,w,y,p,x) &= s_1 G\left(\frac{s}{s_1}\right)+\frac{d}{bn+d}w_1 G\left(\frac{w}{w_1}\right)+\frac{b+d}{bn+\alpha}y_1 G\left(\frac{y}{y_1}\right)\\
&+\frac{\eta_1 s_1 p_1}{\pi y_1}p_1 G\left(\frac{p}{p_1}\right)+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}x_1 G\left(\frac{x}{x_1}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dV_1}{dt} &= \left(1-\frac{s_1}{s}\right)\left(\beta-\delta s-\eta_1 sp-\eta_2 sy\right)+\frac{d}{bn+d}\left(1-\frac{w_1}{w}\right)\left((1-n)(\eta_1 sp+\eta_2 sy)-(b+d)w\right)\\
&+\frac{b+d}{bn+d}\left(1-\frac{y_1}{y}\right)\left(n(\eta_1 sp+\eta_2 sy)+dw-\epsilon y\right)+\frac{\eta_1 s_1 p_1}{\pi y_1}\left(1-\frac{p_1}{p}\right)\left(\pi y-cp-rxp\right)\\
&+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda+\rho xp-mx\right)\\
&=\left(1-\frac{s_1}{s}\right)\left(\beta-\delta s\right)+\eta_1 s_1 p+\eta_2 s_1 y-\frac{\eta_1 d(1-n)}{bn+d}\frac{spw_1}{w}-\frac{\eta_2 d(1-n)}{bn+d}\frac{syw_1}{w}+\frac{d(b+d)}{bn+d}w_1\\
&-\frac{n\eta_1(b+d)}{bn+d}\frac{spy_1}{y}-\frac{n\eta_2(b+d)}{bn+d}\frac{syy_1}{y}-\frac{d(b+d)}{bn+d}\frac{wy_1}{y}-\frac{b+d}{bn+d}\epsilon y+\frac{b+d}{bn+d}\epsilon y_1+\eta_1 s_1 p_1\frac{y}{y_1}-\eta_1 s_1 p_1\frac{yp_1}{y_1 p}\\
&-\frac{\eta_1 s_1 p_1}{\pi y_1}cp+\frac{\eta_1 s_1 p_1}{\pi y_1}cp_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rxp_1-\frac{r\eta_1 s_1 p_1}{\pi y_1}x_1 p+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda-mx\right).
\end{aligned}$$

Applying the conditions for E_1

$$\begin{aligned}
&\beta=\delta s_1+\eta_1 s_1 p_1+\eta_2 s_1 y_1, \quad (b+d)w_1=(1-n)(\eta_1 s_1 p_1+\eta_2 s_1 y_1),\\
&\frac{b+d}{bn+d}\epsilon y_1=\eta_1 s_1 p_1+\eta_2 s_1 y_1, \quad cp_1=\pi y_1-rx_1 p_1, \quad \lambda=mx_1-\rho x_1 p_1
\end{aligned}$$

we get

$$\begin{aligned}
\frac{dV_1}{dt} &= -\delta\frac{(s-s_1)^2}{s} + \frac{d(1-n)}{bn+d}\left(1-\frac{s_1}{s}\right)\left(\eta_1 s_1 p_1 + \eta_2 s_1 y_1\right) \\
&+ \frac{n(b+d)}{bn+d}\left(1-\frac{s_1}{s}\right)\left(\eta_1 s_1 p_1 + \eta_2 s_1 y_1\right) + 3\frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \\
&- \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \frac{spw_1}{s_1 p_1 w} - \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \frac{wy_1}{w_1 y} - \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \frac{yp_1}{y_1 p} \\
&+ 2\frac{d(1-n)}{bn+d}\eta_2 s_1 y_1 - \frac{d(1-n)}{bn+d}\eta_2 s_1 y_1 \frac{syw_1}{s_1 y_1 w} - \frac{d(1-n)}{bn+d}\eta_2 s_1 y_1 \frac{wy_1}{w_1 y} \\
&+ 2\frac{n(b+d)}{bn+d}\eta_1 s_1 p_1 - \frac{n(b+d)}{bn+d}\eta_1 s_1 p_1 \frac{spy_1}{s_1 p_1 y} - \frac{n(b+d)}{bn+d}\eta_1 s_1 p_1 \frac{yp_1}{y_1 p} \\
&+ \frac{n(b+d)}{bn+d}\eta_2 s_1 y_1 - \frac{n(b+d)}{bn+d}\eta_2 s_1 y_1 \frac{s}{s_1} - 2\frac{\eta_1 s_1 p_1}{\pi y_1} r x_1 p_1 \\
&+ \frac{\eta_1 s_1 p_1}{\pi y_1} r x p_1 + \frac{\eta_1 s_1 p_1}{\pi y_1} r x_1 p_1 \frac{x_1}{x} - \frac{m r \eta_1 s_1 p_1}{\rho \pi y_1}\frac{(x-x_1)^2}{x}.
\end{aligned} \tag{21}$$

Eq. (21) can be simplified as:

$$\begin{aligned}
\frac{dV_1}{dt} &= -\delta\frac{(s-s_1)^2}{s} - \frac{n(b+d)}{bn+d}\eta_2 y_1 \frac{(s-s_1)^2}{s} - \frac{\eta_1 s_1 p_1 r\lambda}{\pi y_1 \rho x_1}\frac{(x-x_1)^2}{x} \\
&+ \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1\left[4 - \frac{s_1}{s} - \frac{spw_1}{s_1 p_1 w} - \frac{wy_1}{w_1 y} - \frac{yp_1}{y_1 p}\right] \\
&+ \frac{n(b+d)}{bn+d}\eta_1 s_1 p_1\left[3 - \frac{s_1}{s} - \frac{spy_1}{s_1 p_1 y} - \frac{yp_1}{y_1 p}\right] \\
&+ \frac{d(1-n)}{bn+d}\eta_2 s_1 y_1\left[3 - \frac{s_1}{s} - \frac{syw_1}{s_1 y_1 w} - \frac{y_1 w}{y w_1}\right].
\end{aligned} \tag{22}$$

Using the arithmetic mean-geometric mean inequality we find that the last three terms of Eq. (22) are less that or equal zero. Thus, $\frac{dV_1}{dt} \leq 0$ for all $s, w, y, p, x > 0$ and $\frac{dU_1}{dt} = 0$ at E_1. The global stability of E_1 is induced from LaSalle's invariance principle. ■

4 Numerical Simulations

We perform the numerical simulation of model (6)-(10) using Matlab.

4.1 Effect of the parameters η_1 and η_2

We simulate the system with three different initial values as:

IV1: $s(0) = 18.0, w(0) = 0.2, y(0) = 0.2, p(0) = 1.0$, and $x(0) = 1$,
IV2: $s(0) = 16.0, w(0) = 0.6, y(0) = 1.0, p(0) = 2.0$, and $x(0) = 2.5$,
IV3: $s(0) = 12.0, w(0) = 1.0, y(0) = 1.5, p(0) = 2.5$, and $x(0) = 3.0$.

We fix the value of $n = 0.7$ and the other parameters are given in Table 1. Then we consider two sets of the values of η_1 and η_2 as follows:

Table 1: The parameters's values.

Parameter	Value	Parameter	Value
β	2	δ	0.1
η_1, η_2	varied	ϵ	0.5
π	4	c	0.1
r	0.4	λ	1.4
m	1	ρ	0.2
n	varied	d	0.1
b	0.3		

Set (I): We choose $\eta_1 = \eta_2 = 0.001$. We compute $\mathcal{R}_0 = 0.2189 < 1$. From Figure 1 we can see that, the concentrations of the uninfected monocytes and B cells return to their values $s_0 = \frac{\beta}{\delta} = 20$ and $x_0 = \frac{\lambda}{m} = 1.4$, respectively. On the other hand, the concentrations of latently infected monocytes, actively infected monocytes and CHIKV particles are declining and reaching zero for all the three initial values IV1-IV3. This demonstrates that, there exists one equilibrium E_0 which is globally asymptotically stable. This result agrees the result of Theorem 1.

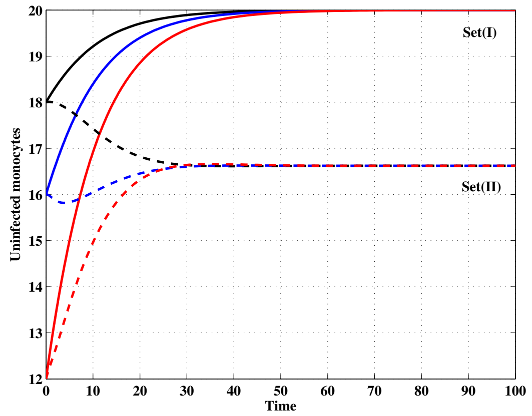
Set (II): We take $\eta_1 = \eta_2 = 0.008$. Then, we calculate $\mathcal{R}_0 = 1.7510 > 1$, $E_0(20.0, 0, 0, 0, 1.4)$ and $E_1 = (16.62, 0.253, 0.523, 2.016, 2.346)$. From Figure 1 we see that when $\mathcal{R}_0 > 1$, the solutions of the system starting at IV1-IV3 will tend to E_1. This agrees the results of Theorem 2.

4.2 Effect of the parameter n

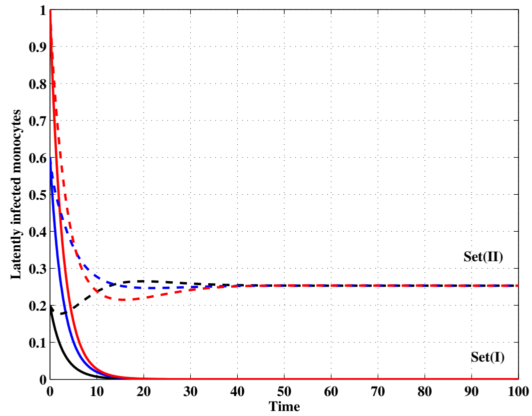
In this case, we use the values of the parameters given in Table 1 and we choose $\eta_1 = \eta_2 = 0.008$ and n is selected. We consider

IV4: $s(0) = 17, w(0) = 0.1, y(0) = 0.4, p(0) = 1.0$, and $x(0) = 2.0$.

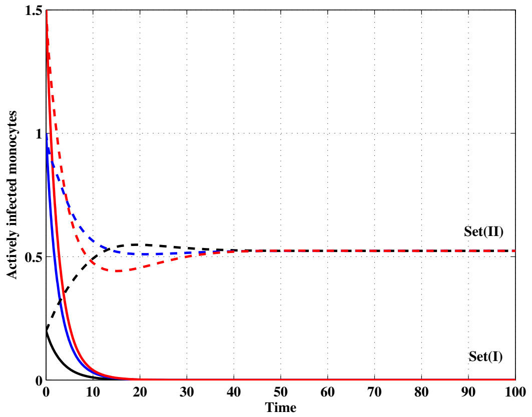
In Table 2, we calclaute the value $\mathcal{R}_0$ and the equilibria for different values of n. From the table we observe the value $\mathcal{R}_0$ is increased as n increased which means the solution of system will converge to E_0 if the values of n are small and they will converge to E_1 if values of n are large. Figure 2 supports the results of Theorem 2



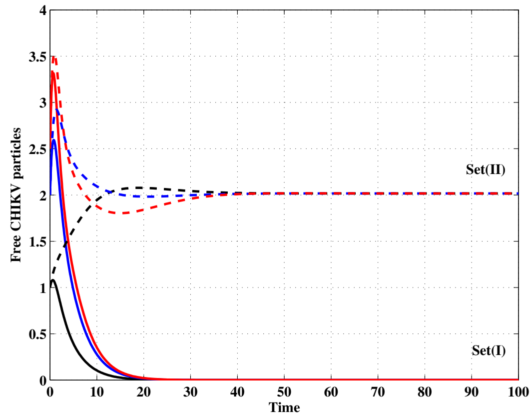
(a) Uninfected monocytes.



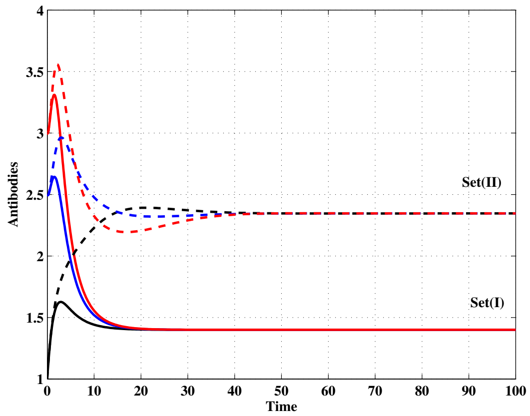
(b) Latently infected monocytes.



(c) Actively infected monocytes.

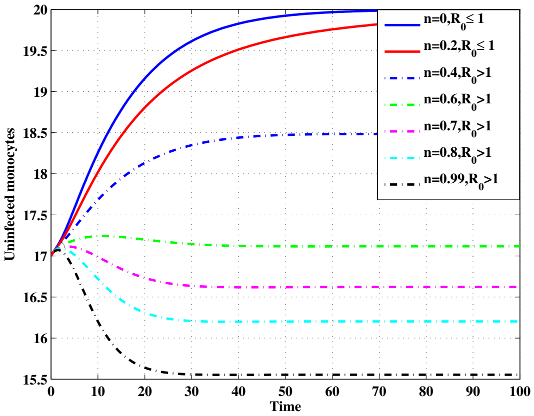


(d) Free CHIKV particles.

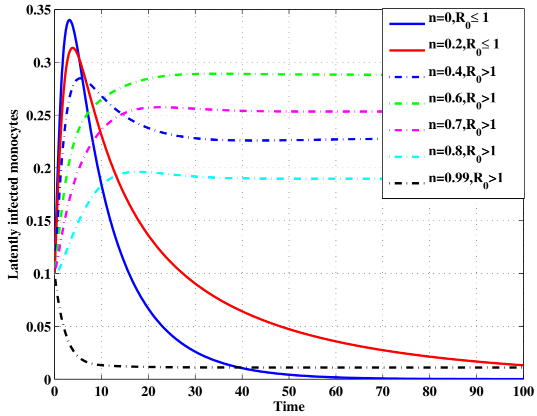


(e) Antibodies.

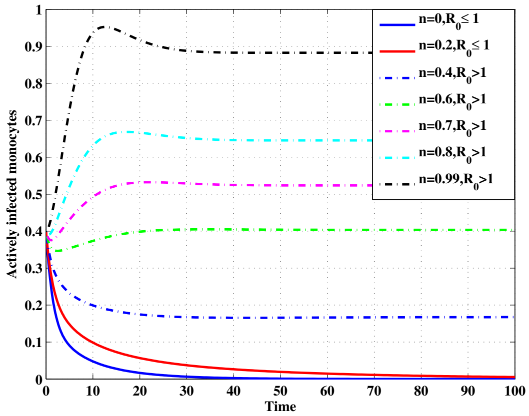
Figure 1: Numerical solutions of system (6)-(10) with selected values of η_1 and η_2.



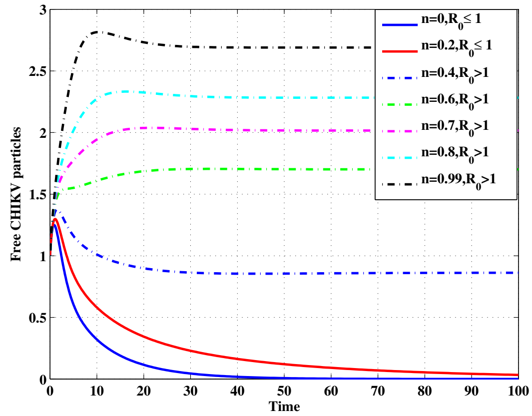
(a) Uninfected monocytes.



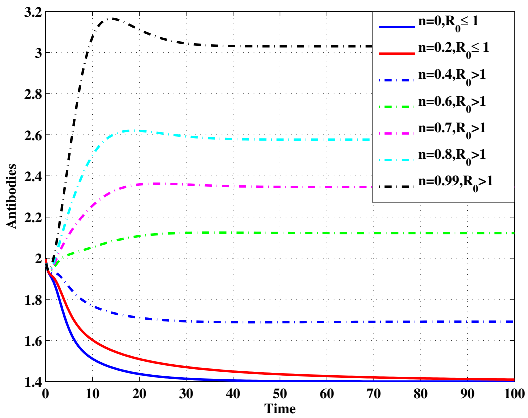
(b) Latently infected monocytes.



(c) Actively infected monocytes.



(d) Free CHIKV particles.



(e) Antibodies.

Figure 2: Numerical solutions of system (6)-(10) with selected values of n.

Table 2: The values of equilibria and $\mathcal{R}_0$ for system (6)-(10) with different values of n.

n	Steady states	$\mathcal{R}_0$
0.000001	$E_0 = (20, 0, 0, 0, 1.4)$	0.5648
0.2	$E_0 = (20, 0, 0, 0, 1.4)$	0.9038
0.256795	$E_0 = (20, 0, 0, 0, 1.4)$	1
0.4	$E_1 = (18.5, 0.2283, 0.1674, 0.8621, 1.6917)$	1.2427
0.6	$E_1 = (17.1178, 0.2882, 0.4035, 1.7011, 2.122)$	1.5816
0.7	$E_1 = (16.6222, 0.2533, 0.5236, 2.0166, 2.3463)$	1.7510
0.8	$E_1 = (16.2037, 0.1898, 0.6454, 2.2832, 2.5766)$	1.9205
0.99	$E_1 = (15.5546, 0.0111, 0.8824, 2.69, 3.0303)$	2.2424

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OPTIMAL BOUNDS FOR TOADER MEAN IN TERMS OF GEOMETRIC AND CONTRAHARMONIC MEANS*

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ABSTRACT. In this paper, we present the best possible parameters α_1, α_2, α_3 and β_1, β_2, β_3 such that the double inequalities

$$C^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b) < T(a,b) < C^{\beta_1}(a,b)G^{1-\beta_1}(a,b),$$

$$\alpha_2 C(a,b) + (1-\alpha_2)G(a,b) < T(a,b) < \beta_2 C(a,b) + (1-\beta_2)G(a,b),$$

$$\frac{\alpha_3}{G(a,b)} + \frac{1-\alpha_3}{C(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_3}{G(a,b)} + \frac{1-\beta_3}{C(a,b)}$$

hold for all $a, b > 0$ with $a \neq b$, where $G(a,b) = \sqrt{ab}$, $C(a,b) = \left(a^2 + b^2\right)/(a+b)$ and $T(a,b) = 2\int_0^{\pi/2} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}dt/\pi$ are the geometric, contraharmonic and Toader means of a and b, respectively.

1. Introduction

The Toader mean $T(a,b)$ [1-5] of two positive real numbers a and b is defined by

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}dt \tag{1.1}$$

$$= \begin{cases} \frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-(b/a)^2}\right), & a > b, \\ \frac{2b}{\pi}\mathcal{E}\left(\sqrt{1-(a/b)^2}\right), & a < b, \\ a, \ a = b, \end{cases}$$

where $\mathcal{E}(r) = \int_0^{\pi/2}\left(1 - r^2\sin^2(t)\right)^{1/2}dt$ $(r \in [0,1])$ is the complete elliptic integral of the second kind [6-30]. The Toader mean $T(a,b)$ is well known in mathematical literature for many years, it satisfies

$$T(a,b) = R_E\left(a^2, b^2\right),$$

where

$$R_E(a,b) = \frac{1}{\pi}\int_0^{\infty} \frac{[a(t+b) + b(t+a)]t}{(t+a)^{3/2}(t+b)^{3/2}}dt$$

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stands for the symmetric complete elliptic integral of the second kind (See [31-33]), therefore it cannot be expressed in terms of the elementary transcendental functions.

Recently, the Toader mean $T(a,b)$ has been the subject of intensive research. In particular, many remarkable inequalities for the Toader mean can be found in the literature [34-41].

Let $G(a,b)=\sqrt{ab}$ [42-48], $A(a,b)=(a+b)/2$ [49-57], $C(a,b)=(a^2+b^2)/(a+b)$ [58-61], and $M_p(a,b)=[(a^p+b^p)/2]^{1/p}$ [62-73] and $M_0(a,b)=\sqrt{ab}$ be respectively the geometric, arithmetic, contraharmonic and pth power means of a and b. Then it is well known that power mean $M_p(a,b)$ is strictly increasing with respect to $p\in\mathbb{R}$ for all fixed $a,b>0$ with $a\neq b$, and the inequalities

$$G(a,b)=M_0(a,b)<A(a,b)=M_1(a,b)<C(a,b)=M_2(a,b) \tag{1.2}$$

hold for all $a,b>0$ with $a\neq b$.

Vuorinen [74] conjectured that

$$T(a,b)>M_{3/2}(a,b) \tag{1.3}$$

for all $a,b>0$ with $a\neq b$. This conjecture was proved by Qiu and Shen [75], and Barnard et al. [76].

Alzer and Qiu [77] proved that the inequality

$$T(a,b)<T_\lambda(a,b) \tag{1.4}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\geq\log 2/(\log\pi-\log 2)=1.5349\cdots$.

From (1.2)-(1.4) we clearly see that

$$G(a,b)<T(a,b)<C(a,b) \tag{1.5}$$

for all $a,b>0$ with $a\neq b$.

Motivated by (1.5), it is natural to ask what are the best possible parameters α_1, α_2, α_3 and β_1, β_2, β_3 such that the double inequalities

$$C^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b)<T(a,b)<C^{\beta_1}(a,b)G^{1-\beta_1}(a,b),$$

$$\alpha_2C(a,b)+(1-\alpha_2)G(a,b)<T(a,b)<\beta_2C(a,b)+(1-\beta_2)G(a,b),$$

$$\frac{\alpha_3}{G(a,b)}+\frac{1-\alpha_3}{C(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_3}{G(a,b)}+\frac{1-\beta_3}{C(a,b)}$$

hold for all $a,b>0$ with $a\neq b$? The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Let $r\in[0,1]$, $\mathcal{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2(t))^{-1/2}dt$ and $\mathcal{E}(r)=\int_0^{\pi/2}\left(1-r^2\sin^2(t)\right)^{1/2}dt$ be respectively the complete elliptic integrals of the first and second kinds. Then it is well known that $\mathcal{K}(r)$ is strictly increasing and $\mathcal{E}(r)$ is strictly decreasing on $[0,1]$,

$$\mathcal{K}(0)=\mathcal{E}(0)=\pi/2,\ \mathcal{K}(1)=\infty,\ \mathcal{E}(1)=1, \tag{2.1}$$

and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the formulas (See[17, Appendix E, pp. 474-475])

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)},\ \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r}, \tag{2.2}$$

$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{1+r}. \tag{2.3}$$

Lemma 2.1. *(See [78]) Let $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions $[f(x)-f(a)]/[g(x)-g(a)]$ and $[f(x)-f(b)]/[g(x)-g(b)]$. If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2. *(See [79]) The function $r\to(1-r^2)^\lambda\mathcal{K}(r)$ is strictly decreasing from $[0,1]$ onto $[0,\pi/2]$ if $\lambda\geq 1$.*

Lemma 2.3. *Let $f_1(r)=[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2$. Then $f_1(r)$ is strictly increasing from $(0,1]$ onto $(\pi/4,1]$.*

Proof. Let $g_1(r)=\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)$ and $g_2(r)=2r$. Then from (2.1) and (2.2) together with Lemma 2.2 we clearly see that

$$f_1(r)=\frac{g_1(r)}{g_2(r)},\ g_1(0)=g_2(0)=0,\ f_1(1)=1, \tag{2.4}$$

$$\frac{g_1'(r)}{g_2'(r)}=\frac{1}{2}\mathcal{K}(r),\ \lim_{r\to 0}f_1(r)=\lim_{r\to 0}\frac{g_1'(r)}{g_2'(r)}=\frac{\pi}{4}. \tag{2.5}$$

Therefore, Lemma 2.3 follows from (2.4) and (2.5) together with Lemma 2.1 and the monotonicity of $\mathcal{K}(r)$ on $[0,1]$. □

Lemma 2.4. *Let $f_2(r)=[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/(1+r^2)$. Then $f_2(r)$ is strictly decreasing from $[0,1]$ onto $[1,\pi/2]$.*

Proof. It follows from (2.1) and Lemma 2.2 that

$$f_2(0)=\frac{\pi}{2},\ f_2(1)=1. \tag{2.6}$$

Differentiating $f_2(r)$ gives

$$f_2'(r)=\frac{r}{(1+r^2)^2}\left[(1-r^2)f_1(r)-2\mathcal{E}(r)\right], \tag{2.7}$$

where $f_1(r)$ is given by Lemma 2.3.

From (2.7) and Lemma 2.3 together with the monotonicity of $\mathcal{E}(r)$ on $[0,1]$ we get

$$f_2'(r)<\frac{r}{(1+r^2)^2}\left[(1-r^2)-2\right]=-\frac{r}{1+r^2}<0 \tag{2.8}$$

for $r\in(0,1)$.

Therefore, Lemma 2.4 follows easily from (2.6) and (2.8). □

Lemma 2.5. *Let $p\in\mathbb{R}$, $f_1(r)$ and $f_2(r)$ be respectively defined by Lemmas 2.3 and 2.4, and*

$$f(r)=\frac{f_1(r)}{f_2(r)}+\frac{2(1-p)}{1-r^2}-(1+p). \tag{2.9}$$

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Then the following statements are true:

(1) $f(r)>0$ for all $r\in(0,1)$ if $p=1/2$;

(2) $f(r)<0$ for all $r\in(0,1)$ if $p=1$;

Proof. Let $f_3(r)=f_1(r)/f_2(r)$, then Lemmas 2.3 and 2.4 lead to

$$f_3(0^+)=\frac{1}{2},\ f_3(1^-)=1, \tag{2.10}$$

and $f_3(r)$ is strictly increasing on $(0,1)$.

For part (1), if $p=1/2$, then (2.9) becomes

$$f(r)=f_3(r)+\frac{1}{1-r^2}-\frac{3}{2}. \tag{2.11}$$

From (2.10) and (2.11) together with the monotonicity of $f_3(r)$ we clearly see that

$$f(r)>f_3(0^+)+1-\frac{3}{2}=0$$

for all $r\in(0,1)$.

For part (2), if $p=1$, then (2.9) becomes

$$f(r)=f_3(r)-2. \tag{2.12}$$

Therefore, $f(r)<1-2=-1<0$ for all $r\in(0,1)$ follows from (2.10) and (2.12) together with the monotonicity of $f_3(r)$. □

Lemma 2.6. *Let $q\in\mathbb{R}$, $f_1(r)$ be defined by Lemma 2.3, and*

$$g(r)=\frac{2}{\pi}f_1(r)+\frac{1-q}{\sqrt{1-r^2}}-2q. \tag{2.13}$$

Then the following statements are true:

(1) $g(r)>0$ for all $r\in(0,1)$ if $q=1/2$;

(2) there exists $r_0\in(0,1)$ such that $g(r)<0$ for $r\in(0,r_0)$ and $g(r)>0$ for $r\in(r_0,1)$ if $q=2/\pi$.

Proof. For part (1), if $q=1/2$, then (2.13) becomes

$$g(r)=\frac{2}{\pi}f_1(r)+\frac{1}{2\sqrt{1-r^2}}-1. \tag{2.14}$$

It follows from Lemma 2.3 and (2.14) that

$$g(r)>\frac{2}{\pi}\times\frac{\pi}{4}+\frac{1}{2}-1=0$$

for all $r\in(0,1)$.

For part (2), if $q=2/\pi$, then Lemma 2.3 and (2.13) lead to

$$g(0^+)=-\frac{3(4-\pi)}{2\pi}<0,\ g(1^-)=\infty, \tag{2.15}$$

and $g(r)$ is strictly increasing on $(0,1)$.

Therefore, part (2) follows from (2.15) and the monotonicity of $g(r)$. □

Lemma 2.7. *Let*

$$h(r)=\frac{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{\pi}-\frac{(2-r^2)(1+r^2)}{4+r^2}$$

Then $h(r)>0$ for all $r\in(0,1)$.

Proof. Simple computations lead to

$$h(0)=0, \tag{2.16}$$

$$\begin{aligned} h'(r)&=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{\pi r}+\frac{2r(r^4+8r^2-2)}{(4+r^2)^2} \\ &=\frac{r}{(4+r^2)^2}\left[\frac{(4+r^2)^2}{\pi}f_1(r)+2(r^4+8r^2-2)\right], \end{aligned} \tag{2.17}$$

where $f_1(r)$ is defined by Lemma 2.3.

It follows from Lemma 2.3 and (2.17) that

$$h'(r)>\frac{r}{(4+r^2)^2}\left[\frac{(4+r^2)^2}{\pi}\times\frac{\pi}{4}+2(r^4+8r^2-2)\right]=\frac{9r^3(8+r^2)}{4(4+r^2)^2}>0 \tag{2.18}$$

for $r\in(0,1)$.

Therefore, Lemma 2.7 follows easily from (2.16) and (2.18). □

3. Main Results

Theorem 3.1. *The double inequality*

$$C^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b)<T(a,b)<C^{\beta_1}(a,b)G^{1-\beta_1}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 1/2$ and $\beta_1\geq 1$.

Proof. Since $C(a,b)$, $T(a,b)$ and $G(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$ and $p\in\mathbb{R}$. Then from (1.1) and (2.3) we get

$$T(a,b)=\frac{2}{\pi}A(a,b)\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right], \tag{3.1}$$

$$G(a,b)=A(a,b)\sqrt{1-r^2},\ C(a,b)=A(a,b)(1+r^2). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\frac{\log[T(a,b)]-\log[G(a,b)]}{\log[C(a,b)]-\log[G(a,b)]}=\frac{\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\frac{1}{2}\log(1-r^2)}{\log(1+r^2)-\frac{1}{2}\log(1-r^2)}, \tag{3.3}$$

$$\begin{aligned} &\log[T(a,b)]-\{p\log[C(a,b)+(1-p)\log[G(a,b)]\} \\ &=\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\left\{p\log(1+r^2)+\frac{1}{2}(1-p)\log(1-r^2)\right\}. \end{aligned} \tag{3.4}$$

Let

$$F(r)=\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\left\{p\log(1+r^2)+\frac{1}{2}(1-p)\log(1-r^2)\right\}. \tag{3.5}$$

Then simple computations lead to

$$F(0)=0, \tag{3.6}$$

$$F'(r)=\frac{r}{1+r^2}f(r), \tag{3.7}$$

where $f(r)$ is defined by (2.9).

We divide the proof into two cases.

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Case 1 $p = 1/2$. Then Lemma 2.5(1) and (3.7) lead to the conclusion that $F(r)$ is strictly increasing on $(0, 1)$. Therefore,

$$T(a,b) > C^{1/2}(a,b)G^{1/2}(a,b) \tag{3.8}$$

follows from (3.4)-(3.6) and the monotonicity of $F(r)$ on $(0, 1)$.

Case 2 $p = 1$. Then Lemma 2.5(2) and (3.7) lead to the conclusion that $F(r)$ is strictly decreasing on $(0, 1)$. Therefore,

$$T(a,b) < C(a,b) \tag{3.9}$$

follows from (3.4)-(3.6) and the monotonicity of $F(r)$ on $(0, 1)$.

Note taht

$$\lim_{r\to0^+} \frac{\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right)\right] - \frac{1}{2}\log(1-r^2)}{\log(1+r^2) - \frac{1}{2}\log(1-r^2)} = \frac{1}{2}, \tag{3.10}$$

$$\lim_{r\to1^-} \frac{\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right)\right] - \frac{1}{2}\log(1-r^2)}{\log(1+r^2) - \frac{1}{2}\log(1-r^2)} = 1. \tag{3.11}$$

Therefore, Theorem 3.2 follows from (3.8) and (3.9) together with the following statements.

• If $p > 1/2$, then (3.3) and (3.10) imply that there exists $\delta_1 \in (0, 1)$ such that

$$T(a,b) < C^p(a,b)G^{1-p}(a,b)$$

for all $(a - b)/(a + b) \in (0, \delta_1)$.

•• If $p < 1$, then (3.3) and (3.11) imply that there exists $\delta_2 \in (0, 1)$ such that

$$T(a,b) > C^p(a,b)G^{1-p}(a,b)$$

for all $(a - b)/(a + b) \in (1 - \delta_2, 1)$. □

Theorem 3.2. *The double inequality*

$$\alpha_2 C(a,b) + (1-\alpha_2)G(a,b) < T(a,b) < \beta_2 C(a,b) + (1-\beta_2)G(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 1/2$ and $\beta_2 \geq 2/\pi$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a - b)/(a + b) \in (0, 1)$ and $q \in \mathbb{R}$. Then from (3.1) and (3.2) we have

$$\frac{T(a,b) - G(a,b)}{C(a,b) - G(a,b)} = \frac{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - \sqrt{1-r^2}}{(1+r^2) - \sqrt{1-r^2}}, \tag{3.12}$$

$$\begin{aligned} &T(a,b) - [qC(a,b) + (1-q)G(a,b)] \\ &= A(a,b)\left\{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - [q(1+r^2) + (1-q)\sqrt{1-r^2}]\right\}. \end{aligned} \tag{3.13}$$

Let

$$G(r) = \frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - [q(1+r^2) + (1-q)\sqrt{1-r^2}]. \tag{3.14}$$

Then simple computations lead to

$$G(0^+) = 0, \tag{3.15}$$

$$G(1^-) = \frac{4}{\pi} - 2q, \tag{3.16}$$

$$G'(r) = rg(r), \tag{3.17}$$

where $g(r)$ is defined by (2.13).

We divide the proof into two cases.

Case 1 $q=1/2$. Then Lemma 2.6(1) and (3.17) lead to the conclusion that $G(r)$ is strictly increasing on $(0,1)$. Therefore,

$$T(a,b)>\frac{1}{2}C(a,b)+\frac{1}{2}G(a,b) \tag{3.18}$$

follows from (3.13)-(3.15) and the monotonicity of $G(r)$.

Case 2 $q=2/\pi$. Then (3.16) becomes

$$G(1^-)=0. \tag{3.19}$$

It follows from Lemma 2.6(2) and (3.17) that there exists $r_0\in(0,1)$ such that $G(r)$ is strictly decreasing on $(0,r_0)$ and strictly increasing on $(r_0,1)$. Therefore,

$$T(a,b)<\frac{2}{\pi}C(a,b)+\left(1-\frac{2}{\pi}\right)G(a,b) \tag{3.20}$$

follows from (3.13)-(3.15) and (3.19) together with the piecewise monotonicity of $G(r)$.

Note that

$$\lim_{r\to 0^+}\frac{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-\sqrt{1-r^2}}{(1+r^2)-\sqrt{1-r^2}}=\frac{1}{2}, \tag{3.21}$$

$$\lim_{r\to 1^-}\frac{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-\sqrt{1-r^2}}{(1+r^2)-\sqrt{1-r^2}}=\frac{2}{\pi}. \tag{3.22}$$

Therefore, Theorem 3.2 follows easily from (3.12), (3.18) and (3.20)-(3.22). □

Theorem 3.3. *The double inequality*

$$\frac{\alpha_3}{G(a,b)}+\frac{1-\alpha_3}{C(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_3}{G(a,b)}+\frac{1-\beta_3}{C(a,b)}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_3\leq 0$ and $\beta_3\geq 1/2$.

Proof. Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$, then from (3.1) and (3.2) we have

$$\begin{aligned}&\frac{1}{T(a,b)}-\frac{1}{2}\left[\frac{1}{G(a,b)}+\frac{1}{C(a,b)}\right] \\ &=\frac{1}{2}\left[\frac{\pi}{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}-\frac{1}{\sqrt{1-r^2}}-\frac{1}{1+r^2}\right].\end{aligned} \tag{3.23}$$

Let

$$H(r)=\frac{\pi}{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}-\frac{1}{\sqrt{1-r^2}}-\frac{1}{1+r^2}. \tag{3.24}$$

Then Lemma 2.7 and $\sqrt{1-r^2}<1-r^2/2$ lead to

$$\begin{aligned}H(r)&<\frac{\pi}{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}-\frac{1}{1-\frac{r^2}{2}}-\frac{1}{1+r^2}. \\ &=\frac{\pi}{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}-\frac{4+r^2}{(2-r^2)(1+r^2)}<0.\end{aligned} \tag{3.25}$$

Therefore,

$$\frac{1}{C(a,b)} < \frac{1}{T(a,b)} < \frac{1}{2}\left[\frac{1}{G(a,b)} + \frac{1}{C(a,b)}\right] \tag{3.26}$$

follows from Theorem 3.1 and (3.23)-(3.25).

Let $\lambda \in \mathbb{R}$ and $r \in (0,1)$. Then making use of (1.1) and Taylor expansion we get

$$\frac{1}{T(1+r,1-r)} = \frac{\pi}{2}\frac{1}{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)} = 1 - \frac{1}{4}r^2 + o(r^2),$$

$$\frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)} = \frac{\lambda}{\sqrt{1-r^2}} + \frac{1-\lambda}{1+r^2} = 1 + \frac{3p-2}{2}r^2 + o(r^2),$$

$$\begin{aligned}\frac{1}{T(1+r,1-r)} - \left[\frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}\right] \\ = -\frac{3}{2}\left(\lambda - \frac{1}{2}\right)r^2 + o(r^2).\end{aligned} \tag{3.27}$$

Note that

$$\begin{aligned}\lim_{r\to 1^-}\left\{\frac{1}{T(1+r,1-r)} - \left[\frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}\right]\right\} \\ = \frac{\pi}{4} - \lim_{r\to 1^-}\left[\frac{\lambda}{\sqrt{1-r^2}} + \frac{1-\lambda}{1+r^2}\right] = -\infty\end{aligned} \tag{3.28}$$

if $\lambda > 0$.

Therefore, Theorem 3.3 follows from (3.26) and the following statements.

• If $\lambda < 1/2$, then (3.27) implies that there exists $\delta_3 \in (0,1)$ such that

$$\frac{1}{T(1+r,1-r)} > \frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}$$

for $r \in (0,\delta_3)$.

•• If $\lambda > 0$, then there exists $\delta_4 \in (0,1)$ such that

$$\frac{1}{T(1+r,1-r)} < \frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}$$

for $r \in (1-\delta_4, 1)$. □

Let $r \in (0,1)$, $a=1$, $b=\sqrt{1-r^2}$. Then Theorems 3.1-3.3 lead to Corollary 3.4.

Corollary 3.4. *The double inequalities*

$$\frac{\pi}{2}\frac{\sqrt{2-r^2}\sqrt[8]{1-r^2}}{\sqrt{1+\sqrt{1-r^2}}} < \mathcal{E}(r) < \frac{\pi}{2}\frac{2-r^2}{\sqrt{1+\sqrt{1-r^2}}},$$

$$\frac{\pi}{4}\frac{2-r^2+\sqrt[4]{1-r^2}(1+\sqrt{1-r^2})}{1+\sqrt{1-r^2}} < \mathcal{E}(r) < \frac{2(2-r^2)+(\pi-2)\sqrt[4]{1-r^2}(1+\sqrt{1-r^2})}{2(1+\sqrt{1-r^2})},$$

$$\frac{\pi(2-r^2)\sqrt[4]{1-r^2}}{(2-r^2)(1+\sqrt{1-r^2})\sqrt[4]{1-r^2}} < \mathcal{E}(r) < \frac{\pi}{2}\frac{2-r^2}{\sqrt{1+\sqrt{1-r^2}}}$$

holds for all $r \in (0,1)$.

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OPTIMAL BOUNDS FOR TOADER-QI MEAN WITH APPLICATIONS*

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ABSTRACT. In the article, we find the best possible parameters α_1, α_2, α_3, β_1, β_2 and β_3 such that the double inequalities

$$A^{\alpha_1}(a,b)H^{1-\alpha_1}(a,b) < TQ(a,b) < \beta_1 A(a,b) + (1-\beta_1)H(a,b),$$

$$\frac{[\alpha_2 A(a,b) + (1-\alpha_2)H(a,b)]A(a,b)}{L(a,b)} < TQ(a,b)$$

$$< \frac{[\beta_2 A(a,b) + (1-\beta_2)H(a,b)]A(a,b)}{L(a,b)},$$

$$\sqrt{[\alpha_3 L(a,b) + (1-\alpha_3)H(a,b)]A(a,b)} < TQ(a,b)$$

$$< \sqrt{[\beta_3 L(a,b) + (1-\beta_3)H(a,b)]A(a,b)}$$

hold for all $a, b > 0$ with $a \neq b$, where $A(a,b) = (a+b)/2$, $H(a,b) = 2ab/(a+b)$, $L(a,b) = (b-a)/(\log b - \log a)$ and $TQ(a,b) = 2\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta/\pi$ are the arithmetic, harmonic, logarithmic and Toader-Qi means of a and b, respectively. As applications, we present new bounds for the modified Bessel function of the first kind $I_0(t) = \sum_{n=0}^{\infty} t^{2n}/[2^{2n}(n!)^2]$.

1. INTRODUCTION

Let $a, b > 0$ with $a \neq b$. Then the arithmetic mean $A(a,b)$ [1-7], harmonic mean $H(a,b)$ [8-16], logarithmic mean $L(a,b)$ [17-22] and Toader-Qi mean $TQ(a,b)$ [23, 24] are defined by

$$A(a,b) = \frac{a+b}{2}, \quad H(a,b) = \frac{2ab}{a+b}, \tag{1.1}$$

$$L(a,b) = \frac{b-a}{\log b - \log a}, \quad TQ(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta, \tag{1.2}$$

respectively. Recently, the arithmetic mean $A(a,b)$, harmonic mean $H(a,b)$, logarithmic mean $L(a,b)$ have attracted the attention of many researchers, and many remarkable inequalities for these means and related special functions can be found in the literature [25-59].

Very recently, Qi et al. [24] proved that the identity

$$TQ(a,b) = \sqrt{ab}\, I_0\left(\log\sqrt{b/a}\right) \tag{1.3}$$

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and the inequalities

$$L(a,b) < TQ(a,b) < \frac{A(a,b)+G(a,b)}{2} < \frac{2A(a,b)+G(a,b)}{3} < I(a,b)$$

hold for all $a,b>0$ with $a\neq b$, where

$$I_\nu(t) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{t}{2}\right)^{2n+\nu} \tag{1.4}$$

is the modified Bessel function of the first kind [60], $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the classical gamma function [61-69], and $G(a,b) = \sqrt{ab}$ [70-72] and $I(a,b) = (b^b/a^a)^{1/(b-a)}/e$ [73-75] are respectively the geometric and inentric means of a and b.

Yang and Chu [76, 77], and Yang, Chu and Song [78] proved that the inequalities

$$\lambda_1\sqrt{L(a,b)A(a,b)} < TQ(a,b) < \mu_1\sqrt{L(a,b)A(a,b)},$$

$$L^{\lambda_2}(a,b)A^{1-\lambda_2}(a,b) < TQ(a,b) < \mu_2 L(a,b) + (1-\mu_2)A(a,b),$$

$$TQ(a,b) > L_p(a,b)$$

$$\lambda_3\sqrt{L(a,b)I(a,b)} < TQ(a,b) < \mu_3\sqrt{L(a,b)I(a,b)},$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1 \leq \sqrt{2/\pi}$, $\mu_1 \geq 1$, $\lambda_2 \geq 3/4$, $\mu_2 \leq 3/4$, $p\leq 3/2$, $\lambda_3 \leq \sqrt{e/\pi}$ and $\mu_3 \geq 1$, where $L_p(a,b) = [(b^p-a^p)/(p(\log b - \log a))]^{1/p}$ is the p-order logarithmic mean of a and b.

In [79], the authors proved that $p_1 = 0$, $q_1 = 1/4$, $p_2 = 0$ and $q_2 = 1/2 - \sqrt{2}/4$ are the best possible parameters on the interval $[0,1/2]$ such that the double inequalities

$$H[p_1a+(1-p_1)b, p_1b+(1-p_1)a] < TQ(a,b) < H[q_1a+(1-q_1)b, q_1b+(1-q_1)a],$$

$$G[p_2a+(1-p_2)b, p_2b+(1-p_2)a] < TQ(a,b) < G[q_2a+(1-q_2)b, q_2b+(1-q_2)a]$$

hold for all $a,b>0$ with $a\neq b$.

The main purpose of the article is to present the best possible parameters α_1, α_2, α_3, β_1, β_2 and β_3 such that the double inequalities

$$A^{\alpha_1}(a,b)H^{1-\alpha_1}(a,b) < TQ(a,b) < \beta_1 A(a,b) + (1-\beta_1)H(a,b),$$

$$\frac{[\alpha_2 A(a,b) + (1-\alpha_2)H(a,b)]A(a,b)}{L(a,b)} < TQ(a,b) < \frac{[\beta_2 A(a,b) + (1-\beta_2)H(a,b)]A(a,b)}{L(a,b)},$$

$$\sqrt{[\alpha_3 L(a,b) + (1-\alpha_3)H(a,b)]A(a,b)} < TQ(a,b) < \sqrt{[\beta_3 L(a,b) + (1-\beta_3)H(a,b)]A(a,b)}$$

hold for all $a,b>0$ with $a\neq b$, and find the new bounds for the modified Bessel function $I_0(t)$.

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2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1. *(See [80, Theorem 2.18]) The identity*

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

holds for all $n\in\mathbb{N}$.

Lemma 2.2. *(See [81]) Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences with $b_n>0$ and $\lim_{n\to\infty}a_n/b_n=s$. Then the power series $\sum_{n=0}^{\infty}a_nt^n$ is convergent for all $t\in\mathbb{R}$ and*

$$\lim_{t\to\infty}\frac{\sum_{n=0}^{\infty}a_nt^n}{\sum_{n=0}^{\infty}b_nt^n}=s$$

if the power series $\sum_{n=0}^{\infty}b_nt^n$ is convergent for all $t\in\mathbb{R}$.

Lemma 2.3. *(See [82, Lemma 2.2]) The double inequality*

$$\frac{1}{(x+a)^{1-a}}<\frac{\Gamma(x+a)}{\Gamma(x+1)}<\frac{1}{x^{1-a}}$$

holds for all $x>0$ and $a\in(0,1)$.

Lemma 2.4. *(See [80, Theorem 1.25]) Let $a,b\in\mathbb{R}$ with $a<b$, $f,g:[a,b]\mapsto\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)},\qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotonic, then the monotonicity in the conclusion is also strict.

Lemma 2.5. *(See [83], [84, Lemma 2.1]) Let $A(t)=\sum_{k=0}^{\infty}a_kt^k$ and $B(t)=\sum_{k=0}^{\infty}b_kt^k$ be two real power series converging on $(-r,r)$ $(r>0)$ with $b_k>0$ for all k. If the non-constant sequence $\{a_k/b_k\}_{k=0}^{\infty}$ is increasing (decreasing) for all k, then the function $t\mapsto A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

Lemma 2.6. *(See [85, (3.5)]) The identity*

$$I_\lambda(t)I_\mu(t)=\sum_{n=0}^{\infty}\frac{\Gamma(2n+\lambda+\mu+1)}{n!\Gamma(n+\lambda+\mu+1)\Gamma(n+\lambda+1)\Gamma(n+\mu+1)}\left(\frac{t}{2}\right)^{2n+\lambda+\mu}$$

holds for all $\lambda,\mu>-1$ and $t\in\mathbb{R}$.

Lemma 2.7. *The identities*

$$\cosh(t)I_0(t)=\sum_{n=0}^{\infty}\frac{(4n)!}{2^{2n}[(2n)!]^3}t^{2n},$$

$$\sinh(t)I_0(t)=\sum_{n=0}^{\infty}\frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3}t^{2n+1},$$

$$\cosh(t)I_1(t)=\sum_{n=0}^{\infty}\frac{(4n+1)!}{2^{2n+1}(n+1)(2n+1)[(2n)!]^3}t^{2n+1}$$

hold for all $t\in\mathbb{R}$, *where* $\sinh(t)=(e^t-e^{-t})/2$ *and* $\cosh(t)=(e^t+e^{-t})/2$ *are the hyperbolic sine and cosine functions, respectively.*

Proof. It follows from (1.4), and Lemmas 2.1 and 2.6 that

$$I_{-1/2}(t)=\sqrt{\frac{2}{\pi t}}\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}=\sqrt{\frac{2}{\pi t}}\cosh(t),$$

$$I_{1/2}(t)=\sqrt{\frac{2}{\pi t}}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}=\sqrt{\frac{2}{\pi t}}\sinh(t),$$

$$\cosh(t)I_0(t)=\sqrt{\frac{\pi t}{2}}I_{-1/2}(t)I_0(t)$$

$$=\sqrt{\frac{\pi t}{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(2n+\frac{1}{2}\right)}{\left[n!\Gamma\left(n+\frac{1}{2}\right)\right]^2}\left(\frac{t}{2}\right)^{2n-1/2}=\sum_{n=0}^{\infty}\frac{(4n)!}{2^{2n}[(2n)!]^3}t^{2n},$$

$$\sinh(t)I_0(t)=\sqrt{\frac{\pi t}{2}}I_{1/2}(t)I_0(t)$$

$$=\sqrt{\frac{\pi t}{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\left[n!\Gamma\left(n+\frac{3}{2}\right)\right]^2}\left(\frac{t}{2}\right)^{2n+1/2}=\sum_{n=0}^{\infty}\frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3}t^{2n+1},$$

$$\cosh(t)I_1(t)=\sqrt{\frac{\pi t}{2}}I_{-1/2}(t)I_1(t)$$

$$=\sqrt{\frac{\pi t}{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{(n+1)\left(n+\frac{3}{2}\right)\left[n!\Gamma\left(n+\frac{1}{2}\right)\right]^2}\left(\frac{t}{2}\right)^{2n+1/2}$$

$$=\sum_{n=0}^{\infty}\frac{(4n+1)!}{2^{2n+1}(n+1)(2n+1)[(2n)!]^3}t^{2n+1}.$$

□

Lemma 2.8. *The function* $f(t)=\log[I_0(t)]/[\log\cosh(t)]$ *is strictly increasing from* $(0,\infty)$ *onto* $(1/2,1)$.

Proof. Let $f_1(t)=\log[I_0(t)]$, $f_2(t)=\log\cosh(t)$, and a_n and b_n be defined by

$$a_n=\frac{(4n+1)!}{2^{2n+1}(n+1)(2n+1)[(2n)!]^3},\quad b_n=\frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3}.\tag{2.1}$$

Then from (1.4), Lemma 2.7 and (2.1) we clearly see that

$$f(t)=\frac{f_1(t)}{f_2(t)}=\frac{f_1(t)-f_1(0^+)}{f_2(t)-f_2(0^+)},\tag{2.2}$$

$$\frac{a_n}{b_n}=1-\frac{1}{2(n+1)},\tag{2.3}$$

$$\frac{a_0}{b_0}=\frac{1}{2},\quad \lim_{n\to\infty}\frac{a_n}{b_n}=1-\lim_{n\to\infty}\frac{1}{2(n+1)}=1,\tag{2.4}$$

$$\frac{f_1'(t)}{f_2'(t)}=\frac{\cosh(t)I_1(t)}{\sinh(t)I_0(t)}=\frac{\sum_{n=0}^{\infty}a_nt^{2n}}{\sum_{n=0}^{\infty}b_nt^{2n}}.\tag{2.5}$$

From Lemma 2.5, (2.3) and (2.5) we know that the function $f_1'(t)/f_2'(t)$ is strictly increasing on $(0, \infty)$. Then Lemma 2.4 and (2.2) lead to the conclusion that $f(t)$ is strictly increasing on $(0, \infty)$.

Therefore, Lemma 2.8 follows from Lemma 2.2 and (2.4) together with the monotonicity of $f(t)$. □

Lemma 2.9. *The function* $g(t) = [\cosh(t)I_0(t) - 1]/[\cosh(2t) - 1]$ *is strictly decreasing from* $(0, \infty)$ *onto* $(0, 3/8)$.

Proof. Let $n \in \mathbb{N}$, and c_n and d_n be defined by

$$c_n = \frac{(4n+4)!}{2^{2n+2}[(2n+2)!]^3}, \qquad d_n = \frac{2^{2n+2}}{(2n+2)!}. \tag{2.6}$$

Then Lemmas 2.1 and 2.7 together with (2.6) lead to

$$\frac{c_0}{d_0} = \frac{3}{8}, \tag{2.7}$$

$$\frac{c_n}{d_n} = \frac{(4n+4)!}{2^{4n+4}\Gamma(2n+3)(2n+2)!} = \frac{\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}\Gamma(2n+3)}, \tag{2.8}$$

$$g(t) = \frac{\sum_{n=0}^{\infty} \frac{(4n)!}{2^{2n}[(2n)!]^3} t^{2n} - 1}{\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n} - 1} = \frac{\sum_{n=0}^{\infty} c_n t^{2n}}{\sum_{n=0}^{\infty} d_n t^{2n}}, \tag{2.9}$$

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = -\frac{(n+2)(2n+3)(8n+13)(4n+4)!}{2^{4n+5}[(2n+4)!]^2} < 0 \tag{2.10}$$

for all $n \in \mathbb{N}$.

It follows from Lemma 2.3 that

$$\frac{1}{\sqrt{\pi}\left(2n+\frac{5}{2}\right)^{1/2}} < \frac{\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}\Gamma(2n+3)} < \frac{1}{\sqrt{\pi}(2n+2)^{1/2}}. \tag{2.11}$$

From Lemma 2.2, Lemma 2.5 and (2.8)-(2.11) we know that $g(t)$ is strictly decreasing on $(0, \infty)$ and

$$\lim_{t \to \infty} g(t) = \lim_{n \to \infty} \frac{c_n}{d_n} = 0. \tag{2.12}$$

Therefore, Lemma 2.9 follows from (2.7), (2.9), (2.12) and the monotonicity of the function $g(t)$ on the interval $(0, \infty)$. □

Lemma 2.10. *The function* $h(t) = [\sinh(t)I_0(t) - t]/[t\cosh(2t) - t]$ *is strictly decreasing from* $(0, \infty)$ *onto* $(0, 5/24)$.

Proof. Let $n \in \mathbb{N}$, u_n and v_n be defined by

$$u_n = \frac{(4n+6)!}{2^{2n+3}[(2n+3)!]^3}, \qquad v_n = \frac{2^{2n+2}}{(2n+2)!}. \tag{2.13}$$

Then from Lemma 2.1, Lemma 2.7 and (2.13) one has

$$\frac{u_0}{v_0} = \frac{5}{24}, \tag{2.14}$$

$$\frac{u_n}{v_n} = \frac{(4n+5)!}{2^{4n+4}[(2n+3)!]^2} = \frac{(4n+5)\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}(2n+3)^2\Gamma(2n+3)}, \tag{2.15}$$

$$h(t)=\frac{\sum_{n=0}^{\infty}\frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3}t^{2n+1}-t}{t\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}-t}=\frac{\sum_{n=0}^{\infty}u_nt^{2n}}{\sum_{n=0}^{\infty}v_nt^{2n}}, \tag{2.16}$$

$$\begin{aligned}\frac{u_{n+1}}{v_{n+1}}-\frac{u_n}{v_n}&=\frac{(4n+9)!}{2^{4n+8}[(2n+5)!]^2}-\frac{(4n+5)!}{2^{4n+4}[(2n+3)!]^2} \\ &=-\frac{(n+2)(48n^2+202n+211)(4n+5)!}{2^{4n+5}[(2n+5)!]^2}<0\end{aligned} \tag{2.17}$$

for all $n\in\mathbb{N}$.

It follows from Lemma 2.3 that

$$\frac{(4n+5)}{\sqrt{\pi}(2n+3)^2\left(2n+\frac{5}{2}\right)^{1/2}}<\frac{(4n+5)\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}(2n+3)^2\Gamma(2n+3)}<\frac{(4n+5)}{\sqrt{\pi}(2n+3)^2(2n+2)^{1/2}}. \tag{2.18}$$

From Lemma 2.2, Lemma 2.5 and (2.15)-(2.18) we clearly see that $h(t)$ is strictly decreasing on $(0,\infty)$ and

$$\lim_{t\to\infty}h(t)=\lim_{n\to\infty}\frac{u_n}{v_n}=0. \tag{2.19}$$

Therefore, Lemma 2.10 follows easily from (2.14), (2.16), (2.19) and the monotonicity of $h(t)$ on the interval $(0,\infty)$. □

Lemma 2.11. *The function $\lambda(t)=[tI_0^2(t)-t]/[\sinh(2t)-2t]$ is strictly decreasing from $(0,\infty)$ onto $(1/\pi,3/8)$.*

Proof. Let $n\in\mathbb{N}$, σ_n and τ_n be defined by

$$\sigma_n=\frac{(2n+2)!}{2^{2n+2}[(n+1)!]^4},\qquad \tau_n=\frac{2^{2n+3}}{(2n+3)!}. \tag{2.20}$$

Then from Lemma 2.1, Lemma 2.3, Lemma 2.6, (2.20) one has

$$\frac{\sigma_0}{\tau_0}=\frac{3}{8}, \tag{2.21}$$

$$I_0^2(t)=\frac{(2n)!}{2^{2n}(n!)^4}t^{2n},$$

$$\lambda(t)=\frac{\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n+1}-t}{\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+1}-2t}=\frac{\sum_{n=0}^{\infty}\sigma_nt^{2n}}{\sum_{n=0}^{\infty}\tau_nt^{2n}}, \tag{2.22}$$

$$\frac{\sigma_n}{\tau_n}=\frac{(2n+3)[(2n+2)!]^2}{2^{4n+5}[(n+1)!]^4}$$

$$=\frac{\left(n+\frac{3}{2}\right)}{\Gamma^2(n+2)}\left[\frac{(2n+2)!}{2^{2n+2}(n+1)!}\right]^2=\frac{n+\frac{3}{2}}{\pi}\left[\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+2)}\right]^2,$$

$$\frac{1}{\pi}<\frac{\sigma_n}{\tau_n}<\frac{n+\frac{3}{2}}{\pi(n+1)},$$

$$\lim_{n\to\infty}\frac{\sigma_n}{\tau_n}=\frac{1}{\pi}, \tag{2.23}$$

$$\frac{\sigma_{n+1}}{\tau_{n+1}}-\frac{\sigma_n}{\tau_n}=-\frac{(2n+3)(n+2)^2[(2n+2)!]^2}{2^{4n+7}[(n+2)!]^4}<0 \tag{2.24}$$

for all $n\in\mathbb{N}$.

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It follows from Lemma 2.2, Lemma 2.5 and (2.22)-(2.24) that $\lambda(t)$ is strictly decreasing on $(0, \infty)$ and

$$\lim_{t \to \infty} \lambda(t) = \frac{1}{\pi}. \tag{2.25}$$

Therefore, Lemma 2.11 follows easily from (2.21), (2.22), (2.25) and the monotonicity of the function $\lambda(t)$ on the interval $(0, \infty)$. □

3. Main Results

Theorem 3.1. *The double inequalities*

$$A^{\alpha_1}(a,b)H^{1-\alpha_1}(a,b) < TQ(a,b) < \beta_1 A(a,b) + (1-\beta_1)H(a,b),$$

$$\frac{[\alpha_2 A(a,b) + (1-\alpha_2)H(a,b)]A(a,b)}{L(a,b)} < TQ(a,b) < \frac{[\beta_2 A(a,b) + (1-\beta_2)H(a,b)]A(a,b)}{L(a,b)},$$

$$\sqrt{[\alpha_3 L(a,b) + (1-\alpha_3)H(a,b)]A(a,b)} < TQ(a,b) < \sqrt{[\beta_3 L(a,b) + (1-\beta_3)H(a,b)]A(a,b)}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 3/4$, $\beta_1 \geq 3/4$, $\alpha_2 \leq 0$, $\beta_2 \geq 5/12$, $\alpha_3 \leq 2/\pi$ and $\beta_3 \geq 3/4$.

Proof. Since $H(a,b)$, $L(a,b)$, $A(a,b)$ and $TQ(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b > a > 0$. Let $t = \log\sqrt{b/a} > 0$, then from (1.1)-(1.3) one has

$$H(a,b) = \frac{\sqrt{ab}}{\cosh(t)}, \quad L(a,b) = \sqrt{ab}\frac{\sinh(t)}{t}, \tag{3.1}$$

$$TQ(a,b) = \sqrt{ab}I_0(t), \quad A(a,b) = \sqrt{ab}\cosh(t), \tag{3.2}$$

$$\frac{\log TQ(a,b) - \log H(a,b)}{\log A(a,b) - \log H(a,b)} = \frac{\log I_0(t) + \log\cosh(t)}{2\log\cosh(t)} = \frac{1}{2}f(t) + \frac{1}{2}, \tag{3.3}$$

$$\frac{TQ(a,b) - H(a,b)}{A(a,b) - H(a,b)} = \frac{I_0(t)\cosh(t) - 1}{\cosh^2(t) - 1} = 2g(t), \tag{3.4}$$

$$\frac{TQ(a,b)L(a,b) - H(a,b)A(a,b)}{A^2(a,b) - H(a,b)A(a,b)} = \frac{\sinh(t)I_0(t) - t}{t[\cosh^2(t) - 1]} = 2h(t), \tag{3.5}$$

$$\frac{TQ^2(a,b) - H(a,b)A(a,b)}{L(a,b)A(a,b) - H(a,b)A(a,b)} = \frac{t[I_0^2(t) - 1]}{\sinh(t)\cosh(t) - t} = 2\lambda(t), \tag{3.6}$$

where, $f(t)$, $g(t)$, $h(t)$ and $\lambda(t)$ are given by Lemma 2.8, Lemma 2.9, Lemma 2.10 and Lemma 2.11, respectively.

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Therefore, Theorem 3.1 follows easily from (3.3)-(3.6), and Lemma 2.8, Lemma 2.9, Lemma 2.10 and Lemma 2.11. □

From Theorem 3.1, (3.1) and (3.2), we get Corollary 3.2 immediately.

Corollary 3.2. *The double inequalities*

$$\cosh^{1/2}(t) < I_0(t) < \frac{3\cosh^2(t)+1}{4\cosh(t)},$$

$$\frac{t}{\sinh(t)} < I_0(t) < \frac{[5\cosh^2(t)+7]t}{12\sinh(t)},$$

$$\sqrt{\frac{\sinh(2t)}{\pi t}+1-\frac{2}{\pi}} < I_0(t) < \sqrt{\frac{3\sinh(2t)}{8t}+\frac{1}{4}}$$

for all $t > 0$.

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Symmetric identities for Dirichlet-type multiple twisted (h,q)-l-function and higher-order generalized twisted (h,q)-Euler polynomials

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Abstract : In this paper we investigate some interesting symmetric identities for multiple twisted (h,q)-l-function and higher-order generalized twisted (h,q)-Euler polynomials in complex field.

Key words : Symmetric properties, power sums, Euler numbers and polynomials, multiple twisted (h,q)-l-function, higher-order generalized twisted (h,q)-Euler numbers and polynomials.

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1. Introduction

Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Many mathematicians have studied in the area of the q- extension of Euler numbers and polynomials(see [1-10]). Y. He studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [2]). D. Kim *et al.*[3] derived some identities of symmetry for (h,q)-extension of higher-order Euler numbers and polynomials. D. V. Dolgy *et al.*[1] derived some identities of symmetry for higher-order generalized q-Euler polynomials. In this paper, we present a systemic study of the generalized twisted (h,q)-Euler numbers and polynomials of higher-order by using the multiple twisted(h,q)-l-function. Throughout this paper, the notations $\mathbb{N},\mathbb{Z},\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. We assume that $q\in\mathbb{C}$ with $|q|<1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1, 2, 3, 5]) }.$$

Note that $\lim_{q\to 1}[x] = x$. Let χ be a Dirichlet character with conductor $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$ and ε be the p^N-th root of unity(see [8, 9, 10]). T. Kim introduced the multiple q-Euler zeta function which interpolates higher-order q-Euler polynomials at negative integers as follows(see [4, 5]):

$$\zeta_{q,r}(s,x) = [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{\sum_{j=1}^r m_j} q^{\sum_{j=1}^r m_j}}{[m_1+\cdots+m_r+x]_q^s}, \tag{1}$$

where $s\in\mathbb{C}$ and $x\in\mathbb{R}$, with $x\neq 0,-1,-2,\ldots$.

Recently, D. V. Dolgy *et al.*[1] considered some symmetric identities for higher-order generalized q-Euler polynomials. The generalized Euler polynomials of order $r\in\mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2\sum_{l=0}^{d-1} \frac{\chi(l)(-1)^l e^{(x+l)t}}{e^{dt}+1}\right)^r = \sum_{m=0}^{\infty} E_{m,\chi}^{(r)}(x)\frac{t^m}{m!}. \tag{2}$$

When $x=0$, $E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the generalized Euler numbers $E_{n,\chi}^{(r)}$ attached to χ.

For $h \in \mathbb{Z}, \alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, we introduced the higher order twisted q-Euler polynomials with weight α as follows(see [7]):

$$\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k|x) = \frac{[2]_q^k}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1+\varepsilon q^{\alpha l+h}) \cdots (1+\varepsilon q^{\alpha l+h-k+1})}.$$

In the special case, $x = 0$, $\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k|0) = \widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k)$ are called the higher-order twisted q-Euler numbers with weight α.

We consider the higher order generalized q-Euler polynomials of order r attached to χ twisted by ramified roots of unity as follows(see [8]):

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-\zeta)^{\sum_{j=0}^{r} m_j} \left(\prod_{i=1}^{r} \chi(m_i) \right) e^{[x+\sum_{j=1}^{r} m_j]_q t}.$$

In the special case $x = 0$, the sequence $E_{n,\chi,\zeta,q}^{(r)}(0) = E_{n,\chi,\zeta,q}^{(r)}$ are called the n-th generalized q-Euler numbers of order r attached to χ twisted by ramified roots of unity.

As is well known, the higher-order generalized twisted (h,q)-Euler polynomials $E_{n,\chi,q,\varepsilon}^{(h,k)}(x)$ attached to χ are defined by the following generating function to be

$$\begin{aligned}\widetilde{F}_{\chi,q,\varepsilon}^{(h,k)}(t,x) = [2]_q^k & \sum_{m_1,\cdots,m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} q^{\sum_{j=1}^{k}(h-j+1)m_j} \varepsilon^{m_1+\cdots+m_k} \\ & \times \left(\prod_{j=1}^{k} \chi(m_j) \right) e^{[m_1+\cdots+m_k+x]_q t} \\ = & \sum_{n=0}^{\infty} E_{n,\chi,q,\varepsilon}^{(h,k)}(x) \frac{t^n}{n!},\end{aligned} \tag{3}$$

where $h \in \mathbb{Z}$ and $k \in \mathbb{N}$. When $x = 0, E_{n,\chi,q,\varepsilon}^{(h,k)} = E_{n,\chi,q,\varepsilon}^{(h,k)}(0)$ are called the higher-order generalized twisted (h,q)-Euler numbers $E_{n,\chi,q,\varepsilon}^{(h,k)}$ attached to χ. Observe that if $q \to 1, \varepsilon \to 1$, then $E_{n,\chi,q,\varepsilon}^{(h,k)} \to E_{n,\chi}^{(k)}$ and $E_{n,\chi,q,\varepsilon}^{(h,k)}(x) \to E_{n,\chi}^{(k)}(x)$. By using (3) and Cauchy product, we have

$$E_{n,\chi,q,\varepsilon}^{(h,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,\chi,q,\varepsilon}^{(h,k)} [x]_q^{n-l} = (q^x E_{\chi,q,\varepsilon}^{(h,k)} + [x]_q)^n, \tag{4}$$

with the usual convention about replacing $(E_{\chi,q,\varepsilon}^{(h,k)})^n$ by $E_{n,\chi,q,\varepsilon}^{(h,k)}$.

By using complex integral and (3), we can also obtain the Dirichlet-type multiple twisted (h,q)-l-function as follows:

$$\begin{aligned} l_{\chi,q,\varepsilon}^{(h,k)}(s,x) &= \frac{1}{\Gamma(s)} \int_0^\infty \widetilde{F}_{\chi,q,\varepsilon}^{(h,k)}(-t,x) t^{s-1} dt \\ &= [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^{k} m_j} \left(\prod_{j=1}^{k} \chi(m_j) \right) q^{\sum_{j=1}^{k}(h-j+1)m_j} \varepsilon^{\sum_{j=1}^{k} m_j}}{[m_1+\cdots+m_k+x]_q^s}, \end{aligned} \tag{5}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

By using Cauchy residue theorem, the value of Dirichlet-type multiple twisted (h,q)-l-function at negative integers is given explicitly by the following theorem:

Theorem 1. Let $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We obtain

$$l_{\chi,q,\varepsilon}^{(h,k)}(-n,x) = E_{n,\chi,q,\varepsilon}^{(h,k)}(x).$$

The purpose of this paper is to obtain some interesting identities of the power sums and the higher-order generalized twisted (h,q)-Euler polynomials $E^{(h,k)}_{n,\chi,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted (h,q)-l-function. In this paper, if we take $\chi^0=1, \varepsilon=1$, then [3] is the special case of this paper. If we take $\varepsilon=1$ in all equations of this article, then [1] are the special case of our results.

2. Symmetry identities for Dirichlet-type multiple twisted (h,q)-l-function

In this section, by using the similar method of [1, 2, 3], expect for obvious modifications, we investigate some symmetric identities for higher-order generalized twisted (h,q)-Euler polynomials $E^{(h,k)}_{n,\chi,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted (h,q)-l-function. We assume that χ is a Dirichlet character with conductor $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$ and ε be the p^N-th root of unity. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1 \pmod 2$. For $h\in\mathbb{Z}, k\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we obtain certain symmetry identities for Dirichlet-type multiple twisted (h,q)-l-function.

Observe that $[xy]_q=[x]_{q^y}[y]_q$ for any $x,y\in\mathbb{C}$. In (5), we derive next result by substitute $w_2x+\dfrac{w_2}{w_1}(j_1+\cdots+j_k)$ for x in and replace q and ε by q^{w_1} and ε^{w_1}, respectively.

$$\begin{aligned}
&\frac{1}{[2]^k_{q^{w_1}}} l^{(h,k)}_{\chi,q^{w_1},\varepsilon^{w_1}}(s, w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k))\\
&=\sum_{m_1,\cdots,m_k=0}^{\infty}\frac{(-1)^{\sum_{j=1}^k m_j}\left(\prod_{j=1}^k\chi(m_j)\right)q^{w_1\sum_{j=1}^k(h-j+1)m_j}\varepsilon^{w_1\sum_{j=1}^k m_j}}{\left[\dfrac{w_1(m_1+\cdots+m_k)+w_1w_2x+w_2(j_1+\cdots+j_k)}{w_1}\right]^s_{q^{w_1}}}\\
&=\sum_{m_1,\cdots,m_k=0}^{\infty}\frac{(-1)^{\sum_{j=1}^k m_j}\left(\prod_{j=1}^k\chi(m_j)\right)q^{w_1\sum_{j=1}^k(h-j+1)m_j}\varepsilon^{w_1\sum_{j=1}^k m_j}}{\dfrac{[w_1(m_1+\cdots+m_k)+w_1w_2x+w_2(j_1+\cdots+j_k)]^s_q}{[w_1]^s_q}}\\
&=[w_1]^s_q\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{i_1,\cdots,i_k=0}^{dw_2-1}\frac{(-1)^{\sum_{j=1}^k m_j}\left(\prod_{j=1}^k\chi(m_j)\right)q^{w_1\sum_{j=1}^k(h-j+1)m_j}\varepsilon^{w_1\sum_{j=1}^k m_j}}{[w_1(m_1+\cdots+m_k)+w_1w_2x+w_2(j_1+\cdots+j_k)]^s_q}\\
&=[w_1]^s_q\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{i_1,\cdots,i_k=0}^{dw_2-1}(-1)^{\sum_{j=1}^k m_j}(-1)^{\sum_{j=1}^k i_j}\left(\prod_{j=1}^k\chi(i_j)\right)\\
&\quad\times q^{dw_1w_2\sum_{j=1}^k(h-j+1)m_j}q^{w_1\sum_{j=1}^k(h-j+1)i_j}\varepsilon^{dw_1w_2\sum_{j=1}^k m_j}\varepsilon^{w_1\sum_{j=1}^k i_j}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(i_1+\cdots+i_k)+w_2(j_1+\cdots+j_k)]^s_q\right)^{-1}
\end{aligned}\tag{6}$$

Thus, from (6), we can derive the following equation.

$$\begin{aligned}
&\frac{[w_2]^s_q}{[2]^k_{q^{w_1}}}\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times l^{(h,k)}_{\chi,q^{w_1},\varepsilon^{w_1}}(s,w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k))\\
&=[w_1]^s_q[w_2]^s_q\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{i_1,\cdots,i_k=0}^{dw_2-1}\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k(j_l+i_l+m_l)}\left(\prod_{l=1}^k\chi(j_l)\right)\left(\prod_{l=1}^k\chi(i_l)\right)\\
&\quad\times q^{dw_1w_2\sum_{l=1}^k(h-l+1)m_l}q^{w_1\sum_{l=1}^k(h-l+1)i_l}q^{w_2\sum_{l=1}^k(h-l+1)j_l}\\
&\quad\times\varepsilon^{dw_1w_2\sum_{l=1}^k m_l}\varepsilon^{w_1\sum_{l=1}^k i_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(i_1+\cdots+i_k)+w_2(j_1+\cdots+j_k)]^s_q\right)^{-1}
\end{aligned}\tag{7}$$

By using the same method as (7), we have

$$\begin{aligned}
&\frac{[w_1]_q^s}{[2]_{q^{w_2}}^k}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times l_{\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k))\\
&=[w_1]_q^s[w_2]_q^s\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}\sum_{i_1,\cdots,i_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k(j_l+i_l+m_l)}\left(\prod_{l=1}^k\chi(j_l)\right)\left(\prod_{l=1}^k\chi(i_l)\right)\\
&\quad\times q^{dw_1w_2\sum_{l=1}^k(h-l+1)m_l}q^{w_2\sum_{l=1}^k(h-l+1)i_l}q^{w_1\sum_{l=1}^k(h-l+1)j_l}\\
&\quad\times \varepsilon^{dw_1w_2\sum_{l=1}^k m_l}\varepsilon^{w_2\sum_{l=1}^k i_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(j_1+\cdots+j_k)+w_2(i_1+\cdots+i_k)]_q^s\right)^{-1}
\end{aligned}\tag{8}$$

Therefore, by (7) and (8), we have the following theorem.

Theorem 2. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times l_{\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(s,w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times l_{\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}\left(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)
\end{aligned}\tag{9}$$

By (9) and Theorem 1, we obtain the following theorem.

Theorem 3. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times E_{n,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right).
\end{aligned}\tag{10}$$

From (4), we note that

$$E_{n,\chi,q,\varepsilon}^{(h,k)}(x+y)=(q^{x+y}E_{n,\chi,q,\varepsilon}^{(h,k)}+[x+y]_q)^n=\sum_{i=0}^n\binom{n}{i}q^{xi}E_{i,\chi,q,\varepsilon}^{(h,k)}(y)[x]_q^{n-i}.\tag{11}$$

with the usual convention about replacing $(E_{\chi,q,\varepsilon}^{(h,k)})^n$ by $E_{n,\chi,q,\varepsilon}^{(h,k)}$.

By (11), we have

$$
\begin{aligned}
&\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\times\sum_{i=0}^n\binom{n}{i}q^{w_2i(j_1+\cdots+j_k)}E_{i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{n-i}\\
&=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\times\sum_{i=0}^n\binom{n}{i}q^{w_2(n-i)\sum_{l=1}^k j_l}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{i}
\end{aligned}
\tag{12}
$$

Hence we have the following theorem.

Theorem 4. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$
\begin{aligned}
&\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=\sum_{i=0}^n\binom{n}{i}[w_2]_q^i[w_1]_q^{-i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\\
&\times\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(n+h-l-i+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}[j_1\cdots+j_k]_{q^{w_2}}^i.
\end{aligned}
$$

For each integer $n \geq 0$, let

$$
\mathcal{S}_{n,i,\chi,q,\varepsilon}^{(h,k)}(w)=\sum_{j_1,\cdots,j_k=0}^{w-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{\sum_{l=1}^k(n+h-l-i+1)j_l}\varepsilon^{\sum_{l=1}^k j_l}[j_1\cdots+j_k]_q^i.
$$

The above sum $\mathcal{S}_{n,i,\chi,q,\varepsilon}^{(h,k)}(w)$ is called the alternating generalized (h,q)-power sums.

By Theorem 4, we have

$$
\begin{aligned}
&[2]_{q^{w_2}}^k[w_1]_q^n\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[2]_{q^{w_2}}^k\sum_{i=0}^n\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(dw_1)
\end{aligned}
\tag{13}
$$

By using the same method as in (13), we obtain

$$
\begin{aligned}
&[2]_{q^{w_1}}^k[w_2]_q^n\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times E_{n,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)\\
&=[2]_{q^{w_1}}^k\sum_{i=0}^n\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1x)\mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(dw_2)
\end{aligned}
\tag{14}
$$

Therefore, by (13), (14), and Theorem 3, we have the following theorem.

Theorem 5. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[2]_{q^{w_2}}^k \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2 x) \mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(dw_1) \\
&= [2]_{q^{w_1}}^k \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1 x) \mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(dw_2).
\end{aligned}$$

By Theorem 5, we obtain the interesting symmetric identity for the higher-order generalized twisted (h,q)-Euler numbers $E_{n,\chi,q,\varepsilon}^{(h,k)}$ in complex field.

Corollary 6. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[2]_{q^{w_2}}^k \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} \mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(dw_1) E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)} \\
&= [2]_{q^{w_1}}^k \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(dw_2) E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}.
\end{aligned}$$

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An efficient m-step Levenberg-Marquardt method for systems of nonlinear equations*

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Abstract

In this paper, we propose an efficient m-step Levenberg-Marquardt method for systems of nonlinear equations. At every iteration, the efficient m-step LM method computes not only the classical LM step, but also $m-1$ approximate LM steps with frozen $(J_k^T J_k + \lambda_k I)^{-1} J_k^T$. Also, we employ $m-1$ line searches for $m-1$ approximate LM steps for better numerical performance. Under the local error bound condition which is weaker than nonsingularity, the efficient m-step LM method has been proved to have $(m+1)$th convergence order. The global convergence has also been given by trust region technique. Numerical results show that the efficient m-step LM method is efficient and could save many calculations of the Jacobian especially for large scale problems.

Keywords: Unconstrained optimization; Systems of nonlinear equations; Levenberg-Marquardt method; Trust region

MSC2010: 65K05; 90C30

1 Introduction

It's a well-known problem in science and engineering that is to find the solutions of systems of nonlinear equations

$$F(x) = 0, \tag{1}$$

where $F : D \subset \mathrm{R}^n \to \mathrm{R}^n$ is continuously differentiable function. Due to the nonlinearity of $F(x)$, (1) may have no solutions. Throughout the paper, we let that the solution set of (1) is nonempty and denote it by X^*, and in all cases $\|\cdot\|$ refers to the 2-norm.

There are many numerical methods to approximate the solutions of (1) because the exact solutions is difficult to find. A classical numerical method is Newton method which computes the trial step

$$d_k^N = -J_k^{-1} F_k$$

at every iteration, where $F_k = F(x_k)$ and $J_k = F'(x_k)$ is the Jacobian. And the Newton method has quadratic rate of convergence under the condition that $J(x)$ is Lipschitz continuous and nonsingular at the solution of (1). However, the Newton method will be failed when J_k is singular or near singular. To overcome these disadvantages, a large number of researchers have presented many modifications of Newton

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method [1]. One of them is the Levenberg-Marquardt method (LM) [2, 3], which is a famous numerical method with computing the linear equation

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_k \tag{2}$$

to obtain the LM trail step

$$d_k = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_k \tag{3}$$

at every iteration, where $\lambda_k \geqslant 0$ is the LM parameter. It is well-known that the LM method has quadratic convergence as the Newton method if the Jacobian matrix is nonsingular and Lipschitz continuous at the solution. A large number of researchers have focused on this system and many efficient solution techniques are available [4–7].

As we all known, the cost of Jacobian computations is expensive when $F(x)$ is complicated or n is quite large. Recently, to save Jacobian calculations and achieve a fast convergence rate, Fan [8] presented a modified Levenberg-Marquardt method (MLM) with cubic convergence. At every iteration, the MLM method solves not only the linear equations (2) to obtain the LM step (3), but also the linear equations

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,1}$$

to obtain the approximate LM step

$$d_{k,1} = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,1} \tag{4}$$

with $F_{k,1} = F\left(x_{k,1}\right)$, $x_{k,1} = x_k + d_k$, $\lambda_k = \mu_k \left\|F_k\right\|^\delta$, $\mu_k > 0$ and $\delta \in [1, 2]$, and the trial step is

$$s_k^{MLM} = d_k + d_{k,1}.$$

Fan use $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ in stead of

$$\left(J\left(x_{k,1}\right)^T J\left(x_{k,1}\right) + \mu_{k+1} \left\|F\left(x_{k,1}\right)\right\|^\delta I\right)^{-1} J\left(x_{k,1}\right)^T \tag{5}$$

in (4), which does not involve the calculation of $J\left(x_{k,1}\right)$. Since J_k has been used in (3), the cost of Jacobian calculations will be saved.

Similarly, to save more Jacobian calculations, based on the MLM method, Yang [9] presented a high-order Levenberg-Marquardt method (HLM) with biquadratic convergence by solving another linear equations

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,2} \tag{6}$$

to obtain another approximate LM step

$$d_{k,2} = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,2} \tag{7}$$

with $F_{k,2} = F\left(x_{k,2}\right)$, $x_{k,2} = x_{k,1}+d_{k,1}$, $\lambda_k = \mu_k \left\|F_k\right\|^\delta$, $\mu_k > 0$ and $\delta \in [1, 2]$. Yang still use $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ in stead of (5) in (4), $\left(J\left(x_{k,2}\right)^T J\left(x_{k,2}\right) + \mu_{k+2} \left\|F\left(x_{k,2}\right)\right\|^\delta I\right)^{-1} J\left(x_{k,2}\right)^T$ in (7) respectively, which does not need to compute $J\left(x_{k,1}\right)$ and $J\left(x_{k,2}\right)$. The trial step of the HLM method is

$$s_k^{HLM} = d_k + d_{k,1} + d_{k,2}.$$

Furthermore, to save more Jacobian calculations and achieve a faster convergence rate, Fan [10] presented a Shamanskii-like Levenberg-Marquardt (SLM) method with $(m+1)$th convergence by solving $m-1$ linear equations

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,i} \quad \text{with} \quad i = 1, \cdots, m-1 \tag{8}$$

to obtain $m-1$ approximate LM steps

$$d_{k,i} = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,i} \tag{9}$$

where $F_{k,i} = F(x_{k,i})$, $x_{k,i} = x_{k,i-1} + d_{k,i-1}$ with $x_{k,0} = x_k$, $d_{k,0} = d_k$, $\lambda_k = \mu_k \|F_k\|^\delta$, $\mu_k > 0$ and $\delta \in [1,2]$. Fan still use $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ in stead of $\left(J(x_{k,i})^T J(x_{k,i}) + \mu_{k+i} \|F(x_{k,i})\|^\delta I\right)^{-1} J(x_{k,i})^T$ in (9), which does not need to compute $J(x_{k,i})$ $(i = 1, 2, \cdots, m-1)$. The trial step of the SLM method is

$$s_k^{SLM} = d_{k,0} + d_{k,1} + \cdots + d_{k,m-1} = \sum_{i=0}^{m-1} d_{k,i}. \tag{10}$$

If we consider the MLM method as two-step Levenberg-Marquardt method and the HLM method as three-step Levenberg-Marquardt method respectively, then, the Shamanskii-like Levenberg-Marquardt method can be considered as m-step Levenberg-Marquardt method. Also, it is easy to see that $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ is computed in all of the classical LM step (3) and the approximate LM step (4), (7), (9) respectively. So, we can consider $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ is frozen in the two-step LM method, three-step LM method and m-step LM method.

To accelerate the MLM method and for better numerical performance, Fan [11] proposed an accelerated version of the MLM (AMLM) method by employing a line search for the approximate LM step $d_{k,1}$ and computed the trial step by

$$s_k^{AMLM} = d_{k,0} + \alpha_{k,1} d_{k,1}, \tag{11}$$

where $\alpha_{k,1} \in [1, \hat{\alpha}_1]$ is step size with $\hat{\alpha}_1 > 1$ is a positive constant. For the same purpose, based on the AMLM method, Chen [12] compute the linear equation (6) with $x_{k,2} = x_{k,1} + \alpha_{k,1} d_{k,1}$ to obtain an approximate LM step $\bar{d}_{k,2}$. By employing another line search for the approximate LM step $\bar{d}_{k,2}$, Chen presented a new modified Levenberg-Marquardt (NMLM) method. The trial step of the NMLM method is

$$s_k^{NMLM} = d_{k,0} + \alpha_{k,1} d_{k,1} + \alpha_{k,2} \bar{d}_{k,2}, \tag{12}$$

where $\alpha_{k,2} \in [1, \hat{\alpha}_2]$ is step size with $\hat{\alpha}_2 > 1$ is a positive constant.

Now, motivated by (10), (11) and (12), we will employ $m-1$ line searches for approximate LM step $d_{k,i}$ by solving linear equation (8) with $x_{k,i} = x_{k,i-1} + \alpha_{k,i-1} d_{k,i-1}$ and present an efficient m-step Levenberg-Marquardt method with trial step as

$$s_k = d_{k,0} + \alpha_{k,1} d_{k,1} + \cdots + \alpha_{k,m-1} d_{k,m-1}, \tag{13}$$

where $\alpha_{k,i} \in [1, \hat{\alpha}]$ are step size with $\hat{\alpha} > 1$ $(i = 1, \cdots, m-1)$ is a positive constants. It is quite clear that the above new LM method will reduce to the classical Levenberg-Marquardt method while $m = 1$, the AMLM method while $m = 2$ and the NMLM method while $m = 3$ respectively.

We will organize the rest of this paper as follow: In Section 2, we first give the new modified Levenberg-Marquardt method which is called efficient m-step Levenberg-Marquardt algorithm. In Section 3, we derive the global convergence of the new algorithm by using trust region technique. Then we derive the convergence order of the algorithm under the local error bound condition in Section 4. Finally, some numerical results of the new algorithm are given in Section 5.

2 The efficient m-step Levenberg-Marquardt algorithm

In this section, we first present the efficient m-step Levenberg-Marquardt algorithm by using trust region technique, then prove the global convergence.

2.1 The motivation

We take

$$\Phi(x) = \|F(x)\|^2 \tag{14}$$

as the merit function for (1). It is easy to see that $d_{k,i}$ $(i = 0, \cdots, m-1)$ is not only the minimizer of the convex minimization problem

$$\min_{d \in \mathrm{R}^n} \|F_{k,i} + J_k d\|^2 + \lambda_k \|d\|^2 \triangleq \varphi_{k,i}(d), \tag{15}$$

but also a solution of the trust region problem

$$\begin{aligned} &\min_{d\in \mathrm{R}^n} \|F_{k,i} + J_k d\|^2 , \\ &s.t. \quad \|d\| \leqslant \Delta_{k,i}, \end{aligned} \tag{16}$$

where $\Delta_{k,i} = \|d_{k,i}\| = \left\| -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,i} \right\|$. From the result given by Powell in [13], we have

$$\|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 \geqslant \left\|J_k^T F_{k,i}\right\| \min \left\{ \|d_{k,i}\| , \frac{\left\|J_k^T F_{k,i}\right\|}{\left\|J_k^T J_k\right\|} \right\}. \tag{17}$$

Moreover, similar to Fan proposed in [11], if $d_{k,i}$ is a descent direction of the merit function $\Phi(x)$ at $x_{k,i}$, then more reduction of $\Phi(x)$ at $x_{k,i}$ could be expected. So we may perform many line searches at $x_{k,i}$ along $d_{k,i}$ by solving the problem

$$\min_{\alpha>0} \|F(x_{k,i} + \alpha d_{k,i})\|^2 .$$

By Taylor extension, replace $J(x_{k,i})$ with J_k for save Jacobian calculations, the above problem could be approximated by

$$\min_{\alpha>0} \|F(x_{k,i}) + \alpha J_k d_{k,i}\|^2 .$$

The above problem is equivalent to

$$\max_{\alpha>0} \|F_{k,i}\|^2 - \|F_{k,i} + \alpha J_k d_{k,i}\|^2 \triangleq \phi(\alpha), \tag{18}$$

where

$$\phi(\alpha) = -d_{k,i}^T J_k^T J_k d_{k,i} \alpha^2 + 2 d_{k,i}^T \left(J_k^T J_k + \lambda_k I\right) d_{k,i} \alpha$$

is a quadratic function of α, and attains its maximum at

$$\tilde{\alpha}_{k,i} = \frac{d_{k,i}^T \left(J_k^T J_k + \lambda_k I\right) d_{k,i}}{d_{k,i}^T J_k^T J_k d_{k,i}} = 1 + \frac{\lambda_k d_{k,i}^T d_{k,i}}{d_{k,i}^T J_k^T J_k d_{k,i}},$$

provided that $J_k d_{k,i} \neq 0$. We bound $\tilde{\alpha}_{k,i} \in [1, \hat{\alpha}]$ with $\hat{\alpha} > 1$ is a positive constant because of $\tilde{\alpha}_{k,i}$ may be very large if $J_k d_{k,i}$ is close to 0. The problem (18) now is equivalent to

$$\max_{\alpha\in[1,\hat{\alpha}]} \|F_{k,i}\|^2 - \|F_{k,i} + \alpha J_k d_{k,i}\|^2 \triangleq \phi(\alpha). \tag{19}$$

And we have

$$\|F_{k,i}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 \geqslant \|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 \tag{20}$$

2.2 The algorithm

Now, we define the actual reduction of $\Phi(x)$ at the kth iteration as

$$\mathrm{Ared}_k = \|F_k\|^2 - \|F(x_k + d_{k,0} + \alpha_{k,1} d_{k,1} + \cdots + \alpha_{k,m-1} d_{k,m-1})\|^2 . \tag{21}$$

where $d_{k,i}$ are computed by (9). Note that the predicted reduction cannot be defined as usual definition $\|F_k\|^2 - \|F_k + J_k (d_{k,0} + \alpha_{k,1} d_{k,1} + \cdots + \alpha_{k,m-1} d_{k,m-1})\|^2$, because it cannot be proven to be nonnegative, which is required for the global convergence in the trust region method. Hence, we define the new modified predicted reduction as

$$\mathrm{Pred}_k = \sum_{i=0}^{m-1} \left(\|F_{k,i}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 \right), \tag{22}$$

with $\alpha_{k,0} = 1$.

Lemma 2.1. *Let the predicted reduction is defined by (22), then*

$$\mathrm{Pred}_k \geqslant \left\|J_k^T F_{k,0}\right\| \min\left\{\|d_{k,0}\|, \frac{\left\|J_k^T F_{k,0}\right\|}{\left\|J_k^T J_k\right\|}\right\}, \tag{23}$$

where $m \geqslant 1$.

Proof. From (17) and (20), we have

$$\begin{aligned}
\mathrm{Pred} &= \sum_{i=0}^{m-1}\left(\|F_{k,i}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2\right) \\
&\geqslant \sum_{i=0}^{m-1}\left(\|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2\right) \\
&\geqslant \sum_{i=0}^{m-1}\left(\left\|J_k^T F_{k,i}\right\| \min\left\{\|d_{k,i}\|, \frac{\left\|J_k^T F_{k,i}\right\|}{\left\|J_k^T J_k\right\|}\right\}\right) \\
&\geqslant \left\|J_k^T F_{k,0}\right\| \min\left\{\|d_{k,0}\|, \frac{\left\|J_k^T F_{k,0}\right\|}{\left\|J_k^T J_k\right\|}\right\}.
\end{aligned}$$

Then (23) holds. The proof is completed. □

Now, we present the efficient m-step Levenberg-Marquardt algorithm.

Algorithm 2.2 (The efficient m-step Levenberg-Marquardt algorithm).
Input: Given $x_0 \in \mathrm{R}^n$, $\mu_1 > \mu > 0$, $0 < p_0 \leqslant p_1 \leqslant p_2 < 1$, $1 \leqslant \delta \leqslant 2$, $\varepsilon > 0$, $\hat{\alpha} > 1$ **and** $m \geqslant 1$.

Step 1. Set $x_{k,0} = x_k$, $d_{k,0} = d_k$ **and** $k := 0$.

Step 2. Compute $F_k = F_{k,0} = F(x_{k,0})$, $J_k = J(x_{k,0})$. **If** $\left\|J_k^T F_k\right\| < \varepsilon$, **then stop. Otherwise compute**

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,i} \quad \textbf{with} \quad \lambda_k = \mu_k \|F_k\|^\delta, \tag{24}$$

where $x_{k,i} = x_{k,i-1} + \alpha_{k,i-1} d_{k,i-1}$ **to obtain** $d_{k,i}$, $i = 0, 1, \cdots, m-1$. **Set**

$$s_k = \sum_{i=0}^{m-1} \alpha_{k,i} d_{k,i}, \tag{25}$$

where $\alpha_{k,0} = 1$, $\alpha_{k,i}$ $(i = 1, \cdots, m-1)$ **is the step size obtained by solving (19).**

Step 3. Compute $r_k = \mathrm{Ared}_k / \mathrm{Pred}_k$. **Set**

$$x_{k+1} = \begin{cases} x_k + s_k, & \textbf{if } r_k \geqslant p_0, \\ x_k, & \textbf{otherwise}. \end{cases} \tag{26}$$

Step 4. Update μ_{k+1} **as**

$$\mu_{k+1} = \begin{cases} 4\mu_k, & \textbf{if } r_k < p_1, \\ \mu_k, & \textbf{if } r_k \in [p_1, p_2], \\ \max\left\{\frac{\mu_k}{4}, \mu\right\}, & \textbf{if } r_k > p_2. \end{cases} \tag{27}$$

Step 5. Set $k = k+1$, **and go to Step 2.**

Remark 2.3. *(a) Notice that,* μ_k *should be no less than a positive constant* μ *to prevent the steps from being too large when the sequence* $\{x_k\}$ *is near the solution.*

(b) Fan set $\delta \in (0, 2]$ *in [11], but here, we still set* $\delta \in [1, 2]$ *as usual in [8–10, 12] for stable and preferable.*

3 The global convergence

To study the global convergence of Algorithm 2.2, we need the following assumptions.

Assumption 3.1. *Let $F(x)$ is continuously differentiable, and both $F(x)$ and its Jacobian $J(x)$ are Lipschitz continuous, i.e., there exist positive constant L_1 and L_2 such that*

$$\|J(y) - J(x)\| \leqslant L_1 \|y - x\|, \quad \forall x, y \in \mathrm{R}^n \tag{28}$$

and

$$\|F(y) - F(x)\| \leqslant L_2 \|y - x\|, \quad \forall x, y \in \mathrm{R}^n. \tag{29}$$

By the Lipschitzness of the Jacobian proposed by (28), we have

$$\|F(y) - F(x) - J(x)(y - x)\| \leqslant L_1 \|y - x\|^2, \quad \forall x, y \in \mathrm{R}^n. \tag{30}$$

Theorem 3.2. *Under the conditions of Assumption 3.1, Algorithm 2.2 will terminates in finite iterations or satisfies*

$$\lim_{k \to \infty} \left\|J_k^T F_k\right\| = 0. \tag{31}$$

Proof. By contradiction, suppose there exist a positive τ and infinite many k such that

$$\left\|J_k^T F_k\right\| \geqslant \tau. \tag{32}$$

Let T_1, T_2 be the sets of the indices as follow:

$$T_1 = \left\{k \mid \left\|J_k^T F_k\right\| \geqslant \tau\right\},$$
$$T_2 = \left\{k \mid \left\|J_k^T F_k\right\| \geqslant \frac{\tau}{2} \quad \text{and} \quad x_{k+1} \neq x_k\right\}.$$

It is easy to see that T_1 is infinite. In the following, we will derive the contradictions whether T_2 is finite or infinite.

Case 1: T_2 is finite. Then the set

$$T_3 = \left\{k \mid \left\|J_k^T F_k\right\| \geqslant \tau \quad \text{and} \quad x_{k+1} \neq x_k\right\}$$

is also finite. Let $\tilde{k}$ be the largest index of T_3. Then it is easy to see that $x_{k+1} = x_k$ holds for all $k \in \left\{k > \tilde{k} \mid k \in T_1\right\}$. Define the indices set

$$T_4 = \left\{k > \tilde{k} \mid \left\|J_k^T F_k\right\| \geqslant \tau \quad \text{and} \quad x_{k+1} = x_k\right\}.$$

If $k \in T_4$, we can deduce that $\left\|J_{k+1}^T F_{k+1}\right\| \geqslant \tau$ and $x_{k+2} = x_{k+1}$. Hence, we have $x_{k+1} \in T_4$. By induction, we know that $\left\|J_k^T F_k\right\| \geqslant \tau$ and $x_{k+1} = x_k$ hold for all $k > \tilde{k}$, which means $r_k < p_0$. Now, we obtain

$$\lambda_k \to +\infty \quad \text{and} \quad \mu_k \to +\infty \tag{33}$$

and, due to (24), (25) and (27),

$$d_{k,0} = \left\|-\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,0}\right\| \to 0.$$

Moreover, it follows from (29) and (30) that

$$\begin{aligned}\|d_{k,i}\| &= \left\|-\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,i}\right\| \\ &\leqslant \left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,0}\right\| + \left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T J_k \sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j}\right\| \\ &\quad + L_1 \left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T\right\| \left\|\sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j}\right\|^2 \\ &\leqslant \|d_{k,0}\| + \left\|\sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j}\right\| + \frac{L_1 L_2}{\lambda_k} \left\|\sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j}\right\|^2 \\ &\leqslant \|d_{k,0}\| + \sum_{j=0}^{i-1} \alpha_{k,j} \|d_{k,j}\| + \frac{L_1 L_2}{\lambda_k} \left(\sum_{j=0}^{i-1} \alpha_{k,j} \|d_{k,j}\|\right)^2\end{aligned}$$

with $i = 1, \cdots, m-1$. Hence, by induction, we obtain

$$\|d_{k,i}\| \leqslant O\left(\|d_{k,0}\|\right). \tag{34}$$

Note that

$$\begin{aligned}&\|F_{k,i+1}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 \\ &\quad = \left(\|F_{k,i+1}\| + \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|\right)\left(\|F_{k,i+1}\| - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|\right) \\ &\quad \leqslant \left(2\left\|F_{k,0} + J_k \sum_{j=0}^{i} \alpha_{k,j} d_{k,j}\right\| + L_1 \left\|\sum_{j=0}^{i} \alpha_{k,j} d_{k,j}\right\|^2 + L_1 \left\|\sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j}\right\|^2\right) \\ &\qquad \times \left(L_1 \left\|\sum_{j=0}^{i} \alpha_{k,j} d_{k,j}\right\|^2 + L_1 \left\|\sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j}\right\|^2\right)\end{aligned} \tag{35}$$

with $i = 0, 1, \cdots, m-1$. It's clear that while $i = 0$ and $\alpha_{k,0} = 1$,

$$\|F_{k,1}\|^2 - \|F_{k,0} + J_k d_{k,0}\|^2 \leqslant 2L_1 \|F_{k,0} + J_k d_{k,0}\| \|d_{k,0}\|^2 + L_1^2 \|d_{k,0}\|^4 = O\left(\|d_{k,0}\|^2\right).$$

It follows from (21), (22), (29), (35) and Lemma 2.1 that

$$\begin{aligned}|r_k - 1| &= \left|\frac{\mathrm{Ared}_k - \mathrm{Pred}_k}{\mathrm{Pred}_k}\right| \\ &\leqslant \left|\frac{\sum_{i=0}^{m-1}\left(\|F_{k,i+1}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2\right)}{\sum_{i=0}^{m-1}\left(\|F_{k,i}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2\right)}\right| \\ &\leqslant \frac{O\left(\|d_{k,0}\|^2\right)}{\left\|J_k^T F_{k,0}\right\| \min\left\{\|d_{k,0}\|, \frac{\|J_k^T F_{k,0}\|}{\|J_k^T J_k\|}\right\}} \to 0,\end{aligned}$$

which implies that $r_k \to 1$. In view of the updating rule of μ_k, we know that there exists a positive constant $\bar{\mu} > \mu$ such that $\mu_k < \bar{\mu}$ holds for all sufficiently large k, which is a contradiction to (33) .

Case 2: T_2 is infinite. It follows from (23) and (29) that

$$\begin{aligned}\|F_1\|^2 &\geqslant \sum_k \left(\|F_k\|^2 - \|F_{k+1}\|^2\right) \geqslant \sum_{k\in T_2} \left(\|F_k\|^2 - \|F_{k+1}\|^2\right) \\ &\geqslant \sum_{k\in T_2} p_0 \mathrm{Pred}_k \geqslant \sum_{k\in T_2} p_0 \left\|J_k^T F_{k,0}\right\| \min\left\{\|d_{k,0}\|, \frac{\left\|J_k^T F_{k,0}\right\|}{\left\|J_k^T J_k\right\|}\right\} \\ &\geqslant \sum_{k\in T_2} \frac{p_0\tau}{2} \min\left\{\|d_{k,0}\|, \frac{\tau}{2L_2^2}\right\}. \end{aligned} \tag{36}$$

which implies

$$\|d_{k,0}\| \to 0, \quad k \in T_2. \tag{37}$$

Then by the definition of $d_{k,0}$, we have

$$\mu_k \to +\infty, \quad k \in T_2. \tag{38}$$

Moreover, it follows from (28), (29), (34) and (36) that

$$\begin{aligned}&\sum_{k\in T_2} \left|\left\|J_k^T F_k\right\| - \left\|J_{k+1}^T F_{k+1}\right\|\right| \\ &\quad\leqslant \sum_{k\in T_2} \left|\left(\left\|J_k^T F_k\right\| - \left\|J_k^T F_{k+1}\right\|\right) - \left(\left\|J_{k+1}^T F_{k+1}\right\| - \left\|J_k^T F_{k+1}\right\|\right)\right| \\ &\quad\leqslant \sum_{k\in T_2} \left|L_2 \left\|J_k^T\right\| \|s_k\| - L_1 \|F_{k+1}\| \|s_k\|\right| \\ &\quad\leqslant L_1 L_2 \dot{c} \sum_{k\in T_2} \|d_{k,0}\| < +\infty,\end{aligned}$$

with some constants $\dot{c} > 0$, which together with (32) implies there exists a sufficiently large $\hat{k}$ such that

$$\left\|J_k^T F_k\right\| \geqslant \tau \quad \text{and} \quad \sum_{k\in T_2} \left|\left\|J_k^T F_k\right\| - \left\|J_{k+1}^T F_{k+1}\right\|\right| < \frac{\tau}{2}.$$

Hence we can derive that $\left\|J_k^T F_k\right\| \geqslant \frac{\tau}{2}$ for all $k \geqslant \hat{k}$. Combining (37) with (38), we have

$$\|d_{k,0}\| \to 0 \quad \text{and} \quad \mu_k \to +\infty. \tag{39}$$

In the same way as proved in Case 1, we can also obtain that

$$r_k \to 1.$$

Hence, there exists a positive constant $\bar{\mu}$ such that $\mu_k < \bar{\mu}$ holds for all sufficiently large k, which is contradicted to (39). The proof is completed. □

4 The local convergence

In this section, we assume that $x_k \to x^* \in X^*$ and the sequence $\{x_k\}$ lies on some neighbourhood of x^*, i.e., there exist a positive constant $b_1 < 1$ such that $x \in N(x^*, b_1)$. We give some assumptions which the local convergence theory required.

Assumption 4.1. *(a) $F(x)$ is continuously differentiable, and Jacobian $J(x)$ is Lipschitz continuous on $N(x^*, b_1)$, i.e., there exist a positive constant L_1 such that*

$$\|J(y) - J(x)\| \leqslant L_1 \|y - x\|, \quad \forall x, y \in N(x^*, b_1) = \{x \mid \|x - x^*\| \leqslant b_1\}. \tag{40}$$

(b) $\|F(x)\|$ *provides a local error bound on some neighborhood of* $x^* \in X^*$*, i.e., there exist a positive constant* $c > 0$ *such that*

$$\|F(x)\| \geqslant c \operatorname{dist}(x, X^*), \quad \forall x \in N(x^*, b_1). \tag{41}$$

Since the condition of nonsingularity of $J(x)$ is too strong, the Assumption 4.1 (b) provides a weak local error bound condition, which implies that the converse is not necessarily true [4].

By (40), we have

$$\|F(y) - F(x) - J(x)(y - x)\| \leqslant L_1 \|y - x\|^2, \quad \forall x, y \in N(x^*, b_1), \tag{42}$$

and

$$\|F(y) - F(x)\| \leqslant L_2 \|y - x\|, \quad \forall x, y \in N(x^*, b_1), \tag{43}$$

where L_2 is a positive constant.

There exists a positive constant $\omega > 0$ if $F(x)$ provides a local error bound which proposed by Behling and Iusem in [14], then

$$\operatorname{rank}(J(\tilde{x})) = \operatorname{rank}(J(x^*)), \quad \forall \tilde{x} \in N(x^*, \omega) \cap X^*.$$

Let $b \in (0, 1)$ and $b_1 = \min\{\omega, b\}$. Without loss of generality, we further assume that $x_{k,i}$, $i = 0, 1, \cdots, m-1$ lie in $N\left(x^*, \frac{b_1}{2}\right)$.

In the following, we denote $\bar{x}_k \in X^*$ such that

$$\|\bar{x}_k - x_k\| = \operatorname{dist}(x_k, X^*) = \inf_{y \in X^*} \|y - x_k\|.$$

Hence, we have

$$\|\bar{x}_k - x^*\| \leqslant \|\bar{x}_k - x_k\| \leqslant \| + \|x_k - x^*\| \leqslant 2\|x_k - x^*\| \leqslant b_1,$$

which implies that $\bar{x}_k \in N(x^*, b_1)$.

Lemma 4.2. *Let Assumption 4.1 hold, then*

$$\left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{-\frac{\delta}{2}}\right). \tag{44}$$

Proof. Suppose $\operatorname{rank}(J(\bar{x}_k)) = r$ for all $\bar{x}_k \in N(x^*, b_1) \cap X^*$ and the SVD of $J(\bar{x}_k)$ is

$$J(\bar{x}_k) = \bar{U}_k \bar{\Sigma}_k \bar{V}_k^T = \left(\bar{U}_{k,1}, \bar{U}_{k,2}\right) \begin{pmatrix} \bar{\Sigma}_{k,1} & \\ & 0 \end{pmatrix} \begin{pmatrix} \bar{V}_{k,1}^T \\ \bar{V}_{k,2}^T \end{pmatrix} = \bar{U}_{k,1} \bar{\Sigma}_{k,1} \bar{V}_{k,1}^T,$$

where $\bar{\Sigma}_{k,1} = \operatorname{diag}(\bar{\sigma}_{k,1}, \bar{\sigma}_{k,2}, \cdots, \bar{\sigma}_{k,r})$ with $\bar{\sigma}_{k,1} \geqslant \bar{\sigma}_{k,2} \geqslant \cdots \geqslant \bar{\sigma}_{k,r} > 0$. The corresponding SVD of J_k is

$$\begin{aligned} J_k =& U_k \Sigma_k V_k^T = (U_{k,1}, U_{k,2}, U_{k,3}) \begin{pmatrix} \Sigma_{k,1} & & \\ & \Sigma_{k,2} & \\ & & 0 \end{pmatrix} \begin{pmatrix} V_{k,1}^T \\ V_{k,2}^T \\ V_{k,3}^T \end{pmatrix} \\ =& U_{k,1} \Sigma_{k,1} V_{k,1}^T + U_{k,2} \Sigma_{k,2} V_{k,2}^T, \end{aligned}$$

where $\Sigma_{k,1} = \operatorname{diag}(\sigma_{k,1}, \sigma_{k,2}, \cdots, \sigma_{k,r})$ with $\sigma_{k,1} \geqslant \sigma_{k,2} \geqslant \cdots \geqslant \sigma_{k,r} > 0$, and $\Sigma_{k,2} = \operatorname{diag}(\sigma_{k,r+1}, \sigma_{k,r+2}, \cdots, \sigma_{k,r+q})$ with $\sigma_{k,r+1} \geqslant \sigma_{k,r+2} \geqslant \cdots \geqslant \sigma_{k,r+q} > 0$. We will neglect the subscript k if the context is clear in the following, and write J_k as

$$J_k = U_1 \Sigma_1 V_1^T + U_2 \Sigma_2 V_2^T. \tag{45}$$

By the theory of matrix perturbation [15] and the Lipschitzness of J_k, we have

$$\left\|\operatorname{diag}\left(\Sigma_1 - \bar{\Sigma}_1, \Sigma_2, 0\right)\right\| \leqslant \left\|J_k - \bar{J}_k\right\| \leqslant L_1 \|\bar{x}_k - x_k\|, \tag{46}$$

which yields

$$\left\|\Sigma_1 - \bar{\Sigma}_1\right\| \leqslant L_1 \|\bar{x}_k - x_k\| \quad \text{and} \quad \|\Sigma_2\| \leqslant L_1 \|\bar{x}_k - x_k\|. \tag{47}$$

Hence

$$\left\|\lambda_k^{-1} \Sigma_2\right\| = \frac{\|\Sigma_2\|}{\mu_k \|F_k\|^\delta} \leqslant \frac{L_1 \|\bar{x}_k - x_k\|}{m c^\delta \|\bar{x}_k - x_k\|^\delta} = L_1 m^{-1} c^{-\delta} \|\bar{x}_k - x_k\|^{1-\delta} \tag{48}$$

Since for any positive σ_i $(i = 1, 2, \cdots, r)$, we have

$$\frac{\sigma_i}{\sigma_i^2 + \lambda_k} \leqslant \frac{\sigma_i}{2\sigma_i\sqrt{\lambda_k}} = \frac{1}{2\sqrt{\lambda_k}},$$

which implies

$$\left\|\left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1\right\| \leqslant \frac{1}{2\sqrt{\mu_k \left\|F_k\right\|^{\delta}}} \leqslant \frac{1}{2} m^{-\frac{1}{2}} c^{-\frac{\delta}{2}} \left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}}. \tag{49}$$

Combining (48) and (49) with $\delta \in [1, 2]$, we have

$$\begin{aligned}
\left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T\right\| &= \left\|(V_1, V_2, V_3) \begin{pmatrix} \left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 & & \\ & \left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} U_1^T \\ U_2^T \\ U_3^T \end{pmatrix}\right\| \\
&\leqslant \left\|\begin{pmatrix} \left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 & & \\ & \left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 & \\ & & 0 \end{pmatrix}\right\| \\
&\leqslant \left\|\left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1\right\| + \left\|\lambda_k^{-1} \Sigma_2\right\| \\
&\leqslant \frac{1}{2} m^{-\frac{1}{2}} c^{-\frac{\delta}{2}} \left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}} + L_1 m^{-1} c^{-\delta} \left\|\bar{x}_k - x_k\right\|^{1-\delta} \\
&\leqslant O\left(\left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}}\right).
\end{aligned}$$

The proof is completed. □

4.1 Properties of the trial step

Firstly, we investigate the properties of $d_{k,i}$, and hence s_k.

Lemma 4.3. *Under the condition of Assumption 4.1, for sufficiently large k, we have*

$$\left\|d_{k,i}\right\| \leqslant c_i \operatorname{dist}\left(x_k, X^*\right), \quad i = 0, 1, \cdots, m-1,$$

where c_i are some positive constants.

Proof. The proof of $d_{k,0}$ can be found in Lemma 1 of [11], thus

$$\left\|d_{k,0}\right\| \leqslant c_0 \operatorname{dist}\left(x_k, X^*\right). \tag{50}$$

Now we prove $i \geqslant 1$. From (24), (42), (44) and (50), we obtain

$$\begin{aligned}
\left\|d_{k,i}\right\| &= \left\|-\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,i}\right\| \\
&\leqslant \left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,0}\right\| + \left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T J_k \sum_{j=0}^{i-1} \alpha_{k,i} d_{k,i}\right\| \\
&\quad + L_1 \left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T\right\| \left\|\sum_{j=0}^{i-1} \alpha_{k,i} d_{k,i}\right\|^2 \\
&\leqslant \left\|d_{k,0}\right\| + \sum_{j=0}^{i-1} \alpha_{k,i} \left\|d_{k,i}\right\| + L_1 \left(\sum_{j=0}^{i-1} \alpha_{k,i} \|d_{k,i}\|\right)^2 O\left(\left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}}\right) \\
&\leqslant c_i \operatorname{dist}\left(x_k, X^*\right),
\end{aligned}$$

with $i = 1, \cdots, m-1$, for some positive constant c_i. The proof is completed. □

Lemma 4.3 indicates that the trail step

$$\|s_k\| = \left\| \sum_{i=0}^{m-1} \alpha_{k,i} d_{k,i} \right\| \leqslant \sum_{i=0}^{m-1} \alpha_{k,i} \|d_{k,i}\| \leqslant \ddot{c} \operatorname{dist}(x_k, X^*),$$

for some positive constants $\ddot{c}$.

4.2 Boundedness of the LM parameter

Lemma 4.4. *Under the conditions of Assumption 4.1, there exists a positive $\bar{\mu} > \mu$ such that $\mu_k \leqslant \bar{\mu}$ holds for all sufficiently large k.*

Proof. Following the result given in [10, Lemma 2], we have the following inequalities for all sufficiently large k,

$$\|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 \geqslant \bar{c}_i \|F_{k,i}\| \min\{\|d_{k,i}\|, \|\bar{x}_{k,i} - x_{k,i}\|\},$$

where $\bar{c}_i$ are some positive constants, $i = 0, 1, \cdots, m-1$.

In fact, if $\|\bar{x}_{k,i} - x_{k,i}\| \leqslant \|d_{k,i}\|$, by (41), (42) and the fact that $d_{k,i}$ is the solution of (16), we have

$$\begin{aligned}
&\|F(x_{k,i})\| - \|F_{k,i} + J_k d_{k,i}\| \\
&\qquad \geqslant \|F_{k,i}\| - \|F_{k,i} + J_k (\bar{x}_{k,i} - x_{k,i})\| \\
&\qquad \geqslant \|F_{k,i}\| - \|F_{k,i} + J_{k,i} (\bar{x}_{k,i} - x_{k,i})\| - \|J_k - J_{k,i}\| \|\bar{x}_{k,i} - x_{k,i}\| \\
&\qquad \geqslant c \|\bar{x}_{k,i} - x_{k,i}\| - L_1 \|\bar{x}_{k,i} - x_{k,i}\|^2 - L_1 \|\bar{x}_{k,i} - x_{k,i}\| \sum_{j=0}^{i-1} \alpha_{k,j} \|d_{k,j}\| \\
&\qquad \geqslant \bar{c}_i \|\bar{x}_{k,i} - x_{k,i}\|,
\end{aligned} \tag{51}$$

for some $\bar{c}_i > 0$ when k is sufficiently large. In the other case when $\|\bar{x}_{k,i} - x_{k,i}\| > \|d_{k,i}\|$, we have

$$\begin{aligned}
\|F_{k,i}\| - \|F_{k,i} + J_k d_{k,i}\| &\geqslant \|F_{k,i}\| - \left\| F_{k,i} + \frac{\|d_{k,i}\|}{\|\bar{x}_{k,i} - x_{k,i}\|} J_k (\bar{x}_{k,i} - x_{k,i}) \right\| \\
&\geqslant \frac{\|d_{k,i}\|}{\|\bar{x}_{k,i} - x_{k,i}\|} \left(\|F_{k,i}\| - \|F_{k,i} + J_k (\bar{x}_{k,i} - x_{k,i})\|\right) \\
&\geqslant \frac{\|d_{k,i}\|}{\|\bar{x}_{k,i} - x_{k,i}\|} \bar{c}_i \|\bar{x}_{k,i} - x_{k,i}\| \\
&\geqslant \bar{c}_i \|d_{k,i}\|.
\end{aligned} \tag{52}$$

Combining (51) with (52), we obtain

$$\begin{aligned}
\|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 &= (\|F_{k,i}\| + \|F_{k,i} + J_k d_{k,i}\|)(\|F_{k,i}\| - \|F_{k,i} + J_k d_{k,i}\|) \\
&\geqslant \bar{c}_i \|F_{k,i}\| \min\{\|d_{k,i}\|, \|\bar{x}_{k,i} - x_{k,i}\|\}.
\end{aligned}$$

Together with (20), we have

$$\begin{aligned}
\|F_{k,i}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 &\geqslant \|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 \\
&\geqslant \bar{c}_i \|F_{k,i}\| \min\{\|d_{k,i}\|, \|\bar{x}_{k,i} - x_{k,i}\|\}.
\end{aligned} \tag{53}$$

Hence, it follows from (22) and Lemma 4.3, we have

$$\operatorname{Pred}_k \geqslant O(\|\bar{x}_k - x_k\| \|d_{k,0}\|).$$

Since $d_{k,0}$ is a minimizer of (15), we have the following results from (43) and Lemma 4.3 that

$$\begin{aligned}
&\|F_{k,0} + J_k (\alpha_{k,0} d_{k,0} + \cdots + \alpha_{k,i} d_{k,i})\| \\
&\qquad \leqslant \|F_{k,0} + \alpha_{k,0} J_k d_{k,0}\| + \|J_k\| (\alpha_{k,1} \|d_{k,1}\| + \cdots + \alpha_{k,i} \|d_{k,i}\|) \\
&\qquad \leqslant \tilde{c}_i \|\bar{x}_k - x_k\|,
\end{aligned}$$

with $i = 1, \cdots, m-1$ for some positive constants $\tilde{c}_i > 0$. Also, follows from (35), we have

$$\|F_{k,i+1}\|^2 - \|F(x_{k,i}) + \alpha_{k,i} J_k d_{k,i}\|^2 \leqslant O\left(\|\bar{x}_k - x_k\| \|d_{k,0}\|^2\right)$$

which implies that

$$|r_k - 1| = \left|\frac{\text{Ared}_k - \text{Pred}_k}{\text{Pred}_k}\right| \leqslant \frac{O\left(\|\bar{x}_k - x_k\| \|d_{k,0}\|^2\right)}{O\left(\|\bar{x}_k - x_k\| \|d_{k,0}\|\right)} \to 0$$

holds for sufficiently large k. Hence

$$r_k \to 1.$$

Therefore there exists a positive $\bar{\mu} > \mu$ such that $\mu_k \leqslant \bar{\mu}$ holds for all sufficiently large k. The proof is completed. □

4.3 Convergence order of m-step Levenberg-Marquardt algorithm

We now prove the convergence order of m-step LM algorithm based on the results obtained in the above two subsections.

By the SVD of J_k proposed in (45), we have

$$d_{k,i} = -V_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 U_1^T F_{k,i} - V_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 U_2^T F_{k,i}, \tag{54}$$

$$\begin{aligned} &F(x_{k,i}) + J_k d_{k,i} \\ &= F_{k,i} - U_1\Sigma_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 U_1^T F_{k,i} - U_2\Sigma_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 U_2^T F_{k,i} \\ &= \lambda_k U_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} U_1^T F_{k,i} + \lambda_k U_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} U_2^T F_{k,i} + U_3 U_3^T F_{k,i}, \end{aligned} \tag{55}$$

with $i = 0, \cdots, m-1$.

Lemma 4.5. *Under the condition of Assumption 4.1, if $x_{k,i} \in N(x^*, b_1/2)$, then we have*

(a) $\left\|U_1 U_1^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+1}\right)$;

(b) $\left\|U_2 U_2^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+2}\right)$;

(c) $\left\|U_3 U_3^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+2}\right)$;

with $i = 0, \cdots, m-1$.

Proof. We will prove this lemma by an induction process.

For $i = 1, 2$, the results have been shown by Fan and Chen respectively (see [11, 12]), and we have

$$\begin{aligned} &\|d_{k,1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^2\right), \quad \|F_{k,1} + J_k d_{k,1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^3\right), \\ &\|d_{k,2}\| \leqslant O\left(\|\bar{x}_k - x_k\|^3\right), \quad \|F_{k,2} + J_k d_{k,2}\| \leqslant O\left(\|\bar{x}_k - x_k\|^4\right). \end{aligned}$$

Assuming the truth for some $i-1$, we obtain the induction hypothesis:

$$\|d_{k,i-1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^i\right), \quad \|F(x_{k,i-1}) + J_k d_{k,i-1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+1}\right).$$

Turning now to the case for i. It follows from above induction hypothesis that

$$\begin{aligned}\|F_{k,i}\| &= \|F(x_{k,i-1}+\alpha_{k,i-1}d_{k,i-1})\| \\ &\leqslant \|F_{k,i-1}+\alpha_{k,i-1}J_{k,i-1}d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\ &\leqslant \|F_{k,i-1}+J_{k,i-1}d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\ &\leqslant \|F_{k,i-1}+J_kd_{k,i-1}\| + \|J_{k,i-1}-J_k\|\|d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\ &\leqslant \|F_{k,i-1}+J_kd_{k,i-1}\| + L_1\left\|\sum_{j=0}^{i-1}\alpha_{k,j}d_{k,j}\right\|\|d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\ &\leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right) + L_1\|\bar{x}_k-x_k\|O\left(\|\bar{x}_k-x_k\|^{i}\right) \\ &\quad + L_1\alpha_{k,i-1}^2O\left(\|\bar{x}_k-x_k\|^{2i}\right) \\ &\leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right).\end{aligned}$$

So, we have

$$\left\|U_1U_1^TF_{k,i}\right\| \leqslant \|F_{k,i}\| \leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right).$$

Moreover, the local error bound condition implies that

$$\|\bar{x}_{k,i}-x_{k,i}\| \leqslant c^{-1}\|F_{k,i}\| \leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right). \tag{56}$$

Let $\bar{q}_k = -J_k^+F_{k,i}$. Then $\bar{q}_k$ is the least squares solution of $\|\min F_{k,i}+J_kq\|$. It follows from (40), (42), (56) and Lemma 4.3 that

$$\begin{aligned}\left\|U_3U_3^TF_{k,i}\right\| &= \|F_{k,i}+J_k\bar{q}_k\| \leqslant \|F_{k,i}+J_k(\bar{x}_{k,i}-x_{k,i})\| \\ &\leqslant \|F_{k,i}+J_{k,i}(\bar{x}_{k,i}-x_{k,i})\| + \|(J_{k,i}-J_k)(\bar{x}_{k,i}-x_{k,i})\| \\ &\leqslant L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + L_1\left\|\sum_{j=0}^{i-1}\alpha_{k,j}d_{k,j}\right\|\|\bar{x}_{k,i}-x_{k,i}\| \\ &\leqslant O\left(\|\bar{x}_k-x_k\|^{2i+2}\right) + O\left(\|\bar{x}_k-x_k\|^{i+2}\right) \\ &= O\left(\|\bar{x}_k-x_k\|^{i+2}\right).\end{aligned} \tag{57}$$

Let $\tilde{J}_k = U_1\Sigma_1V_1^T$ and $\tilde{q}_k = -\tilde{J}_k^+F_{k,i}$. Since $\tilde{q}_k$ is the least squares solution of $\left\|\min F_{k,i}+\tilde{J}_kq\right\|$, deducing from (40), (42),(47), (56) and Lemma 4.3 that

$$\begin{aligned}&\left\|\left(U_2U_2^T+U_3U_3^T\right)F_{k,i}\right\| \\ =&\left\|F_{k,i}+\tilde{J}_k\tilde{q}_k\right\| \leqslant \left\|F_{k,i}+\tilde{J}_k(\bar{x}_{k,i}-x_{k,i})\right\| \\ \leqslant&\|F_{k,i}+J_{k,i}(\bar{x}_{k,i}-x_{k,i})\| + \left\|\left(\tilde{J}_k-J_{k,i}\right)(\bar{x}_{k,i}-x_{k,i})\right\| \\ \leqslant& L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + \left\|\left(J_k-J_{k,i}-U_2\Sigma_2V_2^T\right)(\bar{x}_{k,i}-x_{k,i})\right\| \\ \leqslant& L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + \|(J_k-J_{k,i})(\bar{x}_{k,i}-x_{k,i})\| + \left\|U_2\Sigma_2V_2^T(\bar{x}_{k,i}-x_{k,i})\right\| \\ \leqslant& L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + L_1\left\|\sum_{j=0}^{i-1}\alpha_{k,j}d_{k,j}\right\|\|\bar{x}_{k,i}-x_{k,i}\| + L_1\|\bar{x}_k-x_k\|\|\bar{x}_{k,i}-x_{k,i}\| \\ \leqslant& O\left(\|\bar{x}_k-x_k\|^{2i+2}\right) + O\left(\|\bar{x}_k-x_k\|^{i+2}\right) + O\left(\|\bar{x}_k-x_k\|^{i+2}\right) \\ \leqslant& O\left(\|\bar{x}_k-x_k\|^{i+2}\right).\end{aligned} \tag{58}$$

Due to the orthogonality of U_2 and U_3, combining (57) and (58), we know that

$$\left\|U_2 U_2^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+2}\right).$$

The proof is completed. □

Now, we are ready to give the estimations of $d_{k,m-1}$ and $\|F(x_{k,m-1}) + J_k d_{k,m-1}\|$.

Lemma 4.6. *Under the condition of Assumption 4.1, for sufficiently large k, we have*

(a) $\|d_{k,m-1}\| \leqslant O(\|\bar{x}_k - x_k\|^m)$*;*

(b) $\|F(x_{k,m-1}) + J_k d_{k,m-1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right)$.

Proof. By (46), we have

$$\|\Sigma_1\|^{-1} = \left|\frac{1}{\sigma_r}\right| \leqslant \left|\frac{1}{\bar{\sigma}_r - L_1\|\bar{x}_k - x_k\|}\right|,$$

which implies

$$\|\Sigma_1\|^{-1} \leqslant \frac{2}{\bar{\sigma}_r}.$$

When $\delta \in [1,2]$, it then follows from Lemma 4.4, Lemma 4.5, (48), (54) and (55) that

$$\begin{aligned}
\|d_{k,m-1}\| &= \left\|-V_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1}\Sigma_1 U_1^T F(x_{k,m-1}) - V_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1}\Sigma_2 U_2^T F(x_{k,m-1})\right\| \\
&\leqslant \|\Sigma_1\|^{-1}\left\|U_1^T F(x_{k,m-1})\right\| + \left\|\lambda_k^{-1}\Sigma_2\right\|\left\|U_2^T F(x_{k,m-1})\right\| \\
&\leqslant O(\|\bar{x}_k - x_k\|^m) + O\left(\|\bar{x}_k - x_k\|^{m+2-\delta}\right) \\
&= O(\|\bar{x}_k - x_k\|^m),
\end{aligned}$$

and

$$\begin{aligned}
&\|F(x_{k,m-1}) + J_k d_{k,m-1}\| \\
&= \left\|\lambda_k U_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} U_1^T F(x_{k,m-1}) + \lambda_k U_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} U_2^T F(x_{k,m-1}) + U_3 U_3^T F(x_{k,m-1})\right\| \\
&\leqslant \lambda_k\left\|\Sigma_1^2\right\|^{-1}\left\|U_1^T F(x_{k,m-1})\right\| + \left\|U_2^T F(x_{k,m-1})\right\| + \left\|U_3^T F(x_{k,m-1})\right\| \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+\delta}\right) + O\left(\|\bar{x}_k - x_k\|^{m+1}\right) + O\left(\|\bar{x}_k - x_k\|^{m+1}\right) \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right).
\end{aligned}$$

The proof is completed. □

Based on the results above, we obtain the convergence rate of Algorithm 2.2.

Theorem 4.7. *Under the conditions of Assumptions 4.1, the convergence rate of Algorithm 2.2 is $(m+1)$th.*

Proof. It follows from Lemma 4.3 and Lemma 4.6 that

$$\begin{aligned}
&c\|\bar{x}_{k+1} - x_{k+1}\| \\
&\leqslant \|F(x_{k+1})\| = \|F(x_k + s_k)\| = \|F(x_{k,m-1} + \alpha_{k,m-1}d_{k,m-1})\| \\
&\leqslant \|F(x_{k,m-1}) + \alpha_{k,m-1}J(x_{k,m-1})d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2 \\
&\leqslant \|F(x_{k,m-1}) + J(x_{k,m-1})d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2 \\
&\leqslant \|F(x_{k,m-1}) + J_k d_{k,m-1}\| + \|(J(x_{k,m-1}) - J_k)d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2 \\
&\leqslant \|F(x_{k,m-1}) + J_k d_{k,m-1}\| + L_1\left\|\sum_{j=0}^{m-2}\alpha_{k,j}d_{k,j}\right\|\|d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|F(x_{k,m-1}) + J_k d_{k,m-1}\| + L_1 \sum_{j=0}^{m-2} \alpha_{k,j} \|d_{k,j}\| \|d_{k,m-1}\| + L_1 \alpha_{k,m-1}^2 \|d_{k,m-1}\|^2 \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right) + O\left(\|\bar{x}_k - x_k\|^{m+1}\right) + O\left(\|\bar{x}_k - x_k\|^{2m}\right) \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right),
\end{aligned}
$$

with $m \geqslant 1$. Hence we have

$$
\|\bar{x}_{k+1} - x_{k+1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right), \tag{59}
$$

which means that $\{x_k\}$ generated by m-step LM method converges to the solution set X^* with $(m+1)$th order. The proof is completed. □

Since

$$
\|\bar{x}_k - x_k\| \leqslant \|\bar{x}_{k+1} - x_{k+1}\| + \|s_k\|,
$$

we obtain from (59) that

$$
\|\bar{x}_k - x_k\| \leqslant 2\|s_k\|
$$

holds for sufficiently large k. By Lemma 4.3, we finally have

$$
\|s_{k+1}\| \leqslant O\left(\|s_k\|^{m+1}\right),
$$

which indicates that $\{x_k\}$ converges to some solution of (1) with Q-order $m+1$. This result is stronger than the convergence to the solution set.

5 Numerical results

We will compute some singular problems, which come from [16] with the same forms as in [17], to test Algorithm 2.2, and compare it with the general LM algorithm (LM), the SLM method which has presented in [10] with $m = 4$.

We compute these test problems with different initial points and different size,

$$
\hat{F}(x) = F(x) - J(x^*) A \left(A^T A\right)^{-1} A^T (x - x^*),
$$

where $F(x)$ is the standard nonsingular test function, x^* is its root, and $A \in R^{n\times k}$ has full column rank with $1 \leqslant k \leqslant n$. Obviously, $\hat{F}(x^*) = 0$ and

$$
\hat{J}(x^*) = J(x^*)\left(I - A\left(A^T A\right)^{-1} A^T\right)
$$

has rank $n-k$. A disadvantage of these problems is that $\hat{F}(x)$ may have roots that are not roots of $F(x)$. We chose the rank of $\hat{J}(x^*)$ to be $n-1$ and $n-2$, respectively, by using

$$
A \in R^{n\times 1}, \quad A^T = (1, 1, \cdots, 1)
$$

and

$$
A \in R^{n\times 2}, \quad A^T = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & -1 & \cdots & \pm 1 \end{pmatrix}.
$$

Set $p_0 = 0.0001$, $p_1 = 0.25$, $p_2 = 0.75$, $\tilde{m} = 10^{-8}$, $\mu_1 = 1$, $\delta = 1$ for all the tests. The stopping criteria for the Algorithm is $\left\|J_k^T F_k\right\| < 10^{-5}$ or the iteration number exceeds $100(n+1)$. The points x_0, $10x_0$, $100x_0$ in the third column of the tables are the starting points, where x_0 was suggested by Moré et. al in [16]. "NF" and "NJ" represent the number of function calculations and Jacobian calculations, respectively. If the method failed to find the solution in $100(n+1)$ iterations, we denoted it by the sign "-", and if the iterations had underflows or overflows, we denoted it by "OF". We also denote "TIME" represents the running time of the problem. All codes are written in MATLAB R2012 programming environment on a personal PC with Inter(R) Core(TM) i5-4590 CPU, 3.30GHz, 4GB RAM, using Windows 7 operation system.

Table 1: Results on the first singular test set with rank($F'(x^*)$)= $n-1$

			Algorith LM	Algorithm SLM with $m=4$	Algorithm 2.2 with $m=4$
Problem	n	x_0	NF/NJ/F/TIME	NF/NJ/F/TIME	NF/NJ/F/TIME
8	3000	1	9/9/1.7993e-05/16.0927	17/5/6.0835e-06/27.9715	17/5/1.6373e-05/45.5972
9	3000	1	1/1/6.9349e-06/0.39072	1/1/6.9349e-06/0.38856	1/1/6.9349e-06/0.38503
		10	3/3/4.9157e-03/4.6299	9/3/1.8731e-03/14.9589	9/3/1.468e-03/16.2261
		100	5/5/2.9136e-02/8.8327	13/4/2.8697e-02/22.3504	13/4/2.3369e-02/24.1051
10	3000	1	7/7/1.8841e-05/113.7354	13/4/1.5023e-05/90.6983	13/4/1.7169e-05/96.5679
		10	9/9/1.1683e-05/146.8708	17/5/1.2381e-05/115.8011	21/6/7.057e-06/155.4833
		100	10/10/9.479e-09/163.7237	21/6/1.4677e-10/142.4809	21/6/1.0401e-13/159.7523
11	3000	1	20/10/2.2123e-04/34.464	77/6/2.3676e-04/128.9774	161/15/1.9047e-04/297.2913
		10	38/26/1.9482e-03/68.8421	161/17/1.3971e-03/269.0105	165/17/2.8922e-03/340.8817
		100	37/22/3.0277e-03/65.9771	145/16/2.45e-04/236.8702	129/12/4.9012e-04/272.922
13	3000	1	9/9/1.4397e-04/16.3563	17/5/8.5893e-05/27.4254	17/5/1.8655e-04/44.4526
		10	14/14/1.4123e-04/26.2765	25/7/2.604e-04/41.0561	29/8/5.5618e-05/79.057
		100	17/17/2.5192e-04/32.2734	33/9/8.9702e-05/54.5243	37/10/2.8248e-05/102.0578
14	3000	1	12/12/3.6595e-05/22.8178	25/7/4.4361e-06/41.4525	25/7/1.3946e-05/65.4345
		10	18/18/4.3039e-05/35.2549	37/10/4.9713e-06/62.3313	37/10/2.4413e-05/98.7529
		100	24/24/2.5066e-05/47.6055	49/13/2.9067e-06/82.7621	49/13/2.1752e-05/132.2612

Table 2: Results on the first singular test set with rank($F'(x^*)$)= $n-2$

			Algorith LM	Algorithm SLM with $m=4$	Algorithm 2.2 with $m=4$
Problem	n	x_0	NF/NJ/F/TIME	NF/NJ/F/TIME	NF/NJ/F/TIME
8	3000	1	9/9/1.7993e-05/15.758	17/5/6.0835e-06/27.7172	17/5/1.6373e-05/44.3831
9	3000	1	1/1/6.9349e-06/0.38195	1/1/6.9349e-06/0.38875	1/1/6.9349e-06/0.38878
		10	3/3/4.9157e-03/4.5209	9/3/1.8731e-03/14.8385	9/3/1.468e-03/16.9752
		100	5/5/2.9136e-02/8.6748	13/4/2.8697e-02/21.854	13/4/2.3369e-02/25.3346
10	3000	1	7/7/1.8841e-05/113.166	13/4/1.5023e-05/89.9889	13/4/1.7169e-05/96.959
		10	9/9/1.1683e-05/145.9313	17/5/1.2381e-05/115.1672	21/6/7.057e-06/155.429
		100	17/12/5.585e-06/210.7712	21/6/5.259e-06/141.4457	21/6/5.2587e-06/156.9107
11	3000	1	20/10/2.2124e-04/33.8705	77/6/2.3676e-04/125.7451	149/14/1.9077e-04/288.4942
		10	37/24/2.3572e-03/65.5213	149/16/1.9236e-03/245.8551	165/17/2.8922e-03/353.1612
		100	46/26/3.0342e-03/80.6403	137/15/2.5525e-04/227.3232	129/13/4.9734e-04/280.1359
13	3000	1	9/9/1.4397e-04/16.2237	17/5/8.5893e-05/27.5697	17/5/1.8655e-004/43.5593
		10	14/14/1.4123e-04/26.2398	25/7/2.604e-04/41.4618	29/8/5.5618e-05/77.8504
		100	17/17/2.5192e-04/32.1537	33/9/8.9702e-05/55.034	37/10/2.8248e-05/100.1965
14	3000	1	12/12/3.6595e-05/22.7153	25/7/4.4361e-06/41.5594	25/7/1.3946e-05/67.8233
		10	18/18/4.3039e-05/34.929	37/10/4.9713e-06/62.0526	37/10/2.4413e-05/102.6231
		100	24/24/2.5066e-05/47.1261	49/13/2.9067e-06/82.8124	49/13/2.1752e-05/135.4006

From table 1 and table 2, we can see that, though Algorithm 2.2 take more running time than the SLM method to compute step size $\alpha_{k,i}$, Algorithm 2.2 still almost always outperforms or at least performs as well as the SLM method whether on the first singular test set or on the second test set, which indicate that the line search really makes the method more efficient and contributes a lot to the numerical performance. That would be great helpful for the real application of the method and especially useful for the large scale problems.

6 Conclusions

In this work, to save more Jacobian calculations, we presented the efficient m-step LM method for systems of nonlinear equations. At every iteration, we compute $m-1$ approximate LM steps with frozen $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ and employ $m-1$ line search for better numerical performance. The efficient m-step LM method have been proved to have $(m+1)$th convergence order under the local error bound condition which is weaker than nonsingularity. Numerical results show that the efficient m-step LM method saved more Jacobian calculations although the calculations of function are more.

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OPTIMAL BOUNDS FOR A TOADER TYPE MEAN USING ARITHMETIC AND GEOMETRIC MEANS*

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Abstract. In the aritcle, we prove that the double inequalities $\alpha A(a,b)+(1-\alpha)G(a,b) < T[A(a,b),G(a,b)] < \beta A(a,b)+(1-\beta)G(a,b)$ and $G[\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a] < T[A(a,b),G(a,b)] < G[\mu a+(1-\mu)b, \mu b+(1-\mu)a]$ hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 1/2$, $\beta\geq 2/\pi$, $\lambda\leq (1-\sqrt{1-4/\pi^2})/2$ and $\mu\geq 1/2-\sqrt{2}/4$ if $\alpha,\beta\in\mathbb{R}$ and $\lambda,\mu\in(0,1/2)$, and find new bounds for the complete elliptic integral $\mathcal{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta$ $(0<r<1)$ of the second kind, where $G(a,b)=\sqrt{ab}$, $T(a,b)=2\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta/\pi$ and $A(a,b)=(a+b)/2$ are respectively the geometric, Toader and arithmetic means of a and b.

1. Introduction

Let $r\in(0,1)$ and $a,b>0$. Then the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1-24] of the first and second kind, Toader mean $T(a,b)$ [25-34], geometric mean $G(a,b)$ [35-41] and arithmetic mean $A(a,b)$ [42-50] are respectively given by

$$\mathcal{K}(r)=\int\limits_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta,\quad \mathcal{E}(r)=\int\limits_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta,$$

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta, \tag{1.1}$$

$$G(a,b)=\sqrt{ab},\quad A(a,b)=\frac{a+b}{2}. \tag{1.2}$$

It is well known that

$$\mathcal{K}(0^+)=\mathcal{E}(0^+)=\pi/2,\quad \mathcal{K}(1^-)=+\infty,\quad \mathcal{E}(1^-)=1,$$

$\mathcal{K}(r)$ is strictly increasing and $\mathcal{E}(r)$ is strictly decreasing on $(0,1)$, $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the derivatives formulas [51, Appendix E, p. 474-475]

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)},\quad \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r},$$

and $T(a,b)$ can be rewritten as

$$T(a,b)=\begin{cases}\frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right), & a>b,\\ a, & a=b,\\ \frac{2b}{\pi}\mathcal{E}\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right), & a<b.\end{cases} \tag{1.3}$$

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Recently, the bounds for the Toader mean $T(a,b)$ have attracted the attention of several researchers. Barnard et. al. [52], and Alzer and Qiu [53] proved that the double inequality

$$M_{p_1}(a,b) < T(a,b) < M_{p_2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p_1 \leq 3/2$ and $p_2 \geq \log 2/(\log \pi - \log 2) = 1.5349\cdots$, where $M_p(a,b) = [(a^p + b^p)/2]^{1/p}$ $(p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ is the pth power mean.

Very recently, Song et. al. [54] proved that the double inequality

$$M_{q_1}(a,b) < T[A(a,b), Q(a,b)] < M_{q_2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $q_1 \leq 2\log 2/[2\log \pi - \log 2 - 2\log \mathcal{E}(\sqrt{2}/2)] = 1.3930\cdots$ and $q_2 \geq 3/2$, where $Q(a,b) = \sqrt{(a^2+b^2)/2}$ is the quadratic mean of a and b.

Let $a, b > 0$ with $a \neq b$. Then from (1.1) and (1.2) together with $G(a,b) < A(a,b)$ we clearly see that the function $\lambda \to R(\lambda) = G[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a]$ is continuous and strictly increasing on $[0, 1/2]$, and

$$R(0) = G(a,b) < T[A(a,b), G(a,b)] < A(a,b) = R\left(\frac{1}{2}\right).$$

It is the aim of this article to find the best possible parameters $\alpha, \beta \in \mathbb{R}$ and $\lambda, \mu \in (0, 1/2)$ such that the double inequalities

$$\alpha A(a,b) + (1-\alpha)G(a,b) < T[A(a,b), G(a,b)] < \beta A(a,b) + (1-\beta)G(a,b),$$

$$G[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a] < T[A(a,b), G(a,b)] < G[\mu a + (1-\mu)b, \mu b + (1-\mu)a]$$

hold for all $a, b > 0$ with $a \neq b$ holds for all $a, b > 0$ with $a \neq b$.

2. Lemmas

Lemma 2.1. *(See [51, Theorem 3.21(1)]) The function $r \mapsto [\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4, 1)$.*

Lemma 2.2. *(See [51, Exercise 3.43(11)]) The function $r \mapsto [\mathcal{K}(r) - \mathcal{E}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4, +\infty)$.*

Lemma 2.3. *(See [51, Theorem 3.21(7)]) The function $r \mapsto (1-r^2)^\lambda \mathcal{K}(r)$ is strictly decreasing from $(0,1)$ onto $(0, \pi/2)$ if $\lambda \geq 1/4$.*

Lemma 2.4. *(See [51, Theorem 1.25]) Let $a, b \in \mathbb{R}$ with $a < b$, $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.5. *The function $r \mapsto \sqrt{1-r^2}[\mathcal{E}(r) - \mathcal{K}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(-\pi/4, 0)$.*

Proof. Let

$$f(r) = \frac{\sqrt{1-r^2}[\mathcal{E}(r) - \mathcal{K}(r)]}{r^2}, \tag{2.1}$$

$$g(r) = [\mathcal{K}(r) - \mathcal{E}(r)] - [\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]. \tag{2.2}$$

Then it follows from (2.1), (2.2), L'Hôpital rule, and Lemmas 2.1 and 2.3 that

$$f(1^-) = 0, \quad f(0^+) = \lim_{r \to 0^+} \frac{[\sqrt{1-r^2}(\mathcal{E}(r) - \mathcal{K}(r))]'}{2r} = \lim_{r \to 0^+} \frac{\mathcal{K}(r) - 2\mathcal{E}(r)}{2\sqrt{1-r^2}} = -\frac{\pi}{4}, \tag{2.3}$$

$$f'(r) = \frac{1}{r^3\sqrt{1-r^2}} g(r), \tag{2.4}$$

$$g(0^+) = 0, \tag{2.5}$$

$$g'(r) = \frac{r^3}{1-r^2}\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2} > 0 \tag{2.6}$$

for $r \in (0,1)$.

Therefore, Lemma 2.5 follows easily from (2.3)-(2.6). $\square$

Lemma 2.6. *The function $r \mapsto \mathcal{E}(r)[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi^2/8,+\infty)$.*

Proof. Let

$$h(r) = \frac{\mathcal{E}(r)[\mathcal{K}(r)-\mathcal{E}(r)]}{r^2}, \quad h_1(r) = \mathcal{E}(r) - \sqrt{1-r^2}\mathcal{K}(r). \tag{2.7}$$

Then from Lemma 2.2 and (2.7) we clearly see that

$$h(0^+) = \frac{\pi^2}{8}, \quad h(1^-) = +\infty, \quad h_1(0^+) = 0, \tag{2.8}$$

$$h'(r) = \frac{\mathcal{E}(r)+\sqrt{1-r^2}\mathcal{K}(r)}{r^3(1-r^2)}h_1(r), \tag{2.9}$$

$$h_1'(r) = \frac{r(1-\sqrt{1-r^2})}{\sqrt{1-r^2}}\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2} > 0 \tag{2.10}$$

for $r \in (0,1)$.

Therefore, Lemma 2.6 follows easily from (2.8)-(2.10). $\square$

3. Main Results

Theorem 3.1. *The double inequality*

$$\alpha A(a,b) + (1-\alpha)G(a,b) < T[A(a,b),G(a,b)] < \beta A(a,b) + (1-\beta)G(a,b)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha \leq 1/2$ and $\beta \geq 2/\pi = 0.6366\cdots$.

Proof. Since $A(a,b)$, $T(a,b)$ and $G(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$ and $r=(a-b)/(a+b) \in (0,1)$. Then (1.2) and (1.3) lead to

$$T[A(a,b),G(a,b)] = \frac{2}{\pi}A(a,b)\mathcal{E}(r), \quad G(a,b) = A(a,b)\sqrt{1-r^2},$$

$$\frac{T[A(a,b),G(a,b)]-G(a,b)}{A(a,b)-G(a,b)} = \frac{\frac{2}{\pi}\mathcal{E}(r)-\sqrt{1-r^2}}{1-\sqrt{1-r^2}}. \tag{3.1}$$

Let

$$F_1(r) = \frac{2}{\pi}\mathcal{E}(r) - \sqrt{1-r^2}, \quad F_2(r) = 1-\sqrt{1-r^2}, \tag{3.2}$$

$$F(r) = \frac{F_1(r)}{F_2(r)} = \frac{\frac{2}{\pi}\mathcal{E}(r)-\sqrt{1-r^2}}{1-\sqrt{1-r^2}}. \tag{3.3}$$

Then Lemma 2.5, (3.2) and (3.3) lead to

$$F_1(0^+) = F_2(0^+) = 0, \tag{3.4}$$

$$\frac{F_1'(r)}{F_2'(r)} = \frac{2}{\pi}\frac{\sqrt{1-r^2}[\mathcal{E}(r)-\mathcal{K}(r)]}{r^2} + 1, \tag{3.5}$$

$$F(0^+) = \lim_{r\to 0^+}\frac{F_1'(r)}{F_2'(r)} = \frac{1}{2}, \quad F(1^-) = \frac{2}{\pi}. \tag{3.6}$$

It follows from Lemmas 2.4 and 2.5 together with (3.3)-(3.5) that $F(r)$ is strictly increasing on $(0,1)$. Therefore, Theorem 3.1 follows from (3.1), (3.3) and (3.6) together with the monotonicity of $F(r)$. $\square$

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Theorem 3.2. *Let $\lambda, \mu \in (0, 1/2)$. Then the double inequality*

$$G[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda a)] < T[A(a,b), G(a,b)] < G[\mu a + (1-\mu)b, \mu b + (1-\mu a)]$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq (1-\sqrt{1-4/\pi^2})/2 = 0.1144\cdots$ and $\mu \geq 1/2 - \sqrt{2}/4 = 0.1464\cdots$.

Proof. We assume that $a > b > 0$, $r = (a-b)/(a+b) \in (0,1)$ and $p \in (0, 1/2)$. Then (1.2) and (1.3) lead to

$$\begin{aligned} &G[pa + (1-p)b, pb + (1-pa)] - T[A(a,b), G(a,b)] \qquad (3.7)\\ &= A(a,b)\left[\sqrt{1-(1-2p)^2r^2} - \frac{2}{\pi}\mathcal{E}(r)\right]\\ &= \frac{A(a,b)}{\sqrt{1-(1-2p)^2r^2} + \frac{2}{\pi}\mathcal{E}(r)} H(r), \end{aligned}$$

where

$$H(r) = 1 - (1-2p)^2r^2 - \frac{4}{\pi^2}\mathcal{E}^2(r),$$

$$H(0^+) = 0, \tag{3.8}$$

$$H(1^-) = 4p(1-p) - \frac{4}{\pi^2}, \tag{3.9}$$

$$H'(r) = 2rH_1(r), \tag{3.10}$$

where

$$H_1(r) = \frac{4}{\pi^2}\frac{\mathcal{E}(r)[\mathcal{K}(r) - \mathcal{E}(r)]}{r^2} - (1-2p)^2. \tag{3.11}$$

It follows from Lemma 2.6 and (3.11) that

$$H_1(0^+) = \frac{1}{2} - (1-2p)^2, \tag{3.12}$$

$$H_1(1^-) = +\infty. \tag{3.13}$$

We divide the proof into four cases.

Case 1 $p = \mu_0 = 1/2 - \sqrt{2}/4$. Then (3.12) becomes

$$H_1(0^+) = 0. \tag{3.14}$$

From Lemma 2.6, (3.11) and (3.14) we clearly see that

$$H_1(r) > 0 \tag{3.15}$$

for all $r \in (0, 1)$. Therefore,

$$T[A(a,b), G(a,b)] < G[\mu_0 a + (1-\mu_0)b, \mu_0 b + (1-\mu_0)a]$$

follows from (3.7), (3.8), (3.10) and (3.15).

Case 2 $p = \lambda_0 = (1-\sqrt{1-4/\pi^2})/2$. Then (3.9) and (3.12) lead to

$$H(1^-) = 0, \tag{3.16}$$

$$H_1(0^+) = -\frac{\pi^2 - 8}{2\pi^2} < 0. \tag{3.17}$$

From Lemma 2.6, (3.10), (3.11), (3.13) and (3.17) we know that there exists $r_0 \in (0,1)$ such that $H(r)$ is strictly decreasing on $(0, r_0)$ and strictly increasing on $(r_0, 1)$. Therefore,

$$T[A(a,b), G(a,b)] > G[\lambda_0 a + (1-\lambda_0)b, \lambda_0 b + (1-\lambda_0)a]$$

follows from (3.7), (3.8) and (3.16) together with the piecewise monotonicity of $H(r)$.

Case 3 $0 < p = \mu^* < 1/2 - \sqrt{2}/4$. Then (3.12) leads to

$$H_1(0^+) < 0. \tag{3.18}$$

Equations (3.7), (3.8) and (3.10) together with inequality (3.18) imply that there exists small enough $\delta_0 \in (0,1)$ such that

$$T[A(a,b), G(a,b)] > G[\mu^* a + (1-\mu^*)b, \mu^* b + (1-\mu^*)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_0)$.

Case 4 $(1-\sqrt{1-4/\pi^2})/2 < p = \lambda^* < 1/2$. Then (3.9) leads to

$$H(1^-) > 0. \tag{3.19}$$

Equation (3.7) and inequality (3.19) imply that there exists small enough $\delta_1 \in (0,1)$ such that

$$T[A(a,b), G(a,b)] < G[\lambda^* a + (1-\lambda^*)b, \lambda^* b + (1-\lambda^*)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_1, 1)$. □

From Theorems 3.1 and 3.2 we get Corollary 3.3 immediately.

Corollary 3.3. *The double inequality*

$$\max\left\{\frac{\pi}{4}\left(1+\sqrt{1-r^2}\right), \frac{\pi}{2}\sqrt{1+\left(\frac{4}{\pi^2}-1\right)r^2}\right\} < \mathcal{E}(r)$$

$$< \min\left\{1+\left(\frac{\pi}{2}-1\right)\sqrt{1-r^2}, \frac{\sqrt{2}\pi}{4}\left(2-r^2\right)\right\}$$

holds for all $r \in (0,1)$.

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Addition Theorem For Exton's q–Exponential Functions

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Abstract. In this paper, we study about the q-exponential function which was introduced by Exton. We propose the addition theorem for this q–exponential function and also Continued fraction representation for this q–exponential function is given.
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1 Introduction

The $\widetilde{q}$–derivative (or symmetric q–derivative) of a function $f(x)$ is defined [3] as

$$\widetilde{D}_q\ f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}$$

where $q \neq \pm 1$. This $\widetilde{q}$–derivative is invariant under inversion of basis.
For any number α, the $\widetilde{q}$–derivative of powers of x are given by

$$\widetilde{D}_q\ x^\alpha = [\alpha]_{\widetilde{q}}\ x^{\alpha-1}$$

where $[\alpha]_{\widetilde{q}} = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}$ and it is called symmetric q–number. In the case, if α is a positive integer we have

$$[\alpha]_{\widetilde{q}} = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}} = q^{1-\alpha}(1 + q^2 + q^4 + \cdots + q^{2\alpha-2}).$$

Relation between q–number and symmetric q–number is

$$[\alpha]_{\widetilde{q}} = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}} = q^{1-\alpha}[\alpha]_{q^2} \tag{1}$$

where $[\alpha]_q = \frac{q^\alpha - 1}{q-1}$ is called q–number.
With easy calculation, one can see [5] that

$$[\alpha]_{\widetilde{\frac{1}{q}}} = [\alpha]_{\widetilde{q}}. \tag{2}$$

$$[-\alpha]_{\widetilde{q}} = -[\alpha]_{\widetilde{q}}. \tag{3}$$

$$[\alpha + \beta]_{\widetilde{q}} = q^\beta[\alpha]_{\widetilde{q}} + q^{-\alpha}[\beta]_{\widetilde{q}}. \tag{4}$$

Furthermore, the $\widetilde{q}$–analogue of factorial, denoted by $[n]_{\widetilde{q}}\,!$, is defined [1] as

$$[n]_{\widetilde{q}}\ ! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_{\widetilde{q}} \times [n-1]_{\widetilde{q}} \times \cdots \times [1]_{\widetilde{q}} & \text{if } n = 1, 2, \ldots \end{cases} \tag{5}$$

and by using (1), we may also write the $\widetilde{q}$–factorial as follows

$$[n]_{\widetilde{q}}\ ! = [n]_{q^2}!\ q^{-\binom{n}{2}} \tag{6}$$

where $[n]_q! = [n]_q \times [n-1]_q \times \cdots \times [1]_q$ for $n = 1, 2, \ldots$.
The $\widetilde{q}$–analogue of $(a - x)^n$, denoted by $(a - x)^n_{\widetilde{q}}$, is defined [3] as

$$(a - x)^n_{\widetilde{q}} = \begin{cases} 1 & n = 0, \\ \prod_{i=0}^{n-1} \left(a - xq^{1-n+2i}\right) & n = 1, 2, \ldots. \end{cases} \tag{7}$$

The $\widetilde{q}$–analogue in (7) is invariant under inversion of basis and one can see that

$$(a-x)_{\widetilde{q}}^{n} = (-1)^n (x-a)_{\widetilde{q}}^{n}. \tag{8}$$

The $\widetilde{q}$–derivative of $(x-a)_{\widetilde{q}}^{n}$ is founded [3] as

$$\widetilde{D}_q \ (x-a)_{\widetilde{q}}^{n} = [n]_{\widetilde{q}} \ (x-a)_{\widetilde{q}}^{n-1}. \tag{9}$$

The $\widetilde{q}$–Taylor series expansion of $(a+x)_{\widetilde{q}}^{n}$ about $x=0$ is

$$(a+x)_{\widetilde{q}}^{n} = \sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}} a^{n-k} x^k \tag{10}$$

where $\binom{n}{k}_{\widetilde{q}} = \frac{[n]_{\widetilde{q}}!}{[k]_{\widetilde{q}}! \ [n-k]_{\widetilde{q}}!}$ are called symmetric q–binomial coefficients. Formula (10) is called Gauss's $\widetilde{q}$–binomial formula (see [3], p. 100).

The object of study in this paper is the q–exponential function which was introduced by Exton (see [6] or [4], p. 128) as

$$E(q,x) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n q^{\frac{1}{2}\binom{n}{2}}, \qquad x \in \mathbb{C} \tag{11}$$

where $[n]_q = \frac{q^n-1}{q-1}$. This q–exponential function is invariant under inversion of basis and unfortunately, there is no known addition theorem for it. Our goal is to give the addition theorem for this q–exponential function and also represent it as a continued fractions.

2 Some Identities

Definition 1. *For any number α, we define*

$$(a+x)_{\widetilde{q}}^{\alpha} = \frac{(a+q^{1-\alpha}x)_{q^2}^{\infty}}{(a+q^{1+\alpha}x)_{q^2}^{\infty}} \tag{12}$$

where $(a+x)_q^{\infty} := \lim_{n\to\infty} \prod_{j=0}^{n} (a+q^j x)$.

Theorem 1. *For any numbers α and β,*

$$(a+x)_{\widetilde{q}}^{\alpha+\beta} = (a+q^{-\beta}x)_{\widetilde{q}}^{\alpha} \ (a+q^{\alpha}x)_{\widetilde{q}}^{\beta}.$$

Proof. The result will be obtained directly by using the definition of $(a+x)_{\widetilde{q}}^{\alpha}$, which is given in (12). □

Corollary 1. *For any number α,*

$$(a+x)_{\widetilde{q}}^{-\alpha} = \frac{1}{(a+x)_{\widetilde{q}}^{\alpha}}$$

Proof. The result will be obtained by using (12).

□

Proposition 1. *For $1 \le j \le n-1$, the $\widetilde{q}$–Pascal rule is*

$$\binom{n}{j}_{\widetilde{q}} = q^{n-j}\binom{n-1}{j-1}_{\widetilde{q}} + q^{-j}\binom{n-1}{j}_{\widetilde{q}}$$

Proof. Let us expand the symmetric q–binomial coefficient $\binom{n}{j}_{\widetilde{q}}$, then we have

$$\begin{aligned}
\binom{n}{j}_{\widetilde{q}} &= \frac{[n]_{\widetilde{q}}!}{[j]_{\widetilde{q}}! \ [n-j]_{\widetilde{q}}!} \\
&= \frac{[n-1]_{\widetilde{q}}! \ [n]_{\widetilde{q}}}{[j]_{\widetilde{q}}! \ [n-j]_{\widetilde{q}}!} \\
&= \frac{[n-1]_{\widetilde{q}}! \ (q^{n-j}[j]_{\widetilde{q}} + q^{-j}[n-j]_{\widetilde{q}})}{[j]_{\widetilde{q}}! \ [n-j]_{\widetilde{q}}!} \\
&= q^{n-j}\binom{n-1}{j-1}_{\widetilde{q}} + q^{-j}\binom{n-1}{j}_{\widetilde{q}}
\end{aligned}$$

which completes the proof. We used (4) in the third line. □

Lemma 1. *For any number x and positive integer r,*

$$\binom{-x}{r}_{\widetilde{q}} = (-1)^r \binom{x+r-1}{r}_{\widetilde{q}}$$

Proof. To prove the lemma we make a use of (5) and (3), then we may write

$$\begin{aligned}\binom{-x}{r}_{\widetilde{q}} &= \frac{[-x]_{\widetilde{q}}!}{[r]_{\widetilde{q}}!\ [-x-r]_{\widetilde{q}}!} \\ &= (-1)^r\ \frac{[x]_{\widetilde{q}}\ [x+1]_{\widetilde{q}}\ \dots\ [x+r-1]_{\widetilde{q}}}{[r]_{\widetilde{q}}!} \\ &= (-1)^r\ \frac{[x+r-1]_{\widetilde{q}}!}{[x-1]_{\widetilde{q}}!\ [r]_{\widetilde{q}}!} = (-1)^r \binom{x+r-1}{r}_{\widetilde{q}}\end{aligned}$$

which completes the proof. □

The following theorem is a symmetric version of Heine's q–binomial formula.

Theorem 2. *For any number x and positive integer n, the following equation holds*

$$\frac{1}{(1-x)_{\widetilde{q}}^{n}} = \sum_{j=0}^{\infty} \binom{n+j-1}{j}_{\widetilde{q}} x^j.$$

Proof. To prove the Theorem we make a use of Corollary 1 and Lemma 1, then we may write

$$\frac{1}{(1-x)_{\widetilde{q}}^{n}} = (1-x)_{\widetilde{q}}^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j}_{\widetilde{q}} (-x)^j = \sum_{j=0}^{\infty} \binom{n+j-1}{j}_{\widetilde{q}} x^j$$

which completes the proof. □

In the next theorem, $\widetilde{q}$–analogue of Vandermonde's identity is given.

Theorem 3. *For any $m, n, r \in \mathbb{N}_0$*

$$\binom{m+n}{r}_{\widetilde{q}} = q^{mr} \sum_{k=0}^{r} \binom{m}{k}_{\widetilde{q}} \binom{n}{r-k}_{\widetilde{q}} q^{-(m+n)k}$$

Proof. We make a use of Theorem 1 to write that

$$(1+x)_{\widetilde{q}}^{m+n} = (1+q^{-n}x)_{\widetilde{q}}^{m}\ (1+q^{m}x)_{\widetilde{q}}^{n}.$$

Using the $\widetilde{q}$–binomial formula in (10) for both sides of the above formula, and then we obtain

$$\begin{aligned}\sum_{r=0}^{m+n} \binom{m+n}{r}_{\widetilde{q}} x^r &= \sum_{r=0}^{m} \binom{m}{r}_{\widetilde{q}} (q^{-n}x)^r\ \sum_{r=0}^{n} \binom{n}{r}_{\widetilde{q}} (q^{m}x)^r \\ &= \sum_{r=0}^{m+n} \left(q^{mr} \sum_{k=0}^{r} \binom{m}{k}_{\widetilde{q}} \binom{n}{r-k}_{\widetilde{q}} q^{-(m+n)k} \right) x^r.\end{aligned}$$

The proof is complete by comparing coefficients of x^r. □

The following corollary is the special case of Vandermonde's identity.

Corollary 2. *For any positive integer n,*

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} q^{n(n-2k)} = \binom{2n}{n}_{\widetilde{q}} \tag{13}$$

Proof. Take $m = r = n$ in Theorem 3 and make a use of the identity $\binom{n}{k}_{\widetilde{q}} = \binom{n}{n-k}_{\widetilde{q}}$ to prove the corollary. □

Corollary 3. *For any positive integer n,*

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} [2k]_{\widetilde{q^n}} = [n]_{\widetilde{q^n}} \binom{2n}{n}_{\widetilde{q}}. \tag{14}$$

Proof. Let us change the base q to q^{-1} in Corollary 2 to obtain

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} q^{-n(n-2k)} = \binom{2n}{n}_{\widetilde{q}} \tag{15}$$

because of the identity $\binom{n}{k}_{\widetilde{\frac{1}{q}}} = \binom{n}{k}_{\widetilde{q}}$. Now by comparing equations (13) and (15) we can write

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} \left(q^{n(n-2k)} - q^{-n(n-2k)}\right) = 0 \tag{16}$$

and also

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} [n-2k]_{\widetilde{q^n}} = 0 \tag{17}$$

since $[\alpha]_{\widetilde{q}} = \frac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}}$. Then we make a use of equations (3) and (4) to rewrite the equation (17) as

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} \left(q^{2nk}[n]_{\widetilde{q^n}} + q^{n^2}[-2k]_{\widetilde{q^n}}\right) = 0,\\
&\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} \left(q^{2nk}[n]_{\widetilde{q^n}} - q^{n^2}[2k]_{\widetilde{q^n}}\right) = 0,\\
&\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} [2k]_{\widetilde{q^n}} = [n]_{\widetilde{q^n}} \sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} q^{-n^2+2nk}.
\end{aligned}$$

The proof will be complete if we apply the identity in (15) to the right side of the last equation. □

3 $\widetilde{q}$–Exponential Functions

In this section, we study about the q–exponential functions (11) which was introduced by Exton. Let us consider $E(q^2, x)$, then we have

$$E(q^2, x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q^2}!} x^n q^{\binom{n}{2}}, \qquad x \in \mathbb{C}. \tag{18}$$

Now we make a use of (6) to rewrite the above formula as follows

$$E(q^2, x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} x^n, \qquad x \in \mathbb{C}. \tag{19}$$

We use a different notation for the Exton's q–exponential function as

$$e_{\widetilde{q}}^{x} := E(q^2, x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} x^n, \qquad x \in \mathbb{C}. \tag{20}$$

One can see that this $\widetilde{q}$–exponential function (20) is invariant under inversion of basis and its $\widetilde{q}$–derivative is equal to itself, that means

$$e_{\widetilde{\frac{1}{q}}}^{x} = e_{\widetilde{q}}^{x} \tag{21}$$

$$\widetilde{D}_q \, e_{\widetilde{q}}^{x} = e_{\widetilde{q}}^{x} \tag{22}$$

The next theorem is about the product of two $\widetilde{q}$–exponential functions.

Theorem 4. *For any x and y, the following equation holds*

$$e_{\widetilde{q}}^{x}\ e_{\widetilde{q}}^{y} = e_{\widetilde{q}}^{(x+y)_{\widetilde{q}}} \tag{23}$$

where $(x+y)_{\widetilde{q}}^{n}$ is defined in (10).

Proof. We use (20) to expand both $e_{\widetilde{q}}^{x}$ and $e_{\widetilde{q}}^{y}$, therefore we obtain

$$\begin{aligned} e_{\widetilde{q}}^{x}\ e_{\widetilde{q}}^{y} &= \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} x^n \ \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} y^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{[k]_{\widetilde{q}}!\ [n-k]_{\widetilde{q}}!} x^k y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} \sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}} x^k y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} (x+y)_{\widetilde{q}}^{n} \\ &= e_{\widetilde{q}}^{(x+y)_{\widetilde{q}}} \end{aligned}$$

and the proof is complete. □

4 Continued Fractions

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}},$$

where $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ are two sequences of real or complex numbers. We use the following symbol for the above continued fraction

$$a_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \left[\frac{b_n}{a_n}\right]. \tag{24}$$

The following theorem is the convergent theorem of continued fractions (See [7], p. 126).

Theorem 5. *If $a_n > 0$ for $n > 1$ then the continued fraction $K_{n=1}^{\infty}\left[\frac{1}{a_n}\right]$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ diverges.*

4.1 Continued Fraction Representation of $\widetilde{q}$–Exponential Functions

The q–exponential functions e_q^x and E_q^x can be written as infinite product form as follows

$$e_q^x = \frac{1}{(1-(1-q)x))_q^{\infty}}, \qquad E_q^x = (1+(1-q)x))_q^{\infty}.$$

In this section, we want to show that the $\widetilde{q}$–exponential function also can be written as infinite product form.
Let us consider the $\widetilde{q}$–derivative of a function $f(x)$ which is defined as

$$\widetilde{D}_q f(x) = \frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x}. \tag{25}$$

Take $f(x) = e_{\widetilde{q}}^{qx}$, therefore we can write (25) as follows

$$q\ e_{\widetilde{q}}^{qx} = \frac{e_{\widetilde{q}}^{q^2 x} - e_{\widetilde{q}}^{x}}{(q-q^{-1})x}, \tag{26}$$

because $\widetilde{D}_q\ e_{\widetilde{q}}^{qx} = q\ e_{\widetilde{q}}^{qx}$. Now by easy manipulation, we may write (26) as

$$\frac{e_{\widetilde{q}}^{x}}{e_{\widetilde{q}}^{qx}} - \frac{e_{\widetilde{q}}^{q^2x}}{e_{\widetilde{q}}^{qx}} = (1-q^2)x. \tag{27}$$

Let us define $g(x) := \frac{e_{\widetilde{q}}^{x}}{e_{\widetilde{q}}^{qx}}$ and then we may write (27) as

$$g(x) = (1-q^2)x + \frac{1}{g(qx)}. \tag{28}$$

Iterating the formula in (28) infinity many times to obtain

$$g(x) = (1-q^2)x + \cfrac{1}{(1-q^2)qx + \cfrac{1}{(1-q^2)q^2x + \cfrac{1}{(1-q^2)q^3x + \cdots}}}. \tag{29}$$

Now by using continued fractions symbol which is defined in (24), we may rewrite (29) as follows

$$g(x) = (1-q^2)x + \underset{n=1}{\overset{\infty}{\mathrm{K}}}\left[\frac{1}{(1-q^2)q^n x}\right] \tag{30}$$

or

$$\frac{1}{g(x)} = \underset{n=0}{\overset{\infty}{\mathrm{K}}}\left[\frac{1}{(1-q^2)q^n x}\right]. \tag{31}$$

By using Theorem 5, one can see that the continued fraction in the right hand side of equation (31) is converge, if $x < 0$ and $q > 1$.
Substitute x with $q^{-1}x$ in the equation (31) and then replace $g(x) = \frac{e_{\widetilde{q}}^{x}}{e_{\widetilde{q}}^{qx}}$ to obtain

$$e_{\widetilde{q}}^{x} = \underset{n=0}{\overset{\infty}{\mathrm{K}}}\left[\frac{1}{(1-q^2)q^{n-1} x}\right]\ e_{\widetilde{q}}^{q^{-1}x}. \tag{32}$$

Iterating this formula k times to obtain

$$e_{\widetilde{q}}^{x} = \prod_{j=1}^{k}\underset{n=0}{\overset{\infty}{\mathrm{K}}}\left[\frac{1}{(1-q^2)q^{n-j} x}\right]\ e_{\widetilde{q}}^{q^{-k}x}. \tag{33}$$

In the case, if $k \to \infty$, we have

$$e_{\widetilde{q}}^{x} = \prod_{j=1}^{\infty}\underset{n=0}{\overset{\infty}{\mathrm{K}}}\left[\frac{1}{(1-q^2)q^{n-j} x}\right] \tag{34}$$

because if $q > 1$, then we have $\lim\limits_{k\to\infty} e_{\widetilde{q}}^{q^{-k}x} = 1$.

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