Volume 28, Number 2 March 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

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On invariance and solutions of some fifth-order rational recursive sequences

M. Folly-Gbetoula * and D. Nyirenda †

Abstract

We study the fifth-order difference equations of the form

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}(a_n + b_n x_{n-4}x_{n-2})}, n = 0, 1, \ldots,$$

where a_n and b_n are real sequences, using the method of Lie group analysis. In particular, nontrivial vector fields associated with the group of point transformations are derived and exact solutions obtained. Closed form formulas for the solutions to the recursive sequences are given explicitly. This work is a generalization of a result by Elsayed [E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, J. Computational Analysis and Applications, 15(1) (2013), 73–81].

Keywords: Difference equation; Symmetry; Reduction; Group invariant; Periodicity
Mathematics Subjet Classification: 39A10; 39A13; 39A90

1 Introduction

Over a century ago, Sophus Lie [7] developed an algorithm based on the invariance of the ordinary differential equations under their symmetry group. Maeda [8, 9] observed that the Lie Symmetry approach can be applied to ordinary difference equations. Recently, Hydon [3] utilized a similar method to come up with some interest-provoking results. It is now a foregone conclusion that Lie's method can be used to find symmetries and conservation laws of recursive sequences, even in the context of variational equations.
In this paper, we obtain the vector fields of

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}(a_n + b_n x_{n-4}x_{n-2})}, \tag{1}$$

where a_n and b_n are random real sequences, and then proceed to find the solutions in closed form. Our work extends the work by Elsayed [1], where the formulas of the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}(\pm \pm x_{n-4}x_{n-2})} \quad n = 0, 1, \ldots, \tag{2}$$

in which the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary non-zero real numbers, were obtained.
For related work, see [2, 4, 10].

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1.1 Background on Lie analysis

In this section, we briefly discuss some key ideas on Lie group analysis of difference equations. For a broader comprehension of the concepts, refer to [3, 11]. The definitions and notation are taken from the same source [3, 11].
Let

$$x^\star = X(x;\varepsilon) \tag{3}$$

be a one parameter Lie group of transformations.

Definition 1.1 *An infinitely differentiable function F is an invariant function of the Lie group of point transformation* (3) *if and only if, for any group transformations,*

$$F(x) = F(x^\star). \tag{4}$$

Definition 1.2 *The infinitesimal generator of the one-parameter Lie group of point transformation* (3) *is the operator*

$$X = X(x) = \xi(x) \times \Delta = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}, \tag{5}$$

where Δ is the gradient operator.

Theorem 1.1 *$F(x)$ is invariant under the Lie group of transformations* (3) *if and only if*

$$XF(x) = 0. \tag{6}$$

Consider the forward fifth-order recursive sequence

$$u_{n+5} = \Phi(n, u_n, \ldots, u_{n+4}) \tag{7}$$

for some smooth function Φ. Suppose the one-parameter Lie group of point transformations is of the form

$$n^* = n, \quad u^*_{n+k} = u_{n+k} + \varepsilon S^k \xi(n, u_n) +)(\varepsilon^2), \quad k = 0, \ldots, 5, \tag{8}$$

where ξ denotes the characteristic, ε (ε is small enough) is the group parameter and $S : n \mapsto n+1$ is the shift forward operator. The symmetry condition is given by

$$u^*_{n+5} = \Phi(n, u^*_n, \ldots, u^*_{n+4}), \tag{9}$$

whenever (7) is true. The substitution of (8) in (9) yields the linearized symmetry condition:

$$S^5\xi(n, u_n) - X\Phi = 0 \tag{10}$$

where X, the vector fields of (7), is given by

$$X = \xi(n, u_n)\frac{\partial}{\partial u_n} + \xi(n+1, u_{n+1})\frac{\partial}{\partial u_{n+1}} + \cdots + \xi(n+4, u_{n+4})\frac{\partial}{\partial u_{n+4}}. \tag{11}$$

Despite the fact that (10) looks simple, its solution finding process is highly involving.
In our work, we will use the canonical coordinate [5]

$$S_n = \int \frac{du_n}{\xi(n, u_n)} \tag{12}$$

to lower the order of the difference equation under investigation.

2 Main results

Let

$$u_{n+5} = \Phi = \frac{u_n u_{n+2}}{u_{n+3}(A_n + B_n u_n u_{n+2})}, \tag{13}$$

where A_n and B_n are random real sequences, be the forward recursive equation equivalent to (2).
Substituting (13) in (10), we have that

$$S^5\xi + \frac{u_n u_{n+2}(S^3\xi)}{u_{n+3}^2(A_n + B_n u_n u_{n+2})} - \frac{A_n u_n (S\xi)}{u_{n+3}(A_n + B_n u_n u_{n+2})^2} - \frac{A_n u_{n+2}\xi}{u_{n+3}(A_n + B_n u_n u_{n+2})^2} = 0. \tag{14}$$

We act the differential operator

$$L = \frac{\partial}{\partial u_n} - \frac{\Phi_{u_n}}{\Phi_{u_{n+3}}} \frac{\partial}{\partial u_{n+3}}$$

to eliminate the first term in (14). This leads to

$$(A_n + B_n u_n u_{n+2}) \left[(S^3\xi)' - (S^3\xi)\right] + B_n u_n (S\xi) - (A_n + B_n u_n u_{n+2})\xi' + \frac{A_n}{u_n}\xi = 0 \tag{15}$$

after simplification. The differentiation of (15) with respect to u_n twice, keeping u_{n+3} fixed, yields

$$-(A_n + B_n u_n u_{n+2})\xi^{(3)} + \frac{A_n}{u_n}\xi^{(2)} - \frac{2A_n}{{u_n}^2}\xi' + \frac{2A_n}{{u_n}^3}\xi = 0. \tag{16}$$

Split (16) by comparing powers of u_{n+2}; we have

$$\begin{cases} u_{n+2} \quad \text{term}: & u_n^3\xi^{(3)} - {u_n}^2\xi^{(2)} + 2u_n\xi' - 2\xi = 0, \\ \text{other terms}: & \xi^{(3)} = 0. \end{cases} \tag{17}$$

Equations in (17) further simplify to

$${u_n}^2\xi^{(2)} - 2u_n\xi' + 2\xi = 0. \tag{18}$$

It is clear that the solution of (16) is

$$\xi\,(n, u_n) = f_n u_n + g_n {u_n}^2 \tag{19}$$

for some arbitrary functions f_n and g_n of n. Using characteristic's expression as given in (19), we reduce equation (14) to the following difference equation

$$\begin{aligned} &B_n g_{n+3} u_n u_{n+2} {u_{n+3}}^2 + B_n(f_{n+3} + f_{n+5}) u_n u_{n+2} u_{n+3} - A_n g_n u_n u_{n+3} + g_{n+5} u_n u_{n+2} \\ &- A_n(f_n + f_{n+2} + f_{n+3} + f_{n+5}) u_{n+3} - A_n g_{n+1} u_{n+2} u_{n+3} + A_n g_{n+3} {u_{n+3}}^2 = 0. \end{aligned} \tag{20}$$

which then splits into a system (by comparing products of powers of shifts of u_n) as follows:

$$u_{n+3} \text{ terms} : f_n + f_{n+2} + f_{n+3} + f_{n+5} = 0 \tag{21a}$$
$$u_n u_{n+2} \text{ terms} : g_{n+5} = 0 \tag{21b}$$
$$u_n u_{n+3} \text{ terms} : g_n = 0 \tag{21c}$$
$$u_n u_{n+2} u_{n+3} \text{ terms} : f_{n+3} + f_{n+5} = 0 \tag{21d}$$
$$u_n u_{n+2} {u_{n+3}}^2 \text{ terms} : g_{n+3} = 0 \tag{21e}$$
$$u_{n+2} u_{n+3} \text{ terms} : g_{n+1} = 0 \tag{21f}$$
$${u_{n+3}}^2 \text{ terms} : g_{n+3} = 0. \tag{21g}$$

Thus, the 'final constraint' is given by:

$$f_n + f_{n+2} = 0, \tag{22a}$$
$$g_n = 0. \tag{22b}$$

Solving (22) for f, we obtain two independent solutions given by $\exp(\pm n\pi/2)$. Therefore, the characteristics are

$$\xi_1 = \alpha^n u_n,\ \xi_2 = \bar{\alpha}^n u_n, \tag{23}$$

and so the prolonged infinitesimal generators admitted by (13) are

$$X_1 = \alpha^n u_n \partial_{u_n} + \alpha^{n+1} u_{n+1} \partial_{u_{n+1}} + \alpha^{n+2} u_{n+2} \partial_{u_{n+2}} + \alpha^{n+3} u_{n+3} \partial_{u_{n+3}} + \alpha^{n+4} u_{n+4} \partial_{u_{n+4}}, \tag{24a}$$

$$X_2 = \bar{\alpha}^n u_n \partial_{u_n} + \bar{\alpha}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\alpha}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\alpha}^{n+3} u_{n+3} \partial_{u_{n+3}} + \bar{\alpha}^{n+4} u_{n+4} \partial_{u_{n+4}}. \tag{24b}$$

Observe that $\alpha = \exp(i\pi/2)$ and $\bar{\alpha}$ is its complex conjugate. Using the generator X_1, we have the canonical coordinate

$$S_n = \int \frac{du_n}{\alpha^n u_n} = \frac{1}{\alpha^n} \ln |u_n|. \tag{25}$$

Thanks to the form of (22), the invariant function $\tilde{V}_n$ is constructed as follows

$$\tilde{V}_n = S_n \alpha^n + S_{n+2} \alpha^{n+2} \tag{26}$$

since $X_1 \tilde{V}_n = \alpha^n + \alpha^{n+2} = 0$ and $X_2 \tilde{V}_n = \bar{\alpha}^n + \bar{\alpha}^{n+2} = 0$. For rational difference equations, it is convenience to use

$$|V_n| = \exp\{-\tilde{V}_n\}, \tag{27}$$

i.e., $V_n = \pm 1/(u_n u_{n+2})$ but we will be using the one with plus sign: $V_n = 1/u_n u_{n+2}$. We then substitute (27) into equation (13) to get the third-order linear difference equation

$$V_{n+3} = A_n V_n + B_n. \tag{28}$$

The iteration of equation (28) leads to

$$V_{3n+j} = V_j \left(\prod_{k_1=0}^{n-1} A_{3k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{3l+j} \prod_{k_2=l+1}^{n-1} A_{3k_2+j} \right), \quad j = 0, 1, 2. \tag{29}$$

Invoking (25), (26) and (27), we have that

$$\begin{aligned} |u_n| &= \exp(\alpha_n S_n) \\ &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 - \frac{1}{2} \sum_{k_1=0}^{n-1} \alpha^n \bar{\alpha}^{k_1} \tilde{V}_{k_1} - \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\alpha}^n \alpha^{k_2} \tilde{V}_{k_2} \right) \\ &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \frac{1}{2} \sum_{k_1=0}^{n-1} \alpha^n \bar{\alpha}^{k_1} \ln |V_{k_1}| + \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\alpha}^n \alpha^{k_2} \ln |V_{k_2}| \right) \\ &= \exp\left(H_n + \sum_{k_1=0}^{n-1} \mathrm{Re}(\gamma(n, k_1)) \ln |V_{k_1}| \right), \end{aligned} \tag{30}$$

in which $H_n = \alpha^n c_1 + \bar{\alpha}^n c_2$ and $\gamma(n, k) = \alpha^n \bar{\alpha}^k$.
It is worthwhile to mention that the function γ satisfies the following:

$$\begin{aligned} &\gamma(0,1) = \bar{\alpha}, \gamma(1,0) = \alpha, \gamma(n,n) = 1, \gamma(n+2,k) = -\gamma(n,k), \\ &\gamma(n,k+2) = -\gamma(n,k), \gamma(4n,k) = \gamma(0,k), \gamma(n,4k) = \gamma(n,0). \end{aligned} \tag{31}$$

From the expression of u_n given in (30) and from the above properties (31), note that

$$|u_{4n+j}| = \exp\left(H_j + \sum_{k_1=0}^{4n+j-1} \mathrm{Re}(\gamma(j, k_1)) \ln |V_{k_1}| \right). \tag{32}$$

For $j = 0$, we have

$$\begin{aligned} |u_{4n}| &= \exp(H_0 + \ln |V_0| - \ln |V_2| + \ldots + \ln |V_{4n-4}| - \ln |V_{4n-2}|) \\ &= \exp(H_0) \prod_{s=0}^{n-1} \left| \frac{V_{4s}}{V_{4s+2}} \right|. \end{aligned} \tag{33}$$

By setting $n = 0$ in (30), we get $\exp(H_0) = u_0$ and so

$$u_{4n} = u_0 \prod_{s=0}^{n-1} \frac{V_{4s}}{V_{4s+2}}. \tag{34}$$

We have omitted the absolute function because it can be shown, using (27), that there is no need for it. In a similar way, we have that

$$u_{4n+j} = u_j \prod_{s=0}^{n-1} \frac{V_{4s+j}}{V_{4s+j+2}}, \quad \text{for any } j = 0, 1, 2, 3. \tag{35}$$

This equation implies that

$$\begin{aligned} u_{12n+j} &= u_j \prod_{s=0}^{3n-1} \frac{V_{4s+j}}{V_{4s+j+2}} \\ &= u_j \prod_{s=0}^{n-1} \frac{V_{12s+j}}{V_{12s+j+2}} \frac{V_{12s+4+j}}{V_{12s+j+6}} \frac{V_{12s+j+8}}{V_{12s+j+10}} \end{aligned}$$

which now holds for $j = 0, 1, 2, \ldots, 11$.
For $j = 0$, we have

$$u_{12n} = u_0 \prod_{s=0}^{n-1} \frac{V_{12s}}{V_{12s+2}} \frac{V_{12s+4}}{V_{12s+6}} \frac{V_{12s+8}}{V_{12s+10}}. \tag{36}$$

Using (29) in (36), we have that

$$\begin{aligned} u_{12n} = u_0 \prod_{s=0}^{n-1} & \frac{V_0 \prod\limits_{k_1=0}^{4s-1} A_{3k_1} + \sum\limits_{l=0}^{4s-1} B_{3l} \prod\limits_{k_2=l+1}^{4s-1} A_{3k_2}}{V_2 \prod\limits_{k_1=0}^{4s-1} A_{3k_1+2} + \sum\limits_{l=0}^{4s-1} B_{3l+2} \prod\limits_{k_2=l+1}^{4s-1} A_{3k_2+2}} \frac{V_1 \prod\limits_{k_1=0}^{4s} A_{3k_1+1} + \sum\limits_{l=0}^{4s} B_{3l+1} \prod\limits_{k_2=l+1}^{4s} A_{3k_2+1}}{V_0 \prod\limits_{k_1=0}^{4s+1} A_{3k_1} + \sum\limits_{l=0}^{4s+1} B_{3l} \prod\limits_{k_2=l+1}^{4s+1} A_{3k_2}} \\ & \times \frac{V_2 \prod\limits_{k_1=0}^{4s+1} A_{3k_1+2} + \sum\limits_{l=0}^{4s+1} \left(B_{3l+2} \prod\limits_{k_2=l+1}^{4s+1} A_{3k_2+2} \right)}{V_1 \prod\limits_{k_1=0}^{4s+2} A_{3k_1+1} + \sum\limits_{l=0}^{4s+2} \left(B_{3l+1} \prod\limits_{k_2=l+1}^{4s+2} A_{3k_2+1} \right)} \end{aligned}$$

$$\begin{aligned} = u_0 \prod_{s=0}^{n-1} & \frac{\prod\limits_{k_1=0}^{4s-1} A_{3k_1} + u_0 u_2 \sum\limits_{l=0}^{4s-1} B_{3l} \prod\limits_{k_2=l+1}^{4s-1} A_{3k_2}}{\prod\limits_{k_1=0}^{4s-1} A_{3k_1+2} + u_2 u_4 \sum\limits_{l=0}^{4s-1} B_{3l+2} \prod\limits_{k_2=l+1}^{4s-1} A_{3k_2+2}} \frac{\prod\limits_{k_1=0}^{4s} A_{3k_1+1} + u_1 u_3 \sum\limits_{l=0}^{4s} B_{3l+1} \prod\limits_{k_2=l+1}^{4s} A_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+1} A_{3k_1} + u_0 u_2 \sum\limits_{l=0}^{4s+1} B_{3l} \prod\limits_{k_2=l+1}^{4s+1} A_{3k_2}} \\ & \times \frac{\prod\limits_{k_1=0}^{4s+1} A_{3k_1+2} + u_2 u_4 \sum\limits_{l=0}^{4s+1} \left(B_{3l+2} \prod\limits_{k_2=l+1}^{4s+1} A_{3k_2+2} \right)}{\prod\limits_{k_1=0}^{4s+2} A_{3k_1+1} + u_1 u_3 \sum\limits_{l=0}^{4s+2} \left(B_{3l+1} \prod\limits_{k_2=l+1}^{4s+2} A_{3k_2+1} \right)}. \end{aligned}$$

Hence x_{12n-4} is equal to

$$x_{-4}\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s-1}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s-1}b_{3l}\prod_{k_2=l+1}^{4s-1}a_{3k_2}}{\prod_{k_1=0}^{4s-1}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s-1}b_{3l+2}\prod_{k_2=l+1}^{4s-1}a_{3k_2+2}}\frac{\prod_{k_1=0}^{4s}a_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s}b_{3l+1}\prod_{k_2=l+1}^{4s}a_{3k_2+1}}{\prod_{k_1=0}^{4s+1}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s+1}b_{3l}\prod_{k_2=l+1}^{4s+1}a_{3k_2}}$$
$$\times\frac{\prod_{k_1=0}^{4s+1}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s+1}\left(b_{3l+2}\prod_{k_2=l+1}^{4s+1}a_{3k_2+2}\right)}{\prod_{k_1=0}^{4s+2}a_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s+2}\left(b_{3l+1}\prod_{k_2=l+1}^{4s+2}a_{3k_2+1}\right)}.$$

For $j=1$, we have

$$u_{12n+1}=u_1\prod_{s=0}^{n-1}\frac{V_1\prod_{k_1=0}^{4s-1}A_{3k_1+1}+\sum_{l=0}^{4s-1}B_{3l+1}\prod_{k_2=l+1}^{4s-1}A_{3k_2+1}}{V_0\prod_{k_1=0}^{4s}A_{3k_1}+\sum_{l=0}^{4s}B_{3l}\prod_{k_2=l+1}^{4s}A_{3k_2}}\frac{V_2\prod_{k_1=0}^{4s}A_{3k_1+2}+\sum_{l=0}^{4s}B_{3l+2}\prod_{k_2=l+1}^{4s}A_{3k_2+2}}{V_1\prod_{k_1=0}^{4s+1}A_{3k_1+1}+\sum_{l=0}^{4s+1}B_{3l+1}\prod_{k_2=l+1}^{4s+1}A_{3k_2+1}}$$
$$\times\frac{V_0\prod_{k_1=0}^{4s+2}A_{3k_1}+\sum_{l=0}^{4s+2}B_{3l}\prod_{k_2=l+1}^{4s+2}A_{3k_2}}{V_2\prod_{k_1=0}^{4s+2}A_{3k_1+2}+\sum_{l=0}^{4s+2}B_{3l+2}\prod_{k_2=l+1}^{4s+2}A_{3k_2+2}}$$

so that x_{12n-3} is equal to

$$x_{-3}\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s-1}a_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s-1}b_{3l+1}\prod_{k_2=l+1}^{4s-1}a_{3k_2+1}}{\prod_{k_1=0}^{4s}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s}b_{3l}\prod_{k_2=l+1}^{4s}a_{3k_2}}\frac{\prod_{k_1=0}^{4s}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s}b_{3l+2}\prod_{k_2=l+1}^{4s}A_{3k_2+2}}{\prod_{k_1=0}^{4s+1}A_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s+1}b_{3l+1}\prod_{k_2=l+1}^{4s+1}a_{3k_2+1}}$$
$$\times\frac{\prod_{k_1=0}^{4s+2}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s+2}b_{3l}\prod_{k_2=l+1}^{4s+2}a_{3k_2}}{\prod_{k_1=0}^{4s+2}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s+2}b_{3l+2}\prod_{k_2=l+1}^{4s+2}a_{3k_2+2}}.$$

For $j=2$, we have

$$u_{12n+2}=u_2\prod_{s=0}^{n-1}\frac{V_2\prod_{k_1=0}^{4s-1}A_{3k_1+2}+\sum_{l=0}^{4s-1}B_{3l+2}\prod_{k_2=l+1}^{4s-1}A_{3k_2+2}}{V_1\prod_{k_1=0}^{4s}A_{3k_1+1}+\sum_{l=0}^{4s}B_{3l+1}\prod_{k_2=l+1}^{4s}A_{3k_2+1}}\frac{V_0\prod_{k_1=0}^{4s+1}A_{3k_1}+\sum_{l=0}^{4s+1}B_{3l}\prod_{k_2=l+1}^{4s+1}A_{3k_2}}{V_2\prod_{k_1=0}^{4s+1}A_{3k_1+2}+\sum_{l=0}^{4s+1}B_{3l+2}\prod_{k_2=l+1}^{4s+1}A_{3k_2+2}}$$
$$\times\frac{V_1\prod_{k_1=0}^{4s+2}A_{3k_1+1}+\sum_{l=0}^{4s+2}B_{3l+2}\prod_{k_2=l+1}^{4s+2}A_{3k_2+1}}{V_0\prod_{k_1=0}^{4s+3}A_{3k_1}+\sum_{l=0}^{4s+3}B_{3l}\prod_{k_2=l+1}^{4s+3}A_{3k_2}}$$

so that

$$x_{12n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{4s-1}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s-1}b_{3l+2}\prod\limits_{k_2=l+1}^{4s-1}a_{3k_2+2}}{\prod\limits_{k_1=0}^{4s}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s}b_{3l+1}\prod\limits_{k_2=l+1}^{4s}a_{3k_2+1}}\frac{\prod\limits_{k_1=0}^{4s+1}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+1}b_{3l}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2}}{\prod\limits_{k_1=0}^{4s+1}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+1}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2+2}}$$
$$\times\frac{\prod\limits_{k_1=0}^{4s+2}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+3}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}b_{3l}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2}}.$$

Following similar substitutions as above where $u_i = x_{i-4}$ and $V_i = \frac{1}{x_{i-4}x_{i-2}}$, we deduce that for $x_{12n+j-4}$ with $j = 3, 4, 5, \ldots, 11$;

$$x_{12n-1} =$$
$$x_{-1}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{4s}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s}b_{3l}\prod\limits_{k_2=l+1}^{4s}a_{3k_2}}{\prod\limits_{k_1=0}^{4s}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s}b_{3l+2}\prod\limits_{k_2=l+1}^{4s}a_{3k_2+2}}\frac{\prod\limits_{k_1=0}^{4s+1}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+1}b_{3l+1}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+2}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+2}b_{3l}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2}}$$
$$\times\frac{\prod\limits_{k_1=0}^{4s+2}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+2}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2+2}}{\prod\limits_{k_1=0}^{4s+3}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+3}b_{3l+1}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2+1}},$$

$$x_{12n} =$$
$$x_0\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{4s}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s}b_{3l+1}\prod\limits_{k_2=l+1}^{4s}a_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+1}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+1}b_{3l}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2}}\frac{\prod\limits_{k_1=0}^{4s+1}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+1}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2+2}}{\prod\limits_{k_1=0}^{4s+2}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}b_{3l+1}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2+1}}$$
$$\times\frac{\prod\limits_{k_1=0}^{4s+3}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}b_{3l}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2}}{\prod\limits_{k_1=0}^{4s+3}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+3}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2+2}},$$

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$$x_{12n+1} = x_1 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s} b_{3l+2} \prod_{k_2=l+1}^{4s} a_{3k_2+2}}{\prod_{k_1=0}^{4s+1} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+1} b_{3l+1} \prod_{k_2=l+1}^{4s+1} a_{3k_2+1}} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+2} b_{3l} \prod_{k_2=l+1}^{4s+2} a_{3k_2}}{\prod_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+2} b_{3l+2} \prod_{k_2=l+1}^{4s+2} a_{3k_2+2}} \times \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+3} b_{3l+1} \prod_{k_2=l+1}^{4s+3} a_{3k_2+1}}{\prod_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+4} b_{3l} \prod_{k_2=l+1}^{4s+4} a_{3k_2}},$$

$$x_{12n+2} = x_2 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+1} b_{3l} \prod_{k_2=l+1}^{4s+1} a_{3k_2}}{\prod_{k_1=0}^{4s+1} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+1} b_{3l+2} \prod_{k_2=l+1}^{4s+1} a_{3k_2+2}} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+2} b_{3l+1} \prod_{k_2=l+1}^{4s+2} a_{3k_2+1}}{\prod_{k_1=0}^{4s+3} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+3} b_{3l} \prod_{k_2=l+1}^{4s+3} a_{3k_2}} \times \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+3} b_{3l+2} \prod_{k_2=l+1}^{4s+3} a_{3k_2+2}}{\prod_{k_1=0}^{4s+4} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+4} b_{3l+1} \prod_{k_2=l+1}^{4s+4} a_{3k_2+1}},$$

$$x_{12n+3} = x_3 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+1} b_{3l+1} \prod_{k_2=l+1}^{4s+1} a_{3k_2+1}}{\prod_{k_1=0}^{4s+2} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+2} b_{3l} \prod_{k_2=l+1}^{4s+2} a_{3k_2}} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+2} b_{3l+2} \prod_{k_2=l+1}^{4s+2} a_{3k_2+2}}{\prod_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+3} b_{3l+1} \prod_{k_2=l+1}^{4s+3} a_{3k_2+1}} \times \frac{\prod_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+4} b_{3l} \prod_{k_2=l+1}^{4s+4} a_{3k_2}}{\prod_{k_1=0}^{4s+4} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+4} b_{3l+2} \prod_{k_2=l+1}^{4s+4} a_{3k_2+2}},$$

$$x_{12n+4} =$$
$$x_4 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+1} b_{3l+2} \prod_{k_2=l+1}^{4s+1} a_{3k_2+2}}{\prod_{k_1=0}^{4s+2} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+2} b_{3l+1} \prod_{k_2=l+1}^{4s+2} a_{3k_2+1}} \frac{\prod_{k_1=0}^{4s+3} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+3} b_{3l} \prod_{k_2=l+1}^{4s+3} a_{3k_2}}{\prod_{k_1=0}^{4s+3} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+3} b_{3l+2} \prod_{k_2=l+1}^{4s+3} a_{3k_2+2}}$$
$$\times \frac{\prod_{k_1=0}^{4s+4} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+4} b_{3l+1} \prod_{k_2=l+1}^{4s+4} a_{3k_2+1}}{\prod_{k_1=0}^{4s+5} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+5} b_{3l} \prod_{k_2=l+1}^{4s+5} a_{3k_2}},$$

$$x_{12n+5} =$$
$$x_5 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+2} b_{3l} \prod_{k_2=l+1}^{4s+2} a_{3k_2}}{\prod_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+2} b_{3l+2} \prod_{k_2=l+1}^{4s+2} a_{3k_2+2}} \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+3} b_{3l+1} \prod_{k_2=l+1}^{4s+3} a_{3k_2+1}}{\prod_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+4} b_{3l} \prod_{k_2=l+1}^{4s+4} a_{3k_2}}$$
$$\times \frac{\prod_{k_1=0}^{4s+4} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+4} b_{3l+2} \prod_{k_2=l+1}^{4s+4} a_{3k_2+2}}{\prod_{k_1=0}^{4s+5} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+5} b_{3l+1} \prod_{k_2=l+1}^{4s+5} a_{3k_2+1}},$$

$$x_{12n+6} =$$
$$x_6 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+2} b_{3l+1} \prod_{k_2=l+1}^{4s+2} a_{3k_2+1}}{\prod_{k_1=0}^{4s+3} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+3} b_{3l} \prod_{k_2=l+1}^{4s+3} a_{3k_2}} \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+3} b_{3l+2} \prod_{k_2=l+1}^{4s+3} a_{3k_2+2}}{\prod_{k_1=0}^{4s+4} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+4} b_{3l+1} \prod_{k_2=l+1}^{4s+4} a_{3k_2+1}}$$
$$\times \frac{\prod_{k_1=0}^{4s+5} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+5} b_{3l} \prod_{k_2=l+1}^{4s+5} a_{3k_2}}{\prod_{k_1=0}^{4s+5} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+5} b_{3l+2} \prod_{k_2=l+1}^{4s+5} a_{3k_2+2}},$$

$$x_{12n+7} = x_7 \prod_{s=0}^{n-1} \frac{\prod\limits_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum\limits_{l=0}^{4s+2} b_{3l+2} \prod\limits_{k_2=l+1}^{4s+2} a_{3k_2+2}}{\prod\limits_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum\limits_{l=0}^{4s+3} b_{3l+1} \prod\limits_{k_2=l+1}^{4s+3} a_{3k_2+1}} \frac{\prod\limits_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum\limits_{l=0}^{4s+4} b_{3l} \prod\limits_{k_2=l+1}^{4s+4} a_{3k_2}}{\prod\limits_{k_1=0}^{4s+4} a_{3k_1+2} + x_{-2}x_0 \sum\limits_{l=0}^{4s+4} b_{3l+2} \prod\limits_{k_2=l+1}^{4s+4} a_{3k_2+2}}$$
$$\times \frac{\prod\limits_{k_1=0}^{4s+5} a_{3k_1+1} + x_{-3}x_{-1} \sum\limits_{l=0}^{4s+5} b_{3l+1} \prod\limits_{k_2=l+1}^{4s+5} a_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+6} a_{3k_1} + x_{-4}x_{-2} \sum\limits_{l=0}^{4s+6} b_{3l} \prod\limits_{k_2=l+1}^{4s+6} a_{3k_2}},$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 are given as follows:

$$x_1 = \frac{x_{-4}x_{-2}}{x_{-1}(a_0 + b_0x_{-4}x_{-2})},\ x_2 = \frac{x_{-3}x_{-1}}{x_0(a_1 + b_1x_{-3}x_{-1})},\ x_3 = \frac{x_{-1}x_0(a_0 + b_0x_{-4}x_{-2})}{x_{-4}(a_2 + b_2x_{-2}x_0)},$$

$$x_4 = \frac{x_{-4}x_{-2}x_0(a_1 + b_1x_{-3}x_{-1})}{x_{-3}x_{-1}(a_0a_3 + (b_0a_3 + b_3)x_{-4}x_{-2})},\ x_5 = \frac{x_{-3}x_{-4}(a_2 + b_2x_{-2}x_0)}{x_0(a_0 + b_0x_{-4}x_{-2})(a_1a_4 + (b_1a_4 + b_4)x_{-3}x_{-1})},$$

$$x_6 = \frac{x_{-3}x_{-1}(a_0a_3 + (b_0a_3 + b_3)x_{-4}x_{-2})}{x_{-4}(a_1 + b_1x_{-3}x_{-1})(a_5a_2 + (b_2a_5 + b_5)x_{-2}x_0)},$$

and

$$x_7 = \frac{x_{-2}x_0(a_0 + b_0x_{-4}x_{-2})(a_1a_4 + (b_1a_4 + b_4)x_{-3}x_{-1})}{x_{-3}(a_2 + b_2x_{-2}x_0)(a_6a_3a_0 + (a_6a_3b_0 + a_6b_3 + b_6)x_{-4}x_{-2})}.$$

We now turn our attention to special cases in the subsequent sections.

3 The case a_n and b_n are 1-periodic

Let $a_n = a$ and $b_n = b$, where $a, b \in \mathbb{R}$. We simply carry out a substitution and find the following solution:

$$x_{12n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{a^{4s} + bx_{-4}x_{-2} \sum\limits_{l=0}^{4s-1} a^l}{a^{4s} + bx_{-2}x_0 \sum\limits_{l=0}^{4s-1} a^l} \frac{a^{4s+1} + bx_{-3}x_{-1} \sum\limits_{l=0}^{4s} a^l}{a^{4s+2} + bx_{-4}x_{-2} \sum\limits_{l=0}^{4s+1} a^l} \frac{a^{4s+2} + bx_{-2}x_0 \sum\limits_{l=0}^{4s+1} a^l}{a^{4s+3} + bx_{-3}x_{-1} \sum\limits_{l=0}^{4s+2} a^l},$$

$$x_{12n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{a^{4s} + bx_{-3}x_{-1} \sum\limits_{l=0}^{4s-1} a^l}{a^{4s+1} + bx_{-4}x_{-2} \sum\limits_{l=0}^{4s} a^l} \frac{a^{4s+1} + bx_{-2}x_0 \sum\limits_{l=0}^{4s} a^l}{a^{4s+2} + bx_{-3}x_{-1} \sum\limits_{l=0}^{4s+1} a^l} \frac{a^{4s+3} + bx_{-4}x_{-2} \sum\limits_{l=0}^{4s+2} a^l}{a^{4s+3} + bx_{-2}x_0 \sum\limits_{l=0}^{4s+2} a^l},$$

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$$x_{12n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{a^{4s}+bx_{-2}x_0\sum_{l=0}^{4s-1}a^l}{a^{4s+1}+bx_{-3}x_{-1}\sum_{l=0}^{4s}a^l}\frac{a^{4s+2}+bx_{-4}x_{-2}\sum_{l=0}^{4s+1}a^l}{a^{4s+2}+bx_{-2}x_0\sum_{l=0}^{4s+1}a^l}\frac{a^{4s+3}+bx_{-3}x_{-1}\sum_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-4}x_{-2}\sum_{l=0}^{4s+3}a^l},$$

$$x_{12n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{a^{4s+1}+bx_{-4}x_{-2}\sum_{l=0}^{4s}a^l}{a^{4s+1}+bx_{-2}x_0\sum_{l=0}^{4s}a^l}\frac{a^{4s+2}+bx_{-3}x_{-1}\sum_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-4}x_{-2}\sum_{l=0}^{4s+2}a^l}\frac{a^{4s+3}+bx_{-2}x_0\sum_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-3}x_{-1}\sum_{l=0}^{4s+3}a^l},$$

$$x_{12n} = x_0\prod_{s=0}^{n-1}\frac{a^{4s+1}+bx_{-3}x_{-1}\sum_{l=0}^{4s}a^l}{a^{4s+2}+bx_{-4}x_{-2}\sum_{l=0}^{4s+1}a^l}\frac{a^{4s+2}+bx_{-2}x_0\sum_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-3}x_{-1}\sum_{l=0}^{4s+2}a^l}\frac{a^{4s+4}+bx_{-4}x_{-2}\sum_{l=0}^{4s+3}a^l}{a^{4s+4}+bx_{-2}x_0\sum_{l=0}^{4s+3}a^l},$$

$$x_{12n+1} = x_1\prod_{s=0}^{n-1}\frac{a^{4s+1}+bx_{-2}x_0\sum_{l=0}^{4s}a^l}{a^{4s+2}+bx_{-3}x_{-1}\sum_{l=0}^{4s+1}a^l}\frac{a^{4s+3}+bx_{-4}x_{-2}\sum_{l=0}^{4s+2}a^l}{a^{4s+3}+bx_{-2}x_0\sum_{l=0}^{4s+2}a^l}\frac{a^{4s+4}+bx_{-3}x_{-1}\sum_{l=0}^{4s+3}a^l}{a^{4s+5}+bx_{-4}x_{-2}\sum_{l=0}^{4s+4}a^l},$$

$$x_{12n+2} = x_2\prod_{s=0}^{n-1}\frac{a^{4s+2}+bx_{-4}x_{-2}\sum_{l=0}^{4s+1}a^l}{a^{4s+2}+bx_{-2}x_0\sum_{l=0}^{4s+1}a^l}\frac{a^{4s+3}+bx_{-3}x_{-1}\sum_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-4}x_{-2}\sum_{l=0}^{4s+3}a^l}\frac{a^{4s+4}+bx_{-2}x_0\sum_{l=0}^{4s+3}a^l}{a^{4s+5}+bx_{-3}x_{-1}\sum_{l=0}^{4s+4}a^l},$$

$$x_{12n+3} = x_3\prod_{s=0}^{n-1}\frac{a^{4s+2}+bx_{-3}x_{-1}\sum_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-4}x_{-2}\sum_{l=0}^{4s+2}a^l}\frac{a^{4s+3}+bx_{-2}x_0\sum_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-3}x_{-1}\sum_{l=0}^{4s+3}a^l}\frac{a^{4s+5}+x_{-4}x_{-2}\sum_{l=0}^{4s+4}a^l}{a^{4s+5}+bx_{-2}x_0\sum_{l=0}^{4s+4}a^l},$$

$$x_{12n+4} = x_4\prod_{s=0}^{n-1}\frac{a^{4s+2}+bx_{-2}x_0\sum_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-3}x_{-1}\sum_{l=0}^{4s+2}a^l}\frac{a^{4s+4}+bx_{-4}x_{-2}\sum_{l=0}^{4s+3}a^l}{a^{4s+4}+bx_{-2}x_0\sum_{l=0}^{4s+3}a^l}\frac{a^{4s+5}+bx_{-3}x_{-1}\sum_{l=0}^{4s+4}a^l}{a^{4s+6}+bx_{-4}x_{-2}\sum_{l=0}^{4s+5}a^l},$$

$$x_{12n+5} = x_5\prod_{s=0}^{n-1}\frac{a^{4s+3}+bx_{-4}x_{-2}\sum_{l=0}^{4s+2}a^l}{a^{4s+3}+bx_{-2}x_0\sum_{l=0}^{4s+2}a^l}\frac{a^{4s+4}+bx_{-3}x_{-1}\sum_{l=0}^{4s+3}a^l}{a^{4s+5}+bx_{-4}x_{-2}\sum_{l=0}^{4s+4}a^l}\frac{a^{4s+5}+bx_{-2}x_0\sum_{l=0}^{4s+4}a^l}{a^{4s+6}+bx_{-3}x_{-1}\sum_{l=0}^{4s+5}a^l},$$

$$x_{12n+6} = x_6 \prod_{s=0}^{n-1} \frac{a^{4s+3} + bx_{-3}x_{-1} \sum_{l=0}^{4s+2} a^l}{a^{4s+4} + bx_{-4}x_{-2} \sum_{l=0}^{4s+3} a^l} \frac{a^{4s+4} + bx_{-2}x_0 \sum_{l=0}^{4s+3} a^l}{a^{4s+5} + bx_{-3}x_{-1} \sum_{l=0}^{4s+4} a^l} \frac{a^{4s+6} + bx_{-4}x_{-2} \sum_{l=0}^{4s+5} a^l}{a^{4s+6} + bx_{-2}x_0 \sum_{l=0}^{4s+5} a^l},$$

$$x_{12n+7} = x_7 \prod_{s=0}^{n-1} \frac{a^{4s+3} + bx_{-2}x_0 \sum_{l=0}^{4s+2} a^l}{a^{4s+4} + bx_{-3}x_{-1} \sum_{l=0}^{4s+3} a^l} \frac{a^{4s+5} + bx_{-4}x_{-2} \sum_{l=0}^{4s+4} a^l}{a^{4s+5} + bx_{-2}x_0 \sum_{l=0}^{4s+4} a^l} \frac{a^{4s+6} + bx_{-3}x_{-1} \sum_{l=0}^{4s+5} a^l}{a^{4s+7} + bx_{-4}x_{-2} \sum_{l=0}^{4s+6} a^l},$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are given by

$$x_1 = \frac{x_{-4}x_{-2}}{x_{-1}(a + bx_{-4}x_{-2})}, \quad x_2 = \frac{x_{-3}x_{-1}}{x_0(a + bx_{-3}x_{-1})}, \quad x_3 = \frac{x_{-1}x_0(a + bx_{-4}x_{-2})}{x_{-4}(a + bx_{-2}x_0)},$$

$$x_4 = \frac{x_{-4}x_{-2}x_0(a + bx_{-3}x_{-1})}{x_{-3}x_{-1}(a^2 + (ab + b)x_{-4}x_{-2})}, \quad x_5 = \frac{x_{-4}x_{-3}(a + bx_{-2}x_0)}{x_0(a + bx_{-4}x_{-2})(a^2 + (ab + b)x_{-3}x_{-1})},$$

$$x_6 = \frac{x_{-3}x_{-1}(a^2 + (ab + b)x_{-4}x_{-2})}{x_{-4}(a + bx_{-3}x_{-1})(a^2 + (ab + b)x_{-2}x_0)}$$

and

$$x_7 = \frac{x_{-2}x_0(a + bx_{-4}x_{-2})(a^2 + (ab + b)x_{-3}x_{-1})}{x_{-3}(a + bx_{-2}x_0)(a^3 + (a^2b + ab + b)x_{-4}x_{-2})}.$$

3.1 The case $a = 1$

The solution, which appears for $b = \pm 1$ in Theorems 1 and 6 of [1], is given by

$$x_{12n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{1 + 4sbx_{-4}x_{-2}}{1 + 4sbx_{-2}x_0} \frac{1 + (4s + 1)bx_{-3}x_{-1}}{1 + (4s + 2)bx_{-4}x_{-2}} \frac{1 + (4s + 2)bx_{-2}x_0}{1 + (4s + 3)bx_{-3}x_{-1}},$$

$$x_{12n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{1 + 4sbx_{-3}x_{-1}}{1 + (4s + 1)bx_{-4}x_{-2}} \frac{1 + (4s + 1)bx_{-2}x_0}{1 + (4s + 2)bx_{-3}x_{-1}} \frac{1 + (4s + 3)bx_{-4}x_{-2}}{1 + (4s + 3)bx_{-2}x_0},$$

$$x_{12n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{1 + 4sbx_{-2}x_0}{1 + (4s + 1)bx_{-3}x_{-1}} \frac{1 + (4s + 2)bx_{-4}x_{-2}}{1 + (4s + 2)bx_{-2}x_0} \frac{1 + (4s + 3)bx_{-3}x_{-1}}{1 + (4s + 4)bx_{-4}x_{-2}},$$

$$x_{12n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1 + (4s + 1)bx_{-4}x_{-2}}{1 + (4s + 1)bx_{-2}x_0} \frac{1 + (4s + 2)bx_{-3}x_{-1}}{1 + (4s + 3)bx_{-4}x_{-2}} \frac{1 + (4s + 3)bx_{-2}x_0}{1 + (4s + 4)bx_{-3}x_{-1}},$$

$$x_{12n} = x_0 \prod_{s=0}^{n-1} \frac{1+(4s+1)bx_{-3}x_{-1}}{1+(4s+2)bx_{-4}x_{-2}} \frac{1+(4s+2)bx_{-2}x_0}{1+(4s+3)bx_{-3}x_{-1}} \frac{1+(4s+4)bx_{-4}x_{-2}}{1+(4s+4)bx_{-2}x_0},$$

$$x_{12n+1} = x_1 \prod_{s=0}^{n-1} \frac{1+(4s+1)bx_{-2}x_0}{1+(4s+2)bx_{-3}x_{-1}} \frac{1+(4s+3)bx_{-4}x_{-2}}{1+(4s+3)bx_{-2}x_0} \frac{1+(4s+4)bx_{-3}x_{-1}}{1+(4s+5)bx_{-4}x_{-2}},$$

$$x_{12n+2} = x_2 \prod_{s=0}^{n-1} \frac{1+(4s+2)bx_{-4}x_{-2}}{1+(4s+2)bx_{-2}x_0} \frac{1+(4s+3)bx_{-3}x_{-1}}{1+(4s+4)bx_{-4}x_{-2}} \frac{1+(4s+4)bx_{-2}x_0}{1+(4s+5)bx_{-3}x_{-1}},$$

$$x_{12n+3} = x_3 \prod_{s=0}^{n-1} \frac{1+(4s+2)bx_{-3}x_{-1}}{1+(4s+3)bx_{-4}x_{-2}} \frac{1+(4s+3)bx_{-2}x_0}{1+(4s+4)bx_{-3}x_{-1}} \frac{1+(4s+5)bx_{-4}x_{-2}}{1+(4s+5)bx_{-2}x_0},$$

$$x_{12n+4} = x_4 \prod_{s=0}^{n-1} \frac{1+(4s+2)bx_{-2}x_0}{1+(4s+3)bx_{-3}x_{-1}} \frac{1+(4s+4)bx_{-4}x_{-2}}{1+(4s+4)bx_{-2}x_0} \frac{1+(4s+5)bx_{-3}x_{-1}}{1+(4s+6)bx_{-4}x_{-2}},$$

$$x_{12n+5} = x_5 \prod_{s=0}^{n-1} \frac{1+(4s+3)bx_{-4}x_{-2}}{1+(4s+3)bx_{-2}x_0} \frac{1+(4s+4)bx_{-3}x_{-1}}{1+(4s+5)bx_{-4}x_{-2}} \frac{1+(4s+5)bx_{-2}x_0}{1+(4s+6)bx_{-3}x_{-1}},$$

$$x_{12n+6} = x_6 \prod_{s=0}^{n-1} \frac{1+(4s+3)bx_{-3}x_{-1}}{1+(4s+4)bx_{-4}x_{-2}} \frac{1+(4s+4)bx_{-2}x_0}{1+(4s+5)bx_{-3}x_{-1}} \frac{1+(4s+6)bx_{-4}x_{-2}}{1+(4s+6)bx_{-2}x_0},$$

$$x_{12n+7} = x_7 \prod_{s=0}^{n-1} \frac{1+(4s+3)bx_{-2}x_0}{1+(4s+4)bx_{-3}x_{-1}} \frac{1+(4s+5)bx_{-4}x_{-2}}{1+(4s+5)bx_{-2}x_0} \frac{1+(4s+6)bx_{-3}x_{-1}}{1+(4s+7)bx_{-4}x_{-2}},$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are given by

$$x_1 = \frac{x_{-4}x_{-2}}{x_{-1}(1+bx_{-4}x_{-2})}, \quad x_2 = \frac{x_{-3}x_{-1}}{x_0(1+bx_{-3}x_{-1})}, \quad x_3 = \frac{x_{-1}x_0(1+bx_{-4}x_{-2})}{x_{-4}(1+bx_{-2}x_0)},$$

$$x_4 = \frac{x_{-4}x_{-2}x_0(1+bx_{-3}x_{-1})}{x_{-3}x_{-1}(1+2bx_{-4}x_{-2})}, \quad x_5 = \frac{x_{-4}x_{-3}(1+bx_{-2}x_0)}{x_0(1+bx_{-4}x_{-2})(1+2bx_{-3}x_{-1})},$$

$$x_6 = \frac{x_{-3}x_{-1}(1+2bx_{-4}x_{-2})}{x_{-4}(1+bx_{-3}x_{-1})(1+2bx_{-2}x_0)} \quad \text{and} \quad x_7 = \frac{x_{-2}x_0(1+bx_{-4}x_{-2})(1+2bx_{-3}x_{-1})}{x_{-3}(1+bx_{-2}x_0)(1+3bx_{-4}x_{-2})}.$$

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3.2 The case $a = -1$

The solution, which appears for $b = \pm$ in Theorems 3 and 8 of [1], is given by

$$x_{12n-4} = x_{-4},\ x_{12n-3} = x_{-3},\ x_{12n-2} = x_{-2},\ x_{12n-1} = x_{-1},\ x_{12n} = x_0,$$

$$x_{12n+1} = \frac{x_{-4}x_{-2}}{x_{-1}(-1+bx_{-4}x_{-2})},\ x_{12n+2} = \frac{x_{-3}x_{-1}}{x_0(-1+bx_{-3}x_{-1})},\ x_{12n+3} = \frac{x_{-1}x_0(-1+bx_{-4}x_{-2})}{x_{-4}(-1+bx_{-2}x_0)},$$

$$x_{12n+4} = \frac{x_{-4}x_{-2}x_0(-1+bx_{-3}x_{-1})}{x_{-3}x_{-1}},\ x_{12n+5} = \frac{x_{-4}x_{-3}(-1+bx_{-2}x_0)}{x_0(-1+bx_{-4}x_{-2})},$$

$$x_{12n+6} = \frac{x_{-3}x_{-1}}{x_{-4}(-1+bx_{-3}x_{-1})},\ x_{12n+7} = \frac{x_{-2}x_0}{x_{-3}(-1+bx_{-2}x_0)}.$$

4 The case a_n and b_n are 3-periodic

The 3-periodicity of the sequences yields the following solution:

$$x_{12n-4} = x_{-4}\prod_{s=0}^{n-1} \frac{a_0^{4s} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s-1} a_0^l}{a_2^{4s} + b_2x_{-2}x_0\sum_{l=0}^{4s-1} a_2^l} \frac{a_1^{4s+1} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s} a_1^l}{a_0^{4s+2} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s+1} a_0^l} \frac{a_2^{4s+2} + b_2x_{-2}x_0\sum_{l=0}^{4s+1} a_2^l}{a_1^{4s+3} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s+2} a_1^l},$$

$$x_{12n-3} = x_{-3}\prod_{s=0}^{n-1} \frac{a_1^{4s} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s-1} a_1^l}{a_0^{4s+1} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s} a_0^l} \frac{a_2^{4s+1} + b_2x_{-2}x_0\sum_{l=0}^{4s} a_2^l}{a_1^{4s+2} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s+1} a_1^l} \frac{a_0^{4s+3} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s+2} a_0^l}{a_2^{4s+3} + b_2x_{-2}x_0\sum_{l=0}^{4s+2} a_2^l},$$

$$x_{12n-2} = x_{-2}\prod_{s=0}^{n-1} \frac{a_2^{4s} + b_2x_{-2}x_0\sum_{l=0}^{4s-1} a_2^l}{a_1^{4s+1} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s} a_1^l} \frac{a_0^{4s+2} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s+1} a_0^l}{a_2^{4s+2} + b_2x_{-2}x_0\sum_{l=0}^{4s+1} a_2^l} \frac{a_1^{4s+3} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s+2} a_1^l}{a_0^{4s+4} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s+3} a_0^l},$$

$$x_{12n-1} = x_{-1}\prod_{s=0}^{n-1} \frac{a_0^{4s+1} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s} a_0^l}{a_2^{4s+1} + b_2x_{-2}x_0\sum_{l=0}^{4s} a_2^l} \frac{a_1^{4s+2} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s+1} a_1^l}{a_0^{4s+3} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s+2} a_0^l} \frac{a_2^{4s+3} + b_2x_{-2}x_0\sum_{l=0}^{4s+2} a_2^l}{a_1^{4s+4} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s+3} a_1^l},$$

$$x_{12n} = x_0\prod_{s=0}^{n-1} \frac{a_1^{4s+1} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s} a_1^l}{a_0^{4s+2} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s+1} a_0^l} \frac{a_2^{4s+2} + b_2x_{-2}x_0\sum_{l=0}^{4s+1} a_2^l}{a_1^{4s+3} + b_1x_{-3}x_{-1}\sum_{l=0}^{4s+2} a_1^l} \frac{a_0^{4s+4} + b_0x_{-4}x_{-2}\sum_{l=0}^{4s+3} a_0^l}{a_2^{4s+4} + b_2x_{-2}x_0\sum_{l=0}^{4s+3} a_2^l},$$

$$x_{12n+1} = x_1 \prod_{s=0}^{n-1} \frac{a_2^{4s+1} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s} a_2^l}{a_1^{4s+2} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+1} a_1^l} \frac{a_0^{4s+3} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+2} a_0^l}{a_2^{4s+3} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+2} a_2^l} \frac{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+3} a_1^l}{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+4} a_0^l},$$

$$x_{12n+2} = x_2 \prod_{s=0}^{n-1} \frac{a_0^{4s+2} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+1} a_0^l}{a_2^{4s+2} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+1} a_2^l} \frac{a_1^{4s+3} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+2} a_1^l}{a_0^{4s+4} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+3} a_0^l} \frac{a_2^{4s+4} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+3} a_2^l}{a_1^{4s+5} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+4} a_1^l},$$

$$x_{12n+3} = x_3 \prod_{s=0}^{n-1} \frac{a_1^{4s+2} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+1} a_1^l}{a_0^{4s+3} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+2} a_0^l} \frac{a_2^{4s+3} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+2} a_2^l}{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+3} a_1^l} \frac{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+4} a_0^l}{a_2^{4s+5} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+4} a_2^l},$$

$$x_{12n+4} = x_4 \prod_{s=0}^{n-1} \frac{a_2^{4s+2} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+1} a_2^l}{a_1^{4s+3} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+2} a_1^l} \frac{a_0^{4s+4} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+3} a_0^l}{a_2^{4s+4} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+3} a_2^l} \frac{a_1^{4s+5} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+4} a_1^l}{a_0^{4s+6} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+5} a_0^l},$$

$$x_{12n+5} = x_5 \prod_{s=0}^{n-1} \frac{a_0^{4s+3} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+2} a_0^l}{a_2^{4s+3} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+2} a_2^l} \frac{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+3} a_1^l}{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+4} a_0^l} \frac{a_2^{4s+5} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+4} a_2^l}{a_1^{4s+6} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+5} a_1^l},$$

$$x_{12n+6} = x_6 \prod_{s=0}^{n-1} \frac{a_1^{4s+3} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+2} a_1^l}{a_0^{4s+4} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+3} a_0^l} \frac{a_2^{4s+4} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+3} a_2^l}{a_1^{4s+5} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+4} a_1^l} \frac{a_0^{4s=6} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+5} a_0^l}{a_2^{4s+6} + b_0 x_{-2} x_0 \sum\limits_{l=0}^{4s+5} a_0^l},$$

$$x_{12n+7} = x_7 \prod_{s=0}^{n-1} \frac{a_2^{4s+3} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+2} a_2^l}{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+3} a_1^l} \frac{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+4} a_0^l}{a_2^{4s+5} + b_2 x_{-2} x_0 \sum\limits_{l=0}^{4s+4} a_2^l} \frac{a_1^{4s+6} + b_1 x_{-3} x_{-1} \sum\limits_{l=0}^{4s+5} a_1^l}{a_0^{4s+7} + b_0 x_{-4} x_{-2} \sum\limits_{l=0}^{4s+6} a_0^l}$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 are given as follows:

$$x_1 = \frac{x_{-4}x_{-2}}{x_{-1}(a_0 + b_0x_{-4}x_{-2})}, \quad x_2 = \frac{x_{-3}x_{-1}}{x_0(a_1 + b_1x_{-3}x_{-1})}, \quad x_3 = \frac{x_{-1}x_0(a_0 + b_0x_{-4}x_{-2})}{x_{-4}(a_2 + b_2x_{-2}x_0)},$$

$$x_4 = \frac{x_{-4}x_{-2}x_0(a_1 + b_1x_{-3}x_{-1})}{x_{-3}x_{-1}(a_0^2 + (b_0a_0 + b_0)x_{-4}x_{-2})}, \quad x_5 = \frac{x_{-3}x_{-4}(a_2 + b_2x_{-2}x_0)}{x_0(a_0 + b_0x_{-4}x_{-2})(a_1^2 + (b_1a_1 + b_1)x_{-3}x_{-1})},$$

$$x_6 = \frac{x_{-3}x_{-1}(a_0^2 + (b_0a_0 + b_0)x_{-4}x_{-2})}{x_{-4}(a_1 + b_1x_{-3}x_{-1})(a_2^2 + (b_2a_2 + b_2)x_{-2}x_0)},$$

and

$$x_7 = \frac{x_{-2}x_0(a_0 + b_0x_{-4}x_{-2})(a_1^2 + (b_1a_1 + b_1)x_{-3}x_{-1})}{x_{-3}(a_2 + b_2x_{-2}x_0)(a_0^3 + (a_0^2b_0 + a_0b_0 + b_0)x_{-4}x_{-2})}.$$

5 Conclusion

In this paper, we derived symmetry generators for the difference equations (2) and explicit formulas for the solutions of the equations were also obtained. Our solution generalised Theorems 1, 3, 6 and 8 of Elsayed [1].

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

On some conditions for p-valency

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Abstract

In this paper we consider analytic functions in the unit disc $\mathbb{D}$ such that $|f^{(p)}(z)|$ is bounded in $\mathbb{D}$. We present several sufficient conditions for function to be p-valent starlike, convex or strongly starlike of a certain order.

Key Words and Phrases. univalent functions; starlike; convex; close-to-convex
2010 Mathematics Subject Classification. Primary 30C45, Secondary 30C80

1. Introduction

A function f analytic in a domain $D \in \mathbb{C}$ is called p-valent in D, if for every complex number w, the equation $f(z) = w$ has at most p roots in D, so that there exists a complex number w_0 such that the equation $f(z) = w_0$ has exactly p roots in D. We denote by $\mathcal{H}$ the class of functions $f(z)$ which are holomorphic in the open unit unit $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}(p)$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, the class of functions $f(z) \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}).$$

Let $\mathcal{A} = \mathcal{A}(1)$. Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $\mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha)$ be the subclasses of $\mathcal{A}(p)$ consisting of all p-valent functions which are starlike and convex of order α, $0 \le \alpha < 1$, defined as

$$\mathcal{S}_p^*(\alpha) = \left\{ f(z) \in \mathcal{A}(p) : \mathfrak{Re} \left\{ \frac{zf'(z)}{pf(z)} \right\} > \alpha, \ z \in \mathbb{D} \right\},$$

$$\mathcal{C}_p(\alpha) = \left\{ f(z) \in \mathcal{A}(p) : zf'(z)/p \in \mathcal{S}_p^*(\alpha) \right\}.$$

Note that $\mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{C}_1(0) = \mathcal{C}$, where $\mathcal{S}^*$ and $\mathcal{C}$ are usual classes of starlike and convex functions respectively.

The well-known Noshiro-Warschawski theorem [1, 10], says that if $f \in \mathcal{H}$ satisfies

$$\mathfrak{Re} \left\{ e^{i\alpha} f'(z) \right\} > 0, \quad (z \in \mathbb{D}) \tag{1.1}$$

for some real α, then $f(z)$ is univalent in $\mathbb{D}$. Ozaki [5], generalized the above theorem for $f \in \mathcal{A}(p)$: if

$$\mathfrak{Re} \left\{ e^{i\alpha} f^{(p)}(z) \right\} > 0, \quad (z \in \mathbb{D}) \tag{1.2}$$

for some real α, then $f(z)$ is at most p-valent in $\mathbb{D}$. Also in [3, 454] it was shown that if $f \in \mathcal{A}(p)$, $p \ge 2$, and

$$|\arg\{f^{(p)}(z)\}| < \frac{3\pi}{4} \quad (z \in \mathbb{D}), \tag{1.3}$$

then f is at most p-valent in $\mathbb{D}$.

The above results (1.1), (1.2) and (1.3) describe some consequences of a certain conditions on $\mathfrak{Re}\{f^{(p)}(z)\}$, or $|\arg\{f^{(p)}(z)\}|$. It is the purpose of this paper is to consider analytic functions with bounded modulus of a certain order of derivative, like $|f''(z)|$, and to present some implications of this hypothesis.

2. Main results

A function $f(z) \in \mathcal{H}$ is said to subordinate a function $g \in \mathcal{H}$ in the unit disc E, written $f \prec g$ if and only if there exits an analytic function $w \in \mathcal{H}$ such that $w(0) = 0, |w(z)| < 1$ and $f(z) = g[w(z)]$ for $z \in E$. Therefore $f \prec g$ in E implies $f(E) \subset g(E)$. In particular if g is univalent in E then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. The idea of subordination was used for defining many of the classes of functions studied in geometric function theory. In [9] Tuneski proved the following theorem.

Theorem 2.1. *If $f(z) \in \mathcal{A}$, $0 < k \leq 1$*

$$|f''(z)| \leq k, \quad (z \in \mathbb{D}),$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{kz}{2-k}, \quad (z \in \mathbb{D}).$$

In [6] it was proved a weaker result

$$|f''(z)| \leq 1, \quad (z \in \mathbb{D})$$

implies that $f(z)$ is univalent in $\mathbb{D}$. Applying Theorem 2.1, Tuneski in [9] obtained the following corollaries.

Corollary 2.2. *If $f(z) \in \mathcal{A}$, $0 \leq \alpha < 1$ and*

$$|f''(z)| \leq \frac{2(1-\alpha)}{2-\alpha}, \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathbb{D}).$$

The result is sharp.

Corollary 2.3. *If $f(z) \in \mathcal{A}$, $0 < \alpha \leq 1$ and*

$$|f''(z)| \leq \frac{2\sin(\alpha\pi/2)}{1+\sin(\alpha\pi/2)}, \quad (z \in \mathbb{D}),$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}).$$

The result is sharp.

In [9] Tuneski proved also the following result.

Theorem 2.4. *If $f(z) \in \mathcal{A}$, $0 < k \leq 1$*

$$|f''(z)| \leq k, \quad (z \in \mathbb{D}),$$

then

$$f'(z) \prec 1 + kz, \quad (z \in \mathbb{D}).$$

Theorem 2.4 implies the following corollary.

Corollary 2.5. *If $f(z) \in \mathcal{A}$, $0 \leq \alpha < 1$ and*

$$|f''(z)| \leq \frac{1-\alpha}{2-\alpha}, \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathbb{D}).$$

The result is sharp.

We also need the following result.

Theorem 2.6. [9] *If* $f(z) \in \mathcal{A}$, $0 < \lambda \leq 1$

$$|f'(z) - 1| \leq \lambda, \quad (z \in \mathbb{D}),$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

where

$$\alpha = \frac{2}{\pi}\sin^{-1}\left(\lambda\sqrt{1-(\lambda^2/4)} + \frac{\lambda}{2}\sqrt{1-\lambda^2}\right).$$

In [2] it was proved the following result.

Theorem 2.7. [2] *Let* $f(z) \in \mathcal{A}(p)$. *Suppose that there exists a positive integer* j, $1 \leq j \leq p$, *such that*

$$j + \mathfrak{Re}\left\{\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

Then we have

$$j - 1 + \mathfrak{Re}\left\{\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

3. Main results

Now we are going to make use of Theorem 2.1, Corollary 2.2 and of Theorem 2.7 to obtain the following theorem.

Theorem 3.1. [2] *Let* $f(z) \in \mathcal{A}(p)$. *Suppose that*

$$|f^{(p+1)}(z)| < p!, \quad (z \in \mathbb{D}).$$

Then $f(z)$ *is* p*-valently convex and* p*-valently starlike in* $\mathbb{D}$.

Proof. If we put

$$g(z) = \frac{1}{p!}f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then it follows that

$$|g''(z)| < \frac{|f^{(p+1)}(z)|}{p!} < 1, \quad (z \in \mathbb{D}).$$

From Theorem 2.1 and Corollary 2.2, we have

$$\mathfrak{Re}\left\{\frac{zg'(z)}{g(z)}\right\} = \mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

and so, we have

$$p - 1 + \mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > p - 1 \geq 0, \quad (z \in \mathbb{D}).$$

From Theorem 2.7, it follows that

$$1 + \mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{and} \quad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

This shows that $f(z)$ is p-valently convex and p-valently starlike in $\mathbb{D}$. □

For real α, $0 \leq \alpha < 1$, if $f(z) \in \mathcal{A}(p)$ satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

then $f(z)$ is called a strongly starlike function of order α. Applying Corollary 2.3 and the method of proving from [2, Th.5] give us the following theorems.

Theorem 3.2. *If $f(z) \in \mathcal{A}(p)$ and if there exists a α, $0 < \alpha \leq 1$, such that*

$$|f^{(p+1)}(z)| \leq \frac{2\sin(\alpha\pi/2)}{1+\sin(\alpha\pi/2)}, \quad (z \in \mathbb{D}), \tag{3.1}$$

then

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

or $f^{(p-1)}(z)/p!$ is strongly starlike of order α in $\mathbb{D}$.

Proof. For the case $p = 1$ Theorem 3.2 becomes Tuneski's result 2.3. Suppose that $p \geq 2$.
If we put

$$g(z) = \frac{1}{p!}f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then it follows that

$$\frac{zg'(z)}{g(z)} = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}, \quad (z \in \mathbb{D}).$$

From Corollary 2.3, we have

$$\left|\arg\left\{\frac{zg'(z)}{g(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

and so, we have

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

This shows that $f^{(p-1)}(z)/p!$ is strongly starlike of order α in $\mathbb{D}$. □

Again, applying [2, Th.5] yields us that if $f(z) \in \mathcal{A}(p)$, then for all $z \in \mathbb{D}$, we have

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \Rightarrow \quad \forall k \in \{1, \ldots, p\} : \quad \mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0. \tag{3.2}$$

Therefore, if we put

$$\frac{2f^{(p-2)}(z)}{p!} := G(z) = z^2 + \cdots \in \mathcal{A}(2),$$

then

$$\frac{zG'(z)}{G(z)} = \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}, \quad (z \in \mathbb{D})$$

and so (3.1) also implies that

$$\mathfrak{Re}\left\{\frac{zG'(z)}{G(z)}\right\} = \mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

This shows that $G(z)$ or $2f^{(p-2)}(z)/p!$ is 2-valently starlike in $\mathbb{D}$.

Theorem 3.3. *If $f(z) \in \mathcal{A}(p)$, $0 < \alpha \leq 1$, $1 \leq p$ and*

$$|f^{(p+1)}(z)| \leq \frac{1}{2}, \quad (z \in \mathbb{D}),$$

then

$$k + \mathfrak{Re}\left\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

for all k, $k \in \{1, 2, \ldots, p-1\}$.

Proof. If we put

$$g(z) = \frac{1}{p!} f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then it follows that

$$\frac{zg''(z)}{g'(z)} = \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}, \quad (z \in \mathbb{D}).$$

From Corollary 2.5, we have

$$1 + \mathfrak{Re}\left\{\frac{zg''(z)}{g'(z)}\right\} = 1 + \mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

and so, we have

$$p + \mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

Applying Theorem 2.7 gives finally

$$k + \mathfrak{Re}\left\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

for all k, $k \in \{1, 2, \ldots, p-1\}$. It completes the proof. □

From Theorem 3.3, we have

$$|f^{(p+1)}(z)| \leq \frac{1}{2}, \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{D}),$$

this suggests the following question.

Open problem. What is the best value of $\alpha(p)$ such that

$$|f^{(p+1)}(z)| \leq \frac{1}{2}, \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha(p), \quad (z \in \mathbb{D}).$$

If $p = 1$, then the function $f(z) = z + z^2/4$ shows that the best value of $\alpha(p)$ is 0.

Theorem 3.4. *If $f(z) \in \mathcal{A}(p)$, $0 < \lambda \leq 1$ and if*

$$|f^{(p)}(z) - p!| < p!\lambda, \quad (z \in \mathbb{D}), \tag{3.3}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}), \tag{3.4}$$

where

$$\alpha = \frac{2}{\pi} \sin^{-1}\left(\lambda\sqrt{1 - (\lambda^2/4)} + \frac{\lambda}{2}\sqrt{1 - \lambda^2}\right). \tag{3.5}$$

This means that $f(z)$ is strongly starlike of order α in $\mathbb{D}$.

Proof. If we put

$$g(z) = \frac{1}{p!} f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then from (3.3), we have

$$|g'(z) - 1| = \left|\frac{f^{(p)}(z)}{p!} - 1\right| < \lambda, \quad (z \in \mathbb{D}).$$

From Theorem 2.6, we have

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

where α has the form (3.5). Let us put

$$p(z) = \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}, \quad (z \in \mathbb{D}).$$

Then it follows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}$$

or

$$1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

From Theorem 2.6, we have

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

this gives

$$\left|\arg\left\{1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}). \tag{3.6}$$

If there exists a point $z_0 \in \mathbb{D}$, such that

$$|\arg\{p(z)\}| < \frac{\alpha\pi}{2}, \quad (|z| < |z_0|)$$

and

$$|\arg\{p(z_0)\}| = \frac{\alpha\pi}{2},$$

then from [4], we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k,$$

where k is a real number such that

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$$

when $p(z_0) = ia$, while for $p(z_0) = -ia$, such that

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right),$$

where $p^{1/\alpha}(z_0) = \pm ia$, $a > 0$. For the case $p^{1/\alpha}(z_0) = ia$, we have

$$\begin{aligned} 1 + \frac{z_0 f''(z_0)}{f'(z_0)} &= p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} \\ &= p(z_0)\left\{1 + \frac{z_0 p'(z_0)}{p^2(z_0)}\right\} \\ &= (ia)^\alpha \left\{1 + i\alpha k \frac{1}{(ia)^\alpha}\right\} \\ &= a^\alpha e^{i\alpha\pi/2}\left\{1 + e^{i\pi(1-\alpha)/2}\alpha k \frac{1}{a^\alpha}\right\}. \end{aligned}$$

Thus, it is trivial that

$$\arg\left\{1 + \frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\} \geq \frac{\alpha\pi}{2}$$

since we have

$$\arg\left\{1 + e^{i\pi(1-\alpha)/2}\alpha k \frac{1}{a^\alpha}\right\} > 0,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right).$$

This contradicts (3.6) and for the case $p^{1/\alpha}(z_0) = -ia$, applying the same method as the above, we would have

$$\arg\left\{1 + \frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\} \leq -\frac{\alpha\pi}{2}$$

which also contradicts (3.6). Applying the same method repeatedly once again, we can complete the proof of Theorem 3.4. □

We now note that Pommerenke [7] and Sakaguchi [8] showed the following.

Lemma 3.5. [7] *If f and h are analytic in $\mathbb{D}$, and h is convex and univalent in $\mathbb{D}$, with*

$$\left|\arg\left\{\frac{f'(z)}{h'(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

for some real α, $0 \leq \alpha \leq 1$, then

$$\left|\arg\left\{\frac{f(z_2) - f(z_1)}{h(z_2) - h(z_1)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

for all $z_1, z_2 \in \mathbb{D}$.

Putting $z_1 = 0$, $z_2 = z$ in Lemma 3.5 gives

$$\left|\arg\left\{\frac{f'(z)}{h'(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \quad \Rightarrow \quad \left|\arg\left\{\frac{f(z)}{h(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}). \tag{3.7}$$

Therefore, applying Theorem 3.4 and (3.7) we can deduce the following corollary.

Corollary 3.6. *If $f(z) \in \mathcal{A}(p)$ is such that*

$$\int_0^z \frac{f(t)}{t}\mathrm{d}t$$

is a convex function, and if

$$|f^{(p)}(z) - p!| < p!\lambda, \quad (z \in \mathbb{D}), \tag{3.8}$$

for some λ, $0 < \lambda \leq 1$, then

$$\left|\arg\left\{\frac{f(z)}{\int_0^z \frac{f(t)}{t}\mathrm{d}t}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}), \tag{3.9}$$

where α is given in (3.5).

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Fractional Cauchy Euler Differential Equation

By

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Abstract

In this paper we give general solution of fractional linear differential equations and fractional Cauchy Euler equation. Since there are many definitions for fractional derivatives, we use the conformable derivative to get exact solutions. Factorizing polynomials of the fractional differential operators is the key method to get such solutions. Some specific examples on both types of equations are presented.

Key Words and Phrases:Conformable, Cauchy Euler, Conformable Linear Differential equations, Conformable Cauchy Euler Equation.

AMS Classification Number : 26A33

1. Introduction

Many authors have solved many well known differential equations like the Conformable Fractional Heat equation, Bessel equation, Legendre equation and many more. [1], [4], [5], [6], [7], [9] and [10]. The Cauchy Euler equation is a well known important type of ordinary differential equation. In This paper we give the procedure and justification of how to handle the Cauchy Euler equation, but the fractional one.

However, there are many definitions available in the literature for fractional derivatives. The main ones are the Riemann Liouville definition and the Caputo definition, see [8] .

(i) Riemann - Liouville Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

(ii) Caputo Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Such definitions have many setbacks such as

(i) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0$ ($D_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^\alpha(f/g) = \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^\alpha(f \circ g)(t) = f^{(\alpha)}(g(t))\, g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$, in general.

(vi) All fractional derivatives, specially Caputo definition, assumes that the function f is differentiable.

We refer the reader to [3] for more results on Caputo and Riemann - Liouville Definitions.

Recently, the authors in [2], gave a new definition of fractional derivative which is a natural extension to the usual first derivative. So many papers since then were written, and many equations were solved using such definition. The definition goes as follows:

Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then for all $t > 0$, $\alpha \in (0, 1)$, let

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

T_α is called the **conformable fractional derivative of f of order α**.
Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$.

If f is $\alpha-$differentiable in some $(0, b)$, $b > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, **then define**

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

According to this definition, we have the following properties, [2],

1. $T_\alpha(1) = 0,$

2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R},$

3. $T_\alpha(\sin at) = at^{1-\alpha}\cos at, \quad a \in \mathbb{R},$

4. $T_\alpha(\cos at) = -at^{1-\alpha}\sin at, \quad a \in \mathbb{R}$

5. $T_\alpha(e^{at}) = at^{1-\alpha}e^{at}, \quad a \in \mathbb{R}.$

Further, many functions behave as in the usual derivative. Here are some formulas

$$T_\alpha(\frac{1}{\alpha}t^\alpha) = 1$$
$$T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha},$$
$$T_\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos(\frac{1}{\alpha}t^\alpha),$$
$$T_\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin(\frac{1}{\alpha}t^\alpha).$$

We will use the conformable fractional derivative for the Cauchy Euler equation. But first, we present the linear fractional case with constant coefficients.

2. Conformable Linear Differential equations

Let us write $y^{(n\alpha)}$ to denote the $\alpha-$derivative of y, $n-$times. That is $y^{(n\alpha)} = T_\alpha T_\alpha ... T_\alpha(y)$, n-times.

Theorem 1. *Let*

$$y^{(n\alpha)} + a_{n-1}y^{(n-1)\alpha} + ... + a_1 y^\alpha + a_0 y = 0 \tag{1}$$

Consider the equation

$$r^{(n\alpha)} + a_{n-1}r^{(n-1)\alpha} + ... + a_1 r^\alpha + a_0 = 0 \tag{$*$}$$

If $r_1^\alpha = \lambda_1, ..., r_n^\alpha = \lambda_n$ *are the real roots of* $(*)$ *then* $y_h = c_1 y_1 + ... + c_n y_n$ *where* $y_k = e^{r_k^\alpha e^{\frac{t^\alpha}{\alpha}}}$.

Proof. Let $T^{n\alpha} = T^\alpha T^\alpha ... T^\alpha$ n-times. Then equation (1) can be written in the form

$$(T^{(n\alpha)} + a_{n-1}T^{(n-1)\alpha} + ... + a_1 r^\alpha + a_0 I)y = 0 \tag{2}$$

(where $T^\alpha = \frac{d^\alpha}{dx^\alpha}$).

Now, if we let $D = T^\alpha$ then (2) becomes

$$(D^n + a_{n-1}D^{n-1} + ... + a_0 I)y = 0$$

The polynomial $(D^n + a_{n-1}D^{n-1} + ... + a_0 I)y = 0$, factorizes to

$$(D - \lambda_1)(D - \lambda_2)...(D - \lambda_n)y = 0 \tag{3}$$

Now, y will be a solution to (3) if $y \in \ker(D-\lambda_k) \forall 1 \le k \le n$, noting that $(D-\lambda_i)$ commutes with $(D-\lambda_j)$ for all i and j . Thus y is the solution for (3) if

$$(D-\lambda_1)y = 0 \text{ or } (D-\lambda_2)y = 0 \text{ or ...or } (D-\lambda_n)y = 0$$

However $(D-\lambda_k)y = 0$ implies

$Dy - \lambda_k y = 0$

So $y^{\alpha} - \lambda_k y = 0$

Hence $y_k = e^{\lambda_k e^{\frac{t^\alpha}{\alpha}}}$ if λ_k is real.

Consequently, $y_h = c_1 e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} + ... + c_n e^{\lambda_n e^{\frac{t^\alpha}{\alpha}}}$, if all the roots are real and distinct.

Now, replacing T^α by r^α we get

$$(r^\alpha - \lambda_1)y = 0 \text{ or } (r^\alpha - \lambda_2)y = 0 \text{ or ...or } (r^\alpha - \lambda_n)y = 0$$

Thus the roots are

$$r_1^\alpha = \lambda_1 \ , \ r_2^\alpha = \lambda_2 \ , ..., r_n^\alpha = \lambda_n$$

and the general solution is

$$y_h = c_1 e^{r_1^\alpha e^{\frac{t^\alpha}{\alpha}}} + ... + c_n e^{r_n^\alpha e^{\frac{t^\alpha}{\alpha}}}$$

There are two other cases for the roots to be considered:

(1) (i)If one of the root is repeated, say λ_1, 2-times. That is $(T^\alpha - \lambda_1)^2$ is a factor of (3). Then $y_1 = e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}, y_2 = \frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}$, are two independent solutions for the differential equation (3).

Proof. We have to show

$$(T^\alpha - \lambda_1)^2 \ \frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} = 0$$

Indeed:

$$(T^\alpha - \lambda_1) \ (T^\alpha - \lambda_1) \frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} = 0$$

$$= (T^\alpha - \lambda_1) \ \left[T^\alpha \left(\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}\right) - \lambda_1 \left(\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}\right) = 0 \right]$$

$$= (T^\alpha - \lambda_1) \ \left[e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} + \lambda_1 \left(\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}\right) - \lambda_1 \left(\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}\right) = 0 \right]$$

$$= (T^\alpha - \lambda_1) \ e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} = 0$$

Similarly one can show that if λ_1 is repeated k-times then

$$y_1 = e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}, \frac{t^\alpha}{\alpha} y_1, ..., \left(\frac{t^\alpha}{\alpha}\right)^{k-1} y_1$$

are independent solutions.

(ii)There is a root, say $\lambda_1 = a + ib$, $a, b \in \mathbb{R}$. Then

$$y_1 = e^{a\frac{t^\alpha}{\alpha}} \cos b \frac{t^\alpha}{\alpha} \text{ and } \quad y_2 = e^{a\frac{t^\alpha}{\alpha}} \sin b \frac{t^\alpha}{\alpha}$$

are two solutions of (3) associated with λ_1.

Indeed:

Since $\lambda_1 = a + ib$ is a root, then $\overline{\lambda_1} = a - ib$ is a root.
Then

$$y_1 = e^{(a+ib)\frac{t^\alpha}{\alpha}} \text{ and } \quad y_2 = e^{(a-ib)\frac{t^\alpha}{\alpha}} \text{ are solutions of (3)}$$

But

$$y_1 = e^{a\frac{t^\alpha}{\alpha}}(\cos b\frac{t^\alpha}{\alpha} \;+\; i\sin b\frac{t^\alpha}{\alpha})$$

$$y_2 = e^{a\frac{t^\alpha}{\alpha}}(\cos b\frac{t^\alpha}{\alpha} \;-\; i\sin b\frac{t^\alpha}{\alpha})$$

Place $y_1 + y_2$ is a solution (the equation being homogenous) and $y_1 - y_2$ is a solution too. So

$$\widetilde{y_1} = y_1 + y_2 = 2e^{a\frac{t^\alpha}{\alpha}}\cos b\frac{t^\alpha}{\alpha} \text{ and } \widetilde{y_2} = y_1 - y_2 = 2ie^{a\frac{t^\alpha}{\alpha}}\sin b\frac{t^\alpha}{\alpha}$$

are solutions of the homogenous equation (3).

Consequentely

$$\widetilde{\widetilde{y_1}} = \frac{1}{2}\widetilde{y_1} = e^{a\frac{t^\alpha}{\alpha}}\cos b\frac{t^\alpha}{\alpha} \text{ and } \widetilde{\widetilde{y_2}} = \frac{1}{2i}\widetilde{y_2} = e^{a\frac{t^\alpha}{\alpha}}\sin b\frac{t^\alpha}{\alpha}$$

are two independent solutions for the equation.

Example 1

$$T^{2\alpha}y + T^\alpha y - 2y = 0 \qquad (i)$$

Solution. Consider the associated equation

$$r^{2\alpha} + r^\alpha - 2 = 0$$

$$(r^\alpha - 2)(r^\alpha + 1) = 0$$

Hence $\lambda_1 = 2, \lambda_2 = -1$

Thus $\quad y_1 = c_1 e^{2\frac{t^\alpha}{\alpha}}$ *and* $y_2 = c_2 e^{-\frac{t^\alpha}{\alpha}}$

One can easily check that these are solutions of (i).See figure (1)

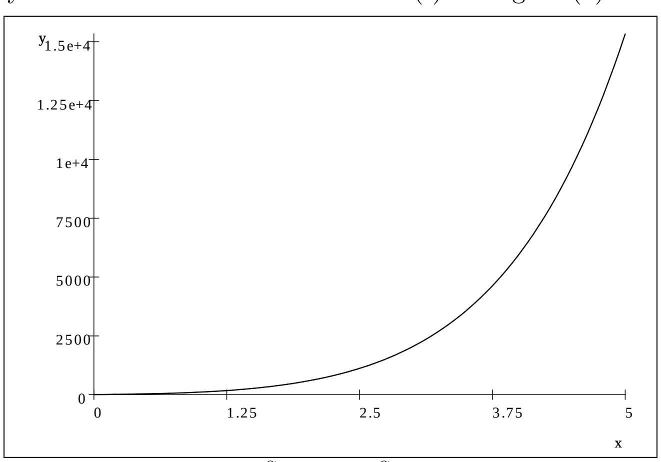

Fig.1 $y = c_1 e^{2\frac{t^\alpha}{\alpha}} + c_2 e^{-\frac{t^\alpha}{\alpha}}, \alpha = 0.5, c_1 > 0$

3. Conformable Cauchy Euler Equation

The standard form of the classical homogenous Cauchy Euler equation of order 2 is:

$$x^2y'' + a_1xy' + a_0y = 0$$

Now the conformable Cauchy Euler equation of order 2 can be written as

$$x^{2\alpha}T^{2\alpha}y + a_1x^{\alpha}T^{\alpha}y + a_0y = 0 \tag{1}$$

Now we will give the procedure how to solve (1).

Procedure

Put $y = x^{\alpha r}$

Then

$$T^{2\alpha}y = T^{\alpha}(T^{\alpha}y)$$
$$= T^{\alpha}(T^{\alpha}x^{\alpha r})$$
$$= T^{\alpha}(\alpha r x^{\alpha r-\alpha})$$
$$= \alpha r(\alpha r - \alpha)x^{\alpha r-2\alpha}$$
$$a_1T^{\alpha}y = a_1(T^{\alpha}x^{\alpha r})$$
$$= a_1\alpha r x^{\alpha r-\alpha}$$

Thus

$$x^{2\alpha}T^{2\alpha}y = x^{2\alpha}(\alpha^2 r(r-1))x^{\alpha r}x^{-2\alpha}$$
$$a_1x^{\alpha}T^{\alpha}y = a_1\alpha r x^{\alpha(r-1)}x^{\alpha r}$$
$$a_0y = a_0x^{\alpha r}$$

Hence

$$x^{2\alpha}(\alpha^2 r(r-1))x^{-2\alpha}.x^{\alpha r} + a_1\alpha r x^{\alpha r}x^{\alpha r}x^{-\alpha}x^{\alpha} + a_0x^{\alpha r} = 0$$

So

$$x^{\alpha r}\left[\alpha^2 r(r-1) + a_1\alpha r + a_0\right] = 0$$

Solve

$$\alpha^2 r(r-1) + a_1\alpha r + a_0 = 0$$

to get $r = r_1, r = r_2$. Assume r_1, r_2 are reals . Then

$y_1 = x^{\alpha r_1},\ y_2 = x^{\alpha r_2}$ are two independent solutions of (1) and

$$y_h = c_1x^{\alpha r_1} + \ c_2x^{\alpha r_2}$$

Remark. The case of conformable Cauchy Euler Equation of any order can be handled in the same way as the case of order 2.

Example 2. Solve

$$x^{2\alpha}y^{(2\alpha)} + x^{2\alpha}y^{(\alpha)} - \frac{y}{2} = 0, y(1) = 1,\ y^{(\alpha)}(1) = 1$$

Solution. Put $y = x^{\alpha r}$ and substitute in the equation to get

$\alpha^2 r(r-1) + \alpha r - \frac{1}{2} = 0$

Take $\alpha = \frac{1}{2}$ we get

$\frac{1}{4} r(r-1) + \frac{1}{2} r - \frac{1}{2} = 0$

$r(r-1) + 2r - 2 = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r_1 = 2, r_2 = 1$

$y_1 = x^{-\frac{2}{2}},\ y_2 = x^{\frac{1}{2}}$ $\qquad (\alpha = \frac{1}{2})$

$y_h = c_1 \frac{1}{x} + c_2 \sqrt[2]{x}$

$y(1) = c_1 + c_2$

$y^{\frac{1}{2}}(x) = c_1(-1)x^{-1-\frac{1}{2}} + c_2 \frac{1}{2}$

So $y^{\frac{1}{2}}(1) = -c_1 + \frac{c_2}{2} = 1$. Hence $\frac{3}{2} c_2 = 2 \Longrightarrow c\ c_1 = -\frac{1}{3}$

$y_h = -\frac{1}{3x} + \frac{4}{3} \sqrt[2]{x}$. See figure (2).

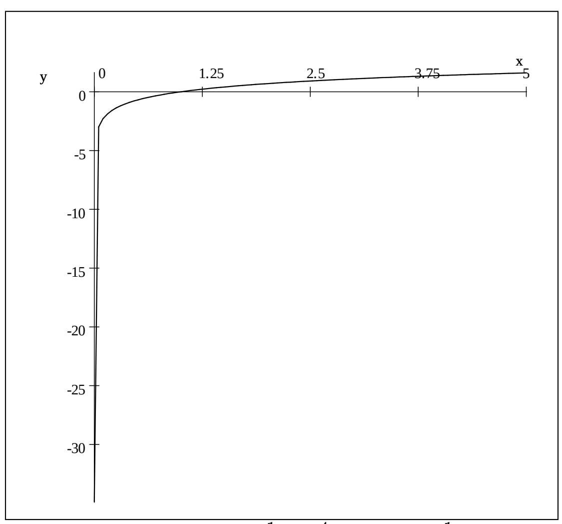

Fig.2 $y_h = -\frac{1}{3x} + \frac{4}{3} \sqrt[2]{x}$, $\alpha = \frac{1}{2}$

4. Conclusion

Conformable fractional derivative can be applied to solve linear differential equation with variable coefficients as an example Cauchy Euler equation.

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Applications of neutrosophic sets in B-algebras

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Abstract. The notions of a neutrosophic subalgebra and a neutrosophic normal subalgebra of a B-algebra are introduced and characterizations of them are discussed. We show that the homomorphic preimage of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra, and the onto homomorphic image of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra.

1. Introduction

Zadeh [12] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). Y. B. Jun, E. H. Roh and H. S. Kim [4] introduced a new notion, called a BH-algebra. J. Neggers and H. S. Kim [9] introduced a new notion, called a B-algebra. C. B. Kim and H. S. Kim [7] introduced the notion of a BG-algebra which is a generalization of B-algebras. S. S. Ahn and H. D. Lee [1] classified the subalgebras by their family of level subalgebras in BG-algebras.

In this paper, we introduce the notions of a neutrosophic subalgebra and a neutrosophic normal subalgebra of a B-algebra and discuss characterizations of them. We show that the homomorphic preimage of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra, and the onto homomorphic image of neutrosophic image of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra.

2. Preliminaries

A *B-algebra* ([9]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B) $(x * y) * z = x * (z * (0 * y))$

for any x, y, z in X. For brevity we call X a *B-algebra.* In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BH-algebra* if it satisfies (B1), (B2) and

(BH) $x * y = y * x = 0$ imply $x = y$ for any $x, y \in X$.

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[0] **2010 Mathematics Subject Classification**: 06F35; 03G25; 03B52.

[0] **Keywords**: B-algebra; (normal) subalgebra; neutrosophic subalgebra; neutrosophic normal subalgebra.

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An algebra $(X;*,0)$ of type $(2,0)$ is called a *BG-algebra* if it satisfies (B1), (B2) and

(BG) $(x*y)*(0*y)=x$ for any $x,y\in X$.

Proposition 2.1.([3, 9]) *Let $(X;*,0)$ be a B-algebra. Then*

(i) *the left cancellation law holds in X, i.e., $x*y=x*z$ implies $y=z$,*
(ii) *if $x*y=0$, then $x=y$ for any $x,y\in X$,*
(iii) *if $0*x=0*y$, then $x=y$ for any $x,y\in X$,*
(iv) *$0*(0*x)=x$, for all $x\in X$,*
(v) *$x*(y*z)=(x*(0*z))*y$ for all $x,y,z\in X$.*

Theorem 2.2.([7]) *If $(X;*,0)$ is a B-algebra, then it is a BG-algebra.*

Proposition 2.3.([7]) *Every BG-algebra is a BH-algebra.*

Let $(X;*_X,0_X)$ and $(Y;*_Y,0_Y)$ be B-algebras. A mapping $\varphi:X\to Y$ is called a *homomorphism* if $\varphi(x*_X y)=\varphi(x)*_Y\varphi(y)$ for any $x,y\in X$. A non-empty subset S of X is called a *subalgebra* of X if $x*y\in S$ for any $x,y\in X$. A non-empty subset N of X is said to be *normal* if $(x*a)*(y*b)\in N$ for any $x*y,a*b\in N$. Then any normal subset N of a B-algebra X is a subalgebra of X, but the converse need not be true ([10]). A non-empty subset X of a B-algebra X is a called a *normal subalgebra* of X if it is both a subalgebra and a normal set.

Definition 2.4. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosophic set A can be denoted by

$$A:=\{\langle x,T_A(x),I_A(x),F_A(x)\rangle|x\in X\},$$

where $T_A(x),I_A(x),F_A(x)\in[0,1]$ for each point x in X. Therefore the sum of $T_A(x),I_A(x)$, and $F_A(x)$ satisfies the condition $0\le T_A(x)+I_A(x)+F_A(x)\le 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.5. Let A be a neutrosophic set in a B-algebra X and $\alpha,\beta,\gamma\in[0,1]$ with $0\le\alpha+\beta+\gamma\le 3$ and an (α,β,γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)}=\{x\in X|T_A(x)\le\alpha,I_A(x)\ge\beta,F_A(x)\le\gamma\}.$$

For any family $\{a_i|i\in\Lambda\}$, we define

$$\bigvee\{a_i|i\in\Lambda\}:=\begin{cases}\max\{a_i|i\in\Lambda\} & \text{if } \Lambda \text{ is finite,}\\ \sup\{a_i|i\in\Lambda\} & \text{otherwise}\end{cases}$$

and

$$\bigwedge\{a_i|i\in\Lambda\}:=\begin{cases}\min\{a_i|i\in\Lambda\} & \text{if } \Lambda \text{ is finite,}\\ \inf\{a_i|i\in\Lambda\} & \text{otherwise.}\end{cases}$$

3. Neutrosophic subalgebras in B-algebras

Definition 3.1. A neutrosophic set A in a B-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{F_A(x), F_A(y)\}$, for any $x, y \in X$.

Proposition 3.2. *Every neutrosophic subalgebra of a B-algebra X satisfies the following conditions:*

(3.1) $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$, *and* $F_A(0) \leq F_A(x)$ *for any* $x \in X$.

Proof. Straightforward. □

Example 3.3. Let $X := \{0, 1, 2, 3\}$ be a B-algebra with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 2\} \\ 0.84, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.82, & \text{if } x \in \{0, 2\} \\ 0.15, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 2\} \\ 0.84, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Theorem 3.4. *Let A be a neutrosophic set in a B-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x) \leq \alpha, T_A(y) \leq \alpha, I_A(x) \geq \beta, I_A(y) \geq \beta$ and $F_A(x) \leq \gamma, F_A(y) \leq \gamma$. Using (NSS), we have $T_A(x*y) \leq \max\{T_A(x), T_A(y)\} \leq \alpha$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\} \geq \beta$, and $F_A(x*y) \leq \max\{F_A(x), F_A(y)\} \leq \gamma$. Hence $x*y \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Assume that there exist $a_t, b_t, a_i, b_i \in X$ and $a_f, b_f \in X$ such that $T_A(a_t * b_t) > \max\{T_A(a_t), T_A(b_t)\}, I_A(a_i * b_i) < \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f * b_f) > \max\{F_A(a_f), F_A(b_f)\}$. Then $T_A(a_t * b_t) > \alpha_1 \geq \max\{T_A(a_t), T_A(b_t)\}, I_A(a_i * b_i) < \beta_1 \leq \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f * b_f) > \gamma_1 \geq \max\{F_A(a_f), F_A(b_f)\}$ for some $\alpha_1, \gamma_1 \in [0, 1)$ and $\beta_1 \in (0, 1]$. Hence $a_t, b_t, a_i, b_i \in A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f, b_f \in A^{(\alpha_1,\beta_1,\gamma_1)}$. But $a_t * b_t, a_i * b_i \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f * b_f \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, which is a contradiction. Hence $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{T_A(x), T_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.5. *If $\{A_i|i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a B-algebra X, then $(\{A_i|i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 3.6. *Let A be a neutrosophic subalgebra of a B-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 0, \lim_{n\to\infty} I_A(a_n) = 1$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$.*

Proof. By Proposition 3.2, we have $T_A(0) \le T_A(x), I_A(0) \ge I_A(x)$, and $F_A(0) \le F_A(x)$ for all $x \in X$. Hence we have $T_A(0) \le T_A(a_n), I_A(0) \ge I_A(a_n)$, and $F_A(0) \le F_A(a_n)$ for every positive integer n. Therefore $0 \le T_A(0) \le \lim_{n\to\infty} T_A(a_n) = 0, 1 = \lim_{n\to\infty} I_A(a_n) \le I_A(0) \le 1$, and $0 \le F_A(0) \le \lim_{n\to\infty} F_A(a_n) = 0$. Thus we have $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$. □

Proposition 3.7. *If every neutrosophic subalgebra A of a B-algebra X satisfies the condition*

(3.2) $T_A(x * y) \le T_A(y), I_A(x * y) \ge I_A(y), F_A(x * y) \le F_A(y)$, *for any $x, y \in X$,*

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.2) that $T_A(x) = T_A(x * 0) \le T_A(0), I_A(x) = I_A(x * 0) \ge I_A(0)$, and $F_A(x) = F_A(x * 0) \le F_A(0)$ for any $x \in X$. By Proposition 3.2, we have $T_A(x) = T_A(0), I_A(x) = I_A(0)$, and $F_A(x) = F_A(0)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Definition 3.8. A neutrosophic set A in a B-algebra X is said to be *neutrosophic normal* of X if it satisfies:

(NSN) $T_A((x*a)*(y*b)) \le \max\{T_A(x*y), T_A(a*b)\}, I_A((x*a)*(y*b)) \ge \min\{I_A(x*y), I_A(a*b)\}$, and $F_A((x*a) * (y * b)) \le \max\{F_A(x * y), F_A(a * b)\}$, for any $x, y, a, b \in X$.

A neutrosophic set A in a B-algebra X is called a *neutrosophic normal subalgebra* of X if it satisfies (NSS) and (NSN).

Example 3.9. Let $X := \{0, 1, 2, 3\}$ be a B-algebra ([8]) with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.12, & \text{if } x \in \{0, 3\} \\ 0.76, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.73, & \text{if } x \in \{0, 3\} \\ 0.14, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.12, & \text{if } x \in \{0, 3\} \\ 0.76, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic normal subalgebra of X.

Proposition 3.10. *Every neutrosophic normal of a B-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be neutrosophic normal of X. Put $y := 0, b := 0$ and $a := y$ in (NSN). Then $T_A((x*y)*(0*0)) \leq \max\{T_A(x*0), T_A(y*0)\}, I_A((x*y)*(0*0)) \geq \min\{I_A(x*0), I_A(y*0)\}$, and $F_A((x*y)*(0*0)) \leq \max\{F_A(x*0), F_A(y*0)\}$. Using (B2) and (B1), we have $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{F_A(x), F_A(y)\}$, for any $x, y \in X$. Hence A is a neutrosophic subalgebra of X. □

The converse of Proposition 3.10 may not be true in general (see Example 3.11).

Example 3.11. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebra ([10]) with the following table:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.12, & \text{if } x = 0 \\ 0.23, & \text{if } x = 5 \\ 0.52 & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.58, & \text{if } x = 0 \\ 0.13, & \text{if } x = 5 \\ 0.11, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.12, & \text{if } x = 0 \\ 0.23, & \text{if } x = 5 \\ 0.52 & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X. But it is not neutrosophic normal of X, since $T_A(1) = T_A((1*3)*(4*2)) = 0.52 \not\leq \max\{T_A(1*4), T_A(3*2)\} = \max\{T_A(5), T_A(5)\} = 0.23$, and/or $I_A(1) = I_A((1*3)*(4*2)) = 0.11 \not\geq \min\{I_A(1*4), I_A(3*2)\} = \min\{I_A(5), I_A(5)\} = 0.13$, and/or $F_A(1) = F_A((1*3)*(4*2)) = 0.52 \not\leq \max\{F_A(1*4), F_A(3*2)\} = \max\{F_A(5), F_A(5)\} = 0.23$.

Theorem 3.12. *Let A be a neutrosophic set in a B-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. Then A is a neutrosophic normal subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are normal subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Similar to Theorem 3.4. □

Proposition 3.13. *Let A be a neutrosophic normal subalgebra of a B-algebra X. Denote that $X_T := \{x \in X | T_A(x) = T_A(0)\}, X_I := \{x \in X | I_A(x) = I_A(0)\}$, and $X_F := \{x \in X | F_A(x) = F_A(0)\}$. Then X_T, X_I, and X_F are normal subalgebras of X.*

Proof. It is sufficient to show that X_T, X_I, and X_F are normal. Let $a, b, x, y \in X$ be such that $x * y, a * b \in X_T$. Then $T_A(x*y) = T_A(0) = T_A(a*b)$. Since A is a neutrosophic normal subalgebra of X, we have $T_A((x*a)*(y*b)) \leq \max\{T_A(x*y), T_A(a*b)\} = T_A(0)$. By Proposition 3.2, we get $T_A((x*a)*(y*b)) = T_A(0)$. Hence $(x*a)*(y*b) \in X_T$. Therefore X_T is a normal subalgebra of X. Similarly, X_I, X_F are normal subalgebras of X. This completes the proof. □

Definition 3.14. Let A and B be neutrosophic sets of a set X. The *union* of A and B is defined to be a neutrosophic set

$$A\tilde{\cup}B := \{\langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle | x \in X\},$$

where $T_{A\cup B}(x) = \min\{T_A(x), T_B(x)\}, I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}, F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$, for all $x \in X$. The *intersection* of A and B is defined to be a neutrosophic set

$$A\tilde{\cap}B := \{\langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle | x \in X\},$$

where $T_{A\cap B}(x) = \max\{T_A(x), T_B(x)\}, I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}, F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$, for all $x \in X$.

Theorem 3.15. *The intersection of two neutrosophic subalgebras of a B-algebra X is a also a neutrosophic subalgebra of X.*

Proof. Let A and B be neutrosophic subalgebras of X. For any $x, y \in X$, we have

$$\begin{aligned} T_{A\cap B}(x*y) &= \max\{T_A(x*y), T_B(x*y)\} \\ &\leq \max\{\max\{T_A(x), T_A(y)\}, \max\{T_B(x), T_B(y)\}\} \\ &= \max\{\max\{T_A(x), T_B(x)\}, \max\{T_A(y), T_B(y)\}\} \\ &= \max\{T_{A\cap B}(x), T_{A\cap B}(y)\}, \\ I_{A\cap B}(x*y) &= \min\{I_A(x*y), I_B(x*y)\} \\ &\geq \min\{\min\{I_A(x), I_A(y)\}, \min\{I_B(x), I_B(y)\}\} \\ &= \min\{\min\{I_A(x), I_B(x)\}, \min\{I_A(y), I_B(y)\}\} \\ &= \min\{I_{A\cap B}(x), I_{A\cap B}(y)\}, \end{aligned}$$

and

$$\begin{aligned} F_{A\cap B}(x*y) &= \max\{F_A(x*y), F_B(x*y)\} \\ &\leq \max\{\max\{F_A(x), F_A(y)\}, \max\{F_B(x), F_B(y)\}\} \\ &= \max\{\max\{F_A(x), F_B(x)\}, \max\{F_A(y), F_B(y)\}\} \\ &= \max\{F_{A\cap B}(x), F_{A\cap B}(y)\}. \end{aligned}$$

Hence $A\tilde{\cap}B$ is a neutrosophic subalgebra of X. □

Corollary 3.16. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosophic subalgebras of a B-algebra X, then so is $\tilde{\cap}_{i\in\mathbb{N}} A_i$.*

The union of any set of neutrosophic subalgebras of a B-algebra X need not be a neutrosophic subalgebra of X.

Example 3.17. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebra as in Example 3.11. Define neutrosophic sets A and B of X as follows:

$$T_A(x) = \begin{cases} 0.11, & \text{if } x \in \{0, 4\} \\ 0.73 & \text{otherwise}, \end{cases}$$

$$I_A(x) = \begin{cases} 0.82, & \text{if } x \in \{0, 4\} \\ 0.12, & \text{otherwise}, \end{cases}$$

$$F_A(x) = \begin{cases} 0.11, & \text{if } x \in \{0, 4\} \\ 0.73 & \text{otherwise}, \end{cases}$$

$$T_B(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 5\} \\ 0.74 & \text{otherwise}, \end{cases}$$

$$I_B(x) = \begin{cases} 0.83, & \text{if } x \in \{0, 5\} \\ 0.13, & \text{otherwise}, \end{cases}$$

and

$$F_B(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 5\} \\ 0.74 & \text{otherwise}. \end{cases}$$

It is easy to check that A and B are neutrosophic subalgebras of X. But $A\tilde{\cup}B$ is not a neutrosophic subalgebra of X, since

$$\begin{aligned} T_{A\cup B}(4 * 5) =& T_{A\cup B}(2) = \min\{T_A(2), T_B(2)\} = 0.73 \\ &\not\leq \max\{T_{A\cup B}(4), T_{A\cup B}(5)\} \\ &= \max\{\min\{T_A(4), T_B(4)\}, \min\{T_A(5), T_B(5)\}\} = 0.13, \end{aligned}$$

and/or

$$\begin{aligned} I_{A\cup B}(4 * 5) =& I_{A\cup B}(2) = \max\{I_A(2), I_B(2)\} = 0.13 \\ &\not\geq \min\{I_{A\cup B}(4), I_{A\cup B}(5)\} \\ &= \min\{\max\{I_A(4), I_B(4)\}, \max\{I_A(5), I_B(5)\}\} = 0.82, \end{aligned}$$

and/or

$$\begin{aligned} F_{A\cup B}(4 * 5) =& F_{A\cup B}(2) = \min\{F_A(2), F_B(2)\} = 0.73 \\ &\not\leq \max\{F_{A\cup B}(4), F_{A\cup B}(5)\} \\ &= \max\{\min\{F_A(4), F_B(4)\}, \min\{F_A(5), F_B(5)\}\} = 0.13. \end{aligned}$$

Let $f : X \to Y$ be a function of sets. If $M = \{\langle y, T_M(y), I_M(y), F_M(y)\rangle | y \in Y\}$ is a neutrosophic set of a set Y, then the preimage of M under f is defined to be a neutrosophic set

$$f^{-1}(M) := \{\langle x, f^{-1}(T_M)(x), f^{-1}(I_M)(x), f^{-1}(F_M)(x)\rangle | x \in X\}$$

of X, where $f^{-1}(T_M)(x) = T_M(f(x)), f^{-1}(I_M)(x) = I_M(f(x))$ and $f^{-1}(F_M)(x) = F_M(f(x))$ for all $x \in X$.

Theorem 3.18. *Let $f : X \to Y$ be a homomorphism of B-algebras. If $M = \{\langle y, T_M(y), I_M(y), F_M(y)\rangle | y \in Y\}$ is a neutrosophic subalgebra of Y, then the preimage of M under f is a neutrosophic subalgebra of X.*

Proof. Let $f^{-1}(M)$ be the preimage of M under f. For any $x, y \in X$, we have

$$\begin{aligned} f^{-1}(T_M(x*y)) =& T_M(f(x*y)) = T_M(f(x)*f(y)) \\ & \le \max\{T_M(f(x)), T_M(f(y))\} = \max\{f^{-1}(T_M)(x), f^{-1}(T_M)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(I_M(x*y)) =& I_M(f(x*y)) = I_M(f(x)*f(y)) \\ & \ge \min\{I_M(f(x)), I_M(f(y))\} = \min\{f^{-1}(I_M)(x), f^{-1}(I_M)(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^{-1}(F_M(x*y)) =& F_M(f(x*y)) = F_M(f(x)*f(y)) \\ & \le \max\{F_M(f(x)), F_M(f(y))\} = \max\{f^{-1}(F_M)(x), f^{-1}(F_M)(y)\}. \end{aligned}$$

Hence $f^{-1}(M)$ is a neutrosophic subalgebra of X. □

Let $f : X \to Y$ be an onto function of sets. If A is a neutrosophic set of X, then the image of A under f is defined to be a neutrosophic set

$$f(A) := \{\langle y, f(T_A)(y), f(I_A)(y), f(F_A)(y)\rangle | y \in Y\}$$

of Y, where $f(T_A)(y) = \bigwedge_{x\in f^{-1}(y)} T_A(x), f(I_A)(y) = \bigvee_{x\in f^{-1}(y)} I_A(x)$, and $f(F_A)(y) = \bigwedge_{x\in f^{-1}(y)} F_A(x)$.

Theorem 3.19. *For an onto homomorphism $f : X \to Y$ of B-algebras, let A be a neutrosophic set of X such that*

(3.3) $(\forall C \subseteq X)(\exists x_0 \in C)(T_A(x_0) = \bigwedge_{z\in C} T_A(z), I_A(x_0) = \bigvee_{z\in C} I_A(z), F_A(x_0) = \bigwedge_{z\in C} F_A(z))$.

If A is a neutrosophic subalgebra of a B-algebra X, then the image of A under f is a neutrosophic subalgebra of Y.

Proof. Let $f(A)$ be the image of A under f. Let $a, b \in Y$. Then $f^{-1}(a) \neq \emptyset$ and $f^{-1}(b) \neq \emptyset$ in X. By (3.3), there exist $x_a \in f^{-1}(a)$ and $x_b \in f^{-1}(b)$ such that

$$T_A(x_a) = \bigwedge_{z\in f^{-1}(a)} T_A(z), I_A(x_a) = \bigvee_{z\in f^{-1}(a)} I_A(z), F_A(x_a) = \bigwedge_{z\in f^{-1}(a)} F_A(z),$$

$$T_A(x_b) = \bigwedge_{w\in f^{-1}(b)} T_A(w), I_A(x_b) = \bigvee_{w\in f^{-1}(b)} I_A(w), F_A(x_b) = \bigwedge_{w\in f^{-1}(b)} F_A(w).$$

Thus

$$\begin{aligned} f(T_A)(a*b) =& \bigwedge_{x\in f^{-1}(a*b)} T_A(x) \le T_A(x_a*x_b) \le \max\{T_A(x_a), T_A(x_b)\} \\ & = \max\{\bigwedge_{z\in f^{-1}(a)} T_A(z), \bigwedge_{w\in f^{-1}(b)} T_A(w)\} = \max\{f(T_A)(a), f(T_A)(b)\}, \end{aligned}$$

$$\begin{aligned} f(I_A)(a*b) =& \bigvee_{x\in f^{-1}(a*b)} I_A(x) \ge I_A(x_a*x_b) \ge \min\{I_A(x_a), I_A(x_b)\} \\ & = \min\{\bigvee_{z\in f^{-1}(a)} I_A(z), \bigvee_{w\in f^{-1}(b)} I_A(w)\} = \min\{f(I_A)(a), f(I_A)(b)\}, \end{aligned}$$

and

$$\begin{aligned} f(F_A)(a*b) = & \bigwedge_{x\in f^{-1}(a*b)} F_A(x) \le F_A(x_a * x_b) \le \max\{F_A(x_a), F_A(x_b)\} \\ &= \max\{\bigwedge_{z\in f^{-1}(a)} F_A(z), \bigwedge_{w\in f^{-1}(b)} F_A(w)\} = \max\{f(F_A)(a), f(F_A)(b)\}. \end{aligned}$$

Hence $f(A)$ is a neutrosophic subalgebra of Y. □

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Investigating Some Properties of a Fourth Order Difference Equation

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ABSTRACT

The principal purpose of this paper is to present some qualitative behavior of the following fourth order difference equation:

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}, \quad n = 0, 1, ...,$$

where the parameters $a,\ b,\ c$ and d are positive real numbers and the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are arbitrary non zero real numbers.

Keywords: stability, periodicity, global attractor, difference equations.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

This paper will provide a detailed study in terms of the local, global stability and obtain the form of the solutions of the following difference equation

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are arbitrary non zero real numbers and a, b, c, d are positive constants..

A huge number of researchers has concentrated on studying and investigating nonlinear difference equations in recent years. In particular, they have highlighted the boundedness, the global attractivity and the periodic behaviour of some certain types of difference equations. For instance: Elsayed et al.[19] studied the global attractor, local stability, periodic solutions and boundedness of the following recursive equation:

$$x_{n+1} = \frac{ax_n x_{n-2}}{bx_n + cx_{n-3}}.$$

Cinar[5] investigated the solution of the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Ibrahim[24] presented some relevant results of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Elsayed[16] analyzed the global stability and examined the periodic solution of the following difference equation:

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-l}}{cx_{n-l} - dx_{n-k}}.$$

Elabbasy et al.[8] investigated the global stability, periodicity character and gave the solution of special case of the difference equation

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Yang et al.[36] examined the global and local stability of the equilibrium points of the following recursive equation:

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1}x_{n-2}}.$$

Simsek et al.[33] obtained the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Abo-Zeid et al.[1] gave a detailed study about the convergence and the periodicity of the solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{(B - Cx_n x_{n-2})}.$$

Tolly et al.[35] illustrated some properties of the solution of the following recursive equation:

$$y_{n+1} = \frac{ay_{n-1}}{by_n y_{n-1} + cy_{n-1}y_{n-2} + d}.$$

Other relevant consequences of rational difference equations can be obtained in refs.[9]-.[12]

Now, some relevant results and definitions will be introduced here to be used in our discussion.

Let I be some interval of real numbers and the function f has continuous partial derivatives on I^{k+1} where $I^{k+1} = I \times I \times \cdots \times I$ $(k+1-$ times). Then, for initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.1. *(Stability)*

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1},..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now assume that the characteristic equation associated with Eq.(3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{4}$$

where $p_i = \dfrac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}$.

Theorem A [12]: *Assume that* $p_i \in R$, $i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \ \ n = 0, 1, ...$$

Next, we introduce a fundamental theorem to prove the global attractor of the fixed points.

Theorem B [26]: Let $g : [a, b]^{k+1} \to [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k + 1$; the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \qquad M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k + 1$, we set

$$m_i = \left\{ \begin{matrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{matrix} \right\}, \quad Mi = \left\{ \begin{matrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{matrix} \right\}.$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (5), and every solution of Equation (5) converges to $\bar{x}$.

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT

This section is devoted to give a detailed description about the local stability of the fixed point.

The equilibrium point of Eq.(1) is given by the following equation:

$$\overline{x} = a\overline{x} - \frac{b\overline{x}}{c\overline{x} - d\overline{x}},$$

from which we have

$$\overline{x} = \frac{b}{(a-1)(c-d)},$$

where $a \neq 1$ and $c \neq d$. Suppose that $f : (0,\infty)^2 \longrightarrow (0,\infty)$ defined as following:

$$f(u,v) = au - \frac{bu}{cu - dv}. \tag{6}$$

Then,

$$\frac{\partial f(u,v)}{\partial u} = a - \frac{b(cu-dv) - bcu}{(cu-dv)^2} = a + \frac{bdv}{(cu-dv)^2}, \tag{7}$$

$$\frac{\partial f(u,v)}{\partial v} = -\frac{-bu(-d)}{(cu-dv)^2} = -\frac{bdu}{(cu-dv)^2}. \tag{8}$$

Next, we calculate equations (7) and (8) at the equilibrium point as follows:

$$\frac{\partial f(\overline{x},\overline{x})}{\partial u} = a + \frac{bd\overline{x}}{(c\overline{x} - d\overline{x})^2} = a + \frac{bd}{(c-d)^2\overline{x}} = a + \frac{d(a-1)}{(c-d)} = -p_0,$$

$$\frac{\partial f(\overline{x},\overline{x})}{\partial v} = -\frac{bd\overline{x}}{(c\overline{x} - d\overline{x})^2} = -\frac{bd}{(c-d)^2\overline{x}} = -\frac{d(a-1)}{(c-d)} = -p_1.$$

Now, the linearized difference equation of Eq.(1) about the fixed point is given by

$$y_{n+1} + p_0 y_{n-1} + p_1 y_{n-3} = 0.$$

Theorem 1. Assume that

$$|ac-d| + d\,|a-1| < |c-d|\,.$$

Then the fixed point of Eq.(1) is locally asymptotically stable.

Proof. By using Theorem A we notice that Eq.(1) is asymptotically stable if

$$|p_0| + |p_1| < 1.$$

Hence, we have

$$\left|a + \frac{d(a-1)}{(c-d)}\right| + \left|-\frac{d(a-1)}{(c-d)}\right| < 1,$$

which can be rearranged as follows:

$$|a(c-d)+d(a-1)|+|-d(a-1)|<|(c-d)|.$$

Therefore,

$$|ac-d|+d\,|a-1|<|c-d|.$$

This completes the proof.

3. GLOBAL STABILITY OF THE EQUILIBRIUM POINT

The global attractivity character of the considered equation will be presented in this section.

Theorem 2. The equilibrium point of Eq.(1) is a global attractor if $a<1$.

Proof. Suppose that p and q are two real numbers and let $f:[p,q]^2\longrightarrow[p,q]$ be a function defined by Eq.(6). Then, equations (7) and (8) tell us that $f(u,v)$ is increasing in u and decreasing in v. Now, we assume that (m,M) is a solution of the following system:

$$m=f(m,M),\quad\text{and}\quad M=f(M,m).$$

Substituting this into Eq.(6) gives

$$\begin{aligned} m &= am-\frac{bm}{cm-dM},\\ M &= aM-\frac{bM}{cM-dm}. \end{aligned}$$

Then,

$$cm^2-dmM=acm^2-admM-bm, \tag{9}$$

$$cM^2-dmM=acM^2-admM-bM. \tag{10}$$

Subtracting Eq.(9) from Eq.(10) yields

$$c(m^2-M^2)=ac(m^2-M^2)+b\,(M-m).$$

Hence, we obtain

$$(m-M)\,[c(1-a)(m+M)+b]=0.$$

Thus, when $a<1$, then we have

$$m=M.$$

We conclude from Theorem B that the equilibrium point is a global attractor of Eq.(1).

4. PERIODICITY OF THE SOLUTION

This section will present a theorem which shows that Eq.(1) has no periodic solution.

Theorem 3. Eq.(1) has no prime period two solutions.

Proof. We will use contradiction to prove this theorem. Assume that Eq.(1) has a positive prime period two solutions given as follows:

$$..., p, q, p, q, ...$$

Then,

$$p = ap - \frac{bp}{cp - dp}. \tag{11}$$

$$q = aq - \frac{bq}{cq - dq}. \tag{12}$$

Equations (11) and (12) can be written as follows:

$$p(a-1) = \frac{b}{c-d},$$

$$q(a-1) = \frac{b}{c-d},$$

which implies that $p = q$ and this contradicts the fact that $p \neq q$.

5. SPECIAL CASE OF EQ.(1)

In this section we will study the solution of the following special case:

$$x_{n+1} = x_{n-1} - \frac{x_{n-1}}{x_{n-1} - x_{n-3}}, \qquad n = 0, 1, 2, ..., \tag{13}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are nonzero real numbers with $x_{-3} \neq x_{-1}$ and $x_{-2} \neq x_0$.

Theorem 4. Let $\{x_n\}_{n=-3}^{\infty}$ be the solution of Eq. (13) satisfying $x_{-3} = r,\ x_{-2} = l,\ x_{-1} = k$ and $x_0 = h$. Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{4n-3} &= nk - (n-1)r - n(n-1) - \frac{nk}{k-r}, \\
x_{4n-2} &= nh - (n-1)l - n(n-1) - \frac{nh}{h-l}, \\
x_{4n-1} &= (n+1)k - nr - n^2 - \frac{nk}{k-r}, \\
x_{4n} &= (n+1)h - nl - n^2 - \frac{nh}{h-l}.
\end{aligned}$$

Proof. For $n = 0$ the result holds. Now, we assume that $n > 0$ and our assumption satisfies for $n-1$. That is

$$x_{4n-7} = (n-1)k - (n-2)r - (n-1)(n-2) - \frac{(n-1)k}{k-r},$$

$$x_{4n-6} = (n-1)h - (n-2)l - (n-1)(n-2) - \frac{(n-1)h}{h-l},$$

$$x_{4n-5} = nk-(n-1)r-(n-1)^2-\frac{(n-1)k}{k-r},$$

$$x_{4n-4} = nh-(n-1)l-(n-1)^2-\frac{(n-1)h}{h-l}.$$

Next, it follows form Eq. (13) that

$$\begin{aligned}
x_{4n-3} &= x_{4n-5}-\frac{x_{4n-5}}{x_{4n-5}-x_{4n-7}},\\
&= nk-(n-1)r-(n-1)^2-\frac{(n-1)k}{k-r}\\
&\quad -\frac{nk-(n-1)r-(n-1)^2-\frac{(n-1)k}{k-r}}{nk-(n-1)r-(n-1)^2-\frac{(n-1)k}{k-r}-((n-1)k-(n-2)r-(n-1)(n-2)-\frac{(n-1)k}{k-r})},\\
&= nk-(n-1)r-(n-1)^2-\frac{(n-1)k}{k-r}-\frac{(nk-nr+r)(k-n-r+1)}{(k-r)(k-n-r+1)},\\
&= nk-(n-1)r-(n-1)^2-\frac{2nk-k-nr+r}{k-r},\\
&= nk-(n-1)r-\frac{n(nk-rn+r)}{k-r},\\
&= nk-(n-1)r-n(n-1)-\frac{nk}{k-r}.
\end{aligned}$$

Also, we obtain from Eq. (13)

$$\begin{aligned}
x_{4n-2} &= x_{4n-4}-\frac{x_{4n-4}}{x_{4n-4}-x_{4n-6}},\\
&= nh-(n-1)l-(n-1)^2-\frac{(n-1)h}{h-l}\\
&\quad -\frac{nh-(n-1)l-(n-1)^2-\frac{(n-1)h}{h-l}}{nh-(n-1)l-(n-1)^2-\frac{(n-1)h}{h-l}-(n-1)h+(n-2)l+(n-1)(n-2)+\frac{(n-1)h}{h-l}},\\
&= nh-(n-1)l-(n-1)^2-\frac{(n-1)h}{h-l}-\frac{(nh-nl+l)(h-l-n+1)}{(h-l)(h-l-n+1)},\\
&= nh-(n-1)l-(n-1)^2-\frac{2nh-h-nl+l}{h-l},\\
&= nh-(n-1)l-\frac{n(nh-nl+l)}{h-l},\\
&= nh-(n-1)l-n(n-1)-\frac{nh}{h-l}.
\end{aligned}$$

Next, we will prove the third part of the theorem. Eq.(13) gives

$$\begin{aligned} x_{4n-1} &= x_{4n-3} - \frac{x_{4n-3}}{x_{4n-3} - x_{4n-5}}, \\ &= nk - (n-1)r - n(n-1) - \frac{nk}{k-r} \\ &\quad - \frac{nk - (n-1)r - n(n-1) - \frac{nk}{k-r}}{[nk - (n-1)r - n(n-1) - \frac{nk}{k-r}] - [nk - (n-1)r - (n-1)^2 - \frac{(n-1)k}{k-r}]}, \\ &= nk - (n-1)r - n(n-1) - \frac{nk}{k-r} + \frac{(k-n-r)(nk-nr+r)(k-r)}{(k-r)(nk-nr+r)}, \\ &= nk - (n-1)r - n(n-1) - \frac{nk}{k-r} + k - n - r = (n+1)k - nr - n^2 - \frac{nk}{k-r}. \end{aligned}$$

Finally, we prove the last part of the theorem. Eq.(13) leads to

$$\begin{aligned} x_{4n} &= x_{4n-2} - \frac{x_{4n-2}}{x_{4n-2} - x_{4n-4}}, \\ &= nh - (n-1)l - n(n-1) - \frac{nh}{h-l} \\ &\quad - \frac{nh - (n-1)l - n(n-1) - \frac{nh}{h-l}}{[nh - (n-1)l - n(n-1) - \frac{nh}{h-l}] - [nh - (n-1)l - (n-1)^2 - \frac{(n-1)h}{h-l}]}, \\ &= nh - (n-1)l - n(n-1) - \frac{nh}{h-l} + \frac{(nh-nl+l)(h-l-n)(h-l)}{(h-l)(nh-nl+1)}, \\ &= nh - (n-1)l - n(n-1) - \frac{nh}{h-l} + h - l - n = (n+1)h - nl - n^2 - \frac{nh}{h-l}. \end{aligned}$$

Hence, the proof has done.

6. NUMERICAL SOLUTIONS

This section shows some numerical examples that confirm the results we obtained in this paper.

Example 1. Let $x_{-3} = 0.2,\ x_{-2} = 5,\ x_{-1} = 1,\ x_0 = 2,\ a = 0.5,\ b = 1,\ c = 6$ and $d = 1$. Then, the local stability is shown as follows:

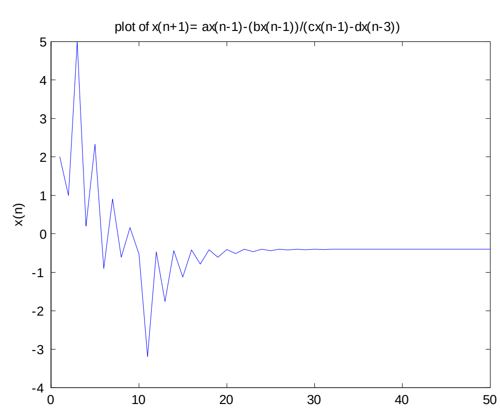

Figure 1. This figure shows the local stability of Eq.(1).

Example 2. Assume that $x_{-3} = 0.2,\ x_{-2} = 3,\ x_{-1} = 0.1,\ x_0 = 2,\ a = 0.1,\ b = 1,\ c = 2$ and $d = 9$. Then, the global stability is illustrated as follows:

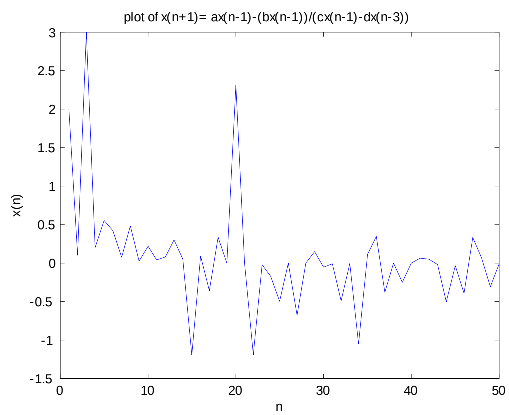

Figure 2. This figure presents a global stability of Eq.(1).

Example 3. This example presents the solution of Eq.(1) when we suppose that $x_{-3} = 0.2,\ x_{-2} = 3,\ x_{-1} = 1,\ x_0 = 0.5,\ a = b = 1,\ c = 0.5$ and $d = 9$. See Figure 3.

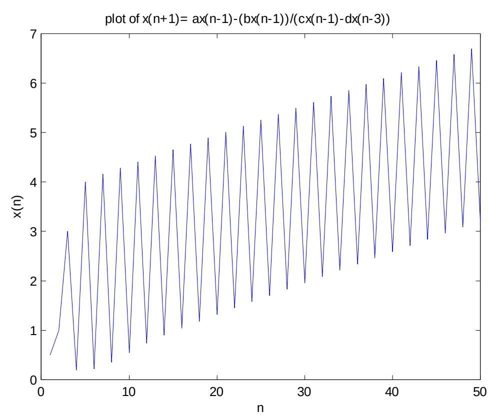

Figure 3. This figure shows the solutions of Eq.(1) when $x_{-3} = 0.2,\ x_{-2} = 3,\ x_{-1} = 1,\ x_0 = 0.5,\ a = b = 1,\ c = 0.5$ and $d = 9$.

Example 4. This example illustrates the solution of Eq.(13) when we assume that $x_{-3} = -7,\ x_{-2} =$

5, $x_{-1} = 0.5$, $x_0 = 8$. See Figure 4.

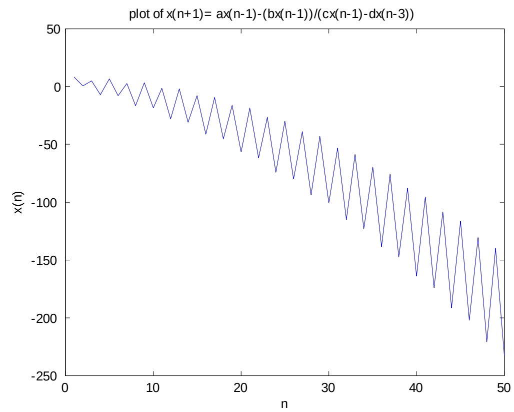

Figure 4.

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A necessary condition for eventually equilibrium or periodic to a system of difference equations

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March 1, 2018

Abstract

In this paper we consider the behavior of a special case of piecewise linear systems of difference equations with initial condition in first quadrant. We found a necessary condition that the solutions become equilibrium point or periodic with prime period 4 without using stability theorems. We constructed inductive statement to represent the behavior of the system and we apply useful lemmas in the proof of main theorem.
Key words: Difference equation, Periodic solution, Stability, Equilibrium point, Piecewise linear system of difference equation.
2010 Mathematics Subject Classification: 39A10 and 65Q10.

1 Introduction

To investigate stability of system of difference equations requires theorems that involve Jacobian matrix. So the functions of the system must are differentiable. Unfortunately, piecewise linear systems of difference equations are the system with absolute value. So we can not apply the stability theorem to the piecewise linear systems. In 1978 Lozi [1] hypothesized a simplified version of Hénon's transformation by using system of difference equation with absolute value and Lozi's Piecewise Linear Model admits a strange attractor with a specific parameter and initial condition. Then, Devaney [2, 3] investigated Gingerbreadman map and he was shown Gingerbreadman map, a map with absolute value, being chaotic in certain regions. Moreover, Ladas's open problem was mentioned an in article [4] as the system of difference equations:

$$x_{n+1} = |x_n| + ay_n + b, y_{n+1} = x_n + c|y_n| + d, n = 0, 1, \dots$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$ and the parameters $a, b, c,$ and $d \in \{-1, 0, 1\}$. He suggests to investigate boundedness character of solutions, the

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global stability, and periodic nature of the solutions.There are several authors studied this open problem such as Grove et. al [4] found that every solution of a specific system is eventually periodic with period 3, Tikjha et. al [5, 6] found that the character of system is eventually periodic with some period and equilibrium point respectively. As mentioned above, we can not apply the stability theorems to this open problem. The common idea of proofs of the above systems of piecewise linear articles is to separate initial condition into few regions and find some characters of solution to the system of each region and then establishing lemmas and finally summarizing the behaviors of each system to be a theorem. Our ultimate goals is to know complete global character of system:

$$x_{n+1} = |x_n| - y_n - b, y_{n+1} = x_n - |y_n| + 1, n = 0, 1, \dots \tag{1}$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$ and the parameters b is any positive number. In this article we will focus to a special case of System(1) when $b = 3$ with initial condition are some points in the first quadrant.

2 Preliminaries

The following definitions [7] are used in this article. A *system of difference equations of the first order* is a system of the form

$$x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n), n = 0, 1, ... \tag{2}$$

where f and g are continuous functions which map $\mathbf{R}^2$ into $\mathbf{R}$.

A *solution* of the System(2) is a sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ which satisfies the system for all $n \geq 0$. If we prescribe an *initial condition* $(x_0, y_0) \in \mathbf{R}^2$ then

$$x_1 = f(x_0, y_0), y_1 = g(x_0, y_0)$$
$$x_2 = f(x_1, y_1), y_2 = g(x_1, y_1)$$
$$\vdots$$

and so the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of the System(2) exists for all $n \geq 0$ and is uniquely determined by the initial condition (x_0, y_0).

A solution of the System(2) which is constant for all $n \geq 0$ is called an *equilibrium solution*. If

$$(x_n, y_n) = (\bar{x}, \bar{y}) \text{ for all } n \geq 0$$

is an equilibrium solution of the System(2), then $(\bar{x}, \bar{y})$ is called an *equilibrium point*, or simply an *equilibrium* of the System(2).

A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called *eventually periodic with prime period p* or *eventually prime period p solution* if there exists an integer $N > 0$ and p is the smallest positive integer such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period p; that is,

$$(x_{n+p}, y_{n+p}) = (x_n, y_n) \text{ for all } n \geq N. \tag{3}$$

The p consecutive point of the solution is called a *p-cycle* of System(2). We denote

$$\begin{pmatrix} x_0, y_0 \\ x_1, y_1 \\ x_2, y_2 \\ x_3, y_3 \end{pmatrix}$$

as 4-cycle which consists of $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and (x_3, y_3) in xy plain.

3 Main Results

In this section we will investigate behaviors of the following system:

$$x_{n+1} = |x_n| - y_n - 3, y_{n+1} = x_n - |y_n| + 1, n = 0, 1, \dots . \tag{4}$$

From System(4) and by simple calculations,

$$P_{4.1} = \begin{pmatrix} -5, & -1 \\ 3, & -5 \\ 5, & -1 \\ 3, & 5 \end{pmatrix} \quad \text{and} \quad P_{4.2} = \begin{pmatrix} 1, & -3 \\ 1, & -1 \\ -1, & 1 \\ -3, & -1 \end{pmatrix}$$

are two 4-cycles of System(4) and equilibrium point is $(-1, -1)$. For convenience in the later part of the proof, we let $S := \{(x, y)|x + \frac{1}{2} \leq y \leq x + 1\}, a_n := \frac{2^{2n+3}-1}{2^{2n+3}}, u_n := \frac{2^{2n+2}+1}{2^{2n+2}}, l_n := \frac{2^{2n+2}-1}{2^{2n+2}}, \delta_n = 2^{2n+4} - 1, B_{n+2} := \{(x, y)|x + \frac{2^{2n+3}-1}{2^{2n+3}} \leq y < x + \frac{2^{2n+4}-1}{2^{2n+4}}\}$. The proof of main theorem requires the following two lemmas.

Lemma 1. *Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a solution of System(4) If there is positive integer N such that $x_N = -y_N - 2 < 0$ and $y_N < 0$ then (x_{N+1}, y_{N+1}) is equilibrium point $(-1, -1)$.*

Proof. The proof is obvious. □

Lemma 2. *Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a solution of System(4) If there is positive integer N such that $x_N = y_N - 2 \geq 0$ then $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ are in $P_{4.1}$.*

Proof. With condition $x_N = y_N - 2 \geq 0$ by simple calculation, we have $(x_{N+1}, y_{N+1}) = (-5, -1) \in P_{4.1}$. □

The following theorem provides a necessary condition of equilibrium point or prime period 4 to System(4) with initial condition in first quadrant.

Theorem 1. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(4) and $x_0, y_0 \geq 0$. If*

$$(x_0, y_0) \in S - B_{n+2} \tag{5}$$

for all integer $n \geq -1$, then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually equilibrium point or the prime period 4 solution($P_{4.1}$ or $P_{4.2}$).

Proof. Let $(x_0, y_0) \in S - B_{n+2}$ for all integer $n \geq -1$. Then $x_0 + \frac{1}{2} \leq y_0$ and $y_0 \leq x_0 + 1$, so we have $x_1 = x_0 - y_0 - 3 < 0$ and $y_1 = x_0 - y_0 + 1 \geq 0$, $x_2 = -2x_0 + 2y_0 - 1 \geq 0$ and $y_2 = -3$, $x_3 = -2x_0 + 2y_0 - 1 \geq 0$ and $y_3 = -2x_0 + 2y_0 - 3$.

If $y_0 \geq x_0 + \frac{3}{2}$ then $y_3 \geq 0$ and so $(x_4, y_4) = (-1, 3)$ and $(x_6, y_6) = (-5, -1) \in P_{4.1}$. Suppose that $y_0 < x_0 + \frac{3}{2}$ then $y_3 < 0$ and so $(x_4, y_4) = (-1, -4x_0 + 4y_0 - 3)$

If $y_4 = -4x_0 + 4y_0 - 3 < 0$ then we have $(x_5, y_5) \in B_1$. This contradicts Condition(5). Suppose that $y_4 \geq 0$, so $x_5 = 4x_0 - 4y_0 + 1 < 0$ and $y_5 = 4x_0 - 4y_0 + 3 \leq 0$, $x_6 = -8x_0 + 8y_0 - 7$ and $y_6 = 8x_0 - 8y_0 + 5 < 0$.

If $x_6 < 0$, that is $x_6 = -y_6 - 2 < 0$, then applying Lemma(1), we have $(x_7, y_7) = (-1, -1)$. Suppose that $x_6 \geq 0$, that is $x_0 + \frac{7}{8} \leq y_0 < x_0 + \frac{3}{2}$, then $(x_7, y_7) = (-16x_0 + 16y_0 - 15, -1)$.

If $x_7 < 0$, then $x_0 + \frac{7}{8} \leq y_0 < x_0 + \frac{15}{16}$, and so $(x_8, y_8) \in B_2$. This contradicts Condition(5). Suppose that $x_7 \geq 0$. That is $x_0 + \frac{15}{16} \leq y_0 < x_0 + \frac{3}{2}$, so $x_8 = -16x_0 + 16y_0 - 17$ and $y_8 = -16x_0 + 16y_0 - 15 \geq 0$.

If $x_8 \geq 0$ that is $x_0 + \frac{17}{16} \leq y_0 < x_0 + \frac{3}{2}$. Applying Lemma(2), $(x_9, y_9) \in P_{4.1}$. Suppose that $x_8 < 0$ that is $x_0 + \frac{15}{16} \leq y_0 < x_0 + \frac{17}{16}$. We have $x_9 = 32x_0 - 32y_0 + 29 < 0$ and $y_9 = -1$, $x_{10} = -32x_0 + 32y_0 - 31$ and $y_{10} = 32x_0 - 32y_0 + 29 < 0$.

If $x_{10} < 0$ that is $x_0 + \frac{15}{16} \leq y_0 < x_0 + \frac{31}{32}$. We have $(x_{11}, y_{11}) = (-1, -1)$. Suppose that $x_{10} \geq 0$ that is $x_0 + \frac{31}{32} \leq y_0 < x_0 + \frac{17}{16}$. We have a closed form of inductive statement on $n \geq 1$ and let $P(n)$ be the following statement:

For $(x_0, y_0) \in R_n = \{(x, y) | x + a_n \leq y < x + u_n\}$, then $x_{4n+6} \geq 0$ and so

$x_{4n+7} = -2^{2n+4}x_0 + 2^{2n+4}y_0 - \delta_n$

$y_{4n+7} = -1$.

If $(x_0, y_0) \in B_{n+2} = \{(x, y) | x + a_n \leq y < x + l_{n+1}\}$, then $x_{4n+7} < 0$.

If $(x_0, y_0) \in R_n - B_{n+2} = \{(x, y) | x + l_{n+1} \leq y < x + u_n\}$, then $x_{4n+7} \geq 0$ and so

$x_{4n+8} = -2^{2n+4}x_0 + 2^{2n+4}y_0 - \delta_n - 2$

$y_{4n+8} = -2^{2n+4}x_0 + 2^{2n+4}y_0 - \delta_n \geq 0$.

If $(x_0, y_0) \in R_n^* = \{(x, y) | x + u_{n+1} \leq y < x + u_n\}$, then $x_{4n+8} \geq 0$ and so

$x_{4n+9} = -5$

$y_{4n+9} = -1$.

If $(x_0, y_0) \in (R_n - B_{n+2}) - R_n^* = \{(x, y) | x + l_{n+1} \leq y < x + u_{n+1}\}$, then $x_{4n+8} < 0$ and so

$x_{4n+9} = 2^{2n+5}x_0 - 2^{2n+5}y_0 + 2\delta_n - 1 < 0$

$y_{4n+9} = -1$

$x_{4n+10} = -2^{2n+5}x_0 + 2^{2n+5}y_0 - 2\delta_n - 1$

$y_{4n+10} = 2^{2n+5}x_0 - 2^{2n+5}y_0 + 2\delta_n - 1 < 0$.

If $(x_0, y_0) \in \tilde{R}_n = \{(x,y)|x + l_{n+1} \leq y < x + a_{n+1}\}$, then $x_{4n+10} < 0$ and so
$x_{4n+11} = -1$
$y_{4n+11} = -1.$

If $(x_0, y_0) \in R_{n+1} = \{(x,y)|x + a_{n+1} \leq y < x + u_{n+1}\}$, then $x_{4n+10} \geq 0$.
We shall first show that $P(1)$ is true. For $(x_0, y_0) \in R_1 = \{(x,y)|x + \frac{31}{32} \leq y < x + \frac{17}{16}\}$ and $\delta_1 = 63$, we have $x_{10} = -32x_0 + 32y_0 - 31 \geq 0$ and so
$x_{4(1)+7} = x_{11} = -2^{2(1)+4}x_0 + 2^{2(1)+4}y_0 - \delta_1$
$y_{4(1)+7} = y_{11} = -1.$

If $(x_0, y_0) \in B_3 = \{(x,y)|x + \frac{31}{32} \leq y < x + \frac{63}{64}\}$, then $x_{11} = -64x_0 + 64y_0 - 63 < 0$.

If $(x_0, y_0) \in R_1 - B_3 = \{(x,y)|x + \frac{63}{64} \leq y < x + \frac{17}{16}\}$, then $x_{11} = -64x_0 + 64y_0 - 63 \geq 0$ and so
$x_{4(1)+8} = x_{12} = -64x_0 + 64y_0 - 65 = -2^{2(1)+4}x_0 + 2^{2(1)+4}y_0 - \delta_1 - 2$
$y_{4(1)+8} = y_{12} = -64x_0 + 64y_0 - 63 = -2^{2(1)+4}x_0 + 2^{2(1)+4}y_0 - \delta_1 \geq 0.$

If $(x_0, y_0) \in R_1^* = \{(x,y)|x + \frac{65}{64} \leq y < x + \frac{17}{16}\}$, then $x_{12} = -64x_0 + 64y_0 - 65 \geq 0$ and so
$x_{4(1)+9} = x_{13} = -5$
$y_{4(1)+9} = y_{13} = -1.$

If $(x_0, y_0) \in (R_1 - B_3) - R_1^* = \{(x,y)|x + \frac{63}{64} \leq y < x + \frac{65}{64}\}$, then $x_{12} = -64x_0 + 64y_0 - 65 < 0$ and so
$x_{4(1)+9} = x_{13} = 128x_0 - 128y_0 + 125 = 2^{2(1)+5}x_0 - 2^{2(1)+5}y_0 + 2\delta_1 - 1 < 0$
$y_{4(1)+9} = y_{13} = -1$
$x_{4(1)+10} = x_{14} = -128x_0 + 128y_0 - 127 = -2^{2(1)+5}x_0 + 2^{2(1)+5}y_0 - 2\delta_1 - 1$
$y_{4(1)+10} = y_{14} = 128x_0 - 128y_0 + 125 = 2^{2(1)+5}x_0 - 2^{2(1)+5}y_0 + 2\delta_1 - 1 < 0.$

If $(x_0, y_0) \in \tilde{R}_1 = \{(x,y)|x + \frac{63}{64} \leq y < x + \frac{127}{128}\}$, then $x_{14} = -128x_0 + 128y_0 - 127 < 0$ and so
$x_{4(1)+11} = x_{15} = -1$
$y_{4(1)+11} = y_{15} = -1.$

If $(x_0, y_0) \in R_2 = \{(x,y)|x + \frac{127}{128} \leq y < x + \frac{65}{64}\}$, then $x_{4(1)+10} = -128x_0 + 128y_0 - 127 \geq 0$. Therefore $P(1)$ is true, as required.

Suppose $P(k)$ is true for a positive integer k. If $(x_0, y_0) \in R_{k+1} = \{(x,y)|x + \frac{2^{2k+5}-1}{2^{2k+5}} \leq y < x + \frac{2^{2k+4}+1}{2^{2k+4}}\}$, then
$x_{4k+10} = -2^{2k+5}x_0 + 2^{2k+5}y_0 - 2\delta_k - 1 \geq 0$ and $y_{4k+10} = 2^{2k+5}x_0 - 2^{2k+5}y_0 + 2\delta_k - 1 < 0$ and so

$$\begin{aligned} x_{4(k+1)+7} = x_{4k+11} &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - (4\delta_k + 3) \\ &= -2^{2(k+1)+4}x_0 + 2^{2(k+1)+4}y_0 - \delta_{k+1} \end{aligned}$$

$y_{4(k+1)+7} = y_{4k+11} = -1.$

If $(x_0, y_0) \in B_{k+3} = \left\{(x,y)|x + \frac{2^{2k+5}-1}{2^{2k+5}} \leq y < x + \frac{2^{2k+6}-1}{2^{2k+6}}\right\}$, then
$x_{4k+11} = -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} < 0.$

If $(x_0, y_0) \in (R_{k+1} - B_{k+3}) = \left\{(x,y)|x + \frac{2^{2k+6}-1}{2^{2k+6}} \leq y < x + \frac{2^{2k+4}+1}{2^{2k+4}}\right\}$, then
$x_{4k+11} = -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} \geq 0$, and so

$$\begin{aligned} x_{4(k+1)+8} = x_{4k+12} &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} - 2 \\ &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - 2^{2k+6} - 1 \end{aligned}$$

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$y_{4(k+1)+8} = y_{4k+12} = -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1}$
$\quad = -2^{2(k+1)+4}x_0 + 2^{2(k+1)+4}y_0 - 2^{2(k+1)+4} + 1 \geq 0.$

If $(x_0, y_0) \in R^*_{k+1} = \{(x,y)|x + u_{k+2} \leq y < x + u_{k+1}\}$
$\quad = \left\{(x,y)|x + \frac{2^{2k+6}+1}{2^{2k+6}} \leq y < x + \frac{2^{2k+4}+1}{2^{2k+4}}\right\}$, then
$x_{4k+12} = -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} - 2 \geq 0$, and so
$x_{4(k+1)+9} = x_{4k+13} = -5$
$y_{4(k+1)+9} = x_{4k+13} = -1.$

If $(x_0, y_0) \in (R_{k+1} - B_{k+3}) - R^*_{k+1} = \left\{(x,y)|x + \frac{2^{2k+6}-1}{2^{2k+6}} \leq y < x + \frac{2^{2k+6}+1}{2^{2k+6}}\right\}$, then
$x_{4k+12} = -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} - 2 < 0$, and so
$x_{4(k+1)+9} = 2\left(2^{2k+6}x_0\right) - 2\left(2^{2k+6}y_0\right) + 2\delta_{k+1} - 1$
$\quad = 2^{2k+7}x_0 - 2^{2k+7}y_0 + 2\delta_{k+1} - 1$
$\quad = 2^{2(k+1)+5}x_0 - 2^{2(k+1)+5}y_0 + 2^{2(k+1)+5} - 3 < 0$
$y_{4(k+1)+9} = x_{4k+13} = -1$
$x_{4(k+1)+10} = -2^{2k+7}x_0 + 2^{2k+7}y_0 - 2\delta_{k+1} - 1$
$\quad = -2^{2(k+1)+5}x_0 + 2^{2(k+1)+5}y_0 - 2^{2(k+1)+5} + 1$
$y_{4(k+1)+10} = x_{4k+14} = 2^{2k+7}x_0 - 2^{2k+7}y_0 + 2\delta_{k+1} - 1$
$\quad = 2^{2(k+1)+5}x_0 - 2^{2(k+1)+5}y_0 + 2^{2(k+1)+5} - 3 < 0.$

If $(x_0, y_0) \in \tilde{R}_{k+1} = \left\{(x,y)|x + \frac{2^{2k+6}-1}{2^{2k+6}} \leq y < x + \frac{2^{2k+7}-1}{2^{2k+7}}\right\}$, then
$x_{4k+14} = -2^{2k+7}x_0 + 2^{2k+7}y_0 - 2\delta_{k+1} - 1 < 0,$
and so
$x_{4(k+1)+11} = x_{4k+15} = -1$
$y_{4(k+1)+11} = x_{4k+15} = -1.$

If $(x_0, y_0) \in R_{k+2} = \{(x,y)|x + a_{k+2} \leq y < x + u_{k+2}\}$
$\quad = \left\{(x,y)|x + \frac{2^{2k+7}-1}{2^{2k+7}} \leq y < x + \frac{2^{2k+6}+1}{2^{2k+6}}\right\}$, then
$x_{4k+14} = -2^{2k+7}x_0 + 2^{2k+7}y_0 - 2\delta_{k+1} - 1 \geq 0.$

Hence $P(k+1)$ is also true. By mathematical induction $P(n)$ is true for any positive integer n. Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n = 1.$$

If $y_0 = x_0 + 1$, then $(x_1, y_1) = (-4, 0)$ and so $(x_2, y_2) = (1, -3) \in P_{4.2}$ and the proof is complete. □

4 Conclusion

In this paper we showed that solution of System(4) with initial condition being a specific region in first quadrant is eventually equilibrium point or prime period 4. We described the behavior of solution to the system by using inductive statement. If initial conditions are in R^*_n then the solution is eventually prime period 4 ($P_{4.1}$). If initial conditions are in $\tilde{R}_1$ then the solution is eventually equilibrium point. The limit of R_n tend to a line $y = x + 1$ and if we choose

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initial condition in the line $y = x + 1$, then solution is eventually prime period 4 ($P_{4.2}$).

Acknowledgement

This work was supported by National Research Council of Thailand and Pibulsongkram Rajabhat University. The first author is supported by the Centre of Excellence in Mathematics, CHE, Thailand.

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BI-UNIVALENT FUNCTIONS ASSOCIATED WITH WRIGHT HYPERGEOMETRIC FUNCTIONS

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Abstract. In this work, using of the Faber polynomial expansions we find upper bounds for $|a_n|$ $(n \geq 3)$ coefficients of functions in subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$, which were defined with Wright hypergeometric functions and quasi-subordinate conditions in the open unit disk. Our results generalize and improve some of the previously known results.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Many derivative and integral operators can be written in terms of convolution of certain analytic functions. This formalism facilitates further mathematical explorations and helps deep understanding of the geometric properties of such operators. For functions $f, h \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$, Hadamard product (or convolution) of $f(z)$ and $h(z)$ is denoted by $f * h$ and is defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (h * f)(z).$$

Now, we recall and state some concepts of the special functions and operators as follows:

For complex parameters $\alpha_1, \ldots, \alpha_\ell$ $(\frac{\alpha_j}{A_j} \neq 0, -1, \ldots; j = 1, 2, \ldots, \ell)$ and $\beta_1, \ldots, \beta_m$ $(\frac{\beta_j}{B_j} \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$, Fox's H-functions (for details, see [19]) which mean the Wright's generalized hypergeometric functions ${}_\ell\Psi_m$ with $A_j, B_j > 0$, give (rather general and typical examples of H-functions, not reducible to G-functions):

$$\begin{aligned} {}_\ell\Psi_m\left(\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_\ell, A_\ell) \\ (\beta_1, B_1), \ldots, (\beta_m, B_m) \end{matrix} : z \right) &= {}_\ell\Psi_m\left[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\right] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \ldots \Gamma(\alpha_\ell + nA_\ell)}{\Gamma(\beta_1 + nB_1) \ldots \Gamma(\beta_m + nB_m)} \frac{z^n}{n!}, \end{aligned}$$

2010 *Mathematics Subject Classification*: Primary 30C45, 30C50; Secondary 30C80

Key words and phrases: Coefficient estimates, Faber polynomial expansion, Wright hypergeometric functions, Subordinate

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where $1+\sum_{n=1}^{m} B_n - \sum_{n=1}^{\ell} A_n \geq 0\,(\ell, m \in \mathbb{N} = \{1, 2, \ldots\})$ and for suitably bounded values of $|z|$.

Now the linear operator is introduced comprising of the generalized hypergeometric function from Srivastava [19] (see [7]) and Wright [24]. Let $\ell, m \in \mathbb{N}$ and suppose that the parameters $\alpha_1, A_1, \ldots, \alpha_\ell, A_\ell$ and $\beta_1, B_1, \ldots, \beta_m, B_m$ are also positive real numbers. Then, corresponding to a function

$$ {}_\ell\Phi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big] $$

defined by

$$ {}_\ell\Phi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big] = \Omega z\, {}_\ell\Psi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big], $$

where $\Omega = \left(\prod_{j=1}^{\ell} \Gamma(\alpha_j)\right)^{-1} \left(\prod_{j=1}^{m} \Gamma(\beta_j)\right)$, we consider a linear operator

$$ \mathcal{W}\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}\big] : \mathcal{A} \longrightarrow \mathcal{A} $$

defined by the following Hadamard product

$$ \mathcal{W}\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}\big] f(z) := z\, {}_\ell\Phi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big] * f(z). $$

We observe that, for $f(z)$ of the form (1.1), we have

$$ \mathcal{W}\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}\big] f(z) := z + \sum_{n=2}^{\infty} \varphi_n a_n z^n, $$

where

$$ \varphi_n = \frac{\Omega\, \Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_\ell + A_\ell(n-1))}{(n-1)!\, \Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_m + B_m(n-1))}. $$

If, for convenience, we write

$$ \mathcal{W}_m^\ell f(z) = \mathcal{W}\big[(\alpha_1, A_1), \ldots, (\alpha_\ell, A_\ell); (\beta_1, B_1), \ldots, (\beta_m, B_m)\big] f(z). $$

The Koebe one-quarter theorem [6] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$ f^{-1}\big(f(z)\big) = z \;\; (z \in \mathbb{U}) \quad \text{and} \quad f\big(f^{-1}(w)\big) = w \;\left(|w| < r_0(f);\; r_0(f) \geqq \frac{1}{4}\right), $$

where

(1.2)

$$ \begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} b_n w^n. \end{aligned} $$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For a brief history and interesting examples in the class Σ, see [13]. Recently, many researchers introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients, see,

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

for example, [4, 15, 20–22, 25, 27]. But the coefficient problem for each of the Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2,3\}$, is still an open problem.

A function $f(z)$ is said to be quasi-subordinate to $\phi(z)$ in the open unit disk $\mathbb{U}$ if there exist analytic functions $\psi(z)$ and $w(z)$, with $w(0) = 0$ such that $|\psi(z)| \leq 1$, $|w(z)| < 1$ and $f(z) = \psi(z)\phi(w(z))$. Denote this quasi-subordination by $f(z) \prec_{\tilde{q}} \phi(z)$. For $\psi(z) = 1$, the quasi-subordination reduces to the subordination (see [17, 18]).

Throughout this paper, we let $\phi(z)$ is analytic function in the unit disk $\mathbb{U}$ with $\phi(0) = 1$ such that

$$\phi(z) = 1 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots \qquad (C_1 > 0)$$

and assume that the function $\psi(z)$ is analytic in the unit disk $\mathbb{U}$ and $|\psi(z)| \leq 1$ such that

$$\psi(z) = D_0 + D_1 z + D_2 z^2 + D_3 z^3 + \cdots .$$

Recently, Cho et al., [5] introduced subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ of Σ and only obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.

Definition 1.1. [5] A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{G}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1\right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w) + \lambda w(\mathcal{W}_m^l g(w))'} - 1\right) \prec_{\tilde{q}} (\phi(w) - 1),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $0 \leq \lambda < 1$, $z, w \in \mathbb{U}$ and the function g is given by (2.1).

Definition 1.2. [5] A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\right) \prec_{\tilde{q}} (\phi(w) - 1),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda \geq 0$, $z, w \in \mathbb{U}$ and the function g is given by (2.1).

Lemma 1.3. [6] *Let $u(z)$ be analytic in the unit disk $\mathbb{U}$ with $u(0) = 0$ and $|u(z)| < 1$ and suppose that $u(z) = \sum_{n=1}^{\infty} p_n z^n$. Then $|p_n| \leqq 1$ $(n \in \mathbb{N})$.*

Lemma 1.4. [9] *Let the function w in the Schwarz function is given by $w(z) = \sum\limits_{n=1}^{\infty} w_n z^n$, where $z \in \mathbb{U}$. Then for every complex number s,*

$$|w_2 + s w_1^2| \leq 1 + (|s| - 1)|w_1^2|.$$

Faber [8] introduced the Faber polynomials, which play an important role in various areas of mathematical sciences, especially in geometric function theory. By using the Faber polynomial expansion of functions $f \in \mathcal{S}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed, (see for details [1] and [2]),

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n, \tag{1.3}$$

where $b_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n)$, and

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &\quad + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \\ &\quad \times [a_5 + (-n+2) a_3^2] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] \\ &\quad + \sum_{j \geq 7} a_2^{n-j} V_j \end{aligned}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$, (see for details [2]). In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2} K_1^{-2} = -a_2, \quad \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \quad \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4).$$

In general, for any $p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, an expansion of K_n^p is (see for details [1, 23] or [2, page 349])

$$K_n^p = p a_{n+1} + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!n!} D_n^n,$$

where

$$D_n^m(a_2, a_3, \cdots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1! \cdots \mu_n!} \tag{1.4}$$

and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m, \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

We noth that it is clear that $D_n^n(a_2, a_3, \cdots, a_n) = a_2^n$.

Lemma 1.5. [2, Equation (1.6) and (1.7)] *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}$, and $k \in \mathbb{Z}$ then we have the following expansion*

$$\begin{aligned} \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^k &= 1 - \sum_{n=1}^{\infty} F_{n-1}^{n+k-1}(a_2, a_3, \ldots, a_n) z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} \left(1 + \frac{n-1}{k} \right) K_{n-1}^k (a_2, a_3, \ldots, a_n) z^{n-1}, \end{aligned}$$

where the first Faber polynomials $F_{n-1}^{n+k-1}(a_2, a_3, \ldots, a_n)$ are given by

$$F_1^{k+1}(a_2) = (1+\lambda) a_2, \quad F_2^{k+2}(a_2, a_3) = \frac{(\lambda-1)(\lambda+2)}{2} a_2^2 + (\lambda+2) a_3, \ldots.$$

Several researchers have solved coefficient estimates problem for various subclasses of bi-univalent functions by using Faber polynomial expansions, see for example [10, 11, 20, 26]. In the present paper, by using the Faber polynomial expansions we obtain estimates of coefficients $|a_n|$ where $n \geq 3$, of functions in the subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ of Σ with various special cases.

2. Main results

First, we can write that the Faber polynomial expansion for $f \in \mathcal{G}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ given by (1.1) is in the form of

(2.1)
$$\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z)+\lambda z(\mathcal{W}_m^l f(z))'}-1\right)=\frac{1}{\gamma}\sum_{n=2}^{\infty}F_{n-1}(\varphi_2a_2,\varphi_3a_3,\cdots,\varphi_na_n)z^{n-1},$$

where

$$F_1(\varphi_2a_2)=(1-\lambda)\varphi_2a_2,\quad F_2(\varphi_2a_2,\varphi_3a_3)=(\lambda^2-1)(\varphi_2a_2)^2+2(1-\lambda)\varphi_3a_3.$$

In general,

$$F_{n-1}(\varphi_2a_2,\varphi_3a_3,\cdots,\varphi_na_n)=(1-\lambda)(n-1)\varphi_na_n$$
$$+\sum_{l=1}^{n-2}K_l^{-1}\Big((1+\lambda)\varphi_2a_2,(1+2\lambda)\varphi_3a_3,\cdots,(1+l\lambda)\varphi_{l+1}a_{l+1}\Big)(1-\lambda)(n-l-1)\varphi_{n-l}a_{n-l}.$$

Then to simplify, we define:

$$F(z)\prec_{\tilde{q}}(\phi(z)-1)\quad\text{and}\quad G(w)\prec_{\tilde{q}}(\phi(w)-1), \tag{2.2}$$

where

$$F(z)=\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z)+\lambda z(\mathcal{W}_m^l f(z))'}-1\right)\text{ and }G(w)=\frac{1}{\gamma}\left(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w)+\lambda w(\mathcal{W}_m^l g(w))'}-1\right),$$
$$F(z)=\frac{1}{\gamma}\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}}-1\right)\text{ and }G(w)=\frac{1}{\gamma}\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}}-1\right).$$

In addition, by definition of quasi-subordinate there exist analytic functions ψ and $u,v:\mathbb{U}\to\mathbb{U}$, where $u(z)=\sum_{n=1}^{\infty}p_nz^n$ and $v(z)=\sum_{n=1}^{\infty}q_nz^n$, so that

$$F(z)=\psi(z)[\phi(u(z))-1]\quad\text{and}\quad G(w)=\psi(w)[\phi(v(w))-1], \tag{2.3}$$

where by equation (1.4) we have

(2.4)
$$\psi(z)[\phi(u(z))-1]=[C_1p_1z+(C_1p_2+C_2p_1^2)z^2+\cdots][D_0+D_1z+D_2z^2+\cdots]$$
$$=\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}C_kD_n^k(p_1,p_2,\cdots,p_n)z^n\right)\sum_{n=0}^{\infty}D_nz^n$$

and

$$\varphi(w)h(v(w))=\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}C_kD_n^k(q_1,q_2,\cdots,q_n)w^n\right)\sum_{n=0}^{\infty}D_nw^n. \tag{2.5}$$

Now, we obtain the following coefficient estimates for these subclasses.

Theorem 2.1. *Let the function $f \in \mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $D_0 \neq 0$. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{|\gamma|\big(C_1 + |D_{n-1}|\big)}{(1-\lambda)(n-1)\varphi_n}, \qquad n \geq 3.$$

Theorem 2.2. *Let the function $f \in \mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $D_0 \neq 0$. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{|\gamma|\big(C_1 + |D_{n-1}|\big)}{\big(\lambda + (n-1)\big)\varphi_n}, \qquad n \geq 3.$$

Theorem 2.3. *Let the function $f \in \mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $C_1 \geq |C_2|$. Then*

$$|a_2| \leq \frac{|\gamma D_0|\ C_1\sqrt{C_1}}{\sqrt{|\gamma D_0| C_1^2 \big|(\lambda^2-1)\varphi_2^2 + 2(1-\lambda)\varphi_3\big| + (C_1 - |C_2|)(1-\lambda)^2\varphi_2^2}}.$$

Theorem 2.4. *Let the function $f \in \mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $C_1 \geq |C_2|$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1\sqrt{2C_1}}{\sqrt{|\gamma D_0| C_1^2 \big|(\lambda-1)(\lambda+2)\varphi_2^2 + 2(\lambda+2)\varphi_3\big| + 2(C_1 - |C_2|)(1+\lambda)^2\varphi_2^2}}.$$

Remark 2.5.
(1) If we take $\psi(z) = 1$ in Theorem 2.1, then we obtain estimates of coefficients $|a_n|$ $(n \geq 3)$ for subclass defined by Murugusundaramoorthy in [14, Theorem 2.2].
(2) If we take $\psi(z) = 1$ in Theorem 2.3, then we obtain an improvement of the estimates obtained for $|a_2|$ by Murugusundaramoorthy in [14, Theorem 2.2].
(3) By setting $\lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $A_1 = A_2 = B_1 = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.3, we get $\varphi_n = 1$ and then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.5].
(4) By setting $\lambda = \gamma = 1$ and $\ell = 2,\ m = 1$ with $A_1 = A_2 = B_1 = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.4, we get $\varphi_n = 1$ and then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.2].
(5) By setting $\lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $\alpha_1 = 2$ and $A_1 = A_2 = B_1 = \alpha_2 = \beta_1 = 1$, $(\mathcal{W}_1^2 f(z) = zf'(z))$ in Theorem 2.3, then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.8].
(6) By setting $\psi(z) = 1, \lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $\alpha_1 = 2$ and $A_1 = A_2 = B_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.3, then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.9].
(7) By setting $\psi(z) = 1, \lambda = 1, \gamma = 1$ and $\ell = 2,\ m = 1$ with $\alpha_1 = a, \alpha_2 = b, \beta_1 = c$, in Theorem 2.4, then we obtain an improvement of the estimates obtained for $|a_2|$ by Omar et al., in [16, Theorem 1].
(8) By setting $\psi(z) = 1, \lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $A_1 = A_2 = B_1 = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.3, we get $\varphi_n = 1$ and then we

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obtain an improvement of the estimates obtained for $|a_2|$ by Ali et al., in [4, Corollary 2.1].

(9) By setting $\psi(z)=1$, $\lambda=\gamma=1$ and $\ell=2,\ m=1$ with $A_1=A_2=B_1=\alpha_1=\alpha_2=\beta_1=1$, in Theorem 2.4, we get $\varphi_n=1$ and then we obtain an improvement of the estimates obtained for $|a_2|$ by Ali et al., in [4, Theorem 2.1].

(10) Theorem 2.3 and Theorem 2.4 are improvements of the results obtained by Cho et al. [5], respectively.

3. Proof of Theorems

Proof of Theorem 2.1. For this work, let

$$F(z)=\frac{1}{\gamma}\Big(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z)+\lambda z(\mathcal{W}_m^l f(z))'}-1\Big)$$

and

$$G(w)=\frac{1}{\gamma}\Big(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w)+\lambda w(\mathcal{W}_m^l g(w))'}-1\Big).$$

For the function $f\in\mathcal{G}_\Sigma^{l,m}(\gamma,\lambda,\phi)$, we have the expansion (2.1) and for the inverse map $g=f^{-1}$, considering (1.2), we get that

$$G(w)=\frac{1}{\gamma}\sum_{n=2}^{\infty}F_{n-1}(\varphi_2 b_2,\varphi_3 b_3,\cdots,\varphi_n b_n)w^{n-1}. \tag{3.1}$$

Comparing the coefficients of (2.1) and (2.4), we conclude

$$\begin{aligned}&\frac{1}{\gamma}\Bigg[(1-\lambda)(n-1)\varphi_n a_n+\sum_{l=1}^{n-2}K_l^{-1}\Big((1+\lambda)\varphi_2 a_2,(1+2\lambda)\varphi_3 a_3,\cdots,(1+l\lambda)\varphi_{l+1}a_{l+1}\Big)\\&\times(1-\lambda)(n-l-1)\varphi_{n-l}a_{n-l}\Bigg]=D_{n-1}+\sum_{t=1}^{n-1}\sum_{k=1}^{t}C_kD_t^k(p_1,p_2,\cdots,p_t)D_{n-(t+1)}.\end{aligned} \tag{3.2}$$

Similarly, from (3.1) and (2.5), we have

$$\begin{aligned}&\frac{1}{\gamma}\Bigg[(1-\lambda)(n-1)\varphi_n b_n+\sum_{l=1}^{n-2}K_l^{-1}\Big((1+\lambda)\varphi_2 b_2,(1+2\lambda)\varphi_3 b_3,\cdots,(1+l\lambda)\varphi_{l+1}b_{l+1}\Big)\\&\times(1-\lambda)(n-l-1)\varphi_{n-l}b_{n-l}\Bigg]=D_{n-1}+\sum_{t=1}^{n-1}\sum_{k=1}^{t}C_kD_t^k(q_1,q_2,\cdots,q_t)D_{n-(t+1)}.\end{aligned} \tag{3.3}$$

For $a_k=0$ where $2\le k\le n-1$ and $D_0\neq 0$, we have $p_2=p_3=\cdots=p_{n-2}=0$ and $q_2=q_3=\cdots=q_{n-2}=0$. So from (3.2) and also from equation (1.3) and (3.3) we get, respectively,

$$\frac{1}{\gamma}(1-\lambda)(n-1)\varphi_n a_n=C_1p_{n-1}+D_{n-1} \tag{3.4}$$

and

$$\frac{1}{\gamma}(1-\lambda)(n-1)\varphi_n b_n=-\frac{1}{\gamma}(1+\lambda)(n-1)\varphi_n a_n=C_1q_{n-1}+D_{n-1}. \tag{3.5}$$

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By solving either of the equations (3.4) and (3.5) for a_n and using Lemma 1.3, we obtain

$$|a_n| = \frac{|\gamma||C_1 p_{n-1} + D_{n-1}|}{(1-\lambda)(n-1)\varphi_n} \leq \frac{|\gamma|\big(C_1 + |D_{n-1}|\big)}{(1-\lambda)(n-1)\varphi_n}$$

and this completes the proof. □

Proof of Theorem 2.2. Let

$$F(z) = \frac{1}{\gamma}\Big(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\Big)$$

and

$$G(w) = \frac{1}{\gamma}\Big(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\Big).$$

For the function $f \in \mathcal{B}_\Sigma^{l,m}(\gamma, \lambda, \phi)$, by Lemma1.5 we have

$$F(z) = \frac{1}{\gamma}\sum_{n=2}^{\infty}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 a_2, \varphi_3 a_3, \ldots, \varphi_n a_n) z^{n-1}. \tag{3.6}$$

For its inverse map $g = f^{-1}$, regarding the equality (1.2) we have

$$G(w) = \frac{1}{\gamma}\sum_{n=2}^{\infty}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 b_2, \varphi_3 b_3, \ldots, \varphi_n b_n) w^{n-1}. \tag{3.7}$$

Comparing the coefficients of (3.6), and (2.4), we conclude that

$$\frac{1}{\gamma}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 a_2, \varphi_3 a_3, \ldots, \varphi_n a_n) = D_{n-1} + \sum_{t=1}^{n-1}\sum_{k=1}^{t} C_k D_t^k(p_1, p_2, \cdots, p_t) D_{n-(t+1)}. \tag{3.8}$$

Similarly, from (3.7) and (2.5), we have

$$\frac{1}{\gamma}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 b_2, \varphi_3 b_3, \ldots, \varphi_n b_n) = D_{n-1} + \sum_{t=1}^{n-1}\sum_{k=1}^{t} C_k D_t^k(q_1, q_2, \cdots, q_t) D_{n-(t+1)}. \tag{3.9}$$

Since $a_k = 0$ where $2 \leq k \leq n-1$, and $D_0 \neq 0$ from (3.8) and (3.9) we get, respectively,

$$\frac{1}{\gamma}\big(\lambda + (n-1)\big)\varphi_n a_n = C_1 p_{n-1} + D_{n-1}$$

and

$$-\quad \frac{1}{\gamma}\big(\lambda + (n-1)\big)\varphi_n a_n = C_1 q_{n-1} + D_{n-1}.$$

By solving either of the above equations for a_n and using Lemma 1.3, we conclude the desired results and this completes the proof. □

Proof of Theorem 2.3. For $n = 2$ and $n = 3$ in (3.2) and (3.3), respectively, we obtain

$$(1-\lambda)\varphi_2 a_2 = \gamma D_0 C_1 p_1, \tag{3.10}$$

$$(\lambda^2-1)\varphi_2^2 a_2^2 + 2(1-\lambda)\varphi_3 a_3 = \gamma D_0[C_1 p_2 + C_2 p_1^2] + \gamma D_1 C_1 p_1, \tag{3.11}$$

$$-(1-\lambda)\varphi_2 a_2 = \gamma D_0 C_1 q_1, \tag{3.12}$$

$$(\lambda^2-1)\varphi_2^2 a_2^2 + 2(1-\lambda)\varphi_3(2a_2^2 - a_3) = \gamma D_0[C_1 q_2 + C_2 q_1^2] + \gamma D_1 C_1 q_1. \tag{3.13}$$

From (3.10) and (3.12), we get

$$p_1 = -q_1. \tag{3.14}$$

Adding (3.11) and (3.13) and using (3.14) we obtain

$$\left[2(\lambda^2-1)\varphi_2^2 + 4(1-\lambda)\varphi_3\right] a_2^2 = \gamma D_0 C_1 \left[p_2 + \frac{C_2}{C_1} p_1^2 + q_2 + \frac{C_2}{C_1} q_1^2\right].$$

By using Lemma 1.4 we have

$$\begin{aligned} \left|2(\lambda^2-1)\varphi_2^2 + 4(1-\lambda)\varphi_3\right| |a_2|^2 &\leq |\gamma D_0| C_1 \left[|p_2 + \frac{|C_2|}{C_1} p_1^2| + |q_2 + \frac{C_2}{C_1} q_1^2|\right] \\ &\leq |\gamma D_0| C_1 \left[2 + 2\left(\frac{|C_2| - C_1}{C_1}\right)|p_1^2|\right] \\ &= |\gamma D_0| C_1 \left[2 + 2\left(\frac{(|C_2| - C_1)(1-\lambda)^2 \varphi_2^2 |a_2^2|}{|\gamma D_0|^2 C_1^3}\right)\right]. \end{aligned}$$

After simplification we have

$$\left(|\gamma D_0| C_1^2 \left|2(\lambda^2-1)\varphi_2^2 + 4(1-\lambda)\varphi_3\right| + 2(C_1 - |C_2|)(1-\lambda)^2 \varphi_2^2\right) |a_2|^2 \leq 2|\gamma D_0|^2 C_1^3,$$

which implies that

$$|a_2|^2 \leq \frac{|\gamma D_0|^2 C_1^3}{|\gamma D_0| C_1^2 \left|(\lambda^2-1)\varphi_2^2 + 2(1-\lambda)\varphi_3\right| + (C_1 - |C_2|)(1-\lambda)^2 \varphi_2^2}$$

and this completes the proof. □

Proof of Theorem 2.4. For $n = 2$ and $n = 3$ in (3.8) and (3.9), respectively, we obtain

$$\begin{aligned} (1+\lambda)\varphi_2 a_2 &= \gamma D_0 C_1 p_1, \\ \frac{(\lambda-1)(\lambda+2)}{2} \varphi_2^2 a_2^2 + (\lambda+2)\varphi_3 a_3 &= \gamma D_0[C_1 p_2 + C_2 p_1^2] + \gamma D_1 C_1 p_1, \\ -(1+\lambda)\varphi_2 a_2 &= \gamma D_0 C_1 q_1, \\ \left(\frac{(\lambda-1)(\lambda+2)}{2} \varphi_2^2 + 2(\lambda+2)\varphi_3\right) a_2^2 - (\lambda+2)\varphi_3 a_3 &= \gamma D_0[C_1 q_2 + C_2 q_1^2] + \gamma D_1 C_1 q_1. \end{aligned}$$

With similar method to Theorem 2.3 we get the desired results and this completes the proof. □

Acknowledgements

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

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Conformable Fractional Approximation by Choquet integrals

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Abstract

Here we present the conformable fractional quantitative approximation of positive sublinear operators to the unit operator. These are given a precise Choquet integral interpretation. Initially we start with the study of the conformable fractional rate of the convergence of the well-known Bernstein-Kantorovich-Choquet and Bernstein-Durrweyer-Choquet polynomial Choquet-integral operators. Then we study in the fractional sense the very general comonotonic positive sublinear operators based on the representation theorem of Schmeidler (1986) [11]. We continue with the conformable fractional approximation by the very general direct Choquet-integral form positive sublinear operators. The case of convexity is also studied throughly and the estimates become much simpler. All approximations are given via inequalities involving the modulus of continuity of the approximated function's higher order conformable fractional derivative.

2010 AMS Mathematics Subject Classification: 26A33, 41A17, 41A25, 41A35, 41A36.

Keywords and Phrases: Jackson type inequality, Choquet integral, Conformable Fractional derivative, comonotonicity of functions and operators, Bernstein-Kantorovich-Choquet and Bernstein-Durrmeyer-Choquet operators, convexity.

1 Introduction

G. Choquet (1953) ([4]), introduced the capacities and his integral. Initially these were applied to statistical mechanics and potential theory, and they gave rise to the study of non-additive measure theory. Slowly but steady these ideas of Choquet started to attract economists especially after the very important work of Shapley (1953) ([13]) in the study of cooperative games. Capacities and Choquet integrals became main stream to Decision theorists since 1989 when D. Schmeidler ([12]) was the first to use them in an axiomatic model of choice with non-additive beliefs. The expected utility results are strengthned by the use of Choquet capacities instead of probability measures.

In now days Choquet integral has wide applications, among others, to decision making under risk and uncertainty, in finance, in economics, in portofolio problems and in insurance.

Our motivation also comes from the foundations of Bayesian decision theory and subjective probability.

Because of the paramount importance of Choquet integral, we decided to research the related positive sublinear operators approximation, part of it is exhibited in this work in the conformable fractional sense.

2 Background - I

Next we present briefly about the Choquet integral, see also [8].

We make

Definition 1 *Consider* $\Omega \neq \varnothing$ *and let* $\mathcal{C}$ *be a* σ*-algebra of subsets in* Ω*.*

(i) (see, e.g., [14], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ *is called a monotone set function (or capacity) if* $\mu(\varnothing) = 0$ *and* $\mu(A) \leq \mu(B)$ *for all* $A, B \in \mathcal{C}$*, with* $A \subset B$*. Also,* μ *is called submodular if*

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C}.$$

μ *is called bounded if* $\mu(\Omega) < +\infty$ *and normalized if* $\mu(\Omega) = 1$*.*

(ii) (see, e.g., [14], p. 233, or [4]) If μ *is a monotone set function on* $\mathcal{C}$ *and if* $f : \Omega \to \mathbb{R}$ *is* $\mathcal{C}$*-measurable (that is, for any Borel subset* $B \subset \mathbb{R}$ *it follows* $f^{-1}(B) \in \mathcal{C}$*), then for any* $A \in \mathcal{C}$*, the Choquet integral is defined by*

$$(C) \int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \cap A) d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \cap A) - \mu(A)] d\beta,$$

where we used the notation $F_\beta(f) = \{\omega \in \Omega : f(\omega) \geq \beta\}$*. Notice that if* $f \geq 0$ *on* A*, then in the above formula we get* $\int_{-\infty}^0 = 0$*.*

The integrals on the right-hand side are the usual Riemann integral.

The function f *will be called Choquet integrable on* A *if* $(C)\int_A f d\mu \in \mathbb{R}$*.*

Next we list some well known properties of the Choquet integral.

Remark 2 *If* $\mu : \mathcal{C} \to [0, +\infty]$ *is a monotone set function, then the following properties hold:*

(i) For all $a \geq 0$ we have $(C)\int_A afd\mu = a \cdot (C)\int_A fd\mu$ (if $f \geq 0$ then see, e.g., [14], Theorem 11.2, (5), p. 228 and if f is arbitrary sign, then see, e.g., [5], p. 64, Proposition 5.1, (ii)).

(ii) For all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [14], pp. 232-233, or [5], p. 65) $(C)\int_A (f+c)\,d\mu = (C)\int_A fd\mu + c\cdot\mu(A)$.

If μ is submodular too, then for all f, g of arbitrary sign and lower bounded, we have (see, e.g., [5], p. 75, Theorem 6.3)

$$(C)\int_A (f+g)\,d\mu \leq (C)\int_A fd\mu + (C)\int_A gd\mu.$$

(iii) If $f \leq g$ on A then $(C)\int_A fd\mu \leq (C)\int_A gd\mu$ (see, e.g., [14], p. 228, Theorem 11.2, (3) if $f, g \geq 0$ and p. 232 if f, g are of arbitrary sign).

(iv) Let $f \geq 0$. If $A \subset B$ then $(C)\int_A fd\mu \leq (C)\int_B fd\mu$. In addition, if μ is finitely subadditive, then

$$(C)\int_{A\cup B} fd\mu \leq (C)\int_A fd\mu + (C)\int_B fd\mu.$$

(v) It is immediate that $(C)\int_A 1\cdot d\mu(t) = \mu(A)$.

(vi) The formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0,1] \to [0,1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and M is a probability measure (or only finitely additive) on a σ-algebra on Ω (that is, $M(\varnothing) = 0$, $M(\Omega) = 1$ and M is countably additive), gives simple examples of normalized, monotone and submodular set functions (see, e.g., [5], pp. 16-17, Example 2.1). Such of set functions μ are also called distorsions of countably normalized, additive measures (or distorted measures). For a simple example, we can take $\gamma(t) = \frac{2t}{1+t}$, $\gamma(t) = \sqrt{t}$.

If the above γ function is increasing, concave and satisfies only $\gamma(0) = 0$, then for any bounded Borel measure m, $\mu(A) = \gamma(m(A))$ gives a simple example of bounded, monotone and submodular set function.

(vii) If μ is a countably additive bounded measure, then the Choquet integral $(C)\int_A fd\mu$ reduces to the usual Lebesgue type integral (see, e.g., [5], p. 62, or [14], p. 226).

(viii) If $f \geq 0$, then $(C)\int_A fd\mu \geq 0$.

(ix) Let $\mu = \sqrt{M}$, where M is the Lebesgue measure on $[0,+\infty)$, then μ is a monotone and submodular set function, furthermore μ is strictly positive, see [7].

(x) If $\Omega = \mathbb{R}^N$, $N \in \mathbb{N}$, we call μ strictly positive if $\mu(A) > 0$, for any open subset $A \subseteq \mathbb{R}^N$.

We need some possibility theory:

Definition 3 *([6]) For the $\Omega \neq \varnothing$, the power set $\mathcal{P}(\Omega)$ denotes the family of all subsets of Ω.*

(i) A function $\lambda : \Omega \to [0,1]$ with the property $\sup\{\lambda(s) : s \in \Omega\} = 1$, is called possibility distribution on Ω.

(ii) $P : \mathcal{P}(\Omega) \to [0,1]$ is called possibility measure, if it satisfies $P(\varnothing) = 0$, $P(\Omega) = 1$, and $P(\cup_{i\in I} A_i) = \sup\{P(A_i) : i \in I\}$ for all $A_i \subset \Omega$, and any I, an at most countable family of indices. Note that if $A, B \subset \Omega$, $A \subset B$, then the last property implies $P(A) \leq P(B)$ and that $P(A \cup B) \leq P(A) + P(B)$.

Any possibility distribution λ on Ω, induces the possibility measure $P_\lambda : \mathcal{P}(\Omega) \to [0,1]$, $P_\lambda(A) = \sup\{\lambda(s) : s \in A\}$, $A \subset \Omega$. Also, if $f : \Omega \to \mathbb{R}_+$, then the possibilistic integral of f on $A \subset \Omega$ with respect to P_λ is defined by $(Pos)\int_A f dP_\lambda = \sup\{f(t)\lambda(t) : t \in A\}$ (see [6], chapter 1).

Note that any possiblity measure μ is normalized, monotone and submodular. From $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ we get monotonicity, and from $\mu(A \cap B) \leq \min\{\mu(A), \mu(B)\}$ we derive the submodularity.

3 Background - II

We make

Definition 4 *([2]) Let $f : [a,b] \subseteq [0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$. We say that f is an α-fractional continuous function, iff $\forall\ \varepsilon > 0\ \exists\ \delta > 0$: for any $x, y \in [a,b]$ such that $|x^\alpha - y^\alpha| \leq \delta$ we get that $|f(x) - f(y)| \leq \varepsilon$.*

We mention

Theorem 5 *([2]) Over $[a,b] \subseteq [0,\infty)$, $\alpha \in [0,1]$, an α-fractional continuous function is a uniformly continuous function and vice versa, a uniformly continuous function is an α-fractional continuous function.*

We need

Definition 6 *([2]) Let $[a,b] \subseteq [0,\infty)$, $\alpha \in [0,1]$. We define the α-fractional modulus of continuity:*

$$\omega_1^\alpha(f,\delta) := \sup_{\substack{x,y\in[a,b]:\\ |x^\alpha - y^\alpha|\leq\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{1}$$

Properties ([2]):

1) $\omega_1^\alpha(f,0) = 0$.

2) $\omega_1^\alpha(f,\delta) \to 0$ as $\delta \downarrow 0$, iff f is in the set of all α-fractional continuous functions, denoted as $f \in C_\alpha([a,b],\mathbb{R})$ $(= C([a,b],\mathbb{R}))$.

3) ω_1^α is ≥ 0 and non-decreasing on $\mathbb{R}_+$.

4) ω_1^α is subadditive:

$$\omega_1^\alpha (f, t_1 + t_2) \leq \omega_1^\alpha (f, t_1) + \omega_1^\alpha (f, t_2). \tag{2}$$

5) ω_1^α is continuous on $\mathbb{R}_+$.

6) Clearly it holds

$$\omega_1^\alpha (f, t_1 + ... + t_n) \leq \omega_1^\alpha (f, t_1) + ... + \omega_1^\alpha (f, t_n), \tag{3}$$

for $t = t_1 = ... = t_n$, we obtain

$$\omega_1^\alpha (f, nt) = n\omega_1^\alpha (f, t). \tag{4}$$

7) Let $\lambda \geq 0$, $\lambda \notin \mathbb{N}$, we get

$$\omega_1^\alpha (f, \lambda t) \leq (\lambda + 1)\, \omega_1^\alpha (f, t). \tag{5}$$

We notice that $\omega_1^\alpha (f, \delta)$ is finite when f is uniformly continuous on $[a, b] \subseteq [0, \infty)$.

We need

Definition 7 *([9], [10]) Let $f : [0, \infty) \to \mathbb{R}$. The conformable α-fractional derivative for $\alpha \in (0, 1]$ is given by*

$$D_\alpha f (t) := \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{6}$$

$$D_\alpha f (0) = \lim_{t \to 0+} D_\alpha f (t).$$

If f is differentiable, then

$$D_\alpha f (t) = t^{1-\alpha} f'(t), \tag{7}$$

where f' is the usual derivative.

We define $D_\alpha^n f = D_\alpha^{n-1} (D_\alpha f)$, $D_\alpha^0 f = f$.

If $f : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then f is continuous at t_0, see [10].

We need

Definition 8 *([2]) Here $C_+ ([a, b]) := \{f : [a, b] \subseteq [0, \infty) \to \mathbb{R}_+$, continuous functions$\}$. Let $L_N : C_+ ([a, b]) \to C_+ ([a, b])$, operators, $\forall\ N \in \mathbb{N}$, such that*

(i)

$$L_N (\alpha f) = \alpha L_N (f),\ \forall \alpha \geq 0, \forall f \in C_+ ([a, b]), \tag{8}$$

(ii) if $f, g \in C_+ ([a, b]) : f \leq g$, then

$$L_N (f) \leq L_N (g),\ \ \forall N \in \mathbb{N}, \tag{9}$$

(iii)

$$L_N (f + g) \leq L_N (f) + L_N (g),\ \ \forall\ f, g \in C_+ ([a, b]). \tag{10}$$

We call $\{L_N\}_{N \in \mathbb{N}}$ positive sublinear operators.

We need

Theorem 9 *([2]) Let $\alpha \in (0,1]$, $[a,b] \subseteq [0,\infty)$. Suppose f is α-conformable fractional differentiable on $[a,b]$. $D_\alpha f$ is continuous on $[a,b]$. Let an $x \in [a,b]$ such that $D_\alpha f(x) = 0$, and L_N from $C_+([a,b])$ into itself, positive sublinear operators. Assume that $L_N(1) = 1$ and $L_N\left(|\cdot - x|^{\alpha+1}\right)(x)$, $L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$.*

Then

$$|L_N(f)(x) - f(x)| \le \frac{\omega_1^\alpha\left(D_\alpha f, \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot$$

$$\left[\left(L_N\left(|\cdot - x|^{\alpha+1}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \quad \forall\ N \in \mathbb{N}. \tag{11}$$

We make

Remark 10 *([2]) By [2], we get that*

$$L_N\left(|\cdot - x|^{\alpha+1}\right)(x) \le \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{1}{2}}. \tag{12}$$

As $N \to +\infty$, by (11) and (12), and $L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x) \to 0$, we obtain that $L_N(f)(x) \to f(x)$.

We need

Theorem 11 *([2]) Let $\alpha \in (0,1]$, $n \in \mathbb{N}$. Suppose f is n times conformable α-fractional differentiable on $[a,b] \subseteq [0,\infty)$, and $D_\alpha^n f$ is continuous on $[a,b]$. For a fixed $x \in [a,b]$ we have $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. Let positive sublinear operators $\{L_N\}_{N\in\mathbb{N}}$ from $C_+([a,b])$ into itself, such that $L_N(1) = 1$, and $L_N\left(|\cdot - x|^{n(\alpha+1)}\right)(x)$, $L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \le \frac{\omega_1^\alpha\left(D_\alpha^n f, \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!} \cdot \tag{13}$$

$$\left[\left(L_N\left(|\cdot - x|^{n(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{(n+1)}\left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right],$$

$\forall\ N \in \mathbb{N}$.

We make

Remark 12 *([2]) By [2], we get that*

$$L_N\left(|\cdot-x|^{n(\alpha+1)}\right)(x)\leq\left(L_N\left(|\cdot-x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{n}{n+1}}. \tag{14}$$

As $N\to+\infty$, *by (13), (14), and* $L_N\left(|\cdot-x|^{(n+1)(\alpha+1)}\right)(x)\to 0$, *we derive that* $L_N(f)(x)\to f(x)$.

We also need

Definition 13 *Let* $f\in C([a,b])$. *We define the usual first modulus of continuity of* f *as:*

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in[a,b]:\\|x-y|\leq\delta}}|f(x)-f(y)|,\quad \delta>0. \tag{15}$$

We need

Theorem 14 *([3]) Let* $\alpha\in(0,1]$ *and* $n\in\mathbb{N}$. *Suppose* $f\in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b]\subseteq[0,\infty)$, *and* $x\in(a,b)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Let* $0<h\leq\min(x-a,b-x)$ *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x)=0$, $k=1,...,n$. *Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $C_+([a,b])$ *into itself, positive sublinear operators such that:* $L_N(1)=1$, $\forall\ N\in\mathbb{N}$. *Then*

$$|L_N(f)(x)-f(x)|\leq\left(\frac{\omega_1(D_\alpha^n f,h)\,b^{1-\alpha}}{(n+1)!\alpha^{n+1}h}\right)L_N\left(|\cdot-x|^{(n+1)\alpha}\right)(x),\quad\forall\ N\in\mathbb{N}. \tag{16}$$

We have

Theorem 15 *([3]) All as in Theorem 14. Additionally assume that* $L_N\left(|\cdot-x|^{(n+1)(\alpha+1)}\right)(x)>0$, $\forall\ N\in\mathbb{N}$. *Then*

$$|L_N(f)(x)-f(x)|\leq\left(\frac{\omega_1(D_\alpha^n f,h)\,b^{1-\alpha}}{(n+1)!\alpha^{n+1}h}\right)\left(L_N\left(|\cdot-x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}, \tag{17}$$

$\forall\ N\in\mathbb{N}$.

An application of Theorem 15 follows:

Theorem 16 *([3])Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $C_+([a,b])$ *into itself, positive sublinear operators:* $L_N(1)=1$, $\forall\ N\in\mathbb{N}$. *Also* $x\in(a,b)$ *and* $L_N\left(|\cdot-x|^{(n+1)(\alpha+1)}\right)(x)>0$, $\forall\ N\in\mathbb{N}$. *Here* $\alpha\in(0,1]$ *and* $n\in\mathbb{N}$. *Suppose* $f\in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b]\subseteq[0,\infty)$, *and* $D_\alpha^n f$ *is* *continuous on* $[a,b]$*. Assume here that* $0 < \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$*, and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$*. Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$*. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}\right)}{(n+1)!\alpha^{n+1}}, \quad (18)$$

$\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$.

If $L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x) \to 0$*, then* $L_N(f)(x) \to f(x)$*, as* $N \to +\infty$.

An application of Theorem 14 follows:

Theorem 17 *([3]) Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $C_+([a,b])$ *into itself, positive sublinear operators:* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$*. Also* $L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$*. Here* $\alpha \in (0,1]$, $n \in \mathbb{N}$ *and* $x \in (a,b)$*;* $[a,b] \subseteq [0,\infty)$*. Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b]$*, and* $D_\alpha^n f$ *is continuous on* $[a,b]$*. Let* $0 < L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x) \leq \min(x-a, b-x)$, $\forall\ N \geq N^*$*;* N*,* $N^* \in \mathbb{N}$*, and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$*. Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$*. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x)\right)}{(n+1)!\alpha^{n+1}}, \quad (19)$$

$\forall\ N \geq N^*$*, where* $N, N^* \in \mathbb{N}$.

If $L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x) \to 0$*, then* $L_N(f)(x) \to f(x)$*, as* $N \to +\infty$.

4 Background - III

We mention

Definition 18 *([7]) Let* $I = [0,1]$*,* $\mathcal{B}_I$ *the* σ*-algebra of all Borel measurable subsets of* I*,* $(\Gamma_{N,x})_{N\in\mathbb{N},\ x\in I}$ *will be the collection of the family* $\Gamma_{N,x} = \{\mu_{N,k,x}\}_{k=0}^N$*, of monotone, submodular and strictly positive set functions* $\mu_{N,k,x}$ *on* $\mathcal{B}_I$*.*

Let $f : [0,1] \to \mathbb{R}_+$ *be a* B_I*-measurable function which is bounded, and call* $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$*, for any* $x \in [0,1]$*.*

The Bernstein-Kantorovich-Choquet operators are defined by the formula

$$K_{N,\Gamma_{N,x}}(f)(x) = \sum_{k=0}^N p_{N,k}(x) \frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}} f(t)\, d\mu_{N,k,x}(t)}{\mu_{N,k,x}\left(\left[\frac{k}{(N+1)}, \frac{(k+1)}{(N+1)}\right]\right)}, \quad \forall\ x \in [0,1]. \quad (20)$$

If $\mu_{N,k,x} = \mu$*, for all* N, x, k*, we will denote* $K_{N,\Gamma_{N,x}}(f) := K_{N,\mu}(f)$*.*

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Theorem 19 *([7]) Suppose that $\mu_{N,k,x} = \mu := \sqrt{M}$, for all N, k and x, where M is the Lebesgue measure on $[0,1]$. Then*

$$\left|K_{N,\mu}(f)(x) - f(x)\right| \leq 2\omega_1\left(f, \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}\right), \tag{21}$$

$\forall\ N \in \mathbb{N}$, $x \in [0,1]$, $f \in C_+([0,1])$, above ω_1 is over $[0,1]$.

Remark 20 *By [7] we have that*

$$K_{N,\mu}(|\cdot - x|)(x) \leq \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\ N \in \mathbb{N}. \tag{22}$$

Let $m > 1$, notice that $|\cdot - x|^{m-1} \leq 1$, therefore

$$|\cdot - x|^m = |\cdot - x|\,|\cdot - x|^{m-1} \leq |\cdot - x|,$$

hence

$$K_{N,\mu}(|\cdot - x|^m)(x) \leq K_{N,\mu}(|\cdot - x|)(x),$$

that is

$$K_{N,\mu}(|\cdot - x|^m)(x) \leq \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\ x \in [0,1],\ N \in \mathbb{N},\ m \geq 1. \tag{23}$$

Notice that $K_{N,\mu}(1) = 1$, $\forall\ N \in \mathbb{N}$.

Clearly $K_{N,\mu}$ operators are positive sublinear operators from $C_+([0,1])$ into itself.

We mention

Definition 21 *([8]) Here we consider measures of possibility. Denoting $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, let us defined*

$$\lambda_{N,k}(t) := \frac{p_{N,k}(t)}{k^k N^{-N}(N-k)^{N-k}\binom{N}{k}} = \frac{t^k(1-t)^{N-k}}{k^k N^{-N}(N-k)^{N-k}}, \quad k = 0, ..., N. \tag{24}$$

By convention we assume that $0^0 = 1$, so that the cases $k = 0$, and $k = N$ make sense. By considering the root $\frac{k}{N}$ of $p'_{N,k}(x)$, it is clear that

$$\max\{p_{N,k}(t) : t \in [0,1]\} = k^k N^{-N}(N-k)^{N-k}\binom{N}{k},$$

which implies that each $\lambda_{N,k}$ is a possibility distribution on $[0,1]$.

Denoting by $P_{\lambda_{N,k}}$ the possibility measure induced by $\lambda_{N,k}$ and $\Gamma_{n,x} := \Gamma_N := \{P_{\lambda_{N,k}}\}_{k=0}^N$ (that is Γ_N is independent of x), we define the nonlinear Bernstein-Durrmeyer-Choquet polynomial operators with respect to the set functions in Γ_N given by the formula

$$D_{N,\Gamma_N}(f)(x) := \sum_{k=0}^{N} p_{N,k}(x) \frac{(C)\int_0^1 f(t)\, t^k (1-t)^{N-k} dP_{\lambda_{N,k}}(t)}{(C)\int_0^1 t^k (1-t)^{N-k} dP_{\lambda_{N,k}}(t)}, \quad (25)$$

$\forall\ x \in [0,1]$, $N \in \mathbb{N}$, $f \in C_+([0,1])$.

Remark 22 *Above $P_{\lambda_{N,k}}$ is bounded, monotone, submodular and strictly positive, $N \in \mathbb{N}$, $k = 0,1,...,N$. Notice that $D_{N,\Gamma_N}(1) = 1$, $\forall\ N \in \mathbb{N}$.*

Clearly D_{N,Γ_N} operators are positive sublinear operators mapping $C_+([0,1])$ into itself.

We mention

Theorem 23 *([8]) For every $f \in C_+([0,1])$, $x \in [0,1]$ and $N \in \mathbb{N} - \{1\}$, we have*

$$|D_{N,\Gamma_N}(f)(x) - f(x)| \leq 2\omega_1 \left(f, \frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N} \right), \quad (26)$$

where ω_1 is on $[0,1]$.

Remark 24 *By [8] we have that*

$$D_{N,\Gamma_N}(|\cdot - x|)(x) \leq \frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\ N \in \mathbb{N} - \{1\}. \quad (27)$$

Let $m > 1$, notice that $|\cdot - x|^{m-1} \leq 1$, therefore

$$|\cdot - x|^m = |\cdot - x|\,|\cdot - x|^{m-1} \leq |\cdot - x|,$$

hence

$$D_{N,\Gamma_N}(|\cdot - x|^m)(x) \leq D_{N,\Gamma_N}(|\cdot - x|)(x),$$

that is

$$D_{N,\Gamma_N}(|\cdot - x|^m)(x) \leq \frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N}, \quad (28)$$

$\forall\ N \in \mathbb{N} - \{1\}$, $\forall\ x \in [0,1]$, $m \geq 1$.

We make

Remark 25 *When* $x \in [0,1]$*, then the* $\max(x(1-x)) = \frac{1}{4}$*, at* $x = \frac{1}{2}$*. Therefore it holds*

$$\frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N} \leq \frac{1}{2\sqrt{N}} + \frac{1}{N}, \tag{29}$$

$\forall\ x \in [0,1],\ \forall\ N \in \mathbb{N}$.

Similarly, it holds

$$\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N} \leq \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}, \tag{30}$$

$\forall\ x \in [0,1],\ \forall\ N \in \mathbb{N} - \{1\}$.

Corollary 26 *(to Theorem 19) It holds*

$$\|K_{N,\mu}(f) - f\|_{\infty} \leq 2\omega_1\left(f, \frac{1}{2\sqrt{N}} + \frac{1}{N}\right), \tag{31}$$

$\forall\ N \in \mathbb{N},\ f \in C_+([0,1])$.

Corollary 27 *(to Theorem 23) It holds*

$$\|D_{N,\Gamma_N}(f) - f\|_{\infty} \leq 2\omega_1\left(f, \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right), \tag{32}$$

$\forall\ N \in \mathbb{N} - \{1\},\ f \in C_+([0,1])$.

5 Main Results

Here first we apply some of the main theorems mentioned in section 3 to the Bernstein-Kantorovich-Choquet operators $K_{N,\mu}$, where $\mu := \sqrt{M}$, with M the Lebesgue measure on $[0,1]$. More precisely here it is

$$K_{N,\mu}(f)(x) = \sum_{k=0}^{N} p_{N,k}(x) \frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}} f(t)\,d\mu(t)}{\mu\left(\left[\frac{k}{(N+1)}, \frac{(k+1)}{(N+1)}\right]\right)}, \tag{33}$$

$\forall\ x \in [0,1],\ \forall\ N \in \mathbb{N},\ f \in C_+([0,1])$.

It follows applications to Bernstein-Durremeyer-Choquet operators D_{N,Γ_N}, see (25).

In particular we need (a variation of Theorem 11):

Theorem 28 *([2]) Let* $\alpha \in (0,1]$ *and* $n \in \mathbb{N} : n\alpha \geq 1$*. That is* $\frac{1}{n} \leq \alpha \leq 1$*. Suppose* f *is* n *times conformable* α*-fractional differentiable on* $[a,b] \subseteq [0,\infty)$*, and* $D_{\alpha}^{n} f$ *is continuous on* $[a,b]$*. For a fixed* $x \in [a,b]$ *we have* $D_{\alpha}^{k} f(x) = 0$*,*

$k = 1, ..., n$. *Let positive sublinear operators* $\{L_N\}_{N \in \mathbb{N}}$ *from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$, *and* $\delta > 0$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha(D_\alpha^n f, \delta)}{\alpha^n n!} \cdot$$

$$\left[L_N(|\cdot - x|^{n\alpha})(x) + \frac{1}{(n+1)\delta} L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x)\right], \tag{34}$$

$\forall\ N \in \mathbb{N}$.

We present

Theorem 29 *Let* $\alpha \in (0,1]$ *and* $n \in \mathbb{N} : n\alpha \geq 1$. *Suppose* f *is* n *times conformable* α*-fractional differentiable on* $[0,1]$, *and* $D_\alpha^n f$ *is continuous on* $[0,1]$. *For a fixed* $x \in [0,1]$ *we have* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|K_{N,\mu}(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha^n f, \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!} \cdot$$

$$\left[\left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right) + \frac{1}{(n+1)}\left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{n}{n+1}}\right] \leq$$

$$\frac{\omega_1^\alpha\left(D_\alpha^n f, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!}\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(n+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{n}{n+1}}\right], \tag{35}$$

$\forall\ N \in \mathbb{N}$.

Notice that $\lim_{N \to \infty} K_{N,\mu}(f)(x) = f(x)$.

Proof. By (34) we have

$$|K_{N,\mu}(f)(x) - f(x)| \leq \frac{\omega_1^\alpha(D_\alpha^n f, \delta)}{\alpha^n n!} \cdot$$

$$\left[K_{N,\mu}(|\cdot - x|^{n\alpha})(x) + \frac{1}{(n+1)\delta} K_{N,\mu}\left(|\cdot - x|^{(n+1)\alpha}\right)(x)\right] \overset{(23)}{\leq}$$

$$\frac{\omega_1^\alpha(D_\alpha^n f, \delta)}{\alpha^n n!}\left[\left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right) + \frac{1}{(n+1)\delta}\left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)\right] \tag{36}$$

(choose $\delta := \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}} > 0$, then $\delta^{n+1} = \sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}$, and $\delta^n = \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{n}{n+1}}$)

$$= \frac{\omega_1^\alpha\left(D_\alpha^n f, \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!} \cdot$$

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$$\left[\left(\sqrt{\frac{x(1-x)}{N}}+\frac{1}{N}\right)+\frac{1}{(n+1)}\left(\sqrt{\frac{x(1-x)}{N}}+\frac{1}{N}\right)^{\frac{n}{n+1}}\right]\overset{(29)}{\leq}$$

$$\frac{\omega_1^{\alpha}\left(D_{\alpha}^n f,\left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!}\left[\left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)+\frac{1}{(n+1)}\left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{n}{n+1}}\right], \tag{37}$$

proving the claim. ■

We continue with

Theorem 30 *All as in Theorem 29. Then*

$$|(D_{N,\Gamma_N}(f))(x)-f(x)|\leq\frac{\omega_1^{\alpha}\left(D_{\alpha}^n f,\left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!}$$

$$\left[\left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)+\right.$$

$$\left.\frac{1}{(n+1)}\left[\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right]^{\frac{n}{n+1}}\right]\leq$$

$$\frac{\omega_1^{\alpha}\left(D_{\alpha}^n f,\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!}\cdot \tag{38}$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)+\frac{1}{(n+1)}\left[\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right]^{\frac{n}{n+1}}\right],$$

$\forall\ N\in\mathbb{N}-\{1\}$.

Notice that $\lim_{N\to+\infty}D_{N,\Gamma_N}(f)(x)=f(x)$.

Proof. By (34) we have

$$|D_{N,\Gamma_N}(f)(x)-f(x)|\leq\frac{\omega_1^{\alpha}(D_{\alpha}^n f,\delta)}{\alpha^n n!}\cdot$$

$$\left[D_{N,\Gamma_N}\left(|\cdot-x|^{n\alpha}\right)(x)+\frac{1}{(n+1)\delta}D_{N,\Gamma_N}\left(|\cdot-x|^{(n+1)\alpha}\right)(x)\right]\overset{(28)}{\leq}$$

$$\frac{\omega_1^{\alpha}(D_{\alpha}^n f,\delta)}{\alpha^n n!}\left[\left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)+\right.$$

$$\left.\frac{1}{(n+1)\delta}\left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)\right] \tag{39}$$

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(choose $\delta := \left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right)^{\frac{1}{n+1}} > 0$, then

$\delta^{n+1} = \frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N}$, and $\delta^{n} = \left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right)^{\frac{n}{n+1}}$)

$$= \frac{\omega_1^{\alpha}\left(D_{\alpha}^{n} f, \left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right)^{\frac{1}{n+1}} \right)}{\alpha^{n} n!} \cdot$$

$$\left[\left(\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right) + \right.$$

$$\left. \frac{1}{(n+1)} \left[\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right]^{\frac{n}{n+1}} \right] \overset{(30)}{\leq} \tag{40}$$

$$\frac{\omega_1^{\alpha}\left(D_{\alpha}^{n} f, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N} \right)^{\frac{1}{n+1}} \right)}{\alpha^{n} n!} \cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N} \right) + \frac{1}{(n+1)} \left[\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N} \right]^{\frac{n}{n+1}} \right],$$

$\forall\ N \in \mathbb{N} - \{1\}$, proving the claim. ■

Next we apply Theorem 14.

We give

Theorem 31 *Let $\alpha \in (0,1]$ and $n \in \mathbb{N}$ such that $(n+1)\alpha \geq 1$, that is $\frac{1}{n+1} \leq \alpha \leq 1$. Suppose $f \in C_{+}([0,1])$ is n times conformable α-fractional differentiable on $[0,1]$, and $x \in (0,1)$, and $D_{\alpha}^{n} f$ is continuous on $[0,1]$. Let $N^{*} \in \mathbb{N}$ such that $\frac{1}{2\sqrt{N^{*}}} + \frac{1}{N^{*}} \leq \min(x, 1-x)$ and assume $|D_{\alpha}^{n} f|$ is convex over $[0,1]$. Furthermore assume that $D_{\alpha}^{k} f(x) = 0$, $k = 1, ..., n$. Then*

$$|(K_{N,\mu}(f))(x) - f(x)| \leq \frac{\omega_1\left(D_{\alpha}^{n} f, \frac{1}{2\sqrt{N}} + \frac{1}{N} \right)}{(n+1)!\alpha^{n+1}}, \tag{41}$$

$\forall\ N \geq N^{*}$, $N \in \mathbb{N}$.

It holds $\lim_{N \to +\infty} K_{N,\mu}(f)(x) = f(x)$.

Proof. By (16) we get

$$|K_{N,\mu}(f)(x) - f(x)| \leq \frac{\omega_1(D_{\alpha}^{n} f, h)}{(n+1)!\alpha^{n+1} h} K_{N,\mu}\left(|\cdot - x|^{(n+1)\alpha} \right)(x) \overset{(23)}{\leq}$$

$$\frac{\omega_1(D_{\alpha}^{n} f, h)}{(n+1)!\alpha^{n+1} h} \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N} \right) \overset{(29)}{\leq}$$

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$$\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1}h}\left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)= \tag{42}$$

(setting $h:=\frac{1}{2\sqrt{N}}+\frac{1}{N}>0$)

$$\frac{\omega_1\left(D_\alpha^n f, \frac{1}{2\sqrt{N}}+\frac{1}{N}\right)}{(n+1)!\alpha^{n+1}},$$

proving the claim. ∎

We continue with

Theorem 32 *Let $x\in(0,1)$ and $N^*\in\mathbb{N}-\{1\}$: $\frac{\left(1+3\sqrt{2}\right)}{2\sqrt{N^*}}+\frac{1}{N^*}\leq\min(x,1-x)$. The rest as in Theorem 31. Then*

$$\left|\left(D_{N,\Gamma_N}(f)\right)(x)-f(x)\right|\leq\frac{\omega_1\left(D_\alpha^n f, \frac{\left(1+3\sqrt{2}\right)}{2\sqrt{N}}+\frac{1}{N}\right)}{(n+1)!\alpha^{n+1}}, \tag{43}$$

$\forall\ N\geq N^*,\ N\in\mathbb{N}-\{1\}$.

It holds $\lim_{N\to+\infty}D_{N,\Gamma_N}(f)(x)=f(x)$.

Proof. We use Theorem 14:

By (16) we get

$$\left|D_{N,\Gamma_N}(f)(x)-f(x)\right|\leq\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1}h}D_{N,\Gamma_N}\left(|\cdot-x|^{(n+1)\alpha}\right)(x)\overset{(28)}{\leq}$$

$$\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1}h}\left(\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)\overset{(30)}{\leq}$$

$$\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1}h}\left(\frac{\left(1+3\sqrt{2}\right)}{2\sqrt{N}}+\frac{1}{N}\right) \tag{44}$$

(setting $h:=\frac{\left(1+3\sqrt{2}\right)}{2\sqrt{N}}+\frac{1}{N}>0$)

$$=\frac{\omega_1\left(D_\alpha^n f, \frac{\left(1+3\sqrt{2}\right)}{2\sqrt{N}}+\frac{1}{N}\right)}{(n+1)!\alpha^{n+1}},$$

proving the claim. ∎

We need

Definition 33 *Let Ω be a set, and let $f,g:\Omega\to\mathbb{R}$ be bounded functions. We say that f and g are comonotonic, if for every $\omega,\omega'\in\Omega$,*

$$\left(f(\omega)-f(\omega')\right)\left(g(\omega)-g(\omega')\right)\geq 0. \tag{45}$$

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We also need the famous Schmeidler's Representation Theorem (Schmeidler 1986)

Theorem 34 *([11]) Denote with $\mathcal{L}_\infty(\mathcal{A})$ the vector space of $\mathcal{A}$-measurable bounded real valued functions on Ω, where $\mathcal{A} \subset 2^\Omega$ is a σ-algebra. Given a real functional $\Gamma : \mathcal{L}_\infty(\mathcal{A}) \to \mathbb{R}$, assume that for $f, g \in \mathcal{L}_\infty(\mathcal{A})$:*

(i) $\Gamma(cf) = c\Gamma(f)$, $\forall\ c > 0$,

(ii) $f \leq g$, implies $\Gamma(f) \leq \Gamma(g)$,

and

(iii) $\Gamma(f+g) = \Gamma(f) + \Gamma(g)$, for any comonotonic f, g.

Then $\gamma(A) := \Gamma(1_A)$, $\forall\ A \in \mathcal{A}$, defines a finite monotone set function on $\mathcal{A}$, and Γ is the Choquet integral with respect to γ, i.e.

$$\Gamma(f) = (C)\int_\Omega f(t)\, d\gamma(t), \quad \forall\ f \in \mathcal{L}_\infty(\mathcal{A}). \tag{46}$$

Above 1_A denotes the characteristic function on A.

Next we give nice interpretations of Theorems 9, 11, 16, 17 involving Choquet integrals and based on Theorem 34.

We make

Remark 35 *Consider here $[a,b] \subset \mathbb{R}_+$, $\mathcal{B} = \mathcal{B}([a,b])$ is the Borel σ-algebra on $[a,b]$, and $\mathcal{L}_\infty(\mathcal{B})$ is the vector space of $\mathcal{B}$-measurable bounded real valued functions on $[a,b]$. Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $\mathcal{L}_\infty(\mathcal{B})$ into $C_+([a,b])$, and $x \in [a,b]$. That is here L_N fulfills the positive homogenuity, monotonicity and subadditivity properties, see (8)-(10).*

Assume $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Clearly here $\mathcal{L}_\infty(\mathcal{B}) \supset C_+([a,b])$, where $[a,b] \subset [0,\infty)$. In particular we treat $L_N|_{C_+([a,b])}$, just denoted for simplicity by L_N, $\forall\ N \in \mathbb{N}$.

It is clear that $L_N(\cdot)(x) : \mathcal{L}_\infty(\mathcal{B}) \to \mathbb{R}$ is a functional, $\forall\ N \in \mathbb{N}$. It has the properties:

(i)

$$L_N(cf)(x) = cL_N(f)(x), \ \forall\ c > 0, \ \forall\ f \in \mathcal{L}_\infty(\mathcal{B}), \tag{47}$$

(ii)

$$f \leq g, \text{ implies } L_N(f)(x) \leq L_N(g)(x), \quad \text{where } f, g \in \mathcal{L}_\infty(\mathcal{B}), \tag{48}$$

and

(iii)

$$L_N(f+g)(x) \leq L_N(f)(x) + L_N(g)(x), \quad \forall\ f, g \in \mathcal{L}_\infty(\mathcal{B}). \tag{49}$$

For comonotonic $f, g \in \mathcal{L}_\infty(\mathcal{B})$, we further assume that

$$L_N(f+g)(x) = L_N(f)(x) + L_N(g)(x). \tag{50}$$

In that case L_N is called comonotonic.

By Theorem 34 we get that:

$$\gamma_{N,x}(A) := L_N(1_A)(x), \ \forall A \in \mathcal{B}, \forall N \in \mathbb{N}, \tag{51}$$

defines a finite monotone set function on $\mathcal{B}$, and

$$L_N(f)(x) = (C)\int_a^b f(t)\,d\gamma_{N,x}(t), \tag{52}$$

$\forall f \in \mathcal{L}_\infty(\mathcal{B})$, $\forall N \in \mathbb{N}$.

In particular (52) is valid for any $f \in C_+([a,b])$. Furthermore $\gamma_{N,x}$ is normalized, that is $\gamma_{N,x}([a,b]) = 1$, $\forall N \in \mathbb{N}$.

We give

Theorem 36 *Let $\alpha \in (0,1]$, $[a,b] \subseteq [0,\infty)$. Suppose f is $\mathbb{R}_+$ valued and is α-conformable fractional differentiable on $[a,b]$, with $D_\alpha f$ being continuous on $[a,b]$. Let $x \in [a,b]$ such that $D_\alpha f(x) = 0$, and $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear comonotonic operators from $\mathcal{L}_\infty(\mathcal{B})$ into $C_+([a,b])$. We assume that $L_N(1) = 1$, and $(C)\int_a^b |t-x|^{\alpha+1} d\gamma_{N,x}(t) > 0$, $(C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t) > 0$, $\forall N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \le \frac{\omega_1^\alpha\left(D_\alpha f, \left((C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha}\cdot$$

$$\left[\left((C)\int_a^b |t-x|^{\alpha+1} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left((C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \tag{53}$$

$\forall N \in \mathbb{N}$.

As $(C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t) \to 0$, $N \to \infty$, we get that $\lim_{N\to+\infty} L_N(f)(x) = f(x)$.

Proof. By Theorems 9, 34. ∎

Theorem 37 *Let $\alpha \in (0,1]$, $n \in \mathbb{N}$. Suppose f is $\mathbb{R}_+$ valued and is n times conformable α-fractional differentiable on $[a,b] \subseteq [0,\infty)$, and $D_\alpha^n f$ is continuous on $[a,b]$. For a fixed $x \in [a,b]$ we have $D_\alpha^k f(x) = 0$, $k = 1,...,n$. Let positive sublinear comonotonic operators $\{L_N\}_{N\in\mathbb{N}}$ from $\mathcal{L}_\infty(\mathcal{B})$ into $C_+([a,b])$, such that $(C)\int_a^b |t-x|^{n(\alpha+1)} d\gamma_{N,x}(t)$, $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t) > 0$, $\forall N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \le \frac{\omega_1^\alpha\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!}\cdot \tag{54}$$

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$$\left[\left((C)\int_a^b |t-x|^{n(\alpha+1)}\, d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \right.$$

$$\left. \frac{1}{(n+1)}\left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\gamma_{N,x}(t)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right],$$

$\forall\ N \in \mathbb{N}$.

As $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\gamma_{N,x}(t) \to 0$, *when* $N \to \infty$, *we get that* $\lim\limits_{N\to+\infty} L_N(f)(x) = f(x)$.

Proof. By Theorems 11, 34. ∎

We continue with

Theorem 38 *Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $\mathcal{L}_\infty(\mathcal{B})$ *into* $C_+([a,b])$ *positive sublinear comonotonic operators, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Additionally assume that* $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\gamma_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$; $x \in (a,b)$. *Here* $\alpha \in (0,1]$, *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Assume here* $0 < \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}}\right)}{(n+1)!\alpha^{n+1}}, \tag{55}$$

$\forall\ N \geq N^*$; $N, N^* \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\gamma_{N,x}(t) \to 0$, *then* $L_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorems 16, 34. ∎

Theorem 39 *Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $\mathcal{L}_\infty(\mathcal{B})$ *into* $C_+([a,b])$ *positive sublinear comonotonic operators, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Additionally assume that* $(C)\int_a^b |t-x|^{(n+1)\alpha}\, d\gamma_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$; $x \in (a,b)$. *Here* $\alpha \in (0,1]$, *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Assume here* $0 < \left((C)\int_a^b |t-x|^{(n+1)\alpha}\, d\gamma_{N,x}(t)\right) \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, (C)\int_a^b |t-x|^{(n+1)\alpha}\, d\gamma_{N,x}(t)\right)}{(n+1)!\alpha^{n+1}}, \tag{56}$$

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$\forall\ N \geq N^*$, *where* $N, N^* \in \mathbb{N}$.

If $(C) \int_a^b |t-x|^{(n+1)\alpha} d\gamma_{N,x}(t) \to 0$, *then* $L_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorems 17, 34. ■

We make

Remark 40 *Consider again* $[a,b] \subset \mathbb{R}_+$, $\mathcal{B} = \mathcal{B}([a,b])$ *the Borel* σ*-algebra on* $[a,b]$. *For each* $N \in \mathbb{N}$ *and each* $x \in [a,b]$ *consider the monotone set functions* $\mu_{N,x}; \mathcal{B} \to \mathbb{R}_+$. *We assume that all* $\mu_{N,x}$ *are normalized, that is* $\mu_{N,x}([a,b]) = 1$, *and submodular. Here we consider the operators* $T_N : C_+([a,b]) \to C_+([a,b])$ *given by the formula*

$$T_N(f)(x) = (C)\int_a^b f(t)\, d\mu_{N,x}(t), \tag{57}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x \in [a,b]$.

Infact here $\mu_{N,x}$ *are chosen so that* $T_N(C_+([a,b])) \subseteq C_+([a,b])$.

We notice here that hold:

(i)

$$T_N(\alpha f)(x) = \alpha T_N(f)(x),\ \forall\ \alpha \geq 0, \tag{58}$$

(ii)

$$f \leq g, \text{ implies } T_N(f)(x) \leq T_N(g)(x), \tag{59}$$

and

(iii)

$$T_N(f+g)(x) \leq T_N(f)(x) + T_N(g)(x), \tag{60}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x \in [a,b]$, $\forall\ f, g \in C_+([a,b])$.

Clearly T_N *are positive sublinear operators, compare to (8)-(10). We also have that* $T_N(1) = 1$, $\forall\ N \in \mathbb{N}$.

We give

Theorem 41 *Let* $\alpha \in (0,1]$, $[a,b] \subseteq [0,\infty)$. *Suppose* f *is* α*-conformable fractional differentiable on* $[a,b]$. $D_\alpha f$ *is continuous on* $[a,b]$. *Let an* $x \in [a,b]$ *such that* $D_\alpha f(x) = 0$. *Assume* $(C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)$, $(C)\int_a^b (t-x)^{2(\alpha+1)} d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha f, \left((C)\int_a^b (t-x)^{2(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot$$

$$\left[\left((C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left((C)\int_a^b (t-x)^{2(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \tag{61}$$

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$\forall\ N \in \mathbb{N}$.

As $N \to \infty$, *and* $(C)\int_a^b (t-x)^{2(\alpha+1)}\, d\mu_{N,x}(t) \to 0$, *we obtain* $T_N(f)(x) \to f(x)$.

Proof. By Theorem 9. ■

Theorem 42 *Let* $\alpha \in (0,1]$, $n \in \mathbb{N}$. *Suppose* f *is* n *times conformable* α-*fractional differentiable on* $[a,b] \subseteq [0,\infty)$ *and takes values on* $\mathbb{R}_+$. $D_\alpha^n f$ *is continuous on* $[a,b]$. *For a fixed* $x \in [a,b]$ *we have* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Assume that* $(C)\int_a^b |t-x|^{n(\alpha+1)}\, d\mu_{N,x}(t)$, $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!}\cdot$$
$$\left[\left((C)\int_a^b |t-x|^{n(\alpha+1)}\, d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \right.$$
$$\left.\frac{1}{(n+1)}\left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right], \tag{62}$$

$\forall\ N \in \mathbb{N}$.

As $N \to \infty$, *and* $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t) \to 0$, *we get* $T_N(f)(x) \to f(x)$.

Proof. By Theorem 11. ■

We continue with

Theorem 43 *Assume* $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$; $x \in (a,b)$. *Here* $\alpha \in (0,1]$, *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α-*fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Assume here that* $0 < \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}}\right)}{(n+1)!\alpha^{n+1}}, \tag{63}$$

$\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)}\, d\mu_{N,x}(t) \to 0$, *then* $T_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorem 16. ■

Theorem 44 *Assume* $(C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$. *Here* $\alpha \in (0,1]$, $n \in \mathbb{N}$ *and* $x \in (a,b)$; $[a,b] \subseteq [0,\infty)$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b]$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Let* $0 < (C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t) \leq \min(x-a, b-x)$, $\forall\ N \geq N^*$; $N, N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, (C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t)\right)}{(n+1)!\alpha^{n+1}}, \quad (64)$$

$\forall\ N \geq N^*$, *where* $N, N^* \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t) \to 0$, *then* $T_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorem 17. ∎

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The Minkowski Inequality and the Brunn-Minkowski Inequality for Dual Orlicz Mixed Affine Quermassintegrals

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Abstract

In this paper, the Orlicz version of the classical dual Cauchy-Kubota formula is given and the concept of dual affine quermassintegrals is extended to dual Orlicz mixed affine quermassintegrals in the framework of Orlicz Brunn-Minkowski theory. Some inequalities for dual Orlicz mixed affine quermassintegrals are obtained, such as dual Orlicz-Minkowski inequality and dual Orlicz-Brunn-Minkowski inequality.

Keywords: Orlicz Brunn-Minkowski theory, integral geometry, dual affine quermassintegral.

1 Introduction

We work in Euclidean space $\mathbb{R}^n$, and use $\mathrm{vol}_i(\cdot)$ to denote the i-dimensional volume. The unit sphere in $\mathbb{R}^n$ is written by S^{n-1}. In the projection of convex body K, quermassintegrals are important geometric invariants and have different definitions in many areas of mathematics. In the theory of mixed volumes quermassintegrals are usually called simple mixed volumes. The reader should refer to [24] and [26] for details. Lutwak [21] introduced the dual quermassintegrals, $\widetilde{W}_{n-i}$, of a star body K. Suppose $\widetilde{W}_0 = \mathrm{vol}_n(K)$ and $\widetilde{W}_n = \omega_n$. If $0 < i < n$, then

$$\widetilde{W}_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{G(n,i)} \mathrm{vol}_i(K \cap \xi) \mathrm{d}\mu_i(\xi), \tag{1.1}$$

where the Grassmann manifold $G(n,i)$ is endowed with the probability Haar measure μ_i, $\mathrm{vol}_i(K \cap \xi)$ is the i-dimensional volume of slice of K by an i-dimensional subspace $\xi \subset \mathbb{R}^n$ and $\omega_i = \pi^{i/2}/\Gamma(1 + i/2)$ denotes the i-dimensional volume of the unit ball in $\mathbb{R}^i$.

The quermassintegrals are connected with the projections of convex bodies, while the dual quermassintegrals are closely related to the cross sections of star bodies, which is proved in [11] that they are the only rotation invariant continuous star valuations with the corresponding homogeneity. Zhang [28] showed that the dual quermassintegrals have the same kind of kinematic formulas as the quermassintegrals.

Affine quermassintegrals [16] is an important geometric invariants in the projection of convex body. Lutwak [15] introduced the dual affine quermassintegrals, $\widetilde{\Phi}_{n-i}(K)$, of a star body K containing the

AMS Subject Classification: 52A39, 52A40, 52A22.

This work was Supported by the National Natural Science Foundation of China(Grant No. 11561020) and was partly supported by the National Natural Science Foundation of China (Grant No. 11371224).

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origin in its interior. Suppose $\widetilde{\Phi}_0(K)=\mathrm{vol}_n(K)$ and $\widetilde{\Phi}_n(K)=\omega_n$. If $0<i<n$, then

$$\widetilde{\Phi}_{n-i}(K)=\frac{\omega_n}{\omega_i}\left(\int_{G(n,i)}\mathrm{vol}_i(K\cap\xi)^n\mathrm{d}\mu_i(\xi)\right)^{\frac{1}{n}}. \tag{1.2}$$

Grinberg [6] showed that both the affine quermassintegrals and the dual affine quermassintegrals are invariant under volume-preserving affine transformations. However, the dual affine quermassintegrals of star bodies received more considerable attention, see [6, 2, 16, 26, 27]. The aim of this paper is to study them further.

Some opened articles [9, 13, 17, 18, 23, 25], Gardner' work [3] and the classical Brunn-Minkowski theory of convex bodies (see, e.g., [4, 26]) were generalized to the Orlicz space, which is called the Orlicz Brunn-Minkowski theory and further extend the L_p-Brunn-Minkowski theory (see, e.g., [19, 20, 12]). We considers a non-zero convex function $\phi:[0,\infty)\to[0,\infty)$ in this paper. It is strictly increasing with $\phi(0)=0$. Suppose that $\mathcal{C}$ is the class of convex and strictly increasing functions $\phi:[0,\infty)\to[0,\infty)$, where $\lim\limits_{t\to\infty}\phi(t)=+\infty$, and $\phi(0)=0$. Note that $\mathcal{S}_o^n$ denotes the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors.

The dual Orlicz mixed volume, $\widetilde{V}_{-\phi}(K,L)$, of $K,L\in\mathcal{S}_o^n$ is defined by

$$\widetilde{V}_{-\phi}(K,L)=\frac{-\phi_r'(1)}{n}\lim_{\varepsilon\to0^+}\frac{\mathrm{vol}_n(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-\mathrm{vol}_n(K)}{\varepsilon}, \tag{1.3}$$

where $\phi_r'(1)$ is the right derivative of a real-valued function ϕ at 1 and $K\widetilde{+}_{-\phi}\varepsilon\diamond L$ denotes the Orlicz radial harmonic combination of K and L. It follows from (1.3) that the dual Orlicz mixed volume $\widetilde{V}_{-\phi}$ has the following integral representation:

$$\widetilde{V}_{-\phi}(K,L)=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho_K}{\rho_L}\right)\rho_K(u)^n\mathrm{d}S(u), \tag{1.4}$$

In [5, 10, 22, 31, 20], the dual mixed volume is extended to the dual L_p-mixed volume. If $\phi(t)=t^p, 1\le p<\infty$, then

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int_{S^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^p\rho_K^n(u)\mathrm{d}S(u). \tag{1.5}$$

Recently, Zhao [30] introduced the notion of dual Orlicz mixed quermassintegrals for $0\le i\le n$ and established its integral representation. If $K,L\in\mathcal{S}_o^n$ and $\phi\in\mathcal{C}$, then

$$\widetilde{W}_{-\phi,i}(K,L)=\frac{-\phi_r'(1)}{n-i}\lim_{\varepsilon\to0^+}\frac{\widetilde{W}_i(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-\widetilde{W}_i(K)}{\varepsilon},\ \text{and} \tag{1.6}$$

$$\widetilde{W}_{-\phi,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho_K}{\rho_L}\right)\rho_K(u)^{n-i}\mathrm{d}S(u),\quad i=0,1,\cdots,n. \tag{1.7}$$

In this paper, we first established the Orlicz version of the classical dual Cauchy-Kubota formula (1.1)

$$\widetilde{W}_{-\phi,n-i}(K,L)=\frac{\omega_n}{\omega_i}\int_{G(n,i)}\widetilde{V}_{-\phi}^{(i)}(K\cap\xi,L\cap\xi)\mathrm{d}\mu_i(\xi), \tag{1.8}$$

where $\widetilde{V}_{-\phi}^{(i)}(K\cap\xi,L\cap\xi)$ is the dual Orlicz mixed volume of the (i)-dimensional star bodies $K\cap\xi$ and $L\cap\xi$ in the subspace $\xi\in G(n,i)$.

For $i=1,2,\cdots,n$, we further consider the following formula.

$$\begin{aligned}\widetilde{\Phi}_{\phi,n-i}(K,L) &= \frac{\omega_n}{\omega_i}\left[\mathbb{E}(\widetilde{V}_{-\phi}^{(i)}(K\cap\xi,L\cap\xi)^n)\right]^{1/n}\\ &= \frac{\omega_n}{\omega_i}\left[\int_{G(n,i)}\widetilde{V}_{-\phi}^{(i)}(K\cap\xi,L\cap\xi)^n\mathrm{d}\mu_i(\xi)\right]^{\frac{1}{n}},\end{aligned} \tag{1.9}$$

and $\widetilde{\Phi}_{\phi,n-i}(K,L)$ is known as the dual Orlicz mixed affine quermassintegrals.

Let $\phi(t)=t^p$ with $p\ge 1$. Then

$$\widetilde{\Phi}_{p,n-i}(K,L)=\frac{\omega_n}{\omega_i}\Bigg[\int\limits_{G(n,i)}\widetilde{V}^{(i)}_{-p}(K\cap\xi,L\cap\xi)^n\mathrm{d}\mu_i(\xi)\Bigg]^{\frac{1}{n}},\tag{1.10}$$

where $\widetilde{V}^{(i)}_{-p}(K\cap\xi,L\cap\xi)$ denotes the dual L_p-mixed volume of $K\cap\xi$ and $L\cap\xi$ in the subspace $\xi\in G(n,i)$.

Taking $L=K$ in (1.9), $\widetilde{\Phi}_{\phi,n-i}(K,K)/\phi(1)=\widetilde{\Phi}_{n-i}(K)$ is just the classical dual affine quermassintegrals of K.

On the basis of the above concepts, one aim of this paper is to establish the following dual Orlicz-Minkowski inequality for dual Orlicz mixed affine quermassintegrals.

Theorem 1.1. *Suppose $K,L\in\mathcal{S}^n_o, n\ge 3$ and $\phi\in\mathcal{C}$. Then for $2\le i\le n$,*

$$\widetilde{\Phi}_{\phi,n-i}(K,L)\ge\widetilde{\Phi}_{n-i}(K)\phi\Bigg(\bigg(\frac{\widetilde{\Phi}_{n-i}(K)}{\widetilde{\Phi}_{n-i}(L)}\bigg)^{\frac{1}{i}}\Bigg).\tag{1.11}$$

If K and L are convex bodies containing the origin in their interiors, then equality holds in the inequality (1.11) if and only if K and L are dilations.

As an application of Theorem 1.1, we prove a uniqueness theorem of convex bodies.

The other aim of this paper is to prove Orlicz radial sum versions of the dual Brunn-Minkowski inequality for dual Orlicz mixed affine quermassintegrals.

Theorem 1.2. *Suppose $K,L\in\mathcal{S}^n_o$ and $\phi\in\mathcal{C}$. Then for $2\le i\le n$,*

$$\phi\Bigg(\bigg(\frac{\widetilde{\Phi}_{n-i}(K\widetilde{+}_{-\phi}L)}{\widetilde{\Phi}_{n-i}(K)}\bigg)^{\frac{1}{i}}\Bigg)+\phi\Bigg(\bigg(\frac{\widetilde{\Phi}_{n-i}(K\widetilde{+}_{-\phi}L)}{\widetilde{\Phi}_{n-i}(L)}\bigg)^{\frac{1}{i}}\Bigg)\le\phi(1).\tag{1.12}$$

If K and L are convex bodies containing the origin in their interiors, then equality holds in the inequality (1.12) if and only if K and L are dilations.

In order to prove Theorems 1.1 and 1.2, we use the integral-geometric technique, motivated by Furstenberg and Tzkoni [1], Grinberg [7], Ma [22], Gardner and Hug, et al. [5] and Zhu et al. [31].

2 Preliminaries

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. We write $\mathcal{K}^n_o$ for the set of convex bodies containing the origin in their interiors. The support function of $K\in\mathcal{K}^n_o$, $h_K=h(K,\cdot):\mathbb{R}^n\backslash\{o\}\to[0,\infty)$, is defined by $h(K,x)=\max\{\langle x,y\rangle:y\in K\}$, where $x\in\mathbb{R}^n\backslash\{o\}$.

For $K\in\mathcal{K}^n_o$, its polar body, $K^*\in\mathcal{K}^n_o$, is defined by $K^*=\{x\in\mathbb{R}^n:\langle x,y\rangle\le 1,\text{for any } y\in K\}$. It is easily known that $(K^*)^*=K$ for $K\in\mathcal{K}^n_o$, and for $c>0$ we have $(cK)^*=c^{-1}K^*$.

If K is a compact set in $\mathbb{R}^n$, then the radial function ρ_K of K is defined by $\rho_K(x)=\max\{\lambda\ge 0:\lambda x\in K\}$ for $x\in\mathbb{R}^n\backslash\{o\}$. If ρ_K is continuous then we call K a star body (about the origin).

Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u\in S^{n-1}$. It is easy to see that for $K,L\in\mathcal{S}^n_o$, $K\subseteq L$ if and only if $\rho_K\le\rho_L$ and for $c>0$ and $x\in\mathbb{R}^n\backslash\{o\}$, $\rho(cK,x)=c\rho(K,x)$. More generally, for $T\in\mathrm{GL}(n)$ the radial function of the image $TK=\{Ty:y\in K\}$ of K is given by (see [26])

$$\rho(TK,x)=\rho(K,T^{-1}x),\quad\text{for } x\in\mathbb{R}^n\backslash\{o\},\tag{2.1}$$

where $\mathrm{GL}(n)$ denotes the linear transformation group on $\mathbb{R}^n$, and T^{-1} is the inverse of T.

For $K, L \in \mathcal{S}_o^n$, $\alpha, \beta \geq 0$ (not both zero) and $\phi \in \mathcal{C}$, the Orlicz radial combination $\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L$ of K and L is defined by (see [22])

$$\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)^{-1} = \inf \left\{ \lambda > 0 : \alpha\phi\Big(\frac{1}{\lambda \rho_K(u)}\Big) + \beta\phi\Big(\frac{1}{\lambda \rho_L(u)}\Big) \leq \phi(1) \right\}, \quad u \in S^{n-1}. \tag{2.2}$$

Note that for all $u \in S^{n-1}$, $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)$ is defined by

$$\alpha\phi\Big(\frac{\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)}{\rho_K(u)}\Big) + \beta\phi\Big(\frac{\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)}{\rho_L(u)}\Big) = \phi(1).$$

If $\phi(t) = t^p$ with $1 \leq p < \infty$, then $\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L$ is the L_p-radial harmonic combination $\alpha \diamond K \widetilde{+}_{-p} \beta \diamond L$, and correspondingly $\widetilde{V}_{-\phi}(K, L)$ is the dual L_p-mixed volume $\widetilde{V}_{-p}(K, L)$. See [20] for more details.

Lemma 2.1. *Let $K, L \in \mathcal{S}_o^n$ and $\alpha, \beta \geq 0$. If $\phi \in \mathcal{C}$, then for $T \in \mathrm{GL}(n)$,*

$$T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L) = \alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL.$$

Proof. From (2.2) and (2.1), we have for $u \in S^{n-1}$,

$$\begin{aligned}
\rho(\alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL, u)^{-1} &= \inf \left\{ \lambda > 0 : \alpha\phi\Big(\frac{1}{\lambda \rho_{TK}(u)}\Big) + \beta\phi\Big(\frac{1}{\lambda \rho_{TL}(u)}\Big) \leq \phi(1) \right\} \\
&= \inf \left\{ \lambda > 0 : \alpha\phi\Big(\frac{1}{\lambda \rho_K(T^{-1}u)}\Big) + \beta\phi\Big(\frac{1}{\lambda \rho_L(T^{-1}u)}\Big) \leq \phi(1) \right\} \\
&= \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, T^{-1}u)^{-1} \\
&= \rho(T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L), u)^{-1}.
\end{aligned}$$

Thus

$$T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L) = \alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL.$$

□

Lemma 2.2. *Let $K, L \in \mathcal{S}_o^n, \phi \in \mathcal{C}$. Then for each $\xi \in G(n, i), i = 1, \cdots, n-1$ and $\varepsilon > 0$,*

$$(K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi = (K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi).$$

Proof. Fixed $\xi \in G(n, i)$, and let $S^{i-1} = S^{n-1} \cap \xi$. For any $u \in S^{i-1}$ and $Q \in \mathcal{S}_o^n$, we get $\rho_Q(u) = \rho_{Q \cap \xi}(u)$. Applying the definition of $K \widetilde{+}_{-\phi} \varepsilon \diamond L$ to $u \in S^{i-1}$, it follows that

$$\phi\Big(\frac{\rho((K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi, u)}{\rho_{K \cap \xi}(u)}\Big) + \varepsilon\phi\Big(\frac{\rho((K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi, u)}{\rho_{L \cap \xi}(u)}\Big) = \phi(1).$$

On the other hand, from $(K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi)$ defined in ξ, we have

$$\phi\Big(\frac{\rho((K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi), u)}{\rho_{K \cap \xi}(u)}\Big) + \varepsilon\phi\Big(\frac{\rho((K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi), u)}{\rho_{L \cap \xi}(u)}\Big) = \phi(1).$$

Thus, $(K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi$ and $(K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi)$ is a same star body in ξ. □

Lemma 2.3. (see [22]) *Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \mathcal{C}$. Then*

$$\widetilde{V}_{-\phi}(K, L) \geq \mathrm{vol}_n(K)\phi\bigg(\Big(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\Big)^{\frac{1}{n}}\bigg), \tag{2.3}$$

with equality if and only if K and L are dilates of each other.

Taking $\phi(t) = t^p$ with $p \geq 1$. The above dual Orlicz-Minkowski inequality is Lutwak's L_p-dual Minkowski inequality (see [20]):

$$\widetilde{V}_{-p}(K,L) \geq \mathrm{vol}_n(K)^{\frac{n+p}{n}} \mathrm{vol}_n(L)^{-\frac{p}{n}}, \tag{2.4}$$

with equality holds if and only if K and L are dilations.

Lemma 2.4. (see [8]) *Suppose that μ is a probability measure on a space X and $f : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi\big(f(x)\big)\mathrm{d}\mu(x) \geq \phi\bigg(\int_X f(x)\mathrm{d}\mu(x)\bigg). \tag{2.5}$$

If ϕ is strictly convex, the equality holds in every inequality if and only if $f(x)$ is constant for μ-almost all $x \in X$.

3 The generalized dual Cauchy-Kubota formula

In this section, we prove the probabilistic essence of dual Orlicz mixed quermassintegrals. We first see the dual Cauchy-Kubota formula. For $K \in \mathcal{S}_o^n$,

$$\widetilde{W}_{n-i}(K) = \frac{\omega_n}{\omega_i}\int_{G(n,i)} \mathrm{vol}_i(K \cap \xi)\mathrm{d}\mu_i(\xi), \quad i = 1, \cdots, n-1. \tag{3.1}$$

Theorem 3.1. *Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \mathcal{C}$. Then for each $i = 1, \cdots, n-1$,*

$$\widetilde{W}_{-\phi,n-i}(K,L) = \frac{\omega_n}{\omega_i}\int_{G(n,i)} \widetilde{V}_{-\phi}^{(i)}(K\cap\xi, L\cap\xi)\mathrm{d}\mu_i(\xi).$$

Proof. By (1.6), (3.1) and Lemma 2.2, we have

$$\begin{aligned}
\widetilde{W}_{-\phi,n-i}(K,L) &= \frac{-\phi_r'(1)}{i}\lim_{\varepsilon\to0^+}\frac{\widetilde{W}_{n-i}(K\widetilde{+}_{-\phi}\varepsilon\diamond L) - \widetilde{W}_{n-i}(K)}{\varepsilon} \\
&= \frac{-\phi_r'(1)}{i}\cdot\frac{\omega_n}{\omega_i}\int_{G(n,i)}\lim_{\varepsilon\to0^+}\frac{\mathrm{vol}_i((K\widetilde{+}_{-\phi}\varepsilon\diamond L)\cap\xi) - \mathrm{vol}_i(K\cap\xi)}{\varepsilon}\mathrm{d}\mu_i(u) \\
&= \frac{-\phi_r'(1)}{i}\cdot\frac{\omega_n}{\omega_i}\int_{G(n,i)}\lim_{\varepsilon\to0^+}\frac{\mathrm{vol}_i((K\cap\xi\widetilde{+}_{-\phi}\varepsilon\diamond L\cap\xi)) - \mathrm{vol}_i(K\cap\xi)}{\varepsilon}\mathrm{d}\mu_i(u).
\end{aligned}$$

From (1.3), we have

$$\widetilde{W}_{-\phi,n-i}(K,L) = \frac{\omega_n}{\omega_i}\int_{G(n,i)} \widetilde{V}_{-\phi}^{(i)}(K\cap\xi, L\cap\xi)\mathrm{d}\mu_i(\xi).$$

□

Up to a constant, the quantity $\widetilde{W}_{-\phi,i}(K,L)$ is the expectation of the random variable

$$\widetilde{V}_{-\phi}^{(i)}(K\cap\cdot, L\cap\cdot) : G(n,i) \to (0,\infty), \quad \xi \mapsto \widetilde{V}_{-\phi}^{(i)}(K\cap\xi, L\cap\xi),$$

which is defined on the probability space $(G(n,i), \mathcal{B}, \mu_i)$ (where $\mathcal{B}$ is the Borel sigma-algebra on $G(n,i)$).

Taking $\phi(t) = t^p$ with $p > 0$ in Theorem 3.1, we have the formula

$$\widetilde{W}_{-p,n-i}(K,L) = \frac{\omega_n}{\omega_i}\int_{G(n,i)} \widetilde{V}_{-p}^{(i)}(K\cap\xi, L\cap\xi)\mathrm{d}\mu_i(\xi).$$

For $K \in \mathcal{S}_o^n$, we extend the dual Cauchy-Kubota formula to $1 \le q \le i < n$,

$$\widetilde{W}_i(K) = \frac{\omega_n}{\omega_{n-q}} \int_{G(n,n-q)} \widetilde{W}_{i-q}^{(n-q)}(K \cap \xi) \mathrm{d}\mu_{n-q}(\xi), \tag{3.2}$$

where $\widetilde{W}_{i-q}^{(n-q)}$ denotes the (i-q)th dual harmonic quermassintegral in the subspace ξ.
It follows from (3.2) and (1.6) that we have the following theorem.

Theorem 3.2. *Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$. Then for $1 \le q \le i < n$,*

$$\widetilde{W}_{-\phi,i}(K,L) = \frac{\omega_n}{\omega_{n-q}} \int_{G(n,n-q)} \widetilde{W}_{-\phi,i-q}^{(n-q)}(K \cap \xi, L \cap \xi) \mathrm{d}\mu_{n-q}(\xi),$$

where $\widetilde{W}_{-\phi,i-q}^{(n-q)}(K \cap \xi, L \cap \xi)$ denotes the dual Orlicz harmonic mixed quermassintegral of the (n-q)-dimensional star bodies $K \cap \xi$ and $L \cap \xi$ in the subspace ξ.

4 Inequalities of dual Orlicz mixed affine quermassintegrals

In this section, we first show that the quantities $\widetilde{\Phi}_{\phi,1}(K,L), \cdots, \widetilde{\Phi}_{\phi,n}(K,L)$ are $\mathrm{SL}(n)$-invariant. Here, $\mathbb{E}(\widetilde{V}_{-\phi}^{(i)}(K \cap \cdot, L \cap \cdot)^n)$ is the expectation of $\widetilde{V}_{-\phi}^{(i)}(K \cap \cdot, L \cap \cdot)^n$.

Theorem 4.1. *Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \mathcal{C}$. Then for $T \in \mathrm{SL}(n)$, there holds*

$$\widetilde{\Phi}_{\phi,i}(TK,TL) = \widetilde{\Phi}_{\phi,i}(K,L), \;\; i = 1, 2, \cdots, n.$$

Proof. Suppose $\xi \in G(n, n-i)$. For $S^{n-i-1} = S^{n-1} \cap \xi$, if

$$T \in \mathrm{SL}(n) = \{T \in \mathrm{GL}(n) : \det T = 1\},$$

then for $u \in S^{n-i-1}$ and $Q \in \mathcal{S}_o^n$, we get $\rho_{TQ}(u) = \rho_{TQ\cap\xi}(u)$. For $x \in \mathbb{R}^n \backslash \{o\}$, let $\langle x \rangle = x/||x||$. From (1.4) and (2.1), we obtain

$$\begin{aligned}
\widetilde{V}_{-\phi}^{(n-i)}(TK \cap \xi, TL \cap \xi) &= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_{TK\cap\xi}(u)}{\rho_{TL\cap\xi}(u)}\right) \rho_{TK\cap\xi}^{n-i}(u) \mathrm{d}S_{n-i-1}(u) \\
&= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_{TK}(u)}{\rho_{TL}(u)}\right) \rho_{TK}^{n-i}(u) \mathrm{d}S_{n-i-1}(u) \\
&= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_K(\langle T^{-1}u \rangle)}{\rho_L(\langle T^{-1}u \rangle)}\right) \rho_K^{n-i}(\langle T^{-1}u \rangle) \mathrm{d}S_{n-i-1}(\langle T^{-1}u \rangle) \\
&= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_{K\cap\xi}(v)}{\rho_{L\cap\xi}(v)}\right) \rho_{K\cap\xi}^{n-i}(v) \mathrm{d}S_{n-i-1}(v) \\
&= \widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi),
\end{aligned}$$

where S_{n-i-1} denotes $n-i-1$-dimensional spherical Lebesgue measure. Thus, from (1.9), it follows that

$$\begin{aligned}
\widetilde{\Phi}_{\phi,i}(TK,TL) &= \frac{\omega_n}{\omega_{n-i}} \left(\int_{G(n,n-i)} \left[\widetilde{V}_{-\phi}^{(n-i)}(TK \cap \xi, TL \cap \xi) \right]^n \mathrm{d}\mu_{n-i}(\xi) \right)^{\frac{1}{n}} \\
&= \frac{\omega_n}{\omega_{n-i}} \left(\int_{G(n,n-i)} \left[\widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi) \right]^n \mathrm{d}\mu_{n-i}(\xi) \right)^{\frac{1}{n}} \\
&= \widetilde{\Phi}_{\phi,i}(K,L).
\end{aligned}$$

□

To prove Theorem 1.1 and Theorem 1.2, the next three lemmas are needed.

Lemma 4.2. (see [14]) *Suppose* $K \in \mathcal{K}_o^n$ *and* $\xi \in G(n,i)$, *then* $K^* \cap \xi = (K|\xi)^*$.

Lemma 4.3. (see [12]) *Suppose* $K_1, K_2 \in \mathcal{K}_o^n$ *and* $2 \leq k \leq n-1$. *If* $K_1|\xi$ *and* $K_1|\xi$ *are dilations for each* $\xi \in G(n,k)$, *then* K_1 *and* K_2 *are dilations.*

Lemma 4.4. *Suppose* $K_1, K_2 \in \mathcal{K}_o^n$ *and* $2 \leq k \leq n-1$. *If* $K_1 \cap \xi$ *and* $K_1 \cap \xi$ *are dilations for each* $\xi \in G(n,k)$, *then* K_1 *and* K_2 *are dilations.*

Proof. If both $K_1 \cap \xi$ and $K_2 \cap \xi$ are dilations for each $\xi \in G(n,k)$, then $K_1 \cap \xi = a(K_2 \cap \xi)$ for $a > 0$. If follows from Lemma 4.2 that $(K_1^*|\xi)^* = a(K_2^*|\xi)^* = (a^{-1}K_2^*|\xi)^*$. Thus, $K_1^*|\xi = a^{-1}K_2^*|\xi$. From Lemma 4.3, we know $K_1^* = \frac{b}{a}K_2^*$. Therefore, $K_1 = cK_2$ for some $c > 0$. □

The normalized dual affine quermassintegrals measure of K are defined by

$$d\widetilde{\Phi}_i^*(K,\cdot) = \left(\frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)}\right)^n [\mathrm{vol}_i(K \cap \cdot)]^n \, d\mu_i, \tag{4.1}$$

where $d\mu_i$ is the normalized Haar measure on $G(n,i)$. Obviously, $\widetilde{\Phi}_i^*(K,\cdot)$ is a probability measure on $G(n,i)$.

Proof of Theorem 1.1. Note that $\widetilde{\Phi}_{\phi,0} = \widetilde{V}_{-\phi}(K,L), \widetilde{\Phi}_0(K) = \mathrm{vol}_n(K)$, and $\widetilde{\Phi}_0(L) = \mathrm{vol}_n(L)$. It follows directly from Lemma 2.3 that the case when $i = n$.

Now, we consider the case when $2 \leq i \leq n-1$. By (1.9), (2.3), (4.1), (2.5) and Hölder's inequality, it follows that

$$\begin{aligned}
&\frac{\widetilde{\Phi}_{\phi,n-i}(K,L)}{\widetilde{\Phi}_{n-i}(K)} \\
=\ & \frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)} \left[\int_{G(n,i)} \left(\widetilde{V}_{-\phi}^{(i)}(K\cap\xi, L\cap\xi)\right)^n d\mu_i(\xi)\right]^{\frac{1}{n}} \\
\geq\ & \frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)} \left[\int_{G(n,i)} (\mathrm{vol}_i(K\cap\xi))^n \phi^n\left(\left(\frac{\mathrm{vol}_i(K\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right) d\mu_i(\xi)\right]^{\frac{1}{n}} \\
=\ & \left[\int_{G(n,i)} \phi^n\left(\left(\frac{\mathrm{vol}_i(L\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right) d\widetilde{\Phi}_i^*(K,\xi)\right]^{\frac{1}{n}} \\
\geq\ & \phi\left[\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}} d\widetilde{\Phi}_i^*(K,\xi)\right] \\
=\ & \phi\left[\left(\frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)}\right)^n \int_{G(n,i)} (\mathrm{vol}_i(K\cap\xi))^{n\left(\frac{ni+1}{ni}\right)} (\mathrm{vol}_i(L\cap\xi))^{n\left(-\frac{1}{ni}\right)} d\mu_i(\xi)\right] \\
\geq\ & \phi\left(\frac{\widetilde{\Phi}_{n-i}(K)^{\frac{ni+1}{i}} \widetilde{\Phi}_{n-i}(L)^{-\frac{1}{i}}}{\widetilde{\Phi}_{n-i}(K)^n}\right) \\
=\ & \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right).
\end{aligned}$$

If K and L are dilations, then the equality holds in (1.11) is obvious. Conversely, let $K, L \in \mathcal{K}_o^n$. Together the equality conditions of the dual Brunn-Minkowski inequality (2.3), Jensen's inequality (2.5) with Hölder's inequality, we know equality holds in inequality (1.11) if and only if $K \cap \xi$ and $L \cap \xi$ are dilations for each $\xi \in G(n, n-i)$. Therefore, Lemma 4.4 can reduce that K and L are dilations. □

Let $\phi(t) = t^p$ with $p \geq 1$. An immediate consequence of Theorem 1.1 is:

Corolloary 4.5. *Suppose $K, L \in \mathcal{S}_o^n$. Then for $p \geq 1$ and $2 \leq i \leq n$,*

$$\widetilde{\Phi}_{p,n-i}(K, L) \geq \widetilde{\Phi}_{n-i}(K)^{1+\frac{p}{i}} \widetilde{\Phi}_{n-i}(L)^{-\frac{p}{i}}. \tag{4.2}$$

If $K, L \in \mathcal{K}_o^n$, then equality holds in the inequality (4.2) if and only if K and L are dilations.

A direct consequence of the dual Orlicz-Minkowski inequality is the following uniqueness.

Corolloary 4.6. *Suppose $\phi \in \mathcal{C}$ with $\phi(1) = 1$, and $\mathfrak{U} \subset \mathcal{K}_o^n$ $(n \geq 3)$ such that $K, L \in \mathfrak{U}$. If for $2 \leq i \leq n$, there holds*

$$\widetilde{\Phi}_{\phi,n-i}(M, K) = \widetilde{\Phi}_{\phi,n-i}(M, L), \quad \text{for all } M \in \mathfrak{U}, \tag{4.3}$$

or

$$\frac{\widetilde{\Phi}_{\phi,n-i}(K, M)}{\widetilde{\Phi}_{n-i}(K)} = \frac{\widetilde{\Phi}_{\phi,n-i}(L, M)}{\widetilde{\Phi}_{n-i}(L)}, \quad \text{for all } M \in \mathfrak{U}, \tag{4.4}$$

then $K = L$.

Proof. Suppose (4.3) holds. If we take K for M, then by (1.9), (1.2), and $\phi(1) = 1$, we have

$$\widetilde{\Phi}_{n-i}(K) = \phi(1)\widetilde{\Phi}_{n-i}(K) = \widetilde{\Phi}_{\phi,n-i}(K, K) = \widetilde{\Phi}_{\phi,n-i}(K, L).$$

Thus

$$1 = \phi(1) \geq \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0, \infty)$, we have $\widetilde{\Phi}_{n-i}(K) \leq \widetilde{\Phi}_{n-i}(L)$, with equality if and only if K and L are dilates of each other.

If let L for M we similarly get $\widetilde{\Phi}_{n-i}(K) \geq \widetilde{\Phi}_{n-i}(L)$. Therefore, $\widetilde{\Phi}_{n-i}(K) = \widetilde{\Phi}_{n-i}(L)$, this obtains $\mathrm{vol}_i(K \cap \xi) = \mathrm{vol}_i(L \cap \xi)$, and from the equality conditions of the dual Orlicz-Minkowski inequality we obtain that K and L are dilates of each other. Since $K \cap \xi$ and $L \cap \xi$ have the same volume, this implies $K = L$.

Further, suppose that (4.4) holds. Similarly, we get

$$1 = \phi(1) = \frac{\widetilde{\Phi}_{\phi,n-i}(K, K)}{\widetilde{\Phi}_{n-i}(K)} = \frac{\widetilde{\Phi}_{\phi,n-i}(L, K)}{\widetilde{\Phi}_{n-i}(L)}.$$

Therefore,

$$1 = \phi(1) \leq \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(L)}{\widetilde{\Phi}_{n-i}(K)}\right)^{\frac{1}{i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0, \infty)$, we have $\widetilde{\Phi}_{n-i}(L) \geq \widetilde{\Phi}_{n-i}(K)$, with equality if and only if K and L are dilates of each other.

Taking L for M, obviously $\widetilde{\Phi}_{n-i}(L) \leq \widetilde{\Phi}_{n-i}(K)$. Therefore, $\widetilde{\Phi}_{n-i}(L) = \widetilde{\Phi}_{n-i}(K)$ can obtain that K and L are dilates of each other. Since $K \cap \xi$ and $L \cap \xi$ have the same volume, this gets $K = L$. □

Proof of Theorem 1.2. For the convenience, define $K_\phi = K \tilde{+}_{-\phi} L$. From Lemma 2.2, we have for $\xi \in G(n, n-i)$, $K_\phi \cap \xi = (K \tilde{+}_{-\phi} L) \cap \xi = (K \cap \xi) \tilde{+}_{-\phi} (L \cap \xi)$. Note that $K_\phi \cap \xi \in \mathcal{S}_o^n$ implies that for $u \in S^{n-i-1}$,

$$\phi\left(\frac{\rho_{K_\phi \cap \xi}(u)}{\rho_{K \cap \xi}(u)}\right) + \phi\left(\frac{\rho_{K_\phi \cap \xi}(u)}{\rho_{L \cap \xi}(u)}\right) = \phi(1). \tag{4.5}$$

Suppose $\lambda_\phi = \omega_n / \big[\omega_i \widetilde{\Phi}_{n-i}(K_\phi)\big]$. By (1.2), (4.5), (1.4), (2.3), (4.1) and (2.5) we obtain

$$\begin{aligned}
&\phi(1)\\
= \;& \lambda_\phi \left[\int_{G(n,i)} (\phi(1)\mathrm{vol}_i(K_\phi\cap\xi))^n \,\mathrm{d}\mu_i(\xi)\right]^{\frac{1}{n}}\\
= \;& \lambda_\phi \left[\int_{G(n,i)} \left(\widetilde{V}^{(i)}_{-\phi}(K_\phi\cap\xi, K\cap\xi) + \widetilde{V}^{(i)}_{-\phi}(K_\phi\cap\xi, L\cap\xi)\right)^n \mathrm{d}\mu_i(\xi)\right]^{\frac{1}{n}}\\
\geq \;& \lambda_\phi \left\{\int_{G(n,i)} \mathrm{vol}_i(K_\phi\cap\xi)^n \left[\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right) + \phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right)\right]^n \mathrm{d}\mu_i(\xi)\right\}^{\frac{1}{n}}\\
= \;& \left\{\int_{G(n,i)} \left[\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right) + \phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right)\right]^n \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right\}^{\frac{1}{n}} \qquad (4.6)\\
\geq \;& \int_{G(n,i)} \left[\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right) + \phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right)\right] \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\\
= \;& \int_{G(n,i)} \phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right) \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi) + \int_{G(n,i)} \phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right) \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\\
\geq \;& \phi\left(\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}} \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right) + \phi\left(\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}} \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right).
\end{aligned}$$

From Hölder inequality and (1.2), we get

$$\begin{aligned}
&\phi\left(\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}} \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right)\\
= \;& \phi\left[\left(\frac{\omega_n}{\omega_i\widetilde{\Phi}_{n-i}(K_\phi)}\right)^n \int_{G(n,i)} (\mathrm{vol}_i(K_\phi\cap\xi))^{n\left(\frac{ni+1}{ni}\right)} (\mathrm{vol}_i(K\cap\xi))^{n\left(-\frac{1}{ni}\right)} \mathrm{d}\mu_i(\xi)\right]\\
\geq \;& \phi\Bigg[\left(\frac{\omega_n}{\omega_i\widetilde{\Phi}_{n-i}(K_\phi)}\right)^n \left(\int_{G(n,i)} (\mathrm{vol}_i(K_\phi\cap\xi))^n \mathrm{d}\mu_i(\xi)\right)^{\frac{ni+1}{ni}} \qquad (4.7)\\
&\times \left(\int_{G(n,i)} (\mathrm{vol}_i(K\cap\xi))^n \mathrm{d}\mu_i(\xi)\right)^{-\frac{1}{ni}}\Bigg]\\
= \;& \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(K)}\right)^{\frac{1}{i}}\right).
\end{aligned}$$

Similarly,

$$\phi\left(\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}} \mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right) \geq \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right). \tag{4.8}$$

Together (4.6), (4.7) with (4.8), this yields

$$\phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(K)}\right)^{\frac{1}{i}}\right) + \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right) \leq \phi(1).$$

Finally, we give the equality conditions. Suppose that K and L are dilations. with equality in (1.12) is obvious.

Conversely, Let $K, L \in \mathcal{K}_o^n$. From the equality conditions of the dual Orlicz-Minkowski inequality of star bodies, Jensen's inequality (2.5) and Hölder's inequality, we obtain that equality holds in inequality (1.12) if and only if $K\cap\xi$ and $L\cap\xi$ are dilations for each $\xi \in G(n, n-i)$. Therefore, Lemma 4.4 can get that K and L are dilations. □

If $\phi(t) = t^p$ with $p \geq 1$, then we get:

Corolloary 4.7. *Let $K, L \in \mathcal{S}_o^n$. If $p \geq 1$ and $2 \leq i \leq n$, then*

$$\widetilde{\Phi}_{n-i}(K\widetilde{+}_{-p}L)^{-\frac{p}{i}} \geq \widetilde{\Phi}_{n-i}(K)^{-\frac{p}{i}} + \widetilde{\Phi}_{n-i}(L)^{-\frac{p}{i}}. \tag{4.9}$$

If $K, L \in \mathcal{K}_o^n$, then equality holds in the inequality (4.9) if and only if K and L are dilations.

An immediate consequence of the inequality (4.9) is:

Corolloary 4.8. *Let $K, L \in \mathcal{S}_o^n$. If $p \geq 1$ and $2 \leq i \leq n$, then*

$$2\widetilde{\Phi}_{n-i}(K\widetilde{+}_{-p}L)^{\frac{p}{i}} \leq \left(\widetilde{\Phi}_{n-i}(K)\widetilde{\Phi}_{n-i}(L)\right)^{\frac{p}{2i}} \leq \frac{1}{2}\left(\widetilde{\Phi}_{n-i}(K)^{\frac{p}{i}} + \widetilde{\Phi}_{n-i}(L)^{\frac{p}{i}}\right). \tag{4.10}$$

If $K, L \in \mathcal{K}_o^n$, with equality in (4.10) if and only if $K = L$.

Proof. By (4.9) and the arithmetic-geometric-harmonic mean inequality, we have

$$\begin{aligned} 2\widetilde{\Phi}_{n-i}(K\widetilde{+}_{-p}L)^{\frac{p}{i}} &\leq \frac{2}{\frac{1}{\widetilde{\Phi}_{n-i}(K)^{\frac{p}{i}}} + \frac{1}{\widetilde{\Phi}_{n-i}(L)^{\frac{p}{i}}}} \\ &\leq \left(\widetilde{\Phi}_{n-i}(K)\widetilde{\Phi}_{n-i}(L)\right)^{\frac{p}{2i}} \\ &\leq \frac{1}{2}\left(\widetilde{\Phi}_{n-i}(K)^{\frac{p}{i}} + \widetilde{\Phi}_{n-i}(L)^{\frac{p}{i}}\right). \end{aligned}$$

We see easily that equality holds in the inequality (4.10) if and only if $K = L$. □

The next result is a relationship between $\widetilde{\Phi}_{\phi,i}(K, L)$ and $\widetilde{W}_{-\phi,i}(K, L)$.

Theorem 4.9. *Suppose $K, L \in \mathcal{S}_o^n$ and $i = 1, 2, \cdots, n-1$. Then*

$$\widetilde{\Phi}_{\phi,i}(K, L) \geq \widetilde{W}_{-\phi,i}(K, L), \tag{4.11}$$

with equality if and only if $\widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi)$ is constant for all $\xi \in G(n, n-i)$.

Proof. Notice that $\widetilde{V}_{-\phi}^{(n-i)}(K\cap\cdot, L\cap\cdot)$ is positive on $G(n, n-i)$ and that μ_{n-i} is a probability measure on $G(n, n-i)$. Hence, it follows from Jensen's inequality (2.5) that

$$\left(\int_{G(n,n-i)} \widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi)^n \mathrm{d}\mu_{n-i}(\xi)\right)^{\frac{1}{n}} \geq \int_{G(n,n-i)} \widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi)\mathrm{d}\mu_{n-i}(\xi),$$

with equality if and only if $\widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi)$ is constant for all $\xi \in G(n, n-i)$. This inequality and the definitions of $\widetilde{\Phi}_{\phi,i}(K, L)$ and $\widetilde{W}_{-\phi,i}(K, L)$ can easily yield the desired inequality. □

Conflict of Interests

The author declare that they have no competing interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved its final version.

Acknowledgment

This work is supported by the National Natural Science Foundations of China (Grant No.11561020) and is partly supported by the National Natural Science Foundations of China (Grant No.11371224).

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Existence and convergence for fixed points of a strict pseudo-contraction in CAT(0) spaces

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Abstract

The purposes of this paper are to introduce and study some existence and convergence theorems for fixed points of a strict pseudo-contraction in the framework of complete CAT(0) spaces. By using available properties in the spaces together with some appropriate conditions of the mapping and under certain assumptions, we can create some suitable sets to be used to construct an iterative projection algorithm to guarantee the existence fixed points for a strict pseudo-contraction. The method allows us to obtain a strong convergence iteration for finding some fixed points of a strict pseudo-contraction in the framework of complete CAT(0) spaces.

Keywords: Strict pseudo-contraction; Iterative projection technique; CAT(0) space

1. Introduction

Let (X, d) be a metric space, and $x, y \in X$ with $l = d(x, y)$. A geodesic path from x to y is an isometry $\gamma : [0, l] \to X$ such that $\gamma(0) = x$ and $\gamma(l) = y$. The image of a geodesic path is called a geodesic segment. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\triangle(x_1, x_2, x_3)$ is the triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean space $\mathbb{E}^2$ such that $d(x_i, x_j) = d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j)$ for all $i, j = 1, 2, 3$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x, y) \leq \ d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies that

$$d^2(x, y_0) \leq \ \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{1.1}$$

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This is the (CN) - inequality of Bruhat and Tits [5]. In fact ([3] p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) - inequality.

The study of CAT(0) spaces, Kirk [15, 16] first studied the fixed point theory in CAT(0) spaces. Since then, many authors have developed the fixed point theory for single-valued and set-valued mappings in the setting of CAT(0) spaces. Dhompongsa et al. [7] proved that a nonexpansive mapping from a nonempty bounded closed convex subset of a CAT(0) space to the family of nonempty compact subsets of the CAT(0) space has a fixed point under suitable conditions. In 2008, Berg and Nikolaev [2] introduced the concept of quasilinearization. In 2010, Kakavandi and Amini [13] introduced the concept of dual space for CAT(0) spaces. In 2012, Dehghan and Rooin [6] presented a characterization of metric projection in CAT(0) spaces. In 2014, Lu et al. [19] establish generalized CAT(0) versions of the Fan-Browder fixed point theorem. In the same year, Ungchittrakool [22] has discovered some significant inequalities for a strict pseudo-contraction in the framework of Hilbert spaces that has resulted in creating the important sets and the iterative shrinking projection technique to ensure the existence for fixed points of a strict pseudo-contraction in the terminology of Browder and Petryshyn [4].

Inspired and motivated by the significance of the problems mentioned above, we will pay attention to investigate and establish the existence theorem for fixed points of the mapping called strict pseudo-contraction mappings and some related mappings in complete CAT(0) spaces by employing suitable structure of certain sets based on the shrinking projection technique.

2. Preliminaries

Recall that a metric space (X, d) is said to be a geodesic space if every two points of X are joining by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y) \quad and \quad d(z, y) = (1-t)d(x, y).$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is $[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}$. A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$. In 1976, Lim in [18] introduced the concept of $\triangle$-convergence, and Kirk and Panyanak [17] has obtained some results in CAT(0) spaces which is every similar for weak convergence in Banach space setting. Next , we present the concept of $\triangle$-convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set
$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$.
The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by
$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$,
the asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by
$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$,
the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set
$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$
and the asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is the set
$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}$.

It is known from Proposition 7 of [8] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A subset of a CAT(0) space equipped with the induced metric, is a CAT(0) space if and only if it is convex ([3], p.167).

Definition 2.1 ([17], Definition 3.1)**.** A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim_{n\to\infty} x_n = x$ and x is called the $\triangle$-limit of $\{x_n\}$.

Lemma 2.2 ([17], Opial's property)**.** *Let X be a complete CAT(0) space and a sequence $\{x_n\}$ in X such that $\{x_n\}$ $\triangle$-converge to x and given $y \in X$ with $y \neq x$. Then we have* $\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$.

It is known from [17] that, the uniqueness of asymtotic center implies that CAT(0) space X satisfies Opial's property.

Let X be a complete CAT(0) space. Bijan Ahmadi Kakavandi [12] introduced the properties of $\triangle$-convergence, i.e., every closed convex subset of X is $\triangle$-closed in the sense that it contains all $\triangle$-limit point of every $\triangle$-convergent sequence.

Lemma 2.3 ([20], Lemma 3.5)**.** *Every bounded closed convex set in a complete CAT(0) space always has a $\triangle$-convergent subsequence.*

Lemma 2.4 ([9], Proposition 2.1)**.** *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Recall that a subset K of a metric space X is said to be $\triangle$- compact if every sequence in K has a $\triangle$- convergent subsequence.

Lemma 2.5 ([17], Proposition 3.6)**.** *Every bounded closed convex set in a complete CAT(0) space is $\triangle$- compact.*

Let C be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. We use the following notation

$\{x_n\} \rightharpoonup w \iff \Phi(w) = \inf_{x\in C} \Phi(x)$ where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

Also, we have $\{x_n\} \rightharpoonup w \iff A_C(\{x_n\}) = \{w\}$.

Lemma 2.6 ([20], Proposition 3.12)**.** *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let C be a closed convex subset of X which contain $\{x_n\}$. Then $\triangle - \lim_{n\to\infty} x_n = x$ implies that $\{x_n\} \rightharpoonup x$.*

Berg and Nikolaev [2] have introduced the concept of quasilinearization as follows. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is the map

$\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$

defined by

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \frac{1}{2}\{d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\} \text{ for all } a, b, c, d \in X. \tag{2.1}$$

It is easily seen that $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{cd}, \overrightarrow{ab} \right\rangle$, $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = -\left\langle \overrightarrow{ba}, \overrightarrow{cd} \right\rangle$ and $\left\langle \overrightarrow{ax}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{xb}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle$ for all $a, b, c, d, x \in X$.

We say that X satisfies the Cauchy - Schwarz inequality if $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \leq d(a, b)d(c, d)$ for all $a, b, c, d \in X$.

It is known ([2], Corollary 3) that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy- Schwarz inequality.

Definition 2.7 ([3], Proposition 2.4)**.** Let (X, d) be a metric space and $C \subseteq X$. The distance function $d(x, C) : X \to C$ is defined by $d(x, C) = \inf_{c\in C} d(x, c)$ for any $x \in X$.

Lemma 2.8 ([3], Proposition 2.4). *Let C be a closed convex subset of a complete CAT(0) space X and $x \in X$. Then there exists a unique point $p \in C$ such that $d(x, C) = d(x, p) = \inf_{y \in C} d(x, y)$.*

Definition 2.9 ([3], Proposition 2.4). Let C be a closed convex subset of a complete CAT(0) space X and $P_C : X \to C$ is defined by $P_C x = p$ such that p satisfies Lemma 2.8. P_C is said to be the metric projection from X onto C.

Dehghan and Rooin [6] presented monotone and a characterization of metric projection in CAT(0) spaces as follows:

A self-mapping T of C where C is a subset of CAT(0) space (X, d) is said to be monotone if $\left\langle \overrightarrow{xy}, \overrightarrow{TxTy} \right\rangle \geq 0$ for all $x, y \in C$. Aslo, it is nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$.

Lemma 2.10 ([6], Lemma 2.1). *Let X be a CAT(0) space, $x, y \in X$, $\lambda \in [0, 1]$ and $z = \lambda x \oplus (1-\lambda) y$. Then $\langle \overrightarrow{zy}, \overrightarrow{zw} \rangle \leq \lambda \langle \overrightarrow{xy}, \overrightarrow{zw} \rangle$ for all $w \in X$.*

Lemma 2.11 ([6], Theorem 2.2). *Let C be a nonempty convex subset of a CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle \geq 0$ for all $y \in C$.*

Lemma 2.12 ([6], Proposition 2.4). *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Then $P_C : X \to C$ is monotone and nonexpansive.*

Lemma 2.13 ([23], Lemma 2.10). *Let X be a CAT(0) space. For any $u, v \in X$ and $t \in [0, 1]$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(1) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t \langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;
(2) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t \langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t \langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Lemma 2.14 ([3], Proposition 2.2). *Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0, 1]$. Then $d[\lambda p \oplus (1-\lambda) q, \lambda r \oplus (1-\lambda) s] \leq \lambda d(p, r) + (1-\lambda) d(q, s)$.*

Lemma 2.15 ([10], Lemma 2.5). *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then $d^2(\lambda x \oplus (1-\lambda) y, z) \leq \lambda d^2(x, z) + (1-\lambda) d^2(y, z) - \lambda(1-\lambda) d^2(x, y)$.*

Definition 2.16 ([1], Definition 3.2.2). Let X be a complete CAT(0) space and let f be a function of X into $(-\infty, \infty]$. Then f is said to be weakly lower semicontinuous on X if and only if for any $x_0 \in X$, $\{x_n\} \rightharpoonup x_0$ implies that $f(x_0) \leq \liminf_{n \to \infty} f(x_n)$.

Lemma 2.17 ([1], Corollary 3.2.4). *Let C be a nonempty closed convex subset of a complete CAT(0) space X. The distance function $d(x, C)$ as well as its square $d^2(x, C)$ are weakly lower semicontinuous.*

We first introduce the definition of k-strict pseudo-contraction in CAT(0) spaces.

Definition 2.18. Let (X, d) be a CAT(0) space and C be a nonempty subset of X. The mapping $T : C \to C$ is said to be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [4] if for all $x, y \in C$ there exists $k \in (-\infty, 1)$ such that

$$d^2(Tx, Ty) \leq d^2(x, y) + k\{d^2(x, Tx) - 2 \left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y, Ty)\}.$$

Lemma 2.19. *Let C be a nonempty closed convex subset of a CAT(0) space X, and $T : C \to C$ be a k-strict pseudo-contraction, then T satisfies the Lipschitz condition with Lipschitz constant $L = \max\{\frac{1+k}{1-k}, 1\}$ for all $x, y \in C$. That is*

$$d(Tx, Ty) \leq \max\{\tfrac{1+k}{1-k}, 1\} d(x, y) \text{ for all } x, y \in C.$$

Proof. Let C be a nonempty closed convex subset of a CAT(0) space X. For $T : C \to C$ be a k-strict pseudo-contraction, we have

$$\begin{aligned} d^2(Tx,Ty) &\leq d^2(x,y) + k\{d^2(x,Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y,Ty)\} \\ &= d^2(x,y) + k\{d^2(x,Tx) - d^2(x,Ty) - d^2(y,Tx) + d^2(x,y) + d^2(Tx,Ty) + d^2(y,Ty)\} \end{aligned} \tag{2.2}$$

By simple calculation from (2.2) we have that

$$\begin{aligned} (1-k)d^2(Tx,Ty) &\leq (1+k)d^2(x,y) + k\{d^2(x,Tx) - d^2(x,Ty) - d^2(y,Tx) + d^2(y,Ty)\} \\ &= (1+k)d^2(x,y) + 2k\left\langle \overrightarrow{xy}, \overrightarrow{TyTx} \right\rangle. \end{aligned} \tag{2.3}$$

Since X satisfies the Cauchy-Schwarz inequality, it follows from (2.3), we get that

$$(1-k)d^2(Tx,Ty) - 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \leq 0. \tag{2.4}$$

Next, we will divide the proof into two cases.

Case 1. $k \leq 0$.
Notice that $k \leq 0 \Leftrightarrow 2k \leq 0 \Leftrightarrow 1+k \leq 1-k \Leftrightarrow \frac{1+k}{1-k} \leq 1 \Leftrightarrow \max\{\frac{1+k}{1-k}, 1\} = 1$. Since $k \leq 0$, from (2.4), we have

$$\begin{aligned} &(1-k)d^2(Tx,Ty) + 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \\ &\qquad \leq (1-k)d^2(Tx,Ty) - 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \leq 0. \end{aligned}$$

Thus $(1-k)d^2(Tx,Ty) + 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \leq 0$.
Solving this quadratic inequality, we obtain
$d(Tx,Ty) \leq d(x,y)$ or $d(Tx,Ty) \leq \{\frac{1+k}{1-k}\}d(x,y)$ for all $x,y \in C$. It implies that
$d(Tx,Ty) \leq d(x,y) = \max\{\frac{1+k}{1-k}, 1\}d(x,y)$ for all $x,y \in C$.

Case 2. $0 \leq k < 1$.
We have $1-k > 0$ and then $k \geq 0 \Leftrightarrow 2k \geq 0 \Leftrightarrow 1+k \geq 1-k \Leftrightarrow \frac{1+k}{1-k} \geq 1 \Leftrightarrow \max\{\frac{1+k}{1-k}, 1\} = \frac{1+k}{1-k}$.
Similarly case 1, we have $(1-k)d^2(Tx,Ty) - 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \leq 0$.
It implies that $d(Tx,Ty) \leq \{\frac{1+k}{1-k}\}d(x,y) = \max\{\frac{1+k}{1-k}, 1\}d(x,y)$ for all $x,y \in C$.
Therefore, the desired result. □

In this paper, we denote that Fix(T) is the set of fixed point of T such that Fix(T) $= \{x \in C : Tx = x\}$.

Lemma 2.20 ([11], Theorem 2.3). *Let C be a closed convex subset of a CAT(0) space X and $T : C \to C$ be a k - strict pseudo-contraction mapping. If Fix(T) $\neq \varnothing$, then Fix(T) is closed and convex so that the projection $P_{Fix(T)}$ is well defined.*

Lemma 2.21 ([19], Lemma 2.2). *Let (E,d) be a complete metric space. Then E is a geodesic space if and only if for every $x,y \in E$, there exists $z \in E$ such that $d(x,z) = d(z,y) = \frac{1}{2}d(x,y)$.*

Lemma 2.22 ([3], p.163). *A geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.*

Let (E,d) be a CAT(0) space and $D \subseteq E$. Niculescu and Roventa [21] introduced the notion of a convex hull of D as follows :

co(D) $= \bigcup_{n=0}^{\infty} D_n$ where $D_0 = D$ and for $n \geq 1$, the set D_n consists of all points in E which lie on geodesics which start and end in D_{n-1}.

$\bar{co}(D)$ denote the closure of the convex hull. It is easy to see that in a CAT(0) space, the closure of the convex hull will be convex and hence it is the smallest closed convex set containing D ([1], p.31).

Definition 2.23 ([19], Definition 2.2)**.** Let D be a nonempty subset of a CAT(0) space (E, d). A set-valued mapping $G : D \to 2^E$ is called to be a KKM mapping if $\mathrm{co}(F) \subset \bigcup_{x\in F} G(x)$ for every $F \in \langle D\rangle$ where $\langle D\rangle$ denotes the class of all nonempty finite subsets of D.

Lemma 2.24 ([14], Lemma 1.8)**.** *Suppose X is a complete CAT(0) space and K is a nonempty subset of X. Let $G : K \to 2^K$ be a mapping such that for each $x \in K$, $G(x)$ be $\triangle$-closed. Suppose that*

(1) *each $x_1, ..., x_m \in K$, $co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$,*

(2) *there exists $x_0 \in K$ such that $G(x_0)$ is $\triangle$- compact.*

Then $\bigcap_{x\in K} G(x) \neq \varnothing$.

Lemma 2.25. *Let (E, d) be a complete CAT(0) space, K be a nonempty $\triangle$-compact subset of E, and $F, G : E \to 2^E$ be two set-valued mappings such that*

(1) *for every $y \in E$, $F(y) \subseteq G(y)$ and $G(y)$ is convex;*
(2) *for every $x \in E$, $F^{-1}(x)$ is open in E;*
(3) *for every $y \in K$, $F(y) \neq \varnothing$;*
(4) *there exists a point $x_0 \in E$ such that $\bar{co}(E\backslash G^{-1}(x_0)) \subseteq K$.*

Then, there exists $\widehat{y} \in E$ such that $\widehat{y} \in G(\widehat{y})$.

Proof. Suppose the contrary. Then, for every $y \in E, y \notin G(y)$. Now let us define two set-valued mappings $\widetilde{G}, \widetilde{F} : E \to 2^E$ by

$$\widetilde{G}(x) = \bar{\mathrm{co}}(E\backslash G^{-1}(x)) \text{ and } \widetilde{F}(x) = \mathrm{co}(E\backslash F^{-1}(x)) \text{ for all } x \in E.$$

By using (1) and $F(y) \subseteq G(y)$ for every $y \in E$, we have

$$F^{-1}(x) = \{y \in E : x \in F(y)\} \subseteq \{y \in E : x \in G(y)\} = G^{-1}(x).$$

Then, $E\backslash G^{-1}(x) \subseteq E\backslash F^{-1}(x)$ for every $x \in E$. It implies that $\mathrm{co}(E\backslash G^{-1}(x)) \subseteq \mathrm{co}(E\backslash F^{-1}(x))$. By using (2), we have $\mathrm{co}(E\backslash F^{-1}(x))$ is closed in E. Since $\bar{\mathrm{co}}(E\backslash G^{-1}(x))$ is the smallest closed set containing $\mathrm{co}(E\backslash G^{-1}(x))$. Then $\bar{\mathrm{co}}(E\backslash G^{-1}(x)) \subseteq \mathrm{co}(E\backslash F^{-1}(x))$. Therefore $\widetilde{G}(x) \subseteq \widetilde{F}(x)$ for every $x \in E$.

We next show that $\widetilde{G}$ is a KKM mapping. That is, for every $A \in \langle E\rangle$, $\mathrm{co}(A) \subseteq \bigcup_{x\in A} \widetilde{G}(x)$. Otherwise, there exist $A \in \langle E\rangle$ and a point $y \in \mathrm{co}(A)$ such that $y \notin \bigcup_{x\in A} \widetilde{G}(x) = \bigcup_{x\in A} (\bar{\mathrm{co}}(E\backslash G^{-1}(x)))$. For $(E\backslash G^{-1}(x)) \subseteq E$, we have $\mathrm{co}(E\backslash G^{-1}(x)) = \bigcup_{n=0}^{\infty} (E\backslash G^{-1}(x))_n$ where $(E\backslash G^{-1}(x))_0 = E\backslash G^{-1}(x)$ and for $n \geq 1$, $(E\backslash G^{-1}(x))_n$ consists of all points in E which lie on geodesics which start and end in $(E\backslash G^{-1}(x))_{n-1}$.

Let us consider, $y \notin \bigcup_{x\in A} \widetilde{G}(x) = \bigcup_{x\in A} (\bar{\mathrm{co}}(E\backslash G^{-1}(x))) = \bigcup_{x\in A} \mathrm{cl}_E\{\bigcup_{n=0}^{\infty} (E\backslash G^{-1}(x))_n\}$. Since $\bigcup_{x\in A} \mathrm{cl}_E(E\backslash G^{-1}(x)) \subseteq \bigcup_{x\in A} \mathrm{cl}_E\{\bigcup_{n=0}^{\infty} (E\backslash G^{-1}(x))_n\}$. It implies that

$$y \notin \bigcup_{x\in A} \mathrm{cl}_E(E\backslash G^{-1}(x)) = E\backslash \bigcap_{x\in A} \mathrm{int}_E G^{-1}(x).$$

It follows that $y \in \bigcap_{x\in A} G^{-1}(x)$. Therefore $A \subseteq G(y)$. Since $G(y)$ is convex by (1), and $\text{co}(A)$ is the smallest convex set containing A. We get that $y \in \text{co}(A) \subseteq G(y)$, which is a contradiction. Hence $\widetilde{G}$ is a KKM mapping. By the definition of $\widetilde{G}$, $\widetilde{G}(x)$ is $\triangle$-closed in E for every $x \in E$. By using (4), there exists a point $x_0 \in E$ such that $\widetilde{G}(x_0) = \bar{\text{co}}(E\backslash G^{-1}(x_0)) \subseteq K$, it implies that $\widetilde{G}(x_0)$ is $\triangle$-compact. Then, by Lemma 2.24, we get that $\varnothing \neq \bigcap_{x\in E} \widetilde{G}(x) \subseteq \widetilde{G}(x_0) \subseteq K$. Therefore, we have

$$\varnothing \neq K \cap (\bigcap_{x\in E} \widetilde{G}(x)) \subseteq K \cap (\bigcap_{x\in E} \widetilde{F}(x)).$$

Taking $y_0 \in K \cap (\bigcap_{x\in E} \widetilde{F}(x))$, we have $y_0 \in K$ and $x \notin F(y_0)$ for every $x \in E$. Hence, we have $F(y_0) = \varnothing$ which contradicts (3). Therefore, there exists $\widehat{y} \in E$ such that $\widehat{y} \in G(\widehat{y})$. This completes our proof. □

Remark 2.26. If $F = G$, then (4) of Lemma 2.25 can be replaced by the following equivalent condition:

$(4)^*$ there exists a point $x_0 \in E$ such that $\text{co}(E\backslash F^{-1}(x_0)) \subseteq K$.

3. Main Results

In this section, motivated by Ungchittrakool [22]. We discuss the existence and convergence for fixed point of a strict pseudo-contraction in the terminology of Browder and Petryshyn in the framework of complete CAT(0) spaces.

Lemma 3.1. *Let C be a bounded closed convex subset of a complete CAT(0) space (X, d). Then (C, d) is a complete CAT(0) space.*

Proof. Let C be a bounded closed convex subset of complete CAT(0) space (X, d). Notice that, a subset of a CAT(0) space equipped with the induced metric, is a CAT(0) space if and only if it is convex. This implies that (C, d) is a CAT(0) space. Since C is closed subset of complete metric space (X, d), then (C, d) is complete metric space. Therefore, we have (C, d) is a complete CAT(0) space. □

Lemma 3.2. *Let C be a bounded closed convex subset of a complete CAT(0) space X. Let T be a k- strict pseudo-contraction in the terminology of Browder and Petryshyn. Then, there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$.*

Proof. Let C be a bounded closed convex subset of a complete CAT(0) space (X, d). We claim that there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$. For any $y \in C$, we assume that the set $\{x \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle < 0\}$ is nonempty. We also define two set-valued mappings $F, G : C \to 2^C$ by $F(y) = G(y) = \{x \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle < 0\}$.

We first show that $G(y)$ is convex and $F^{-1}(x)$ is an open set.

Step1. To show that $G(y)$ is convex.

Let $x_1, x_2 \in G(y)$ and $u_t = tx_1 \oplus (1-t)x_2$ such that $t \in [0, 1]$. So, we have $x_1, x_2 \in C$, that $u_t \in C$.

Let us consider $\left\langle \overrightarrow{u_t y}, \overrightarrow{u_t Ty} \right\rangle$, by Lemma 2.13, we get that

$$\begin{aligned}
&\left\langle \overrightarrow{u_t y}, \overrightarrow{u_t Ty} \right\rangle \\
&\leq t \left\langle \overrightarrow{x_1 y}, \overrightarrow{u_t Ty} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 y}, \overrightarrow{u_t Ty} \right\rangle = t \left\langle \overrightarrow{u_t Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{u_t Ty}, \overrightarrow{x_2 y} \right\rangle \\
&\leq t\{t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_1 y} \right\rangle\} + (1-t)\{t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_2 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle\} \\
&= t^2 \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t)^2 \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_2 y} \right\rangle \\
&= t^2 \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + t(1-t)\{\langle \overrightarrow{x_2 x_1}, \overrightarrow{x_1 y} \rangle + \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle\} \\
&\quad + (1-t)^2 \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle + t(1-t)\{\langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle + \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle\} \\
&= t^2 \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t)^2 \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle \\
&\quad + t(1-t)\{\langle \overrightarrow{x_2 x_1}, \overrightarrow{x_1 x_2} \rangle + \langle \overrightarrow{x_2 x_1}, \overrightarrow{x_2 y} \rangle\} + t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle \\
&= t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle - t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_1 x_2} \rangle \\
&\quad - t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle + t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle \\
&\leq t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle \\
&< 0.
\end{aligned}$$

Therefore $u_t \in G(y)$, that is $G(y)$ is convex.
Step 2. To show that $F^{-1}(x)$ is an open set.
For $F^{-1}(x) = \{y \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle < 0\}$, we show that $C \backslash F^{-1}(x) = \{y \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle \geq 0\}$ is a closed set. Let $\{y_n\} \subseteq C \backslash F^{-1}(x)$ such that $y_n \to y_0$. Then $\left\langle \overrightarrow{xy_n}, \overrightarrow{xTy_n} \right\rangle \geq 0$. We will show that $y_0 \in C \backslash F^{-1}(x)$. By Lemma 2.19, T is a Lipschitzian map. Inparticular, T is continuous. It follows that

$$\begin{aligned}
0 &\leq \left\langle \overrightarrow{xy_n}, \overrightarrow{xTy_n} \right\rangle = \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_n} \right\rangle + \left\langle \overrightarrow{y_0 y_n}, \overrightarrow{xTy_n} \right\rangle = \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle + \left\langle \overrightarrow{xy_0}, \overrightarrow{Ty_0 Ty_n} \right\rangle + \left\langle \overrightarrow{y_0 y_n}, \overrightarrow{xTy_n} \right\rangle \\
&\leq \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle + d(x, y_0) d(Ty_0, Ty_n) + d(y_0, y_n) d(x, Ty_n) \\
&\leq \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle + \max\{\frac{1+k}{1-k}, 1\} d(x, y_0) d(y_0, y_n) + d(y_0, y_n) d(x, Ty_n),
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit in both sides, we get that $\left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle \geq 0$.
That is $y_0 \in C \backslash F^{-1}(x)$. Hence $C \backslash F^{-1}(x)$ is a closed set in C, therefore $F^{-1}(x)$ is an open set.

We next show that there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$. By assumption, we have $F(y) \neq \varnothing$ for every $y \in C$, and by Lemma 2.5, we have C is $\triangle$- compact. Notice that there exists a point $z \in C$ such that $C \backslash F^{-1}(z) \subseteq C$. Also, $co(C \backslash F^{-1}(z))$ is the smallest convex set containing $C \backslash F^{-1}(z)$. Then, we get that there exists a point $z \in C$ such that $co(C \backslash F^{-1}(z)) \subseteq C$ where C is a nonempty $\triangle$-compact subset of C. Also, by Lemma 3.1, we have (C, d) is a complete CAT(0) space. By Lemma 2.25 and Remark 2.26, we have $x_0 \in C$ such that $x_0 \in G(x_0)$. This implies that $0 = \left\langle \overrightarrow{x_0 x_0}, \overrightarrow{x_0 Tx_0} \right\rangle < 0$. This is a contradiction. We obtain that $\{x \in C : \left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle < 0\} = \varnothing$. Therefore $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$. □

Lemma 3.3. *Let C be a bounded closed convex subset of a complete CAT(0) space X. Let T be a k- strict pseudo-contraction in the terminology of Browder and Petryshyn. Then, there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{x_0 Tx_0} \right\rangle \geq 0$ for all $x \in C$.*

Proof. Let C be a bounded closed convex subset of complete CAT(0) space (X,d). By Lemma 3.1, we have (C,d) is a complete CAT(0) space. By Lemma 3.2, we have

$$x_0 \in C \text{ such that } \left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0 \text{ for all } x \in C. \tag{3.1}$$

Also, for any $u, z \in C$ and $0 < t < 1$ and since C is convex, we have $y_t = (1-t)u \oplus tz \in C$. Then, for $x_0 \in C$ we have

$$y_t = (1-t)x_0 \oplus tz \in C. \tag{3.2}$$

By using (3.1) and (3.2), we have $0 \leq \left\langle \overrightarrow{\{(1-t)x_0 \oplus tz\}x_0}, \overrightarrow{y_tTx_0} \right\rangle$. By Lemma 2.10, we have $0 \leq \left\langle \overrightarrow{\{(1-t)x_0 \oplus tz\}x_0}, \overrightarrow{y_tTx_0} \right\rangle \leq t \left\langle \overrightarrow{zx_0}, \overrightarrow{y_tTx_0} \right\rangle$. Since $t > 0$, it follows that $0 \leq \left\langle \overrightarrow{zx_0}, \overrightarrow{y_tTx_0} \right\rangle$. By Lemma 2.19, T is a Lipschitzian map. Inparticular, T is continuous. Also $y_t \to x_0$ as $t \to 0$. It follows that

$$0 \leq \left\langle \overrightarrow{zx_0}, \overrightarrow{y_tTx_0} \right\rangle = \langle \overrightarrow{zx_0}, \overrightarrow{y_tx_0} \rangle + \left\langle \overrightarrow{zx_0}, \overrightarrow{x_0Tx_0} \right\rangle \leq d(z,x_0)d(y_t,x_0) + \left\langle \overrightarrow{zx_0}, \overrightarrow{x_0Tx_0} \right\rangle,$$

for $0 < t < 1$. Taking the limit in both sides, we get that $\left\langle \overrightarrow{zx_0}, \overrightarrow{x_0Tx_0} \right\rangle \geq 0$ as $t \to 0$.
Therefore, $\left\langle \overrightarrow{xx_0}, \overrightarrow{x_0Tx_0} \right\rangle \geq 0$ for all $x \in C$. □

Lemma 3.4. *Let (X,d) be a complete CAT(0) space and T be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn with domain $D(T)$ and range $R(T)$. Then for all $x, y \in D(T)$ the following inequalities hold and are equivalent :*

(1) $d^2(x,Tx) + d^2(y,Ty) \leq \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{xTx} \right\rangle - \frac{2k}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{yTy} \right\rangle - 2\left\langle \overrightarrow{Txy}, \overrightarrow{yTy} \right\rangle$;

(2) $d^2(x,Tx) + d^2(y,Ty) \leq \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{yTy} \right\rangle + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle$;

(3) $d^2(x,Tx) + d^2(y,Ty) \leq \frac{2}{1-k}\left\langle \overrightarrow{xTy}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{Txy}, \overrightarrow{yTy} \right\rangle - 2(\frac{1+k}{1-k})\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle$;

(4) $d^2(x,Tx) + d^2(y,Ty)$
$\leq \left\langle \overrightarrow{xTy}, \overrightarrow{xTx} \right\rangle - \left\langle \overrightarrow{Txy}, \overrightarrow{yTy} \right\rangle + \frac{1+k}{2}\{d^2(x,Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y,Ty)\}$.

Proof. We first show that (2) holds.

$$\begin{aligned} d^2(x,Tx) + d^2(y,Ty) &= d^2(x,Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y,Ty) + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle \\ &= d^2(x,Tx) - d^2(x,Ty) - d^2(Tx,y) + d^2(x,y) + d^2(Tx,Ty) \\ &\quad + d^2(y,Ty) + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle \\ &\leq d^2(x,Tx) - d^2(x,Ty) - d^2(Tx,y) + d^2(x,y) + d^2(x,y) \\ &\quad + kd^2(x,Tx) - 2k\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + kd^2(y,Ty) + d^2(y,Ty) + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle. \end{aligned}$$

By simple calculation from the inequality above we get that

$$\begin{aligned} (1-k)\{d^2(x,Tx) + d^2(y,Ty)\} &\leq d^2(x,Tx) + d^2(y,Ty) + d^2(y,x) + d^2(x,y) - d^2(y,Tx) \\ &\quad - d^2(x,Ty) + 2(1-k)\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle. \end{aligned}$$

Dividing throughout with $(1-k)$ we have that

$$\begin{aligned} d^2(x,Tx) + d^2(y,Ty) &\leq \frac{1}{1-k}\{d^2(x,Tx) + d^2(y,Ty) + d^2(y,x) + d^2(x,y) \\ &\quad - d^2(y,Tx) - d^2(x,Ty)\} + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle \\ &= \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{yTy} \right\rangle + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle. \end{aligned} \tag{3.3}$$

Then, (2) is true. Next, we observe that

$$\begin{aligned}-\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle &= -\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{yTx},\overrightarrow{yTy}\right\rangle\\ &=\{2-\frac{2}{1-k}\}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle-2\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle\\ &=\frac{-2k}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle-2\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle. \qquad (3.4)\end{aligned}$$

Substituting (3.4) in (3.3), we get that (1) holds; that is

$d^2(x,Tx)+d^2(y,Ty)\le\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle-\frac{2k}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle-2\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle$,

and hence (1) and (2) are equivalent.

We next show that (3) is true. Let us consider

$$\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle=\frac{2}{1-k}\{\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle+\left\langle\overrightarrow{Tyy},\overrightarrow{xTx}\right\rangle\}=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle. \qquad (3.5)$$

$$-\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle=-\frac{2}{1-k}\{\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle+\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\}=-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle. \qquad (3.6)$$

Combining (3.5) and (3.6), we have

$$\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-\frac{4}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle.$$

We get that

$$\begin{aligned}&\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\\ &=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-\frac{4}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\\ &=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-2(\frac{1+k}{1-k})\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle.\end{aligned}$$

This shows that (3) is true. We get that (2) and (3) are equivalent. Next, we will show that (3) and (4) are equivalent. We will show that (3) implies (4). Since $\frac{1-k}{2}>0$, for (3) is true, we get that

$$\begin{aligned}(\frac{1-k}{2})\{d^2(x,Tx)+d^2(y,Ty)\}&\le\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-(1+k)\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\\ [1-(\frac{1+k}{2})]\{d^2(x,Tx)+d^2(y,Ty)\}&\le\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-(1+k)\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\\ d^2(x,Tx)+d^2(y,Ty)&\le\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-(1+k)\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\\ &\quad+(\frac{1+k}{2})\{d^2(x,Tx)+d^2(y,Ty)\}\\ &=\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle\\ &\quad+(\frac{1+k}{2})\{d^2(x,Tx)+d^2(y,Ty)-2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\}.\end{aligned}$$

Then, we get that (4) holds. By using a similar method, we get that (4) implis (3). That is, we get that (3) and (4) are equivalent. This completes our proof. □

Lemma 3.5. *Let (X, d) be a complete CAT(0) space and T be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn with domain $D(T)$ and range $R(T)$. If, there exists $u \in D(T)$ such that $\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle \geq 0$ and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$ for some $x \in D(T)$, the following inequalities hold :*

$$d^2(x,Tx) \leq \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle, & \begin{cases} \text{if } k \in [0,1); \\ \text{or if } \left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \leq 0; \end{cases} \\ \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle \\ \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle \end{cases}, & \text{if } \left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0 \text{ and } k \in [0,1); \\ \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle, & \text{if } \left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0 \text{ and } k \in [-1,0); \\ \left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle, & \text{if } k \in (-\infty,-1]. \end{cases}$$

Proof. If $k \in [0,1)$, then $k < 1 \Leftrightarrow 0 < 1-k$ and note that $0 \leq 2k$, so we have $\frac{2k}{1-k} \geq 0 \Leftrightarrow -\frac{2k}{1-k} \leq 0$. By Lemma 3.4(1), $\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle \geq 0$ and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that

$$\begin{aligned} d^2(x,Tx) &\leq d^2(x,Tx) + d^2(u,Tu) \leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle - \frac{2k}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle - 2\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \\ &\leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle. \end{aligned} \tag{3.7}$$

If $\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \leq 0$, then by Lemma 3.4(2) and $\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that

$$\begin{aligned} d^2(x,Tx) &\leq d^2(x,Tx) + d^2(u,Tu) \leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle + 2\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \\ &\leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle. \end{aligned}$$

Before we prove the next case, let us consider the following

$$k \in [-1,1) \Leftrightarrow -1 \leq k < 1 \begin{cases} \Leftrightarrow 0 \leq 1+k < 2. \\ \Leftrightarrow 1 \geq -k > -1 \Leftrightarrow 2 \geq 1-k > 0 \Leftrightarrow \frac{1}{1-k} \geq \frac{1}{2}. \end{cases}$$

Therefore, we have $2(\frac{1+k}{1-k}) \geq 1+k \geq 0$ and then

$$-2(\frac{1+k}{1-k}) \leq 0 \quad \text{whenever} \quad k \in [-1,1). \tag{3.8}$$

If $\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0$ and $k \in [0,1)$, then it follows from (3.8), Lemma 3.4(3) and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that

$$\begin{aligned} d^2(x,Tx) &\leq d^2(x,Tx) + d^2(u,Tu) \leq \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle - 2(\frac{1+k}{1-k})\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \\ &\leq \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle. \end{aligned}$$

From (3.7), we can conclude in this case that $d^2(x,Tx) \leq \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle \\ \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle \end{cases}.$ If $\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0$ and $k \in [-1,0)$, then by (3.8), Lemma 3.4(3) and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that $d^2(x,Tx) \leq$

$\frac{2}{1-k}\left\langle\overrightarrow{xTu},\overrightarrow{xTx}\right\rangle$. Finally, if $k\in(-\infty,-1]$, then by using lemma 3.4(4) and $\left\langle\overrightarrow{Txu},\overrightarrow{uTu}\right\rangle\geq 0$, we get that

$$\begin{aligned}d^2(x,Tx)&\leq d^2(x,Tx)+d^2(u,Tu)\leq\left\langle\overrightarrow{xTu},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txu},\overrightarrow{uTu}\right\rangle+\frac{1+k}{2}d^2(xTx,uTu)\\&\leq\left\langle\overrightarrow{xTu},\overrightarrow{xTx}\right\rangle.\end{aligned}$$

This completes our proof. □

Every iteration process generated by the shrinking projection method for a k-strict pseudo-contraction T in the terminology of Browder and Petryshn is well defined even if T is fixed point free.

Lemma 3.6. *Let (X,d) be a complete CAT(0) space and C be nonempty closed and convex subset of X. Let $T:C\to C$ be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn, that is for all $x,y\in X$ there exists an element $k\in(-\infty,1)$ such that*

$$d^2(Tx,Ty)\leq d^2(x,y)+k[d^2(x,Tx)-2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle+d^2(y,Ty)].$$

Let $x_0\in X$, $C_1=C$ and $\{x_n\}$ be a sequence in C generated by

$$\begin{cases}x_n&=P_{C_n}(x_0),\\C_{n+1}&=\left\{z\in C_n:d^2(x_n,Tx_n)\leq max\,\{\frac{2}{1-k},1\}\left\langle\overrightarrow{x_nz},\overrightarrow{x_nTx_n}\right\rangle\right\},\end{cases}\tag{3.9}$$

for all $n\in\mathbb{N}$. Then, C_n is nonempty closed convex subsets of X and consequently, $\{x_n\}$ is well defined for every $n\in\mathbb{N}$.

Proof. Clearly, C_1 is nonempty. Suppose that C_m is nonempty for some $m\in N$. We wish to show that C_{m+1} is nonempty. Since $C_m\subset C_{m-1}\subset...\subset C_1$, we have that $C_1,C_2,...,C_m$ are nonempty. Next, we will show that $C_1,C_2,...,C_m$ are closed and convex. It is sufficient to show that C_m is closed and convex. It is not hard to show that for any $\{z_k\}\subseteq C_m$ such that $z_k\to z_0$, we have $z_0\in C_m$. We get that C_m is closed.

We next show that C_m is convex. Notice that a subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space if and only if it is convex. Thus, we will show that (C_m,d) is a complete CAT(0) space. Let each $x,y\in C_m$, we have $x,y\in C$. By Lemma 3.1, we have (C,d) is a complete CAT(0) space and thus, it is a geodesic space, hence x,y are joined by a geodesic. Since x,y are arbitrary, thus we have $x,y\in C_m$ are joined by a geodesic. Hence C_m is a geodesic space. Since C_m is closed subset of complete metric space (C,d), then (C_m,d) is a complete metric space. It follows from Lemma 2.21 that, for every $y,z\in C_m$, there exists $p\in C_m$ such that $d(y,p)=d(p,z)=\frac{1}{2}d(y,z)$. Now, we claim that C_m satisfies the (CN) inequality. In fact, let $x,y,z\in C_m$ and $p\in C_m$ with $d(y,p)=d(p,z)=\frac{1}{2}d(y,z)$. Let α and β be two numbers satisfy $\alpha+\beta\geq 1$. Then $\alpha^2+\beta^2\geq\frac{1}{2}(\alpha+\beta)^2\geq\frac{1}{2}$ with equality if and only if $\alpha=\beta=\frac{1}{2}$. By this fact and by the triangle inequality, we get that $(\dfrac{d(y,p)}{d(y,z)})^2+(\dfrac{d(p,z)}{d(y,z)})^2\geq\dfrac{1}{2}$. That is, $\frac{1}{2}d^2(y,z)\leq d^2(y,p)+d^2(p,z)$. It follows from the above inequality that, setting $x=p$, we get that $d^2(x,y)+d^2(x,z)\geq 2d^2(x,p)+\frac{1}{2}d^2(y,z)$, this implies that C_m satisfies the (CN) inequality. By Lemma 2.22, we know that (C_m,d) is a CAT(0) space. By above, we have (C_m,d) is a complete metric space. Then (C_m,d) is a complete CAT(0) space. This implies that C_m is convex subset of X. Thus, we have C_m is closed and convex. Finally, put $r=\max\{d(x_0,x_i),d(x_0,Tx_i):i=1,2,...,m\}$ and $B_r=\{z\in X:d(x_0,z)\leq r\}$. Obviously $C\cap B_r$ is a nonempty bounded closed convex subset of X. It follows from Lemma 3.3 that there exists an element $u\in C\cap B_r$ such that $\left\langle\overrightarrow{yu},\overrightarrow{uTu}\right\rangle\geq 0$ for all $y\in C\cap B_r$. In particular, we have

$$\left\langle\overrightarrow{x_iu},\overrightarrow{uTu}\right\rangle\geq 0\quad\text{and}\quad\left\langle\overrightarrow{Tx_iu},\overrightarrow{uTu}\right\rangle\geq 0\tag{3.10}$$

for every $i = 1, 2, ..., m$.
Case I. max $\{\frac{2}{1-k}, 1\} = \frac{2}{1-k}$.
Notice that max $\{\frac{2}{1-k}, 1\} = \frac{2}{1-k} \Leftrightarrow 1 \leq \frac{2}{1-k} \Leftrightarrow 1-k \leq 2 \Leftrightarrow -1 \leq k \Leftrightarrow k \in [-1, 1)$, it follows from (3.10) and Lemma 3.5 that

$$d^2(x_i, Tx_i) \leq \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{x_i u}, \overrightarrow{x_i Tx_i}\right\rangle, & \begin{cases} \text{if } k \in [0,1); \\ \text{or if } \left\langle \overrightarrow{x_i Tx_i}, \overrightarrow{uTu}\right\rangle \leq 0; \end{cases} \\ \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{x_i u}, \overrightarrow{x_i Tx_i}\right\rangle \\ \frac{2}{1-k}\left\langle \overrightarrow{x_i Tu}, \overrightarrow{x_i Tx_i}\right\rangle \end{cases}, & \text{if } \left\langle \overrightarrow{x_i Tx_i}, \overrightarrow{uTu}\right\rangle \geq 0 \text{ and } k \in [0,1); \\ \frac{2}{1-k}\left\langle \overrightarrow{x_i Tu}, \overrightarrow{x_i Tx_i}\right\rangle, & \text{if } \left\langle \overrightarrow{x_i Tx_i}, \overrightarrow{uTu}\right\rangle \geq 0 \text{ and } k \in [-1,0) \end{cases}$$

for every $i = 1, 2, ..., m$. This shows that $u \vee Tu \in C_{m+1}$.
Case II. max $\{\frac{2}{1-k}, 1\} = 1$.
Notice that max $\{\frac{2}{1-k}, 1\} = \frac{2}{1-k} \leq 1 \Leftrightarrow 2 \leq 1-k \Leftrightarrow k \leq -1 \Leftrightarrow k \in (-\infty, -1]$, it follows from (3.9) and Lemma 3.5 that $d^2(x_i, Tx_i) \leq \left\langle \overrightarrow{x_i Tu}, \overrightarrow{x_i Tx_i}\right\rangle$, if $k \in (-\infty, -1]$ for every $i = 1, 2, ..., m$. This shows that $Tu \in C_{m+1}$. By Case I and Case II, we can conclude that $u \vee Tu \in C_{m+1}$. Hence C_{m+1} is nonempty. By induction on n, therefore the desired result. ☐

Theorem 3.7. *Let all the assumptions be the same as in Lemma 3.6. Then, the following are equivalent :*

(1) $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
(2) $\{x_n\}$ *is bounded;*
(3) $Fix(T)$ *is nonempty.*

Proof. [(1) ⇒ (2)] Let $u \in \bigcap_{n=1}^{\infty} C_n$. By Lemma 2.12, it follows from the nonexpansiveness of P_{C_n} that $d(x_n, u) = d(P_{C_n}x_0, P_{C_n}u) \leq d(x_0, u)$. This shows that x_n is bounded.

[(2) ⇒ (3)] Suppose that x_n is bounded, we first claim that $0 \leq d^2(x_{n+1}, x_n) \leq d^2(x_{n+1}, x_0) - d^2(x_n, x_0)$. Since $x_n = P_{C_n}x_0$, by Lemma 2.11, we have $\langle \overrightarrow{x_0 x_n}, \overrightarrow{x_n x_{n+1}}\rangle \geq 0$ for all $x_{n+1} \in C_n$. So, we have $\langle \overrightarrow{x_0 x_n}, \overrightarrow{x_0 x_{n+1}}\rangle - \langle \overrightarrow{x_0 x_n}, \overrightarrow{x_0 x_n}\rangle = \langle \overrightarrow{x_0 x_n}, \overrightarrow{x_n x_{n+1}}\rangle \geq 0$ and hence $d^2(x_0, x_n) = \langle \overrightarrow{x_0 x_n}, \overrightarrow{x_0 x_n}\rangle \leq \langle \overrightarrow{x_0 x_n}, \overrightarrow{x_0 x_{n+1}}\rangle$. By Lemma 2.14, 2.15 and using (2.1), we have

$$\begin{aligned} d^2(x_n, x_{n+1}) &\leq 2d^2(x_n, x_{n+1}) = 4d^2(\frac{1}{2}x_0 \oplus \frac{1}{2}x_n, \frac{1}{2}x_0 \oplus \frac{1}{2}x_{n+1}) \\ &\leq d^2(x_n, x_0) + d^2(x_{n+1}, x_0) + d^2(x_0, x_0) + d^2(x_n, x_{n+1}) - d^2(x_0, x_{n+1}) - d^2(x_0, x_n) \\ &= d^2(x_n, x_0) + d^2(x_{n+1}, x_0) + 2\langle \overrightarrow{x_{n+1}x_0}, \overrightarrow{x_0 x_n}\rangle \\ &= d^2(x_n, x_0) + d^2(x_{n+1}, x_0) - 2\langle \overrightarrow{x_0 x_{n+1}}, \overrightarrow{x_0 x_n}\rangle \\ &\leq d^2(x_n, x_0) + d^2(x_{n+1}, x_0) - 2d^2(x_0, x_n). \end{aligned} \tag{3.11}$$

This shows that $\{d(x_n, x_0)\}$ is nondecreasing and with is the bounded of $\{x_n\}$, we have $\lim_{n\to\infty} d(x_n, x_0)$ exists. From (3.11), we get that $d^2(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Thus $\overrightarrow{x_n x_{n+1}} = 0$. Since $x_{n+1} \in C_{n+1}$, we have $d^2(x_n, Tx_n) \leq \max\{\frac{2}{1-k}, 1\}\left\langle \overrightarrow{x_n x_{n+1}}, \overrightarrow{x_n Tx_n}\right\rangle \leq \max\{\frac{2}{1-k}, 1\} d(x_n, x_{n+1}) d(x_n, Tx_n)$. Thus $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded and by Lemma 2.3, we have $\triangle - \lim_{j\to\infty} x_{n_j} = w$. Since $d(x_{n_j}, Tx_{n_j}) \to 0$ as $j \to \infty$, then we get that

$$\Phi(x) = \limsup_{j\to\infty} d(x_{n_j}, x) = \limsup_{j\to\infty} d(Tx_{n_j}, x) \text{ for all } x \in C. \tag{3.12}$$

By taking $x = Tw$ in (3.12), we have

$$\begin{aligned}
\Phi(Tw)^2 &= \limsup_{j\to\infty}\ d^2(Tx_{n_j}, Tw)\\
&\leq \limsup_{j\to\infty}\{d^2(x_{n_j}, w) + k[d^2(x_{n_j}, Tx_{n_j}) - 2\left\langle \overrightarrow{x_{n_j}Tx_{n_j}}, \overrightarrow{wTw}\right\rangle + d^2(w, Tw)]\}\\
&= \limsup_{j\to\infty}\ d^2(x_{n_j}, w) + k\limsup_{j\to\infty}[d^2(x_{n_j}, Tx_{n_j}) + 2\left\langle \overrightarrow{Tx_{n_j}x_{n_j}}, \overrightarrow{wTw}\right\rangle + d^2(w, Tw)]\\
&\leq \limsup_{j\to\infty} d^2(x_{n_j}, w) + k\limsup_{j\to\infty} d^2(x_{n_j}, Tx_{n_j})\\
&\quad + 2k\limsup_{j\to\infty}[d(Tx_{n_j}, x_{n_j})d(w, Tw)] + k\limsup_{j\to\infty} d^2(w, Tw)\\
&= \Phi(w)^2 + k\ d^2(w, Tw). \qquad (3.13)
\end{aligned}$$

Since $-\infty < k < 1$, we can choose a real number $\lambda \in [0, 1]$ be such that $\max\{0, k\} < \lambda < 1$. By Lemma 2.15, we have

$$d^2\left(x_{n_j}, \lambda w \oplus (1-\lambda)Tw\right) \leq \lambda d^2\left(x_{n_j}, w\right) + (1-\lambda)d^2\left(x_{n_j}, Tw\right) - \lambda(1-\lambda)d^2\left(w, Tw\right)$$

Taking the superior limit on both sides of the above inequality, we get that

$$\Phi(\lambda w \oplus (1-\lambda)Tw)^2 \leq \lambda\Phi(w)^2 + (1-\lambda)\Phi(Tw)^2 - \lambda(1-\lambda)d^2\left(w, Tw\right)$$

Since $\triangle - \lim_{j\to\infty} x_{n_j} = w$. By using (3.13) and Lemma 2.2, we have

$$\begin{aligned}
\Phi(w)^2 &\leq \Phi(\lambda w \oplus (1-\lambda)Tw)^2 \leq \lambda\Phi(w)^2 + (1-\lambda)\Phi(Tw)^2 - \lambda(1-\lambda)d^2\left(w, Tw\right)\\
&\leq \lambda\Phi(w)^2 + (1-\lambda)\left(\Phi(w)^2 + kd^2\left(w, Tw\right)\right) - \lambda(1-\lambda)d^2\left(w, Tw\right)\\
&= \lambda\Phi(w)^2 + (1-\lambda)\Phi(w)^2 + (1-\lambda)kd^2\left(w, Tw\right) - \lambda(1-\lambda)d^2\left(w, Tw\right)\\
&= \Phi(w)^2 + (1-\lambda)(k-\lambda)d^2\left(w, Tw\right).
\end{aligned}$$

This implies that $(1-\lambda)(\lambda-k)d^2\left(w, Tw\right) \leq 0$. Since $\max\{0, k\} < \lambda < 1$, we have $(1-\lambda)(\lambda-k) > 0$. This implies that $Tw = w$, that is $w \in Fix(T) \neq \varnothing$.

[(3) $\Rightarrow$ (1)] Suppose that $Fix(T) \neq \varnothing$. We claim that $Fix(T) \subset C_n$ for all $n \in N$. If $w \in Fix(T)$, then we have $\left\langle \overrightarrow{ab}, \overrightarrow{wTw}\right\rangle = 0$ for all $a, b \in X$. Taking $u = w$ in the proof of Lemma 3.6, it is not hard to observe that all inequalities are satisfied. This implies that $w \in C_n$ for all $n \in N$. Therefore $Fix(T) \subset \bigcap_{n=1}^{\infty} C_n \neq \varnothing$. ☐

Theorem 3.8. *Let all the assumptions be the same as in Theorem 3.7 Then, if* $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$ ($\Leftrightarrow$ $\{x_n\}$ *is bounded* $\Leftrightarrow$ $Fix(T) \neq \varnothing$)*,then the sequence* $\{x_n\}$ *generated by* (3.9) *converges strongly to some points of* C *and its strong limit point is a member of* $Fix(T)$*, that is* $\lim_{n\to\infty} x_n = P_{Fix(T)}x_0 \in Fix(T)$.

Proof. If $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$, then Theorem 3.7 ensures that $\{x_n\}$ is bounded sequence in C. By Lemma 2.3, we have $\{x_{n_j}\} \subseteq \{x_n\}$ such that $\triangle - \lim_{j\to\infty} x_{n_j} = u$. By the proof of Theorem 3.7[(2) $\Rightarrow$ (3)], we have $u \in Fix(T)$. By Lemma 2.20, we have $P_{Fix(T)}$ is well defined. Also, $P_{Fix(T)}x_0 \in Fix(T) \subset C_n$, we observe that

$$d(x_n, x_0) = d(P_{C_n}x_0, x_0) \leq d(P_{Fix(T)}x_0, x_0) \qquad (3.14)$$

for all $n \in N$. Since $\{d(x_n, x_0)\}$ is nondecreasing, we get that $\lim_{n\to\infty} d(x_n, x_0)$ exists. Since $\triangle - \lim_{j\to\infty} x_{n_j} = u$, by using Lemma 2.6, we have $\{x_{n_j}\} \rightharpoonup u$. By Definition 2.16 and Lemma 2.17, we get that $d(u, \{x_0\}) \le \liminf_{j\to\infty} d(x_{n_j}, \{x_0\})$. By Lemma 2.8, there exists an $x_0 \in \{x_0\}$ such that

$$d(u, x_0) = d(u, \{x_0\}) \le \liminf_{j\to\infty} d(x_{n_j}, x_0) \tag{3.15}$$

By using (3.14) and (3.15), we get that

$$d(u, x_0) \le \liminf_{j\to\infty} d(x_{n_j}, x_0) = \lim_{n\to\infty} d(x_n, x_0) \le d(P_{Fix(T)}x_0, x_0). \tag{3.16}$$

Taking into account $u \in Fix(T)$, from (3.16), we have $d(u, x_0) \le d(P_{Fix(T)}x_0, x_0) \le d(u, x_0)$. This implies that $d(u, x_0) = d(P_{Fix(T)}x_0, x_0)$. By Lemma 2.8, we obtain that $u = P_{Fix(T)}x_0$. Therefore $\{x_n\} \rightharpoonup P_{Fix(T)}x_0$ and $d(x_n, x_0) \to d(P_{Fix(T)}x_0, x_0)$. Consequently, from (3.11), we get that $d^2(x_n, P_{Fix(T)}x_0) \le d^2(P_{Fix(T)}x_0, x_0) - d^2(x_n, x_0) \to 0$ as $n \to \infty$. This completes our proof. □

Remark 3.9. The results in this section extend and improve the corresponding Theorem 3.4 and 3.5 in [22] in the case of an iterative projection technique in a Hilbert space.

4. Conclusion

In the present paper, we study some existence and convergence theorems for fixed points of a strict pseudo-contraction by using an iterative projection technique with some suitable conditions. We obtain the sufficient conditions for the existence and convergence theorem for the fixed points of strict pseudo-contraction mappings in complete CAT(0) spaces.

Acknowledgements

The authors would like to thank, Naresuan University, Phitsanulok 65000, Thailand.

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ON THE CEU-DEGREE OF SIMILARITY IN INTERNATIONAL TRADE BY USING THE CHOQUET INTEGRAL EXPECTED UTILITY

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Abstract. Recently, we considered the Choquet integrals with respected to a fuzzy measure and the Choquet-expected utility(CEU)which was represented by preference functionals. We note that the CEU provides a useful tool to calculate the subjective capacity of trade values between Korea and some countries in Wood-Jang [4,10].

In this paper, by using the Choquet-expected utility in Wood-Jang [4] and the degree of similarity in Biswas [1], we define the CEU-degree of the similarity related with the CEU of trade values between Korea and some countries. In particular, we investigate some applications of the CEU-degree of similarity related with the CEU of trade values.

1. Introduction

By using fuzzy sets and Choquet integrals in [1,2,4,5,6,10], many researchers have studied the concept of Choquet intgeral expected utility and its related areas(see[3,4,8.9.11,12]). Recently, Wood-Jang [6,7] studied some applications of the Choquet integral as imprecise market premium functionals with respect to an imprecise set function which was an interval-valued measure of risk and the Choquet integral with respect to a fuzzy measure of a utility function. In 1995, Biswas [1] investigated a student's evaluation on the space of fuzzy sets which include data information for the students respective classes.

In this paper, by using the degree of similarity in Biswas [1], we define the CEU-degree of the similarity which is related to the CEU for the trade values that exist between Korea and some of its important trading partners (such as Korea-USA, Korea-New Zealand, Korea-India, and Korea-Turkey). In particular, we investigate the evaluation of the CEU-degree of similarity which is related with the CEU of trade values $CEU(u(a))$ of a utility u from an act a on S for specified HS product codes for animal product exports between Korea and selected trading partners for years 2010-2013. We note that we include the dates used in our previous studies [4,10].

In particular, we investigate the following applications:

1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. Choquet integral, Choquet expected utility, fuzzy neasure, the degree of similarity.

(1) we calculate CEU-degree of contribution from an economic value perspective, for animal exports with HS product code $i = 1, 2, 3, 4, 5$ between Korea and selected trading partners for years 2010-2013 and

(2) we compare these values with the USA and other trading partners in terms of CEU-degrees (13), (14), and (15) of the similarity which is related to the relationships and characterizations involved in the value of international trade between Korea and each of the four countries analyzed in this study(see[14]).

2. Preliminaries and definitions

Let S be a finite set of states of nature and $F(S)$ be the set of all fuzzy sets $A = \{(s, m_A(s)) \mid s \in S,\ m_A \longrightarrow [0,1] \text{ is a function}\}$. Recall that m_A is called a membership function of A.

Definition 2.1. ([4-7,9,10,11,13])

(1) A real-valued function μ on S the subsets of is called a fuzzy measure if it satisfies

$$\begin{array}{ll} \text{(i)} & \mu(\emptyset) = 0,\ \mu(S) = 1, \\ \text{(ii)} & A \subset B \Rightarrow \mu(A) \leq \mu(B). \end{array} \tag{1}$$

(2) The Choquet integrals with respect to a fuzzy measure μ of $A \in F(S)$ is defined by

$$(C)\int f_A d\mu = \int_0^1 \mu(\{s \in S | f_A(s) \geq \alpha\}) d\alpha, \tag{2}$$

Definition 2.2. ([4-7,9,10,11,13]) (1) Let $A \in F(S)$. The Choquet integrals with respect to a fuzzy measure μ of a fuzzy set $A = (S, f_A)$ is defined by

where the integral on the right-hand side is an ordinary one.

(2) Let $S = \{s_1, s_2, \cdots, s_n\}$ be a finite set. The discrete Choquet integral with respect to a fuzzy measure μ is defined by

$$(C)\int m_A d\mu = \sum_{i=1}^{n} f_A(s^{(i)}) \left[\mu(E^{(i)}) - \mu(E^{(i+1)})\right], \tag{3}$$

where $E^{(i)} = \{s \in S | m_A(s) \geq m_A(s^i)\}$ for $i = 1, 2, \cdots, n$. By convention, let $E^{n+1} = \emptyset$.

By using the Choquet integral, we consider the Choquet expected utility(CEU) of a utility u from an act a as follows.

Definition 2.3. ([4]) Let $u : X \longrightarrow [0,1]$ be a utility and a be an act from S to X. The Choquet expected utility(CEU) with respect to a fuzzy measure μ of utility u from act a is defined by

$$CEU(u(a)) = (C)\int u(a(s)) d\mu(s). \tag{4}$$

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We note that if $m_A(s) = u(a(s))$ and $A = (S, m_A)$, then $A \in F(S)$, that is, A is a fuzzy set. From Definition 2.1(3) and Definition 2.2 with a finite set S, we get $CEU(u(a))$ as follows:

$$CEU(u(a)) = \sum_{i=1}^{n} u(a(s^{(i)})) \left[\mu(E^{(i)}) - \mu(E^{(i+1)})\right]. \tag{5}$$

where $E^{(i)} = \{s \in S | u(a(s)) \geq u(a(s^{(i)})\}$ for all $i = 1, 2, \cdots, n$.

3. CEU-fuzzy marks and CEU-degree of similarity

In this section, we consider the CEU of a utility on a set of trade values (in USD) that represent the trading relationship that Korea shares with selected trading partners(i.e. Korea-USA, Korea-New Zealand, Korea-India, and Korea-Turkey). We also examine these respective trading relationships by incorporating a clearly defined set of Harmonized System (HS) product code product categories (i.e. HS Codes $i = 1, 2, 3, 4, 5$) for each individual year that is under review (i.e. $2010, 2011, 2012, 2013$). We note that the product code definitions have been provided by the UN Comtrade's online database and the relevant categories are defined as follows(see[14]):

1. Live animals; animal products.
2. Meat and edible meat offal.
3. Fish and crustaceans, mollusks and other aquatic invertebrates.
4. Dairy produce; birds' eggs; natural honey; edible products of animal origin, not elsewhere specified or included.
5. Products of animal origin, not elsewhere specified or included.

Firstly, we denote that HSPC=HS Product Code, s=Year, $a(s)$=Trade Value, $u(a(s))$=the utility of $a(s)$, $CEU(u, a)$=the Choquet Expected Utility of u from a. By using the trade values in tables A1 A4, we can calculate the Choquet integral of an utility on the set of trade values (in USD) that represent Korea's trading relationship with a particular country for years $2010, 2012, 2012, 2013$. Let $s_1 = 2010, s_2 = 2011, s_3 = 2012, s_4 = 2013$. If we define a fuzzy measure μ on S as follows(see[4]):

$$\begin{aligned} &\mu(E^{(4)}) = \mu_1(\{s^{(4)}\}) = 0.1, \;\; \mu(E^{(3)}) = \mu_1(\{s^{(3)}, s^{(4)}\}) = 0.3, \\ &\mu(E^{(2)}) = \mu_1(\{s^{(2)}, s^{(3)}, s^{(4)}\}) = 0.6, \;\; \mu(E^{(1)}) = \mu_1(\{s^{(4)}, s^{(3)}, s^{(2)}, s^{(1)}\}) = 1, \end{aligned} \tag{6}$$

and if $a(s)$ is the trade value of s and $u(a) = \sqrt{\frac{a}{100141401}}$, then we obtain the following $CEU(u(a))$ as follows:

$$\begin{aligned} CEU(u(a)) &= \sum_{i=1}^{4} u(a(s^{(i)})) \left(\mu(E^{(i)}) - (\mu(E^{(i+1)})\right) \\ &= 0.4u(a(s^{(1)})) + 0.3u(a(s^{(2)})) + 0.2u(a(s^{(3)})) + 0.1u(a(s^{(4)})). \end{aligned} \tag{7}$$

By using (5), we calculate the four tables $A_1 \sim A_4$ as follows(see [4]): By using four tables, we get the four X-fuzzy sets $X : \{1, 2, 3, 4, 5\} \to [0, 1]$ by $X = \{(i, m_X(i)) | i = 1, 2, 3, 4, 5\}$ (i.e., USA-fuzzy set U, NZ-fuzzy set N, IN-fuzzy set I, TR-fuzzy set T) defined by

$$U = \{(1, 0.05664), \; (2, 0.04483), \; (3, 0.93879), \; (4, 0.20821), \; (5, 0.04858)\} \tag{8}$$

$$N = \{(1, 0.00533), \; (2, 0.00000), \; (3, 0.78873), \; (4, 0.15976), \; (5, 0.01557)\} \tag{9}$$

$$I = \{(1, 0.00154),\ (2, 0.00000),\ (3, 0.04570),\ (4, 0.00000),\ (5, 0.00000)\} \tag{10}$$

$$T = \{(1, 0.00264),\ (2, 0.00887),\ (3, 0.00368),\ (4, 0.00470),\ (5, 0.00000)\} \tag{11}$$

Definition 3.1. ([1]) The degree of similarity between X-fuzzy set and Y-fuzzy set is defined by

$$S(X,Y) = \frac{\hat{X}\cdot\hat{Y}}{\max\{\hat{X}\cdot\hat{X}, \hat{Y}\cdot\hat{Y}\}} \tag{12}$$

where

$$\begin{aligned}\hat{X} &=< m_X(1), m_X(2), m_X(3), m_X(4), m_X(5) >,\\ \hat{Y} &=< m_Y(1), m_Y(2), m_Y(3), m_Y(4), m_Y(5) >\end{aligned} \tag{13}$$

are vectors and

$$\begin{aligned}&\hat{X}\cdot\hat{Y}\\ &= m_X(1)\cdot m_Y(1) + m_X(2)\cdot m_Y(2) + m_X(3)\cdot m_Y(3) + m_X(4)\cdot m_Y(4) + m_X(5)\cdot m_Y(5).\end{aligned} \tag{14}$$

By using Definition 3.1, we define the degree of similarity between X-fuzzy set and Y-fuzzy set is called the CEU-degree of similarity as follows.

Definition 3.2. If X and Y are elements of $\{U, N, I, T\}$, then the degree of similarity between X-fuzzy set and Y-fuzzy set is called the CEU-degree of similarity.

From Definition 3.1 and Definition 3.2, we get the CEU-degree of similarity between X-fuzzy set and Y-fuzzy set where X and Y are elements of $\{U, N, I, T\}$.

Example 3.1. (1) From Definition 3.1 and Definition 3.2, we get the CEU-degree of similarity between U-fuzzy set and N-fuzzy set as follows:

$$S(U,N) = \frac{\hat{U}\cdot\hat{N}}{\max\{\hat{U}\cdot\hat{U}, \hat{N}\cdot\hat{N}\}} = \frac{0.7747737841}{\max\{0.932255903, 0.6478891043\}} = 0.83174152. \tag{15}$$

(2) From (8) and (10), we get the CEU-degree of similarity between U-fuzzy set and I-fuzzy set as follows:

$$S(U,I) = \frac{\hat{U}\cdot\hat{I}}{\max\{\hat{U}\cdot\hat{U}, \hat{I}\cdot\hat{I}\}} = \frac{0.0429899286}{\max\{0.932255903, 0.0020908616\}} = 0.0461138712. \tag{16}$$

(3) From (8) and (11), we get the CEU-degree of similarity between U-fuzzy set and T-fuzzy set as follows:

$$S(U,T) = \frac{\hat{U}\cdot\hat{T}}{\max\{\hat{U}\cdot\hat{U}, \hat{T}\cdot\hat{T}\}} = \frac{0.004979328}{\max\{0.932255903, 0.0001219413\}} = 0.0053411601. \tag{17}$$

By using four information with those of the CEU-degrees of similarity (13), (14), and (15), we understand the exact difference of similarity between the USA and each of the other three trading partners. By using the CEU-degrees of similarity between USA and another country, we are able to provide a useful plan to find a more effective method of improving the value of international trade between Korea and each of the four countries analyzed in this study. We provide information that may well be of interest to international business practitioners that want a clearer understanding of the relationship and characterizations related to the value of international trade between Korea and each of the four countries measured.

Table A1: The CEU for animal product exports between Korea and the USA for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU_{(i,USA)}(u(a))$
1	s_1	$286892 = a(s^{(1)})$	0.05352	0.05664
	s_2	$330299 = a(s^{(2)})$	0.05743	
	s_3	$358496 = a(s^{(3)})$	0.05983	
	s_4	$364918 = a(s^{(4)})$	0.06037	
2	s_1	$997539 = a(s^{(4)})$	0.09981	0.04483
	s_2	$376805 = a(s^{(3)})$	0.06034	
	s_3	$30005 = a(s^{(1)})$	0.01731	
	s_4	$272884 = a(s^{(2)})$	0.05220	
3	s_1	$74866073 = a(s^{(1)})$	0.86464	0.93879
	s_2	$95654573 = a(s^{(2)})$	0.97734	
	s_3	$100141401 = a(s^{(4)})$	1.00000	
	s_4	$99871717 = a(s^{(3)})$	0.99865	
4	s_1	$3722326 = a(s^{(1)})$	0.19280	0.20821
	s_2	$4323214 = a(s^{(2)})$	0.20778	
	s_3	$5016833 = a(s^{(4)})$	0.22382	
	s_4	$4910771 = a(s^{(3)})$	0.22145	
5	s_1	$235669 = a(s^{(2)})$	0.04851	0.04858
	s_2	$359747 = a(s^{(3)})$	0.05994	
	s_3	$101795 = a(s^{(1)})$	0.05994	
	s_4	$863858 = a(s^{(4)})$	0.09088	

Remark 3.1. *As demonstrated in (13) (14) and (15) this study compares the similarities that exist between Korea and its respective trading partners. As such, our study details the following information:*

$$\text{Korea} - \text{USA} : \text{Korea} - \text{NZ} : \text{Korea} - \text{India} : \text{Korea} - \text{Turkey} = 1 : 0.832 : 0.046 : 0.005 \tag{18}$$

(2) Given a situation whereby Korea spends 10 *million USD as a means of developing a strong trading relationship between itself and its US trading partner, we are able to also ascertain the level of support that is needed to develop effective trading ties with other countries, for example:*

$$\begin{aligned} &\text{NewZealand} : 8,320,000\text{USD} \\ &\text{India} : 460,000\text{USD} \\ &\text{Turkey}50,000\text{USD}. \end{aligned} \tag{19}$$

Table A2: The CEU for animal product exports between Korea and New Zealand for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU_{(i,NZ)}(u(a))$
1	s_1	$6650 = a(s^{(4)})$	0.00815	0.00533
	s_2	$4497 = a(s^{(3)})$	0.00670	
	s_3	$1589 = a(s^{(1)})$	0.00398	
	s_4	$2779 = a(s^{(2)})$	0.00527	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$70759196 = a(s^{(2)})$	0.84059	0.78873
	s_2	$91263506 = a(s^{(4)})$	0.95464	
	s_3	$70763937 = a(s^{(3)})$	0.84062	
	s_4	$46632301 = a(s^{(1)})$	0.68240	
4	s_1	$165773 = a(s^{(3)})$	0.04069	0.15976
	s_2	$113751 = a(s^{(1)})$	0.03370	
	s_3	$148756 = a(s^{(2)})$	0.03854	
	s_4	$277350 = a(s^{(4)})$	0.05263	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.01557
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$218022 = a(s^{(3)})$	0.04666	
	s_4	$393025 = a(s^{(4)})$	0.00265	

Table A3: the CEU for Animal product expert between Korea and India for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$1050 = a(s^{(3)})$	0.00324	0.00264
	s_2	$1300 = a(s^{(4)})$	0.00360	
	s_3	$450 = a(s^{(1)})$	0.00212	
	s_4	$700 = a(s^{(2)})$	0.00264	
2	s_1	$35432 = a(s^{(3)})$	0.01881	0.00887
	s_2	$50639 = a(s^{(4)})$	0.02249	
	s_3	$2656 = a(s^{(1)})$	0.00515	
	s_4	$8230 = a(s^{(2)})$	0.00907	
3	s_1	$8695 = a(s^{(4)})$	0.009318	0.00368
	s_2	$5247 = a(s^{(3)})$	0.00724	
	s_3	$0 = a(s^{(1)})$	0.00000	
	s_4	$1865 = a(s^{(2)})$	0.00432	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00470
	s_2	$21614 = a(s^{(3)})$	0.01469	
	s_3	$30938 = a(s^{(4)})$	0.01758	
	s_4	$0 = a(s^{(2)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

Table A4: The CEU for animal product exports between Korea and Turkey for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$0 = a(s^{(1)})$	0.00000	0.00154
	s_2	$6900 = a(s^{(4)})$	0.00830	
	s_3	$150 = a(s^{(2)})$	0.00122	
	S_4	$300 = a(s^{(3)})$	0.00173	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$0 = a(s^{(1)})$	0.00000	0.04570
	s_2	$672952 = a(s^{(3)})$	0.08198	
	s_3	$2532837 = a(s^{(4)})$	0.15904	
	s_4	$199874 = a(s^{(2)})$	0.04468	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

4. Conclusions

The Choquet expected utility(see Definition 2.3) is a useful tool which can be used to calculate the evaluation of the contribution of animal exports between Korea and selected trading partners. By using the Choquet expected utility, we obtained Tables $A1 \sim A4$ in [10]. From these Tables $A1 \sim A4$, we gave four X- fuzzy sets (8),(9),(10),(11) which are representations of the evaluation of contribution to animal exports for HP product codes $i = 1, 2, 3, 4, 5$ between Korea and selected trading partners for years 2010-2013.

By using these X-fuzzy sets, we obtained three CEU-degrees (13), (14), and (15) of similarity. From three CEU-degrees (13), (14), and (15) of similarity, we can clearly understand the difference of similarity that exists between the USA and each of three countries measured in the study. By using CEU-degrees of similarity between the USA and a respective trading partner, we are able to provide a more effective method of improving the value of international trade between Korea and its trading partners. We also provide valuable information that can be used to compare the USA and another countries as was the case with the three CEU-degrees (13), (14), and (15) of similarity that is related with the relationship and characterizations of the international trade values that exist between Korea and its respective trading partner.

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The general solution of a mixed cubic-quartic functional equation and the Ulam stability of matrix fuzzy normed spaces

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Abstract In this paper, we consider the following new type cubic-quartic (CQ) functional equation

$$\begin{aligned} f(\lambda x+y)+f(\lambda x-y) &= \frac{\lambda^2+\lambda}{2}[f(x+y)+f(x-y)]+\frac{\lambda^2-\lambda}{2}[f(-x-y)+f(y-x)] \\ &\quad +(\lambda^4+\lambda^3-\lambda^2-\lambda)f(x)+(\lambda^4-\lambda^3-\lambda^2+\lambda)f(-x)+(1-\lambda^2)[f(y)+f(-y)], \end{aligned}$$

where $\lambda \geq 2$ is a fixed integer. We investigate the general solution of the functional equation, and then, using the fixed point method, we prove some stability results for this functional equation in matrix fuzzy normed spaces.

Keywords Ulam stability; Cubic-quartic mapping; Cubic-quartic functional equation; Matrix fuzzy normed spaces.

Mathematics Subject Classification(2010) 39B82; 39B52; 46H25.

1 Introduction

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of reals and complex numbers, respectively. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, and $\mathbb{N}_{m_0}$ denotes the set of all positive integers greater than or equal to a given $m_0 \in \mathbb{N}$.

The study of stability problems for functional equations is related to a question of Ulam [14] concerning the stability of group homomorphisms. Subsequently, the partial result of Ulam's problem was proved by Hyers [8], The solution of Hyers was generalized by Rassias [13] for approximate linear mappings by allowing the Cauchy difference $||f(x+y) - f(x) - f(y)||$ to be controlled by $\epsilon\,(\|x\|^p + \|y\|^p)$. In 1994, a further generalization was obtained by Găvruţa [10], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. This new idea is known as the Hyers-Ulam-Rassias stability of functional equations.

Park [7] considered the following cubic-quartic functional equation

$$f(2x+y)+f(2x-y) = 3f(x+y)+3f(x-y)+f(-x-y)+f(y-x)+18f(x)+6f(-x)-3f(y)-3f(-y), \quad (1.1)$$

and investigated the orthogonally stability of (1.1). Very recently, Song [5] proved Ulam stability of this equation (1.1) in matrix intuitionistic fuzzy normed spaces. For more interesting discussions and generalizations of the original problem of Ulam have been investigated, see for instance [1, 2, 9, 11, 12, 15] and the references therein.

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In the present paper, we introduce a new mixed type cubic and quartic functional equation:

$$\begin{aligned} f(\lambda x+y)+f(\lambda x-y)=&\frac{\lambda^2+\lambda}{2}[f(x+y)+f(x-y)]+\frac{\lambda^2-\lambda}{2}[f(-x-y)+f(y-x)] \\ &+(\lambda^4+\lambda^3-\lambda^2-\lambda)f(x)+(\lambda^4-\lambda^3-\lambda^2+\lambda)f(-x)+(1-\lambda^2)[f(y)+f(-y)], \end{aligned} \tag{1.2}$$

where $\lambda \geq 2$ is a fixed integer. One can see that the functional equation (1.1) is a special case of (1.2) when we take the integer $\lambda = 2$. Every solution of the functional equation (1.2) is said to be a cubic-quartic mapping.

The aim of this paper is to discuss the general solution and then establish the Ulam stability of (1.2). More precisely, we discuss the Ulam stability of (1.2) in matrix fuzzy normed spaces by applying the fixed point method.

2 Preliminaries

In this section, we recall some basic facts concerning fuzzy normed spaces, matrix fuzzy normed spaces and some useful results.

Definition 2.1 ([4]) Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:
(1) $N(x, t) = 0$ for $t \leq 0$; (2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$; (3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$; (4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$; (5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$; (6) $N(x, \cdot)$ is continuous on $\mathbb{R}$ for $x \neq 0$.

In this case (X, N) is called a fuzzy normed vector space.

Definition 2.2 ([4]) Let (X, N) be a fuzzy normed space. A sequence x_n in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1 (t > 0)$. A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $N(x_m - x_n, t) > 1 - \epsilon$ $(m, n \geq n_0)$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

We will use the following notations: $M_{m,n}(X)$ is the set of all $m \times n$ matrices in X; When $m = n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$; $e_j \in M_{1,n}(\mathbb{R})$ denote the row vector whose jth component is 1 and the other components are zero; $E_{ij} \in M_n(\mathbb{R})$ is that (i, j)-component is 1 and the other components are zero; $E_{ij} \otimes x \in M_n(X)$ is that (i, j)-component is x and the other components are zero.

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\|\,\|B\|\,\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{R}), x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

For $x \in M_n(X), y \in M_k(X), x \oplus y := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, we introduce the concept of matrix fuzzy normed spaces.

Let X, Y be vector space. For a given mapping $h : X \to Y$ and a given positive integer n, define $h_n : M_n(X) \to M_n(Y)$ by $h_n([x_{ij}]) := [h(x_{ij})]$ for all $[x_{ij}] \in M_n(X)$.

Definition 2.3 ([6, 15]) Let (X, N) be a fuzzy normed space.

(1) $(X, \{N_n\})$ is called a matrix fuzzy normed space if for each positive integer n, $(M_n(X), N_n)$ is a fuzzy normed space and $N_k(AxB, t) \geq N_n(x, \frac{t}{\|A\|\cdot\|B\|})$ for all $t > 0, A \in M_{k,n}(\mathbb{R}), x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $||A|| \cdot ||B|| \neq 0$.

(2) $(X, \{N_n\})$ is called a matrix fuzzy Banach space if (X, N) is a fuzzy Banach space and $(X, \{N_n\})$ is a matrix fuzzy normed space.

Lemma 2.1 ([6]) *Let $(X, \{N_n\})$ be a matrix fuzzy normed space. Then*

(1) $N_n(E_{kl} \otimes x, t) = N(x, t)$ *for all* $t > 0, x \in X$,

(2) *For all* $[x_{ij}] \in M_n(X)$ *and* $t = \sum_{i,j=1}^{n} t_{ij}$,

$$N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min \{N(x_{ij}, t_{ij}) : i, j = 1, 2, \ldots, n\},$$
$$N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min \left\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\right\}.$$

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.

Theorem 2.1 ([3]) *Let (E, d) be a complete generalized metric space and $J : E \to E$ be a strictly contractive mapping, that is*

$$d(Jx, Jy) \leq Ld(x, y), \forall x, y \in E$$

for some $0 < L < 1$. Then, for each given element $x \in E$, either $d(J^n x, J^{n+1}x) = +\infty, \forall n \geq 0$ or $d(J^n x, J^{n+1}x) < \infty, \forall n \geq n_0$, for some natural number n_0. Moreover, if the second alternative holds, then

(1) *The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*

(2) *y^* is the unique fixed point of J in the set $E' = \{y \in E | d(J^{n_0}x, y) < +\infty\}$ and $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in E'$.*

3 General solution of the functional equation (1.2)

In this section, we investigate the general solution of the mixed cubic-quartic functional equation (1.2). Throughout this section, let X be a vector space over $\mathbb{Q}$, Y be a vector space, and $\lambda \in \mathbb{N}_2$. Some basic facts on n-additive symmetric mappings can be found in [12].

Lemma 3.1 *If an odd mapping $f : X \to Y$ satisfies (1.2), then f is of the form $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of the 3-additive symmetric map $A_3 : X^3 \to Y$.*

Proof. Using the oddness of f, we have $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. (1.2) with $y = 0$ yields

$$f(\lambda x) = \lambda^3 f(x). \tag{3.1}$$

Applying (3.1) to (1.2), we obtain

$$f(\lambda x + y) + f(\lambda x - y) = \lambda[f(x + y) + f(x - y)] + 2\lambda(\lambda^2 - 1)f(x), \tag{3.2}$$

From (3.2), by Theorems 3.4 and 3.5 in [12], f is a generalized polynomial function of degree at most 3:

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x), \tag{3.3}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and A^i is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3$. By $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and $A^2(x) = 0$ for all $x \in X$. By $f(\lambda x) = \lambda^3 f(x)$ and $A^i(rx) = r^i A^i(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain $A^1(x) = 0$ for all $x \in X$. Therefore, $f(x) = A^3(x)$ for all $x \in X$. □

Lemma 3.2 *If an even mapping $f : X \to Y$ satisfies (1.2), then f is of the form $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of the 4-additive symmetric map $A_4 : X^4 \to Y$.*

Proof. In view of the evenness of f, we have $f(-x) = f(x)$ for all $x \in X$. Let $y = 0$ in (1.2), we obtain

$$f(\lambda x) = \lambda^4 f(x). \tag{3.4}$$

The rest of the proof is similar to the proof of Lemma 3.1.

□

Theorem 3.1 *A mapping $f : X \to Y$ satisfies (1.2) for all $x, y \in X$ if and only if f is the form*

$$f(x) = A^4(x) + A^3(x), \tag{3.5}$$

where A^i is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 3, 4$.

Proof. Assume that f satisfies the functional equation (1.2), we decompose f into the odd part and the even part by putting

$$f_o(x) = \frac{f(x) - f(-x)}{2}, f_e(x) = \frac{f(x) + f(-x)}{2}, \tag{3.6}$$

then, $f(x) = f_o(x) + f_e(x)$ for all $x \in X$. It is easy to show that the mapping f_o and f_e satisfy (1.2). Therefore our assertion follows immediately from Lemmas 3.1 and 3.2. Conversely, assume that $f(x) = A^4(x) + A^3(x)$ for all $x \in X$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 3, 4$. Using

$$\begin{aligned} &A^4(x+y) + A^4(x-y) = 2A^4(x) + 2A^4(y) + 12A^{2,2}(x,y), \\ &A^3(x+y) + A^3(x-y) = 2A^3(x) + 6A^{1,2}(x,y), \\ &A^i(rx) = r^i A^i(x), i \in \{3,4\}, r \in \mathbb{Q}, \\ &A^{i,j}(rx, sy) = r^i s^j A^{i,j}(x,y), i \in \{1,2\}, r, s \in \mathbb{Q}, \end{aligned} \tag{3.7}$$

by a simple computation, one can see that f satisfies (1.2), which complete the proof of Theorem 3.1. □

4 Stability of the functional equation (1.2)

Throughout this section, let $(X, \{N_n\})$ be a matrix fuzzy normed space, $(Y, \{N_n\})$ be a matrix fuzzy Banach space, $\lambda \in \mathbb{N}_2$ and $n \in \mathbb{N}$. Using the fixed point method, we prove the Ulam stability of the CQ-functional equation (1.2) in matrix fuzzy normed spaces.

Now before taking up the main subject, for a given mapping $f : X \to Y$, we define the difference operator $Df : X^2 \to Y$, and $Df_n : M_n(X^2) \to M_n(Y)$.

$$\begin{aligned} (Df)(a,b) :=& f(\lambda a + b) + f(\lambda a - b) - \frac{\lambda^2+\lambda}{2}[f(a+b) + f(a-b)] - \frac{\lambda^2-\lambda}{2}[f(-a-b) + f(b-a)] \\ &- (\lambda^4 + \lambda^3 - \lambda^2 - \lambda) f(a) - (\lambda^4 - \lambda^3 - \lambda^2 + \lambda) f(-a) - (1-\lambda^2)[f(b) + f(-b)], \end{aligned}$$

$$\begin{aligned} (Df_n)([x_{ij}],[y_{ij}]) :=& f_n(\lambda[x_{ij}] + [y_{ij}]) + f_n(\lambda[x_{ij}] - [y_{ij}]) - \frac{\lambda^2+\lambda}{2}[f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}])] \\ &- \frac{\lambda^2-\lambda}{2}[f_n(-[x_{ij}] - [y_{ij}]) + f_n([y_{ij}] - [x_{ij}])] - (\lambda^4 + \lambda^3 - \lambda^2 - \lambda) f_n([x_{ij}]) \\ &- (\lambda^4 - \lambda^3 - \lambda^2 + \lambda) f_n(-[x_{ij}]) - (1-\lambda^2)[f_n([y_{ij}]) + f_n(-[y_{ij}])] \end{aligned}$$

for all $a, b \in X, x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 4.1 *Let $\varphi_1 : X^2 \to [0, \infty)$ be a function such that for some real number α with $0 < \alpha < 1$,*

$$\varphi_1\left(\frac{a}{\lambda}, \frac{b}{\lambda}\right) \le \frac{\alpha}{\lambda^4} \varphi_1(a,b), \qquad a, b \in X. \tag{4.1}$$

Suppose that $f : X \to Y$ is an even function with $f(0) = 0$ and such that

$$N_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\varphi_1(x_{ij},y_{ij})}, \qquad t>0, x=[x_{ij}], y=[y_{ij}] \in M_n(X). \tag{4.2}$$

Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N_n\left(f_n([x_{ij}]) - Q_n([x_{ij}]),t\right) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t+\alpha n^2\sum_{i,j=1}^{n}\varphi_1(x_{ij},0)}, \qquad t>0, x=[x_{ij}] \in M_n(X).$$

Proof. When $n = 1$, (4.2) is equivalent to

$$N(Df(a,b),t) \geq \frac{t}{t+\varphi_1(a,b)}, \qquad t>0, a,b \in X. \tag{4.3}$$

Putting $b = 0$ in (4.3), we obtain that

$$N(2f(\lambda a) - 2\lambda^4 f(a),t) \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X. \tag{4.4}$$

Hence

$$N(f(a) - \lambda^4 f(\frac{a}{\lambda}),\frac{t}{2}) \geq \frac{t}{t+\varphi_1(\frac{a}{\lambda},0)}, \qquad t>0, a \in X. \tag{4.5}$$

Using (4.1) we get

$$N(f(a) - \lambda^4 f(\frac{a}{\lambda}),t) \geq \frac{t}{t+\frac{\alpha}{2\lambda^4}\varphi_1(a,0)}, \qquad t>0, a \in X. \tag{4.6}$$

Consider the set $E_1 = \{g : X \to Y, g(0) = 0\}$, and introduce the generalized metric d_1:

$$d_1(g,h) := \inf\left\{\epsilon \in \mathbb{R}_+ : N(g(a) - h(a),\epsilon t) \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X\right\},$$

where, as usual, $\inf\emptyset = +\infty$. It is easy to prove that (E_1, d_1) is a complete generalized metric space.

Now, let us consider the linear mapping $J_1 : E_1 \to E_1$ such that

$$J_1 g(a) = \lambda^4 g(\frac{a}{\lambda}), \qquad g \in E_1, a \in X.$$

It is easy to see that J_1 is a strictly contractive self-mapping of E_1 with the Lipschitz constant $L = \alpha$. Indeed, given $g, h \in E_1$, let $\epsilon \in (0,\infty)$ be an arbitrary constant with $d_1(g,h) = \epsilon$. From the definition of d_1, it follows that

$$N(g(a) - h(a),\epsilon t) \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X.$$

Hence

$$\begin{aligned} N(J_1g(a) - J_1h(a),\alpha\epsilon t) &= N(\lambda^4 g(\frac{a}{\lambda}) - \lambda^4 h(\frac{a}{\lambda}),\alpha\epsilon t) = N(g(\frac{a}{\lambda}) - h(\frac{a}{\lambda}),\frac{\alpha\epsilon t}{\lambda^4}) \\ &\geq \frac{\frac{\alpha}{\lambda^4}t}{\frac{\alpha}{\lambda^4}t+\varphi_1(\frac{a}{\lambda},0)} \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X. \end{aligned}$$

So, $d_1(g,h) = \epsilon$ implies that $d(J_1g, J_1h) \leq \alpha\epsilon$. This means that $d_1(J_1g, J_1h) \leq \alpha d_1(g,h)$ for all $g, h \in E_1$, thus J_1 is a strictly contractive self-mapping, and the Lipschitz constant $L = \alpha$.

It follows from (4.6) that

$$N(f(a) - J_1f(a),t) \geq \frac{t}{t+\frac{\alpha}{2\lambda^4}\varphi_1(a,0)}, \qquad t>0, a \in X,$$

thus we have that $d_1(f, J_1f) \leq \frac{\alpha}{2\lambda^4} < +\infty$.

According to Theorem 2.1, we deduce the existence of a fixed point of J_1, that is, the existence of a mapping $Q : X \to Y$ such that $Q(a) = J_1Q(a) = \lambda^4 Q(\frac{a}{\lambda})$, i.e., $Q(\frac{a}{\lambda}) = \frac{1}{\lambda^4}Q(a)$ for each $a \in X$. Moreover, we have $d_1(J_1^l f, Q) \to 0 (l \to +\infty)$, which implies

$$\lim_{l\to+\infty} N(J_1^l f(a) - Q(a), t) = 1, \qquad t > 0, a \in X. \tag{4.7}$$

Also, $d_1(f, Q) \leq \frac{1}{1-L} d_1(J_1 f, f)$ implies the inequality $d_1(f, Q) \leq \frac{\alpha}{2\lambda^4(1-\alpha)}$, which means that

$$N(f(a) - Q(a), t) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha\varphi_1(a, 0)}, \qquad t > 0, a \in X. \tag{4.8}$$

Replacing a and b by $\frac{a}{\lambda^l}$ and $\frac{b}{\lambda^l}$ in (4.3), respectively, we have

$$N\left(\lambda^{4l} Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), t\right) = N\left(Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), \frac{t}{\lambda^{4l}}\right) \geq \frac{\frac{t}{\lambda^{4l}}}{\frac{t}{\lambda^{4l}} + \varphi_1(\frac{a}{\lambda^l}, \frac{b}{\lambda^l})}, \qquad t > 0, a, b \in X. \tag{4.9}$$

It follows from (4.1) that

$$\varphi_1(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}) \leq \frac{\alpha^l}{\lambda^{4l}}\varphi_1(a, b), \qquad a, b \in X,$$

thus

$$N\left(\lambda^{4l} Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), t\right) \geq \frac{t}{t + \alpha^l \varphi_1(a, b)}, \qquad t > 0, a, b \in X. \tag{4.10}$$

Letting $l \to +\infty$ in (4.10), we obtain

$$N\left(\lambda^{4l} Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), t\right) \to 1, \qquad t > 0, a, b \in X, \tag{4.11}$$

which means

$$N(DQ(a, b), t) = 1, \qquad t > 0, a, b \in X. \tag{4.12}$$

Thus, $DQ(a, b) = 0$ for all $a, b \in X$. By the definition of Q, it is clear that $Q(-a) = Q(a)$ for all $a \in X$. Then by Lemma 3.1, the mapping Q is quartic.

Assume that there exists another quartic function $F : X \to Y$ which satisfies (4.8). Then it is clear that $F(\frac{a}{\lambda}) = \frac{1}{\lambda^4}F(a)$, and while $a = 0$, we have $F(a) = 0$, thus $J_1F(a) = \lambda^4 F(\frac{a}{\lambda}) = F(a)$ for all $a \in X$, i.e., F is a fixed point of J_1. By (4.8) we get

$$N(f(a) - F(a), t) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha\varphi_1(a, 0)}, \qquad t > 0, a \in X.$$

Hence, $d_1(f, F) \leq \frac{\alpha}{2\lambda^4(1-\alpha)}$. So, $F \in E_1^{'} = \{g \in E_1, d_1(f, g) < \infty\}$. By Theorem 2.1, Q is the unique fixed point in E_1, which means that $Q = F$.

By Lemma 2.1 and (4.8), we have

$$\begin{aligned} N(f_n([x_{ij}]) - Q_n([x_{ij}]), t) &\geq \min\left\{N(f(x_{ij}) - Q(x_{ij}), \frac{t}{n^2}) : \quad i, j = 1, 2, ..., n\right\} \\ &\geq \min\left\{\frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha n^2\varphi_1(x_{ij}, 0)} : \quad i, j = 1, 2, ..., n\right\} \\ &\geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha n^2 \sum_{i,j=1}^n \varphi_1(x_{ij}, 0)} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X), t > 0$. This completes the proof. □

Theorem 4.2 *Let $\varphi_2 : X^2 \to [0, \infty)$ be a function such that for some real number α with $0 < \alpha < \lambda$,*

$$\varphi_2(\frac{a}{\lambda}, \frac{b}{\lambda}) \leq \frac{\alpha}{\lambda^4}\varphi_2(a, b), \qquad a, b \in X. \tag{4.13}$$

Suppose that $f : X \to Y$ is an odd function such that

$$N_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\varphi_2(x_{ij},y_{ij})}, \qquad t>0, x=[x_{ij}], y=[y_{ij}]\in M_n(X). \tag{4.14}$$

Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N_n\left(f_n([x_{ij}])-C_n([x_{ij}]),t\right) \geq \frac{2\lambda^3(\lambda-\alpha)t}{2\lambda^3(\lambda-\alpha)t+\alpha n^2\sum_{i,j=1}^{n}\varphi_2(x_{ij},0)}, \qquad t>0, x=[x_{ij}]\in M_n(X).$$

Proof. The proof is similar to the proof of Theorem 4.1.

□

Theorem 4.3 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that for some real number α with $0<\alpha<1$,*

$$\varphi(\frac{a}{\lambda},\frac{b}{\lambda}) \leq \frac{\alpha}{\lambda^4}\varphi(a,b), \qquad a,b\in X. \tag{4.15}$$

Suppose that $f : X \to Y$ is a function such that $f(0)=0$, and for all $x=[x_{ij}], y=[y_{ij}]\in M_n(X)$, satisfying

$$N_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\varphi(x_{ij},y_{ij})}, \qquad t>0. \tag{4.16}$$

Then there exist a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that

$$N_n\left(f_n([x_{ij}])-Q_n([x_{ij}])-C_n([x_{ij}]),t\right) \geq \frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t+\alpha n^2\sum_{i,j=1}^{n}\psi(x_{ij},0)},$$

where $\psi(a,b) := \varphi(a,b)+\varphi(-a,-b)$ for all $a,b\in X$.

Proof. Let $f_e(a)=\frac{1}{2}(f(a)+f(-a))$, it is easy to see that $f_e(0)=0, f_e(-a)=f_e(a)$.

$$\begin{aligned} N(Df_e(a,b),t) &= N(\frac{1}{2}Df(a,b)+\frac{1}{2}Df(-a,-b),t) = N(Df(a,b)+Df(-a,-b),2t) \\ &\geq \min\{N(Df(a,b),t),N(Df(-a,-b),t)\} \geq \frac{t}{t+\psi(a,b)}. \end{aligned}$$

Let $f_o(a)=\frac{1}{2}(f(a)-f(-a))$, we can get $N(Df_o(a,b),t)\geq\frac{t}{t+\psi(a,b)}$. From (4.15), it follows that $\psi(\frac{a}{\lambda},\frac{b}{\lambda}) \leq \frac{\alpha}{\lambda^4}\psi(a,b)$. It is easy to check that all conditions of Theorems 4.1 and 4.2 hold, by the proofs of Theorems 4.1 and 4.2, we know that there exist a quartic mapping $Q : X \to Y$ and a cubic mapping $C : X \to Y$ such that

$$N\left(f_e(a)-Q(a),t\right) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t+\alpha\psi(a,0)}, \qquad t>0, a\in X,$$

and

$$N\left(f_o(a)-C(a),t\right) \geq \frac{2\lambda^3(\lambda-\alpha)t}{2\lambda^3(\lambda-\alpha)t+\alpha\psi(a,0)}, \qquad t>0, a\in X.$$

Therefore

$$\begin{aligned} N\left(f(a)-C(a)-Q(a),t\right) &= N(f_e(a)-Q(a)+f_o(a)-C(a),t) \\ &\geq \min\{N(f_e(a)-Q(a),\frac{t}{2}),N(f_o(a)-C(a),\frac{t}{2})\} \\ &\geq \min\{\frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t+\alpha\psi(a,0)},\frac{\lambda^3(\lambda-\alpha)t}{\lambda^3(\lambda-\alpha)t+\alpha\psi(a,0)}\} \\ &= \frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t+\alpha\psi(a,0)}, \qquad t>0, a\in X. \end{aligned} \tag{4.17}$$

Using a proof method similar to Theorem 3.10 in [11], we can prove the uniqueness of C and Q. By Lemma 2.1 and (4.17), we have

$$\begin{aligned} N\left(f_n([x_{ij}]) - Q_n([x_{ij}] - C_n([x_{ij}]), t\right) &\geq \min\left\{N(f(x_{ij}) - Q(x_{ij}) - C(x_{ij}), \frac{t}{n^2}) : \quad i,j = 1,2,...,n\right\} \\ &\geq \min\left\{\frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t + \alpha n^2 \psi(x_{ij},0)} : \quad i,j = 1,2,...,n\right\} \\ &\geq \frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t + \alpha n^2 \sum_{i,j=1}^{n} \psi(x_{ij},0)} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X), t > 0$. This completes the proof. □

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A High-Accuracy Collocation Method for Solving Mixed Boundary Value Problems on Nonsmooth Boundaries*

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Abstract

By potential theory, the mixed Dirichlet-Neumann boundary value problem for the Laplacian is converted into the boundary integral equations (BIEs) with logarithmic singularity. Then the resulting system of the integral equations is solved by the Sidi-Israeli quadrature method (SIQM) with a Sigmoidal transformation. The convergence of numerical solutions by SIQM is proved based on Anselone's collective compact theory. Furthermore, a convergence estimate of the solution error is presented, which possesses high accuracy order $O\ (h^3_{\max})$, where $h_{\max}$ is the mesh size. Finally, The efficiency of the method is illustrated by examples.

Keyword: Boundary value problem, collective compact theory, singularity, integral equations

1 Introduction

Consider the following mixed Dirichlet-Neumann boundary value problem for the Laplacian

$$\begin{cases} \Delta u = 0, \quad \text{in} \ \ \Omega, \\ u|_{\Gamma_{D_i}} = f_i, \quad i = 1, 2, \cdots, p, \\ \dfrac{\partial u}{\partial n}|_{\Gamma_{N_j}} = g_j, \quad j = 1, 2, \cdots, q, \end{cases} \tag{1.1}$$

where Ω is a simply connected region with the piecewise-smooth boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, and $\Gamma_D = \cup_{i=1}^{p} \Gamma_{D_i}$ and $\Gamma_N = \cup_{j=1}^{q} \Gamma_{N_j}$. Here, f_i and g_j are given on Γ_{D_i} and

*This work is supported by Project (NO. KYTZ201505) Supported by the Scientific Research Foundation of CUIT

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Γ_{N_j} respectively, and $\partial u/\partial n$ denotes the derivative of u with respect to the outward normal vector n.

By the potential theory [17], the solution of Eq. (1.1) can be represented as a single-layer potential of the form

$$u(P) = -\frac{1}{\pi}\int_{\Gamma} \ln|P-Q| z(Q)\mathrm{d}S_Q, \quad P \in \Omega, \tag{1.2}$$

where z is an unknown function called the "the single layer" density. From the jump condition for the normal derivative of the single layer potential at the boundary, we then have the following boundary integral equations (BIEs)

$$\begin{cases} -\dfrac{1}{\pi}\displaystyle\sum_{j=1}^{p}\int_{\Gamma_{D_j}} \ln|P-Q| z_{D_j}(Q)\mathrm{d}S_Q \\ -\dfrac{1}{\pi}\displaystyle\sum_{j=1}^{q}\int_{\Gamma_{N_j}} \ln|P-Q| z_{N_j}(Q)\mathrm{d}S_Q = f_i, \quad P\in\Gamma_{D_i}, \quad i=1,2,\cdots,p, \\ z_{N_i}(P) - \dfrac{1}{\pi}\displaystyle\sum_{j=1}^{p}\int_{\Gamma_{D_j}} \frac{\partial \ln|P-Q|}{\partial n_P} z_{D_j}(Q)\mathrm{d}S_Q \\ -\dfrac{1}{\pi}\displaystyle\sum_{j=1}^{q}\int_{\Gamma_{N_j}} \frac{\partial \ln|P-Q|}{\partial n_P} z_{N_j}(Q)\mathrm{d}S_Q = g_i, \quad P\in\Gamma_{N_i}, \quad i=1,2,\cdots,q, \end{cases} \tag{1.3}$$

where $z_{D_j} := z|_{\Gamma_{D_j}}$ and $z_{N_j} := z|_{\Gamma_{N_j}}$ are sought on Γ_{D_i} and Γ_{N_j}, respectively. Once z_{D_j} and z_{N_j} are solved from the Eq. (1.3), the solution $u(P)$ can be computed by

$$u(P) = -\frac{1}{\pi}\sum_{j=1}^{p}\int_{\Gamma_{D_j}} \ln|P-Q| z_{D_j}(Q)\mathrm{d}S_Q - \frac{1}{\pi}\sum_{j=1}^{q}\int_{\Gamma_{N_j}} \ln|P-Q| z_{N_j}(Q)\mathrm{d}S_Q, \quad P\in\Omega. \tag{1.4}$$

Even for the boundary data f_i and g_i are smooth, the solutions z_{D_j} and z_{N_j} may not be smooth. We denote by $P_i,\ i=0,1$ of the two interface points of the boundary Γ and by β_i with $0<\beta_i<2\pi,\ i=0,1$ the interior angle of Γ at P_i. In fact, from [1, 2] it follows that around P_i we have

$$u(P) = c(\Theta) r^{\pi/(2\beta_i)} + \text{smoother terms}, \quad P\in\Omega, \tag{1.5}$$

where (r,Θ) are the polar coordinates centered at P_i. Then , using (1.2) to define a potential not only in Ω but also in $R^2\backslash\bar{\Omega}$, the single z is the difference between the normal derivatives of u on Γ from inside and outside Γ. Therefore, near $P_i,\ i=0,1$, we get

$$z(P) = c r^{\min\{\pi/(2\beta_i),\ \pi/(4\pi-2\beta_i)\}-1} + \text{smoother terms}, \quad P\in\Omega. \tag{1.6}$$

Hence, z_{D_j} and z_{N_j} have this behavior near the corners P_i. To smooth these irregularities, in the next section we will introduce a smoothing parameterization $\psi_\gamma(t)$, which improves the behavior of the unknown function z by incorporating the Jacobian of the transformation. In fact, the new unknown function will be $z(\psi_\gamma(t))|\psi'_\gamma(t)|$, whose smoothness degree at the corner depends upon a smoothing parameter: the larger its value, the smoother the transformed density. There exist numerical methods for approximately solving mixed value problems on polygonal domains by means of boundary integral equations (see [18, 19]). They are based on the collocation method, and in general no error estimates are available [20]. After that, the proof of asymptotic error estimates for the finite element Galerkin approximation of the boundary integral equations for a mixed Dirichlet-Neumann boundary value problem for the Laplacian in a plane polygonal domain is given in [21]. This was a generalization of [22], where the case of a domain with a smooth boundary was treated. In [6], the trigonometric collocation method which uses a mesh grading transformation and a cosine approximating space is proposed for solving the mixed boundary value problems on domains with curved polygonal boundaries, the complete stability and solvability analysis of the transformed integral equations is given by use of a Mellin transform technique, in which each arc of the polygon has associated with it a periodic Sobolev space. Inspired by the technique developed in [6], A collocation method using Chebyshev polynomial expansions as approximants and the zeros of Chebyshev polynomials as collocation nodes is applied to solved (1.3) [2]. From [5], we know that the Sidi transformation [3] is the important one of "integral" sigmoidal transformations, which can yield fast convergence of the collocation solution by smoothing the singularities of the exact solution. Hence, we apply Sidi-Israeli quadrature method [4] and trapezoidal rule with Sidi transformation [3, 16] to calculate the integrals with weakly singular kernels and continuous kernels in (1.3) respectively.

This paper is organized as follows: in Section 2, the convergence analysis is carried out based on the theory of collectively compact operators [7, 8, 9, 10] for closed curved polygons. in Section 3, a convergence estimate of the solution error is given. Numerical examples are provided to verify the theoretical results in Section 4, and conclusions are made in Section 5.

2 Collocation method for the boundary integral equations

2.1 Discretization for integral operators

In [4], high-accuracy numerical quadrature methods based on the appropriate Euler-Maclaurin expansions of trapezoidal rule approximations are proposed for the singular and weakly singular Fredholm integral equations. These integral equations are used in the solution of planar elliptic boundary value problems such as those that arise in free surface flows, elasticity, potential theory, conformal mapping, etc. Let the functions $G(x,t) = \log|x-t|g(x) + \tilde{g}(x)$ are periodic with period $T = b-a$, and that they are $2m$ times differentiable on $R\backslash\{t+kT\}_{k=-\infty}^{\infty}$. Then the Sidi-Israeli quadrature formula [4] for integrals with kernel $G(x,t)$ can be described by

$$Q_n[G(x,t)] = h\Big\{\sum_{\substack{j=1\\x_j\neq t}}^{n} G(x_j,t)+\tilde{g}(t)+\log(\frac{h}{2\pi})g(t)\Big\},\ h=(b-a)/n,\ x_j=a+jh,\quad (2.1)$$

and

$$\int_a^b G(x,t)dx - Q_n[G(x,t)] = 2\sum_{\mu=1}^{m-1}\frac{\zeta'(-2\mu)}{(2\mu)!}g^{(2\mu)}h^{2\mu+1}+O\ (h^{2m}),\quad \text{as}\quad h\to 0,$$

where $\zeta(z)$ is a Riemann function.

Define the following boundary integral operators on Γ_{D_j} and Γ_{N_j}

$$U_{ij}z_j(x) = -\frac{1}{\pi}\int_{\Gamma_{D_j}} \ln|x-y|z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{D_i},\ i,j=1,\cdots,p,$$

$$M_{ij}z_j(x) = -\frac{1}{\pi}\int_{\Gamma_{N_j}} \frac{\partial\ln|x-y|}{\partial n_x}z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{N_i},\ i,\ j=1,\cdots,q,$$

$$V_{ij}z_j(x) = -\frac{1}{\pi}\int_{\Gamma_{N_j}} \ln|x-y|z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{D_i},\ i=1,\cdots,p,\ j=1,\cdots,q,$$

$$W_{ij}z_j(x) = -\frac{1}{\pi}\int_{\Gamma_{D_j}} \frac{\partial\ln|x-y|}{\partial n_x}z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{N_i},\ i=1,\cdots,q,\ j=1,\cdots,p.$$

Assume that Γ_{D_j} or Γ_{N_j} can be described by the parameter mapping: $x_j(t) = (x_{j1}(t),\ x_{j2}(t)) : [0,1]\to\Gamma_{D_j}(\text{or }\Gamma_{N_j})$ with $|x_j'(t)| = [|x_{j1}'(t)|^2+\ |x_{j2}'(t)|^2]^{1/2} > 0$. In order to degrade the singularities at corners, we apply the Sidi transformation [3, 16] to the parameter mapping, which is defined by

$$\psi_\gamma(t) = \frac{\int_0^t(\sin\pi\tau)^\gamma\mathrm{d}\tau}{\int_0^1(\sin\pi\tau)^\gamma\mathrm{d}\tau} : [0,1]\to[0,1],\quad \gamma\geq 1. \quad (2.2)$$

Define the following "smoothing parameterization"

$$\alpha(t) = \begin{cases}\alpha_i^{(1)}(t) = x_i(\psi_\gamma(t))\in\Gamma_{D_i} & t\in[-1,1],\\ \alpha_i^{(2)}(t) = x_i(\psi_\gamma(t))\in\Gamma_{N_i} & t\in[-1,1],\end{cases} \quad (2.3)$$

Thus, we can rewrite equations (1.3) as a $p\times q$ matrix integral equation system

$$\begin{cases}\sum_{j=1}^{p}\int_0^1 u(t,s)\bar{z}_j^{(1)}(s)\mathrm{d}s+\sum_{j=1}^{q}\int_0^1 v(t,s)\bar{z}_j^{(2)}(s)\mathrm{d}s = f_i(t),\ t\in[0,1],\ i=1,2,\cdots,p,\\ z_i(t)+\sum_{j=1}^{p}\int_0^1 w(t,s)\bar{z}_j^{(1)}(s)\mathrm{d}s+\sum_{j=1}^{q}\int_0^1 m(t,s)\bar{z}_j^{(2)}(s)\mathrm{d}s = g_i(t),\ i=1,2,\cdots,q,\end{cases} \quad (2.4)$$

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where

$$u(t,s) = -\frac{1}{\pi}\ln|\alpha_i^{(1)}(t) - \alpha_j^{(1)}(s)|, \tag{2.5}$$

$$v(t,s) = -\frac{1}{\pi}\ln|\alpha_i^{(1)}(t) - \alpha_j^{(2)}(s)|, \tag{2.6}$$

$$w(t,s) = -\frac{1}{\pi}\frac{\alpha_{i2}^{(2)'}(t)[\alpha_{i1}^{(2)}(t) - \alpha_{j1}^{(1)}(s)] - \alpha_{i1}^{(2)'}(t)[\alpha_{i2}^{(2)}(t) - \alpha_{j2}^{(1)}(s)]}{[\alpha_{i1}^{(2)}(t) - \alpha_{j1}^{(1)}(s)]^2 + [\alpha_{i2}^{(2)}(t) - \alpha_{j2}^{(1)}(s)]^2}, \tag{2.7}$$

$$m(t,s) = \begin{cases} -\frac{1}{\pi}\frac{\alpha_{i2}^{(2)'}(t)[\alpha_{i1}^{(2)}(t)-\alpha_{j1}^{(2)}(s)]-\alpha_{i1}^{(2)'}(t)[\alpha_{i2}^{(2)}(t)-\alpha_{j2}^{(2)}(s)]}{[\alpha_{i1}^{(2)}(t)-\alpha_{j1}^{(2)}(s)]^2+[\alpha_{i2}^{(2)}(t)-\alpha_{j2}^{(2)}(s)]^2}, & \text{as } t \neq s, \\ -\frac{1}{2\pi}\frac{\alpha_{i2}^{(2)''}(t)\alpha_{i1}^{(2)'}(t)-\alpha_{i1}^{(2)''}(t)\alpha_{i2}^{(2)'}(t)}{(\alpha_{i1}^{(2)'}(t))^2+(\alpha_{i2}^{(2)'}(t))^2}, & \text{as } t = s. \end{cases} \tag{2.8}$$

and

$$\begin{aligned} &\bar{z}_j^{(1)}(s) = z_{D_j}(x_j(\psi_\gamma(s)))|x_j'(\psi_\gamma(s))|\psi_\gamma'(s), \\ &\bar{z}_j^{(2)}(s) = z_{N_j}(x_j(\psi_\gamma(s)))|x_j'(\psi_\gamma(s))|\psi_\gamma'(s), \\ &f_i(t) = f(\alpha_i^{(1)}(t)),\ \ g_i(t) = g(\alpha_i^{(2)}(t)),\ \ \alpha_i^{(k)}(t) = (\alpha_{i1}^{(k)}(t), \alpha_{i2}^{(k)}(t)),\ k = 1, 2. \end{aligned}$$

Lemma 2.1. Although $z_{D_j}(s)$ and $z_{N_j}(s)$ have singularities at endpoints $s = 0$ and $s = 1$, $\bar{z}_j^{(1)}(s)$ and $\bar{z}_j^{(2)}(s)$ have no singularities by Sidi transformation at $s = 0$ and $s = 1$.

Proof. Let $d_j = \min\{\pi/(2\beta_j), \pi/(4\pi - 2\beta_j)\} - 1$ in (1.6), then we have $-1/2 \leq d_j < 0$. Suppose that $z_{DN}(s) = s^{d_j}\varphi_j(s)$ near $s = 0$, where $z_{DN} = z_{D_j}$ or z_{N_j}, and the function $\varphi_j(s)$ is differentiable enough on $[0, 1]$ with $\varphi_j(0) \neq 0$. Using Taylor's formula, we can obtain

$$z_{DN}(s) = \sum_{i=0}^{l}\frac{\varphi_j^{(i)}(0)}{i!}s^{i+d_j} + O(s^{l+d_j+1}), \quad \text{as} \quad s \to 0^+. \tag{2.9}$$

From [3], we have

$$\psi_\gamma(s) \sim \sum_{i=0}^{\infty}\epsilon_i s^{\gamma+2i+1}, \quad \psi_\gamma'(s) \sim \sum_{i=0}^{\infty}\delta_i s^{\gamma+2i}, \quad \text{as} \quad s \to 0^+, \quad \epsilon_0,\ \delta_0 > 0. \tag{2.10}$$

By substituting (2.9) and (2.10) into the expression of $\bar{z}_j^{(k)}(s)$, $k = 1, 2$, we have

$$\bar{z}_j^{(k)}(s) = c_1\varphi_j(0)s^{(\gamma+1)d_j+\gamma}(1 + O(s^2)) \ \ \text{as} \ \ s \to 0^+, \tag{2.11}$$

where c_1 is a constant. Also assume that $z_{DN}(s) = (1-s)^{d_j}\varphi_j(s)$ near $s = 1$. Similarly, we have

$$\bar{z}_j^{(k)}(s) = c_2\varphi_j(0)(1 - s)^{(\gamma+1)d_j+\gamma}(1 + O((1 - s)^2)) \ \ \text{as} \ \ s \to 1^-, \tag{2.12}$$

where c_2 is a constant independent of s. By (2.11), (2.12) and $d_j \geq -\frac{1}{2}$, we can obtain $(\gamma+1)(d_j+1)-1 \geq 0$ for $\gamma \geq 1$. The proof is completed. $\square$

Now we can rewrite the Eqs. (2.4) as follows

$$\begin{bmatrix} U & V \\ W & I+M \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{2.13}$$

where

$$f=(f_1,f_2,\cdots,f_p),\ g=(g_1,g_2,\cdots,g_q),$$
$$z^{(1)}=(\bar{z}_1^{(1)},\bar{z}_2^{(1)},\cdots,\bar{z}_p^{(1)})^T,\ z^{(2)}=(\bar{z}_1^{(2)},\bar{z}_2^{(2)},\cdots,\bar{z}_q^{(2)})^T,$$
$$U=[U_{ij}]_{i,j=1}^{p,p},\ V=[V_{ij}]_{i,j=1}^{p,q},\ W=[W_{ij}]_{i,j=1}^{q,p},\ M=[M_{ij}]_{i,j=1}^{q,q}.$$

Let $U=A+B$, where $A=\mathrm{diag}(A_{11},A_{22},\cdots,A_{pp})$ and $B=[B_{ij}]_{i,j=1}^{p,p}$, where

$$A_{ii}\bar{z}_i^{(1)}(t)=\int_0^1 a(t,s)\bar{z}_i^{(1)}(s)\mathrm{d}s,$$

with the kernel

$$a_{ii}(t,s)=-\frac{1}{\pi}\ln|2e^{-1/2}\sin(\pi(t-s))|,$$

and

$$B_{ij}\bar{z}_j^{(1)}(t)=\int_0^1 b(t,s)\bar{z}_j^{(1)}(s)\mathrm{d}s,$$

with the kernel

$$b_{ij}(t,s)=\begin{cases} -\frac{1}{\pi}\ln|\frac{\alpha_i^{(1)}(t)-\alpha_j^{(1)}(s)}{2e^{-1/2}\sin(\pi(t-s))}|, & \text{as} \quad i=j, \\ \\ -\frac{1}{\pi}\ln|\alpha_i^{(1)}(t)-\alpha_j^{(1)}(s)|, & \text{as} \quad i\neq j. \end{cases} \tag{2.14}$$

In the subsequent analysis we will focus on the singularity of the kernels $b_{ij}(t,s)$. Obviously, if $\Gamma_{D_i}\cap\Gamma_{D_j}=\emptyset$, $b_{ij}(t,s)$ are continuous in $[0,1]^2$, and if $\Gamma_{D_i}\cap\Gamma_{D_j}\neq\emptyset$, $b_{ij}(t,s)$ have singularities at the points $(t,s)=(0,1)$ and $(t,s)=(1,0)$. For convenience of analysis, we only discuss the case in which $(t,s)=(1,0)$. Defining the following function

$$\tilde{b}_{ij}(t,s)=b_{ij}(t,s)\sin^{\gamma}(\pi t), \quad \gamma\geq 1, \quad \Gamma_{D_i}\cap\Gamma_{D_j}\neq\emptyset. \tag{2.15}$$

Lemma 2.2. Let $\tilde{b}_{ij}(t,s)$ be defined by (2.15), then $\tilde{b}_{ij}(t,s)$ and $\frac{\partial^k\tilde{b}_{ij}(t,s)}{\partial t^k}$ $(k=1,2)$ are smooth on $[0,1]^2$.

Proof. By the continuity of $\tilde{b}_{ii}(t,s)$ in (2.14) and the boundness of $\sin^{\gamma}(\pi t)$, we can immediately complete the proof for the case $i=j$. Hence, we only consider the case in which $j-i=1$. Let $\Gamma_{D_{i-1}}\cap\Gamma_{D_i}=P_i=(0,0)$ and $\theta_i\in(0,2\pi)$ be the corresponding interior angle. Then we have

$$\ln|\alpha_i^{(1)}(t)-\alpha_{i-1}^{(1)}(s)|=\frac{1}{2}\ln[(|\alpha_i^{(1)}(t)|-|\alpha_{i-1}^{(1)}(s)|)^2+4|\alpha_i^{(1)}(t)||\alpha_{i-1}^{(1)}(s)|\sin^2(\theta_i/2)] \tag{2.16}$$

which shows the kernel $b_{i-1,i}(t,s)$ has a logarithmic singularity at $(t,s) = (1,0)$. Suppose that $a_0(t) = |\alpha_{i-1}^{(1)}(t)|$ and $a_1(s) = |\alpha_i^{(1)}(s)|$, we have $a_0(0) = a_1(0) = 0$. If $\theta_i \in (0,\pi) \cup (\pi, 2\pi)$, then $b_{i-1,i}(1,0) = 0$. If $(t,s) \neq (1,0)$, then we can obtain

$$\tilde{b}_{i-1,i}(t,s) = -\frac{1}{2\pi}\sin^\gamma(\pi t)\ln[a_0^2(t) + a_1^2(s) - 2a_0(t)a_1(s)\cos\theta_{i-1}] \tag{2.17}$$
$$= -\frac{1}{2\pi}\sin^\gamma(\pi t)\ln[a_0^2(t) + a_1^2(s)]$$
$$-\frac{1}{2\pi}\sin^\gamma(\pi t)\ln[1 - 2a_0(t)a_1(s)\cos\theta_{i-1}/(a_0^2(t) + a_1^2(s))]$$
$$= \varpi_1(t,s) + \varpi_2(t,s).$$

Since

$$|2a_0(t)a_1(s)\cos\theta_{i-1}/(a_0^2(t) + a_1^2(s))| \leq |\cos\theta_{i-1}| < 1,$$

the function $\varpi_2(t,s)$ and its second derivative are bounded. Noting that

$$\psi_\gamma^{(k)}(0) = \psi_\gamma^{(k)}(1) = 0, \quad k = 0, \cdots, \gamma,$$

we have

$$a_{\bar{i}}^{(k)}(0) = a_{\bar{i}}^{(k)}(1) = 0, \quad \bar{i} = i-1 \text{ or } i, \quad k = 1, \cdots, \gamma.$$

Let $(t,s) \in [\varepsilon/2, \varepsilon] \times [1-\varepsilon, 1-\varepsilon/2]$ for all $\varepsilon > 0$, we have $|\varpi_1(t,s)| = O(\varepsilon^\gamma |\ln\varepsilon|)$, so $\varpi_1(t,s)$ is also bounded. In addition, we have

$$|\frac{\partial}{\partial t}\varpi_1(t,s)| \leq \frac{1}{2\pi}|\sin^\gamma(\pi t)\frac{2a_0(s)|\alpha_{i-1}^{(1)'}(\psi_\gamma(s))|\psi_\gamma'(s)}{a_0^2(t) + a_1^2(s)}|$$
$$= O(\varepsilon^\gamma)O(\varepsilon^{2\gamma})/O(\varepsilon^{2\gamma}) = O(\varepsilon^\gamma)$$

and

$$|\frac{\partial^2}{\partial t^2}\varpi_1(t,s)| = O(\varepsilon^{\gamma-1}).$$

This shows $\frac{\partial^k \tilde{b}_{ij}(t,s)}{\partial t^k}$ $(k = 0,1,2)$ are also continuous in $[0,1]^2$. At last, if $\theta_{i-1} = \pi$, then

$$\tilde{b}_{i-1,i}(t,s) = -\frac{1}{\pi}\sin^\gamma(\pi t)\ln(a_0(t) + a_1(s)), \tag{2.18}$$

we can use the same method mentioned above to prove $\tilde{b}_{i-1,i}(t,s)$ and its second derivative are bounded. The proof of Lemma 2.2 is completed. □

Let $h_j = 1/n_j$ $(n_j \in N)$ and $t_j = s_j = (j-1/2)h_j$ $(j = 1, ..., n_j)$ be the mesh sizes and nodes respectively. By the trapezoidal or the midpoint rule [11] we construct the Nyström's approximation operator $B_{ij}^{h_j}$ of the integral operator B_{ij}, defined by

$$(B_{ij}^{h_j}\bar{z}_j^{(1)})(t) = h_j\sum_{j=1}^{n_j} b_{ij}(t,s_j)\bar{z}_j^{(1)}(s_j), \quad t \in [0,1], \quad i = 1, ..., p, \tag{2.19}$$

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which has the error bounds [3, 11]

$$(B_{ij}\bar{z}_j^{(1)})(t) - (B_{ij}^{h_j}\bar{z}_j^{(1)})(t) = O\,(h_j^{2\ell}), \quad \text{for} \quad \Gamma_{D_i} \cap \Gamma_{D_j} = \emptyset, \quad \ell \in N, \tag{2.20}$$

and

$$(B_{ij}\bar{z}_j^{(1)})(t) - (B_{ij}^{h_j}\bar{z}_j^{(1)})(t) = O\,(h_j^{\omega}), \quad \text{for} \quad \Gamma_{D_i} \cap \Gamma_{D_j} \in \{P_j\}, \tag{2.21}$$

where (see [3])

$$\omega = \begin{cases} \min\{(\gamma+1)(d_j+1), \gamma+1\}, & \gamma \;\; \text{odd}, \\ \min\{(\gamma+1)(d_j+1), 2(\gamma+1)\}, & \gamma \;\; \text{even}. \end{cases} \tag{2.22}$$

For the logarithmically singular operators A_{ii}, by the Sidi-Israeli quadrature formula, we can also construct the approximate operator $A_{ii}^{h_i}$,

$$\begin{aligned}(A_{ii}^{h_i}\bar{z}_j^{(1)})(t) = &-\frac{h_i}{\pi}\left\{\sum_{\substack{j=1\\ s_j\neq t}}^{n_j} \ln\left|2e^{-1/2}\sin\pi(t-s_j)\right|\bar{z}_j^{(1)}(s_j)\right\} \\ &-\frac{h_i}{\pi}\left\{\ln\left(2\pi e^{-1/2}h_i/(2\pi)\right)\bar{z}_j^{(1)}(t)\right\} \quad (i=1,...,n_i),\end{aligned} \tag{2.23}$$

which has the error bounds [4]

$$(A_{ii}^{h_i}\bar{z}_j^{(1)})(t) - (A_{ii}\bar{z}_j^{(1)})(t) = -\frac{2}{\pi}\sum_{\mu=1}^{2\ell-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[\bar{z}_j^{(1)}]^{(2\mu)}h_i^{2\mu+1} + O(h_i^{2\ell}), \quad t \in \{t_i\},$$

where $\zeta'(t)$ is the derivative of the Riemann zeta function.

By the trapezoidal or the midpoint rule, we can also construct the Nyström's approximation operators $V_{ij}^{h_j}$, $W_{ij}^{h_j}$ and $M_{ij}^{h_j}$ for the continuous operators V_{ij}, W_{ij} and M_{ij}, that is,

$$(\Xi_{ij}^{h_j}\bar{z}_j)(t) = h_j\sum_{j=1}^{n_j}\chi_{ij}(t,s_j)\bar{z}_j(s_j), \quad t \in [0,1], \tag{2.24}$$

which have the error bounds $O(h_j^{2\ell})$ or $O(h_j^{\omega})$. Here, $\Xi_{ij} = V_{ij}, W_{ij}$ or M_{ij}, $\chi_{ij}(t,s) = v_{ij}(t,s), w_{ij}(t,s)$ or $m_{ij}(t,s)$.

Now we write the discrete equations for (2.13) are

$$\begin{bmatrix} U^h & V^h \\ W^h & I^h + M^h \end{bmatrix}\begin{bmatrix} z^{(1)h} \\ z^{(2)h} \end{bmatrix} = \begin{bmatrix} f^h \\ g^h \end{bmatrix}, \tag{2.25}$$

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where

$$\begin{aligned}
&U^h = A^h + B^h,\ A^h = \mathrm{diag}(A_{11}^{h_1}, ..., A_{pp}^{h_p}), \quad A_{ii}^{h_i} = [a(t_i, s_j)]_{i,j=1}^{n_p},\\
&B^h = [B_{ij}^{h_j}]_{i,j=1}^{p}, \quad B_{ij}^{h_j} = [b_{ij}(t_i, s_j)]_{i,j=1}^{n_p,n_p}, \quad V^h = [V_{ij}^{h_j}]_{i,j=1}^{p,q},\\
&V_{ij}^{h_j} = [v_{ij}(t_i, s_j)]_{i,j=1}^{n_p,n_q}, \quad W^h = [W_{ij}^{h_j}]_{i,j=1}^{q,p}, \quad W_{ij}^{h_j} = [w_{ij}(t_i, s_j)]_{i,j=1}^{n_q,n_p},\\
&M^h = [M_{ij}^{h_j}]_{i,j=1}^{p,q}, \quad M_{ij}^{h_j} = [m_{ij}(t_i, s_j)]_{i,j=1}^{n_q,n_q},\\
&z^{(1)h} = (z_1^{(1)h_1}(t_1), ..., z_1^{(1)h_1}(t_{n_1}), ..., z_p^{(1)h_p}(t_1), ..., z_p^{(1)h_p}(t_{n_p}))^T,\\
&z^{(2)h} = (z_1^{(2)h_1}(t_1), ..., z_1^{(2)h_1}(t_{n_1}), ..., z_q^{(2)h_q}(t_1), ..., z_q^{(2)h_q}(t_{n_q}))^T,\\
&f^h = (f_1^{h_1}(t_1), ..., f_1^{h_1}(t_{n_1}), ..., f_p^{h_p}(t_1), ..., f_p^{h_p}(t_{n_p}))^T,\\
&g^h = (g_1^{h_1}(t_1), ..., g_1^{h_1}(t_{n_1}), ..., g_q^{h_q}(t_1), ..., g_q^{h_q}(t_{n_q}))^T.
\end{aligned}$$

Let

$$\begin{bmatrix} U^h & V^h \\ W^h & I^h + M^h \end{bmatrix} = \begin{bmatrix} A^h & 0 \\ 0 & I^h \end{bmatrix} + \begin{bmatrix} B^h & V^h \\ W^h & M^h \end{bmatrix}, \tag{2.26}$$

then (2.25) is equivalent to

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (A^h)^{-1}B^h & (A^h)^{-1}V^h \\ W^h & M^h \end{bmatrix} \right) \begin{bmatrix} z^{(1)h} \\ z^{(2)h} \end{bmatrix} = \begin{bmatrix} (A^h)^{-1}f^h \\ g^h \end{bmatrix}. \tag{2.27}$$

2.2 The collectively compact convergence

For the convenience of the analysis of the existence and convergence of numerical solutions, we first introduce the subspaces and some special operators to be used. Define the subspace $C_0[0,1] = \{v(t) \in C[0,1] : v(t)(\sin^\gamma(\pi t))^{-1} \in C[0,1]\}$ of the space $C[0,1]$ with the norm $||v||^* = \max_{0\le t\le 1} |v(t)(\sin^\gamma(\pi t)))^{-1}|$. Let $S^{h_j} = \mathrm{span}\{e_j(t), j = 1, ..., n_j\} \subset C_0[0,1]$ be a piecewise linear function subspace with the basis nodes $\{t_i\}_{i=1}^{n_j}$, where $e_j(t)$ are the basis functions satisfying $e_j(t_i) = \delta_{ji}$. Also define a prolongation operator $\Pi_1^{h_j} : \Re^{n_j} \to S^{h_j}$ satisfying $\Pi_1^{h_j} v = \sum_{j=1}^{n_j} v_j e_j(t), \forall v = (v_1, ..., v_{n_j}) \in \Re^{n_j}$, and a restricted operator $\Pi_2^{h_j} : C_0[0,1] \to \Re^{n_j}$ satisfying $\Pi_2^{h_j} v = (v(t_1), ..., v(t_{n_j})) \in \Re^{n_j}, \forall v \in C_0[0,1]$.

Lemma 2.3. Let $\Gamma = \Gamma_D \cup \Gamma_N$ satisfy $C_\Gamma \neq 1$, and also let

$$\bar{B}_{ij}^{h_j} = \begin{cases} B_{ij}^{h_j}, & \Gamma_{D_i} = \Gamma_{D_j} \text{ or } \Gamma_{D_i} \cap \Gamma_{D_j} = \emptyset, \\ \tilde{B}_{ij}^{h_j}, & \Gamma_{D_i} \cap \Gamma_{D_j} \in \{P_j\}, \end{cases}$$

where the kernel $\tilde{b}_{ij}(t,s)$ of $\tilde{B}_{ij}$ is defined by (2.15). Then under the transformation (2.2), we have

$$\|(A_{ij})^{-1} \bar{B}_{ij}^{h_j}\|_{2,0} \le M \tag{2.28}$$

and

$$\Pi_1^{h_i} (A_{ii}^{h_i})^{-1} \Pi_2^{h_j} \bar{B}_{ij}^{h_j} \xrightarrow{c.c} (A_{ij})^{-1} B_{ij}, \quad \text{in } C[0,1] \to C[0,1], \tag{2.29}$$

where M is a constant and $\xrightarrow{c.c}$ denotes the collectively compact convergence.

Proof. From [14] and by Lemma 2.2, $b_{ij}(t,s)$ and $\tilde{b}_{ij}(t,s)$ are continuous on $[0,1]^2$, and then we have (2.28). Using the following result [14],

$$\Pi_1^{h_i}(A_{ii}^{h_i})^{-1}\Pi_2^{h_i}A_{ii} \xrightarrow{p} I, \quad \text{in} \quad C^2[0,1] \to C[0,1],$$

where I is the embedding operator and $\xrightarrow{p}$ denotes the pointwisely convergence, and by

$$\begin{aligned}\|\Pi_1^{h_i}(A_{ii}^{h_i})^{-1}\Pi_2^{h_i}\bar{B}_{ij}^{h_j}\|_{0,0} &= \|(\Pi_1^{h_i}(A_{ij}^{h_i})^{-1}\Pi_2^{h_i}A_{ij})((A_{ij})^{-1}\bar{B}_{ij}^{h_j})\|_{0,0}\\ &\le \|\Pi_1^{h_i}(A_{ij}^{h_i})^{-1}\Pi_2^{h_i}A_{ij}\|_{0,2}\|(A_{ij})^{-1}\bar{B}_{ij}^{h_j}\|_{2,0}\\ &\le C,\end{aligned}$$

where C is a constant. Thus, we complete the proof of Lemma 2.3. □

Replacing $(A_{ii}^{h_i})^{-1}$, $B_{ij}^{h_j}$ $(i,j{=}1,...,p)$, $V_{ij}^{h_j}$ $(i=1,...,p, j=1,...,q)$, $W_{ij}^{h_j}$ $(i=1,...,q, j=1,...,p)$ and $M_{ij}^{h_j}$ $(i,j=1,...,q)$ by $\Pi_1^{h_i}(A_{ii}^{h_i})^{-1}\Pi_2^{h_i}$, $\Pi_1^{h_i}B_{ij}^{h_j}\Pi_2^{h_j}$, $\Pi_1^{h_i}V_{ij}^{h_j}\Pi_2^{h_j}$, $\Pi_1^{h_i}W_{ij}^{h_j}\Pi_2^{h_j}$, and $\Pi_1^{h_i}M_{ij}^{h_j}\Pi_2^{h_j}$, respectively.

Define the following operators

$$(\hat{A}^h)^{-1}\hat{B}^h,\ \hat{W}^h : (C_0[0,1])^p \to \cup_{j=1}^p S^{h_j}, \quad (\hat{A}^h)^{-1}\hat{V}^h,\ \hat{M}^h : (C[0,1])^q \to \cup_{j=1}^q S^{h_j},$$

where

$$\begin{array}{ll}(\hat{A}^h)^{-1}\hat{B}^h = \Pi_{11}^h(A^h)^{-1}\Pi_{21}^h B^h, & (\hat{A}^h)^{-1}\hat{V}^h = \Pi_{11}^h(A^h)^{-1}\Pi_{22}^h V^h,\\ \hat{W}^h = \Pi_{11}^h(A^h)^{-1}\Pi_{21}^h W^h, & \hat{M}^h = \Pi_{12}^h(A^h)^{-1}\Pi_{22}^h M^h,\\ \Pi_{11}^h = \mathrm{diag}(\Pi_1^{h_1},...,\Pi_1^{h_p}), & \Pi_{12}^h = \mathrm{diag}(\Pi_1^{h_1},...,\Pi_1^{h_q}),\\ \Pi_{21}^h = \mathrm{diag}(\Pi_2^{h_1},...,\Pi_2^{h_p}), & \Pi_{22}^h = \mathrm{diag}(\Pi_2^{h_1},...,\Pi_2^{h_q}).\end{array}$$

Hence, we can write (2.27) as the following operator equation

$$\left(\begin{bmatrix} E_1^h & 0\\ 0 & E_2^h\end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1}\hat{B}^h & (\hat{A}^h)^{-1}\hat{V}^h\\ \hat{W}^h & \hat{M}^h\end{bmatrix}\right)\begin{bmatrix}\hat{z}^{(1)h}\\ \hat{z}^{(2)h}\end{bmatrix} = \begin{bmatrix}(\hat{A}^h)^{-1}\hat{f}^{(1)h}\\ \hat{f}^{(2)h}\end{bmatrix}.$$

Theorem 2.4. (see [12, 15]) Assume $\Gamma_D = \cup_{j=1}^p \Gamma_{D_j}$ satisfy $C_{\Gamma_D} \neq 1$, and Γ_{D_j} $(j = 1,\ ...,\ p)$ are smooth curves. Then the operator sequence$\{(\hat{A}^h)^{-1}\hat{B}^h\}$ is collectively compact convergent to $A^{-1}B$ in $V = (C_0[0,1])^p$. That is, we have

$$(\hat{A}^h)^{-1}\hat{B}^h \xrightarrow{c.c} A^{-1}B. \tag{2.30}$$

Consider the integral

$$Q(g) = \int_\Omega g(x)\mathrm{d}x,$$

where Ω is the bounded domain. Supposed that the quadrature formulae for $Q(g)$ is

$$Q_n(g) = \sum_{j=1}^n \omega_j^{(n)} g(x_j^{(n)}),$$

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where the weights $\omega_j^{(n)}$ satisfy to the following condition

$$\sum_{j=1}^{n} |\omega_j^{(n)}| \leq C, \tag{2.31}$$

where C is a constant.

Theorem 2.5. [8, 13] Assume that the kernel $k(x, y)$ of K is continuous on $\Omega \times \Omega$ and the K_n is the Nyström's approximation operator for K, and the condition (2.31) holds. Then we have

$$K_n \overset{c.c}{\to} K$$

By the Theorem 2.5, we can immediately obtain the following theorem.

Theorem 2.6. Let $\Gamma = \Gamma_D \cup \Gamma_N$ satisfy $C_\Gamma \neq 1$, Γ_{D_j} $(j = 1, ..., p)$ and Γ_{N_j} $(j = 1, ..., q)$are smooth curves, then we have

$$(\hat{A}^h)^{-1}\hat{V}^h \overset{c.c}{\to} A^{-1}V, \quad \hat{W}^h \overset{c.c}{\to} W, \quad \hat{M}^h \overset{c.c}{\to} M. \tag{2.32}$$

3 Errors analysis

In this section, we give the following theorem, which provides a convergence estimate of the solution error.

Theorem 3.1. Assume $\Gamma = \Gamma_D \cup \Gamma_N$ satisfy $C_\Gamma \neq 1$, $f_j = f|_{\Gamma_{D_j}} \in C^6(\Gamma_{D_j})$ and $g_j = g|_{\Gamma_{N_j}} \in C^6(\Gamma_{N_j})$, then when we choose an appropriate number γ in (2.22) such that $\omega > 3$, the following estimate hold

$$\|\hat{z}^{(1)h} - z^{(1)}\|_\infty = O(h_{\max}^3), \quad \|\hat{z}^{(2)h} - z^{(2)}\|_\infty = O(h_{\max}^3) \tag{3.1}$$

where $h_{\max} = \max_{1 \leq j \leq \max\{p,q\}} h_j$.

Proof. By the trapezoidal rule, the asymptotic expansion holds

$$\begin{bmatrix} \hat{U}^h & \hat{V}^h \\ \hat{W}^h & I^h + \hat{M}^h \end{bmatrix} \left(\begin{bmatrix} \hat{z}^{(1)h} \\ \hat{z}^{(2)h} \end{bmatrix} - \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix} \right)$$

$$= \begin{bmatrix} \Pi_{11}^h U \Pi_{21}^h & \Pi_{12}^h V \Pi_{22}^h \\ \Pi_{11}^h W \Pi_{21}^h & \Pi_{21}^h I \Pi_{22}^h + \Pi_{21}^h M \Pi_{22}^h \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix} - \begin{bmatrix} U^h & V^h \\ W^h & I^h + M^h \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} \Pi_{11}^h \Pi_{21}^h \Sigma_1 \psi_1 + \Pi_{12}^h \Pi_{22}^h \Sigma_2 \psi_2 \\ \Pi_{11}^h \Pi_{21}^h \Sigma_1 \psi_3 + \Pi_{12}^h \Pi_{22}^h \Sigma_2 \psi_4 \end{bmatrix} + O(h^{3.5}) \cdot I_{p+q}$$

where $\Sigma_1 = \text{diag}(h_1^3, \cdots, h_p^3)$, $\Sigma_2 = \text{diag}(h_1^3, \cdots, h_q^3)$, $\psi_1 = (\psi_{11}, \psi_{12}, \cdots, \psi_{1p})^T$, $\psi_2 = (\psi_{21}, \psi_{22}, \cdots, \psi_{2q})^T$, $\psi_3 = (\psi_{31}, \psi_{32}, \cdots, \psi_{3p})^T$ and $\psi_4 = (\psi_{41}, \psi_{42}, \cdots, \psi_{4q})^T$. Hence, we have

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1}\hat{B}^h & (\hat{A}^h)^{-1}\hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right) \begin{bmatrix} \hat{z}^{(1)h} - z^{(1)} \\ \hat{z}^{(2)h} - z^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} \Pi_{11}^h \Pi_{21}^h (\hat{A}^h)^{-1}\Sigma_1\psi_1 + \Pi_{12}^h\Pi_{22}^h(\hat{A}^h)^{-1}\Sigma_2\psi_2 \\ \Pi_{11}^h\Pi_{21}^h\Sigma_1\psi_3 + \Pi_{12}^h\Pi_{22}^h\Sigma_2\psi_4 \end{bmatrix} + o(h^5)\cdot I_{p+q} \qquad (3.2)$$

Define the auxiliary equation

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (A^h)^{-1}B^h & (A^h)^{-1}V^h \\ W^h & M^h \end{bmatrix} \right) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} A^{-1}\psi_1 + A^{-1}\psi_2 \\ \psi_3 + \psi_4 \end{bmatrix} \qquad (3.3)$$

and its approximate equation

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1}\hat{B}^h & (\hat{A}^h)^{-1}\hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right) \begin{bmatrix} \Phi_1^h \\ \Phi_2^h \end{bmatrix}$$
$$= \begin{bmatrix} \Pi_{11}^h \Pi_{21}^h (\hat{A}^h)^{-1}\Sigma_1\psi_1 + \Pi_{12}^h\Pi_{22}^h(\hat{A}^h)^{-1}\Sigma_2\psi_2 \\ \Pi_{11}^h\Pi_{21}^h\Sigma_1\psi_3 + \Pi_{21}^h\Pi_{22}^h\Sigma_2\psi_4 \end{bmatrix} \qquad (3.4)$$

Substituting (3.4) into (3.2), we can obtain

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1}\hat{B}^h & (\hat{A}^h)^{-1}\hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right)$$
$$\times \left(\begin{bmatrix} \hat{z}^{(1)h} - z^{(1)} \\ \hat{z}^{(2)h} - z^{(2)} \end{bmatrix} - \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{bmatrix} \begin{bmatrix} \Phi_1^h \\ \Phi_2^h \end{bmatrix} \right) = o(h^5). \qquad (3.5)$$

Since

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1}\hat{B}^h & (\hat{A}^h)^{-1}\hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right)^{-1}$$

is bounded. we have

$$\begin{bmatrix} \hat{z}^{(1)h} - z^{(1)} \\ \hat{z}^{(2)h} - z^{(2)} \end{bmatrix} - \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_1^h \\ \tilde{\Phi}_2^h \end{bmatrix} = o(h^5)$$

that is

$$\|\hat{z}^{(1)h} - z^{(1)}\|_\infty = O(h_{\max}^3), \quad \|\hat{z}^{(2)h} - z^{(2)}\|_\infty = O(h_{\max}^3) \qquad (3.6)$$

where $h_{\max} = \max_{1\le j\le \max\{p,q\}} h_j$. □

4 Numerical experiments

In this section, we will test the SIQM proposed in this paper for the numerical solution of the mixed problem (1.1) via the boundary integral equations (1.3).

Let $err_n^u(P) = \left|u(P) - u_n(P)\right|$ be the errors by SIQM using n boundary nodes, and let $EOC = \log(err_n/err_{2n})/\log 2$ be the estimated order of convergence.

Example 1. [2] Ω is a domain with a re-entrant corner, enclosed by the curve:

$$\Gamma: \quad (-\frac{1}{2}\sin(\frac{3\pi}{2}x),\ -\sin(\pi x)), \quad 0 \le x \le 2,$$

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and the Dirichlet and Neumman arcs Γ_D and Γ_N are parameterized by the interval $[0,1]$ and $[1,2]$, respectively. Setting $u(x_1,x_2) = x_1^2 - x_2^2$. Because u is the real part of an analytic function, u satisfies the Laplace equation in Ω. Let $u = \bar{f}_1$ on Γ_D and $\partial u/\partial n = \bar{f}_2$ on Γ_N. Let each boundary be divided into 2^k $(k = 3, \cdots, 8)$ segments. The errors and error ratio of the interior points $P_1 = (0.4,\ 0)$, $P_2 = (0,\ 0.6)$ and $P_3 = (0.1,\ 0.5)$ using n $(= 2 \times 2^k,\ k = 3, \cdots, 8)$ nodes by transformation $\psi_6(t)$ are listed in Table 1. In addition, the numerical solution u of the interior points along the line $x_2 = x_1 - 0.4$ are computed, where $x_1 = -0.4 : 0.01 : 0.4$. The plots of computed errors are shown in Figure 1 (b) to Figure 2.

Table 1: The Errors of u.

n	2×2^3	2×2^4	2×2^5	2×2^6	2×2^7	2×2^8
$err_n^u(P_1)$	5.007-04	3.872-03	2.040-04	2.581-05	3.222-06	4.026-07
$EOC(P_1)$	–	−2.951	4.246	2.982	3.002	3.000
$err_n^u(P_2)$	2.973-01	5.533-02	1.737-03	1.878-05	2.008-06	2.509-07
$EOC(P_2)$	–	2.426	4.993	6.531	3.226	3.000
$err_n^u(P_3)$	2.414-01	1.931-02	4.130-04	1.687-05	2.035-06	2.543-07
$EOC(P_3)$	–	3.644	5.547	4.613	3.052	3.000

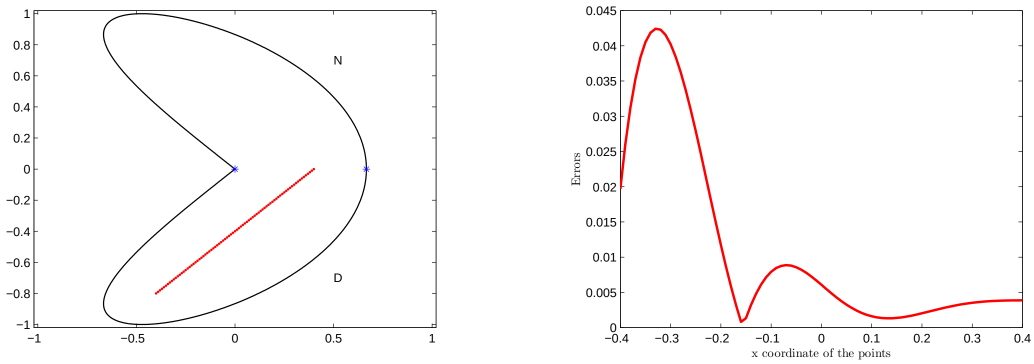

Figure 1: *Left: The contour* Γ *for Example 1; Right: Errors of* u *by* 2×2^4 *boundary nodes.*

Example 2. Consider the following problem where the domain is a quarter-circle.

$$\Delta u = 0 \quad \text{for } x_1 > 0,\ x_2 > 0, \text{ and } x_1^2 + x_2^2 < 1,$$

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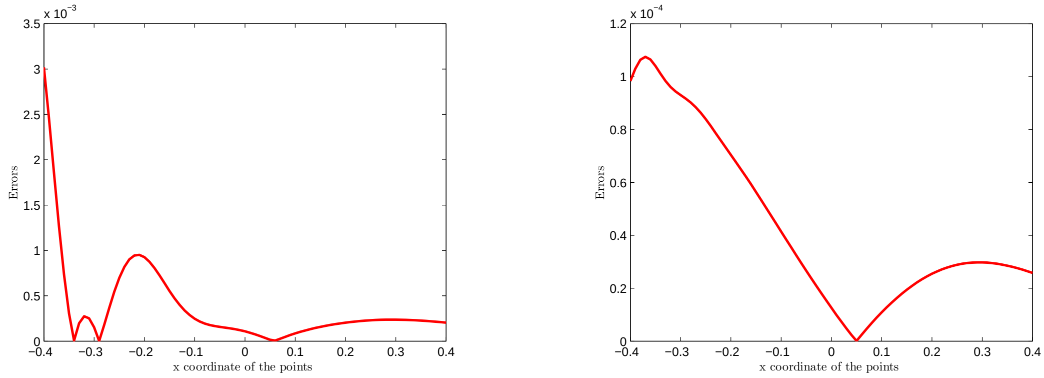

Figure 2: *Left: Errors of* u *by* 2×2^5 *boundary nodes; Right: Errors of* u *by* 2×2^6 *boundary nodes.*

subject to the boundary conditions

$$\Gamma_{D_1}:\ u=1,\quad \text{on } x_1^2+x_2^2=1,\quad \text{for } x_1>0,\ x_2>0;$$
$$\Gamma_{D_2}:\ u=0,\quad x_2=0;$$
$$\Gamma_{N_1}:\ \frac{\partial u}{\partial n}=0,\quad x_1=0.$$

The analytical solution of this problem is $u=\frac{2}{\pi}\mathrm{arctan}(\frac{2x_2}{1-x_1^2-x_2^2})$.

Let each boundary be divided into 2^k $(k=3,\cdots,8)$ segments. The errors and error ratio of the interior points $P_1=(0.1,\ 0.1)$, $P_2=(0.8,\ 0.1)$ and $P_3=(0.1,\ 0.7)$ using n $(=3\times 2^k,\ k=3,\cdots,8)$ nodes by transformation $\psi_6(t)$ are listed in Table 2. In addition, the numerical solution u of the interior points along the curve segment $L:\ x_1=0.7\mathrm{cos}(\frac{\pi}{2}t),\ x_2=0.7\mathrm{sin}(\frac{\pi}{2}t)$ are computed, where $t=0.05:0.01:0.95$. The plots of computed errors are shown in Figure 3 (b) to Figure 4. From the numerical results of Table 1 and Table 2 we can see that $EOC\approx 3$.

Table 2: The Errors of u .

n	3×2^3	3×2^4	3×2^5	3×2^6	3×2^7	3×2^8
$err_n^u(P_1)$	1.167-03	1.223-04	1.341-05	1.675-06	2.093-07	2.617-08
$EOC(P_1)$	–	3.255	3.188	3.001	3.000	3.000
$err_n^u(P_2)$	7.409-03	9.133-04	1.691-05	2.421-06	3.025-07	3.781-08
$EOC(P_2)$	–	3.020	5.755	2.805	3.000	3.000
$err_n^u(P_3)$	1.997-02	1.694-03	3.282-05	1.062-06	1.313-07	1.641-08
$EOC(P_3)$	–	3.559	5.689	4.950	3.016	3.000

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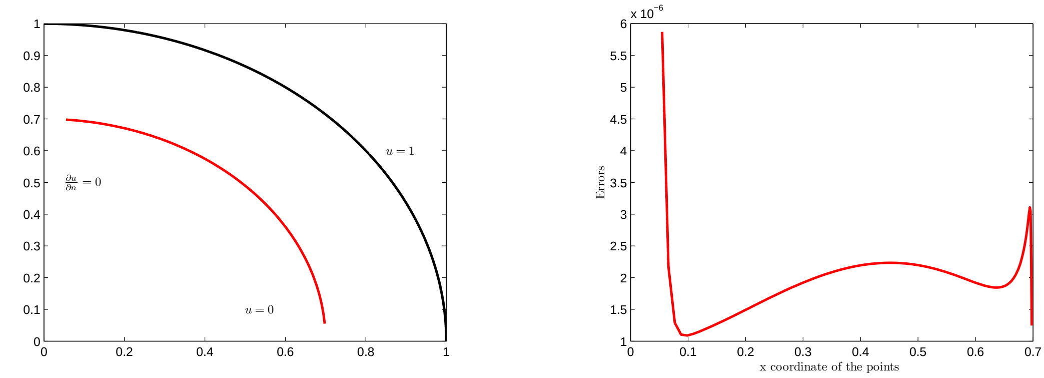

Figure 3: *Left: The contour* Γ *for Example 2; Right: Errors of* u *by* 3×2^6 *boundary nodes.*

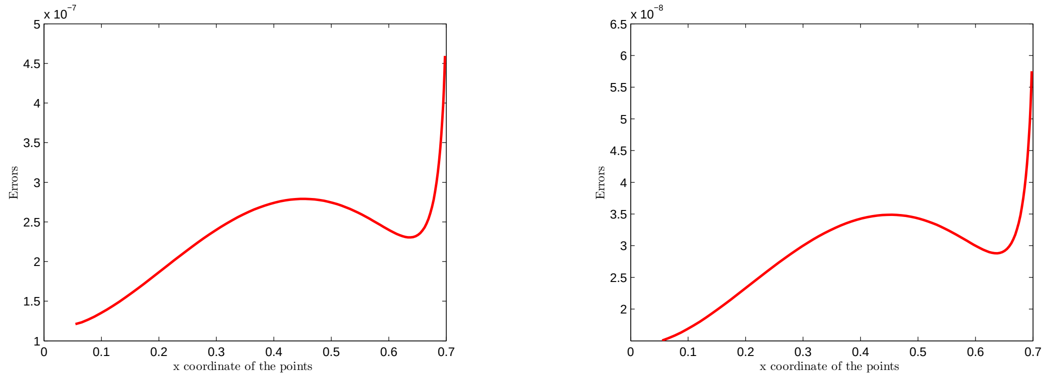

Figure 4: *Left: Errors of* u *by* 3×2^7 *boundary nodes; Right: Errors of* u *by* 3×2^8 *boundary nodes.*

5 Conclusions

In this paper, the convergence and error of SIQM for the boundary integral equations of the mixed Dirichlet-Neumann boundary value problem for the Laplacian are studied on nonsmooth boundaries. Especially, in order to provide a good accuracy in the solution near the singular points, the Sidi transformation is used for the boundary integral equations of problems (1.1). The numerical results show that the presented algorithm has a high accuracy of $O\ (h_{\max}^3)$, which coincides with our theoretical analysis.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Adaptive Modified Function Projective Synchronization of Chaotic Dynamical System with Different Order

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Abstract: This work present the adaptive modified function projective synchronization of two systems with different order, which is a further extension of many existing synchronization schemes, such as function projection synchronization, modified projective synchronization and so on. Based on Lyapunov direct method of stability, an adaptive control is proposed to realize the modified function projective synchronization. Finally, numerical results are provided to illustrate the effectiveness of the obtained result.

1 Introduction

In the last few years, control and synchronization of chaos have generate much interest according to its application in secure communications [2]. Synchronization of chaotic systems means that two or more systems adjust each other to a common dynamical behavior. Up to now, many different kind of synchronization were studied such as: complete and anti synchronization, generalized synchronization, projective synchronization [9]-[39]. Recently, projective synchronization has a lot of attention because it obtain faster communication. Modifief projective synchronization is one of the important projective synchronization methods. It means that the drive and response systems could be synchronized up to constant scaling matrix [28]-[31] .Later, a new projective synchronization method called function projective synchronization where the responses of the synchronized dynamical states synchronize up to a scaling function [32]-[37]. More recently, researcher introduces a new type of synchronization phenomenon, modified function projective synchronization ,where the drive and response systems could be synchronized up to a desired scaling function matrix [38]-[39]. In recent years, most of researches for the synchronization assumed that the drive and response are identical or different systems with the same order. But in the real systems, especially in biology and social systems the synchronization is applied even though the oscillators haven't the same order. Hence, studying the synchronization of two systems with different order plays significant role in application.

The rest of this paper is as the following: The Liu chaotic and hyperchaotic dynamical systems are introduced in Section 2. Section 3 gives the definition of MFPS. In Section 4, an adaptive modified projective synchronization of Liu chaotic and hyperchaotic systems is proposed based on Lyapunov direct method of stability. Section 5 gives the numerical result and the conclusion is obtained in the last Section.

2 The Liu (chaotic and hyperchaotic) systems

The Liu hyperchaotic system is defined by:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx + kxz + ew, \\ \dot{z} = -cz - hx^2 + mw, \\ \dot{w} = -dy, \end{cases} \tag{1}$$

where $x,\ y,\ z$ and w are the state vectors, and $a,\ b,\ c,\ d,\ e,\ k,\ h$ and m are constant parameters.It can be generate a chaotic attractor for the parameters $a = 10,\ b = 40,\ c = 2.5,\ d = 2.5,\ e = 1,\ k = 1,\ h = 4$, and $m = 1$ in Figure 1 and the chaotic motions of Liu system are illustrated in Figure 2.

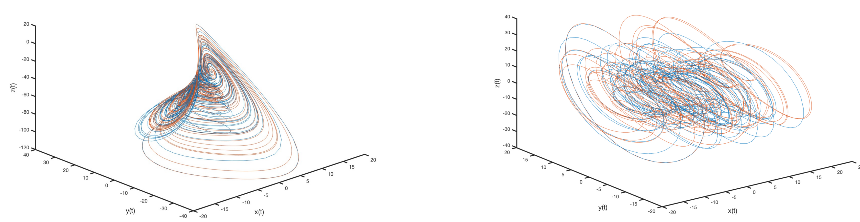

Figure 1: Liu hyperchaotic system at a=10, b=40, c=2.5, d=2.5, e=1, m=1, k=1 and h=4

The Liu chaotic system is given by:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx - kxz, \\ \dot{z} = -cz + hx^2, \end{cases} \tag{2}$$

where $x,\ y$, and z are the state vectors, and the parameters a, b, c, h and k are positive real constants. A chaotic attractor for the parameters $a = 10,\ b = 40,\ c = 2.5,\ k = 1$ and $h = 4$ is shown in Figure 3, and the system states responses in time domain are shown in Figure 4.

3 The modified function projective synchronization scheme

We define the drive and the response systems as follows:

$$\begin{aligned} \dot{x} &= \chi(x), \\ \dot{y} &= \Psi(y) + U(t, x, y), \end{aligned}$$

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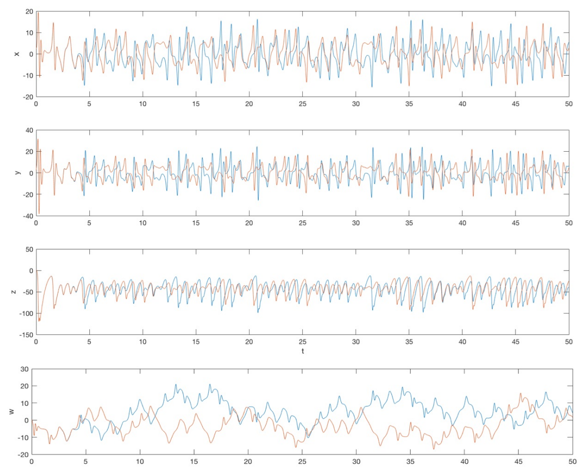

Figure 2: The behavior of the trajectories of the Liu hyper chaotic system

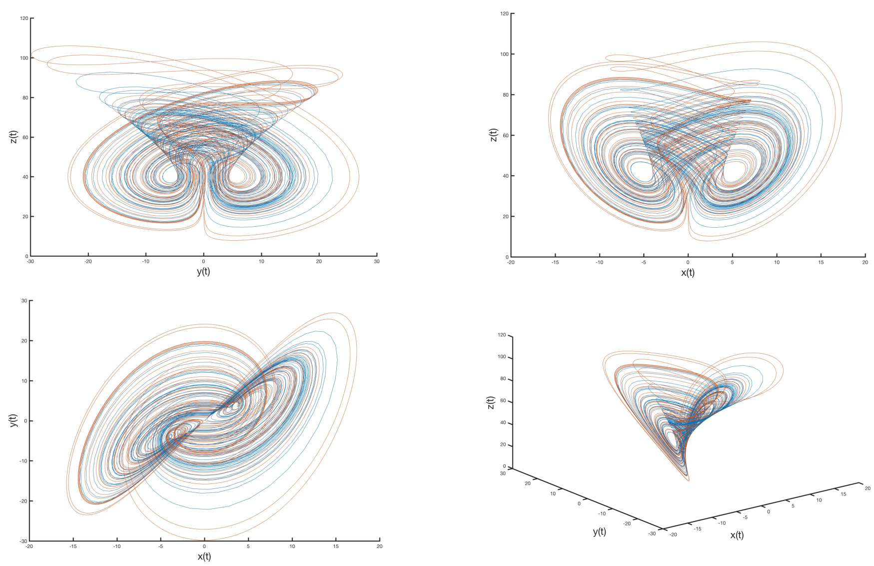

Figure 3: Phase portrait of Liu chaotic system at a=10, b=40, c=2.5, k=1 and h=4.

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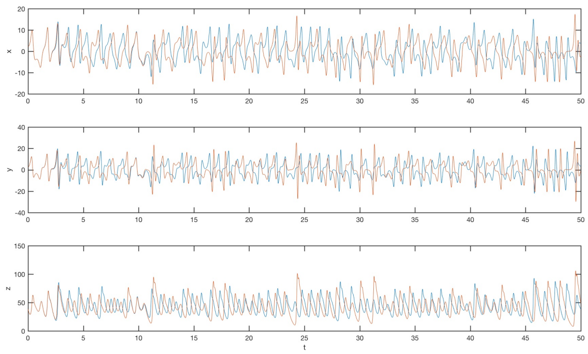

Figure 4: The behavior of the trajectories of the Liu chaotic system.

where x, y are the state variables, $\chi, \Psi : R^n \to R^n$ are continuous nonlinear functions and $U(t, x, y)$ is a control function.

Let the error state be $e = y - \Lambda(t)x$ where $\Lambda(t) = diag\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$ is n-order diagonal matrix where $\beta_i = \eta_{i1}x + \eta_{i2},\ (i = 1, 2, \ldots, n),\ \eta \in R.$

Definition 1. (MFPS)
We say that the drive system and the response system are modified function projective synchronization (MFPS), if there is a scaling function $\Lambda(t)$, such that

$$\lim_{t\to+\infty} \|e\| = 0.$$

4 Modified function projective synchronization between Liu chaotic and hyperchaotic systems

Following the scheme of Zheng in [39], we apply this scheme to achieve the MFPS between Liu chaotic and hyperchaotic systems with different order. The Liu hyperchaotic system is defined below as a drive (or master) system:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1), \\ \dot{y}_1 = bx_1 + kx_1z_1 + ew_1, \\ \dot{z}_1 = -cz_1 - hx_1^2 + mw_1, \\ \dot{w}_1 = -dy_1, \end{cases} \tag{3}$$

where $x_1,\ y_1,\ z_1$ and w_1 are the state vectors. Moreover, the Liu system as the response (or slave) system is given by::

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + u_1, \\ \dot{y}_2 = bx_2 - kx_2z_2 + u_2, \\ \dot{z}_2 = -cz_2 + hx_2^2 + u_3, \end{cases} \tag{4}$$

4

where $x_2,\ y_2$ and z_2 are the state vectors, and $u_i,\ (i=1,2,3)$ are the controller to be determined later.

Since the order of the drive system is greater than the response system, we must increase the order of the response system by structure a state vector. Based on the method in [39], we structure a state variable $w_2=\frac{1}{2}x_2^2$, then the response system become:

$$\begin{cases}\dot{x}_2=a(y_2-x_2)+u_1,\\ \dot{y}_2=bx_2-kx_2z_2+u_2,\\ \dot{z}_2=-cz_2+hx_2^2+u_3,\\ \dot{w}_2=a(y_2-x_2)x_2+u_4.\end{cases}\tag{5}$$

Let the error state vector be expressed by:

$$\begin{cases}e_1=x_2-(\eta_{11}x_1+\eta_{12})x_1,\\ e_2=y_2-(\eta_{21}y_1+\eta_{22})y_1,\\ e_3=z_2-(\eta_{31}z_1+\eta_{32})z_1,\\ e_4=w_2-(\eta_{41}w_1+\eta_{42})w_1,\end{cases}\tag{6}$$

Moreover, the error dynamical system can be described by:

$$\begin{cases}\dot{e}_1=ay_2-ax_2-2\eta_{11}ax_1y_1+2\eta_{11}ax_1^2-\eta_{21}ay_1+\eta_{12}ax_1+u_1,\\ \dot{e}_2=bx_2-kx_2z_2-2\eta_{21}bx_1y_1-2\eta_{21}kx_1y_1z_1-2\eta_{21}ey_1w_1-\eta_{22}bx_1-\eta_{22}kx_1z_1-\eta_{22}ew_1+u_2,\\ \dot{e}_3=-cz_2+hx_2^2+2\eta_{31}cz_1^2+2\eta_{31}hz_1x_1^2-2\eta_{31}mz_1w_1+\eta_{32}cz_1+\eta_{32}hx_1^2-\eta_{32}mw_1+u_3,\\ \dot{e}_4=ay_2x_2-ax_2^2+2\eta_{41}dy_1w_1+\eta_{42}y_1+u_4.\end{cases}\tag{7}$$

Now, the aim is to design the control function $u_i(t),\ (i=1,2,3,4)$ to achieve the MFPS.

Consider the following Lyapunov function:

$$V=\frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2),$$

which is a positive definite function, then the time derivative of the Lyapunov function is given as follows:

$$\dot{V}=e_1\dot{e_1}+e_2\dot{e_2}+e_3\dot{e_3}+e_4\dot{e_4}.$$

Moreover,

$$\begin{aligned}\dot{V}=&\,e_1(ay_2-ax_2-2\eta_{11}ax_1y_1+2\eta_{11}ax_1^2-\eta_{12}ay_1+\eta_{12}ax_1+u_1),\\ &+e_2(bx_2-kx_2z_2-2\eta_{21}bx_1y_1-2\eta_{21}kx_1y_1z_1-2\eta_{21}ey_1w_1-\eta_{22}bx_1-\eta_{22}kx_1z_1-\eta_{22}ew_1+u_2),\\ &+e_3(-cz_2+hx_2^2+2\eta_{31}cz_1^2+2\eta_{31}hz_1x_1^2-2\eta_{31}mz_1w_1+\eta_{32}cz_1+\eta_{32}hx_1^2-\eta_{32}mw_1+u_3),\\ &+e_4(ay_2x_2-ax_2^2+2\eta_{41}dy_1w_1+\eta_{42}y_1+u_4).\end{aligned}\tag{8}$$

Thus, we choose the controller as the following:

$$\begin{cases} u_1 = -ay_2 + 2\eta_{11}ax_1y_1 - \eta_{11}ax_1^2 + \eta_{12}ay_1, \\ u_2 = -bx_2 + kx_2z_2 + 2\eta_{21}bx_1y_1 + 2\eta_{21}kx_1y_1z_1 + 2\eta_{21}ey_1w_1 + \eta_{22}bx_1 + \eta_{22}kx_1z_1 \\ \quad + \eta_{22}ew_1 - by_2 + \eta_{21}by_1^2 + \eta_{22}by_1, \\ u_3 = -hx_2^2 - \eta_{31}cz_1^2 - 2\eta_{31}hz_1x_1^2 + 2\eta_{31}mz_1w_1 - \eta_{32}hx_1^2 + \eta_{32}mw_1, \\ u_4 = -ay_2x_2 + ax_2^2 - 2\eta_{41}dy_1w_1 - \eta_{42}y_1 - dw_2 + d\eta_{41}w_1^2 + d\eta_{42}w_1, \end{cases} \tag{9}$$

by this choice, the time derivative of Lyapunov function is:

$$\begin{aligned} \dot{V} &= e_1(-ax_2 + \eta_{11}ax_1^2 + \eta_{12}ax_1) + e_2(-by_2 + \eta_{21}by_1^2 + \eta_{22}by_1) \\ &+ e_3(-cz_2 + \eta_{31}cz_1^2 + \eta_{32}cz_1) + e_4(-dw_2 + \eta_{41}dw_1^2 + \eta_{42}dw_1), \\ &= -(ae_1^2 + be_2^2 + ce_3^2 + de_4^2), \\ &= -e^T Pe, \end{aligned} \tag{10}$$

where $P = diag[a,\ b,\ c,\ d]$.

Obviously, the origin of the error dynamical system is asymptotically stable since $\dot{V}$ is negative definite. Thus, the drive and the response systems are achieving the MFPS.

5 Numerical results

In this section, we show a numerical simulation to verify the influence of the synchronization controller (9). We assume that the initial states of the drive and the response systems are

$[x_1(0),\ y_1(0),\ z_1(0),\ w_1(0)]^T = [2.4,\ 2.2,\ 0.8,\ 0]^T$ and $[x_2(0),\ y_2(0),\ z_2(0),\ w_2(0)]^T = [0.2,\ 0.1,\ 3,\ 6]^T$.

These numerical simulation are presented in Figure 5. Firstly, when the scaling functions are given by:

$$\beta_1 = 3x_1 + 4,\ \beta_2 = 1.5y_1 + 2,\ \beta_3 = 2z_1 + 4 \text{ and } \beta_4 = w_1 + 7,$$

we get adaptive modified function projective synchronization (MFPS) in Figure 5 (a) . Furthermore, Figure 5 (b) shows the generalized function projective synchronization (GFPS) when the scaling functions are given by $\beta_1 = 3x_1,\ \beta_2 = 1.5y_1,\ \beta_3 = 2z_1$, and $\beta_4 = w_1$. Also, we get the modified projective synchronization (MPS) according to the constants $\beta_1 = 4,\ \beta_2 = 2,\ \beta_3 = 1$, and $\beta_4 = 7$ shown in Figure 5 (c) . The complete synchronization error of the drive and response systems are displayed in Figure 5 (d) when the scaling function is simplified to $\beta_i = +1,\ (i = 1, 2, 3, 4)$ with $\eta_{i1} = 0,\ \eta_{i2} = +1,\ (i = 1, 2, 3, 4)$. Finally, if we choose the scaling function $\beta_i = -1, (i = 1, 2, 3, 4)$ in which $\eta_{i1} = 0$, and $\eta_{i2} = -1,\ (i = 1, 2, 3, 4)$ we gained the anti-phase synchronization between the two systems in Figure 5 (e). From these results, they clearly show that the synchronization errors $e = [e_1, e_2, e_3, e_4]^T$ are converge to zero as time goes to infinity.

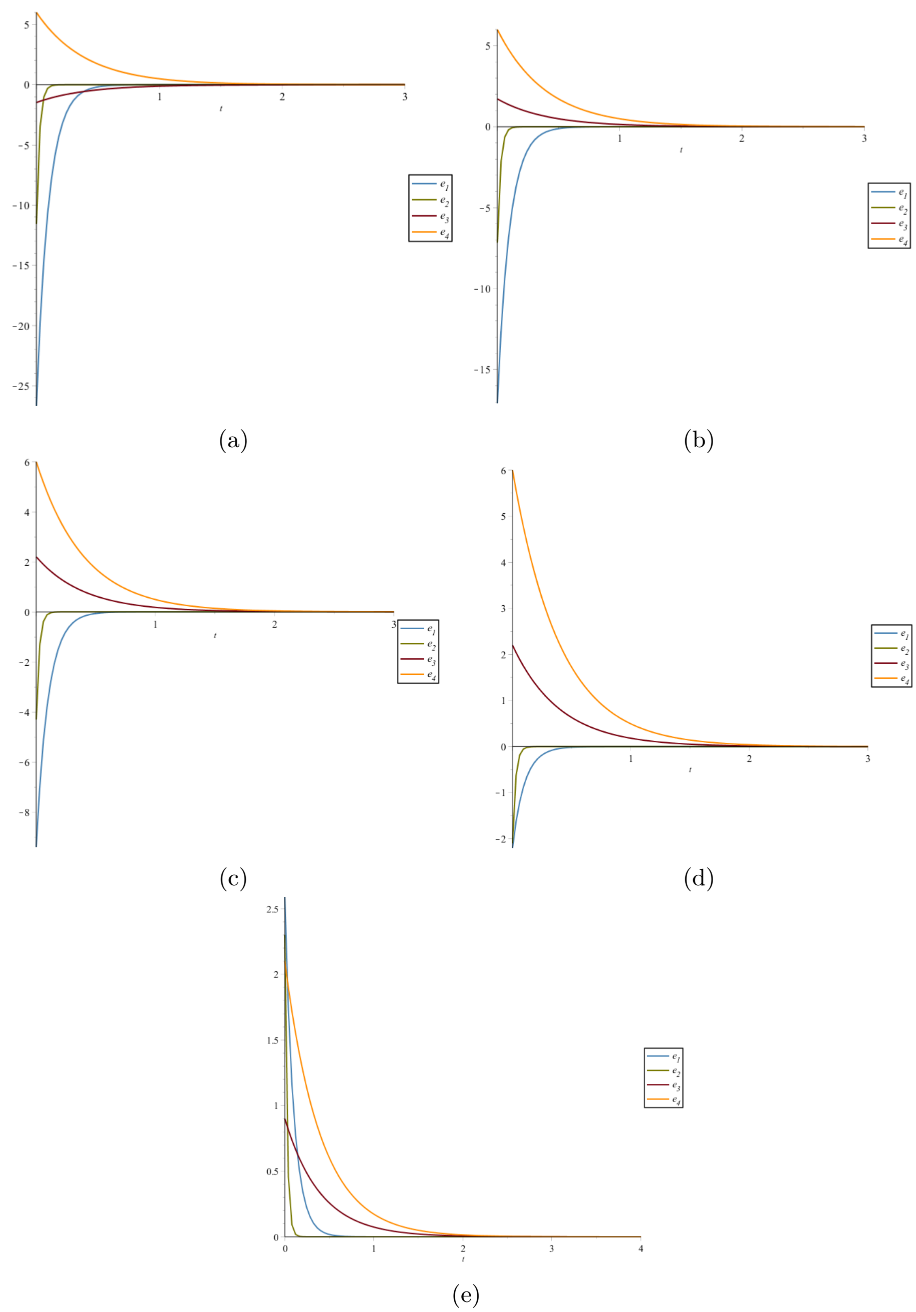

Figure 5: The errors between Liu (chaotic and hyperchaotic) systems for (a) MFPS (b) GFPS (c) MPS (d) Complete synchronization (e) Anti-phase synchronization.

6 Conclusion

In this paper, we have introduced a modified function projective synchronization between two chaotic systems with different dimensional. The Liu chaotic system (third order) and Liu hyperchaotic system (fourth order) are chosen to illustrate the proposed technique. The results show that we can apply the MFPS between the two systems if we increased the order. By using adaptive control method, some conditions are derived for the stability of the error proved according to Lyapunov direct method of stability. Finally, the graphical presentation of the numerical results with error states tending to zero as time becomes large, clearly exhibit that the applied adaptive control method is effective and convenient to achieve global synchronization among non identical chaotic systems with different order.

Acknowledgement

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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Dual log-Minkowski inequality for star bodies

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Abstract

We validate a modified dual log-Minkowski inequality and prove some variants of the dual log-Minkowski inequality for star bodies in $\mathbb{R}^n$ containing the origin in their interior. In addition, we point out that the equivalence between the dual log-Minkowski inequality and the dual log-Brunn-Minkowski inequality.

Keywords: Dual cone-volume measure, L_0-Minkowski problem, dual log-Brunn-Minkowski inequality, dual log-Minkowski inequality.

1 Introduction

The classical Brunn-Minkowski theory of convex bodies was placed in a larger theory by Lutwak's L_p-Minkowski problem [13, 14]. Therefore, many classical results for convex bodies became a part of the extended L_p-Brunn-Minkowski-Firey theory, while many other results of the extended theory bring new and original insight in convex geometric analysis.

One such strikingly new behavior is due to the log-Brunn-Minkowski inequality [2]. That is, let K, L be convex bodies that contain the origin in their interiors and $0 \leq \lambda \leq 1$, the log-Minkowski combination which is defined by

$$(1-\lambda)\cdot K +_0 \lambda \cdot L = \cap_{u\in\mathbb{S}^{n-1}}\{x \in \mathbb{R}^n : x\cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\}, \tag{1.1}$$

where, $x \cdot u$ denotes the standard inner product of x and u in $\mathbb{R}^n$, $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$ and h_K denotes the support function of convex body. Böröczky, Lutwak, Yang and Zhang [2] conjectured that for origin-symmetric convex bodies K and L in $\mathbb{R}^n$ with $0 \leq \lambda \leq 1$,

$$\mathrm{vol}_n((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq \mathrm{vol}_n(K)^{1-\lambda}\mathrm{vol}_n(L)^{\lambda}, \tag{1.2}$$

where $\mathrm{vol}_n(\cdot)$ denotes the n-dimensional volume of body in $\mathbb{R}^n$. They call (1.2) as the log-Brunn-Minkowski inequality. Note that while the inequality (1.2) is not true for general convex bodies, it implies the classical Brunn-Minkowski inequality for origin-symmetric convex bodies. In [2], Böröczky, et al. proved the inequality (1.2) when $n = 2$ and showed that (1.2) is equivalent to the logarithmic Minkowski inequality (log-Minkowski inequality) for all n, that is

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) \mathrm{d}\bar{v}_L(u) \geq \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.3}$$

AMS Subject Classification: 52A30, 52A40.

This work was Supported by the National Natural Science Foundation of China(Grant No.11561020) and was partly supported by the National Natural Science Foundation of China (Grant No.11371224).

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where $dv_L(u) = \frac{1}{n}h_L(u)dS_L(u)$ is the cone-volume measure of L, $d\bar{v}_L(u) = \frac{1}{\mathrm{vol}_n(L)} dv_L(u)$ and S_L is surface area measure of L on $\mathbb{S}^{n-1}$.

In [23], Stancu proved some variants of the log-Minkowski inequality for general convex bodies without the symmetry assumption.

The dual L_p-Brunn-Minkowski theory for star bodies developed by Lutwak [15, 16] and received considerable attention, see [1, 4, 6, 10, 11, 17, 20, 21, 22, 25]. Recently, Gardner, et al. [7] established dual log-Minkowski inequality as follows. If K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior, then

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) d\bar{\widetilde{v}}_K(u) \le \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.4}$$

with equality if and only if K and L are dilatates, where $d\bar{\widetilde{v}}_K$ is the dual cone-volume probability measure of K (see definition (2.11)). In the present paper, we prove a modified dual log-Minkowski inequality and obtain the double dual log-Minkowski inequality through the Gibbs' inequality. Secondly, we prove an analogue of the dual log-Minkowski inequality. In addition, we point out the equivalence between the dual log-Minkowski inequality and the dual log-Brunn-Minkowski inequality.

Our first result is the following dual log-Minkowski inequality:

Theorem 1.1. *Let K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior. Then*

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) d\bar{\widetilde{v}}_{-1}(K,L;u) \ge \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right) \ge \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.5}$$

with equality if and only if K and L are dilates, where $d\widetilde{v}_{-1}(K,L;\cdot)$ is the dual mixed volume measure $\widetilde{V}_{-1}(K,L) = \int_{\mathbb{S}^{n-1}} d\widetilde{v}_{-1}(K,L;u)$ and $d\bar{\widetilde{v}}_{-1}(K,L;u) = \frac{1}{\widetilde{V}_{-1}(K,L)} d\widetilde{v}_{-1}(K,L;u)$.

Secondly, we obtain the following double log-Minkowski inequality.

Theorem 1.2. *Let K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior. Then*

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) d\bar{\widetilde{v}}_K(u) \le \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right) \le \int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) d\bar{\widetilde{v}}_{-1}(K,L;u), \tag{1.6}$$

with equality in inequality if and only if K and L are dilates.

Further, we prove an analogue of the dual log-Minkowski inequality. In what follows, we will denote

$$\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}} = \frac{\int_{\mathbb{S}^{n-1}} \frac{\rho_K(u)}{\rho_L(u)} d\widetilde{v}_K(u)}{\int_{\mathbb{S}^{n-1}} d\widetilde{v}_K(u)},$$

$$\left(\frac{\rho_K}{\rho_L}\right)_{\max} = \max_{u\in\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right) \text{ and } \left(\frac{\rho_K}{\rho_L}\right)_{\min} = \min_{u\in\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right).$$

Theorem 1.3. *Let K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior with $L \subseteq K$. Then*

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) d\bar{\widetilde{v}}_K(u) \ge \frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}} \cdot \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.7}$$

with equality if and only if $K = \lambda L$, where $0 < \lambda \le 1$.

In general, if $K, L \in \mathcal{S}_o^n$, then

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\overline{\widetilde{v}}_K(u) \geq \frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}} \cdot \frac{1}{n} \ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)$$
$$+ \ln\left[\left(\frac{\rho_K}{\rho_L}\right)_{\min}\right] \cdot \left[1 - \frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\right], \tag{1.8}$$

with equality if and only if K is homothetic to L.

Finally, we point out the equivalence between the dual log-Minkowski inequality (1.4) and the dual log-Brunn-Minkowski inequality (2.8). We give a different proof with Wang and Liu [24].

2 Notation and preliminaries

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$, of a compact, convex set $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max(x \cdot y : y \in K), \tag{2.1}$$

and uniquely determines the convex set. Let $\mathcal{K}_o^n$ be the set of convex bodies in $\mathbb{R}^n$ containing the origin in their interior.

If L is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_L = \rho(L, \cdot) : \mathbb{R}^n \backslash \{o\} \to [0, +\infty)$, is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in L\}, \quad x \in \mathbb{R}^n \backslash \{o\}. \tag{2.2}$$

If ρ_L is positive and continuous, then L will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denotes the set of star bodies in $\mathbb{R}^n$ containing the origin in their interior. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in \mathbb{S}^{n-1}$. Obviously, for a pair $K, L \in \mathcal{S}_o^n$, we have

$$\rho_K \leq \rho_L, \quad \text{if and only if}, \quad K \subseteq L. \tag{2.3}$$

If $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \geq 0$ (not both zero), then, for $p \geq 1$, the harmonic L_p-combination, $\lambda \diamond K \widehat{+}_p \mu \diamond L \in \mathcal{S}_o^n$ is defined by (see [14])

$$\rho(\lambda \diamond K \widehat{+}_p \mu \diamond L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \tag{2.4}$$

For $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, the dual mixed volume, $\widetilde{V}_{-p}(K, L)$, is defined

$$-\frac{n}{p} \widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0} \frac{\mathrm{vol}_n(K \widehat{+}_p \varepsilon \diamond L) - \mathrm{vol}_n(K)}{\varepsilon}.$$

The following integral representation for the dual mixed volume $\widetilde{V}_{-p}$ is obtained (see [14]): If $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} \mathrm{d}S(u),$$

where $\mathrm{d}S$ is the spherical Lebesgue measure on $\mathbb{S}^{n-1}$. This integral representation, together the Hölder inequality with the polar coordinate formula, immediately gives the dual L_p-Minkowski inequality: If $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_{-p}(K, L)^n \geq \mathrm{vol}_n(K)^{n+p} \mathrm{vol}_n(L)^{-p}, \tag{2.5}$$

with equality if and only if K and L are dilates.

Using the dual L_p-Minkowski inequality, we can obtain the following dual L_p-Brunn-Minkowski inequality (see [14]). Suppose $K, L \in \mathcal{S}_o^n$, $\lambda, \mu > 0$ and $p \geq 1$, then

$$\mathrm{vol}_n(\lambda \diamond K \widehat{+}_p \mu \diamond L)^{-p/n} \geq \lambda \mathrm{vol}_n(K)^{-p/n} + \mu \mathrm{vol}_n(L)^{-p/n}, \tag{2.6}$$

with equality if and only if K and L are dilates.

Note that definition (2.4) makes sense for all $p > 0$. The case $p = 0$ is the limiting case given by

$$\rho((1-\lambda) \diamond K \hat{+}_0 \lambda \diamond L, \cdot) = \rho(K, \cdot)^{1-\lambda} \rho(L, \cdot)^{\lambda}, \quad 0 \leq \lambda \leq 1, \tag{2.7}$$

it is called the radial log-Minkowski-combination.

Similarly, the inequality (2.6) makes sense for all $p > 0$. The case $p = 0$ is the limiting case given by an dual log-Brunn-Minkowski inequality. Namely, if $K, L \in \mathcal{S}_o^n$, then for all $\lambda \in [0, 1]$,

$$\mathrm{vol}_n((1-\lambda) \diamond K \hat{+}_0 \lambda \diamond L) \leq \mathrm{vol}_n(K)^{1-\lambda} \mathrm{vol}_n(L)^{\lambda}, \tag{2.8}$$

with equality if and only if K and L are dilates.

If $K \in \mathcal{S}_o^n$, then

$$\mathrm{d}\widetilde{v}_K(u) = \frac{1}{n} \rho_K^n(u) \mathrm{d}S(u) \tag{2.9}$$

is the dual cone-volume measure of K and

$$\mathrm{d}\widetilde{v}_{-1}(K, L; u) = \frac{1}{n} \rho_K^{n+1}(u) \rho_L^{-1}(u) \mathrm{d}S(u) \tag{2.10}$$

is the dual mixed volume measure with $(n+1)$ copies of K and (-1) copies of L. Note that we usually write $\widetilde{V}_{-1}(K, L) = \int_{\mathbb{S}^{n-1}} \mathrm{d}\widetilde{v}_{-1}(K, L; u)$. The dual cone-volume measure of a star body K in $\mathbb{R}^n$ with $\mathrm{vol}_n(K)$ is the Borel probability measure $\overline{\widetilde{v}}_K$ in $\mathbb{S}^{n-1}$ defined by

$$\mathrm{d}\overline{\widetilde{v}}_K = \frac{\rho_K^n(u)}{n \mathrm{vol}_n(K)} \mathrm{d}S(u). \tag{2.11}$$

And the normalized dual mixed cone measure of a star bodies K, L in $\mathbb{R}^n$ with $\widetilde{V}_{-1}(K, L)$ is the Borel probability measure $\overline{\widetilde{v}}_{-1}(K, L; \cdot)$ on $\mathbb{S}^{n-1}$ defined by

$$\mathrm{d}\overline{\widetilde{v}}_{-1}(K, L; u) = \frac{1}{\widetilde{V}_{-1}(K, L)} \mathrm{d}\widetilde{v}_{-1}(K, L; u). \tag{2.12}$$

3 Proofs of dual log-Minkowski type results

In this section, we will prove the theorems mentioned in Section 1.

Proof of Theorem 1.1. Consider the function $G_{K,L}(p) : [1, \infty] \to \mathbb{R}$ defined by

$$G_{K,L}(p) = \frac{1}{\widetilde{V}_{-1}(K, L)} \int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} \right)^{\frac{p}{n+p}} \mathrm{d}\widetilde{v}_K(u).$$

Through using L'Hôpital's rule, we obtain

$$\begin{aligned}
\lim_{p\to\infty}\ln(G_{K,L}(p))^{n+p} &= \lim_{p\to\infty}\ln\left(\frac{\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{p}{n+p}}\mathrm{d}\widetilde{v}_K(u)}{\widetilde{V}_{-1}(K,L)}\right)^{n+p}\\
&= \lim_{p\to\infty}\ln\left(\frac{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{-\frac{n}{n+p}}\mathrm{d}S(u)}{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\mathrm{d}S(u)}\right)^{n+p}\\
&= \ln\exp\left(\frac{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{-n}\mathrm{d}S(u)}{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\mathrm{d}S(u)}\right)\\
&= \frac{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{-n}\mathrm{d}S(u)}{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\mathrm{d}S(u)}\\
&= -\frac{n}{\widetilde{V}_{-1}(K,L)}\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\widetilde{v}_K(u).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\exp\left[-\frac{n}{\widetilde{V}_{-1}(K,L)}\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\widetilde{v}_K(u)\right]\\
=\ &\lim_{p\to\infty}\left[\frac{1}{\widetilde{V}_{-1}(K,L)}\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{p}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right]^{p+n}, \qquad (3.1)
\end{aligned}$$

and it follows from Hölder's inequality that

$$\begin{aligned}
&\left(\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{p}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right)^{\frac{p+n}{p}}\left(\int_{\mathbb{S}^{n-1}}\mathrm{d}\widetilde{v}_K(u)\right)^{-\frac{n}{p}}\\
\le\ &\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\mathrm{d}\widetilde{v}_K(u)=\widetilde{V}_{-1}(K,L). \qquad (3.2)
\end{aligned}$$

Note that $\int_{\mathbb{S}^{n-1}}\mathrm{d}\widetilde{v}_K(u)=\mathrm{vol}_n(K)$, (2.10) and (2.12), together (3.1) with (3.2), we have

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u)\ge\ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right).$$

According to the condition of equality in Hölder's inequality, we easily see that with equality in the above inequality if and only if K and L are dilates.

Using dual Minkowski's inequality (2.5), we have the second inequality in the theorem, this is,

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u)\ge\ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right)\ge\frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right). \qquad (3.3)$$

From the condition of equality in dual Minkowski's inequality, we know that with equality if and only if K and L are dilates. Which completes the proof of the theorem. □

Remark 3.1. *Our first inequality in* (1.5) *can be written as*

$$\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u)\ge\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right). \qquad (3.4)$$

Use dual Minkowski's inequality in (3.4)*, we have*

$$\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u)\ge\frac{1}{n}\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)^{1/n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right). \qquad (3.5)$$

Proof of Theorem 1.2. We consider Gibbs' inequality from information theory (see [3], (8.57), p. 252-253): If p and q are probability density functions on a measure space (X,ν), then

$$\int p\ln p\mathrm{d}\nu \geq \int p\ln q\mathrm{d}\nu, \tag{3.6}$$

with equality if and only if $p=q$ almost everywhere (a.e.).

By taking

$$p\mathrm{d}\nu = \frac{\rho_L(u)}{\rho_K(u)}\cdot\frac{1}{\mathrm{vol}_n(K)}\mathrm{d}\widetilde{v}_{-1}(K,L;u) \text{ and } q\mathrm{d}\nu = \frac{1}{\widetilde{V}_{-1}(K,L)}\mathrm{d}\widetilde{v}_{-1}(K,L;u)$$

(and later reversing the two measures above so that the first is $q\mathrm{d}\nu$ and the second is $p\mathrm{d}\nu$), we obtain the double inequality as follows.

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u) \leq \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right) \leq \int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u). \tag{3.7}$$

According to the condition of equality in Gibbs' inequality (3.6), we obtain that with equality in inequality (3.7) if and only if

$$\frac{\rho_L(u)}{\rho_K(u)} = \frac{\mathrm{vol}_n(K)}{\widetilde{V}_{-1}(K,L)} \Longleftrightarrow \frac{1}{n}\rho_L^n(u) = \frac{1}{n}\left(\frac{\mathrm{vol}_n(K)}{\widetilde{V}_{-1}(K,L)}\right)^n\rho_K^n(u)$$

almost everywhere (a.e.) on $\mathbb{S}^{n-1}$. Integrating both sides of the last equation over $\mathbb{S}^{n-1}$ with the sphere Lebesgue measure $\mathrm{d}S(u)$, we get

$$\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)} = \left(\frac{\mathrm{vol}_n(K)}{\widetilde{V}_{-1}(K,L)}\right)^n.$$

From the condition of equality in the dual L_p-Minkowski inequality (2.5) ($p=1$), we see that with equality in inequality (3.7) if and only if K and L are dilates. □

Remark 3.2. *The proof of Theorem 1.2 can be seen that we provide a new proof for the dual Minkowski inequality itself. In fact, it is consistent with the idea of splitting mentioned by Gardner, Hug and Weil in [8] and [9].*

A natural idea is to give a proof of the dual log-Minkowski inequality similar to the proof of the Theorem 1.1. However, as such, we obtain again the left-hand side inequality of (1.6) due to the following lemma:

Lemma 3.3. *Let $K,L\in\mathcal{S}_o^n$, then*

$$\begin{aligned}&\exp\left(\int_{\mathbb{S}^{n-1}}\ln\frac{\rho_K(u)}{\rho_L(u)}\mathrm{d}\overline{\widetilde{v}}_K(u)\right)\\ &= \lim_{p\to\infty}\left(\frac{1}{\mathrm{vol}_n(K)}\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{1}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right)^{p+n}.\end{aligned} \tag{3.8}$$

The proof follows the same idea used in deriving (3.1).

From Hölder's inequality, we have

$$\begin{aligned}&\left(\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{1}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right)^{p+n}\left(\int_{\mathbb{S}^{n-1}}\mathrm{d}\widetilde{v}_K(u)\right)^{1-(p+n)}\\ &\leq \int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\mathrm{d}\widetilde{v}_K(u) = \widetilde{V}_{-1}(K,L).\end{aligned} \tag{3.9}$$

Lemma 3.3, together with (3.9), implies that

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\overline{\widetilde{v}}_K(u) \le \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right).$$

It will be convenient to invoke the logarithmic mean $L(x,y)$ of two positive numbers x, y, which is given by

$$L(x,y) = \begin{cases} \frac{x-y}{lnx - \ln y}, & \text{for } x \neq y \\ x, & \text{for } x = y. \end{cases} \tag{3.10}$$

To prove Theorem 1.3, the following Hadamard type inequality for positive log-convex functions will be used [12].

Lemma 3.4. *Let f be a positive, integrable, log-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le L\left(f(a), f(b)\right). \tag{3.11}$$

Suppose f has two derivative. The equality holds in the inequality (3.11) if and only if $f(t) = c$ almost everywhere (a.e.) or $\frac{f'(t)}{f(t)} = c$ almost everywhere (a.e.), where c is the constant.

The condition of the equality holds in the inequality (3.11) is that we supplements. Indeed, since f is log-convex function on $[a,b]$, and then $f(t)$ and $\frac{f'(t)}{f(t)}$ are monotonically increasing at the same time. So, we have

$$\begin{aligned} L(f(a), f(b)) &= \frac{f(b) - f(a)}{\ln f(b) - \ln f(b)} = \frac{\int_a^b f'(x)\mathrm{d}x}{\int_{f(a)}^{f(b)} \frac{1}{x}\mathrm{d}x} \\ &\overset{\text{x=f(t)}}{=} \frac{\int_a^b f'(x)\mathrm{d}x}{\int_a^b \frac{f'(t)}{f(t)}\mathrm{d}t} \ge \frac{1}{b-a}\int_a^b f(t)\mathrm{d}x \\ &= \frac{\int_a^b f(t)\mathrm{d}t}{\int_a^b 1\mathrm{d}t}. \end{aligned} \tag{3.12}$$

Thus, the inequality (3.11) is transformed into

$$\int_a^b f'(x)\mathrm{d}x \int_a^b 1\mathrm{d}x \ge \int_a^b f(t)\mathrm{d}t \int_a^b \frac{f'(t)}{f(t)}\mathrm{d}t. \tag{3.13}$$

Note that $f(t)$ and $\frac{f'(t)}{f(t)}$ are monotonically increasing at the same time, According to the condition of equality in Chebyshev's inequality, we see with equality in inequality (3.11) if and only if $f(t) = c$ or $\frac{f'(t)}{f(t)} = c$. Namely, $f(t) = c$ or $f(t) = e^{ct}$. □

Proof of Theorem 1.3. Consider the case $L \subseteq K$ and the function

$$F(q) : q \mapsto \int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right)^q \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\widetilde{v}_K(u), \quad q \in \mathbb{R}.$$

Apparently, $F(q)$ is non-negative. If $u \mapsto \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)$ is zero on $\mathbb{S}^{n-1}$, then $F(q)$ is identically zero. Now, we assume that this is not the case, which also implies $F(1) \ge F(0) > 0$. If $F(1) = F(0)$, the conclusion is trivial (as using (3.7), K must be equal to L), and then, we assume $F(1) > F(0)$.

A simple verification shows that $F(q)$ is a log-convex function, this is because $\frac{\mathrm{d}^2}{\mathrm{d}q^2} \ln F(q) \ge 0$. By employing Hadamard type inequality (3.11) for positive log-convex functions [12], we have that

$$\frac{F(1) - F(0)}{\ln\big(F(1)/F(0)\big)} \ge \int_0^1 \left[\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right)^q \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\widetilde{v}_K(u)\right] \mathrm{d}q. \tag{3.14}$$

Using Fubini-Tonelli's theorem, the following inequality

$$F(0) \geq F(1) \cdot \exp\left[- \frac{F(1) - F(0)}{\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)} \right] \tag{3.15}$$

is true. Note that

$$\begin{aligned} \frac{F(1) - F(0)}{\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)} &= \frac{\int_{\mathbb{S}^{n-1}} \ln \left(\frac{\rho_K(u)}{\rho_L(u)} \cdot \right) \cdot \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)}{\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)} \\ &\leq \ln \left(\frac{\rho_K}{\rho_L} \right)_{\max}, \end{aligned} \tag{3.16}$$

then combining (3.15) and (3.16), we have

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \ln \left(\frac{\rho_K(u)}{\rho_L(u)} \right) \mathrm{d}\overline{\widetilde{v}}_K(u) \\ &\geq \exp\left[- \ln \left(\frac{\rho_K}{\rho_L} \right)_{\max} \right] \cdot \frac{\widetilde{V}_{-1}(K, L)}{\mathrm{vol}_n(K)} \int_{\mathbb{S}^{n-1}} \ln \left(\frac{\rho_K(u)}{\rho_L(u)} \right) \mathrm{d}\overline{\widetilde{v}}_{-1}(K, L; u), \end{aligned} \tag{3.17}$$

it follows from (3.3) that

$$\int_{\mathbb{S}^{n-1}} \ln \left(\frac{\rho_K}{\rho_L} \right) \mathrm{d}\overline{\widetilde{v}}_K(u) \geq \frac{\left(\frac{\rho_K}{\rho_L} \right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L} \right)_{\max}} \frac{1}{n} \ln \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)} \right).$$

Now we discuss the conditions of equality in inequality (1.7), and the discussion is split into two cases. Assuming that $F(q)$ is identically zero, then $\rho_K(u) = \rho_L(u)$ for all u's with respect to $\mathbb{S}^{n-1}$ if and only if $K = L$.

Case 1. According to the conditions of equality in Hadamard type inequality (3.11) and inequality (3.3), we see with equality in inequality (1.7) if and only if

$$\begin{cases} F(q) = c \text{ for } q \geq 1, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.18}$$

From the definition of function $F(q)$, (3.18) is equivalent to

$$\begin{cases} \frac{\rho_K(u)}{\rho_L(u)} = 1 \text{ for any } u \in \mathbb{S}^{n-1}, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.19}$$

Namely, $K = L$.

Case 2. According to the conditions of equality in Hadamard type inequality (3.11) and inequality (3.3), we see with equality in inequality (1.7) if and only if

$$\begin{cases} \frac{F'(q)}{F(q)} = c \text{ for } q \geq 1, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.20}$$

Using mean value theorem for multiple integral [5, 19], there is a $u_0 \in \mathbb{S}^{n-1}$, such that (3.18) is equivalent to

$$\begin{cases} \frac{\rho_K(u_0)}{\rho_L(u_0)} = cq \text{ for a } u_0 \in \mathbb{S}^{n-1}, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.21}$$

Since $L \subseteq K$, $K = \lambda L$ with $0 < \lambda \leq 1$.

As mentioned above, we see with equality in inequality (1.7) if and only if $K = \lambda L$ with $0 < \lambda \leq 1$.

Assume now that K and L are arbitrary star bodies. If L is not included in K, there exists a λ, $0<\lambda<1$, such that $\tilde{L}:=\lambda L\subseteq K$. By using (1.7) for $\tilde{L}$ and K. Thus,

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u)-\ln\lambda\geq\frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\cdot\frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\lambda^n\mathrm{vol}_n(L)}\right) \tag{3.22}$$

or

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u)\geq\frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\cdot\frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)+\ln\lambda\cdot\left(1-\frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\right). \tag{3.23}$$

Taking $\lambda=\min_{u\in\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)$ will suffice, we now obtain the second inequality.

The claim that the homothety of K and L is the only case of equality follows from the first part. □

Remark3.5. *Note that, if* $L\subseteq K$ *then* (1.8) *implies* (1.7)*. Also,* $\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}$, $\left(\frac{\rho_K}{\rho_L}\right)_{\max}$ *and* $\left(\frac{\rho_K}{\rho_L}\right)_{\min}$ *depend only on the values of the ratio* $\left(\frac{\rho_K(u)}{\rho_L(u)}\right)$ *on* $\mathbb{S}^{n-1}$.

We conclude this paper by pointing out that the equivalence between inequalities (1.4) and (2.8). We give a different proof with Wang and Liu [24]. For any $K\in\mathcal{S}^n_o$, define the real numbers R_K and r_K by

$$R_K=\max_{u\in\mathbb{S}^{n-1}}\rho_K(u),\qquad r_K=\min_{u\in\mathbb{S}^{n-1}}\rho_K(u). \tag{3.24}$$

Note that the definition of $\mathcal{S}^n_o$ is such that $0<r_K\leq R_K<\infty$, for all $K\in\mathcal{S}^n_o$.

Theorem 3.6. *For* $K,L\in\mathcal{S}^n_o$*, the dual log-Brunn-Minkowski inequality* (2.8) *and the dual log-Minkowski inequality* (1.4) *are equivalent.*

Proof. Suppose that K and L are fixed star bodies in $\mathcal{S}^n_o$. For $0\leq\lambda\leq1$, let

$$Q_\lambda=(1-\lambda)\diamond K\hat{+}_0\lambda\diamond L,$$

i.e., the radial function of star body Q_λ is $q_\lambda:=\rho_{Q_\lambda}=\rho_K^{1-\lambda}\rho_L^\lambda$. Since q_0 and q_1 are the radial functions of star bodies, we have $Q_0=K$ and $Q_1=L$.

Suppose that we have the dual log-Minkowski inequality (1.4) for K and L. Now $\rho_{Q_\lambda}=\rho_K^{1-\lambda}\rho_L^\lambda$ a.e. with respect to $\mathcal{S}^{n-1}$, and thus

$$\begin{aligned}
0&=\frac{1}{n\mathrm{vol}_n(Q_\lambda)}\int_{\mathbb{S}^{n-1}}\rho_{Q_\lambda}(u)^n\ln\frac{\rho_K(u)^{1-\lambda}\rho_L(u)^\lambda}{\rho_{Q_\lambda}(u)}\mathrm{d}S(u)\\
&=(1-\lambda)\frac{1}{n\mathrm{vol}_n(Q_\lambda)}\int_{\mathbb{S}^{n-1}}\rho_{Q_\lambda}(u)^n\ln\frac{\rho_K(u)}{\rho_{Q_\lambda}(u)}\mathrm{d}S(u)\\
&\quad+\lambda\frac{1}{n\mathrm{vol}_n(Q_\lambda)}\int_{\mathbb{S}^{n-1}}\rho_{Q_\lambda}(u)^n\ln\frac{\rho_L(u)}{\rho_{Q_\lambda}(u)}\mathrm{d}S(u)\\
&=-(1-\lambda)\int_{\mathbb{S}^{n-1}}\ln\frac{\rho_{Q_\lambda}(u)}{\rho_K(u)}\mathrm{d}\overline{\widetilde{v}}_{Q_\lambda}(u)-\lambda\int_{\mathbb{S}^{n-1}}\ln\frac{\rho_{Q_\lambda}(u)}{\rho_L(u)}\mathrm{d}\overline{\widetilde{v}}_{Q_\lambda}(u)\\
&\leq-(1-\lambda)\frac{1}{n}\ln\frac{\mathrm{vol}_n(Q_\lambda)}{\mathrm{vol}_n(K)}-\lambda\frac{1}{n}\ln\frac{\mathrm{vol}_n(Q_\lambda)}{\mathrm{vol}_n(L)}\\
&=\frac{1}{n}\ln\frac{\mathrm{vol}_n(K)^{1-\lambda}\mathrm{vol}_n(L)^\lambda}{\mathrm{vol}_n(Q_\lambda)}.
\end{aligned} \tag{3.25}$$

This gives the dual log-Brunn-Minkowski inequality (2.8).

Suppose now that we have the dual log-Brunn-Minkowski inequality (2.8) for K and L. Namely,

$$\mathrm{vol}_n((1-\lambda)\diamond K\hat{+}_0\lambda\diamond L)\leq \mathrm{vol}_n(K)\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}. \tag{3.26}$$

Using the polar coordinates formula of volume, the radial log-Minkowski-combination (2.7) and the Borel probability measure (2.11), it follows from (3.26) that

$$\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)\leq\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}. \tag{3.27}$$

Therefore

$$\frac{\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)-1}{\lambda}\leq\frac{\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}-1}{\lambda}.$$

Taking the limit on both sides of the last inequality as $\lambda\to 0$, we get

$$\lim_{\lambda\to 0}\frac{\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)-1}{\lambda}\leq\lim_{\lambda\to 0}\frac{\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}-1}{\lambda}.$$

We are easy to prove the function $f(x)=\frac{a^x-1}{x}$ is uniformly continuous on $(0,\infty)$ for $0<a\leq 1$, and the Bernoulli's inequality leads to the function $f(x)=\frac{a^x-1}{x}$ is uniform boundness for $a>1$. From the definition (3.24), we have

$$\frac{\left(\frac{\rho_L}{\rho_K}\right)^{n\lambda}-1}{\lambda}\leq\frac{\left(\frac{R_L}{r_K}\right)^{n\lambda}-1}{\lambda}.$$

Using Lebesgue dominated convergence theorem we know that the order of the integral and the limit can be changed. Therefore, we can obtain

$$\int_{\mathbb{S}^{n-1}}\lim_{\lambda\to 0}\frac{\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}-1}{\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)\leq\lim_{\lambda\to 0}\frac{\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}-1}{\lambda}.$$

Since $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$, then

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n}\mathrm{d}\overline{\widetilde{v}}_K(u)\leq\ln\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right). \tag{3.28}$$

This is the dual log-Minkowski inequality (1.4), which completes the proof.

Acknowledgment

The authors are deeply indebted to Professor Li Yin, Professor Xiaoming Zhang, Doctor Xuezhong Wang, Doctor Denghui Wu and Doctor Yibin Feng for many invaluable suggestions in preparing the manuscript. The authors also would like to thank the referee for the many suggested improvements.

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Subalgebra and ideal-type hyper values in BCK/BCI-algebras

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Abstract. The notions of subalgebra-type hyper value and ideal-type hyper value are introduced, and related properties are investigated. The relation between subalgebra-type hyper value and ideal-type hyper value is considered. Conditions for a pair (α, β) in $[0,1] \times [0,1]$ to be subalgebra-type hyper value and ideal-type hyper value are discussed. For a hyperfuzzy structure, conditions for its level sets to be S-energetic, I-energetic, right vanished and right stable are founded.

1. Introduction

Jun et al. [3] introduced the notion of energetic (resp. right vanish, right stable) subsets in BCK/BCI-algebras, and investigated several related properties. Ghosh et al. [1] introduced the concept of hyperfuzzy sets which is a generalization of fuzzy sets and interval-valued fuzzy sets. Jun et al. [4] and Song et al. [6] applied hyper structure to BCK/BCI-algebras, and discussed hyperfuzzy subalgebras and hyperfuzzy ideals in BCK/BCI-algebras.

In this article, we introduce the concepts of subalgebra-type hyper value and ideal-type hyper value, and investigate several properties. We discuss the relation between subalgebra-type hyper value and ideal-type hyper value. We provide an example to show that any subalgebra-type hyper value is not an ideal-type hyper value. We consider conditions for a pair (α, β) in $[0,1] \times [0,1]$ to be subalgebra-type hyper value and ideal-type hyper value. Given a hyperfuzzy structure, we find conditions for its level sets to be S-energetic, I-energetic, right vanished and right stable.

2. Preliminaries

By a *BCI-algebra* we mean a system $X := (X, *, 0)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = y * x = 0 \;\Rightarrow\; x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 0).$$

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[0] **2010 Mathematics Subject Classification**: 08F35; 03G25; 03B52.

[0] **Keywords**: hyperfuzzy subalgebra; hyperfuzzy ideal; subalgebra-type hyper value; ideal-type hyper value; energetic subset; right vanished subset; right stable subset.

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In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x * 0 = x), \tag{2.1}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{2.2}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if

$$0 \in I, \tag{2.3}$$

$$(\forall x \in X)(\forall y \in I)\,(x * y \in I\ \Rightarrow\ x \in I)\,. \tag{2.4}$$

We refer the reader to the books [2] and [5] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Let X be a nonempty set. A mapping $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$ is called a *hyperfuzzy set* over X (see [1]), where $\tilde{\mathcal{P}}([0, 1])$ is the family of all nonempty subsets of $[0, 1]$. An ordered pair $(X, \tilde{\mu})$ is called a *hyper structure* over X.

Given a hyper structure $(X, \tilde{\mu})$ over a nonempty set X, we consider two fuzzy structures $(X, \tilde{\mu}_{\inf})$ and $(X, \tilde{\mu}_{\sup})$ over X in which

$$\tilde{\mu}_{\inf} : X \to [0, 1],\ x \mapsto \inf\{\tilde{\mu}(x)\},$$

$$\tilde{\mu}_{\sup} : X \to [0, 1],\ x \mapsto \sup\{\tilde{\mu}(x)\}.$$

Given a nonempty set X, let $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$ denote the collection of all BCK-algebras and all BCI-algebras, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$. In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ unless otherwise specified.

Definition 2.1 ([4])**.** For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- *fuzzy subalgebra* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{2.5}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{2.6}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{2.7}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{2.8}$$

It is clear that every 3-fuzzy subalgebra is a 1-fuzzy subalgebra and every 2-fuzzy subalgebra is a 4-fuzzy subalgebra.

Definition 2.2 ([4])**.** For any $(X, *, 0) \in \mathcal{B}(X)$ and $i, j \in \{1, 2, 3, 4\}$, a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$ is called an (i, j)-*hyperfuzzy subalgebra* of $(X, *, 0)$ if $(X, \tilde{\mu}_{\inf})$ is an i-fuzzy subalgebra of $(X, *, 0)$ and $(X, \tilde{\mu}_{\sup})$ is a j-fuzzy subalgebra of $(X, *, 0)$.

Given a hyper structure $(X, \tilde{\mu})$ over X and $\alpha, \beta \in [0,1]$, we consider the following sets (see [6]):

$$\begin{aligned}
U(\tilde{\mu}_{\inf}; \alpha) &:= \{x \in X \mid \tilde{\mu}_{\inf}(x) \geq \alpha\}, \\
L(\tilde{\mu}_{\inf}; \alpha) &:= \{x \in X \mid \tilde{\mu}_{\inf}(x) \leq \alpha\}, \\
U(\tilde{\mu}_{\sup}; \beta) &:= \{x \in X \mid \tilde{\mu}_{\sup}(x) \geq \beta\}, \\
L(\tilde{\mu}_{\sup}; \beta) &:= \{x \in X \mid \tilde{\mu}_{\sup}(x) \leq \beta\}.
\end{aligned}$$

Definition 2.3 ([6]). A fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- *fuzzy ideal* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy ideal* of $(X, *, 0)$) if

$$(\forall x \in X)\,(\mu(0) \geq \mu(x))\,, \tag{2.9}$$

$$(\forall x, y \in X)\,(\mu(x) \geq \min\{\mu(x * y), \mu(y)\})\,, \tag{2.10}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy ideal* of $(X, *, 0)$) if

$$(\forall x \in X)\,(\mu(0) \leq \mu(x))\,, \tag{2.11}$$

$$(\forall x, y \in X)\,(\mu(x) \leq \min\{\mu(x * y), \mu(y)\})\,, \tag{2.12}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy ideal* of $(X, *, 0)$) if it satisfies (2.9) and

$$(\forall x, y \in X)\,(\mu(x) \geq \max\{\mu(x * y), \mu(y)\})\,, \tag{2.13}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy ideal* of $(X, *, 0)$) if it satisfies (2.11) and

$$(\forall x, y \in X)\,(\mu(x) \leq \max\{\mu(x * y), \mu(y)\})\,. \tag{2.14}$$

It is clear that every 3-fuzzy ideal is a 1-fuzzy ideal and every 2-fuzzy ideal is a 4-fuzzy ideal.

Definition 2.4 ([6]). For any $i, j \in \{1, 2, 3, 4\}$, a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$ is called an (i, j)-*hyperfuzzy ideal* of $(X, *, 0)$ if $(X, \tilde{\mu}_{\inf})$ is an i-fuzzy ideal of $(X, *, 0)$ and $(X, \tilde{\mu}_{\sup})$ is a j-fuzzy ideal of $(X, *, 0)$.

3. Subalgebra and ideal-type hyper values

Definition 3.1 ([3]). A nonempty subset A of $(X, *, 0)$ is said to be S-*energetic* if it satisfies:

$$(\forall a, b \in X)\,(a * b \in A \;\Rightarrow\; \{a, b\} \cap A \neq \emptyset)\,.$$

Let A be a proper subset of X containing 0. Then there exists $a \in X \setminus A$, and so $a * a = 0 \in A$ but $\{a\}$ and A are disjoint. Thus every proper subset A of X containing 0 cannot be S-energetic.

Theorem 3.2. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, if it is a $(4, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are S-energetic subsets of $(X, *, 0)$ for all $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0, 1] \times [0, 1]$.*

Proof. Assume that $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are nonempty for every $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$. If $x*y\in U(\tilde{\mu}_{\inf};\alpha)$ and $a*b\in L(\tilde{\mu}_{\sup};\beta)$ for all $x,y,a,b\in X$, then

$$\alpha\le\tilde{\mu}_{\inf}(x*y)\le\max\{\tilde{\mu}_{\inf}(x),\tilde{\mu}_{\inf}(y)\}$$

and

$$\beta\ge\tilde{\mu}_{\sup}(a*b)\ge\min\{\tilde{\mu}_{\sup}(a),\tilde{\mu}_{\sup}(b)\}.$$

It follows that

$$\tilde{\mu}_{\inf}(x)\ge\alpha \text{ or } \tilde{\mu}_{\inf}(y)\ge\alpha, \text{ that is, } x\in U(\tilde{\mu}_{\inf};\alpha) \text{ or } y\in U(\tilde{\mu}_{\inf};\alpha)$$

and

$$\tilde{\mu}_{\sup}(a)\le\beta \text{ or } \tilde{\mu}_{\sup}(b)\le\beta, \text{ that is, } a\in L(\tilde{\mu}_{\sup};\beta) \text{ or } b\in L(\tilde{\mu}_{\sup};\beta).$$

Hence $\{x,y\}\cap U(\tilde{\mu}_{\inf};\alpha)\ne\emptyset$ and $\{a,b\}\cap L(\tilde{\mu}_{\sup};\beta)\ne\emptyset$. Therefore $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are S-energetic subsets of $(X,*,0)$ for all $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$. □

Corollary 3.3. *Given a hyper structure $(X,\tilde{\mu})$ over $(X,*,0)$, if it is a $(2,1)$-hyperfuzzy (resp., $(2,3)$-hyperfuzzy and $(4,3)$-hyperfuzzy) subalgebra of $(X,*,0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are S-energetic subsets of $(X,*,0)$ for all $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$.*

Proof. Straightforward. □

Definition 3.4 ([3])**.** A nonempty subset A of $(X,*,0)$ is said to be *I-energetic* if it satisfies:

$$(\forall x,y\in X)\,(y\in A\ \Rightarrow\ \{x,y*x\}\cap A\ne\emptyset)\,.$$

Theorem 3.5. *Given a hyper structure $(X,\tilde{\mu})$ over $(X,*,0)$, if it is a $(4,1)$-hyperfuzzy ideal of $(X,*,0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are I-energetic subsets of $(X,*,0)$ for all $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$.*

Proof. Let $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$ be such that $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are nonempty. Let $x,y,a,b\in X$ be such that $y\in U(\tilde{\mu}_{\inf};\alpha)$ and $b\in L(\tilde{\mu}_{\sup};\beta)$. Then

$$\alpha\le\tilde{\mu}_{\inf}(y)\le\max\{\tilde{\mu}_{\inf}(y*x),\tilde{\mu}_{\inf}(x)\}$$

and

$$\beta\ge\tilde{\mu}_{\sup}(b)\ge\min\{\tilde{\mu}_{\sup}(b*a),\tilde{\mu}_{\sup}(a)\}.$$

Hence

$$\tilde{\mu}_{\inf}(y*x)\ge\alpha \text{ or } \tilde{\mu}_{\inf}(x)\ge\alpha, \text{ i.e., } y*x\in U(\tilde{\mu}_{\inf};\alpha) \text{ or } x\in U(\tilde{\mu}_{\inf};\alpha)$$

and

$$\tilde{\mu}_{\sup}(b*a)\le\beta \text{ or } \tilde{\mu}_{\sup}(a)\le\beta, \text{ i.e., } b*a\in L(\tilde{\mu}_{\sup};\beta) \text{ or } a\in L(\tilde{\mu}_{\sup};\beta).$$

It follows that $\{x,y*x\}\cap U(\tilde{\mu}_{\inf};\alpha)\ne\emptyset$ and $\{a,b*a\}\cap L(\tilde{\mu}_{\sup};\beta)\ne\emptyset$. Therefore $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are I-energetic subsets of $(X,*,0)$ for all $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$. □

Corollary 3.6. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, if it is a $(2, 1)$-hyperfuzzy (resp., $(2, 3)$-hyperfuzzy and $(4, 3)$-hyperfuzzy) subalgebra of $(X, *, 0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are I-energetic subsets of $(X, *, 0)$ for all $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0, 1] \times [0, 1]$.*

Proof. Straightforward. □

Definition 3.7. Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0, 1] \times [0, 1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. Then (α, β) is called a *subalgebra-type hyper value* for $(X, \tilde{\mu})$ if the following assertion is valid.

$$(\forall x, y \in X) \left(\begin{array}{l} \tilde{\mu}_{\inf}(x * y) \leq \alpha \ \Rightarrow \ \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \leq \alpha, \\ \tilde{\mu}_{\sup}(x * y) \geq \beta \ \Rightarrow \ \max\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\} \geq \beta \end{array} \right). \tag{3.1}$$

Example 3.8. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [5]). Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1]), \ x \mapsto \begin{cases} [0.5, 0.53) & \text{if } x = 0, \\ (0.3, 0.58] & \text{if } x = 1, \\ [0.3, 0.44) \cup [0.45, 0.58) & \text{if } x = 2, \\ (0.4, 0.5] \cup [0.60, 0.68] & \text{if } x = 3, \\ [0.2, 0.63] & \text{if } x = 4. \end{cases}$$

It is routine to verify that every pair $(\alpha, \beta) \in [0.2, 0.5] \times [0.53, 0.68)$ is a subalgebra-type hyper value for $(X, \tilde{\mu})$.

Theorem 3.9. *For a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0, 1] \times [0, 1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$, then (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$.*

Proof. Let $x, y, a, b \in X$ be such that $\tilde{\mu}_{\inf}(x * y) \leq \alpha$ and $\tilde{\mu}_{\sup}(a * b) \geq \beta$. Since $(X, \tilde{\mu})$ is a $(1, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$, we have

$$\alpha \geq \tilde{\mu}_{\inf}(x * y) \geq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\beta \leq \tilde{\mu}_{\sup}(a * b) \leq \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\}.$$

Hence (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$. □

Corollary 3.10. *For a hyper structure $(X,\tilde{\mu})$ over $(X,*,0)$, let $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$ be such that $L(\tilde{\mu}_{\inf};\alpha)$ and $U(\tilde{\mu}_{\sup};\beta)$ are nonempty. If $(X,\tilde{\mu})$ is a $(1,2)$-hyperfuzzy (resp., $(3,2)$-hyperfuzzy and $(3,4)$-hyperfuzzy) subalgebra of $(X,*,0)$, then (α,β) is a subalgebra-type hyper value for $(X,\tilde{\mu})$.*

Proof. Straightforward. □

Theorem 3.11. *Let $(X,\tilde{\mu})$ be a hyper structure over $(X,*,0)$. If (α,β) is a subalgebra-type hyper value for $(X,\tilde{\mu})$, then $L(\tilde{\mu}_{\inf};\alpha)$ and $U(\tilde{\mu}_{\sup};\beta)$ are S-energetic subsets of $(X,*,0)$.*

Proof. Let $x,y,a,b\in X$ be such that $x*y\in L(\tilde{\mu}_{\inf};\alpha)$ and $a*b\in U(\tilde{\mu}_{\sup};\beta)$. Then $\tilde{\mu}_{\inf}(x*y)\leq\alpha$ and $\tilde{\mu}_{\sup}(a*b)\geq\beta$. Since (α,β) is a subalgebra-type hyper value for $(X,\tilde{\mu})$, it follows from (3.1) that $\min\{\tilde{\mu}_{\inf}(x),\tilde{\mu}_{\inf}(y)\}\leq\alpha$ and $\max\{\tilde{\mu}_{\sup}(a),\tilde{\mu}_{\sup}(b)\}\geq\beta$. Hence

$$\tilde{\mu}_{\inf}(x)\leq\alpha \text{ or } \tilde{\mu}_{\inf}(y)\leq\alpha$$

and

$$\tilde{\mu}_{\sup}(a)\geq\beta \text{ or } \tilde{\mu}_{\sup}(b)\geq\beta,$$

that is,

$$x\in L(\tilde{\mu}_{\inf};\alpha) \text{ or } y\in L(\tilde{\mu}_{\inf};\alpha)$$

and

$$a\in U(\tilde{\mu}_{\sup};\beta) \text{ or } b\in U(\tilde{\mu}_{\sup};\beta).$$

Thus $\{x,y\}\cap L(\tilde{\mu}_{\inf};\alpha)\neq\emptyset$ and $\{a,b\}\cap U(\tilde{\mu}_{\sup};\beta)\neq\emptyset$, and therefore $L(\tilde{\mu}_{\inf};\alpha)$ and $U(\tilde{\mu}_{\sup};\beta)$ are S-energetic subsets of $(X,*,0)$. □

Combining Theorems 3.9 and 3.11, we have the following corollary.

Corollary 3.12. *For a hyper structure $(X,\tilde{\mu})$ over $(X,*,0)$, let $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$ be such that $L(\tilde{\mu}_{\inf};\alpha)$ and $U(\tilde{\mu}_{\sup};\beta)$ are nonempty. If $(X,\tilde{\mu})$ is a $(1,4)$-hyperfuzzy subalgebra of $(X,*,0)$, then $L(\tilde{\mu}_{\inf};\alpha)$ and $U(\tilde{\mu}_{\sup};\beta)$ are S-energetic subsets of $(X,*,0)$.*

Definition 3.13. Given a hyper structure $(X,\tilde{\mu})$ over $(X,*,0)$, let $(\alpha,\beta)\in\Lambda_\alpha\times\Lambda_\beta\subseteq[0,1]\times[0,1]$ be such that $L(\tilde{\mu}_{\inf};\alpha)$ and $U(\tilde{\mu}_{\sup};\beta)$ are nonempty. Then (α,β) is called an *ideal-type hyper value* for $(X,\tilde{\mu})$ if the following assertion is valid.

$$(\forall x,y\in X)\left(\begin{array}{l}\tilde{\mu}_{\inf}(y)\leq\alpha \;\Rightarrow\; \min\{\tilde{\mu}_{\inf}(y*x),\tilde{\mu}_{\inf}(x)\}\leq\alpha,\\ \tilde{\mu}_{\sup}(y)\geq\beta \;\Rightarrow\; \max\{\tilde{\mu}_{\sup}(y*x),\tilde{\mu}_{\sup}(x)\}\geq\beta\end{array}\right). \tag{3.2}$$

Example 3.14. In Example 3.8, the pair (α,β) is an ideal-type hyper value for $(X,\tilde{\mu})$.

Example 3.15. Let $X=\{0,1,2,a,b\}$ be a set with the binary operation $*$ which is given in Table 2.

Table 2. Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then $(X, *, 0)$ is a BCI-algebra (see [5]). Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1]),\ x \mapsto \begin{cases} [0.54, 0.72) & \text{if } x = 0, \\ (0.58, 0.64] & \text{if } x = 1, \\ [0.56, 0.72) & \text{if } x = 2, \\ (0.60, 0.68] & \text{if } x = a, \\ [0.60, 0.64) & \text{if } x = b. \end{cases}$$

If we take $(\alpha, \beta) \in (0.54, 0.60] \times [0.64, 0.72)$, then (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$.

We consider a relation between subalgebra-type hyper value and ideal-type hyper value.

Theorem 3.16. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, \tilde{\mu}) \in \mathcal{B}_K(X)$ such that*

$$(\forall x \in X)\left(\tilde{\mu}_{\inf}(0) \geq \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(0) \leq \tilde{\mu}_{\sup}(x)\right). \tag{3.3}$$

Then every ideal-type hyper value for $(X, \tilde{\mu})$ is a subalgebra-type hyper value for $(X, \tilde{\mu})$.

Proof. Let (α, β) be an ideal-type hyper value for $(X, \tilde{\mu})$. Assume that $\tilde{\mu}_{\inf}(x * y) \leq \alpha$ and $\tilde{\mu}_{\sup}(a * b) \geq \beta$ for $x, y, a, b \in X$. Using (3.2), (2.2) and (3.3), we have

$$\begin{aligned} \alpha &\geq \min\{\tilde{\mu}_{\inf}((x * y) * x), \tilde{\mu}_{\inf}(x)\} \\ &= \min\{\tilde{\mu}_{\inf}((x * x) * y), \tilde{\mu}_{\inf}(x)\} \\ &= \min\{\tilde{\mu}_{\inf}(0 * y), \tilde{\mu}_{\inf}(x)\} \\ &= \min\{\tilde{\mu}_{\inf}(0), \tilde{\mu}_{\inf}(x)\} = \tilde{\mu}_{\inf}(x) \end{aligned}$$

and

$$\begin{aligned} \beta &\leq \max\{\tilde{\mu}_{\sup}((a * b) * a), \tilde{\mu}_{\sup}(a)\} \\ &= \max\{\tilde{\mu}_{\sup}((a * a) * b), \tilde{\mu}_{\sup}(a)\} \\ &= \max\{\tilde{\mu}_{\sup}(0 * b), \tilde{\mu}_{\sup}(a)\} \\ &= \max\{\tilde{\mu}_{\sup}(0), \tilde{\mu}_{\sup}(a)\} = \tilde{\mu}_{\sup}(a). \end{aligned}$$

It follows that

$$\min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \leq \tilde{\mu}_{\inf}(x) \leq \alpha \text{ and } \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\} \geq \tilde{\mu}_{\sup}(a) \geq \beta.$$

Therefore (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$. □

The converse of Theorem 3.16 is not true in general as seen in the following example.

Example 3.17. Let $X = \{0, 1, a, b, c\}$ be a set with the binary operation $*$ which is given in Table 3.

Table 3. Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

Then $(X, *, 0)$ is a BCI-algebra (see [5]). Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} [0.51, 0.55) & \text{if } x = 0, \\ (0.48, 0.63] & \text{if } x = 1, \\ [0.45, 0.58) & \text{if } x = a, \\ (0.41, 0.5] \cup [0.60, 0.63] & \text{if } x = b, \\ [0.35, 0.65] & \text{if } x = c. \end{cases}$$

If we take $(\alpha, \beta) \in (0.41, 0.45) \times (0.63, 0.65]$, then (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$, but it is not an ideal-type hyper value for $(X, \tilde{\mu})$ since

$$\tilde{\mu}_{\inf}(b) = 0.41 \leq \alpha \text{ and } \min\{\tilde{\mu}_{\inf}(b * a), \tilde{\mu}_{\inf}(a)\} = 0.45 \nleq \alpha$$

and/or

$$\tilde{\mu}_{\sup}(c) = 0.65 \geq \beta \text{ and } \max\{\tilde{\mu}_{\sup}(c * a), \tilde{\mu}_{\sup}(a)\} = 0.63 \ngeq \beta.$$

We provide conditions for a pair (α, β) to be an ideal-type hyper value.

Theorem 3.18. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0, 1] \times [0, 1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1, 4)$-hyperfuzzy ideal of $(X, *, 0)$, then (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$.*

Proof. Let $x, y, a, b \in X$ be such that $\tilde{\mu}_{\inf}(y) \leq \alpha$ and $\tilde{\mu}_{\sup}(b) \geq \beta$. Since $(X, \tilde{\mu})$ is a $(1, 4)$-hyperfuzzy ideal of $(X, *, 0)$, it follows that

$$\alpha \geq \tilde{\mu}_{\inf}(y) \geq \min\{\tilde{\mu}_{\inf}(y * x), \tilde{\mu}_{\inf}(x)\}$$

and

$$\beta \leq \tilde{\mu}_{\sup}(b) \leq \max\{\tilde{\mu}_{\sup}(b * a), \tilde{\mu}_{\sup}(a)\}.$$

Therefore (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$. □

Corollary 3.19. *Given a hyper structure* $(X, \tilde{\mu})$ *over* $(X, *, 0)$*, let* $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ *be such that* $L(\tilde{\mu}_{\inf}; \alpha)$ *and* $U(\tilde{\mu}_{\sup}; \beta)$ *are nonempty. If* $(X, \tilde{\mu})$ *is a* $(1,2)$*-hyperfuzzy (resp.,* $(3,2)$*-hyperfuzzy and* $(3,4)$*-hyperfuzzy) ideal of* $(X, *, 0)$*, then* (α, β) *is an ideal-type hyper value for* $(X, \tilde{\mu})$*.*

Proof. Straightforward. □

Theorem 3.20. *Let* $(X, \tilde{\mu})$ *be a hyper structure over* $(X, *, 0)$*. If* (α, β) *is an ideal-type hyper value for* $(X, \tilde{\mu})$*, then* $L(\tilde{\mu}_{\inf}; \alpha)$ *and* $U(\tilde{\mu}_{\sup}; \beta)$ *are* I*-energetic subsets of* $(X, *, 0)$*.*

Proof. Let $x, y, a, b \in X$ be such that $y \in L(\tilde{\mu}_{\inf}; \alpha)$ and $b \in U(\tilde{\mu}_{\sup}; \beta)$. Then $\tilde{\mu}_{\inf}(y) \leq \alpha$ and $\tilde{\mu}_{\sup}(b) \geq \beta$. Since (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$, it follows from (3.2) that $\min\{\tilde{\mu}_{\inf}(y * x), \tilde{\mu}_{\inf}(x)\} \leq \alpha$ and $\max\{\tilde{\mu}_{\sup}(b * a), \tilde{\mu}_{\sup}(a)\} \geq \beta$. Hence

$$\tilde{\mu}_{\inf}(y * x) \leq \alpha \text{ or } \tilde{\mu}_{\inf}(x) \leq \alpha$$

and

$$\tilde{\mu}_{\sup}(b * a) \geq \beta \text{ or } \tilde{\mu}_{\sup}(a) \geq \beta,$$

that is,

$$y * x \in L(\tilde{\mu}_{\inf}; \alpha) \text{ or } x \in L(\tilde{\mu}_{\inf}; \alpha)$$

and

$$b * a \in U(\tilde{\mu}_{\sup}; \beta) \text{ or } a \in U(\tilde{\mu}_{\sup}; \beta).$$

Thus $\{y * x, x\} \cap L(\tilde{\mu}_{\inf}; \alpha) \neq \emptyset$ and $\{b * a, a\} \cap U(\tilde{\mu}_{\sup}; \beta) \neq \emptyset$, and therefore $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are I-energetic subsets of $(X, *, 0)$. □

Combining Theorems 3.18 and 3.20, we have the following corollary.

Corollary 3.21. *Given a hyper structure* $(X, \tilde{\mu})$ *over* $(X, *, 0)$*, let* $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ *be such that* $L(\tilde{\mu}_{\inf}; \alpha)$ *and* $U(\tilde{\mu}_{\sup}; \beta)$ *are nonempty. If* $(X, \tilde{\mu})$ *is a* $(1,4)$*-hyperfuzzy ideal of* $(X, *, 0)$*, then* $L(\tilde{\mu}_{\inf}; \alpha)$ *and* $U(\tilde{\mu}_{\sup}; \beta)$ *are* I*-energetic subsets of* $(X, *, 0)$*.*

Definition 3.22 ([3])**.** A nonempty subset A of $(X, *, 0)$ is said to be *right vanished* if it satisfies:

$$(\forall x, y \in X)\,(x * y \in A \;\Rightarrow\; x \in A)\,. \tag{3.4}$$

A is said to be *right stable* if $A * X := \{a * x \mid a \in A,\, x \in X\} \subseteq A$.

Lemma 3.23 ([6])**.** *If* $(X, \tilde{\mu})$ *is a* $(4,1)$*-hyperfuzzy ideal of* $(X, *, 0)$*, then*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; \tilde{\mu}_{\inf}(x) \leq \tilde{\mu}_{\inf}(y),\, \tilde{\mu}_{\sup}(x) \geq \tilde{\mu}_{\sup}(y))\,. \tag{3.5}$$

Theorem 3.24. *Given a hyper structure* $(X, \tilde{\mu})$ *over* $(X, *, 0) \in \mathcal{B}_K(X)$ *and* $(\alpha, \beta) \in [0,1] \times [0,1]$*, if* $(X, \tilde{\mu})$ *is a* $(4,1)$*-hyperfuzzy ideal of* $(X, *, 0)$*, then* $L(\tilde{\mu}_{\inf}; \alpha)$ *and* $U(\tilde{\mu}_{\sup}; \beta)$ *are right stable subsets of* $(X, *, 0)$ *whenever they are nonempty.*

Proof. Let $(\alpha, \beta) \in [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. Let $x, a, b \in X$ be such that $a \in L(\tilde{\mu}_{\inf}; \alpha)$ and $b \in U(\tilde{\mu}_{\sup}; \beta)$. Then $\tilde{\mu}_{\inf}(a) \leq \alpha$ and $\tilde{\mu}_{\sup}(b) \geq \beta$. Since $a * x \leq a$ and $b * x \leq b$, it follows from Lemma 3.23 that $\tilde{\mu}_{\inf}(a * x) \leq \tilde{\mu}_{\inf}(a) \leq \alpha$ and $\tilde{\mu}_{\sup}(b * x) \geq \tilde{\mu}_{\sup}(b) \geq \beta$, that is, $a * x \in L(\tilde{\mu}_{\inf}; \alpha)$ and $b * x \in U(\tilde{\mu}_{\sup}; \beta)$. Hence $L(\tilde{\mu}_{\inf}; \alpha) * X \subseteq L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta) * X \subseteq U(\tilde{\mu}_{\sup}; \beta)$. Therefore $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are right stable subsets of $(X, *, 0)$. □

Corollary 3.25. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0) \in \mathcal{B}_K(X)$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, if $(X, \tilde{\mu})$ is a $(2,1)$-hyperfuzzy (resp., $(2,3)$-hyperfuzzy and $(4,3)$-hyperfuzzy) ideal of $(X, *, 0)$, then $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are right stable subsets of $(X, *, 0)$ whenever they are nonempty.*

Proof. Straightforward. □

Theorem 3.26. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0) \in \mathcal{B}_K(X)$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, if $(X, \tilde{\mu})$ is a $(4,1)$-hyperfuzzy ideal of $(X, *, 0)$, then $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are right vanished subsets of $(X, *, 0)$ whenever they are nonempty.*

Proof. Let $(\alpha, \beta) \in [0,1] \times [0,1]$ be such that $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are nonempty. Assume that $x * y \in U(\tilde{\mu}_{\inf}; \alpha)$ and $a * b \in L(\tilde{\mu}_{\sup}; \beta)$ for any $x, y, a, b \in X$. Using Lemma 3.23 implies that

$$\alpha \leq \tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(x), \text{ that is, } x \in U(\tilde{\mu}_{\inf}; \alpha)$$

and

$$\beta \geq \tilde{\mu}_{\sup}(a * b) \geq \tilde{\mu}_{\sup}(a), \text{ that is, } a \in L(\tilde{\mu}_{\sup}; \beta).$$

Hence $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are right vanished subsets of $(X, *, 0)$. □

Corollary 3.27. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0) \in \mathcal{B}_K(X)$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, if $(X, \tilde{\mu})$ is a $(2,1)$-hyperfuzzy (resp., $(2,3)$-hyperfuzzy and $(4,3)$-hyperfuzzy) ideal of $(X, *, 0)$, then $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are right vanished subsets of $(X, *, 0)$ whenever they are nonempty.*

Proof. Straightforward. □

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