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GEHRING INEQUALITIES ON TIME SCALES

MARTIN BOHNER AND SAMIR H. SAKER

Abstract. In this paper, we first prove a new dynamic inequality based on an application of the time scales version of a Hardy-type inequality. Second, by employing the obtained inequality, we prove several Gehring-type inequalities on time scales. As an application of our Gehring-type inequalities, we present some interpolation and higher integrability theorems on time scales. The results as special cases, when the time scale is equal to the set of all real numbers, contain some known results, and when the time scale is equal to the set of all integers, the results are essentially new.

1. Introduction

Let I be a fixed cube with sides parallel to the coordinate axes and let f and g be nonnegative measurable functions defined on I. The classical integral Hölder inequality

$$\int_I |f(x)g(x)|\mathrm{d}x \leq \left[\int_I |f(x)|^p \mathrm{d}x\right]^{\frac{1}{p}} \left[\int_I |g(x)|^q \mathrm{d}x\right]^{\frac{1}{q}},$$

where $1/p + 1/q = 1$, shows that there is a natural scale of inclusion for the $\mathrm{L}^p(I)$-spaces, when the underlying space I has a finite measure $|I|$.

In 1972, Muckenhoupt [14] proved the first simplest reverse integral (mean) inequality, which can be considered as a reverse inclusion, of the form

$$\frac{1}{|I|}\int_I w(x)\mathrm{d}x \leq \kappa \operatorname{essinf}_{x\in I} w(x), \tag{1.1}$$

where w is a nonnegative measurable function defined on I. A function verifying (1.1) is called an A_1-weight Muckenhoupt function. In [14] (see also [13]), it is proved that any A_1-weight Muckenhoupt function belongs to $\mathrm{L}^r(I)$, for $1 \leq r < s$ and s depending on κ and the dimension of the space.

In 1973, Gehring [8] extended the result of Muckenhoupt for reverse mean inequalities. We say that w satisfies a Gehring condition (or a reverse Hölder inequality) if there exists $p > 1$ and a constant $\kappa > 0$ such that for every cube I with sides parallel to the coordinate axes, we have

$$\left(\frac{1}{|I|}\int_I w^p(x)\mathrm{d}x\right)^{1/p} \leq \frac{\kappa}{|I|}\int_I w(x)\mathrm{d}x.$$

In this case we write $w \in \mathrm{RH}_p$. A well known result obtained by Gehring [8] states that if $w \in \mathrm{RH}_p$, then w satisfies a higher integrability condition, namely

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for sufficiently small $\varepsilon > 0$, $q = p + \varepsilon$, we have for any cube I,

$$\left(\frac{1}{|I|}\int_I w^q(x)\mathrm{d}x\right)^{1/q} \leq \left(\frac{\kappa}{|I|}\int_I w^p(x)\mathrm{d}x\right)^{1/p}.$$

In other words, Gehring's result states that $w \in \mathrm{RH}_p$ implies that there exists $\varepsilon > 0$ such that $w \in \mathrm{RH}_{p+\varepsilon}$. The proof of Gehring's inequality is based on the use of the Calderón–Zygmund decomposition and the scale structure of L^p-spaces. In [12], the author extended Gehring's inequality by means of connecting it to the real method of interpolation by considering maximal operators, and via rearrangements reinterpreted the underlying estimates through the use of K-functionals. This technique allowed to quantify in a precise way, via reiteration, how Calderón–Zygmund decompositions have to be reparameterized in order to characterize different L^p-spaces.

Reverse integral inequalities (cf. [8, 9]) and its many variants and extensions are important in qualitative analysis of nonlinear PDEs, in the study of weighted norm inequalities for classical operators of harmonic analysis, as well as in functional analysis. These inequalities also appear in different fields of analysis such as quasiconformal mappings, weighted Sobolev embedding theorems, and regularity theory of variational problems (see [11]).

In recent years, the study of dynamic inequalities on time scales has received a lot of attention. For details, we refer to the books [2, 3, 5, 6] and the recent paper [1] and the references cited therein. The general idea in studying dynamic inequalities on time scales is to prove a result for an inequality, where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. This idea goes back to its founder Stefan Hilger [10]. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ with $q > 1$. The study of dynamic inequalities on time scales helps avoid proving results twice – once for differential inequalities and once again for difference inequalities.

Following this trend and to develop the study of dynamic inequalities on time scales, we aim in this paper to prove Gehring-type inequalities on time scales, which contain the classical integral inequalities of Gehring's type and their discrete versions as special cases. We believe that the reverse dynamic inequalities on time scales will be, just like in the classical case, similarly important for the analysis of dynamic equations on time scales.

The rest of the paper is organized as follows: In Section 2, we recall some definitions and notations related to time scales which will be used throughout the paper. Section 3 features some auxiliary results, in particular, a time scales version of Hardy's inequality. In Section 4, we present the proofs of our Gehring-type inequalities on time scales and give some interpolation results as well as some higher integrability theorems for monotone nonincreasing functions on time scales, see Section 5. As special cases, we offer discrete versions of the Gehring inequalities. To the best of the authors' knowledge, nothing is known regarding the discrete analogues of Gehring inequalities or even their extensions, and thus the presented discrete inequalities are essentially new.

2. Time Scales Preliminaries

We assume that the reader is familiar with time scales as presented in the monographs [5, 6]. For concepts concerning general measure and integration on time scales, see [6, Chapter 5] and [4, 7]. Here, we only state four facts that are essentially used in the proofs of our results. For a function $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is a time scale, we denote the delta derivative by f^{Δ} and the forward shift by $f^{\sigma} = f \circ \sigma$, where σ is the time scales jump operator. The time scales *product rule* says that for two differentiable functions f and g, the product fg is differentiable with

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta}. \tag{2.1}$$

On the other hand, the time scales *integration by parts rule* says that for two integrable functions $f, g : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$, we have

$$\int_a^b f^{\Delta}(t) g(\sigma(t)) \Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f(t) g^{\Delta}(t) \Delta t. \tag{2.2}$$

We also need the time scales *chain rule* which says that if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable with

$$(f \circ g)^{\Delta} = g^{\Delta} \int_0^1 f'(hg^{\sigma} + (1-h)g) \mathrm{d}h. \tag{2.3}$$

Finally, we need the time scales *Hölder inequality* which says that for two nonnegative integrable functions $f, g : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$ and $p, q > 1$ with $1/p + 1/q = 1$, we have

$$\int_a^b f(t) g(t) \Delta t \le \left[\int_a^b f^p(t) \Delta t \right]^{1/p} \left[\int_a^b f^q(t) \Delta t \right]^{1/q}, \tag{2.4}$$

and p, q are called the corresponding *exponents*.

Throughout this paper, we assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions, delta differentiable, locally delta integrable, and the integrals considered are assumed to exist (finite, i.e., convergent).

3. Auxiliary Results

In this section, we give some auxiliary results that are used in the proofs of our main results.

Definition 3.1. Throughout this paper, we suppose that $\mathbb{T}$ is a time scale with $0 \in \mathbb{T}$, and we let $T > 0$ with $T \in \mathbb{T}$. For any function $f : (0, T] \to \mathbb{R}$ which is Δ-integrable, nonnegative, and nonincreasing, we define the *average* function $\mathcal{A}f : (0, T] \to \mathbb{R}$ by

$$\mathcal{A}f(t) := \frac{1}{t} \int_0^t f(s) \Delta s \quad \text{for all} \quad t \in (0, T]. \tag{3.1}$$

Some simple facts about $\mathcal{A}f$ are given next.

Lemma 3.2. *If $f : (0, T] \to \mathbb{R}$ is Δ-integrable, nonnegative, and nonincreasing, then*

$$\mathcal{A}f \ge f. \tag{3.2}$$

Proof. Due to

$$\mathcal{A}f(t) = \frac{1}{t}\int_0^t f(s)\Delta s \geq \frac{1}{t}\int_0^t f(t)\Delta s = f(t),$$

(3.2) follows immediately. □

Lemma 3.3. *If $f : (0, T] \to \mathbb{R}$ is Δ-integrable, nonnegative, and nonincreasing, then so is $\mathcal{A}f$.*

Proof. In this proof, we write $F = \mathcal{A}f$ for brevity. We show that F inherits the nonincreasing nature of f. Let $t_1 < t_2$. Then

$$\begin{aligned} F(t_1) - F(t_2) &= \frac{1}{t_1}\int_0^{t_1} f(s)\Delta s - \frac{1}{t_2}\left[\int_0^{t_1} f(s)\Delta s + \int_{t_1}^{t_2} f(s)\Delta s\right] \\ &= \frac{t_2 - t_1}{t_2}\left[\frac{1}{t_1}\int_0^{t_1} f(s)\Delta s - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(s)\Delta s\right] \\ &\geq \frac{t_2 - t_1}{t_2}\left[\frac{1}{t_1}\int_0^{t_1} f(t_1)\Delta s - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t_1)\Delta s\right] = 0, \end{aligned}$$

completing the proof. □

Now we present a Hardy inequality (see also [3, Corollary 1.5.1]) which, for completeness, we prove in our special setting.

Theorem 3.4. *If $q > 1$ and $f : (0, T] \to \mathbb{R}$ is Δ-integrable, nonnegative, and nonincreasing, then*

$$\mathcal{A}\left[(\mathcal{A}f)^\sigma\right]^q \leq \left(\frac{q}{q-1}\right)^q \mathcal{A}f^q. \tag{3.3}$$

Proof. In this proof, we write $F = \mathcal{A}f$ for brevity. Using Lemma 3.3, the chain rule shows that

$$\begin{aligned} (F^q)^\Delta &\overset{(2.3)}{=} qF^\Delta \int_0^1 (hF^\sigma + (1-h)F)^{q-1}\,\mathrm{d}h \\ &\leq qF^\Delta \int_0^1 (hF^\sigma + (1-h)F^\sigma)^{q-1}\,\mathrm{d}h = qF^\Delta (F^\sigma)^{q-1}. \end{aligned} \tag{3.4}$$

Moreover, since

$$tF(t) = \int_0^t f(s)\Delta s,$$

the product rule yields

$$f(t) \overset{(2.1)}{=} F(\sigma(t)) + tF^\Delta(t). \tag{3.5}$$

Now, putting $u(t) = t$ and $v(t) = F^q(t)$, we use integration by parts to find

$$\begin{aligned} \int_0^t (F(\sigma(s)))^q \Delta s &= \int_0^t u^\Delta(s) v(\sigma(s))\Delta s \\ &\overset{(2.2)}{=} u(t)v(t) - \lim_{s\to 0^+} u(s)v(s) - \int_0^t u(s)v^\Delta(s)\Delta s \\ &= tF^q(t) - \int_0^t s v^\Delta(s)\Delta s \end{aligned}$$

$$\begin{aligned}
&\geq -\int_0^t s v^{\Delta}(s)\Delta s \\
&\overset{(3.4)}{\geq} -q\int_0^t sF^{\Delta}(s)F^{q-1}(\sigma(s))\Delta s \\
&\overset{(3.5)}{=} -q\int_0^t \left[f(s)-F(\sigma(s))\right]F^{q-1}(\sigma(s))\Delta s \\
&= -q\int_0^t f(s)F^{q-1}(\sigma(s))\Delta s + q\int_0^t (F(\sigma(s)))^q\Delta s
\end{aligned}$$

so that, by using Hölder's inequality with exponents q and $q/(q-1)$,

$$\begin{aligned}
(q-1)\int_0^t (F(\sigma(s)))^q\Delta s &\leq q\int_0^t f(s)(F(\sigma(s)))^{q-1}\Delta s \\
&\overset{(2.4)}{\leq} q\left[\int_0^t (f(s))^q\Delta s\right]^{1/q}\left[\int_0^t (F(\sigma(s)))^q\Delta s\right]^{(q-1)/q},
\end{aligned}$$

resulting in (3.3). □

In the main results of this paper, we assume that there exists a constant $\lambda \geq 1$ such that

(3.6) $$\sigma(t) \leq \lambda t \quad \text{for all} \quad t \in \mathbb{T}.$$

We now apply the time scales chain rule to obtain some estimates that will be used later.

Lemma 3.5. *Let $x(t) = t$. If $0 < \gamma < 1$, then*

(3.7) $$\left(x^{1-\gamma}\right)^{\Delta} \geq \frac{1-\gamma}{\sigma^{\gamma}},$$

and if $\gamma > 1$ and (3.6) holds, then

(3.8) $$\left(x^{1-\gamma}\right)^{\Delta} \geq \frac{(1-\gamma)\lambda^{\gamma}}{\sigma^{\gamma}}.$$

Proof. By the chain rule, we obtain

$$\begin{aligned}
\left(x^{1-\gamma}\right)^{\Delta}(t) &\overset{(2.3)}{=} (1-\gamma)x^{\Delta}(t)\int_0^1 \frac{\mathrm{d}h}{(hx(\sigma(t))+(1-h)x(t))^{\gamma}} \\
&= (1-\gamma)\int_0^1 \frac{\mathrm{d}h}{(h\sigma(t)+(1-h)t)^{\gamma}}.
\end{aligned}$$

Thus, if $0 < \gamma < 1$, then

$$\left(x^{1-\gamma}\right)^{\Delta}(t) \geq (1-\gamma)\int_0^1 \frac{\mathrm{d}h}{(h\sigma(t)+(1-h)\sigma(t))^{\gamma}} = \frac{1-\gamma}{(\sigma(t))^{\gamma}},$$

which is (3.7), and if $\gamma > 1$ and (3.6) holds, then

$$\left(x^{1-\gamma}\right)^{\Delta}(t) \geq (1-\gamma)\int_0^1 \frac{\mathrm{d}h}{(ht+(1-h)t)^{\gamma}} = \frac{1-\gamma}{t^{\gamma}} \overset{(3.6)}{\geq} \frac{(1-\gamma)\lambda^{\gamma}}{(\sigma(t))^{\gamma}},$$

which is (3.8). □

Lemma 3.6. *If F is nonnegative and nondecreasing and $\gamma > 1$, then*

(3.9) $$(F^{\gamma})^{\Delta} \geq \gamma F^{\Delta}F^{\gamma-1}.$$

Proof. Again we apply the chain rule to see that

$$\begin{aligned}(F^\gamma)^\Delta &\overset{(2.3)}{=} \gamma F^\Delta \int_0^1 (hF^\sigma + (1-h)F)^{\gamma-1} \mathrm{d}h \\ &\geq \gamma F^\Delta \int_0^1 (hF + (1-h)F)^{\gamma-1} \mathrm{d}h \\ &= \gamma F^\Delta F^{\gamma-1},\end{aligned}$$

which shows (3.9). □

4. Main Results

We say that $f : (0,T] \to \mathbb{R}$ belongs to $\mathrm{L}^p_\Delta(0,T]$ provided $\int_0^T |f(t)|^p \Delta t < \infty$. The first theorem will be used later in the proof of the Gehring inequality.

Theorem 4.1. *If $f \in \mathrm{L}^p_\Delta(0,T]$ for $p > 1$ is nonnegative and nonincreasing, then, for any $q \in (0,p)$, we have*

$$\mathcal{A}f^p \leq \frac{q}{p} [\mathcal{A}f^q]^{p/q} + \frac{(p-q)\lambda^{p/q}}{p} \mathcal{A}\left[(\mathcal{A}f^q)^\sigma\right]^{p/q}. \tag{4.1}$$

Proof. From the Hardy inequality, see (3.3), we see that the second integral on the right-hand side of (4.1) is finite. Now, we consider this integral. Then, for $0 < q < p$, we put

$$\gamma = \frac{p}{q} > 1 \quad \text{and} \quad F(t) = \int_0^t f^q(s)\Delta s.$$

Using the notation from Lemma 3.5, we have

$$\begin{aligned}&\frac{(p-q)\lambda^{p/q}}{pt} \int_0^t \left[\frac{1}{\sigma(s)} \int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^{p/q} \Delta s \\ &= \frac{(\gamma-1)\lambda^\gamma}{\gamma t} \int_0^t \left[\frac{1}{\sigma(s)} \int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^\gamma \Delta s \\ &= \frac{(\gamma-1)\lambda^\gamma}{\gamma t} \int_0^t \left[\frac{F(\sigma(s))}{\sigma(s)}\right]^\gamma \Delta s \\ &= -\frac{1}{\gamma t} \int_0^t F^\gamma(\sigma(s)) \frac{(1-\gamma)\lambda^\gamma}{(\sigma(s))^\gamma} \Delta s \\ &\overset{(3.8)}{\geq} -\frac{1}{\gamma t} \int_0^t F^\gamma(\sigma(s)) \left(x^{1-\gamma}\right)^\Delta (s)\Delta s \\ &\overset{(2.2)}{=} \lim_{s\to 0^+} \frac{F^\gamma(s)x^{1-\gamma}(s)}{\gamma t} - \frac{F^\gamma(t)x^{1-\gamma}(t)}{\gamma t} + \frac{1}{\gamma t}\int_0^t (F^\gamma)^\Delta (s) x^{1-\gamma}(s)\Delta s \\ &= \frac{1}{\gamma t} \int_0^t s^{1-\gamma} (F^\gamma)^\Delta (s)\Delta s + \frac{1}{\gamma t} \lim_{s\to 0^+} \left[s\left(\frac{F(s)}{s}\right)^\gamma\right] - \frac{1}{\gamma}\left(\frac{F(t)}{t}\right)^\gamma \\ &\overset{(3.9)}{\geq} \frac{1}{\gamma t} \int_0^t \frac{\gamma F^\Delta(s) F^{\gamma-1}(s)}{s^{\gamma-1}} \Delta s - \frac{1}{\gamma}\left(\frac{F(t)}{t}\right)^\gamma \\ &= \frac{1}{t} \int_0^t f^q(s) \left[\mathcal{A}f^q(s)\right]^{\gamma-1} \Delta s - \frac{1}{\gamma}\left[\mathcal{A}f^q(t)\right]^\gamma\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.2)}{\geq} \frac{1}{t}\int_0^t f^q(s)\,[f^q(s)]^{\gamma-1}\,\Delta s - \frac{1}{\gamma}\,[\mathcal{A}f^q(t)]^\gamma \\
&= \frac{1}{t}\int_0^t [f^q(s)]^{\gamma}\,\Delta s - \frac{1}{\gamma}\,[\mathcal{A}f^q(t)]^\gamma \\
&= \mathcal{A}f^p(t) - \frac{q}{p}\,[\mathcal{A}f^q(t)]^{p/q}
\end{aligned}
$$

from which (4.1) follows. □

Now, we are ready to state and prove our first time scales version of Gehring's mean inequality for monotone functions.

Theorem 4.2 (Gehring Inequality I). *Assume* (3.6). *If* $f \in \mathrm{L}^q_\Delta(0,T]$ *for* $q > 1$ *is nonnegative and nonincreasing such that*

$$\mathcal{A}f^q \leq \kappa\,[\mathcal{A}f]^q \quad \text{for some} \quad \kappa > 0, \tag{4.2}$$

then $f \in \mathrm{L}^p_\Delta(0,T]$ *for any* $p > q$ *satisfying*

$$\tilde{\kappa} := \frac{q\kappa^{p/q}}{p - (p-q)(\lambda\kappa)^{p/q}\left(\frac{p}{p-1}\right)^p} > 0, \tag{4.3}$$

and in this case,

$$\mathcal{A}f^p \leq \tilde{\kappa}\,[\mathcal{A}f]^p\,. \tag{4.4}$$

Proof. Assuming (4.2), we find

$$
\begin{aligned}
\frac{1}{t}\int_0^t f^p(s)\Delta s &\overset{(4.1)}{\leq} \frac{q}{p}\left[\frac{1}{t}\int_0^t f^q(s)\Delta s\right]^{p/q} \\
&\quad + \frac{(p-q)\lambda^{p/q}}{pt}\int_0^t \left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^{p/q}\Delta s \\
&\overset{(4.2)}{\leq} \frac{q}{p}\kappa^{p/q}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p + \frac{(p-q)\lambda^{p/q}}{pt}\int_0^t \kappa^{p/q}\left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\right]^p\Delta s \\
&\overset{(3.3)}{\leq} \frac{q}{p}\kappa^{p/q}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p + \frac{(p-q)(\lambda\kappa)^{p/q}}{pt}\left(\frac{p}{p-1}\right)^p\int_0^t f^p(s)\Delta s
\end{aligned}
$$

so that, due to (4.3),

$$\frac{1}{t}\int_0^t f^p(s)\Delta s \leq \tilde{\kappa}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p,$$

from which (4.4) follows. □

As a special case of Theorem 4.2 when $\mathbb{T} = \mathbb{R}$, we get the classical Gehring inequality (see Section 1) with $\lambda = 1$. In the case when $\mathbb{T} = \mathbb{N}$, we have the following result with $\lambda = 2$.

Corollary 4.3 (Discrete Gehring Inequality I). *Let* $q > 1$ *and* $\{a_n\}_{n\in\mathbb{N}_0}$ *be a nonnegative and nonincreasing sequence such that*

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^q \leq \kappa\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i\right)^q.$$

Then, for $p > q$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^p \le \tilde{\kappa}\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i\right)^p,$$

provided

$$\tilde{\kappa} := \frac{q\kappa^{p/q}}{p-(p-q)(2\kappa)^{p/q}\left(\frac{p}{p-1}\right)^p} > 0.$$

It is natural to ask what happens if in (4.4) we fix $p > 1$ and consider the improvement to this inequality that would result from lowering the exponent on the right-hand side. The following result gives an answer.

Theorem 4.4. *Suppose that the assumptions of Theorem 4.2 hold and define $\tilde{\kappa}$ as in* (4.3)*. Then, for all $0 < r < 1$, we have*

$$\mathcal{A}f^p \le \overline{\kappa}\,[\mathcal{A}f^r]^{p/r}, \quad \textit{where} \quad \overline{\kappa} := \tilde{\kappa}^{1/\theta} \quad \textit{with} \quad \theta := \frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}. \tag{4.5}$$

Proof. Note first that $\theta \in (0,1)$ and

$$\frac{1-\theta}{p} + \frac{\theta}{r} = 1.$$

Then, by the Hölder inequality with exponents $p/(1-\theta)$ and r/θ, we have

$$\begin{aligned}
\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{1/p} &\overset{(4.4)}{\le} \frac{\tilde{\kappa}^{1/p}}{t}\int_0^t f(s)\Delta s \\
&= \frac{\tilde{\kappa}^{1/p}}{t}\int_0^t f^{1-\theta}(s)f^{\theta}(s)\Delta s \\
&\overset{(2.4)}{\le} \frac{\tilde{\kappa}^{1/p}}{t}\left[\int_0^t f^p(s)\Delta s\right]^{(1-\theta)/p}\left[\int_0^t f^r(s)\Delta s\right)^{\theta/r} \\
&= \tilde{\kappa}^{1/p}\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{(1-\theta)/p}\left[\frac{1}{t}\int_0^t f^r(s)\Delta s\right]^{\theta/r}
\end{aligned}$$

so that, by dividing, we find

$$\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{\theta/p} \le \tilde{\kappa}^{1/p}\left[\frac{1}{t}\int_0^t f^r(s)\Delta s\right]^{\theta/r},$$

i.e., (4.5). □

By Theorem 4.4, under the assumptions of Theorem 4.2, if $f \in \mathrm{L}^r_\Delta(0,T]$ for $0 < r < 1$, then $f \in \mathrm{L}^p_\Delta(0,T]$ for $p > 1$. But in the general case when $p \ne r$, $\mathrm{L}^p_\Delta(0,T]$ neither includes nor is included in $\mathrm{L}^r_\Delta(0,T]$. The following theorem gives some results for $\mathrm{L}^p_\Delta(0,T]$-interpolation.

Theorem 4.5. *Suppose that $0 < p_0 < p_1 < \infty$ and that $0 < \theta < 1$.*

(i) *If $p = (1-\theta)p_0 + \theta p_1$ and $f \in \mathrm{L}^{p_0}_\Delta(0,T] \cap \mathrm{L}^{p_1}_\Delta(0,T]$, then $f \in \mathrm{L}^p_\Delta(0,T]$ and*

$$\mathcal{A}f^p \le [\mathcal{A}f^{p_0}]^{1-\theta}[\mathcal{A}f^{p_1}]^{\theta}.$$

(ii) *If* $p = \frac{1}{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}}$ *and* $f \in \mathrm{L}_\Delta^{p_0}(0,T] \cap \mathrm{L}_\Delta^{p_1}(0,T]$*, then* $f \in \mathrm{L}_\Delta^{p}(0,T]$ *and*

$$\mathcal{A}f^p \le [\mathcal{A}f^{p_0}]^{(1-\theta)p/p_0} [\mathcal{A}f^{p_1}]^{\theta p/p_1}.$$

Proof. For (i), we apply the Hölder inequality with exponents $1/(1-\theta)$ and $1/\theta$ to see that

$$\begin{aligned}\frac{1}{t}\int_0^t f^p(s)\Delta s &= \frac{1}{t}\int_0^t f^{(1-\theta)p_0}(s) f^{\theta p_1}(s)\Delta s \\ &\overset{(2.4)}{\le} \left[\frac{1}{t}\int_0^t f^{p_0}(s)\Delta s\right]^{1-\theta}\left[\frac{1}{t}\int_0^t f^{p_1}(s)\Delta s\right]^{\theta},\end{aligned}$$

which shows (i). For (ii), we apply the Hölder inequality with exponents $1/(1-\gamma)$ and $1/\gamma$, where

$$\gamma := \frac{\theta p}{p_1} \quad \text{so that} \quad 1-\gamma = \frac{(1-\theta)p}{p_0},$$

to see that

$$\begin{aligned}\frac{1}{t}\int_0^t f^p(s)\Delta s &= \frac{1}{t}\int_0^t f^{(1-\theta)p}(s) f^{\theta p}(s)\Delta s \\ &\overset{(2.4)}{\le} \left[\frac{1}{t}\int_0^t f^{(1-\theta)p/(1-\gamma)}(s)\Delta s\right]^{1-\gamma}\left[\frac{1}{t}\int_0^t f^{\theta p/\gamma}(s)\Delta s\right]^{\gamma} \\ &= \left[\frac{1}{t}\int_0^t f^{p_0}(s)\Delta s\right]^{(1-\theta)p/p_0}\left[\frac{1}{t}\int_0^t f^{p_1}(s)\Delta s\right]^{\theta p/p_1},\end{aligned}$$

which shows (ii). □

In the following, we give a new proof of Gehring's mean inequality on time scales. The inequality will be proved by using a condition similar to the condition (1.1) due to Muckenhoupt. In fact, we do not assume that the reverse Hölder inequality holds.

Theorem 4.6 (Gehring Inequality II)**.** *Assume* (3.6)*. If* $f : (0,T] \to \mathbb{T}$ *is nonnegative and nonincreasing such that*

$$\mathcal{A}f^\sigma \le \nu f \quad \textit{for some} \quad \nu > 1, \tag{4.6}$$

then $f \in \mathrm{L}_\Delta^p(0,T]$ *for* $p \in [1, \alpha/(\alpha-1))$*, where* $\alpha = \lambda\nu$*, and we have*

$$\mathcal{A}(f^p)^\sigma \le \tilde{\nu}\,[\mathcal{A}f^\sigma]^p, \quad \textit{where} \quad \tilde{\nu} := \frac{\alpha}{\alpha - p(\alpha-1)} > 0. \tag{4.7}$$

Proof. For this proof, we put

$$F(t) := \int_0^t f^\sigma(s)\Delta s, \quad l(t) = \log(t), \quad L(t) = \log(F(t)).$$

By the chain rule, we get

$$\begin{aligned}\frac{1}{\alpha} l^\Delta(t) &\overset{(2.3)}{=} \frac{1}{\lambda\nu}\int_0^1 \frac{\mathrm{d}h}{h\sigma(t)+(1-h)t} \\ &\overset{(3.6)}{\le} \frac{1}{\lambda\nu}\int_0^1 \frac{\mathrm{d}h}{h\sigma(t)+(1-h)\frac{\sigma(t)}{\lambda}} \\ &\le \frac{1}{\lambda\nu}\cdot\frac{\lambda}{\sigma(t)} = \frac{1}{\nu\sigma(t)}\end{aligned}$$

$$
\begin{aligned}
&\overset{(4.6)}{\leq} \frac{f(\sigma(t))}{F(\sigma(t))} = \frac{F^{\Delta}(t)}{F(\sigma(t))}\\
&= F^{\Delta}(t)\int_0^1 \frac{\mathrm{d}h}{hF(\sigma(t))+(1-h)F(\sigma(t))}\\
&\leq F^{\Delta}(t)\int_0^1 \frac{\mathrm{d}h}{hF(\sigma(t))+(1-h)F(t)}\\
&\overset{(2.3)}{=} L^{\Delta}(t),
\end{aligned}
$$

and hence, by integrating,

$$\log\left(\frac{t}{\sigma(s)}\right)^{1/\alpha} = \frac{1}{\alpha}l(t) - \frac{1}{\alpha}l(\sigma(s)) \leq L(t) - L(\sigma(s)) = \log\left(\frac{F(t)}{F(\sigma(s))}\right)$$

so that

$$f(\sigma(s)) \leq \frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\sigma(\tau))\Delta\tau = \frac{F(\sigma(s)}{\sigma(s)} \leq \left(\frac{\sigma(s)}{t}\right)^{1/\alpha}\frac{F(t)}{\sigma(s)},$$

and by integrating again, putting $\gamma := p(1-1/\alpha) \in (0,1)$, and using the notation from Lemma 3.5, we obtain

$$
\begin{aligned}
\frac{1}{t}\int_0^t f^p(\sigma(s))\Delta s &\leq \frac{F^p(t)}{t^{1+p/\alpha}}\int_0^t \frac{\Delta s}{(\sigma(s))^{p(1-1/\alpha)}}\\
&\overset{(3.7)}{\leq} \frac{F^p(t)}{(1-\gamma)t^{1+p/\alpha}}\int_0^t \left(x^{1-\gamma}\right)^{\Delta}(s)\Delta s\\
&= \frac{t^{1-\gamma}F^p(t)}{(1-\gamma)t^{1+p/\alpha}} = \frac{1}{1-\gamma}\left(\frac{F(t)}{t}\right)^p,
\end{aligned}
$$

proving (4.7). □

As a special case of Theorem 4.6 when $\mathbb{T} = \mathbb{N}$, we have the following result.

Corollary 4.7 (Discrete Gehring Inequality II). *Let $\{a_n\}_{n\in\mathbb{N}_0}$ be a nonnegative and nonincreasing sequence. If there exists a constant $\nu > 1$ such that*

$$\frac{1}{n}\sum_{i=1}^n a_i \leq \nu a_n,$$

then, for $p \in [1, \alpha/(\alpha-1)]$, where $\alpha = 2\nu$, we have

$$\frac{1}{n}\sum_{i=1}^n a_i^p \leq \tilde{\nu}\left[\frac{1}{n}\sum_{i=1}^n a_i\right]^p, \quad \textit{where} \quad \tilde{\nu} := \frac{\alpha}{\alpha - p(\alpha-1)}.$$

5. Higher Integrability

In the following, as an application of Gehring's inequality (4.7), we prove a higher integrability theorem for monotone nonincreasing functions. First notice that for all nonnegative and nonincreasing functions $f \in \mathrm{L}^q_{\Delta}(0,T]$ with $q > 1$, we always have

$$\mathcal{A}f^q(t) = \frac{1}{t}\int_0^t f^q(s)\Delta s = \frac{1}{t}\int_0^t f^{q-1}(s)f(s)\Delta s \geq \frac{f^{q-1}(t)}{t}\int_0^t f(s)\Delta s. \tag{5.1}$$

Let us now consider the class of nonnegative and nonincreasing functions $f \in \mathrm{L}^q_\Delta(0,T]$ that satisfy the reverse of (5.1), namely

$$\mathcal{A}f^q \le \eta f^{q-1}\mathcal{A}f \quad \text{for some} \quad \eta > 1. \tag{5.2}$$

Theorem 5.1. *Assume* (3.6). *If* $f \in \mathrm{L}^q_\Delta(0,T]$ *for* $q>1$ *is nonnegative and nonincreasing such that* (5.2) *holds, then* $f \in \mathrm{L}^p_\Delta(0,T]$ *for* $p \in [q, q+c]$, $c \in (q,\eta)$, *and we have*

$$\mathcal{A}(f^p)^\sigma \le \tilde{\eta}\,[\mathcal{A}f^q]^{p/q}, \quad \textit{where} \quad \tilde{\eta} := \frac{\lambda {\eta_q}^{1+p/q}}{\lambda\eta_q - \frac{p}{q}(\lambda\eta_q - 1)} \quad \textit{with} \quad \eta_q = \frac{\eta q}{q-1}. \tag{5.3}$$

Proof. In this proof, we write $F = \mathcal{A}f^q$ for brevity. By using the Hölder inequality with exponents $q/(q-1)$ and q, we obtain

$$\begin{aligned}
\frac{1}{t}\int_0^t F(\sigma(s))\Delta s &\overset{(5.2)}{\le} \frac{\eta}{t}\int_0^t (f(\sigma(s)))^{q-1}\cdot\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\Delta s \\
&\overset{(2.4)}{\le} \frac{\eta}{t}\left[\int_0^t (f(\sigma(s)))^q\Delta s\right]^{(q-1)/q}\left[\int_0^t\left(\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\right)^q\Delta s\right]^{1/q} \\
&\overset{(3.3)}{\le} \frac{\eta q}{(q-1)t}\left[\int_0^t (f(s))^q\Delta s\right]^{(q-1)/q}\left[\int_0^t (f(s))^q\Delta s\right]^{1/q} \\
&= \frac{\eta_q}{t}\int_0^t f^q(s)\Delta s = \eta_q F(t),
\end{aligned}$$

i.e.,

$$\mathcal{A}F^\sigma \le \eta_q F. \tag{5.4}$$

Since F is also nonnegative and nonincreasing (see Lemma 3.3), it satisfies the assumptions of Theorem 4.6, and thus

$$\frac{1}{t}\int_0^t [F(\sigma(s))]^r\,\Delta s \le \tilde{\eta}_q\left[\frac{1}{t}\int_0^t F(\sigma(s))\Delta s\right]^r \tag{5.5}$$

with

$$\tilde{\eta}_q = \frac{\alpha_q}{\alpha_q - r(\alpha_q - 1)} \quad \text{and} \quad \alpha_q = \lambda\eta_q \quad \text{for} \quad r = \frac{p}{q} \in \left[1, \frac{\alpha_q}{\alpha_q - 1}\right).$$

Noting that

$$F(t) = \frac{1}{t}\int_0^t f^q(s)\Delta s \ge f^q(t), \tag{5.6}$$

we obtain

$$\begin{aligned}
\frac{1}{t}\int_0^t (f(\sigma(s)))^p\Delta s &= \frac{1}{t}\int_0^t (f^q(\sigma(s)))^r\Delta s \\
&\overset{(5.6)}{\le} \frac{1}{t}\int_0^t (F(\sigma(s)))^r\Delta s \\
&\overset{(5.5)}{\le} \tilde{\eta}_q\left(\frac{1}{t}\int_0^t F^\sigma(s)\Delta s\right)^r \\
&\overset{(5.4)}{\le} \tilde{\eta}_q{\eta_q}^r\,[F(t)]^r = \tilde{\eta}\,[F(t)]^r
\end{aligned}$$

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$$= \tilde{\eta}\left[\frac{1}{t}\int_0^t f^q(s)\Delta s\right]^{p/q},$$

proving (5.3). □

In Theorem 5.1, if $\mathbb{T}=\mathbb{R}$, then we have that $\sigma(t)=t$, $\alpha_q=\eta_q$, and we get the following result.

Corollary 5.2. *Let $\eta>1$ and $q>1$. Then every nonnegative nonincreasing function f satisfying*

$$\int_0^t f^q(x)\mathrm{d}x \le \eta f^{q-1}(t)\int_0^t f(x)\mathrm{d}x$$

belongs to $\mathrm{L}^p_\Delta(0,T](0,T]$ for $p\in[q,q+c]$ and $c\in(q,\eta)$, and we have

$$\frac{1}{t}\int_0^t f^p(x)\mathrm{d}x \le \tilde{\eta}\left(\frac{1}{t}\int_0^t f^q(x)\mathrm{d}x\right)^{p/q},$$

where

$$\tilde{\eta} := \frac{\left(\frac{\eta q}{q-1}\right)^{\frac{p}{q}+1}}{\frac{\eta q}{q-1}-\frac{p}{q}\left(\frac{\eta q}{q-1}-1\right)}.$$

In Theorem 5.1, if $\mathbb{T}=\mathbb{N}$, then we have that $\sigma(t)=t+1$, and by choosing $\lambda=2$, we get the following result.

Corollary 5.3. *Let $\eta>1$ and $q>1$. Suppose $\{a_n\}_{n\in\mathbb{N}_0}$ is a nonnegative and nonincreasing sequence satisfying*

$$\sum_{i=0}^{n-1} a_i^q \le \eta a_n^{q-1}\sum_{i=0}^{n-1} a_i.$$

Then, for $p\in[q,q+c]$, $c\in(q,\eta)$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^p \le \tilde{\eta}\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i^q\right)^{p/q},$$

where

$$\tilde{\eta} := \frac{2\left(\frac{\eta q}{q-1}\right)^{\frac{p}{q}+1}}{2\frac{\eta q}{q-1}-\frac{p}{q}\left(2\frac{\eta q}{q-1}-1\right)}.$$

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Basin of attraction of the fixed point and period-two solutions of a certain anti-competitive map

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Abstract

We investigate the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}^2}{C x_{n-1}^2 + x_n}$$

where the parameters γ, C are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. We determine the basin of attraction of fixed point and period-two solutions. The associated map is not defined at the $(0,0)$. However, we show that there exist period two solutions on the axis that are locally asymptotically stable and two continuous invariant curves passing through the point $(0,0)$, which are boundaries of the basins of attractions of these period two solutions, such that every solution starting on these two curves or in the region between these two curves is attracted to the point $(0,0)$.

Key Words: Basin of attraction; difference equation; global attractivity; global stable manifold; monotonicity;
MSC(2010): Primary: 39A10,39A23, 39A30; Secondary: 37E05

1 Introduction and Preliminaries

In this paper we consider the following quadratic rational difference equation of second order

$$x_{n+1} = \frac{\gamma x_{n-1}^2}{C x_{n-1}^2 + x_n} \tag{1}$$

We assume that $\gamma, C > 0$ and initial conditions x_{-1}, x_0 are positive real numbers, such that $x_0 + x_{-1} > 0$. Notice that the map associated to this equation is not defined at the point $(0, 0)$. The second iterate of the map associated to Equation (1) is *competitive map*. We call such map *anti-competitive*. See [7, 8]. Theory of competitive systems and maps in the plane have been extensively developed and main results are given in [2, 6, 11, 13, 14]. Equation (1) is a special case of the difference equation

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n}, \quad n = 0, 1, 2, \ldots \tag{2}$$

where the parameters $\beta, \gamma, \delta, B, C, D$ are nonnegative numbers which satisfy $B + C + D > 0$ and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n > 0$ for all $n \geq 0$. Locally stability of the equilibrium points of (2) has been studied in [10]. In this paper we describe global behavior of solutions of Equation (1).

Equation (1) is related to the difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}}{B x_n + C x_{n-1}}, \; n = 0, 1, \ldots \tag{3}$$

where the parameters γ, B and C are positive real numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$, see [1, 9].

As we will see in this paper Equation (1) has very different behaviour than Equation (3) showing that introduction of quadratic terms can significantly change behaviour of the equation. We prove that parametric space splits into four regions given by $0 < \gamma < 1$, $1 < \gamma < 3, \gamma = 3$ and $\gamma > 3$. By using results from [3, 13] we obtain global result in each of these four regions, different than global results for Equation (3). For example in Section 3 we show that there exist two increasing continues invariant curves passing through the point $(0, 0)$ which are the boundaries of basins of attractions of the period-two solutions such that every solution that starts on these two curves or in the region between these two curves is attracted to the point $(0, 0)$.

We now present some basic notation about competitive map in the plane.
Consider a first order system of difference equations of the form

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, 2, \ldots, \quad (x_{-1}, x_0) \in \mathcal{I} \times \mathcal{I} \tag{4}$$

where $f, g : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ are continuous functions on an interval $\mathcal{I} \subset \mathbb{R}$, $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y. Such system is called *competitive*. One may associate a competitive map T to a competitive system (4) by setting $T = (f, g)$ and considering T on $\mathcal{B} = \mathcal{I} \times \mathcal{I}$.

A point $\mathbf{x} \in \mathcal{B}$ is a *fixed point* of T if $T(\mathbf{x}) = \mathbf{x}$, and a *minimal period-two point* if $T^2(\mathbf{x}) = \mathbf{x}$ and $T(\mathbf{x}) \neq \mathbf{x}$. A *period-two point* is either a fixed point or a minimal period-two point. In a similar fashion one can define a minimal period p point. The *orbit of* $\mathbf{x} \in \mathcal{B}$ is the sequence $\{T^\ell(\mathbf{x})\}_{\ell=0}^\infty$. A *minimal period-two orbit* is an orbit $\{\mathbf{x}_\ell\}_{\ell=0}^\infty$ for which $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 = \mathbf{x}_2$. The *basin of attraction* of a fixed point $\mathbf{x}$ is the set of all $\mathbf{y}$ such that $T^n(\mathbf{y}) \to \mathbf{x}$. A fixed point $\mathbf{x}$ is a *global attractor* on a set $\mathcal{A}$ if $\mathcal{A}$ is a subset of the basin of attraction of $\mathbf{x}$. A fixed point $\mathbf{x}$ is a *saddle point* if T is differentiable at $\mathbf{x}$, and the eigenvalues of the Jacobian matrix of T at $\mathbf{x}$ are such that one of them lies in the interior of the unit circle in $\mathbb{R}^2$, while the other eigenvalue lies in the exterior of the unit circle. If $T = (T_1, T_2)$ is a map on $\mathcal{R} \subset \mathbb{R}^2$, define the sets $\mathcal{R}_T(-,+) := \{(x, y) \in \mathcal{R} : T_1(x, y) \leq x, \; T_2(x, y) \geq y \}$ and $\mathcal{R}_T(+,-) := \{(x, y) \in \mathcal{R} : T_1(x, y) \geq x, \; T_2(x, y) \leq y \}$.

If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, i.e., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 \; : x \geq u, \; y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 \; : x \leq u, \; y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{ne} (s, t)$ if and only if $x \leq s$ and $y \leq t$. A stronger inequality may be defined as $\mathbf{v} = (v_1, v_2) \ll \mathbf{w} = (w_1, w_2)$ if $\mathbf{v} \preceq \mathbf{w}$ with $v_1 \neq w_1$ and $v_2 \neq w_2$. For $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^2$, the *order interval* $[\![\mathbf{u}, \mathbf{v}]\!]$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{u} \preceq \mathbf{x} \preceq \mathbf{v}$.

A map T is *competitive* if $T(\mathbf{x}) \preceq_{se} T(\mathbf{y})$ whenever $\mathbf{x} \preceq_{se} \mathbf{y}$, and T is *strongly competitive* if $\mathbf{x} \preceq_{se} \mathbf{y}$ implies $T(\mathbf{x}) - T(\mathbf{y}) \in \{(u, v) : u > 0,\ v < 0\}$. If T is differentiable, a sufficient condition for T to be strongly competitive is that the Jacobian matrix of T at any $\mathbf{x} \in \mathcal{B}$ has the sign configuration

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

For additional definitions and results (e.g., repeller, hyperbolic fixed points, stability, asymptotic stability, stable and unstable manifolds) see [6, 14] for competitive maps, and [9, 11] for difference equations.

This paper is structured as follows. In Section 2 we prove linearized stability results. Depending on parameter γ we determine the nature of equilibrium point and period-two solutions and then we prove convergence result for period-two solution. In Section 3 we describe completely global behaviour of Equation (1).

2 Linearized stability analysis and convergence result

In this section we prove linearized stability and convergence results for Equation(1).

Theorem 1 *If $\gamma > 1$ then Equation (1) has the unique equilibrium point $\bar{x}$ which is given by*

$$\bar{x} = \frac{\gamma - 1}{C}$$

and $\bar{x}$ is

a) locally asymptotically stable if $\gamma > 3$.

b) a non-hyperbolic point if $\gamma = 3$;

c) a saddle point if $1 < \gamma < 3$;

Proof. The proof follows from the well known linearized stability theorem, see [10]. □

Theorem 2 *For the Equation (1) the following holds:*

(a) For all values of parameters Equation (1) has prime period-two solution

$$\left\{0, \frac{\gamma}{C}\right\}$$

which is locally asymptotically stable.

(b) If $\gamma > 3$ then Equation (1) has prime period-two solution

$$\left\{\frac{\gamma - \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}, \frac{\gamma + \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}\right\}$$

which is a saddle point.

Proof.

(a) It is easy to check that $\{0, \frac{\gamma}{C}\}$ is period two solution for all values of parameters. This period two solution always exists.

(b) Assume that $\gamma > 3$. If $\ldots a, b, a, a, b, \ldots$ is a period two solution, then this solution satisfies the following system of algebraic equations

$$\begin{aligned} b &= \frac{\gamma b^2}{Cb^2 + a} \\ a &= \frac{\gamma a^2}{Ca^2 + b}. \end{aligned}$$

Straightforward calculations shows that under the condition $\gamma > 3$ the unique solution of this system is given by

$$a = \frac{\gamma - \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}, \quad b = \frac{\gamma + \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}.$$

By using linearized stability theorem, it is easy to see that this period two solution is a saddle point, see [10].

Note that it is not possible to obtain period two solution $\{0, \frac{\gamma}{C}\}$ by solving the previous system of algebraic equations.

□

Now, we show that every solution of Equation (1) converges to a period-two solution (not necessarily minimal).
Let

$$F(u, v) = \frac{\gamma v^2}{Cv^2 + u}.$$

It is easy to see that

$$F'_x = -\frac{\gamma v^2}{(Cv^2 + u)^2} \text{ and } F'_y = \frac{2\gamma uv}{(Cv^2 + u)^2}$$

Set

$$u_n = x_{n-1} \quad \text{and} \quad v_n = x_n \quad \text{for } n = 0, 1, \ldots \tag{5}$$

We can rewrite Equation (1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{\gamma u_n^2}{Cu_n^2 + v_n} \end{aligned} \tag{6}$$

for $n = 0, 1, \ldots$.
Let T be the map associated to Equation (1):

$$T(u, v) = (v, F(v, u)) = \left(v, \frac{\gamma u^2}{Cu^2 + v}\right). \tag{7}$$

then

$$(u_{n+1}, v_{n+1}) = T(u_n, v_n) \tag{8}$$

It is easy to see that

$$T^2(u, v) = T(T(u, v)) = (T_{21}(u, v), T_{22}(u, v)) \left(\frac{\gamma u^2}{Cu^2 + v}, \frac{\gamma v^2 \left(Cu^2 + v\right)}{C^2u^2v^2 + Cv^3 + \gamma u^2}\right)$$

from which it follows that

$$(u_{2n+2}, v_{2n+2}) = T^2(u_{2n}, v_{2n}) \tag{9}$$

which is equivalent to

$$(x_{2n+1}, x_{2n+2}) = T^2(x_{2n-1}, x_{2n}).$$

The Jacobian matrix of the map T has the form:

$$J_T(u, v) = \begin{pmatrix} 0 & 1 \\ \frac{2uv\gamma}{(cu^2+v)^2} & -\frac{u^2\gamma}{(cu^2+v)^2} \end{pmatrix} \quad (10)$$

The determinant of (10) is given by

$$detJ_T(u, v) = -\frac{2\gamma uv}{(cu^2 + v)^2} \quad (11)$$

The Jacobian matrix of the map T^2 has the form:

$$J_{T^2}(u, v) = \begin{pmatrix} \frac{2uv\gamma}{(Cu^2+v)^2} & -\frac{u^2\gamma}{(Cu^2+v)^2} \\ -\frac{2uv^3\gamma^2}{(\gamma u^2+Cv^2(Cu^2+v))^2} & \frac{u^2v(2Cu^2+3v)\gamma^2}{(\gamma u^2+Vv^2(Cu^2+v))^2} \end{pmatrix} \quad (12)$$

The determinant of (12) is given by

$$detJ_{T^2}(u, v) = \frac{4\gamma^3u^3v^2}{(Cu^2 + v)(Cv^2(Cu^2 + v) + \gamma u^2)^2} \quad (13)$$

The equilibrium curves of the map T^2 are given by

$$\mathcal{C}_1 := \{(x, y) \in [0, \infty)^2 : T_{21}(x, y) = x\} = \{(x, y) \in [0, \infty)^2 : y = \gamma x - Cx^2\}$$

and

$$\mathcal{C}_2 := \{(x, y) \in [0, \infty)^2 : T_{22}(x, y) = y\} = \left\{(x, y) \in [0, \infty)^2 : x = \frac{y\sqrt{\gamma - Cy}}{\sqrt{Cy(Cy - \gamma) + \gamma}}\right\}$$

By direct inspection of Equation (13) we obtain the following result:

Lemma 1 *The map T^2 is competitive on $[0, \infty)^2 \setminus \{(0, 0)\}$ and strongly competitive on $(0, \infty)^2$.*

It is easy to see that the following holds.

Lemma 2 *For all $x_{-1}, x_0 \in [0, \infty)$, such that $x_{-1} + x_0 > 0$, the following holds $x_n \le \frac{\gamma}{C}$ for $n \ge 1$.*

By using very powerful Theorem 1.5 from [4] and Lemma 2, we obtain the following convergence result.

Theorem 3 *Every solution of Equation (1) converges to a period-two solution or to zeros.*

3 Global behavior

In this section we consider the following four parametric regions $\gamma > 3$, $1 < \gamma < 3, \gamma = 3$ and $0 < \gamma < 1$. We completely describe the global behaviour of Equation (1) in these regions.

The following theorem details the case $\gamma > 3$.

Theorem 4 *Assume that*

$$\gamma > 3.$$

Then system (8) has a unique equilibrium point $E(\bar{u}, \bar{u})$ which is locally asymptotically stable and there exist two prime period-two solutions: $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is locally asymptotically stable and $\{P_3(\bar{u}_2, \bar{v}_2), P_4(\bar{v}_2, \bar{u}_2)\}$ which is a saddle point, where

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{\gamma}{C} \text{ and } \bar{u} = \frac{\gamma - 1}{C}$$

and

$$\bar{u}_2 = \frac{\gamma + 1 - \sqrt{(\gamma - 3)(\gamma + 1)}}{2C} \text{ and } \bar{v}_2 = \frac{\gamma + 1 + \sqrt{(\gamma - 3)(\gamma + 1)}}{2C}$$

Furthermore, global stable manifold of the periodic solution $\{P_3, P_4\}$ is given by $\mathcal{W}^s(\{P_3, P_4\}) = \mathcal{W}^s(P_3) \cup \mathcal{W}^s(P_4)$ where $\mathcal{W}^s(P_3)$ and $\mathcal{W}^s(P_4)$ are continuous increasing curves, invariant under the map T^2 and $T(\mathcal{W}^s(P_3)) = \mathcal{W}^s(P_4)$, and divide the first quadrant into two connected components, namely

$$\mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_3) : \exists y \in \mathcal{W}^s(P_3) \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}_1^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_3) : \exists y \in \mathcal{W}^s(P_3) \text{ with } x \preceq_{se} y\}$$

and

$$\mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_4) : \exists y \in \mathcal{W}^s(P_4) \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}_2^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_4) : \exists y \in \mathcal{W}^s(P_4) \text{ with } x \preceq_{se} y\}$$

respectively. In addition, $\mathcal{W}^s(P_3)$ is passing through the point P_3 and $\mathcal{W}^s(P_4)$ is passing through the point P_4 and the following holds:

i) If $(u_0, v_0) \in \mathcal{W}^s(P_3)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_3, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_4.

ii) If $(u_0, v_0) \in \mathcal{W}^s(P_4)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_4, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_3.

iii) If $(u_0, v_0) \in \mathcal{W}_1^+$ (the region above $\mathcal{W}^s(P_3)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

iv) If $(u_0, v_0) \in \mathcal{W}_2^-$ (the region below $\mathcal{W}^s(P_4)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_2, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

v) If $(u_0, v_0) \in \mathcal{W}_1^- \cap \mathcal{W}_2^+$ (the region between $\mathcal{W}_1^-$ and $\mathcal{W}_2^+$) then the sequence $\{(u_n, v_n)\}$ is attracted to E.

See Figure 1.

Proof. Theorem 1 implies that there exists a unique equilibrium point $E(\bar{x}, \bar{x})$ which is locally asymptotically stable. Theorem 2 implies that the periodic solution $\{P_1, P_2\}$ is locally asymptotically stable and $\{P_3, P_4\}$ is a saddle point. In view of (12) the map $T^2(u, v) = T(T(u, v))$ is competitive on $\mathcal{R} = \mathbb{R}_+^2 \setminus \{(0, 0)\}$ and strongly competitive on $int(\mathcal{R})$. It is easy to see that at each point, the Jacobian matrix of T^2 has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively.

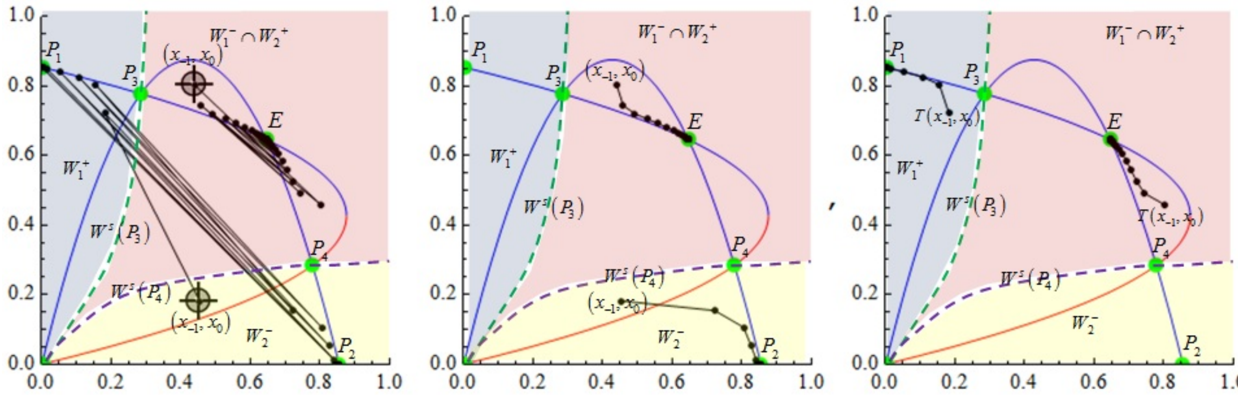

Figure 1: Visual illustration of Theorem 4.

In view of Theorem 3 we have that all solutions converge to period-two solution. Hence, all conditions of Theorem 4 in [13] are satisfied, which yields the existence of the global stable manifolds $\mathcal{W}^s(P_3)$ and $\mathcal{W}^s(P_4)$ which are the graphs of the strictly decreasing functions of the first coordinate on an interval.

By Theorem 4 in [13], we have that if $(u_0, v_0) \in \mathcal{W}^s(P_3)$ then $(u_{2n}, v_{2n}) = T^{2n}(u_0, v_0) \to P_3$ as $n \to \infty$ which implies that $(u_{2n+1}, v_{2n+1}) = T(T^{2n}(u_0, v_0)) \to T(P_3) = P_4$ as $n \to \infty$ from which it follows the statement i). The proof of the statement ii) is similar to the proof of the statement i).

Take $(u_0, v_0) \in \mathcal{W}_1^+ \cap \mathcal{R}$. By Theorem 4 in [13], we have that there exists $n_0 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_2(P_3) \cap \mathcal{R}), \quad n > n_0$. In view of Theorem 1 in [11], since P_3 is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_2(P_3) \cap \mathcal{R})$, there exists $r_0 > 0$ such that $(u_0, v_0) \preceq_{se} P_3 - r_0\mathbf{v}_1$ and $T^2(P_3 - r_0\mathbf{v}_1) \preceq_{se} P_3 - r_0\mathbf{v}_1$. By monotonicity $T^{2n+2}(P_3 - r_0\mathbf{v}_1) \preceq_{se} T^{2n}(P_3 - r_0\mathbf{v}_1) \ll P_3$. In view of Lemma 2 we have that

$$T\left([0,\infty)^2 \setminus \{(0,0)\}\right) \subset \left[0, \frac{\gamma}{C}\right)^2 \setminus \{(0,0)\}.$$

From this and the fact that $P_1 \ll P_3 \ll E \ll P_4 \ll P_2$ we have that $T^{2n}(P_3 - r_0\mathbf{v}_1) \to P_1$ as $n \to \infty$. By monotonicity we have that $P_1 \preceq_{se} T^{2n}(u_0, v_0) \preceq_{se} T^{2n}(P_3 - r_0\mathbf{v}_1) \ll P_3$ which implies that $T^{2n}(u_0, v_0) \to P_1$ and $T^{2n+1}(u_0, v_0) = T(T^{2n}(u_0, v_0)) \to T(P_1) = P_2$ as $n \to \infty$ which proves the statement iii).

Take $(u_0, v_0) \in \mathcal{W}_2^- \cap \mathcal{R}$. By Theorem 4 in [13], we have that there exists $n_1 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_4(P_4) \cap \mathcal{R}), \quad n > n_1$. In view of Theorem 1 in [11], since P_4 is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_4(P_4) \cap \mathcal{R})$, there exists $r_1 > 0$ such that $P_4 + r_1\mathbf{v}_1 \preceq_{se} (u_0, v_0)$ and $P_4 + r_1\mathbf{v}_1 \preceq_{se} T^2(P_4 + r_1\mathbf{v}_1)$. The rest of the proof of the statement iv) is similar to the proof of the statement iii) and we skip it here.

Now, we show that each orbit starting in the region $\mathcal{W}_1^- \cap \mathcal{W}_2^+$ converges to E. Take $(u_0, v_0) \in \mathcal{W}_1^- \cap \mathcal{W}_2^+$. By Theorem 4 in [13],we have that there exists $n_2 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_4(P_3) \cap Q_2(P_4) \cap \mathcal{R}) = [[P_3, P_4]]$, for $n > n_2$. Since P_3 and P_4 are the saddle points and E is locally asymptotically stable, in view of Corollary 2 [12] we have that $T^{2n}(u', v') \to E$ and $T^{2n+1}(u', v') = T(T^{2n}(u', v')) \to T(E) = E$ as $n \to \infty$ for all $(u', v') \in [[P_3, E]]$ and that $T^{2n}(u'', v'') \to E$ and $T^{2n+1}(u'', v'') = T(T^{2n}(u'', v'')) \to T(E) = E$ as $n \to \infty$ for all $(u'', v'') \in [[E, P_4]]$. Then there exist the points $(u_0', v_0') \in [[P_3, E]]$ and $(u_0'', v_0'') \in [[E, P_4]]$ such that $(u_0', v_0') \preceq_{se} T^{2n_2+2}(u_0, v_0) \preceq_{se} (u_0'', v_0'')$. By monotonicity of the map T^2 we have that $T^{2n}(u_0, v_0) \to E$ and $T^{2n+1}(u_0, v_0) = T(T^{2n}(u_0, v_0)) \to T(E) = E$ as $n \to \infty$ for all $(u_0, v_0) \in \mathcal{W}_1^- \cap \mathcal{W}_2^+$. This completes the proof of statement v) of the Theorem.

□

The following theorem considers the case $1 < \gamma < 3$.

Theorem 5 *Assume that*

$$1 < \gamma < 3.$$

Then system (8) *has a unique equilibrium point* $E(\bar{u}, \bar{u})$ *which is a saddle point and prime period-two solution* $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ *which is locally asymptotically stable, where*

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{\gamma}{C} \text{ and } \bar{u} = \frac{\gamma - 1}{C}.$$

Global stable manifold $\mathcal{W}^s(E)$, *which is continuous increasing curve, divides the first quadrant into two connected components*

$$\mathcal{W}^-(E) := \{x \in \mathcal{R} \setminus \mathcal{W}^s(E) : \exists y \in \mathcal{W}^s(E) \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}^+(E) := \{x \in \mathcal{R} \setminus \mathcal{W}^s(E) : \exists y \in \mathcal{W}^s(E) \text{ with } x \preceq_{se} y\}$$

such that

$$\mathbb{R}_+^2 = \mathcal{W}^-(E) \cup \mathcal{W}^+(E) \cup \mathcal{W}^s(E).$$

In addition, $\mathcal{W}^s(E)$ *passing through the point* E *and the following holds:*

i) Every initial point (u_0, v_0) *in* $\mathcal{W}^s(E)$ *is attracted to* E.

ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ *(the region below* $\mathcal{W}^s(E)$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *is attracted to* P_2, *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *is attracted to* P_1.

iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(x_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

See Figure 1.

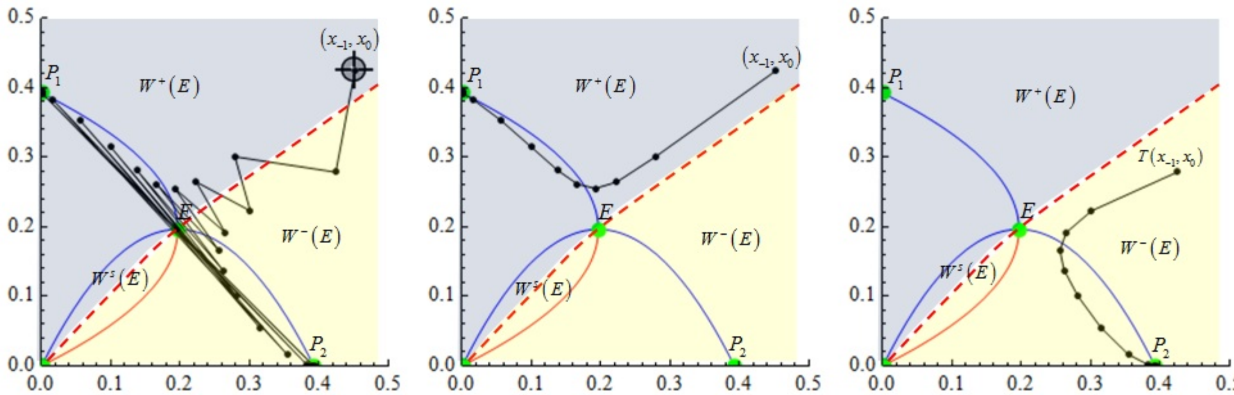

Figure 2: Visual illustration of Theorem 5 .

Proof. Theorem 1 implies that there exists a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point. Theorem 2 implies that the period-two solution $\{P_1, P_2\}$ is locally asymptotically stable. Similar as in the proof of Theorem 4 all conditions of Theorem 4 in [13] are satisfied, which yields the existence of the global stable manifold $\mathcal{W}^s(E)$ which is the graph of the strictly increasing function.

Take $(u_0, v_0) \in \mathcal{W}^+ \cap \mathcal{R}$. By Theorem 4 in [13], we have that there exists $n_0 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_2(E) \cap \mathcal{R}), \quad n > n_0$. In view of Theorem 1 in [11], since E is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_2(E) \cap \mathcal{R})$, there exists $r_0 > 0$ such that $(u_0, v_0) \preceq_{se} E - r_0\mathbf{v}_1 \preceq_{se} E$ and $T^2(E - r_0\mathbf{v}_1) \preceq_{se} E - r_0\mathbf{v}_1$. By monotonicity $T^{2n+2}(E - r_0\mathbf{v}_1) \preceq_{se} T^{2n}(E - r_0\mathbf{v}_1) \ll E$. In view of Lemma 2 we have that $T^n(u, v) \in [0, \gamma/C)^2 \setminus \{(0, 0)\}$. From this and the fact that $P_1 \ll E \ll P_2$ we have that $T^{2n}(E - r_0\mathbf{v}_1) \to P_1$ as $n \to \infty$. By monotonicity, $P_1 \preceq_{se} T^{2n}(u_0, v_0) \preceq_{se} T^{2n}(E - r_0\mathbf{v}_1) \ll E$ which implies that $T^{2n}(u_0, v_0) \to E$ and $T^{2n+1}(u_0, v_0) = T(T^{2n}(u_0, v_0)) \to T(E) = E$ as $n \to \infty$ which proves the statement ii).

The proof of the statement iii) is similar and we skip it here.

□

Now, we assume that $\gamma = 3$. The following theorem holds.

Theorem 6 *Assume that*

$$\gamma = 3.$$

Then System (8) has a unique equilibrium point $E(\bar{u}, \bar{u})$ which is a non-hyperbolic and prime period-two solution $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is locally asymptotically stable, where

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{3}{C} \text{ and } \bar{u} = \frac{2}{C}.$$

There exists a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and it divides the first quadrant into two connected invariant components

$$\mathcal{W}^-(E) := \{x \in \mathcal{R} \setminus \mathcal{C}_E : \exists y \in \mathcal{C}_E \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}^+(E) := \{x \in \mathcal{R} \setminus \mathcal{C}_E : \exists y \in \mathcal{C}_E \text{ with } x \preceq_{se} y\}$$

such that the following holds:

i) Every initial point (u_0, v_0) in $\mathcal{C}_E$ is attracted to E.

ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region above $\mathcal{C}_E$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

ii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region below $\mathcal{C}_E$) then the subsequence of even-indexed terms $\{(x_{2n}, v_{2n})\}$ is attracted to P_2, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

See Figure 3.

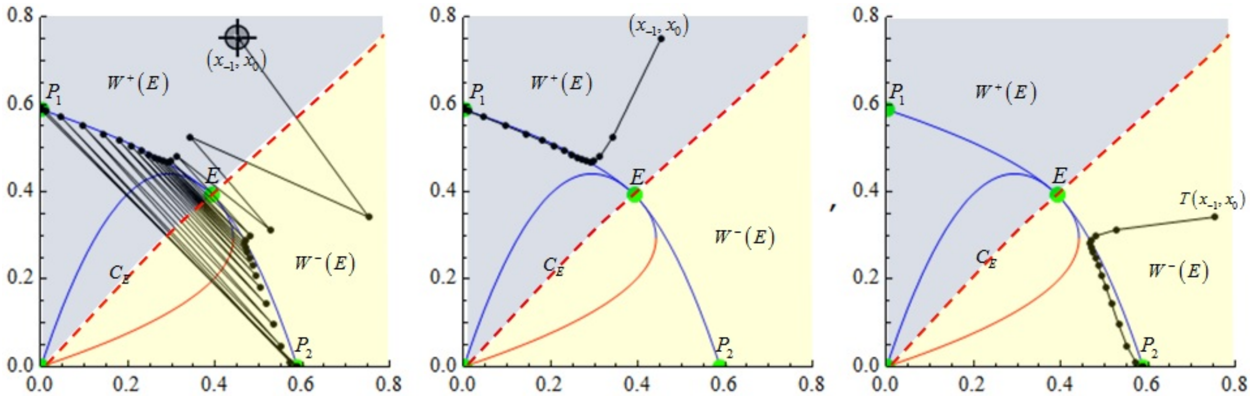

Figure 3: Visual illustration of Theorem 6 .

Proof. Theorem 1 implies that there exists a unique equilibrium point $E(\bar{x}, \bar{x})$ which is non-hyperbolic. Theorem 1(c) implies that the periodic solution $\{P_1, P_2\}$ is locally asymptotically stable. Similar as in the proof of Theorem 4 all conditions of Theorem 4 in [13] are satisfied, which yields the existence a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and for every x $\in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \mathrm{int}\, \mathcal{Q}_2(E)$ for $n \geq n_0$ and for every x $\in \mathcal{W}_-$ there exists $n_1 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \mathrm{int}\, \mathcal{Q}_4(E)$ for $n \geq n_1$.

Set $U(t) = t(3 - Ct)$. It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t \in [\frac{3}{2C}, \bar{u}]$ and $E \preceq_{se} (t, U(t))$ if $t \in [\bar{u}, \frac{3}{C}]$ and $U(\bar{u}) = \bar{u}$. In view of Lemma 2 we have that

$$T\left([0,\infty)^2 \setminus \{(0,0)\}\right) \subset \left[0, \frac{\gamma}{C}\right)^2 \setminus \{(0,0)\}.$$

One can show that

$$T^2(t, U(t)) - (t, U(t)) = \left(0, \frac{t(Ct-3)(Ct-2)^3}{Ct(Ct-3)^2+1}\right)$$

which implies that $T^2(t, U(t)) \preceq_{se} (t, U(t))$ if $t < \bar{u}$ and $(t, U(t)) \preceq_{se} T^2(t, U(t))$ if $t > \bar{u}$. By monotonicity if $t < \bar{u}$ then we obtain that $T^{2n}(t, U(t)) \to P_1$ as $n \to \infty$ and if $t > \bar{u}$ then we have that $T^{2n}(t, U(t)) \to P_2$ as $n \to \infty$.

If $(u', v') \in int\mathcal{Q}_2(E)$ then there exists t_1 such that $P_1 \preceq_{se} (u', v') \preceq_{se} (t_1, U(t_1)) \ll_{se} E$. By monotonicity of the map T^2 we obtain that $P_1 \preceq_{se} T^{2n}(u', v') \preceq_{se} T^{2n}(t_1, U(t_1)) \ll_{se} E$ which implies that $T^{2n}(u', v') \to P_1$ and $T^{2n+1}(u', v') \to T(P_1) = P_2$ as $n \to \infty$ which proves the statement ii).

If $(u'', v'') \in int\mathcal{Q}_4(E)$ then there exists t_1 such that $E \ll (t_2, U(t_2)) \preceq_{se} (u'', v'') \preceq_{se} P_2$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(t_2, U(t_2)) \preceq_{se} T^{2n}(u'', v'') \ll_{se} P_2$ which implies that $T^{2n}(u'', v'') \to P_2$ and $T^{2n+1}(u'', v'') \to T(P_2) = P_1$ as $n \to \infty$ which proves the statement iii), and completes the proof of the Theorem.

□

First we notice the following. Theorem 3 and Lemma 2 imply that $T^{2n}(x_0, y_0)$ is asymptotic to either $P_1 = (0, \frac{\gamma}{C})$ or $P_2 = (\frac{\gamma}{C}, 0)$ or $(0, 0)$, for all $(x_0, y_0) \in \mathcal{R} \setminus \{(0, 0)\}$. Let $\mathcal{B}(P_1)$ be the basin of attraction of P_1 and $\mathcal{B}(P_2)$ be the basin of attraction of P_2 with respect to the map T^2. Let $\mathcal{C}^+$ denote the boundary of $\mathcal{B}(P_1)$ considered as a subset of *int* $Q_1(0, 0)$ (the first quadrant relative to $(0, 0)$) and $\mathcal{C}^-$ denote the boundary of $\mathcal{B}(P_2)$ considered as a subset of *int* $Q_1(0, 0)$. It is easy to see that $(0, 0) \in \mathcal{C}^+$ and $(0, 0) \in \mathcal{C}^-$.

Now, similarly to the proof of the of Claim 1 and Claim 2 in [5], one can prove that the following lemma holds.

Lemma 3 *Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sets defined above. Then the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ are invariant under the map T^2 and they are the graphs of continuous strictly increasing functions. Further, $\mathcal{C}^+ \cup \mathcal{C}^- \subset \mathcal{B}(0,0)$.*

The following theorem details the existence two invariant strictly increasing curves passing through the point $(0,0)$, such that every solution that stars on these two curves or in the region between these two curves is attracted to the point $(0,0)$.

Theorem 7 *Assume that*

$$0 < \gamma < 1.$$

Then there exists prime period-two solution $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is locally asymptotically stable, where

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{\gamma}{C}$$

Furthermore, there exist sets $\mathcal{C}^+$ and $\mathcal{C}^-$ which are continuous increasing curves, invariant under the map T^2 and $T(\mathcal{C}^+) = \mathcal{C}^-$, and divide the first quadrant into two connected components, namely

$$\mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{C}^+ : \exists y \in \mathcal{C}^+ \text{ with } y \preceq_{se} x\} \quad \text{and} \quad \mathcal{W}_1^+ := \{x \in \mathcal{R} \setminus \mathcal{C}^+ : \exists y \in \mathcal{C}^+ \text{ with } x \preceq_{se} y\}$$

and

$$\mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{C}^- : \exists y \in \mathcal{C}^- \text{ with } y \preceq_{se} x\} \quad \text{and} \quad \mathcal{W}_2^+ := \{x \in \mathcal{R} \setminus \mathcal{C}^- : \exists y \in \mathcal{C}^- \text{ with } x \preceq_{se} y\}$$

respectively. In addition, $\mathcal{C}^+$ and $\mathcal{C}^-$ passing through the point $(0,0)$ and the following holds:

i) If $(u_0, v_0) \in \mathcal{W}_1^+$ (the region above $\mathcal{C}^+$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

ii) If $(u_0, v_0) \in \mathcal{W}_2^-$ (the region below $\mathcal{C}^-$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_2, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

iii) If $(u_0, v_0) \in (\mathcal{C}^+ \cup \mathcal{C}^+) \cup (\mathcal{W}_1^- \cap \mathcal{W}_2^+)$ (the region between $\mathcal{C}^+$ and $\mathcal{C}^-$) then the sequence $\{(u_n, v_n)\}$ is attracted to $(0,0)$.

Proof. The proof follows from Lemma 3, and it is similar to the proof of Theorem 4, so we skip it. □

Based on a series of numerical simulations we pose the following hypothesis.

Conjecture 1 *Suppose that all assumptions of the Theorem 7 are satisfied, then the following holds: $\mathcal{C}^+ = \mathcal{C}^-$.*

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Fourier Approximation Schemes of Stochastic Pseudo-Hyperbolic Equations with Cubic Nonlinearity and Regular Noise

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Abstract

The pseudo-hyperbolic equation with cubic nonlinearity and additive space-time noise is discussed. The space-time noise is assumed to be Gaussian in time and possesses a Fourier series expansion in space. First, we prove the existence and uniqueness of the approximate strong solutions of the equation and show that the truncated Fourier solution which can be approximated by the truncated finite-dimensional system, is an approximate solution. Second, a new transformation is used to convert pseudo-hyperbolic equation into a system of equations, which can construct an infinitesimal generator with good properties. After analyzing the related total energy evolution, we obtain that the energy growth will not blow-up in the limited time. Finally, we present a Fourier scheme of a procedure for its numerical approximation and give the stability and convergence analysis of the scheme.

keyword: thermal convection equation, Fourier coefficients, cubic-type nonlinearities; stochastic; energy

1 Introduction

Stochastic differential equations (SDEs) can model many natural phenomena with white noise and engineering applications, such as epidemiology, economics and so on [15, 1, 14, 3, 43, 23, 22]. SDEs hold for the important original work

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of Ito [12] as well as books [7, 27]. The shorter accounts of stochastic dynamic systems on stability, filtering, and control [18, 13] are rather unsuited for the study. Stability of SDEs has been well studied by researchers [25, 16, 19]. Since the analytical solution is difficult to obtain, different numerical methods have been introduced such as [30, 33, 8, 29]. The common theoretical basis is the stochastic Ito-Taylor expansion in terms of multiple Wiener integrals [15].

The analysis of the linear SDEs is well investigated such as [20, 6, 28, 24]. In the recent past, the nonlinear SDEs are researched. In [21] a class of fully nonlinear SDEs is studied by using the stochastic characteristic method. In [11] a strong convergence result under less restrictive conditions is proved by using Euler-Maruyama method. In [10], the exponential stability of the multidimensional nonlinear SDEs with variable delays is investigated. Nonlinear filtering equations have developed based on a classification where the measure term is either deterministic or random [39].

Consider the semi-linear stochastic pseudo-hyperbolic equation with cubic-type nonlinearities perturbed by additive space-time random noise W [40]:

$$\begin{cases} d(u+u_t) = \sigma^2 \dfrac{\partial^2 (u+u_t)}{\partial^2 x} dt + B(u+u_t)dt + b\cdot dW(t,x), \\ u(0,x) = u_0, \; u_t(0,x) = u_{t_0}, \quad 0 < x < L, \\ u(t,0) = u(t,L) = 0, \; u_t(t,0) = u_t(t,L) = 0, \quad 0 < t < T, \end{cases} \tag{1.1}$$

where $b \in \mathbb{R}^1$ is an overall noise intensity parameter. $B(u) = u(a_1 - a_2\|u\|^2_{L^2})$ is cubic-type with real parameters $a_2 > 0$ and a_1 [38]. The space-time Q-regular noise $W(t,x)$ is as follows:

$$W(t,x) = \sum_{n=1}^{+\infty} \alpha_n W_n(t) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{+\infty} \alpha_n W_n(t) e_n(x) \tag{1.2}$$

with independent and identically distributed Wiener process $W_n \in \mathcal{N}(0,t)$, where $trace(Q) = \sum_{n=1}^{+\infty} \alpha_n^2 < +\infty$. We know that $e_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), n \geq 1$ are the eigenfunctions of the Laplace operator which form an orthonormal system in $H = L^2(0,L)$ and satisfy in one-dimensional, $\Delta e_n(x) = -\frac{n^2\pi^2}{L^2} e_n$. The main contribution of this paper is to discuss the Fourier solution $u(t,x)$ and its numerical approximations by truncated Fourier series [41]. We construct an infinitesimal generator with good properties and convert into the equations which can be easily solved.

The rest of the paper is organized as follows. In section 2, we verify the existence and uniqueness of solution and give a finite-dimensional system of the SDEs. In section 3, we estimate the truncated total energy. In section 4 we show numerical methods to find those Fourier coefficients. In the last section 5 numerical experiments are provided which support our results.

2 Existence and Uniqueness of Approximate Strong Solutions and Fourier-Series Solutions

In general, it is difficult to solve nonlinear equations. However, taking the equations into system of equations can avoid lots of complex calculations in infinitesimal generators and energy estimation. Let $v = u + u_t$, Eqs (1.1) becomes

$$\begin{cases} v = u + u_t, \\ \dfrac{dv}{dt} = \left[\sigma^2 \dfrac{\partial^2 v}{\partial x^2} + v\left(a_1 - a_2\|v\|_{L^2}^2\right)\right] + b \cdot \dfrac{dW(t,x)}{dt}. \end{cases} \tag{2.1}$$

It can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & \frac{\sigma^2 \partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ v \end{pmatrix}(a_1 - a_2 \left\|\begin{pmatrix} 0 \\ v \end{pmatrix}\right\|_{L^2}^2) + \begin{pmatrix} 0 \\ b \end{pmatrix}\frac{dW}{dt}. \tag{2.2}$$

From the definitions of strong solution and approximate strong solution[36], we obtain that conditions of the strong solutions of (2.1) exist and the uniqueness is that all operators are globally Lipschitz-continuous. Under conditions weaker than global Lipschitz-continuity, we can also achieve a result of the strong solutions.

Lemma 2.1. *For all $a_2 \geq 0$, the mapping $v \in H \mapsto B(v) = v(a_1 - a_2\|v\|_{L^2}^2)$ satisfies the angle condition on H. In other words, for all $u, v \in H$, we have*

$$F(u,v) := < B(u) - B(v), u - v >_H \leq a_1\|u - v\|_H^2, \tag{2.3}$$

specially

$$< B(v), v >_H \leq \left(a_1 - a_2\frac{\|v\|_H^2}{2}\right)\|v\|_H^2 \leq a_1\|v\|_H^2.$$

Proof. Denoting $f(u) := \|u\|_H^2 u$ and $g(u,v) := < f(u) - f(v), u - v >_H$ which is symmetric. Then we obtain that

$$2g(u,v) = \left(\|u\|_H^2 + \|v\|_H^2\right)\|u - v\|_H^2 + \left(\|u\|_H^2 - \|v\|_H^2\right)^2\left(\|u\|_H^2 + \|v\|_H^2\right)\|u - v\|_H^2,$$
$$g(u,v) \geq \frac{\|u\|_H^2 + \|v\|_H^2}{2}\|u - v\|_H^2.$$

Now using (2.3), the above inequality and the definition of B, we have

$$F(u,v) \leq -a_2\frac{\|u\|_H^2 + \|v\|_H^2}{2}\|u - v\|_H^2 + a_1\|u - v\|_H^2 \leq a_1\|u - v\|_H^2,$$
$$< B(v), v >_H \leq -a_2\frac{\|v\|_H^2}{2}\|v\|_H^2 + a_1\|v\|_H^2 \leq a_1\|v\|_H^2, \text{by setting } u = (0,0).$$

Then the proof is completed. □

3

From Lemma 2.1 and Theorem 3 in [36], the unique approximate strong and continuous solution of Eqs (2.2) exists.

Theorem 2.2. *Assumptions of definitions of strong and approximate strong solution [36] are satisfied with* $\mathbb{E}\|v(0,\cdot)\|_H^2<\infty$*, for* $B(0,L)\times\mathcal{F}_0$*-measurable initial data* $v(0,\cdot)\in H$*. The approximate strong global solution* v *of Eqs (2.2) exists.*

Next, we propose our method to solve the SDEs. There are many methods such as Galerkin-type method [9, 26, 5], Monte-Carlo [4, 31], collocation method [42], projection methods [32]. The method presented in this paper is Fourier-series solutions. The existence of separated solutions is established in [17]. Solutions of this type are used in [2]. The difficulty lies in computing Fourier-series solutions and in finding a good infinitesimal generator to estimate the energy of the system.

Using the principle of linear superposition, the Fourier series is

$$u(t,x)=\sum_{n=1}^{+\infty}c_{un}(t)e_n(x),\ \ v(t,x)=\sum_{n=1}^{+\infty}c_{vn}(t)e_n(x). \tag{2.4}$$

We truncate the series as follows:

$$\widetilde{u}(t,x)=\sum_{n=1}^{N}c_{un}(t)e_n(x),\ \ \widetilde{v}(t,x)=\sum_{n=1}^{N}c_{vn}(t)e_n(x). \tag{2.5}$$

which form the strong solutions of Eqs (2.2).

Theorem 2.3. *The Fourier coefficients of Eqs (2.4) satisfy (P-a.s.) the infinite-dimensional system, for* $k=1,2,\cdots$ *and* $b_k=b\alpha_k$*,*

$$\begin{cases} c'_{uk}(t)=c_{vk}(t)-c_{uk}(t), \\ dc_{vk}=\left(-\sigma^2\dfrac{k^2\pi^2}{L^2}+a_1-a_2\displaystyle\sum_{n=1}^{+\infty}c_{vn}^2(t)\right)c_{vk}dt+b_kdW_k, \end{cases} \tag{2.6}$$

Proof. By plugging Eqs (2.4) into Eqs (2.1), we achieve that for $0\le t\le T$,

$$\int_0^L u'(t,x)e_k(x)dx=\sum_{n=1}^{+\infty}c'_{un}(t)\int_0^L e_n(x)e_k(x)dx=c_{vk}(t)-c_{uk}(t),$$

$$\int_0^L dv(t,x)e_k(x)dx=c_{vk}\left(-\frac{\sigma^2k^2\pi^2}{L^2}+a_1-a_2\sum_{n=1}^{+\infty}\left[c_{vn}(t)\right]^2\right)dt+b_kdW_k(t).$$

As we know that the u,v is the unique strong solution of (2.1) with

$$\|u(t,\cdot)\|_H^2=\sum_{k=1}^{\infty}\left[c_{uk}(t)\right]^2<\infty,\ \ \|v(t,\cdot)\|_H^2=\sum_{k=1}^{\infty}\left[c_{vk}(t)\right]^2<\infty,$$

and have Fourier coefficient $c_{uk},\ c_{vk}$ which can be approximated by the truncated finite-dimensional system. So the above computations work. □

We note that for stochastic systems with additive noise, the stochastic integration leads to the same type of stochastic integral. See more details in [37]. Therefore, we can calculate each $c_{uk},\ c_{vk}$ and $\widetilde{u}(t,x)=\sum_{n=1}^{N} c_{un}(t)e_n(x)$.

3 Total Energy Evolution

For the case of sufficiently strong diffusion with $\sigma^2\pi^2 > L^2(a_1+1)$, we investigate the behavior of related energy functional, which is defined at time $t \geq 0$ by

$$\mathcal{E}(t)=\frac{\sigma^2}{2}\|v_x(t,\cdot)\|_H^2-\frac{a_1+1}{2}\|v(t,\cdot)\|_H^2+\frac{a_2}{4}\|v(t,\cdot)\|_H^4. \tag{3.1}$$

This energy functional is indeed nonnegative and finite almost surely (a.s.) as one can see from the following theorem. For its proof, we take the functional in terms of its Fourier coefficients c_k by

$$V(t):=V(c_{vk}(t)\!:k\in N)=\frac{1}{2}\sum_{n=1}^{+\infty}\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1-1\right)c_{vn}^2(t)+\frac{a_2}{4}\left(\sum_{n=1}^{+\infty}c_{vn}^2(t)\right)^2,$$

for $t \geq 0$. It is easy to know that $V \geq 0$ for all sequences $(c_{vk}(t))_k$ and acts as a Lyapunov functional. Besides, $\mathcal{E}(t)=V(t)$ for all $t \geq 0$.

Theorem 3.1. *Assume that $e(0)=\mathbb{E}V(c_{vk}(0):k\in N)<\infty$, $\sigma^2\pi^2 \geq L^2(a_1+1)$ and trace$(Q)=\sum_{n=1}^{\infty}\alpha_n^2<\infty$. Then, the total expected energy of the original system (2.1) is linearly bounded in time by*

$$e(t)=\mathbb{E}V(c_{vk}(t):k\in N)\leq e(0)+2\left[\sum_{n=1}^{+\infty}[\frac{\sigma^2n^2\pi^2}{L^2}-a_1-1]c_{vn}^2(t)+\sqrt{a_2}(b^2\beta^2)^2\right]t,$$

where $\beta^2=\sum_{n=1}^{\infty}\alpha_n^2+2\max_{n\in\mathbb{N}}\alpha_n^2$.

Proof. The truncated infinitesimal generator can be rewritten

$$L=\sum_{n=1}^{N}[c_{vn}-c_{un}]\frac{\partial}{\partial c_{un}}+\frac{b^2}{2}\sum_{n=1}^{N}\alpha_n^2\frac{\partial^2}{\partial c_{vn}^2}+\sum_{n=1}^{N}\left[-\frac{\sigma^2n^2\pi^2}{L}+a_1-a_2\sum_{k=1}^{N}c_{vk}^2\right]c_{vn}\frac{\partial}{\partial c_{vn}}.$$

We express Eqs (3.1) in terms of its truncated Fourier coefficients c_{vk} by

$$\widetilde{V}(t):=\widetilde{V}(c_{vk}(t):k\in N)=\frac{1}{2}\sum_{n=1}^{N}[\frac{\sigma^2n^2\pi^2}{L^2}-a_1-1]c_{vn}^2(t)+\frac{a_2}{4}\left(\sum_{n=1}^{N}c_{vn}^2(t)\right)^2,$$

for $t \geq 0$. Then, after calculating the infinitesimal generator, we estimate the energy of the system (3.1) as follow:

$$L\widetilde{V} = L\widetilde{V}_1 + L\widetilde{V}_2 = \sum_{n=1}^{N} (c_{vn} - c_{un})\, c_{un} - \left[\sum_{n=1}^{N} \left(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 + a_2 \sum_{k=1}^{N} c_{vk}^2 \right) c_{vn} \right]^2$$
$$+ \frac{1}{2} \sum_{n=1}^{N} \alpha_n^2 \sum_{n=1}^{N} \left(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 \right) + \frac{b^2}{2} \sum_{n=1}^{N} \alpha_n^2 a_2 \sum_{n=1}^{N} c_{vn}^2(t) + 2c_{vn}^2(t).$$

$$L\widetilde{V}_1 \leq \sum_{n=1}^{N} \frac{1}{4} \left(c_{un}^{'} + c_{un} \right)^2 \leq \frac{5b^2}{12\sqrt{3a_2}} \sum_{n=1}^{+\infty} \alpha_n^2 \left(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 + a_2 \left(b^2 \beta_N^2 \right)^{\frac{3}{2}} \right).$$

From the estimate in [38] and denoting $\beta^2 = \sum_{n=1}^{+\infty} \alpha_n^2 + 2 \max\limits_{n \in \mathbb{N}} \alpha_n^2$, we obtain

$$L\widetilde{V}_2 \leq b^2 \sum_{n=1}^{+\infty} \alpha_n^2 \left(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 + a_2 \left(b^2 \beta_N^2 \right)^{\frac{3}{2}} \right) \left(\frac{1}{12a_2} \right)^{\frac{1}{2}} \frac{5}{6}.$$

Consequently, Dynkin formula says that

$$e_N(t) = \mathbb{E}\left[\widetilde{V}(t) \right] \leq e(0) + 2b^2 \sum_{n=1}^{+\infty} \alpha_n^2 \left(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 \right) + 2a_2 \left(b^2 \beta_N^2 \right)^{\frac{3}{2}} \left(\frac{1}{12a_2} \right)^{\frac{1}{2}} \frac{5}{6},$$

for t ≥ 0. Since $e_N \geq 0$ is increasing in N and uniformly bounded in time t for any $t \in [0, T]$, we know that $lim_{N \to +\infty} e_n(t) = e(t)$, and

$$0 \leq e(t) \leq e(0) + 2b^2 \sum_{n=1}^{+\infty} \alpha_n^2 \left(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 \right) t + 2a_2 \left(b^2 \beta_N^2 \right)^{\frac{3}{2}} \left(\frac{1}{12a_2} \right)^{\frac{1}{2}} \frac{5}{6} t,$$

as $e(0) < \infty$, $\sigma^2 \pi^2 \geq L^2(a_1 + 1)$ and $trace(Q) = \sum_{n=1}^{\infty} \alpha_n^2 < \infty$. □

More precisely, for $T < \infty, \forall\ 0 \leq t \leq\ T, \exists\ K_1, K_2 \geq 0$

$$(E\|v(t,\cdot)\|_H^2 + K_0) e^{K_1 T} \geq E\|v(t,\cdot)\|_H^2 \geq E\|u(t,\cdot)\|_H^2.$$

In fact, if $\sigma^2 \pi^2 > L^2(a_1 + 1)$, we can know that the following mentioned estimates of second moments have linearly bounded ones (in time). For $T < \infty$, $\exists\ c \geq 0$, $0 \leq t \leq\ T$, $E\|u(t,\cdot)\|_H^2\ \leq\ E\|v(t,\cdot)\|_H^2\ \leq\ E\|v(0,\cdot)\|_H^2 + ct$.

4 Numerical Methods for Fourier Coefficients

The truncated Fourier series $\widetilde{u}$, $\widetilde{v}$ in Eqs (2.5) satisfy the Eqs (2.1). Since the explicit solution is unknown, we take advantage of numerical approximations.

Along partitions $0 = t_0 < t_1 < t_2 < \cdots < t_{n_T} = T$ of interval $[0, T]$ with the step sizes $h_n = t_{n+1} - t_n$, and $0 = x_0 < x_1 < x_2 < \cdots < x_{n_L} = L$ of interval $[0, L]$ with the step sizes $d_n = x_{n+1} - x_n$.

For each fixed x_m, let us consider the forward Euler method for c_{vk}

$$c_{vk}(n+1) = h_n c_{vk}(n)\left(\frac{-\sigma^2 k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^{N} c_{vl}^2(n)\right) + c_{vk}(n) + b_k \triangle W_n^k, \tag{4.1}$$

where $\triangle W_n^k = W_k(t_{n+1}) - W_k(t_n) \in \mathcal{N}(0, h_n)$. Other one is backward Euler method

$$c_{vk}(n{+}1) = h_n c_{vk}(n{+}1)\left(\frac{-\sigma^2 k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^{N} c_{vl}^2(n{+}1)\right) + c_{vk}(n) + b_k \triangle W_n^k. \tag{4.2}$$

In our opinion, the best approach is linear-implicit Euler-type method

$$c_{vk}(n{+}1) = h_n c_{vk}(n{+}1)\left(\frac{-\sigma^2 k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^{N} c_{vl}^2(n)\right) + c_{vk}(n) + b_k \triangle W_n^k. \tag{4.3}$$

After calculating the c_{vk}, we can obtain $\widetilde{v}(t_{n+1}, x_m) = \sum_{n=1}^{N} c_{vn}(t) e_n(x)$. Then u_N can be calculated

$$\widetilde{u}(t_{n+1}, x_{m+1}) = d_n\left(\widetilde{v}(t_{n+1}, x_m) - \widetilde{u}(t_{n+1}, x_m)\right) + \widetilde{u}(t_{n+1}, x_m).$$

We note that Eqs (4.1) has a disadvantage that is lacking of stability and monotonicity deficits. A slight disadvantage of Eqs (4.2) is that we have to solve locally implicit algebraic equations at each iteration step n, which results in a lot of calculation and time. An advantage of methods (4.2) and (4.3) is very well stability and moment dissipativity behavior, and they keep some monotonicity properties [34, 35].

Theorem 4.1. *Consider the forward Euler method that*

$$c_{vk}(n+1) = \frac{c_{vk}(n) + b_k \triangle W_n^k}{1 + h_n\left(\frac{\sigma^2 k^2 \pi^2}{L^2} - a_1 + a_2\sum_{l=1}^{N} c_{vl}^2(n)\right)}, \tag{4.4}$$

where $n \in \mathbb{N}, b_k = b\alpha_k$ *and* $\triangle w_n^k \in N(0, h_n)$. *If* $\sigma^2\pi^2 \geq L^2(a_1+1)$, *their second moments is linearly bounded in time,*

$$\mathbb{E}\left[\|u(t_n, \cdot)\|_H^2\right] < +\infty.$$

Proof. Suppose that $1 + h_n\left[\frac{\sigma^2\pi^2}{L^2} - (a_1+1)\right] > 0$. The Eqs (4.4) is finite due to the linear-implicit character of method (4.3). From Eqs (2.6), it follows

$$\begin{cases} \dfrac{c_{uk}(n+1) - c_{uk}(n)}{h_n} = \left[c_{vk}(t) - c_{uk}(t)\right], \\ \dfrac{c_{vk}(n+1) - c_{vk}(n)}{h_n} = \left[\dfrac{-\sigma^2 k^2\pi^2}{L^2} + a_1 - a_2\displaystyle\sum_{l=1}^{+\infty} c_{vl}^2(t)\right] c_{vk} + b_k\dfrac{W_k(t_n+1) - W_k(t_n)}{h_n}. \end{cases}$$

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It remains to consider the second moments. We estimate c_{vk} by Eqs (4.4)

$$c_{vk}(n+1) = \frac{c_{vk}(n) + b_k \triangle w_n^k}{1 + h_n \left[\frac{\sigma^2 k^2 \pi^2}{L^2} - a_1 + a_2 \sum_{l=1}^N c_{vl}^2(n)\right]},$$

$$\mathbb{E}\left[c_k(n+1)\right]^2 = \mathbb{E}\left[\frac{\left[c_k(n)\right]^2 + b_k^2 h_n}{\left[1 + h_n\left(\frac{\sigma^2 k^2 \pi^2}{L^2} - a_1 + a_2 \sum_{l=1}^N c_{vl}^2(n)\right)\right]^2}\right] .$$

Since denominator is less than one, we have

$$\mathbb{E}\left[c_{vk}(n+1)\right]^2 \leq \mathbb{E}\left[c_{vk}(n)\right]^2 + (b_k)^2 h_n \leq \mathbb{E}\left[c_{vk}(0)\right]^2 + (b_k)^2 t_{n+1}.$$

From Eqs (2.5), we obtain

$$\sum_{k=1}^N \mathbb{E}\left[\|c_{vk}(n)\|^2\right] \leq \sum_{k=1}^N \mathbb{E}\left[\|c_{vk}(0)\|^2\right] + b^2 \sum_{k=1}^N \alpha_k^2 t_n.$$

Since $\sum_{k=1}^{+\infty} \alpha_k^2$, $\sum_{k=1}^{+\infty} \|c_{vk}(0)\|^2 < \infty$, we obtain that, as $N \to \infty$ and $h \to 0$

$$\mathbb{E}\left[\|v(t_n,\cdot)\|_H^2\right] = \sum_{k=1}^{+\infty} \mathbb{E}\left[\|c_{vk}(n)\|^2\right] \leq \sum_{k=1}^{+\infty} \mathbb{E}\left[\|c_{vk}(0)\|^2\right] + b^2 \sum_{k=1}^{+\infty} \alpha_k^2 t_n < \infty.$$

□

Recall the definition in [38], let c_k^h denote the numerical approximation of the k-th Fourier coefficients c_k. The numerical approximation $c_h = (c_k^h)_{k=1,2,\cdots,N}$ is said to be mean consistent with rate r_0 iff there are a constant $C_0 = C_0(T)$ and a positive continuous function or functional V such that

$$\forall n{=}0,1,\cdots,n_T-1 : \|\mathbb{E}[c(n+1)] - \mathbb{E}[c^h(n+1)]\|_N \leq C_0 V(c(n)) h_n^{r_0}$$

along any (nonrandom) partitions with sufficiently small step sizes $h_n \leq \delta \leq 1$, where $\|\cdot\|$ is the Euclidean vector norm in $\mathbb{R}^N$, provided that one has nonrandom data $c(n) = c^h(n)$.

Lemma 4.2. *The method (LIM) governed by Eqs (4.3) is mean consistent with rate* $r_0 = 1.5$.

The similar results may be found in [38].

5 Numerical Experiments

Under the condition that $\sigma^2\pi^2 > L^2(a_1 + 1)$, we present the results of systematic numerical simulations for solutions of the SDEs. The order is defined by $order = \lg\left(\|\mathbb{E}[c(n+1)] - \mathbb{E}[c^h(n+1)]\|_N\right)$. The ratio is defined by

$ratio = \left|1 - \frac{\|E\|}{\|E+\epsilon\|}\right|$, where E is the total energy of the system at $t = 0$ and ϵ is the noise.

Case 5.1. *We consider the simple initial data with*

$$u(0,x) = \begin{cases} x, & x < \frac{1}{2}L, \\ L - x, & x > \frac{1}{2}L. \end{cases}$$

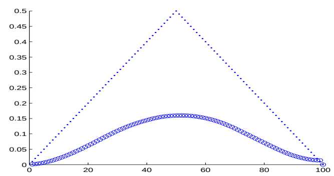

Figure 1: The numerical results at the times $t = 2$ with ratio $\approx 1\%$

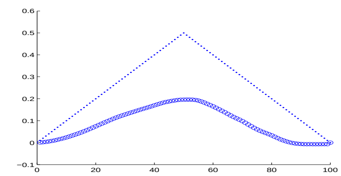

Figure 2: The numerical results at the times $t = 2$ with ratio$\approx 5\%$

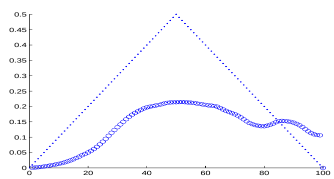

Figure 3: The numerical results at the times $t = 2$ with ratio $> 10\%$

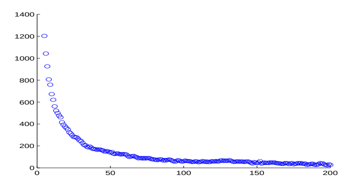

Figure 4: The total energy with different random terms at $t = 10$ with ratio $\approx 1\%$

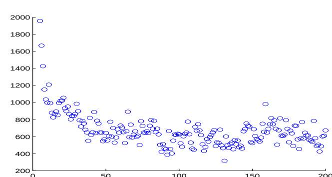

Figure 5: The total energy with different random terms at $t = 10$ with ratio$\approx 5\%$

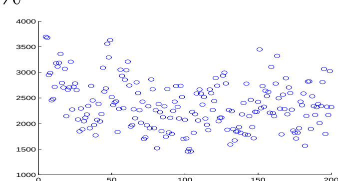

Figure 6: The total energy with different random terms at $t = 10$ with ratio $> 10\%$

The parameters $T = 2$, $\Delta t = 0.05$, $\Delta x = 0.01, L = 1$, $a_1 = 0.1, a_2 = 1$ and $\sigma = 9$ are chosen over the region $[0, 1]$. In Figure 1, 2 and 3, the lines of "$\cdot$" and "$\circ$" respectively denote the initial value $u(0, x)$ and the terminal value $u(2, x)$. Figure 1 shows that the wave dissipates at time $t = 2$. Figure 2, 3 show

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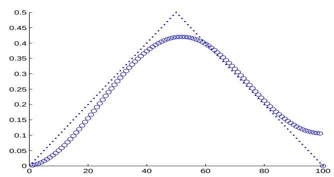

Figure 7: The numerical results at the times $t = 10$ with ratio $\approx 1\%$

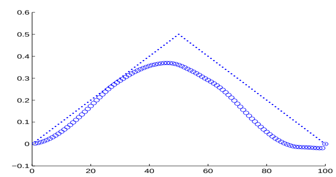

Figure 8: The numerical results at the times $t = 10$ with ratio$\approx 5\%$

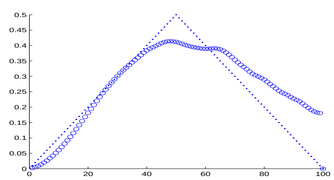

Figure 9: The numerical results at the times $t = 10$ with ratio $> 10\%$

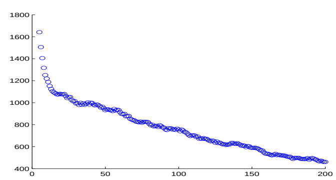

Figure 10: The total energy with different random terms at $t = 10$ with ratio $\approx 1\%$

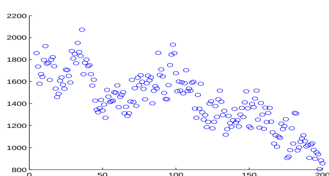

Figure 11: The total energy with different random terms at $t = 10$ with ratio$\approx 5\%$

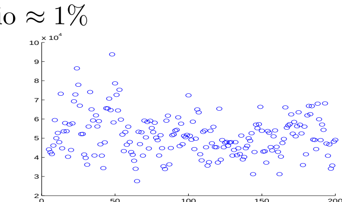

Figure 12: The total energy with different random terms at $t = 10$ with ratio $> 10\%$

Table 1: Order of convergence in space and time for the Euclidean vector norm

Δx	Δt	ratio	order	ratio	order
0.01	0.05	1%	4.1677	5%	3.0281
0.01	0.1	1%	4.014	5%	2.6964
0.05	0.05	1%	4.5498	5%	3.4972
0.05	0.1	1%	4.9379	5%	2.6441

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the influence of noise enhancement on waveform. When the ratio $> 10\%$, the waveform was destroyed. Figure 4, 5 and 6 present the total energy evolution and show that the total energy stablely declines. However the total energy is linearly bounded in time. These results are in good agreement with the Theorem 3.1. Similarly, Table 1 presents the numerical results of the linear-implicit Euler-type schemes, which are in good agreement with Lemma 4.2.

Case 5.2. *In the second case, the initial data is same to the case 1. Using* $L = 1,\ a_1 = 1,\ a_2 = 1,\ b = 1$ *and* $\sigma = 9$*, we present the numerical solution at the terminal time* $T = 10$*.*

Table 2: Order of convergence in space and time for the Euclidean vector norm

Δx	Δt	ratio	order	ratio	order
0.01	0.05	1%	4.5093	5%	3.3822
0.01	0.1	1%	4.1886	5%	2.5853
0.05	0.05	1%	4.0957	5%	3.8375
0.05	0.1	1%	4.4245	5%	3.2689

Figure 7 shows that the wave dissipates at time $t = 10$. Figure 8, 9 show the influence of noise enhancement on waveform. When the ratio $> 10\%$, the waveform was destroyed. Figure 11, 10 and 12 present the numerical results of the total energy evolution and show that the total energy stablely declines. With the increase of the noise, the downward trend is not significant but vibrate. These results are in good agreement with the Theorem 4.2. Similarly, Table 2 presents the numerical results of the linear-implicit Euler-type schemes, which are in good agreement with Lemma 4.2.

Acknowledgements

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An iterative algorithm for solving split feasibility problems and fixed point problems in p-uniformly convex and smooth Banach spaces

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Abstract

In this paper, we introduce an iterative process for approximation of a common fixed point for a finite family of multi-valued Bregman relatively nonexpansive mappings with a solution of the split feasibility problems in p-uniformly convex and uniformly smooth Banach spaces. We prove the strong convergence theorems of the proposed iterative process in p-uniformly convex and uniformly smooth Banach spaces and present the numerical results to verify the efficiency and implementation of our results.

Keywords: Bregman relatively nonexpansive mappings; strong convergence theorems; uniformly convex Banach spaces; uniformly smooth Banach spaces; split feasibility problems.

1 Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. The split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q. \tag{1.1}$$

Note that the inverse image of the set Q under A is a convex set. Hence the problem 1.1 can be written in case that the intersection $C \cap A^{-1}(Q)$ is nonempty. We will denote the nonempty solution set of (1.1) by $\Omega = C \cap A^{-1}(Q)$. Therefore Ω is a closed convex subset of E_1.

In 1994, Censor and Elfving [8] introduced the SFP (1.1) in finite-dimensional Hilbert spaces for modelling inverse problems which arise from phase retrievals, medical image reconstruction. Various algorithms

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have been invented to solve the SFP (1.1) (see [2, 6, 11, 25, 28, 29] and the references therein). In particular, Byrne [6] introduced a so-called CQ algorithm, taking an initial point x_0 arbitrarily and construct the sequence $\{x_n\}$ by

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), n \geq 1,$$

where $0 < \gamma < \frac{2}{\|A\|^2}$, and P_C denotes the projection onto a set C. That is, $P_C(x) = \arg\min_{y \in C} \|x - y\|$. Recently, Schöpfer et al. [19] solved the SFP (1.1) in p-uniformly convex real Banach spaces which are also uniformly smooth using the following algorithm: for $x_1 \in E_1$ and $n \geq 1$, set

$$x_{n+1} = \Pi_C J^q_{E_1^*}[J^p_{E_1} x_n - t_n A^* J^p_{E_2}(Ax_n - P_Q(Ax_n))], \tag{1.2}$$

where Π_C denotes the the Bregman projection and J the duality mapping. Clearly the above algorithm covers the Byrne' CQ algorithm [6]. They used algorithm (1.2) for obtaining the weak convergence result in a p-uniformly convex real Banach spaces which are uniformly smooth with the condition that the duality mapping of E is sequentially weak-to-weak-continuous. In 2014, Wang [26] studied the following multiple-sets split feasibility problem (MSSFP) (see [11]): find $x \in E_1$ satisfying

$$x \in \bigcap_{i=1}^{r} C_i, \;\; Ax \in \bigcap_{j=r+1}^{r+s} Q_j \tag{1.3}$$

where r, s are two given integers, $C_i, i = 1, ..., r$, is a closed convex subset of E_1, and Q_j, $j = r+1, ..., r+s$, is a closed convex subset in E_2. Wang [26] modified the above algorithm (1.2) and proved the strong convergence theorem using an idea appeared in [13] and the following algorithm: for any initial guess x_0, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n x_n \\ D_n = \{u \in E : \Delta_p(y_n, u) \leq \Delta_p(x_n, u)\} \\ E_n = \{u \in E : \langle x_n - u, J^p_E(x_0) - J^p_E(x_n)\rangle \geq 0\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(x_0), \end{cases} \tag{1.4}$$

where T_n is defined, for each $n \in \mathbb{N}$, by

$$T_n x = \begin{cases} \Pi_{C_{i(n)}}(x), \;\; 1 \leq i(n) \leq r \\ J^q_{E_1^*}[J^p_{E_1}(x) - t_n A^* J^p_{E_2}(I - P_{Q_{i(n)}})Ax], r+1 \leq i(n) \leq r+s, \end{cases} \tag{1.5}$$

$i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \mod (r+s) + 1,$$

and t_n satisfies

$$0 < t \leq t_n \leq \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}.$$

For better comparison of (1.5) with (1.2), we state a version of (1.2) for solving problem (1.3):

$$x_{n+1} = \Pi_{C_{i(n)}} J^q_{E_1^*}[J^p_{E_1}(x_n) - t_n A^* J^p_{E_2}(Ax_n - P_{Q_{i(n)}}(Ax_n))], \tag{1.6}$$

where $i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \mod (r+s) + 1.$$

In 1967, Bregman [3] has discovered an elegant and effective technique for the use of the Bregman distance function Δ_p in the process of designing and analyzing feasibility and optimization algorithms.

This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequality problems, equilibrium problems, fixed point problems for nonlinear mappings, and so on (see [16, 4, 17] , and the references therein).

Recently, Shehu et al. [22] studied split feasibility problems and fixed point problems concerning left Bregman strongly nonexpansive mappings: find an element $x \in E_1$ satisfying

$$x \in C \cap F(T) \text{ such that } Ax \in Q. \tag{1.7}$$

Shehu et al. [22] proposed the following algorithm: for a fixed $u \in E_1$, let $\{x_n\}_{n=1}^{\infty}$ be iteratively generated by $u_1 \in E_1$,

$$\begin{cases} x_n = \Pi_C J_{E_1^*}^q [J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_C J_{E_1^*}^q(\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(Tx_n)), \ \ n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Moreover Shehu et al. [22] proved the strong convergence of the sequence generated by (1.8) for solving problem (1.7) in p-uniformly convex real Banach spaces which are also uniformly smooth.

In 2014, Pang et al. [9] showed that the class of Bregman relatively nonexpansive mappings embraces properly the class of Bregman strongly nonexpansive mappings. Very recently, Shahzad and Zegeye [21] introduced the class of multi-valued Bregman relatively nonexpansive mappings which includes the class of single-valued Bregman relatively nonexpansive mappings. Hence, the class of multi-valued Bregman relatively nonexpansive mappings is a more general class of mappings and gave a example of a multi-valued Bregman relatively nonexpansive mappings. Moreover, Shahzad and Zegeye [21] proved that if C is a nonempty closed convex subset of $int(domf)$ where $f : E \to \mathbb{R}$ is a uniformly Frechet differentiable and totally convex on bounded subsets of E and $T : C \to CB(C)$ is a Bregman relatively nonexpansive mapping, then $F(T)$ is closed and convex.

Our aim in this paper is to construct an iterative scheme for solving problem (1.7) which is also a fixed point of a multi-valued Bregman relatively nonexpansive mapping T in p-uniformly convex real Banach spaces which are also uniformly smooth and then prove the strong convergence theorems of the sequences generated by our scheme under some suitable assumptions.

2 Preliminaries

Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of smoothness of E is the function $\rho_E(\tau) : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq 1\Big\}.$$

E is called to be uniformly smooth if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$$

and E is called to be q-uniformly smooth if there exists a $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\Big\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\Big\}.$$

E is called to be uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$ and p-uniformly convex if there is a $C_p > 0$ so that $\delta_E(\varepsilon) \geq C_p\varepsilon^p$ for any $\varepsilon \in (0, 2]$. The L_p space is 2-uniformly convex for $1 < p \leq 2$ and p-uniformly convex for $p \geq 2$.

Lemma 2.1. *[27] Let $x, y \in E$. If E is q-uniformly smooth, then there exists a $C_q > 0$ such that*

$$\|x - y\|\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

It is known that if E is p-uniformly convex and uniformly smooth, then its dual E^* is q-uniformly smooth and uniformly convex. Moreover the duality mapping J_E^p is one-to-one, single-valued and $J_E^p = (J_{E^*}^q)^{-1}$ where $J_{E^*}^q$ is the duality mapping of E^* (see [10, 14]).

Definition 2.2. The duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\|^p = \|x\|^{p-1}\}.$$

The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^p x_n, y\rangle \to \langle J_E^p x, y\rangle$$

holds for any $y \in E$. We observe that $l_p(p > 1)$ has such a property, but $L_p(p > 2)$ does not have this property.

Let $f : E \to (-\infty, +\infty]$ be a convex function and $x \in int(dom)f$. The function f is said to be Gâteaux differentiable at x if

$$\lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t} \text{ exists for any } y \in E.$$

Definition 2.3. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function. The Bregman distance with respect to f is defined as:

$$\Delta_f(x, y) = f(y) - f(x) - \langle f^{'}(x), y - x\rangle, \quad x, y \in E.$$

It is worth noting that the duality mapping J_E^p is in fact the derivative of the function $f_p(x) = (\frac{1}{p})\|x\|^p$. Then the Bregman distance with respect to f_p is given by

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x - y\rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, x\rangle.\end{aligned}$$

In general, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties.

The following are some of important properties of the Bregman distance which are needed in the sequel

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_E^p x - J_E^p z\rangle, \tag{2.1}$$

and

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle. \tag{2.2}$$

For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [20]):

$$\tau\|x-y\|^p \leq \Delta_p(x,y) \leq \langle x-y, J_E^p x - J_E^p y\rangle, \tag{2.3}$$

where $\tau > 0$ is some fixed number.
Let C be a nonempty closed convex subset of E. The metric projection

$$P_C x = \arg\min_{y\in C} \|x-y\|, \;\; x \in E,$$

is the unique minimizer of the norm distance which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0, \;\; \forall z \in C. \tag{2.4}$$

Similar to the metric projection, the Bregman projection is defined as

$$\Pi_C x = \arg\min_{y\in C} \Delta_p(x,y), \;\; x \in E,$$

which is well-defined and the minimizer of it is unique (for more details see [19]). The Bregman projection can also be characterized by a variational inequality:

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0, \;\; \forall z \in C, \tag{2.5}$$

from which one has

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x,z) - \Delta_p(x, \Pi_C x), \;\; \forall z \in C. \tag{2.6}$$

Following [1] and [7], we use of the function $V_p : E^* \times E \to [0,+\infty)$ associated with f_p which is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p, \;\; \forall x \in E, \;\bar{x} \in E^*.$$

Then V_p is nonnegative and $V_p(\bar{x}, x) = \Delta_p(J_{E^*}^q(\bar{x}), x)$ for all $x \in E^*$ and $y \in E$.
Moreover, by the subdifferential inequality,

$$\langle f^{'}(\bar{x}), x - \bar{x}\rangle \leq f(\bar{x}) - f(x). \tag{2.7}$$

With $f(x) = \frac{1}{q}\|x\|^q$, $x \in E^*$, then $f^{'}(x) = J_{E^*}^q$, we have

$$\langle J_{E^*}^q(x), y\rangle \leq \frac{1}{q}\|x-y\|^q - \frac{1}{q}\|x\|^q, \forall x,y \in E^*. \tag{2.8}$$

Using (2.8), we have for all $\bar{x}, \bar{y} \in E^*$ and $x \in E$ that

$$\begin{aligned}
V_p(\bar{x}+\bar{y}, x) &= \frac{1}{q}\|\bar{x}+\bar{y}\|^q - \langle \bar{x}+\bar{y}, x\rangle + \frac{1}{p}\|x\|^p \\
&\geq \frac{1}{q}\|\bar{x}\|^q + \langle \bar{y}, J_{E^*}^q(\bar{x})\rangle - \langle \bar{x}+\bar{y}, x\rangle + \frac{1}{p}\|x\|^p \\
&= \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p + \langle \bar{y}, J_{E^*}^q(\bar{x})\rangle \\
&+ \langle \bar{y}, J_{E^*}^q(\bar{x})\rangle - \langle \bar{y}, x\rangle \\
&= \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p + \langle \bar{y}, J_{E^*}^q(\bar{x}) - x\rangle \\
&= V_p(\bar{x}, x) + \langle \bar{y}, J_{E^*}^q(\bar{x}) - x\rangle.
\end{aligned}$$

In other words,

$$V_p(\bar{x}, x) + \langle \bar{y}, J^q_{E^*}(\bar{x}) - x\rangle \leq V_p(\bar{x} + \bar{y}, x), \tag{2.9}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see, for example, [23],[24]).

Let C be a nonempty closed convex subset of a smooth Banach space E and let T be a mapping from C into itself. A point $p \in C$ is said to be an asymptotic fixed point [16] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.4. Let C be a nonempty convex subset of $int(domf)$. A mapping $T : C \to int(domf)$ with $F(T) \neq \emptyset$ is called to be

(i) Bregman quasi-nonexpansive if

$$\Delta_p(Tx, \bar{x}) \leq \Delta_p(x, \bar{x}),\ \forall x \in C, \bar{x} \in F(T);$$

(ii) Bregman relatively nonexpansive if $F(T) = \hat{F}(T)$,

$$\Delta_p(Tx, \bar{x}) \leq \Delta_p(x, \bar{x}),\ \forall x \in C, \bar{x} \in F(T);$$

(iii) left Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$ if

$$\Delta_p(Tx, \bar{x}) \leq \Delta_p(x, \bar{x}),\ \forall x \in C, \bar{x} \in \hat{F}(T),$$

and if whenever $\{x_n\} \subset C$ is bounded, $\bar{x} \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (\Delta_p(x_n, \bar{x}) - \Delta_p(Tx_n, \bar{x})) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

It is obvious that any left Bregman strongly nonexpansive mapping is a Bregman relatively nonexpansive mapping, but the converse is not true in general. Pang et al. [9] showed that there exists a Bregman relatively nonexpansive mapping which is not a Bregman strongly nonexpansive mapping.

Let $N(C)$ and $CB(C)$ denote the families of nonempty subsets and nonempty closed bounded subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\},$$

for all $A, B \in CB(C)$ where $\operatorname{dist}(x, B) = \inf\{\|x - y\| : y \in B\}$ is the distance from a point x to a subset B.

Recall that a multi-valued mapping $T : C \to CB(C)$ is said to be

(i) nonexpansive if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in C$;

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$.

Let $T : C \to CB(C)$. A point $p \in C$ is said to be a fixed point of T if $p \in F(T)$ where $F(T) = \{p \in T : p \in Tp\}$. A point $p \in C$ is said to be an asymptotic fixed point [16] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$.

Definition 2.5. *[21]* Let $T : C \to CB(C)$ is said to be Bregman relatively nonexpansive if the following conditions are satisfied:

(A1) $F(T)$ is nonempty;

(A2) $\Delta_p(z, \bar{x}) \le \Delta_p(x, \bar{x})$ for $z \in Tx$, $x \in C$ and $\bar{x} \in F(T)$;

(A3) $F(T) = \hat{F}(T)$.

The following is the example of a multi-valued Bregman relatively nonexpansive mapping appeared in [21]:

Example 2.6. [21] Let $I = [0, 1]$, $X = L^p(I)$, $1 < p < \infty$ and $C = \{f \in X : f(x) \ge 0, \forall x \in I\}$. Let $T : C \to CB(C)$ be defined by

$$\begin{cases} \{h \in C : f(x) - \frac{1}{2} \le h(x) \le f(x) - \frac{1}{4}, \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\}, \text{ otherwise.} \end{cases} \tag{2.10}$$

Then T is defined by (2.10) is a multi-valued Bregman relatively nonexpansive mapping.

We next state the following lemmas which will be used in the sequel.

Lemma 2.7. *[5] Let E be a Banach space and $f : E \to \mathbb{R}$ a Gâteaux differentiable function which is locally uniformly convex on E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent*

(i) $\lim_{n\to\infty} D_f(x_n, y_n) = 0$;

(ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.8. *[12] Let E be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then*

$$f\Big(\sum_{k=0}^{n} \alpha_k x_k\Big) \le \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - y_i\|),$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in B_r$, $\alpha_k \in (0, 1)$, and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f .

Lemma 2.9. *[27] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \;\; n \ge 1,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \le 0$; (iii) $\gamma_n \ge 0$; $(n \ge 1)$, $\sum \gamma_n < 0$. Then, $a_n \to 0$ as $n \to \infty$.

3 Main results

In this section, we introduce an iterative process for approximation of a common fixed point for a finite family of multi-valued Bregman relatively nonexpansive mappings with a solution of the split feasibility problems in p-uniformly convex and uniformly smooth Banach spaces and prove the strong convergence theorems of the proposed iterative process in p-uniformly convex and uniformly smooth Banach spaces

Theorem 3.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP has a nonempty solution set Ω. Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued bregman relative nonexpansive mappings of C into $CB(C)$ such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. Let $u_1 \in E_1$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_n = \Pi_c J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c J^q_{E_1^*}(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J^p_{E_1}(z_n^{(i)})) \ \ , z_n^{(i)} \in T_i x_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n^{(i)}\} \subset [a.b] \subset (0,1)$ for all $i = 0, 1, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$. Suppose the following conditions are satisfied:

(i) $\Sigma_{n=1}^\infty \alpha_n^{(i)} = 0$ *for all* $i = 0, 1, ..., N$.

(ii) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $x^* \in \mathcal{F}$.

Proof. Let $x^* \in \Omega$. Suppose that $w_n = Au_n - P_Q(Au_n)$ and $v_n = J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))]$, $\forall n \geq 1$. Therefore $x_n = \Pi_c v_n$, $\forall n \geq 1$. It follows that

$$\begin{aligned} \langle J^p_{E_2}(w_n), Au_n - Ax^*\rangle &= \|Au_n - P_Q(Au_n)\|^p + \langle J^p_{E_2}(w_n), P_Q(Au_n) - Ax^*\rangle \\ &\geq \|Au_n - P_Q(Au_n)\|^p = \|w_n\|^p. \end{aligned} \tag{3.2}$$

By Lemma 2.1, we obtain that

$$\begin{aligned} \Delta_p(x_n, x^*) &\leq \Delta_p(v_n, x^*) \\ &= \Delta_p(J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(w_n)], x^*) \\ &= \frac{1}{q}\|J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(w_n)\|^q - \langle J^p_{E_1}(u_n), x^*\rangle + t_n\langle J^p_{E_2}(w_n), Ax^*\rangle + \frac{1}{p}\|x^*\|^p \\ &\leq \frac{1}{q}\|J^p_{E_1}(u_n)\|^q - t_n\langle Au_n, J^p_{E_2}(w_n)\rangle + \frac{C_q(t_n\|A\|)^q}{q}\|J^p_{E_2}(w_n)\|^q \\ &- \langle J^p_{E_1}(u_n), x^*\rangle + t_n\langle J^p_{E_2}(w_n), Ax^*\rangle + \frac{1}{p}\|x^*\|^p \\ &= \frac{1}{q}\|u_n\|^p - \langle J^p_{E_1}(u_n), x^*\rangle + \frac{1}{p}\|x^*\|^p + t_n\langle Au_n, J^p_{E_2}(w_n)\rangle \\ &+ \frac{C_q(t_n\|A\|)^q}{q}\|J^p_{E_2}(w_n)\|^q \\ &= \Delta_p(u_n, x^*) + t_n\langle J^p_{E_2}(w_n), Ax^* - Au_n\rangle + \frac{C_q(t_n\|A\|)^q}{q}\|J^p_{E_2}(w_n)\|^q \\ &= \Delta_p(u_n, x^*) + \Big(t_n - \frac{C_q(t_n\|A\|)^q}{q}\Big)\|w_n\|^p. \end{aligned} \tag{3.3}$$

Using the condition(ii), we have

$$\Delta_p(x_n, x^*) \leq \Delta_p(u_n, x^*) \ \ \forall n \geq 1.$$

Now, using (3.1), we have

$$\begin{aligned}\Delta_p(x_{n+1}, x^*) \leq \Delta_p(u_{n+1}, x^*) &\leq \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(z_n^{(i)}, x^*) \\ &\leq \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(x_n, x^*) \\ &= \Delta_p(x_n, x^*). \end{aligned} \tag{3.4}$$

This shows that $\{\Delta_p(x_n, x^*)\}$ is a bounded decreasing sequence. Hence the $\lim_{n\to\infty} \Delta_p(x_n, x^*)$ exists and thus $\lim_{n\to\infty}(\Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*)) = 0$. Let $y_n = J^q_{E_1^*}(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J^p_{E_1}(z_n^{(i)}))$, $n \geq 1$. Therefore

$$\begin{aligned}\Delta_p(x_{n+1}, x^*) &\leq \Delta_p(u_{n+1}, x^*) \\ &= V_p(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^{N} \alpha_n^{(i)} J^p_{E_1}(z_n^{(i)}), x^*) \\ &\leq V_p(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^{N} \alpha_n^{(i)} J^p_{E_1}(z_n^{(i)}) - \alpha_n^{(0)}(J^p_{E_1}(x_n) - J^p_{E_1}(x^*)), x^*) \\ &+ \alpha_n^{(0)} \langle J^p_{E_1}(x_n) - J^p_{E_1}(x^*), y_n - x^* \rangle \\ &= V_p(\alpha_n^{(0)} J^p_{E_1}(x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} J^p_{E_1}(z_n^{(i)}), x^*) + \alpha_n^{(i)} \langle J^p_{E_1}(x_n) - J^p_{E_1}(x^*), y_n - x^* \rangle \\ &= \alpha_n^{(0)} V_p(J^p_{E_1}(x^*), x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} V_p(J^p_{E_1}(z_n^{(i)}), x^*) \\ &+ \alpha_n^{(0)} \langle J^p_{E_1}(x_n) - J^p_{E_1}(x^*), y_n - x^* \rangle \\ &= \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(z_n^{(i)}, x^*) + \alpha_n^{(0)} \langle J^p_{E_1}(x_n) - J^p_{E_1}(x^*), y_n - x^* \rangle \\ &\leq \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(x_n, x^*) + \alpha_n^{(0)} \langle J^p_{E_1}(x_n) - J^p_{E_1}(x^*), y_n - x^* \rangle. \end{aligned} \tag{3.5}$$

By Lemma 2.8, we obtain that

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &\leq \Delta_p(u_{n+1}, x^*) \\
&= V_p(\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}), x^*) \\
&= \frac{1}{q}\|x^*\|^q - \langle \alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}), x^* \rangle \\
&+ \frac{1}{p}\|\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)})\|^p \\
&= \frac{1}{q}\|x^*\|^q - \alpha_n^{(0)} \langle J_{E_1}^p(x_n), x^* \rangle - \sum_{i=1}^N \alpha_n^{(i)} \langle J_{E_1}^p(z_n^{(i)}), x^* \rangle \\
&+ \frac{1}{p}\|\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)})\|^p
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{q}\|x^*\|^q - \alpha_n^{(0)} \langle J_{E_1}^p(x_n), x^* \rangle - \sum_{i=1}^N \alpha_n^{(i)} \langle J_{E_1}^p(z_n^{(i)}), x^* \rangle \\
&+ \alpha_n^{(0)} \frac{1}{p}\|J_{E_1}^p(x_n)\|^p + \sum_{i=1}^N \alpha_n^{(i)} \frac{1}{p}\|J_{E_1}^p(z_n^{(i)})\|^p - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&= \alpha_n^{(0)} V_p(J_{E_1}^p(x_n), x^*) + \sum_{i=1}^N \alpha_n^{(i)} V_p(J_{E_1}^p(z_n^{(i)}), x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&= \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^N \alpha_n^{(i)} \Delta_p(z_n^{(i)}, x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&\leq \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^N \alpha_n^{(i)} \Delta_p(x_n, x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&= \Delta_p(x_n, x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n)^p - J_{E_1}^p(z_n^{(i)})\|).
\end{aligned}$$

Thus

$$\alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n)\|^p - J_{E_1}^p(z_n^{(i)})\|) \leq \Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*). \tag{3.6}$$

Then, from (3.6), we have

$$\alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \to 0, \;\; n \to \infty.$$

By the property of ρ_r, we have

$$\lim_{n\to\infty} \|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\| = 0.$$

Since $J_{E_1^*}^q$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|x_n - z_n^{(i)}\| = 0.$$

Since $d(x_n, T_i x_n) \leq \|x_n - z_n^{(i)}\|$, we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0,$$

for each $i = \{1, 2, ..., N\}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to z. Since T_i is a multi-valued Bregman relative nonexpansive mapping, we obtain $z \in F(T_i)$, for each $i \in \{1, 2, ..., N\}$ and hence $z \in \cap_{i=1}^N F(T_i)$.
We now show that $z \in \Omega$. From (3.3), we obtain that

$$\Big(\frac{C_q(t_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*). \tag{3.7}$$

From (3.4), we have

$$\Delta_p(u_{n+1}, x^*) \leq \Delta_p(x_n, x^*). \tag{3.8}$$

Putting (3.7) into (3.8), we have

$$\Big(\frac{C_q(t_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \leq \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*). \tag{3.9}$$

By condition (ii) and (3.9), we have

$$\begin{aligned} 0 &< t\Big(1 - \frac{C_q k^{q-1}\|A\|^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \\ &\leq \Big(t_n - \frac{C_q(t_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \\ &\leq \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*). \end{aligned}$$

Hence, we obtain that

$$\lim_{n\to\infty} \|Au_n - P_Q(Au_n)\| = 0. \tag{3.10}$$

Since $v_n = J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))]$, $\forall n \geq 1$, then we have

$$\begin{aligned} 0 \leq \|J^p_{E_1}(v_n) - J^p_{E_1}(u_n)\| &\leq t_n\|A^*\|\|J^p_{E_2}(Au_n - P_Q(Au_n))\| \\ &\leq \Big(\frac{q}{C_q\|A\|^q}\Big)^{q-1}\|A^*\|\|Au_n - P_Q(Au_n)\|^{p-1}. \end{aligned} \tag{3.11}$$

It follows that

$$\lim_{n\to\infty} \|J^p_{E_1}(v_n) - J^p_{E_1}(u_n)\| = 0.$$

Since $J^q_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Furthermore,

$$\|J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] - u_n\| = \|v_n - u_n\| \to 0, n \to \infty.$$

Since J_{E_1} is norm-to-norm uniformly continuous on bounded subsets of E_1, then

$$\begin{aligned} t\|A^* J^p_{E_2}(Au_n - P_Q(Au_n))\| &\leq t_n\|A^* J^p_{E_2}(Au_n - P_Q(Au_n))\| \\ &= \|J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n)) - J^p_{E_1}(u_n)\|. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \|A^* J^p_{E_2}(Au_n - P_Q(Au_n))\| = 0.$$

From (2.6) and (3.4), we obtain that

$$\begin{aligned}\Delta_p(v_n, x_n) &= \Delta_p(v_n, \Pi_c v_n)\\ &\le \Delta_p(v_n, x^*) - \Delta_p(x_n, x^*)\\ &\le \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*)\\ &\le \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

Hence

$$\|x_n - u_n\| = \|v_n - u_n\| + \|v_n - x_n\| \to 0 \text{ as } n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in \omega_\omega(x_n)$. Since $x_{n_j} \rightharpoonup z$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, we obtain that $u_{n_j} \rightharpoonup z$. From (2.2), (2.5) and (2.3), we have

$$\begin{aligned}\Delta_p(z, \Pi_c z) &\le \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), z - \Pi_c z\rangle\\ &= \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), z - u_{n_j}\rangle + \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), u_{n_j} - \Pi_c u_{n_j}\rangle\\ &+ \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), \Pi_c u_{n_j} - \Pi_c z\rangle\\ &\le \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), z - u_{n_j}\rangle + \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), u_{n_j} - \Pi_c u_{n_j}\rangle.\end{aligned}$$

As $j \to \infty$, we obtain that $\Delta_p(z, \Pi_c z) = 0$. Thus $z \in C$. Let us now fix $x \in C$. Then $Ax \in Q$ and

$$\begin{aligned}\|(I - P_Q)Au_{n_j}\|^p &= \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Ax_n - P_Q(Au_{n_j})\rangle\\ &= \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Ax_n - Ax\rangle + \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Ax_n - P_Q(Au_{n_j})\rangle\\ &\le \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Au_{n_j} - Ax\rangle\\ &\le M\|A^*(I - P_Q)Au_{n_j}\|^{p-1} \to 0, n \to \infty,\end{aligned}$$

where $M > 0$ is sufficiently large number. It then follows from (2.4) that

$$\begin{aligned}\|(I - P_Q)Az\|^p &= \langle J^p_{E_2}(Az - P_Q(Az)), Az - P_Q(Az)\rangle\\ &= \langle J^p_{E_2}(Az - P_Q(Az)), Az - Au_{n_j}\rangle + \langle J^p_{E_2}(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle\\ &+ \langle J^p_{E_2}(Az - P_Q(Az)), P_Q(Au_{n_j}) - P_Q(Az)\rangle\\ &\le \langle J^p_{E_2}(Az - P_Q(Az)), Az - Au_{n_j}\rangle + \langle J^p_{E_2}(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle.\end{aligned}$$

Also, since $Au_{n_j} \rightharpoonup Az$, we have that

$$\lim_{n\to\infty} \|(I - P_Q)Az\| = 0.$$

Thus $Az \in Q$. This implies that $z \in \Omega$ and hence $z \in F(T) \cap \Omega$.Furthermore, we have

$$\Delta_p(x_n, y_n) \le \alpha_n^{(0)} \Delta_p(x_n, x_n) + \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(x_n, z_n^{(i)}). \tag{3.12}$$

Since $\|x_n - z_n^{(i)}\| \to 0$ as $n \to \infty$ and $\{z_n^i\}$ is a bounded sequence. By Lemma 2.7, we obtain that $\lim_{n\to\infty} \Delta_p(x_n, z_n^{(i)}) = 0$. From (3.12), it follows that $\|x_n - y_n\| \to 0, n \to \infty$.

Let $p \in F(T) \cap \Omega$. We next show that $\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), y_n - p\rangle \leq 0$. To show the inequality $\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), y_n - p\rangle \leq 0$, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), x_n - p\rangle = \lim_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), x_{n_j} - p\rangle = 0.$$

Since $\|x_n - y_n\| \to 0$ as $n \to \infty$ and (2.5), we obtain that

$$\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), y_n - p\rangle \leq \limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), x_n - p\rangle = 0. \tag{3.13}$$

Using (3.13), (3.5) and Lemma 2.9, we obtain that $\Delta_p(x_n, p) \to 0, n \to \infty$. Hence, $x_n \to p$ as $n \to \infty$. □

Corollary 3.2. *Let E_1 and E_2 be two L_p spaces with $2 \leq p < \infty$. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP has a nonempty solution set Ω. Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued Bregman relative nonexpansive mappings of C into $CB(C)$ such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. Let $u_1 \in E_1$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_n = \Pi_c J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c J^q_{E_1^*}(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J^p_{E_1}(z_n^{(i)})) \ \ , z_n^{(i)} \in T_i x_n, \end{cases} \tag{3.14}$$

where $\{\alpha_n^{(i)}\} \subset [a.b] \subset (0,1)$ for all $i = 0,1,...,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$. Suppose the following conditions are satisfied:

(i) $\Sigma_{n=1}^\infty \alpha_n^{(i)} = 0$ *for all* $i = 0,1,...,N$

(ii) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $x^* \in \mathcal{F}$.

If we assume that each T_i, $i = 1,2,...,N$, in Theorem 3.1 is a Bregman relative nonexpansive single-valued mapping, we obtain the following corollary:

Corollary 3.3. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP has a nonempty solution set Ω. Let $\{T_i\}_{i=1}^N$ be a finite family of single-valued Bregman relative nonexpansive mappings of C into C such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. Let $u_1 \in E_1$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_n = \Pi_c J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c J^q_{E_1^*}(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J^p_{E_1}(T_i x_n)), \end{cases} \tag{3.15}$$

where $\{\alpha_n^{(i)}\} \subset [a.b] \subset (0,1)$ for all $i = 0,1,...,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$. Suppose the following conditions are satisfied:

(i) $\Sigma_{n=1}^\infty \alpha_n^{(i)} = 0$ *for all* $i = 0,1,...,N$

(ii) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $x^* \in \mathcal{F}$.

4 Numerical Example

In this section, we present the numerical example supporting our main result. All codes are written in Matlab2013b.

Example 4.1. Let $E_1 = L_2([0,1]) = E_2$ with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(t)(g(t)dt.$$

Suppose that

$$C := \{x \in L_2([0,1]) : \langle x, a\rangle = b\},$$

where $a = 2t^2$ and $b = 0$. Therefore

$$P_C(x) = \max\Big\{0, \frac{b - \langle a, x\rangle}{\|a\|_2^2}\Big\}a + x.$$

Let

$$Q := \{x \in L_2([0,1]) : \langle x, c\rangle \geq d\},$$

where $c = \frac{t}{3}$ and $d = -2$. It follows that

$$P_Q(x) := \frac{d - \langle c, x\rangle}{\|c\|_2^2}c + x.$$

Define

$$A : L_2([0,1] \to L_2([0,1] \text{ by } (Ax)(t) = \frac{x(t)}{2}.$$

Then A is a bounded linear operator with $\|A\| = 2$ and $A^* = A$. Suppose that

$$T_1(f)\begin{cases} \{h \in C : f(x) - \frac{3}{4} \leq h(x) \leq f(x) - \frac{1}{3}, \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\}, \text{ otherwise,} \end{cases} \tag{4.1}$$

and

$$T_2(f)\begin{cases} \{g \in C : f(x) - \frac{1}{2} \leq g(x) \leq f(x) - \frac{1}{4}, \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\}, \text{ otherwise.} \end{cases} \tag{4.2}$$

In [21], we obtain that T_1 and T_2 are multi-valued Bregman relative nonexpansive mappings. Consider the problem:

$$\text{find } x \in F(T) \cap C \text{ such that } Ax \in Q. \tag{4.3}$$

We see that the set of solutions of problem (4.3) is nonempty, since $x = 0$ is in the set of solutions. Let $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, and $\alpha_n^{(2)} = \frac{12n-1}{18n}$ for all $n \geq 1$. Put $z_n^{(1)} = x_n - \frac{3}{4}$ and $z_n^{(2)} = x_n - \frac{1}{2}$. Using the iterative method (3.1), we obtain that

$$\begin{cases} x_n = \Pi_c[u_n - t_n A^*(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c(\frac{1}{12n}(x_n) + \frac{12n-1}{36n}(x_n - \frac{3}{4}) + \frac{12n-1}{18n}(x_n - \frac{1}{2})), \ \ n \geq 1. \end{cases} \tag{4.4}$$

We make different choices of u_1 and t_n and take $\frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-6}$ as our stopping criterion.

Case 1 $t_n = 0.001$ and $u_1 = t$. We have the numerical analysis tabulated in Table 1 and show in Figure 1.

Table 1 Example 4.1: Case 1

No.of iteration	$\|x_{n+1} - x_n\|_2$	$\|u_{n+1} - u_n\|_2$
2	0.45960659	0.45871914
3	0.03706339	0.03979071
4	0.00089775	0.00150921
5	0.00002339	0.00002339
6	0.00000070	0.00000210
7	0.00000043	0.00000141
8	0.00000030	0.00000100
9	0.00000023	0.00000075

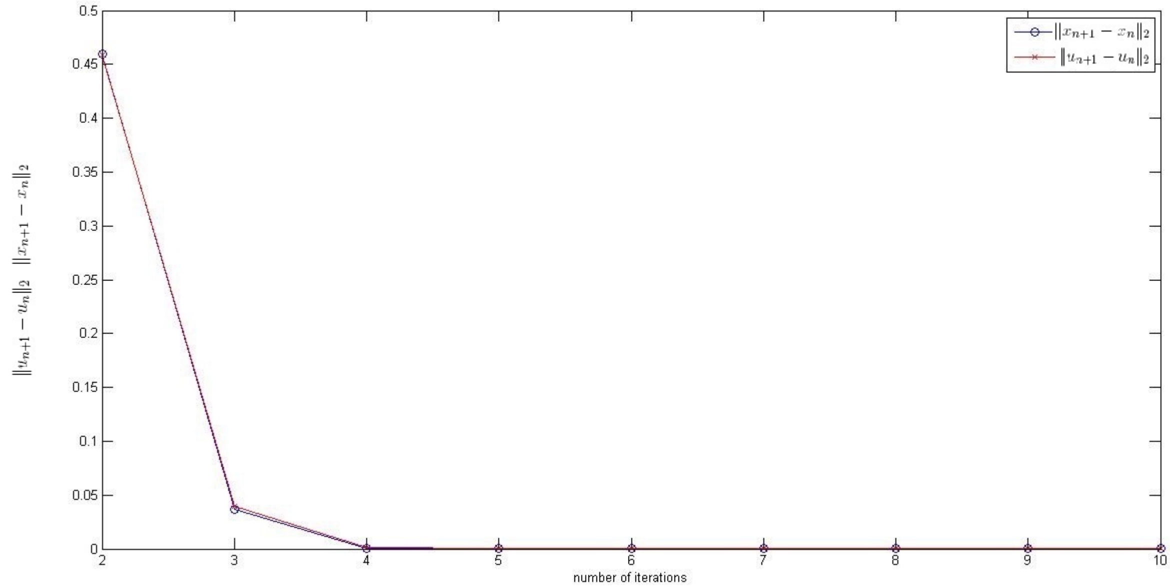
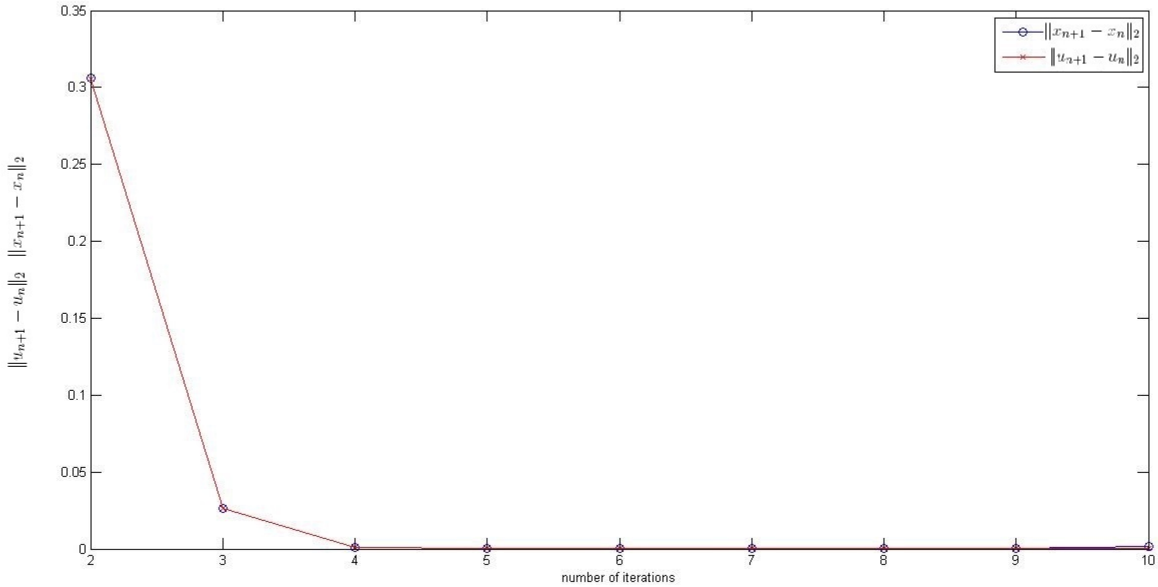


Figure 1. Example 4.1: Case 1.

Case 2 $t_n = 0.0002$ and $u_1 = t^2$. We have the numerical analysis tabulated in Table 2 and show in Figure 2.

Table 2 Example 4.1: Case 2

No.of iteration	$\|x_{n+1} - x_n\|_2$	$\|u_{n+1} - u_n\|_2$
2	0.30581518	0.30563219
3	0.02659138	0.02659287
4	0.0008344	0.00110931
5	0.00002409	0.00002388
6	0.00000053	0.00000064
7	0.00000009	0.00000028
8	0.00000006	0.00000020
9	0.00000005	0.00000015



4.jpg 4.bb

Figure 2. Example 4.1: Case 2.

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Expressions and Dynamical Behavior of Rational Recursive Sequences

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ABSTRACT

In this paper, we study the qualitative behavior of the rational recursive sequences

$$x_{n+1}=\frac{x_{n-11}}{\pm 1\pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}},\quad n=0,1,2,...,$$

where the initial conditions are arbitrary real numbers. Also, we give the numerical examples of some cases of difference equations and obtained some related graphs and figures using by Matlab.

Keywords: Difference Equation, Recursive sequence, Local stability, Periodicity.

Mathematics Subject Classification: 39A10.

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1. INTRODUCTION

Difference equations and dynamic equations on time scales have an immense possibility for applications in engineering, physics, biology, economics, etc. Lately, considerable attentiveness has been devoted to the oscillation theory of the various classes of equations,see e.g. [1]-[42] and the references cited therein.

In this study, we are interested with the behavior of the solution of difference equations

$$x_{n+1}=\frac{x_{n-11}}{\pm 1\pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}},\quad n=0,1,2,...,\tag{1}$$

where the initial conditions are arbitrary real numbers. For some outcome in this study for examples: Cinar [8-10] obtained the solutions of the difference equations

$$x_{n+1}=\frac{x_{n-1}}{1+x_nx_{n-1}},\quad x_{n+1}=\frac{x_{n-1}}{-1+x_nx_{n-1}},\quad x_{n+1}=\frac{ax_{n-1}}{1+bx_nx_{n-1}}.$$

Cinar et al. [11] gave the form of the solution of the difference equation

$$x_{n+1}=\frac{x_{n-3}}{-1+x_nx_{n-1}x_{n-2}x_{n-3}}.$$

Elabbasy et al. [13] solved the following problem

$$x_{n+1}=\frac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k}x_{n-i}}.$$

In [14] Elsayed studied the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}.$$

Elsayed [21-22] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-4}x_{n-9}}.$$

Elsayed [23] investigated the Solution of difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}.$$

Elsayed and Iricanin [24] has got the solution of the difference equation

$$x_{n+1} = \max\left\{A_n/x_n, x_{n-1}\right\}.$$

Ibrahim [26] studied the third order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}\left(a + b x_n x_{n-2}\right)}.$$

In [30] Kent et al studied the Behavior of solutions of the difference equation

$$x_{n+1} = x_n x_{n-2} - 1.$$

Let I be some interval of real numbers and let $F : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial condition $x_{-k},\ x_{-k+1},\ ...,\ x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if $\overline{x} = F(\overline{x})$, that is,

$$x_n = \overline{x} \ for\ all \quad n \geq -k.$$

is a solution of Eq.(2), or equivalently, $\overline{x}$ is **a fixed point** of F.

Definition 2. (Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Linearized Stability Analysis

Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point x^*. Let

$$p_i = \frac{\partial F}{\partial u_i}\left(\overline{x}, \overline{x}, ..., \overline{x}\right) \quad for \quad i = 0, 1, ..., k,$$

denote the partial derivatives of $F(u_0, u_1, ...u_k)$ evaluated at the equilibrium $\overline{x}$ of Eq.(2).

Then the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + ... + p_k y_{n-k}\ , \quad n = 0, 1, ..., \tag{3}$$

is called **the linearized equation associated** of Eq.(2) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^{k+1} - p_0\lambda^k - ... - p_{k-1}\lambda - p_k = 0, \tag{4}$$

is called the characteristic equation of Eq.(3) about $\overline{x}$.

The following result known as the Linear Stability Theorem is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(2).

Definition 3. The equilibrium point $\overline{x}$ is said to be **hyperbolic** if $|F(\overline{x})| \neq 1$.

If $|F(\overline{x})| = 1$, $\overline{x}$ is **non hyperbolic.**

Theorem A. [31] Assume that $p_0, p_2, ..., p_k$ are real numbers such that

$$|p_0| + |p_1| + ... + |p_k| < 1, \qquad \text{or} \quad \sum_{i=1}^{k} |p_i| < 1.$$

Then all roots of Eq.(4) lie inside the unit disk.

2. THE FIRST EQUATION $X_{N+1} = \frac{X_{N-11}}{1+X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

In this part, we obtain the following special case of Eq.(1) in the form:

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \tag{5}$$

where the initial values are arbitrary real numbers.

Theorem 2.1. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation (5). Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{12n-11} &= p\prod_{i=0}^{n-1}\frac{1+4ipkfc}{1+(4i+1)\,pkfc},\ x_{12n-10} = m\prod_{i=0}^{n-1}\frac{1+4imheb}{1+(4i+1)\,mheb},\ x_{12n-9} = l\prod_{i=0}^{n-1}\frac{1+4ildga}{1+(4i+1)\,ldga},\\
x_{12n-8} &= k\prod_{i=0}^{n-1}\frac{1+(4i+1)\,pkfc}{1+(4i+2)\,pkfc},\ x_{12n-7} = h\prod_{i=0}^{n-1}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb},\ x_{12n-6} = g\prod_{i=0}^{n-1}\frac{1+(4i+1)\,ldga}{1+(4i+2)\,ldga},\\
x_{12n-5} &= f\prod_{i=0}^{n-1}\frac{1+(4i+2)\,pkfc}{1+(4i+3)\,pkfc},\ x_{12n-4} = e\prod_{i=0}^{n-1}\frac{1+(4i+2)\,mheb}{1+(4i+3)\,mheb},\ x_{12n-3} = d\prod_{i=0}^{n-1}\frac{1+(4i+2)\,ldga}{1+(4i+3)\,ldga},\\
x_{12n-2} &= c\prod_{i=0}^{n-1}\frac{1+(4i+3)\,pkfc}{1+(4i+4)\,pkfc},\ x_{12n-1} = b\prod_{i=0}^{n-1}\frac{1+(4i+3)\,mheb}{1+(4i+4)\,mheb},\ x_{12n} = a\prod_{i=0}^{n-1}\frac{1+(4i+3)\,ldga}{1+(4i+4)\,ldga},
\end{aligned}$$

where $x_{-11} = p,\ x_{-10} = m,\ x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_5 = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,$ $x_{-1} = b,\ x_0 = a$ and $\prod_{i=0}^{-1}\alpha_i = 1.$

Proof. For $n = 0$, the result holds. Now, assume that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{12n-23} &= p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)\,pkfc},\ x_{12n-22} = m\prod_{i=0}^{n-2}\frac{1+4imheb}{1+(4i+1)\,mheb},\ x_{12n-21} = l\prod_{i=0}^{n-2}\frac{1+4ildga}{1+(4i+1)\,ldga},\\
x_{12n-20} &= k\prod_{i=0}^{n-2}\frac{1+(4i+1)\,pkfc}{1+(4i+2)\,pkfc},\ x_{12n-19} = h\prod_{i=0}^{n-2}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb},\ x_{12n-18} = g\prod_{i=0}^{n-2}\frac{1+(4i+1)\,ldga}{1+(4i+2)\,ldga},\\
x_{12n-17} &= f\prod_{i=0}^{n-2}\frac{1+(4i+2)\,pkfc}{1+(4i+3)\,pkfc},\ x_{12n-16} = e\prod_{i=0}^{n-2}\frac{1+(4i+2)\,mheb}{1+(4i+3)\,mheb},\ x_{12n-15} = d\prod_{i=0}^{n-2}\frac{1+(4i+2)\,ldga}{1+(4i+3)\,ldga},
\end{aligned}$$

$$x_{12n-14}=c\prod_{i=0}^{n-2}\frac{1+(4i+3)\,pkfc}{1+(4i+4)\,pkfc},\;x_{12n-13}=b\prod_{i=0}^{n-2}\frac{1+(4i+3)\,mheb}{1+(4i+4)\,mheb},\;x_{12n-12}=a\prod_{i=0}^{n-2}\frac{1+(4i+3)\,ldga}{1+(4i+4)\,ldga}.$$

Now, it follows from Eq. (5) that

$$\begin{aligned}
x_{12n-11} &= \frac{x_{12n-23}}{1+x_{12n-14}x_{12n-17}x_{12n-20}x_{12n-23}} \\
&= \frac{p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}}{1+c\prod_{i=0}^{n-2}\frac{1+(4i+3)pkfc}{1+(4i+4)pkfc}f\prod_{i=0}^{n-2}\frac{1+(4i+2)pkfc}{1+(4i+3)pkfc}k\prod_{i=0}^{n-2}\frac{1+(4i+1)pkfc}{1+(4i+2)pkfc}p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}} \\
&= \frac{p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}}{1+pkfc\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+4)pkfc}} = p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}\left(\frac{1}{1+\frac{pkfc}{1+(4n-4)pkfc}}\right) \\
&= p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)\,pkfc}\left(\frac{1+(4n-4)\,pkfc}{1+(4n-3)\,pkfc}\right)
\end{aligned}$$

Therefore, we have

$$x_{12n-11}=p\prod_{i=0}^{n-1}\frac{1+4ipkfc}{1+(4i+1)\,pkfc}.$$

Similarly

$$\begin{aligned}
x_{12n-7} &= \frac{x_{12n-19}}{1+x_{12n-10}x_{12n-13}x_{12n-16}x_{12n-19}} \\
&= \frac{h\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+2)mheb}}{1+m\prod_{i=0}^{n-1}\frac{1+4imheb}{1+(4i+1)mheb}b\prod_{i=0}^{n-2}\frac{1+(4i+3)mheb}{1+(4i+4)mheb}e\prod_{i=0}^{n-2}\frac{1+(4i+2)mheb}{1+(4i+3)mheb}h\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+2)mheb}} \\
&= \frac{h\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+2)mheb}}{1+mheb\prod_{i=0}^{n-1}\frac{1+4imheb}{1+(4i+1)mheb}\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+4)mheb}} \\
&= h\prod_{i=0}^{n-2}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}\left(\frac{1}{1+\frac{mheb}{1+(4n-3)mheb}}\right) \\
&= h\prod_{i=0}^{n-2}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}\left(\frac{1+(4n-3)\,mheb}{1+(4n-2)\,mheb}\right).
\end{aligned}$$

Hence, we have

$$x_{12n-7}=h\prod_{i=0}^{n-1}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}.$$

Similarly, other relations can be obtained and thus, the proof has been proved.

Theorem 2.2. Eq.(5) has unique equilibrium point which is the number zero and this equilibrium is not locally asymptotically stable. Also, $\overline{x}$ is non hyperbolic.

Proof. For the equilibrium points of Eq.(5), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^4},$$

Then

$$\overline{x} + \overline{x}^5 = \overline{x},$$

or $\overline{x}^5 = 0$. Then the unique equilibrium point of Eq.(5) is $\overline{x} = 0$.

Let $f : (0,\infty)^4 \to (0,\infty)$ be a function defined by

$$F(u,v,w,t) = \frac{u}{1+uvwt}.$$

Then it follows that,

$$\begin{aligned} F_u(u,v,w,t) &= \frac{1}{(1+uvwt)^2},\ F_v(u,v,w,t) = \frac{-u^2wt}{(1+uvwt)^2}, \\ F_w(u,v,w,t) &= \frac{-u^2vt}{(1+uvwt)^2}, F_t(u,v,w,t) = \frac{-u^2vw}{(1+uvwt)^2}, \end{aligned}$$

we see that

$$F_u(\overline{x},\overline{x},\overline{x},\overline{x}) = 1,\ F_v(\overline{x},\overline{x},\overline{x},\overline{x}) = 0,\ F_w(\overline{x},\overline{x},\overline{x},\overline{x}) = 0,\ F_t(\overline{x},\overline{x},\overline{x},\overline{x}) = 0.$$

The proof follows by using Theorem A. By Definition 3, $\overline{x}$ is non hyperbolic.

Theorem 2.3. Every positive solution of Eq.(5) is bounded and $\lim_{n\to\infty} x_n = 0$.

Proof. It is following by Eq.(5) that

$$x_{n+1} = \frac{x_{n-11}}{1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}} \le x_{n-11}.$$

Then

$$x_{n+1} < x_{n-11}, \quad for\ all \quad n \ge 0$$

Then the subsequences $\{x_{12n-11}\}_{n=0}^{\infty}$, $\{x_{12n-10}\}_{n=0}^{\infty}$, $\{x_{12n-9}\}_{n=0}^{\infty}$, ..., $\{x_{12n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\{x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}.$$

3. THE SECOND EQUATION $X_{N+1} = \frac{X_{N-11}}{-1+X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

In this part, we give the solution of the recursive equation in the form:

$$x_{n+1} = \frac{x_{n-11}}{-1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \tag{6}$$

where the initial values are arbitrary real numbers with $x_{-2}x_{-5}x_{-8}x_{-11} \neq 1$, $x_{-1}x_{-4}x_{-7}x_{-10} \neq 1$, $x_0x_{-3}x_{-6}x_{-9} \neq 1$.

Theorem 3.1. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation (6). Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{12n-11} &= \frac{p}{(-1+pkfc)^n},\ x_{12n-10} = \frac{m}{(-1+mheb)^n},\ x_{12n-9} = \frac{l}{(-1+ldga)^n},\\
x_{12n-8} &= k(-1+pkfc)^n,\ x_{12n-7} = h(-1+mheb)^n,\ x_{12n-6} = g(-1+ldga)^n,\\
x_{12n-5} &= \frac{f}{(-1+pkfc)^n},\ x_{12n-4} = \frac{e}{(-1+mheb)^n},\ x_{12n-3} = \frac{d}{(-1+ldga)^n},\\
x_{12n-2} &= c(-1+pkfc)^n,\ x_{12n-1} = b(-1+mheb)^n,\ x_{12n} = a(-1+ldga)^n,
\end{aligned}$$

where $x_{-11} = p,\ x_{-10} = m,\ x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_5 = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,$ $x_{-1} = b,\ x_0 = a.$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is

$$\begin{aligned}
x_{12n-23} &= \frac{p}{(-1+pkfc)^{n-1}},\ x_{12n-22} = \frac{m}{(-1+mheb)^{n-1}},\ x_{12n-21} = \frac{l}{(-1+ldga)^{n-1}},\\
x_{12n-20} &= k(-1+pkfc)^{n-1},\ x_{12n-19} = h(-1+mheb)^{n-1},\ x_{12n-18} = g(-1+ldga)^{n-1},\\
x_{12n-17} &= \frac{f}{(-1+pkfc)^{n-1}},\ x_{12n-116} = \frac{e}{(-1+mheb)^{n-1}},\ x_{12n-15} = \frac{d}{(-1+ldga)^{n-1}},\\
x_{12n-14} &= c(-1+pkfc)^{n-1},\ x_{12n-13} = b(-1+mheb)^{n-1},\ x_{12n-12} = a(-1+ldga)^{n-1}.
\end{aligned}$$

Now, it follows from Eq.(6) that

$$\begin{aligned}
x_{12n-11} &= \frac{x_{12n-23}}{-1+x_{12n-14}x_{12n-17}x_{12n-20}x_{12n-23}}\\
&= \frac{\frac{p}{(-1+pkfc)^{n-1}}}{-1+c(-1+pkfc)^{n-1}\frac{f}{(-1+pkfc)^{n-1}}k(-1+pkfc)^{n-1}\frac{p}{(-1+pkfc)^{n-1}}}\\
&= \frac{p}{(-1+pkfc)^{n-1}(-1+pkfc)}.
\end{aligned}$$

Then

$$x_{12n-11} = \frac{p}{(-1+pkfc)^n}.$$

Similarly

$$\begin{aligned}
x_{12n-6} &= \frac{x_{12n-18}}{-1+x_{12n-9}x_{12n-12}x_{12n-15}x_{12n-18}}\\
&= \frac{g(-1+ldga)^{n-1}}{-1+\frac{l}{(-1+ldga)^n}a(-1+ldga)^{n-1}\frac{d}{(-1+ldga)^{n-1}}g(-1+ldga)^{n-1}}\\
&= \frac{g(-1+ldga)^{n-1}}{-1+ldga(-1+ldga)^{-1}}.
\end{aligned}$$

Therefore, we have

$$x_{12n-6} = g(-1+ldga)^n.$$

The same other relations can be proved and thus, the proof has been completed.

Theorem 3.2. Eq.(6) has three equilibrium points which are $0, \pm\sqrt[4]{2}$ and these equilibrium points are not locally asymptotically stable.

Proof. The proof is the same as Theorem 2.2.

Theorem 3.3. Eq.(6) has a periodic solutions of period twelve iff $pkfc = mheb = ldga = 2$ and will be take the form

$$\{p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...\}.$$

Proof. Assume that there exists a prime twelve solutions

$$p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...,$$

of Eq.(6) ,we have from Eq.(6) that

$$\begin{aligned} p &= \frac{p}{(-1+pkfc)^n},\ m = \frac{m}{(-1+mheb)^n},\ l = \frac{l}{(-1+ldga)^n}, \\ k &= k(-1+pkfc)^n,\ h = h(-1+mheb)^n,\ g = g(-1+ldga)^n, \\ f &= \frac{f}{(-1+pkfc)^n},\ e = \frac{e}{(-1+mheb)^n},\ d = \frac{d}{(-1+ldga)^n}, \\ c &= c(-1+pkfc)^n,\ b = b(-1+mheb)^n,\ a = a(-1+ldga)^n, \end{aligned}$$

or

$$(-1+pkfc)^n = 1, \qquad (-1+mheb)^n = 1, \qquad (-1+ldga)^n = 1$$

Then

$$pkfc = mheb = ldga = 2.$$

Second let $pkfc = mheb = ldga = 2$. Then we have from Eq.(6) that

$$\begin{aligned} x_{12n-11} &= p,\ x_{12n-10} = m,\ x_{12n-9} = l,\ x_{12n-8} = k, \\ x_{12n-7} &= h,\ x_{12n-6} = g,\ x_{12n-5} = f,\ x_{12n-4} = e, \\ x_{12n-3} &= d,\ x_{12n-2} = c,\ x_{12n-1} = b, x_{12n} = a. \end{aligned}$$

Therefore we have a period twelve solutions and the proof is complete.

4. THE THIRD EQUATION $X_{N+1} = \frac{X_{N-11}}{1-X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

In this section we examine the following equation

$$x_{n+1} = \frac{x_{n-11}}{1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \tag{7}$$

where the initial conditions are arbitrary positive real numbers.

Theorem 4.1. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation (7). Then for $n = 0, 1, ...$

$$\begin{aligned} x_{12n-11} &= p\prod_{i=0}^{n-1}\frac{1-4ipkfc}{1-(4i+1)pkfc},\ x_{12n-10} = m\prod_{i=0}^{n-1}\frac{1-4imheb}{1-(4i+1)mheb},\ x_{12n-9} = l\prod_{i=0}^{n-1}\frac{1-4ildga}{1-(4i+1)ldga}, \\ x_{12n-8} &= k\prod_{i=0}^{n-1}\frac{1-(4i+1)pkfc}{1-(4i+2)pkfc},\ x_{12n-7} = h\prod_{i=0}^{n-1}\frac{1-(4i+1)mheb}{1-(4i+2)mheb},\ x_{12n-6} = g\prod_{i=0}^{n-1}\frac{1-(4i+1)ldga}{1-(4i+2)ldga}, \\ x_{12n-5} &= f\prod_{i=0}^{n-1}\frac{1-(4i+2)pkfc}{1-(4i+3)pkfc},\ x_{12n-4} = e\prod_{i=0}^{n-1}\frac{1-(4i+2)mheb}{1-(4i+3)mheb},\ x_{12n-3} = d\prod_{i=0}^{n-1}\frac{1-(4i+2)ldga}{1-(4i+3)ldga}, \\ x_{12n-2} &= c\prod_{i=0}^{n-1}\frac{1-(4i+3)pkfc}{1-(4i+4)pkfc},\ x_{12n-1} = b\prod_{i=0}^{n-1}\frac{1-(4i+3)mheb}{1-(4i+4)mheb},\ x_{12n} = a\prod_{i=0}^{n-1}\frac{1-(4i+3)ldga}{1-(4i+4)ldga}, \end{aligned}$$

where $x_{-11} = p, x_{-10} = m, x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g\ , x_5 = f, x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,$ $x_{-1} = b, x_0 = a$ and $\delta pkfc \neq 1, \delta mheb \neq 1, \delta ldga \neq 1$ for $\delta = 1, 2, 3, ...$

Proof. The proof is similar as the proof of the Theorem 2.1.

Theorem 4.2. Eq.(7) has unique equilibrium point which is the number zero and this equilibrium is not locally asymptotically stable.

5. THE FOURTH EQUATION $X_{N+1} = \frac{X_{N-11}}{-1-X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

Here we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-11}}{-1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}} \tag{8}$$

where the initial values are arbitrary non zero real numbers with $x_{-2}x_{-5}x_{-8}x_{-11} \neq -1$, $x_{-1}x_{-4}x_{-7}x_{-10} \neq -1$, $x_0x_{-3}x_{-6}x_{-9} \neq -1$.

Theorem 5.1. Suppose $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation $x_{n+1} = \frac{x_{n-11}}{-1-x_{n-2}x_{n-5}x_{n-8}x_{n-11}}$,Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{12n-11} &= \frac{p}{(-1-pkfc)^n},\ x_{12n-10} = \frac{m}{(-1-mheb)^n},\ x_{12n-9} = \frac{l}{(-1-ldga)^n}, \\
x_{12n-8} &= k(-1-pkfc)^n,\ x_{12n-7} = h(-1-mheb)^n,\ x_{12n-6} = g(-1-ldga)^n, \\
x_{12n-5} &= \frac{f}{(-1-pkfc)^n},\ x_{12n-4} = \frac{e}{(-1-mheb)^n},\ x_{12n-3} = \frac{d}{(-1-ldga)^n}, \\
x_{12n-2} &= c(-1-pkfc)^n,\ x_{12n-1} = b(-1-mheb)^n,\ x_{12n} = a(-1-ldga)^n,
\end{aligned}$$

where $x_{-11} = p, x_{-10} = m, x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g\ , x_5 = f, x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,$ and $x_0 = a$.

Theorem 5.2 Eq.(8) has three equilibrium points which are $0, \pm\sqrt[4]{-2}$ and these equilibrium points are not locally asymptotically stable.

Proof. The proof as the proof of Theorem 3.3.

Theorem 5.3. Eq.(8) has a periodic solutions of period twelve iff $pkfc = mheb = ldga = -2$ and will be take the form

$$\{p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...\}.$$

6. NUMERICAL EXAMPLES

To verify the results of this paper, we consider some numerical examples as follows.

Example 6.1 The graph of the difference equation (5) and the case when $x_{-11} = 3.3,\ x_{-10} = 1.7,\ x_{-9} = 2.6,$ $x_{-8} = 5,\ x_{-7} = 3,\ x_{-6} = 11,\ x_5 = 6,\ x_{-4} = 2,\ x_{-3} = 7,\ x_{-2} = 9,\ x_{-1} = 4.6$ and $x_0 = 1.6$.shown in Figure 1.

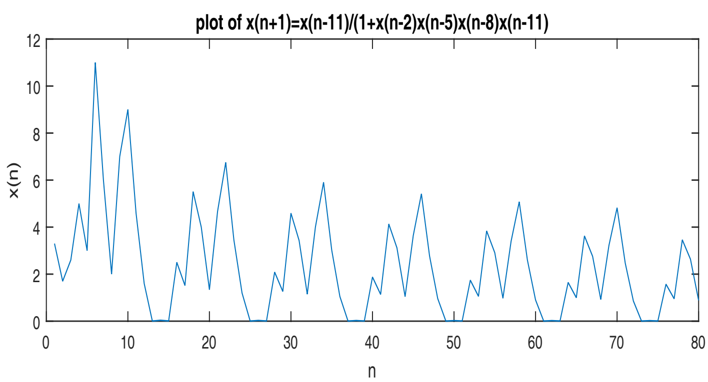

Figure 1.

Example 6.2. In Figure 2, we show that for Eq.(5) that $x_{-11} = 4.1,\ x_{-10} = 2,\ x_{-9} = 3.2,\ x_{-8} = 6,\ x_{-7} = -1,$ $x_{-6} = 2.4,\ x_5 = 1,\ x_{-4} = 4.2,\ x_{-3} = 7,\ x_{-2} = 11,\ x_{-1} = 4$ and $x_0 = -2$.

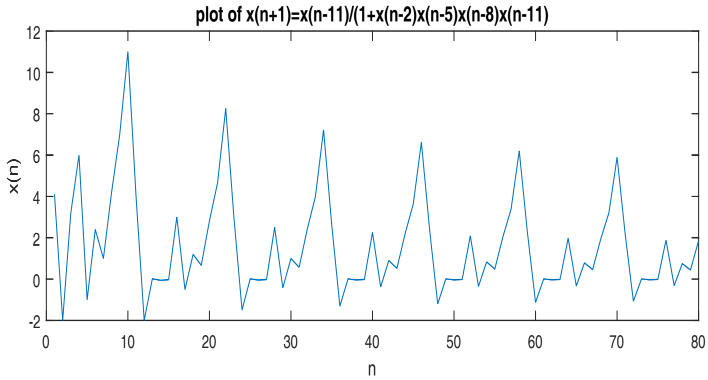

Figure 2.

Example 6.3. The graph is shown of the solutions of Eq.(6) where $x_{-11} = 3,\ x_{-10} = -2,\ x_{-9} = 9,\ x_{-8} = -5,$ $x_{-7} = 8,\ x_{-6} = 2,\ x_5 = 4,\ x_{-4} = 4,\ x_{-3} = -4,\ x_{-2} = -1/30,\ x_{-1} = -1/32$ and $x_0 = -1/36$.in Figure 3.

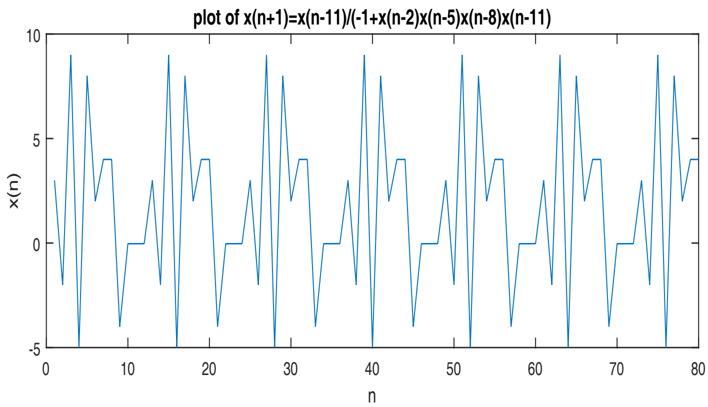

Figure 3.

Example 6.4. Figure 4 shows the behavior of difference equation.(6) when we choose $x_{-11} = 5,\ x_{-10} = -2,\ x_{-9} = 6,\ x_{-8} = -1,\ x_{-7} = 4,\ x_{-6} = -11,\ x_5 = 6,\ x_{-4} = 2,\ x_{-3} = 7,\ x_{-2} = -1/15,\ x_{-1} = -1/8$ and $x_0 = -1/231$.

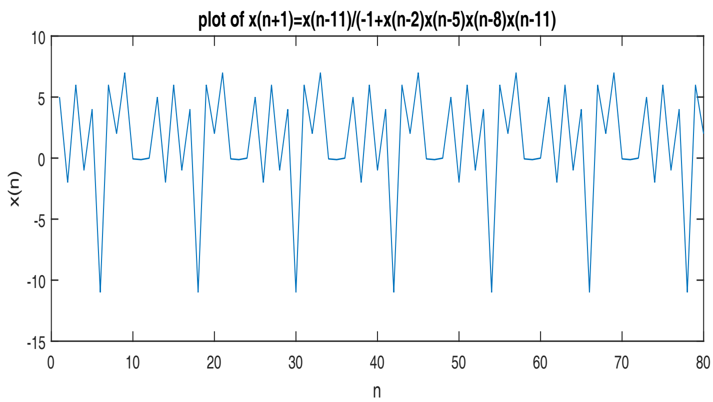

Figure 4.

Example 6.5. The diagram of the difference equation defined by $x_{n+1} = \frac{x_{n-11}}{1-x_{n-2}x_{n-5}x_{n-8}x_{n-11}}$ shows the period thirty six solutions since $x_{-11} = 1,\ x_{-10} = 3.5,\ x_{-9} = -4,\ x_{-8} = 6,\ x_{-7} = -2,\ x_{-6} = 2.4,\ x_5 = -1,\ x_{-4} = 1.2,$ $x_{-3} = 8,\ x_{-2} = 10,\ x_{-1} = -3$ and $x_0 = 4$. in Figure 5

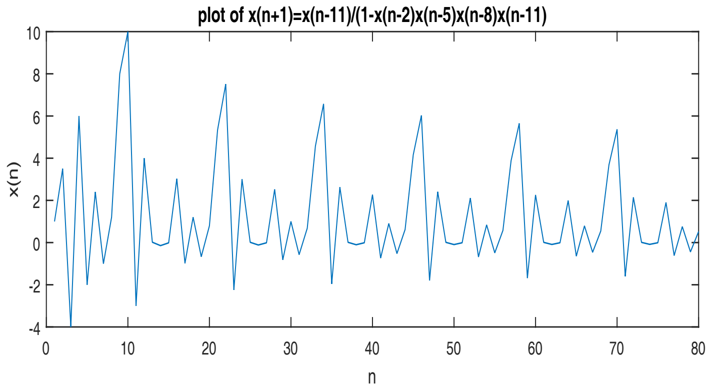

Figure 5.

Example 6.6. See Figure 6, we suppose for Eq.(7), that $x_{-11} = 4.3,\ x_{-10} = 8.1,\ x_{-9} = -3,\ x_{-8} = 2.7,$ $x_{-7} = -1,\ x_{-6} = 2.4,\ x_5 = 3,\ x_{-4} = 1.5,\ x_{-3} = 11,\ x_{-2} = -2,\ x_{-1} = 5$ and $x_0 = -2$.

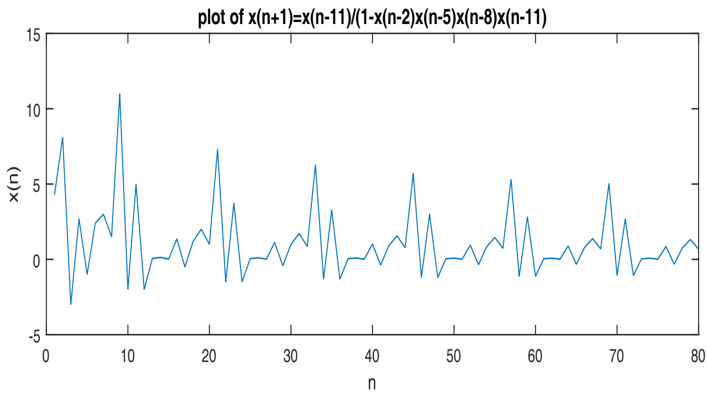

Figure 6.

Example 6.7.(see Figure 7) shows the period thirty six solutions of Eq.(8) since $x_{-11} = 3,\ x_{-10} = 9,\ x_{-9} = -6,$ $x_{-8} = 2,\ x_{-7} = 1,\ x_{-6} = 4,\ x_5 = 5,\ x_{-4} = -4,\ x_{-3} = 3,\ x_{-2} = -1/15,\ x_{-1} = 1/18$ and $x_0 = 1/36$.

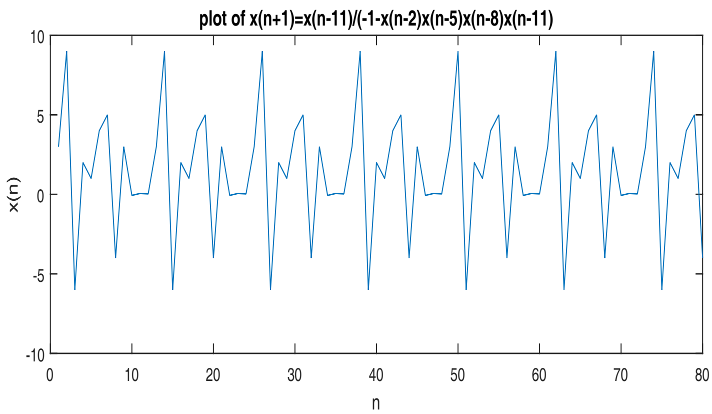

Figure 7.

Example 6.8. (See Figure 8) , we suppose for difference equation (8), that $x_{-11} = 11$, $x_{-10} = 3$, $x_{-9} = -5$, $x_{-8} = -2$, $x_{-7} = 4$, $x_{-6} = 2$, $x_5 = 9$, $x_{-4} = -2$, $x_{-3} = 7$, $x_{-2} = 1/99$, $x_{-1} = 1/12$ and $x_0 = 1/35$.

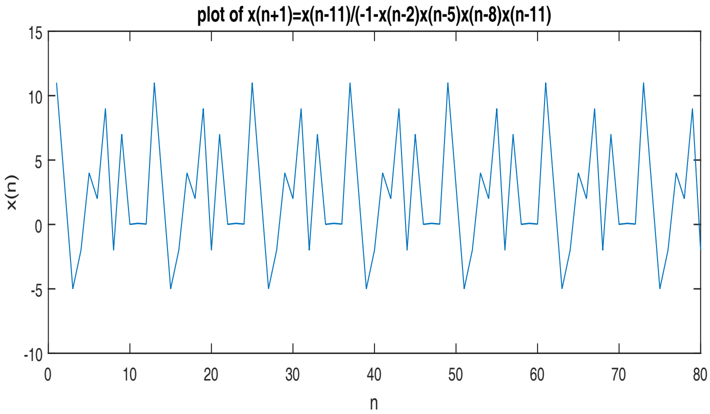

Figure 8.

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Some fixed point theorems of non-self contractive mappings in complete metric spaces

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Abstract In this paper, we establish some fixed point theorems for non-self mappings, which solve the problem 1 in [1], satisfying special contractive conditions in complete metric spaces.

Keyword Fixed point; non-self mapping; contractive mapping; complete metric spaces

1 Introduction

The aim of this paper is to answer an open problem of Rus [1]. We give a non-self mapping T satisfying receptively four contractive conditions such that T has a unique fixed point. This is a solution for the open problem.

An open problem in [1] as following:

Let (X,d) be a metric space, Y a non-empty bounded and closed subset of X and $T : Y \to X$ a non-self operator. We suppose that there exists a sequence $(x_n)_{n\in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. In which additional conditions on T we have:

(a) $F_T \neq \emptyset$?

(b) $F_T = \{x^*\}$?

where $F_T := \{x \in X | x = Tx\}$.

In this paper, we give following marks.

(1) $M_T(Y) = \sup\{d(x,y)|x,y \in Y\}$;

(2) $E_T(Y) = \sup\{d(x,Tx)|x \in Y\}$;

(3) $N_T(y) = \sup\{d(x,Ty)|x,y \in Y\}$.

In (2), we can easy to obtain: i) if $X \subset Y$, then $E_T(X) \leq E_T(Y)$; ii) $E_T(Y) = E_T(\overline{Y})$.

Lemma 1 [4] Let $a_n, b_n \in R_+, n \in N$. We suppose that:

(i)$\Sigma_{k=0}^{\infty} a_k < \infty$;

(ii)$b_n \to 0$ as $n \to \infty$.

Then

$$\Sigma_{k=0}^{\infty} a_{n-k} b_k \to 0 \text{ as } n \to \infty.$$

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2 Fixed point theorems

In this section, we give some non-self contractions as follows.

Let (X,d) be a metric space, Y be a non-empty bounded and closed subset of X. Suppose that $T: Y \to X$ be a non-self mapping satisfied following condition:

(W1) $d(Tx,Ty) \le ad(x,y) + bd(x,Tx) + cd(y,Ty)$, for all $x,y \in Y$, where $a,b,c \in R_+$ and $a+b+c<1$;

(W2) $d(Tx,Ty) \le bd(x,Ty) + cd(y,Tx)$, for all $x,y \in Y$, where $b,c \in R_+$;

(W3) $d(Tx,Ty) \le ad(x,y) + bd(x,Ty) + cd(y,Tx)$, for all $x,y \in Y$, where $a,b,c \in R_+$ and $a<1$;

(W4) $d(Tx,Ty) \le a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty)$, for all $x,y \in Y$, where $a_1,a_2,a_3,a_4 \in R_+$ and $a_1+a_2+a_3<1$.

Lemma 2 Let (X,d) be a metric space, Y be a bounded and non-empty closed subset of X. If $T: Y \to X$ satisfying (W1), then T is a non-self α-graphic contraction with $\alpha = a+c$.

Proof Let $x \in Y$ such that $Tx \in Y$, we get

$$d(T^2x,Tx) \le ad(Tx,x) + bd(Tx,T^2x) + cd(x,Tx),$$

so

$$d(T^2x,Tx) \le \frac{a+c}{1-b} d(x,Tx).$$

Theorem 1 Let (X,d) be a metric space, Y be a non-empty bounded and closed subset of X. $T: Y \to X$ be a non-self mapping satisfying (W1). We suppose that there exists a sequence $(x_n)_{n\in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. Then

(i) T has a unique fixed point;

(ii) $T^{n-1}(x_n) \to x^*$ *and* $T^n(x_n) \to x^*$ *as* $n \to +\infty$;

(iii) $d(x,x^*) \le \frac{1+b}{1-a} d(x,Tx),\ \forall\ x \in\ Y$, i.e. $M_T(Y) \le \frac{1+b}{1-a} E_T(Y)$.

Proof (i)+(ii) Let $Y_1 := \overline{T(Y)}$, $Y_2 := \overline{T(Y_1 \cap Y)}$, $\cdots$, $Y_{n+1} := \overline{T(Y_n \cap Y)}$, $n \in N^*$. We remark that:

(1) $Y_{n+1} \subset Y_n$, $\forall\ n \in N^*$;

(2) $T^n(x_n) \in Y_n$, $\forall\ n \in N^*$, so $Y_n \neq \varnothing$.

Since T satisfying (W1), we have that:

$$\begin{aligned} M(Y_{n+1}) &= M(\overline{T(Y_n \cap Y)}) = M(T(Y_n \cap Y)) \\ &\le aM(Y_n \cap Y) + (b+c)E_T(Y_n \cap Y) \le \cdots \\ &\le a^{n+1}M(Y) + a^n(b+c)E_T(Y) + \cdots + a(b+c)E_T(Y_{n-1} \cap Y) + (b+c)E_T(Y_n \cap Y). \end{aligned} \tag{2.1}$$

On the other hand, from Lemma 2, we get

$$\begin{aligned}E_T(Y_n \cap Y) &= E_T(\overline{T(Y_{n-1} \cap Y} \cap Y) = E_T(T(Y_{n-1} \cap Y) \cap Y)\\ &= \sup\{d(Tx, T^2x)|x \in Y_{n-1} \cap Y, Tx \in Y\} \leq \frac{a+c}{1-b} E_T(Y_{n-1} \cap Y)\\ &\leq \cdots \leq (\frac{a+c}{1-b})^n E_T(Y),\ n \in N^*.\end{aligned}$$

Because of $a+b+c < 1$, so $(\frac{a+c}{1-b})^n \to 0,\ n \to +\infty$, i.e. $E_T(Y_n \cap Y) \to 0,\ n \to +\infty$.

Let $a_n = a^n$ and $b_n = (b+c)E_T(Y_n \cap Y)$, by lemma 1, we have

$$M(Y_{n+1}) \to 0\ as\ n \to +\infty.$$

From Cantor intersection lemma, we get

$$Y_\infty := \cap_{n \in N} Y_n \neq \emptyset,\ M(Y_\infty) = 0 \text{ and } T(Y_\infty \cap Y) \subset Y_\infty.$$

From $Y_\infty \neq \emptyset$ and $M(Y_\infty) = 0$, we have that $Y_\infty = x^*$, i.e. Y_∞ be a single point set. Otherwise, $T^n(x_n) \in Y_n$ and $T^{n-1}(x_n) \in Y_{n-1} \cap Y$, this implies that $\{T^n(x_n)\}_{n \in N}$ and $\{T^{n-1}(x_n)\}_{n \in N}$ are fundamental sequences.

Since $Y_n, n \in N$ are closed, so we get

$$T^{n-1}(x_n) \to x^*\ and\ T^n(x_n) \to x^*\ as\ n \to +\infty.$$

Also because of T is continuous, then $T^n(x_n) \to T(x^*)$. Therefore, $T(x^*) = x^*$.

(iii) Let $x \in Y$, by using (W1) we have

$$d(x, x^*) \leq d(x, Tx) + d(Tx, x^*) \leq d(x, Tx) + ad(x, x^*) + bd(x, Tx) + cd(d(x, Tx), Td(x, Tx)),$$

so

$$d(x, x^*) \leq \frac{1+b}{1-a} d(x, Tx),\ \forall\ x \in\ Y.$$

Remark 1 Let $b = c$ in Theorem 1, then T is a non-self $\acute{C}iri\acute{c} - Reich - Rus$ operator. And then, Theorem 1 generalizes Theorem 5 in Rus [1]. At the same time, this theorem gives an answer to the Problem 1 of [1].

For (W4), we give a Lemma as following:

Lemma 3 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. Define $T : Y \to X$ be a non-self mapping. Then $N_T(Y_n \cap Y) \to 0,\ as\ n \to \infty$, where $Y_n = \overline{T(Y_{n-1} \cap Y)}$.

Proof From the definitions of N_T and Y_n, we have

$$\begin{aligned}\sup\{d(x, Ty)|x, y \in Y_n \cap Y\} &= N_T(Y_n \cap Y) = N_T(\overline{T(Y_{n-1} \cap Y)} \cap Y)\\ &= N_T(T(Y_{n-1} \cap Y) \cap Y) = \sup\{d(Tx, T^2y)|x, y \in Y_{n-1} \cap Y\}.\end{aligned}$$

Since $Y_{n-1} \cap Y \subset Y_n \cap Y$, so $d(Tx, T^2y) \leq d(x, Ty)$, for all $x, y \in Y_{n-1} \cap Y$. Hence, $N_T(Y_n \cap Y) \leq N_T(Y_{n-1} \cap Y)$.

By the density of real numbers we can get, there exists $k \in R^+$ and $k < 1$, such that $N_T(Y_n \cap Y) \leq kN_T(Y_{n-1} \cap Y)$.

And then,

$$N_T(Y_n \cap Y) \leq kN_T(Y_{n-1} \cap Y) \leq \cdots \leq k^n N_T(Y) \to 0, \text{ as } n \to +\infty.$$

Theorem 2 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. $T : Y \to X$ be a non-self mapping satisfying (W3). We suppose that there exists a sequence $(x_n)_{n\in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. Then

(i) T has a unique fixed point;

(ii) $T^{n-1}(x_n) \to x^*$ and $T^n(x_n) \to x^*$ *as* $n \to +\infty$.

Proof Let $Y_1 := \overline{T(Y)}$, $Y_2 := \overline{T(Y_1 \cap Y)}$, $\cdots$, $Y_{n+1} := \overline{T(Y_n \cap Y)}$, $n \in N^*$. We remark that:

(1) $Y_{n+1} \subset Y_n$, $\forall\, n \in N^*$;

(2) $T^n(x_n) \in Y_n$, $\forall\, n \in N^*$, so $Y_n \neq \varnothing$.

Since T satisfying (W3), we have that:

$$\begin{aligned}
&M(Y_{n+1}) = M(\overline{T(Y_n \cap Y)}) = M(T(Y_n \cap Y))\\
&\leq aM(Y_n \cap Y) + (b + c)N_T(Y_n \cap Y)\\
&\leq aM(Y_n) + (b + c)N_T(Y_n \cap Y) \leq \cdots\\
&\leq a^{n+1}M(Y) + a^n(b + c)N_T(Y) + \cdots + a(b + c)N_T(Y_{n-1} \cap Y) + (b + c)N_T(Y_n \cap Y).
\end{aligned}$$

Let $a_n = a^n$ and $b_n = (b + c)N_T(Y_n \cap Y)$, by lemma 3 we have

$$M(Y_{n+1}) \to 0 \ as\ n \to +\infty,$$

and the proof is similar with the proof of Theorem 1. This is the complete proof.

For (W3), when $a = 0$, it becomes condition (W2). Thence, we have the following Corollary:

Corollary 1 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. Define $T : Y \to X$ be a non-self mapping satisfying (W3), the conclusions of Theorem 2 remain holds.

Theorem 3 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. $T : Y \to X$ be a non-self mapping satisfying (W4). We suppose that there exists a sequence $(x_n)_{n\in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. Then

(i) T has a unique fixed point;

(ii) $T^{n-1}(x_n) \to x^*$ *and* $T^n(x_n) \to x^*$ *as* $n \to +\infty$;

(iii) $d(x, x^*) \leq \frac{1+b}{1-a} d(x, Tx)$, $\forall\, x \in Y$, i.e. $M_T(Y) \leq \frac{1+b}{1-a} E_T(Y)$.

Proof (i)+(ii) Let $Y_1 := \overline{T(Y)}$, $Y_2 := \overline{T(Y_1 \cap Y)}$, $\cdots$, $Y_{n+1} := \overline{T(Y_n \cap Y)}$, $n \in N^*$. We remark that:

(1) $Y_{n+1} \subset Y_n$, $\forall\ n \in N^*$;

(2) $T^n(x_n) \in Y_n$, $\forall\ n \in N^*$, so $Y_n \neq \varnothing$.

Since T satisfying (W1), we have that:

$$M(Y_{n+1}) = M(\overline{T(Y_n \cap Y)}) = M(T(Y_n \cap Y))$$
$$\leq a_1 M(Y_n \cap Y) + (a_2 + a_3)E_T(Y_n \cap Y) + a_4 N_T(Y_n \cap Y)$$
$$\leq \cdots \leq$$
$$\leq a_1^{n+1} M(Y) + [a_1^n (a_2 + a_3)E_T(Y) + \cdots + a_1(a_2 + a_3)E_T(Y_{n-1} \cap Y) + (a_2 + a_3)E_T(Y_n \cap Y)]$$
$$+ [a_1^n \cdot a_4 N_T(Y) + \cdots + a_1 \cdot a_4 N_T(Y_{n-1} \cap Y) + a_4 N_T(Y_n \cap Y)]$$
$$= a_1^{n+1} M(Y) + \Phi_{E_T} + \Phi_{N_T},$$

where $\Phi_{E_T} = a_1^n (a_2 + a_3)E_T(Y) + \cdots + a_1(a_2 + a_3)E_T(Y_{n-1} \cap Y) + (a_2 + a_3)E_T(Y_n \cap Y)$,
$\Phi_{N_T} = a_1^n \cdot a_4 N_T(Y) + \cdots + a_1 \cdot a_4 N_T(Y_{n-1} \cap Y) + a_4 N_T(Y_n \cap Y)$.

For $x \in Y$, such that $Tx \in T(Y)$, we have

$$d(T^2x, Tx) \leq a_1 d(Tx, x) + a_2 d(Tx, T^2x) + a_3 d(x, Tx) + a_4 d(Tx, Tx),$$

so

$$d(T^2x, Tx) \leq \frac{a_1 + a_3}{1 - a_2} d(x, Tx). \tag{2.2}$$

Thence

$$E_T(Y_n \cap Y) = E_T(\overline{T(Y_{n-1} \cap Y} \cap Y) = E_T(T(Y_{n-1} \cap Y) \cap Y)$$
$$= \sup\{d(Tx, T^2x) | x \in Y_{n-1} \cap Y, Tx \in Y\} \leq \frac{a_1 + a_3}{1 - a_2} E_T(Y_{n-1} \cap Y)$$
$$\leq \cdots \leq (\frac{a_1 + a_3}{1 - a_2})^n E_T(Y),\ \ n \in N^*.$$

Because of $a + b + c < 1$, so $(\frac{a_1 + a_3}{1 - a_2})^n \to 0,\ n \to +\infty$, i.e. $E_T(Y_n \cap Y) \to 0,\ n \to +\infty$.

Let $a_n = a_1^n$ and $b_n = (a_2 + a_3)E_T(Y_n \cap Y)$, by lemma 1 we have $\Phi_{E_T} \to 0$ as $n \to +\infty$.

From Lemma 3, we know, $N_T(Y_n \cap Y) \to 0$ as $n \to +\infty$.

Let $a_n^{'} = a_1^n$ and $b_n^{'} = a_4 N_T(Y_n \cap Y)$, by lemma 1 we obtain $\Phi_{N_T} \to 0$ as $n \to +\infty$.

In summary, we get $M(Y_{n+1}) \to 0$ as $n \to +\infty$, and the proof is similar with the proof of Theorem 1.

Although let $a = 0$ in (W4), it becomes (W2), because different proof process details of the transformation, so we give separately the proof of Theorem 1 and Theorem 3.

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Double-framed soft sets in B-algebras

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Abstract. The notion of a double-framed soft (normal) subalgebra in a B-algebra is introduced and related properties are investigated. We consider characterizations of a double-framed soft (normal) subalgebra and establish a new double-framed soft subalgebra from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras is a double framed soft subalgebra.

1. Introduction

Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [10] described the application of soft set theory to a decision making problem. Jun [5] discussed the union soft sets with applications in BCK/BCI-algebras. Jun et al. [6] introduced the notion of double-framed soft sets, and applied it to BCK/BCI-algebras. They discussed double-frame soft algebras and investigated some related properties.

We refer the reader to the papers [3, 4, 14] for further information regarding algebraic structures/properties of soft set theory. On the while, Y. B. Jun, E. H. Roh and H. S. Kim [7] introduced a new notion, called a BH-algebra. J. Neggers and H. S. Kim [12] introduced a new notion, called a B-algebra. C. B. Kim and H. S. Kim [9] introduced the notion of a BG-algebra which is a generalization of B-algebras. S. S. Ahn and H. D. Lee [1] classified the subalgebras by their family of level subalgebras in BG-algebras.

In this paper, we introduce the notion of a double-framed soft (normal) subalgebra in a B-algebra and investigate some related properties. We consider characterizations of a double-framed soft (normal) subalgebra and establish a new double-framed soft subalgebra from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras is a double framed soft subalgebra.

2. Preliminaries

A *B-algebra* [12] is a non-empty set X with a constant 0 and a binary operation “ $*$ ” satisfying axioms:

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(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B) $(x * y) * z = x * (z * (0 * y))$

for any x, y, z in X. For brevity we call X a *B-algebra*. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BH-algebra* if it satisfies (B1), (B2) and

(BH) $x * y = y * x = 0$ imply $x = y$ for any $x, y \in X$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BG-algebra* if it satisfies (B1), (B2) and

(BG) $(x * y) * (0 * y) = x$ for any $x, y \in X$.

Proposition 2.1. [2, 12] *Let $(X; *, 0)$ be a B-algebra. Then*

(i) *the left cancellation law holds in X, i.e., $x * y = x * z$ implies $y = z$,*

(ii) *if $x * y = 0$, then $x = y$ for any $x, y \in X$,*

(iii) *if $0 * x = 0 * y$, then $x = y$ for any $x, y \in X$,*

(iv) *$0 * (0 * x) = x$, for all $x \in X$,*

(v) *$x * (y * z) = (x * (0 * z)) * y$ for all $x, y, z \in X$.*

A non-empty subset S of a B-algebra X is called a *subalgebra* of X if $x * y \in S$ for any $x, y \in S$. A non-empty subset N of X is said to be *normal* if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. Then any normal subset N of a B-algebra X is a subalgebra of X, but the converse need not be true ([13]). A non-empty subset X of a B-algebra X is a called a *normal subalgebra* of X if it is both a subalgebra and normal.

Molodtsov [11] defined the soft set in the following way: Let U be an initial universe set and let E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

A fair $(\tilde{f}, A)$ is called a *soft set* over U, where $\tilde{f}$ is a mapping given by $\tilde{f} : X \to \mathscr{P}(U)$. In other words, a soft set over U is parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the set $(\tilde{f}, A)$. A soft set over U can be represented by the set of ordered pairs:

$$(\tilde{f}, A) = \{(x, \tilde{f}(x)) | x \in A, \tilde{f}(x) \in \mathscr{P}(U)\},$$

where $\tilde{f} : X \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$. Clearly, a soft set is not a set.

3. **Double-framed soft normal subalgebras**

In what follows let X denote a B-algebra unless otherwise specified.

Definition 3.1. A double-framed pair $\langle(\alpha, \beta); X\rangle$ is called a *double-framed soft set* over U, where α and β are mappings from X to $\mathscr{P}(U)$.

Definition 3.2. A double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U is called a *double-framed soft subalgebra* over U if it satisfies :

$$(3.1)\quad (\forall x, y \in X)\,(\alpha(x * y) \supseteq \alpha(x) \cap \alpha(y),\ \beta(x * y) \subseteq \beta(x) \cup \beta(y))\,.$$

Example 3.3. Let X be the set of parameters where $X := \{0, 1, 2, 3\}$ is a B-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Let $\langle(\alpha, \beta); X\rangle$ be a double-framed soft set over U defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_3 & \text{if } x = 0, \\ \tau_1 & \text{if } x = 3, \\ \tau_2 & \text{if } x = \{1, 2\}, \end{cases}$$

and

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = 3, \\ \gamma_1 & \text{if } x = \{1, 2\} \end{cases}$$

where $\tau_1, \tau_2, \tau_3, \gamma_1, \gamma_2$ and γ_3 are subsets of U with $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$ and $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$ It is easy to show that $\langle(\alpha, \beta); X\rangle$ is a double-framed soft subalgebra over U.

Lemma 3.4. *Every double-framed soft subalgebra $\langle(\alpha, \beta); X\rangle$ over U satisfies the following condition:*

$$(3.2)\quad (\forall x \in X)\,(\alpha(x) \subseteq \alpha(0),\ \beta(x) \supseteq \beta(0))\,.$$

Proof. Straightforward. □

Proposition 3.5. *For a double-framed soft subalgebra $\langle(\alpha, \beta); X\rangle$ over U, the following are equivalent:*

(i) $(\forall x \in X)\,(\alpha(x) = \alpha(0),\ \beta(x) = \beta(0))\,.$

(ii) $(\forall x, y \in X)\,(\alpha(y) \subseteq \alpha(x * y),\ \beta(y) \supseteq \beta(x * y))\,.$

Proof. Assume that (ii) is valid. Taking $y := 0$ in (ii) and using (B2), we have $\alpha(0) \subseteq \alpha(x * 0) = \alpha(x)$ and $\beta(0) \supseteq \beta(x * 0) = \beta(x)$. It follows from Lemma 3.4 that $\alpha(x) = \alpha(0)$ and $\beta(x) = \beta(0)$.

Conversely, suppose that $\alpha(x) = \alpha(0)$ and $\beta(x) = \beta(0)$ for all $x \in X$. Using (3.1), we have

$$\alpha(y) = \alpha(0) \cap \alpha(y) = \alpha(x) \cap \alpha(y) \subseteq \alpha(x * y),$$
$$\beta(y) = \beta(0) \cup \beta(y) = \beta(x) \cup \beta(y) \supseteq \beta(x * y)$$

for all $x, y \in X$. This completes the proof. □

For two double-framed soft sets $\langle(\alpha, \beta); X\rangle$ and $\langle(f, g); X\rangle$ over U, the *double-framed soft int-uni set* of $\langle(\alpha, \beta); X\rangle$ and $\langle(f, g); X\rangle$ is defined to be a double-framed soft set $\langle(\alpha \tilde{\cap} f, \beta \tilde{\cup} g); X\rangle$ where

$$\alpha \tilde{\cap} f : X \to \mathscr{P}(U),\ x \mapsto \alpha(x) \cap f(x),$$
$$\beta \tilde{\cup} g : X \to \mathscr{P}(U),\ x \mapsto \beta(x) \cup g(x).$$

It is denoted by $\langle(\alpha, \beta); X\rangle \sqcap \langle(f, g); X\rangle = \langle(\alpha \tilde{\cap} f, \beta \tilde{\cup} g); X\rangle$.

Theorem 3.6. *The double-framed soft int-uni set of two double-framed soft subalgebras* $\langle(\alpha,\beta);X\rangle$ *and* $\langle(f,g);X\rangle$ *over* U *is a double-framed soft subalgebra over* U.

Proof. For any $x, y \in X$, we have

$$\begin{aligned}(\alpha\tilde{\cap}f)(x*y) =&\alpha(x*y)\cap f(x*y) \supseteq (\alpha(x)\cap\alpha(y))\cap(f(x)\cap f(y))\\ =&(\alpha(x)\cap f(x))\cap(\alpha(y)\cap f(y)) = (\alpha\tilde{\cap}f)(x)\cap(\alpha\tilde{\cap}f)(y)\end{aligned}$$

and

$$\begin{aligned}(\beta\tilde{\cup}g)(x*y) =&\beta(x*y)\cup g(x*y) \subseteq (\beta(x)\cup\beta(y))\cup(g(x)\cup g(y))\\ =&(\beta(x)\cup g(x))\cup(\beta(y)\cup g(y)) = (\beta\tilde{\cup}g)(x)\cup(\beta\tilde{\cup}g)(y).\end{aligned}$$

Therefore $\langle(\alpha,\beta);X\rangle \sqcap \langle(f,g);X\rangle$ is a double-framed soft subalgebra over U. □

For two double-framed soft sets $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U, the *double-framed soft uni-int set* of $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ is defined to be a double-framed soft set $\langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$ where

$$\begin{aligned}&\alpha\tilde{\cup}f : X\to\mathscr{P}(U),\ x\mapsto\alpha(x)\cup f(x),\\ &\beta\tilde{\cap}g : X\to\mathscr{P}(U),\ x\mapsto\beta(x)\cap g(x).\end{aligned}$$

It is denoted by $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$.

The following example shows that the double-framed soft uni-int set of two double-framed soft subalgebras $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U may not be a double-framed soft subalgebra over U.

Example 3.7. Let $E = X$ be the set of parameters, and let $U = \mathbb{Z}$ be the initial universe set, where $X = \{0,1,2,3,4,5\}$ is a B-algebra [12] with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Let $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ be double-framed soft sets over U defined, respectively, as follows:

$$\alpha : X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\mathbb{Z} & \text{if } x\in\{0,4\},\\ 9\mathbb{Z} & \text{if } x\in\{1,2,3,5\},\end{cases}$$

$$\beta : X\to\mathscr{P}(U),\ x\mapsto\begin{cases}7\mathbb{Z} & \text{if } x\in\{0,4\},\\ \mathbb{Z} & \text{if } x\in\{1,2,3,5\},\end{cases}$$

$$f : X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\mathbb{Z} & \text{if } x\in\{0,5\},\\ 3\mathbb{Z} & \text{if } x\in\{1,2,3,4\},\end{cases}$$

and

$$g : X\to\mathscr{P}(U),\ x\mapsto\begin{cases}2\mathbb{Z} & \text{if } x\in\{0,5\},\\ \mathbb{Z} & \text{if } x\in\{1,2,3,4\},\end{cases}$$

It is routine to verify that $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ are double-framed soft subalgebras over U. But $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f, \beta\tilde{\cap}g);X\rangle$ is not a double-framed soft subalgebra over U, since $(\alpha\tilde{\cup}f)(4*5) = (\alpha\tilde{\cup}f)(2) = \alpha(2)\cup f(2) = 9\mathbb{Z}\cup 3\mathbb{Z} = 3\mathbb{Z} \not\supseteq \mathbb{Z} = (\alpha\tilde{\cup}f)(4)\cap(\alpha\tilde{\cup}f)(5)$ and/or $(\beta\tilde{\cap}g)(4*5) = (\beta\tilde{\cap}g)(2) = \mathbb{Z}\cap\mathbb{Z} = \mathbb{Z} \not\subseteq 7\mathbb{Z}\cup 2\mathbb{Z} = (\beta\tilde{\cap}g)(4)\cup(\beta\tilde{\cap}g)(5)$.

Let $\langle(\alpha,\beta);A\rangle$ and $\langle(f,g);B\rangle$ be double-framed soft sets over a common universe U. Then $\langle(\alpha,\beta);A\rangle$ is called a *double-framed soft subset* of $\langle(f,g);B\rangle$, denoted by $\langle(\alpha,\beta);A\rangle \tilde{\sqsubseteq} \langle(f,g);B\rangle$, if

(i) $A \subseteq B$,

(ii) $(\forall e \in A)\left(\begin{array}{l}\alpha(e) \text{ and } f(e) \text{ are identical approximations,} \\ \beta(e) \text{ and } g(e) \text{ are identical approximations.}\end{array}\right)$.

Theorem 3.8. *Let $\langle(\alpha,\beta);A\rangle$ be a double-framed soft subset of a double-framed soft set $\langle(f,g);B\rangle$. If $\langle(f,g);B\rangle$ is a double-framed soft subalgebra over U, then so is $\langle(\alpha,\beta);A\rangle$.*

Proof. Let $x, y \in A$. Then $x, y \in B$, and so

$$\alpha(x)\cap\alpha(y) = f(x)\cap f(y) \subseteq f(x*y) = \alpha(x*y),$$
$$\beta(x)\cup\beta(y) = g(x)\cup g(y) \supseteq g(x*y) = \beta(x*y).$$

Hence $\langle(\alpha,\beta);A\rangle$ is a double-framed soft subalgebra over U. □

The converse of Theorem 3.8 is not true as seen in the following example.

Example 3.9. Let $(U = \mathbb{Z}, X)$ where $X = \{0,1,2,3\}$ is a B-algebra as in Example 3.3. For a subalgebra $\{0,3\}$, define a double-framed soft set $\langle(\alpha,\beta);\{0,3\}\rangle$ over U as follows:

$$\alpha : \{0,3\} \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0, \\ 2\mathbb{Z} & \text{if } x = 3, \end{cases}$$

and

$$\beta : \{0,3\} \to \mathscr{P}(U),\ x \mapsto \begin{cases} 27\mathbb{Z} & \text{if } x = 0, \\ 9\mathbb{Z} & \text{if } x = 3, \end{cases}$$

Then $\langle(\alpha,\beta);\{0,3\}\rangle$ is a double-framed soft subalgebra over U. Take $B := X$ and define a double-framed soft set $\langle(f,g);B\rangle$ over U as follows:

$$f : B \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0, \\ 72\mathbb{Z} & \text{if } x = 1, \\ 4\mathbb{Z} & \text{if } x = 2, \\ 2\mathbb{Z} & \text{if } x = 3, \end{cases}$$

and

$$g : B \to \mathscr{P}(U),\ x \mapsto \begin{cases} 27\mathbb{Z} & \text{if } x = 0, \\ 3\mathbb{Z} & \text{if } x = 1, \\ \mathbb{Z} & \text{if } x = 2, \\ 9\mathbb{Z} & \text{if } x = 3. \end{cases}$$

Then $\langle(f,g);B\rangle$ is not a double-framed soft subalgebra over U since $f(0*2) = f(1) = 72\mathbb{Z} \not\supseteq f(0)\cap f(2) = \mathbb{Z}\cap 4\mathbb{Z} = 4\mathbb{Z}$ and/or $g(1*3) = g(2) = \mathbb{Z} \not\subseteq g(1)\cup g(3) = 3\mathbb{Z}\cup 9\mathbb{Z} = 3\mathbb{Z}$.

For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U and two subsets γ and δ of U, the *γ-inclusive set* and the *δ-exclusive set* of $\langle(\alpha,\beta);X\rangle$, denoted by $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$, respectively, are defined as follows: $i_X(\alpha;\gamma) := \{x\in X \mid \gamma\subseteq\alpha(x)\}$ and $e_X(\beta;\delta) := \{x\in X\mid \delta\supseteq\beta(x)\}$, respectively. The set $DF_X(\alpha,\beta)_{(\gamma,\delta)} := \{x\in X\mid \gamma\subseteq\alpha(x),\ \delta\supseteq\beta(x)\}$ is called a *double-framed including set* of $\langle(\alpha,\beta);X\rangle$. It is clear that $DF_X(\alpha,\beta)_{(\gamma,\delta)} = i_X(\alpha;\gamma)\cap e_X(\beta;\delta)$.

Theorem 3.10. *For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, the following are equivalent:*

(i) *$\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U.*

(ii) *For every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$, the γ-inclusive set and the δ-exclusive set of $\langle(\alpha,\beta);X\rangle$ are subalgebras of X.*

Proof. Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. Let $x,y\in X$ be such that $x,y\in i_X(\alpha;\gamma)$ and $x,y\in e_X(\beta;\delta)$ for every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$. It follows from (3.1) that

$$\alpha(x*y)\supseteq\alpha(x)\cap\alpha(y)\supseteq\gamma \text{ and } \beta(x*y)\subseteq\beta(x)\cup\beta(y)\subseteq\delta.$$

Hence $x*y\in i_X(\alpha;\gamma)$ and $x*y\in e_X(\beta;\delta)$, and therefore $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of X.

Conversely, suppose that (ii) is valid. Let $x,y\in X$ be such that $\alpha(x)=\gamma_x$, $\alpha(y)=\gamma_y$, $\beta(x)=\delta_x$ and $\beta(y)=\delta_y$. Taking $\gamma=\gamma_x\cap\gamma_y$ and $\delta=\delta_x\cup\delta_y$ imply that $x,y\in i_X(\alpha;\gamma)$ and $x,y\in e_X(\beta;\delta)$. Hence $x*y\in i_X(\alpha;\gamma)$ and $x*y\in e_X(\beta;\delta)$, which imply that $\alpha(x*y)\supseteq\gamma=\gamma_x\cap\gamma_y=\alpha(x)\cap\alpha(y)$ and $\beta(x*y)\subseteq\delta=\delta_x\cup\delta_y=\beta(x)\cup\beta(y)$. Therefore $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. □

Corollary 3.11. *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft algebra over U, then the double-framed including set of $\langle(\alpha,\beta);X\rangle$ is a subalgebra X.*

For any double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, let $\langle(\alpha^*,\beta^*);X\rangle$ be a double-framed soft set over U defined by

$$\alpha^*: X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\alpha(x) & \text{if } x\in i_X(\alpha;\gamma),\\ \eta & \text{otherwise,}\end{cases}$$

$$\beta^*: X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\beta(x) & \text{if } x\in e_X(\beta;\delta),\\ \rho & \text{otherwise,}\end{cases}$$

where γ,δ,η and ρ are subsets of U with $\eta\subsetneq\alpha(x)$ and $\rho\supsetneq\beta(x)$.

Theorem 3.12. *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U, then so is $\langle(\alpha^*,\beta^*);X\rangle$.*

Proof. Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. Then $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of X for every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$, by Theorem 3.10. Let $x,y\in X$. If $x,y\in i_X(\alpha;\gamma)$, then $x*y\in i_X(\alpha;\gamma)$. Thus

$$\alpha^*(x*y)=\alpha(x*y)\supseteq\alpha(x)\cap\alpha(y)=\alpha^*(x)\cap\alpha^*(y).$$

If $x\notin i_X(\alpha;\gamma)$ or $y\notin i_X(\alpha;\gamma)$, then $\alpha^*(x)=\eta$ or $\alpha^*(y)=\eta$. Hence

$$\alpha^*(x*y)\supseteq\eta=\alpha^*(x)\cap\alpha^*(y).$$

Now, if $x, y \in e_X(\beta;\delta)$, then $x * y \in e_X(\beta;\delta)$. Thus

$$\beta^*(x*y) = \beta(x*y) \subseteq \beta(x) \cup \beta(y) = \beta^*(x) \cup \beta^*(y).$$

If $x \notin e_X(\beta;\delta)$ or $y \notin e_X(\beta;\delta)$, then $\beta^*(x) = \rho$ or $\beta^*(y) = \rho$. Hence

$$\beta^*(x*y) \subseteq \rho = \beta^*(x) \cup \beta^*(y).$$

Therefore $\langle(\alpha^*, \beta^*); X\rangle$ is a double-framed soft subalgebra over U. □

Let $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ be double-framed soft sets over U, where X, Y are B-algebras. The $(\alpha_\wedge,\ \beta_\vee)$-*product* of $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ is defined to be a double-framed soft set $\langle(\alpha_{X\wedge Y}, \beta_{X\vee Y}); X \times Y\rangle$ over U in which

$$\alpha_{X\wedge Y} : X \times Y \to \mathscr{P}(U),\ (x, y) \mapsto \alpha(x) \cap \alpha(y),$$

$$\beta_{X\vee Y} : X \times Y \to \mathscr{P}(U),\ (x, y) \mapsto \beta(x) \cup \beta(y).$$

Theorem 3.13. *For any B-algebras X and Y as sets of parameters, let $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ be double-framed soft subalgebras over U. Then the $(\alpha_\wedge,\ \beta_\vee)$-product of $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ is also a double-framed soft subalgebra over U.*

Proof. Note that $(X \times Y, \circledast, (0, 0))$ is a B-algebra. For any $(x, y), (a, b) \in X \times Y$, we have

$$\begin{aligned}
&\alpha_{X\wedge Y}\left((x, y) \circledast (a, b)\right) = \alpha_{X\wedge Y}\left(x * a, y * b\right)\\
&= \alpha(x * a) \cap \alpha(y * b) \supseteq (\alpha(x) \cap \alpha(a)) \cap (\alpha(y) \cap \alpha(b))\\
&= (\alpha(x) \cap \alpha(y)) \cap (\alpha(a) \cap \alpha(b))\\
&= \alpha_{X\wedge Y}\left(x, y\right) \cap \alpha_{X\wedge Y}\left(a, b\right)
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{X\vee Y}\left((x, y) \circledast (a, b)\right) = \beta_{X\vee Y}\left(x * a, y * b\right)\\
&= \beta(x * a) \cup \beta(y * b) \subseteq (\beta(x) \cup \beta(a)) \cup (\beta(y) \cup \beta(b))\\
&= (\beta(x) \cup \beta(y)) \cup (\beta(a) \cup \beta(b))\\
&= \beta_{X\vee Y}\left(x, y\right) \cup \beta_{X\vee Y}\left(a, b\right)
\end{aligned}$$

Hence $\langle(\alpha_{X\wedge Y}, \beta_{X\vee Y}); E \times F\rangle$ is a double-framed soft subalgebra over U. □

Definition 3.14. A double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U is said to be *double-framed soft normal* of a B-algebra X if it satisfies:

(3.3) $(\forall x, y, a, b \in X)(\alpha((x * a) * (y * b)) \supseteq \alpha(x * y) \cap \alpha(a * b), \beta((x * a) * (y * b)) \subseteq \beta(x * y) \cup \beta(a * b))$.

A double-framed soft $\langle(\alpha, \beta); X\rangle$ over U is called a *double-framed soft normal subalgebra* of a B-algebra X if it satisfies (3.1) and (3.3).

Example 3.15. Let $(U = \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a B-algebra as in Example 3.3. Let $\langle(\alpha, \beta); X\rangle$ be a double-framed soft set over U defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 3\}, \\ 2\mathbb{Z} & \text{if } x \in \{1, 2\}, \end{cases}$$

and

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 3\mathbb{Z} & \text{if } x \in \{0, 3\}, \\ \mathbb{Z} & \text{if } x \in \{1, 2\}, \end{cases}$$

It is easy to check that $\langle(\alpha, \beta); X\rangle$ is a double-framed soft normal over U.

Proposition 3.16. *Every double-framed soft normal $(\tilde{f}, X)$ of a B-algebra X is a double-framed soft subalgebra of X.*

Proof. Put $y := 0, b := 0$ and $a := y$ in (3.3). Then $\alpha((x * y) * (0 * 0)) \supseteq \alpha(x * 0) \cap \alpha(y * 0)$ and $\beta((x * y) * (0 * 0)) \subseteq \beta(x * 0) \cup \beta(y * 0)$ for any $x, y \in X$. Using (B2) and (B1), we have $\alpha(x * y) \supseteq \alpha(x) \cap \alpha(y)$ and $\beta(x * y) \subseteq \beta(x) \cup \beta(y)$. Hence $\langle(\alpha, \beta); X\rangle$ is a double-framed soft subalgebra over U. □

The converse of Proposition 3.16 may not be true in general (Example 3.17).

Example 3.17. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set, where $X = \{0, 1, 2, 3, 4, 5\}$ is a B-algebra as in Example 3.7. Let $\langle(\alpha, \beta); X\rangle$ be double-framed soft set over U defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = 5, \\ \gamma_1 & \text{if } x \in \{1, 2, 3, 4\}, \end{cases}$$

and

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_1 & \text{if } x = 0, \\ \tau_2 & \text{if } x = 5, \\ \tau_3 & \text{if } x \in \{1, 2, 3, 4\}, \end{cases}$$

where $\gamma_1, \gamma_2, \gamma_3, \tau_1, \tau_2$ and τ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$ and $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. It is routine to verify that $\langle(\alpha, \beta); X\rangle$ is a double-framed soft subalgebra over U. But it is not double-framed soft normal over U since since $\alpha(1) = \alpha((1 * 3) * (4 * 2)) = \gamma_1 \not\supseteq \alpha(1 * 4) \cap \alpha(3 * 2) = \alpha(5) \cap \alpha(5) = \gamma_2$ and/or $\beta(1) = \beta((1 * 3) * (4 * 2)) = \tau_3 \not\subseteq \beta(1 * 4) \cup \beta(3 * 2) = \beta(5) \cup \beta(5) = \tau_2$.

Theorem 3.18. *For a double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U, the following are equivalent:*

(i) *$\langle(\alpha, \beta); X\rangle$ is a double-framed soft normal subalgebra over U.*

(ii) *For every subsets γ and δ of U with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$, the γ-inclusive set and the δ-exclusive set of $\langle(\alpha, \beta); X\rangle$ are normal subalgebras of X.*

Proof. Similar to Theorem 3.10. □

Proposition 3.19. *Let a double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U of a B-algebra X be double-framed soft normal. Then $\alpha(x * y) = \alpha(y * x)$ and $\beta(x * y) = \beta(y * x)$ for any $x, y \in X$.*

Proof. Let $x, y \in X$. By (B1) and (B2), we have $\alpha(x*y) = \alpha((x*y)*(x*x)) \supseteq \alpha(x*x) \cap \alpha(y*x) = \alpha(0) \cap \alpha(y*x) = \alpha(y*x)$. Interchanging x with y, we obtain $\alpha(y*x) \supseteq \alpha(x*y)$.

By (B1) and (B2), we have $\beta(x*y) = \beta((x*y)*(x*x)) \subseteq \beta(x*x) \cup \beta(y*x) = \beta(0) \cup \beta(y*x) = \beta(y*x)$. Interchanging x with y, we obtain $\beta(y*x) \subseteq \beta(x*y)$. □

Theorem 3.20. *Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft normal subalgebra of a B-algebra X. Then the set $X_{(\alpha,\beta)} := \{x \in X | \alpha(x) = \alpha(0), \beta(x) = \beta(0)\}$ is a normal subalgebra of X.*

Proof. It is sufficient to show that $X_{(\alpha,\beta)}$ is normal. Let $a, b, x, y \in X$ be such that $x*y \in X_{(\alpha,\beta)}$ and $a*b \in X_{(\alpha,\beta)}$. Then $\alpha(x*y) = \alpha(0) = \alpha(a*b), \beta(x*y) = \beta(0) = \beta(a*b)$. Since $\langle(\alpha,\beta);X\rangle$ is a double-framed soft normal subalgebra of X, we have $\alpha((x*a)*(y*b)) \supseteq \alpha(x*y) \cap \alpha(a*b) = \alpha(0)$ and $\beta((x*a)*(y*b)) \subseteq \beta(x*y) \cup \beta(a*b) = \beta(0)$. Using (3.2), we conclude that $\alpha((x*a)*(y*b)) = \alpha(0)$ and $\beta((x*a)*(y*b)) = \beta(0)$. Hence $(x*a)*(y*b) \in X_{(\alpha,\beta)}$. This completes the proof. □

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APPLICATIONS OF DOUBLE DIFFERENCE FRACTIONAL ORDER OPERATORS TO ORIGINATE SOME SPACES OF SEQUENCES

ANU CHOUDHARY AND KULDIP RAJ

Abstract. In the present article, we introduce and study some sequence spaces by means of double difference fractional order operators, Orlicz function and four dimensional bounded regular matrix. We make an effort to study some topological and algebraic properties of these sequence spaces. Some inclusion relations between newly formed sequence spaces are also establish. Finally, we study several results under the suitable choice of order γ.

1. Introduction and Preliminaries

Let $(\varpi_{k,l}, \nu_{k,l})$ be a double sequence of seminormed spaces such that $\varpi_{k-1,l-1} \subseteq \varpi_{k,l}$ for all non-negative integers k and l. A sequence space X is called solid or normal if and only if it contains all such sequences $y = (y_{k,l})$ corresponding to each of which there is a sequence $x = (x_{k,l}) \in X$ such that $|y_{k,l}| \leq |x_{k,l}|$ for all non negative integers k and l. Let Q be a normal sequence space and Ω^2 denotes the set of all double complex sequences. Define a linear space

$$\Omega^2(\varpi_{k,l}) = \{x = (x_{k,l}) \in \Omega^2 : x_{k,l} \in \varpi_{k,l} \text{ for all non-negative integers } k \text{ and } l\}.$$

Let ν and ν' be seminorms on a linear space X. Then ν is said to be stronger than ν' if whenever $(x_{k,l})$ is a sequence such that $\nu(x_{k,l}) \to 0$, then also $\nu'(x_{k,l}) \to 0$. If each is stronger than the other, then ν and ν' are said to be equivalent.

A double sequence has Pringsheim limit L (denoted by $P-\lim x = L$) provided that given $\epsilon > 0$ there exist $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ (see [11]). A double sequence $x = (x_{k,l})$ is bounded if there exists a positive number n such that $|x_{k,l}| < n$ for all k and l.

Some initial works on double sequences is due to Bromwich [5]. Later on, the double sequences were studied in (see [12], [13]) and operators on sequence spaces were studied in (see [1], [9]).

The fractional difference operator $\Delta^{(\gamma)}$ for a positive proper fraction γ on single sequence is defined as

$$\Delta^{(\gamma)}(x_k) = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\gamma+1)}{m!\Gamma(\gamma-m+1)} x_{k-m},$$

where $\Gamma(\gamma)$ denotes the Euler gamma function of a real number γ or generalized factorial function (see [2], [3]). For $\gamma \notin \{0, -1, -2, -3, \cdots\}$, $\Gamma(\gamma)$ can be expressed as an improper integral,

$$\Gamma(\gamma) = \int_0^{\infty} e^{-s} s^{\gamma-1} ds.$$

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For $x \in \Omega^2$ and a positive proper fraction γ, the double difference operator of fractional order γ is defined as

$$\Delta_2^{(\gamma)}(x_{k,l}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\frac{\Gamma(\gamma+1)^2}{m!n!\Gamma(\gamma-m+1)\Gamma(\gamma-n+1)}x_{k-m,l-n}. \tag{1.1}$$

The above defined infinite series can be reduced to finite series if γ is a positive integer (see [4]). Throughout the text it is assumed that $(x_{k,l}) = 0$ for any negative integers k and l.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $R > 0$ such that $M(2u) \leq RM(u), u \geq 0$.

The idea of Orlicz function was used by Lindenstrauss and Tzafriri [7] to define the following sequence space:

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

known as an *Orlicz sequence space*. The space l_M is a Banach space with the norm,

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (see [8], [10]).

Remark 1.1. (1) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function and q be a non-negative integer. Then for a real number $d \in [0, \infty)$, we have
(i)$\mathcal{M}(qx) \leq q\mathcal{M}(x)$
(ii) $\mathcal{M}(dx) \leq (1 + \lfloor d\rfloor)\mathcal{M}(x)$, where $\lfloor . \rfloor$ denotes the greatest integer function.
(2) For a complex number α,

$$|\alpha|^{p_{k,l}} \leq \max\{1, |\alpha|^L\}$$

and

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}),$$

where $L = \sup_{k,l} p_{k,l} < \infty$ and $D = \max(1, 2^{L-1})$.

Let $\mathcal{A} = (a_{ijkl})$ be a four-dimensional infinite matrix of scalars. For all $i, j \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the sum

$$y_{i,j} = \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}x_{k,l}$$

is called the $\mathcal{A}$-*means* of the double sequence $(x_{k,l})$. A double sequence $(x_{k,l})$ is said to be $\mathcal{A}$-*summable* to the limit L if the $\mathcal{A}$-means exist for all i, j in the sense of Pringsheim's convergence

$$P\text{-}\lim_{p,q\to\infty}\sum_{k,l=0,0}^{p,q} a_{ijkl}x_{k,l} = y_{i,j} \text{ and } P\text{-}\lim_{i,j\to\infty} y_{i,j} = L.$$

A four-dimensional matrix $\mathcal{A}$ is said to be *bounded-regular* (or *RH-regular*) if every bounded P-convergent sequence is $\mathcal{A}$-summable to the same limit and the $\mathcal{A}$-means are also bounded.

Theorem 1.2. *(Robison* [14] *and Hamilton* [6]*) The four dimensional matrix $\mathcal{A}$ is RH-regular if and only if*

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(RH$_1$) $P\text{-}\lim_{i,j} a_{ijkl} = 0$ *for each* k *and* l,

(RH$_2$) $P\text{-}\lim_{i,j} \sum_{k,l=1,1}^{\infty,\infty} |a_{ijkl}| = 1$,

(RH$_3$) $P\text{-}\lim_{i,j} \sum_{k=1}^{\infty} |a_{ijkl}| = 0$ *for each* l,

(RH$_4$) $P\text{-}\lim_{i,j} \sum_{l=1}^{\infty} |a_{ijkl}| = 0$ *for each* k,

(RH$_5$) $\sum_{k,l=1,1}^{\infty,\infty} |a_{ijkl}| < \infty$ *for all* $i, j \in \mathbb{N}$.

A real valued function g defined on a linear space X is called a paranorm, if it satisfies the following conditions for all $x, y \in X$ and for all scalars β
(i) $g(\lambda) = 0$, where λ is the zero element of X
(ii) $g(-x) = g(x)$
(iii) $g(x + y) \leq g(x) + g(y)$
(iv) If (β_n) is a sequence of scalars with $\beta_n \to 0$ as $n \to 0$ and x_n is a sequence in X such that $g(x_n - x) \to 0$ as $n \to \infty$ for some $x \in X$, then $g(\beta_n x_n - \beta x) \to 0$ as $n \to \infty$.
Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix, $u = (u_{k,l})$ be any double sequence of strictly positive real numbers, $p = (p_{k,l})$ be a bounded double sequence of positive real numbers, $\Delta_2^{(\gamma)}$ denotes the double difference operator of fractional order γ. In this paper we define the following sequence space
$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] =$

$$\Big\{x = (x_{k,l}) \in \Omega^2(\varpi_{k,l}) : \Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j} \in Q, \text{ for some } \rho > 0\Big\}.$$

Remark 1.3. (1) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function and $\rho = \rho_1 + \rho_2$. Then for $x = (x_{k,l})$ and $y = (y_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l}) + u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$

$$\leq D\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho_1}\Big)\Big)\Big]^{p_{k,l}} + \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)} y_{k,l}}{\rho_2}\Big)\Big)\Big]^{p_{k,l}}\Big)$$

for all non-negative integers i and j and for some $\rho_1, \rho_2 > 0$.

(2) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function and $d \in \mathbb{C}$. Then for $L = \sup_{k,l} p_{k,l} < \infty$, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(dx_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$\leq \max\{1,(1+\lfloor|d|\rfloor)^L\}\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)$$

for all non-negative integers i and j and for some $\rho > 0$.

(3) Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two Musielak Orlicz functions. Then

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[(M_{k,l}+M'_{k,l})\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$\leq D\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$+ \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)$$

for all non-negative integers i and j and for some $\rho > 0$.

(4) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function. Let $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two sequences of seminorms. Then

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big((\nu_{k,l}+\nu'_{k,l})\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$\leq D\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$+ \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu'_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)$$

for all non-negative integers i and j and for some $\rho > 0$.

The main goal of this paper is to introduce the double difference operator $\Delta_2^{(\gamma)}$ of fractional order γ. In this study, being an application of double difference operator $\Delta_2^{(\gamma)}$, some new difference double sequence spaces of fractional order have been introduced and subsequently, their topological and algebraic properties have been discussed in detail. Infact, this study involves new results obtained under different suitable choice of γ.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\nu = (\nu_{k,l})$ be a sequence of seminorms and $u = (u_{k,l})$ be a double sequence of strictly positive real numbers. Then the sequence space $Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$ is a linear space over the complex field $\mathbb{C}$.*

Proof. This is a routine matter, so we omit it. □

Theorem 2.2. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\nu = (\nu_{k,l})$ be a sequence of seminorms and $u = (u_{k,l})$ be a double sequence of strictly positive real numbers. Then the* *sequence space* $Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$ *is a paranormed space with paranorm g defined by*

$$g(x) = \inf\left\{(\rho)^{\frac{p_{k,l}}{L}} : \left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \le 1,\ \textit{for some } \rho > 0\right\},$$

where $N = \max\{1, L\}$ *and* $L = \sup_{k,l} p_{k,l} < \infty$.

Proof. (i) Clearly $g(x) \ge 0$, for $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$.
(ii) $g(-x) = g(x)$.
(iii) Let $x = (x_{k,l}), y = (y_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$, then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho_1}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \le 1$$

and

$$\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho_2}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \le 1.$$

Now for $\rho = \rho_1 + \rho_2$ and by using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l}) + u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho_1+\rho_2}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \\
&\le \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho_1}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \\
&+ \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}y_{k,l}}{\rho_2}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \\
&\le 1.
\end{aligned}$$

Hence, $g(x+y)$

$$\begin{aligned}
&= \inf\left\{(\rho_1+\rho_2)^{\frac{p_{k,l}}{L}} : \left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l}) + u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \le 1,\ \text{for some } \rho > 0\right\} \\
&\le \inf\left\{(\rho_1)^{\frac{p_{k,l}}{L}} : \left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho_1}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \le 1,\ \text{for some } \rho_1 > 0\right\} \\
&+ \inf\left\{(\rho_2)^{\frac{p_{k,l}}{L}} : \left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho_2}\right)\right)\right]^{p_{k,l}}\right)^{\frac{1}{N}} \le 1,\ \text{for some } \rho_2 > 0\right\} \\
&= g(x) + g(y).
\end{aligned}$$

(iv) Finally, we show that scalar multiplication is continuous. In order to show this, let us consider a complex number σ. Then by definition we have

$$
\begin{aligned}
&g(\sigma x)\\
&= \inf\Big\{(\rho)^{\frac{p_{k,l}}{L}} : \Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{\sigma u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{N}} \leq 1, \rho > 0\Big\},\\
&= \inf\Big\{(|\sigma|t)^{\frac{p_{k,l}}{L}} : \Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{t}\Big)\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{N}} \leq 1, t > 0\Big\},
\end{aligned}
$$

where $t = \frac{\rho}{|\sigma|}$. Hence the proof. □

Theorem 2.3. *Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two Musielak Orlicz functions, $u = (u_{k,l})$ be a double sequence of strictly positive real numbers and $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \cap Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}'] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M} + \mathcal{M}'].$$

Proof. Suppose $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \cap Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}']$. This implies that

$$\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j}$$

and

$$\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j}$$

both are in Q. Now by using part (3) of Remark 1.3, we have

$$
\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[(M_{k,l} + M'_{k,l})\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\\
&\qquad \leq D\Big\{\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\\
&\qquad + \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big\}.
\end{aligned}
$$

Since Q is normal, $\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[(M_{k,l} + M'_{k,l})\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j} \in Q$.

Then $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M} + \mathcal{M}']$. Hence the proof. □

Theorem 2.4. *Suppose that $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two double sequences of seminorms. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \cap Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu + \nu', u, \mathcal{A}, \mathcal{M}].$$

Proof. One can easily obtain the proof by using part (4) of Remark 1.3. So, we omit it. □

Theorem 2.5. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function. If $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two double sequences of seminorms such that $(\nu_{k,l})$ is stronger than $(\nu'_{k,l})$ for each k and l, then $Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}]$.*

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Proof. Consider a double sequence $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. Then

$$\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right)\right)\right]^{p_{k,l}} \right)_{i,j} \in Q.$$

Since each $\nu_{k,l}$ is stronger than corresponding $\nu'_{k,l}$, we have a natural number $N_{k,l}$ corresponding to each pair of non-negative integer k and l such that $\nu'_{k,l}(w) \leq N_{k,l}\nu_{k,l}(w)$. Let $N = \max\{N_{k,l}\}$. Then $\nu'_{k,l}(w) \leq N\nu_{k,l}(w)$ for all non-negative integers k and l. Thus,

$$\nu'_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right) \leq N\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right).$$

From Remark 1.1, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl} \left[M_{k,l}\left(\nu'_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right)\right)\right]^{p_{k,l}}$$

$$\leq \max\{1, N^L\} \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right)\right)\right]^{p_{k,l}}.$$

Since Q is normal, $\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl} \left[M_{k,l}\left(\nu'_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right)\right)\right]^{p_{k,l}} \right)_{i,j} \in Q.$

This implies $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}]$. □

Corollary 2.6. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function. If $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two double sequences of seminorms such that $(\nu_{k,l})$ is equivalent to $(\nu'_{k,l})$ for each k and l. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] = Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}].$$

Theorem 2.7. *Suppose $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two Musielak Orlicz function such that $M_{k,l}(1)$ is finite for each k and l. Let $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}'] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M} \circ \mathcal{M}'].$$

Proof. Consider $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}']$. So,

$$\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl} \left[M'_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right)\right)\right]^{p_{k,l}} \right)_{i,j} \in Q.$$

Since each $(M_{k,l})$ is continuous and $M_{k,l}(0) = 0$ for each k and l, we choose $\varsigma \in (0, 1)$ corresponding to an arbitrary $\epsilon > 0$ such that $M_{k,l}(s) < \epsilon$ for $0 \leq s \leq \varsigma$. Let us take

$$s_{k,l} = M'_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho} \right)\right)$$

and

$$\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl} \left[M_{k,l}(s_{k,l}) \right]^{p_{k,l}} = \sum_1 a_{ijkl} \left[M_{k,l}(s_{k,l}) \right]^{p_{k,l}} + \sum_2 a_{ijkl} \left[M_{k,l}(s_{k,l}) \right]^{p_{k,l}}, \tag{2.1}$$

where the first summation is over $s_{k,l} \leq \varsigma$ and the second is taken over $s_{k,l} > \varsigma$. For $s_{k,l} \leq \varsigma$, we have $M_{k,l}(s_{k,l}) < \epsilon$ and hence

$$\sum_1 a_{ijkl} \left[M_{k,l}(s_{k,l}) \right]^{p_{k,l}} < \sum_1 a_{ijkl} [\epsilon]^{p_{k,l}}.$$

Now by using Part (2) of Remark 1.1, we have

$$\sum_{1} a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}} < \max\{1,\epsilon^L\}\sum_{1} a_{ijkl} \le \max\{1,\epsilon^L\}\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}. \tag{2.2}$$

For $s_{k,l} > \varsigma$, we have $s_{k,l} < \left(\frac{s_{k,l}}{\varsigma}\right)$. So, from Part (1) of Remark 1.1, we have

$$M_{k,l}(s_{k,l}) < M_{k,l}\left(\frac{s_{k,l}}{\varsigma}\right) \le \left(1+\left\lfloor\frac{s_{k,l}}{\varsigma}\right\rfloor\right)M_{k,l}(1) \le 2M_{k,l}(1)\frac{s_{k,l}}{\varsigma}.$$

Let $\xi = \max_{k,l}(M_{k,l}(1))$, then $M_{k,l}(s_{k,l}) < 2\xi\frac{s_{k,l}}{\varsigma}$. By using Part (2) of Remark 1.1, we get

$$\sum_{2} a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}} \le \max\left(1,\left(\frac{2\xi}{\varsigma}\right)^L\right)\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}[s_{k,l}]^{p_{k,l}}. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we have

$$\begin{aligned}\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl}[M_{k,l}(s_{k,l})]^{p_{k,l}} &\le \max(1,\epsilon^L)\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\\ &+ \max\left(1,\left(\frac{2\xi}{\varsigma}\right)^L\right)\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}[s_{k,l}]^{p_{k,l}}.\end{aligned}$$

Since Q is normal,

$$\left(\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl}[M_{k,l}(s_{k,l})]^{p_{k,l}}\right)_{i,j}$$

$$= \left(\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl}\left[(M_{k,l}\circ M'_{k,l})\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}\right)_{i,j} \in Q.$$

Thus, $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}\circ\mathcal{M}']$. Hence the proof. ☐

Theorem 2.8. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function such that $M_{k,l}(s) \le M_{k-1,l-1}(s)$ for all $s \in [0,\infty)$, $\nu = (\nu_{k,l})$ be a double sequence of seminorm such that $\nu_{k,l}(s) \le \nu_{k-1,l-1}(s)$ for all s. Let $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix such that $a_{ijkl} \le a_{ij(k-1)(l-1)}$ for all non-negative integers i, j, k and l and suppose $p = (p_{k,l} \equiv p)$ is a constant sequence of positive real number. Then*

$$Q[\Delta_2^{(\gamma-1)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \subset Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}].$$

Proof. Suppose $x = (x_{k,l})$ is a double sequence in $Q[\Delta_2^{(\gamma-1)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. Then

$$\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k,l}}{\rho}\right)\right)\right]^{p}\right)_{i,j} \in Q.$$

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Since $M_{k,l}(s) \le M_{k-1,l-1}(s), \nu_{k,l}(s) \le \nu_{k-1,l-1}(s)$ and $a_{ijkl} \le a_{ij(k-1)(l-1)}$, we have

$$\begin{aligned}&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\right)\right)\right]^p\\ \le&\sum_{k,l=0,0}^{\infty,\infty} a_{ij(k-1)(l-1)}\left[M_{k-1,l-1}\left(\nu_{k-1,l-1}\left(\frac{u_{k-1,l-1}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\right)\right)\right]^p\\ =&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k,l}}{\rho}\right)\right)\right]^p.\end{aligned}$$

Since Q is normal ,$\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\right)\right)\right]^p\right)_{i,j} \in Q$. Now,

$$\begin{aligned}&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^p\\ =&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}(x_{k,l}-x_{k-1,l-1})}{\rho}\right)\right)\right]^p\\ \le& D\Bigg\{\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k,l}}{\rho}\right)\right)\right]^p\\ +&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\right)\right)\right]^p\Bigg\}.\end{aligned}$$

Again Q is normal. So, $\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^p\right)_{i,j} \in Q$.

Thus, $x=(x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. □

In order to show the strictness of the above inclusion let us consider the following example.

Example 2.9. *Consider $\mathcal{M}=(M_{k,l})$ be a Musielak Orlicz function with $M_{k,l}(x)=x, (\nu_{k,l})$ be a sequence of seminorm with $\nu_{k,l}(x)=|x|, p_{k,l}=1, \mathcal{A}=I$ be an identity matrix of infinite order, $\rho=1, u_{k,l}(x)=x$ and $(x_{k,l})=1$ for all non-negative integers k and l. Then*

$$\sup_{i,j}\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}} = \sup_{i,j}|\mathcal{J}_{i,j}(\gamma)| \text{ (say)},$$

where $\mathcal{J}_{i,j}(\gamma)$ is the expansion of the series (1.1) for $x_{k,l}=1$. For $\gamma=\frac{1}{2}$, $\sup_{i,j}|\mathcal{J}_{i,j}(\gamma)|<\infty$ whereas for $\gamma=-\frac{1}{2}$, $\sup_{i,j}|\mathcal{J}_{i,j}(\gamma)|=\infty$. Thus, $x=(x_{k,l}) \in \ell_\infty[\Delta_2^{(1/2)}, p, \nu, u, I, \mathcal{M}]$ but $x=(x_{k,l}) \notin \ell_\infty[\Delta_2^{(-1/2)}, p, \nu, u, I, \mathcal{M}]$. Therefore, the inclusion relation is strict in general.

Theorem 2.10. *Let $\mathcal{M}=(M_{k,l})$ be a Musielak Orlicz function and P be a nonnegative four-dimensional bounded-regular matrix whose all entries are 1. If $0<\inf_{k,l} h_{k,l} < h_{k,l} \le c_{k,l} < \sup_{k,l} c_{k,l} < \infty$ for all non-negative integers k and l. Then*

$$\ell_\infty[\Delta_2^{(\gamma)}, h, \nu, u, P, \mathcal{M}] \subseteq \ell_\infty[\Delta_2^{(\gamma)}, c, \nu, u, P, \mathcal{M}].$$

Proof. Consider that $x = (x_{k,l}) \in \ell_\infty[\Delta_2^{(\gamma)}, h, \nu, u, P, \mathcal{M}]$. This implies

$$\sup_{i,j} \sum_{k,l=0,0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} = \sum_{k,l=0,0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} < \infty.$$

Then for sufficiently large k say k_0 and sufficiently large l say l_0, we have

$$\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} \leq 1$$

for all $k \geq k_0$ and $l \geq l_0$. Hence,

$$\left\{\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}}\right\}^{\frac{c_{k,l}}{h_{k,l}}} \leq \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}}$$

for all $k \geq k_0$ and $l \geq l_0$. Now by taking summation from k_0 to ∞ and l_0 to ∞ on both sides, we have

$$\sum_{k,l=k_0,l_0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{c_{k,l}} \leq \sum_{k,l=k_0,l_0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} < \infty.$$

Thus, $\sup_{i,j} \sum_{k,l=0,0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{c_{k,l}} < \infty.$

Therefore, $x = (x_{k,l}) \in \ell_\infty[\Delta_2^{(\gamma)}, c, \nu, u, P, \mathcal{M}]$. Hence the proof. □

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On the dynamics of higher-order anti-competitive system: $x_{n+1} = A + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}}$, $y_{n+1} = B + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}$

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Abstract

We study the boundedness and persistence, asymptotic stability, existence and uniqueness of positive equilibrium point, and rate of convergence of an anti-competitive system of higher-order difference equations. The proposed work is considerably extended and improve some existing results in the literature.

Keywords: difference equations; boundedness; persistence; asymptotic stability; rate of convergence

2010 AMS Mathematics subject classifications: 39A10, 40A05.

1 Introduction

Difference equations or systems of difference equations play a vital role in the development of different sciences ranging from life to decision sciences (see [1–7] and references cited therein). DeVault *et al.* [5] have investigated that every positive solution of the difference equation: $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$, $n = 0, 1, \cdots$, converges to a period two solution. Abu-Saris and DeVault [6] have investigated the global stability of the positive equilibrium of the difference equation: $x_{n+1} = A + \frac{x_n}{x_{n-k}}$, $n = 0, 1, \cdots$. Zhang *et al.* [7] have studied the global dynamics of the difference equation: $x_{n+1} = A + \frac{x_n}{\sum_{i=1}^{2} y_{n-i}}$, $y_{n+1} = B + \frac{y_n}{\sum_{i=1}^{2} x_{n-i}}$, $n = 0, 1, \cdots$. In this paper, our goal is to investigate the dynamics of following higher-order anti-competitive system of difference equations:

$$x_{n+1} = A + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}},\ y_{n+1} = B + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}},\ n = 0, 1, \cdots, \tag{1}$$

where initial conditions $x_{-p},\ y_{-p},\ p = k,\ k-1,\ k-2, \cdots, 1, 0$ and $A,\ B$ are positive.

2 Main results

Hereafter we will prove main results for under consideration system.

Theorem 1. *If $ABk^2 > 1$, then the following statements holds:*

(i) Every positive solution $\{(x_n, y_n)\}$ of (1) is bounded and persists.

(ii) The interval $\left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ is invariant set for (1).

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Proof. (i) If $\{(x_n, y_n)\}$ be a positive solution of (1) then

$$x_n \geq A,\ y_n \geq B,\ n = 0, 1, \cdots. \tag{2}$$

From (1) and (2), one gets

$$x_{n+1} \leq A + \frac{1}{kA} y_n,\ y_{n+1} \leq B + \frac{1}{kB} x_n. \tag{3}$$

Moreover, from (3), one gets

$$x_{n+1} \leq A + \frac{B}{kA} + \frac{1}{k^2 AB} x_{n-1},\ y_{n+1} \leq B + \frac{A}{kB} + \frac{1}{k^2 AB} y_{n-1}. \tag{4}$$

Now consider

$$\varsigma_{n+1} = A + \frac{B}{kA} + \frac{1}{k^2 AB} \varsigma_{n-1},\ \varrho_{n+1} = B + \frac{A}{kB} + \frac{1}{k^2 AB} \varrho_{n-1}. \tag{5}$$

Therefore, solution $\{(\varsigma_n, \varrho_n)\}$ of (5) is given by

$$\begin{aligned} \varsigma_n &= c_1 \left(\sqrt{\frac{1}{k^2 AB}}\right)^n + c_2 \left(-\sqrt{\frac{1}{k^2 AB}}\right)^n + \frac{kB\left(kA^2 + B\right)}{k^2 AB - 1}, \\ \varrho_n &= d_1 \left(\sqrt{\frac{1}{k^2 AB}}\right)^n + d_2 \left(-\sqrt{\frac{1}{k^2 AB}}\right)^n + \frac{kA\left(kB^2 + A\right)}{k^2 AB - 1}, \end{aligned} \tag{6}$$

where $c_1,\ c_2,\ d_1,\ d_2$ depend upon $\varsigma_{-1},\ \varsigma_0,\ \varrho_{-1},\ \varrho_0$. Assuming $ABk^2 > 1$, then (6) implies that $\{\varsigma_n\}$ and $\{\varrho_n\}$ are bounded. Now considering solution $\{(\varsigma_n, \varrho_n)\}$ of (6) for which

$$\varsigma_{-1} = x_{-1},\ \varsigma_0 = x_0,\ \varrho_{-1} = y_{-1},\ \varrho_0 = y_0, \tag{7}$$

where $x_{-1}, x_0 \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$ and $y_{-1}, y_0 \in \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$. From (4) and (7) one gets

$$x_n \leq \frac{kB\left(kA^2 + B\right)}{k^2 AB - 1},\ y_n \leq \frac{kA\left(kB^2 + A\right)}{k^2 AB - 1}. \tag{8}$$

From (2) and (8), we get

$$A \leq x_n \leq \frac{kB\left(kA^2 + B\right)}{k^2 AB - 1},\ B \leq y_n \leq \frac{kA\left(kB^2 + A\right)}{k^2 AB - 1},\ n = 0, 1, \cdots.$$

□

Proof. (ii) Follows from induction. □

Theorem 2. *System (1) has a unique positive equilibrium point* $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ *if*

$$k^2 \left(\frac{2k^2 B(kA^2 + B)}{k^2 AB - 1} \left(\frac{kB(kA^2 + B)}{k^2 AB - 1} - A\right) - B\right) \left(\frac{2kB(kA^2 + B)}{k^2 AB - 1} - A\right) < 1. \tag{9}$$

Proof. Consider

$$x = A + \frac{y}{kx},\ y = B + \frac{x}{ky}. \tag{10}$$

From (10),

$$y = kx(x - A),\ x = ky(y - B).$$

Defining

$$S(x) = ks(x)(s(x) - B) - x, \tag{11}$$

where

$$s(x) = kx(x - A), \tag{12}$$

and $x \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$. We claim that $S(x)=0$ has a unique solution $x \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$. From (11) and (12) one gets

$$S'(x) = 2ks(x)s'(x) - kBs'(x) - 1, \tag{13}$$

and

$$s'(x) = 2kx - kA. \tag{14}$$

Now if $\bar{x} \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$ be a solution of $S(x)=0$ then from (11) and (12) one gets

$$ks(\bar{x})(s(\bar{x}) - B) = \bar{x}, \tag{15}$$

where

$$s(\bar{x}) = k\bar{x}(\bar{x} - A). \tag{16}$$

In view of (14), (15) and (16), equation (13) becomes

$$\begin{aligned} S'(x) &= k^2\left(2kx(x-A)-B\right)(2x-A)-1, \\ &\leq k^2\left(\frac{2k^2B(kA^2+B)}{k^2AB-1}\left(\frac{kB(kA^2+B)}{k^2AB-1}-A\right)-B\right)\left(\frac{2kB(kA^2+B)}{k^2AB-1}-A\right)-1. \end{aligned} \tag{17}$$

Now assume that (9) hold then from (17) one gets $S'(x) < 0$. Hence $S(x) = 0$ has a unique positive solution $\bar{x} \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$. □

Theorem 3. *If*

$$\frac{1}{kA} + \frac{kB^2+A}{A(k^2AB-1)} < 1, \ \frac{1}{kB} + \frac{kA^2+B}{B(k^2AB-1)} < 1, \tag{18}$$

then equilibrium $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ *of the system (1) is locally asymptotically stable.*

Proof. The linearized system of (1) about $(\bar{x}, \bar{y})$ is

$$\Phi_{n+1} = E\Phi_n,$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}, \ E = \begin{pmatrix} 0 & -\frac{\bar{y}}{k^2\bar{x}^2} & \dots & -\frac{\bar{y}}{k^2\bar{x}^2} & -\frac{\bar{y}}{k^2\bar{x}^2} & \frac{1}{k\bar{x}} & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{k\bar{y}} & 0 & \dots & 0 & 0 & 0 & -\frac{\bar{x}}{k^2\bar{y}^2} & \dots & -\frac{\bar{x}}{k^2\bar{y}^2} & -\frac{\bar{x}}{k^2\bar{y}^2} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let us denote $2k+2$ eigenvalues of E as $\kappa_1, \ \kappa_2, \ \dots, \ \kappa_{2k+2}$ and $D = diag(m_1, \ m_2, \ \dots, \ m_{2k+2})$ be a diagonal matrix, where $m_1 = m_{k+2} = 1$, $m_i = m_{k+1+i} = 1 - i\epsilon, \ i = 2, \ 3, \ \cdots, \ k+1$, and

$$0 < \epsilon < \min\left\{\frac{1}{k+1}\left(1 - \frac{1}{kA} - \frac{kB^2+A}{A(K^2AB-1)}\right), \ \ \frac{1}{k+1}\left(1 - \frac{1}{kB} - \frac{kA^2+B}{B(K^2AB-1)}\right)\right\} < 1.$$

Since D is invertible and by computing DED^{-1}, one gets

$$DED^{-1} = \begin{pmatrix} 0 & -\frac{\bar{y}}{k^2\bar{x}^2}m_1m_2^{-1} & \dots & -\frac{\bar{y}}{k^2\bar{x}^2}m_1m_k^{-1} & -\frac{\bar{y}}{k^2\bar{x}^2}m_1m_{k+1}^{-1} & \frac{1}{k\bar{x}}m_1m_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ m_2m_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & m_{k+1}m_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{k\bar{y}}m_{k+2}m_1^{-1} & 0 & \dots & 0 & 0 & 0 & -\frac{\bar{x}}{k^2\bar{y}^2}m_{k+2}m_{k+3}^{-1} & \dots & -\frac{\bar{x}}{k^2\bar{y}^2}m_{k+2}m_{2k+1}^{-1} & -\frac{\bar{x}}{k^2\bar{y}^2}m_{k+2}m_{2k+2}^{-1} \\ 0 & 0 & \dots & 0 & 0 & m_{k+3}m_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & m_{2k+2}m_{2k+1}^{-1} & 0 \end{pmatrix}. \quad (19)$$

From $m_1 > m_2 > \cdots > m_{k+1} > 0$ and $m_{k+2} > m_{k+3} > \cdots > m_{2k+2} > 0$, one has

$$m_2m_1^{-1} < 1,\ m_3m_2^{-1} < 1,\ \cdots,\ m_{k+1}m_k^{-1} < 1, m_{k+3}m_{k+2}^{-1} < 1,\ m_{k+4}m_{k+3}^{-1} < 1,\ \cdots,\ m_{2k+2}m_{2k+1}^{-1} < 1.$$

Also,

$$\begin{aligned} \frac{\bar{y}}{k^2\bar{x}^2}m_1m_2^{-1} + \cdots + \frac{\bar{y}}{k^2\bar{x}^2}m_1m_{k+1}^{-1} + \frac{1}{k\bar{x}}m_1m_{k+2}^{-1} &= \frac{1}{k\bar{x}} + \frac{\bar{y}}{k^2\bar{x}^2}\left(\frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-(k+1)\epsilon}\right), \\ &< \left(\frac{1}{k\bar{x}} + \frac{\bar{y}}{k\bar{x}^2}\right)\frac{1}{1-(k+1)\epsilon}, \\ &< \left(\frac{1}{kA} + \frac{kB^2+A}{A\left(k^2AB-1\right)}\right)\frac{1}{1-(k+1)\epsilon} < 1. \end{aligned}$$

And

$$\begin{aligned} \frac{1}{k\bar{y}}m_{k+2}m_1^{-1} + \frac{\bar{x}}{k^2\bar{y}^2}m_{k+2}m_{k+3}^{-1} + \cdots + \frac{\bar{x}}{k^2\bar{y}^2}m_{k+2}m_{2k+2}^{-1} &= \frac{1}{k\bar{y}} + \frac{\bar{x}}{k^2\bar{y}^2}\left(\frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-(k+1)\epsilon}\right), \\ &< \left(\frac{1}{k\bar{y}} + \frac{\bar{x}}{k\bar{y}^2}\right)\frac{1}{1-(k+1)\epsilon}, \\ &< \left(\frac{1}{kB} + \frac{kA^2+B}{B\left(k^2AB-1\right)}\right)\frac{1}{1-(k+1)\epsilon} < 1. \end{aligned}$$

Since E has the same eigenvalues as DED^{-1} and hence

$$\begin{aligned} \max_{1\le n\le 2k+2}|\kappa_n| \le \|DED^{-1}\|_\infty &= \max\{m_2m_1^{-1}, \cdots, m_{k+1}m_k^{-1}, m_{k+3}m_{k+2}^{-1}, \cdots, m_{2k+2}m_{2k+1}^{-1}, \\ &\quad \frac{1}{k\bar{x}} + \frac{\bar{y}}{k^2\bar{x}^2}\left(\frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-(k+1)\epsilon}\right), \frac{1}{k\bar{y}} + \\ &\quad \frac{\bar{x}}{k^2\bar{y}^2}\left(\frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-(k+1)\epsilon}\right)\} < 1. \end{aligned} \quad (20)$$

Thus equation (20) implies that $(\bar{x},\bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ of (1) is locally asymptotically stable. □

Theorem 4. *Equilibrium* $(\bar{x},\bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ *of (1) is globally asymptotically stable.*

Proof. Let $\{(x_n, y_n)\}$ be arbitrary solution of (1). Also let $\lim_{n\to\infty}\sup x_n = L_1,\ \lim_{n\to\infty}\inf x_n = l_1,\ \lim_{n\to\infty}\sup y_n = L_2,\ \lim_{n\to\infty}\inf y_n = l_2$ where $l_i, L_i \in (0,\infty),\ i = 1,2$. Then from (1) one gets

$$L_1 \le A + \frac{L_2}{kl_1},\ l_1 \ge A + \frac{l_2}{kL_1}. \quad (21)$$

And

$$L_2 \leq B + \frac{L_1}{kl_2},\ l_2 \geq B + \frac{l_1}{kL_2}. \tag{22}$$

From (21), we have

$$Ak(L_1 - l_1) \leq L_2 - l_2. \tag{23}$$

From (22), we get

$$Bk(L_2 - l_2) \leq L_1 - l_1. \tag{24}$$

From (23) and (24), we get

$$(ABk^2 - 1)(L_1 - l_1) \leq 0,$$

which implies that $l_1 = L_1$. Similarly it is easy to prove that $l_2 = L_2$. □

Theorem 5. *Assuming $\{(x_n, y_n)\}$ be a positive solution of (1) such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$, where $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$. Then, the error vector ξ_n satisfying*

$$\lim_{n\to\infty} \sqrt[n]{\|\xi_n\|} = |\kappa E|,\ \lim_{n\to\infty} \frac{\|\xi_{n+1}\|}{\|\xi_n\|} = |\kappa E|,$$

where κE are the characteristic roots of E.

Proof. If $\{(x_n, y_n)\}$ be any solution of (1) such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$. To find error term one has

$$x_{n+1} - \bar{x} = \frac{y_n}{\sum_{i=1}^{k} x_{n-i}} - \frac{\bar{y}}{k\bar{x}} = -\sum_{i=1}^{k} \frac{\bar{y}}{k\bar{x}\left(\sum_{i=1}^{k} x_{n-i}\right)} (x_{n-i} - \bar{x}) + \frac{1}{\sum_{i=1}^{k} x_{n-i}} (y_n - \bar{y}),$$

$$y_{n+1} - \bar{y} = \frac{x_n}{\sum_{i=1}^{k} y_{n-i}} - \frac{\bar{x}}{k\bar{y}} = \frac{1}{\sum_{i=1}^{k} y_{n-i}} (x_n - \bar{x}) - \sum_{i=1}^{k} \frac{\bar{x}}{k\bar{y}\left(\sum_{i=1}^{k} y_{n-i}\right)} (y_{n-i} - \bar{y}).$$

Denote $\epsilon_n^1 = x_n - \bar{x}$ and $\epsilon_n^2 = y_n - \bar{y}$, one has

$$\epsilon_{n+1}^1 = \sum_{i=1}^{k} A_{ni}\epsilon_{n-i}^1 + B_n\epsilon_n^2,\ \epsilon_{n+1}^2 = C_n\epsilon_n^1 + \sum_{i=1}^{k} D_{ni}\epsilon_{n-i}^2,$$

where

$$A_{n1} = A_{n2} = \cdots = A_{nk} = -\frac{\bar{y}}{k\bar{x}\left(\sum_{i=1}^{k} x_{n-i}\right)},\ B_n = \frac{1}{\sum_{i=1}^{k} x_{n-i}},$$

$$C_n = \frac{1}{\sum_{i=1}^{k} y_{n-i}},\ D_{n1} = D_{n2} = \cdots = D_{nk} = -\sum_{i=1}^{k} \frac{\bar{x}}{k\bar{y}\left(\sum_{i=1}^{k} y_{n-i}\right)}.$$

Taking the limits, we obtain

$$\lim_{n\to\infty} A_{n1} = \lim_{n\to\infty} A_{n2} = \cdots = \lim_{n\to\infty} A_{nk} = -\frac{\bar{y}}{k^2\bar{x}^2},\ \lim_{n\to\infty} B_n = \frac{1}{k\bar{x}},$$

$$\lim_{n\to\infty} C_n = \frac{1}{k\bar{y}},\ \lim_{n\to\infty} D_{n1} = \lim_{n\to\infty} D_{n2} = \cdots = \lim_{n\to\infty} D_{nk} = -\frac{\bar{x}}{k^2\bar{y}^2}.$$

Hence we have system (1.10) of [8] where

$$\xi_{n+1} = E\xi_n, \tag{25}$$

where $\xi_n = \begin{pmatrix} \epsilon_n^1 \\ \epsilon_{n-1}^1 \\ \vdots \\ \epsilon_{n-k}^1 \\ \epsilon_n^2 \\ \epsilon_{n-1}^2 \\ \vdots \\ \epsilon_{n-k}^2 \end{pmatrix}$, $E = \begin{pmatrix} 0 & -\frac{\bar{y}}{k^2\bar{x}^2} & \dots & -\frac{\bar{y}}{k^2\bar{x}^2} & -\frac{\bar{y}}{k^2\bar{x}^2} & \frac{1}{k\bar{x}} & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{k\bar{y}} & 0 & \dots & 0 & 0 & 0 & -\frac{\bar{x}}{k^2\bar{y}^2} & \dots & -\frac{\bar{x}}{k^2\bar{y}^2} & -\frac{\bar{x}}{k^2\bar{y}^2} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$. This is similar to linearized system of (1) about $(\bar{x}, \bar{y})$. □

3 Conclusion

In the present work, dynamics of following higher-order anti-competitive system is studied:

$$x_{n+1} = A + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}},\ y_{n+1} = B + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}.$$

Our investigations reveal that if $ABk^2 > 1$, then $\{(x_n, y_n)\}$ of this system is bounded and persists and the region $\left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ is invariant set. It is proved that if $\frac{1}{kA} + \frac{kB^2+A}{A(k^2AB-1)} < 1$ and $\frac{1}{kB} + \frac{kA^2+B}{B(k^2AB-1)} < 1$ then equilibrium $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ of the system is locally asymptotically stable. Finally global dynamics and rate of convergence that converges to $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ of (1) are also demonstrated.

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Stability of a modified within-host HIV dynamics model with antibodies

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Abstract

We investigate a modified HIV infection model with antibodies and latency. The model consider saturated HIV-CD4$^+$ T cells and HIV-macrophages incidence rates. We show that the solutions of the proposed model are nonnegative and bounded. We established that the global stability of the three steady states of the model depend on threshold parameters R_0 and R_1. Using Lyapunov function, we established the global stability of the steady states of the model. The theoretical results are confirmed by numerical simulations. The results show that antibodies can reduce the HIV infection.

1 Introduction

Constructing and analyzing of within-host human immunodeficiency virus (HIV) dynamics models have become one of the hot topics during the last decades [1]-[18]. These works can help researchers for better understanding the HIV dynamical behavior and providing new suggestions for clinical treatment. A vast of the mathematical models presented in the literature have focused on modeling the interaction between three main compartments, uninfected CD4$^+$ T cells (s), infected cells (u) and free HIV particles (p). Other models have differentiated between latent and active infected cells [19]-[23], an HIV mathematical model has been presented by inroducing a new variable (w) for the latently infected cells as:

$$\dot{s} = \rho - \delta s - \lambda sp, \tag{1}$$

$$\dot{w} = \lambda sp - (\alpha + \beta) w, \tag{2}$$

$$\dot{u} = \beta w - au, \tag{3}$$

$$\dot{p} = ku - gp, \tag{4}$$

where, ρ is the creation rate of the uninfected CD4$^+$ T cells, δ, α, a and g are the death rate constants of the four compartments s, w, u and p, respectively. The term βw represents the activation rate of the latently infected cells. The HIV-CD4$^+$T cell incidence rate is given by λsp. Parameter k represents the rate constant of free virus production. Sun et. al. [24] have modified the above model by considering the saturated infection rate

$\frac{\lambda sp}{s+p}$ as:

$$\dot{s} = \rho - \delta s - \frac{\lambda sp}{s+p}, \tag{5}$$

$$\dot{w} = \frac{\lambda sp}{s+p} - (\alpha+\beta)\, w, \tag{6}$$

$$\dot{u} = \beta w - au, \tag{7}$$

$$\dot{p} = ku - gp, \tag{8}$$

Model (5)-(8) consider one type of target cells (CD4$^+$ T cells). Moreover, the model does not acount the presence of the antibodies which are important in reducing the HIV infection. To have more accurate HIV model we improve model (5)-(8) by taking into account the dynamics of HIV with two target cells, CD4$^+$T cells and macrophages and antibodies. The global stability of the model is proven by using Lyapunov method.

2 The modified HIV

We propose the following model:

$$\dot{s}_i = \rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i+p}, \qquad i = 1,2, \tag{9}$$

$$\dot{w}_i = \frac{\lambda_i s_i p}{s_i+p} - (\alpha_i+\beta_i)\, w_i, \qquad i = 1,2, \tag{10}$$

$$\dot{u}_i = \beta_i w_i - a_i u_i, \qquad i = 1,2, \tag{11}$$

$$\dot{p} = \sum_{i=1}^{2} k_i u_i - gp - \mu pz, \tag{12}$$

$$\dot{z} = rpz - \zeta z. \tag{13}$$

where, $z(t)$ represents the populations of the antibody immune cells. The antibodies are proliferated and die at rates rpz and ζz, respectively. The HIV particles are killed by antibodies at rate μpz.

2.1 Preliminaries.

Lemma 1. The solutions of model (9)-(13) with the initial conditions $s_i(0), w_i(0), u_i(0), p(0)$ and $z(0)$ are nonnegativite and bounded for $t \geq 0$.

Proof. We have

$$\dot{s}_i\mid_{s_i=0} = \rho_i > 0, \qquad \dot{w}_i\mid_{w_i=0} = \frac{\lambda_i s_i p}{s_i+p} \geq 0 \;\; \forall\; s_i \geq 0, p \geq 0, \qquad \dot{u}_i\mid_{u_i=0} = \beta_i w_i \geq 0 \quad \forall\; w_i \geq 0,\; i = 1,2$$

$$\dot{p}\mid_{p=0} = \sum_{i=1}^{2} k_i u_i \geq 0 \;\; \forall \;\; u_i \geq 0, \qquad \dot{z}\mid_{z=0} = 0.$$

This shows the nonnegativity of the model's solutions. Now we let $G_i(t) = s_i(t) + w_i(t) + u_i(t)$, then

$$\dot{G}_i = \rho_i - \delta_i s_i - \alpha_i w_i - a_i u_i \leq \rho_i - \kappa_i (s_i + w_i + u_i) = \rho_i - \kappa_i G_i,$$

where $\kappa_i = \min\{\delta_i, \alpha_i, a_i\}, i = 1,2$. Hence $0 \leq G_i(t) \leq M_i$ where, $M_i = \frac{\rho_i}{\kappa_i}$. therefore $s_i(t), w_i(t)$ and $u_i(t)$ are all bonded. Let $G_3(t) = p(t) + \frac{\mu}{r} z(t)$, then

$$\dot{G}_3(t) = \sum_{i=1}^{2} k_i u_i - gp - \frac{\mu\zeta}{r} z \leq \sum_{i=1}^{2} k_i M_i - \kappa_3 \left(p + \frac{\mu}{r} z\right) = \sum_{i=1}^{2} k_i M_i - \kappa_3 G_3(t),$$

where $\kappa_3 = \min\{g, \zeta\}$. Hence $p(t) \leq M_3$ and $z(t) \leq M_4$ for $t \geq 0$ where, $M_3 = \frac{1}{\kappa_3}\sum_{i=1}^{2} k_i M_i$ and $M_4 = \frac{rM_3}{\mu}$. So that, there is a bounded subset of D

$$\Gamma = \{(s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in H : 0 \leq s_i + w_i + u_i \leq M_i, 0 \leq p \leq M_3, 0 \leq w \leq M_4\}.$$

is positively invariant with respect to system (9)-(13).

Lemma 2. For system (9)-(13) there exist two bifurcation parameters R_0 and R_1 with $R_0 > R_1$ such that
(i) if $R_0 \leq 1$, then the system has only one steady state Π_0,
(ii) if $R_1 \leq 1 < R_0$, then the system has only two steady states Π_0 and Π_1,
(iii) if $R_1 > 1$, then the system has three steady states Π_0,$\Pi_1$and Π_2.
Proof. Let

$$\rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i + p} = 0, \tag{14}$$

$$\frac{\lambda_i s_i p}{s_i + p} - (\alpha_i + \beta_i) w_i = 0, \tag{15}$$

$$\beta_i w_i - a_i u_i = 0, \tag{16}$$

$$\sum_{i=1}^{2} k_i u_i - gp - \mu p z = 0, \tag{17}$$

$$rpz - \zeta z = 0. \tag{18}$$

Eq. (18) we obtain two possible solutions, $z = 0$ or $p = \frac{\zeta}{r}$. First, we consider the case $z = 0$, then from Eqs. (15)-(16) we can get:

$$w_i = \frac{\lambda_i s_i p}{(\alpha_i + \beta_i)(s_i + p)}, \quad u_i = \frac{\lambda_i \beta_i s_i p}{a_i (\alpha_i + \beta_i)(s_i + p)}, \tag{19}$$

where $s_i^0 = \frac{\rho_i}{\delta_i}$. From Eq. (17) we obtain

$$\left(\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i s_i}{a_i g (\alpha_i + \beta_i)(s_i + p)} - 1\right) gp = 0. \tag{20}$$

Eq. (20) has two possible solutions $p = 0$ or $\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i s_i}{a_i g(\alpha_i+\beta_i)(s_i+p)} = 1$.

If $p = 0$, then substituting it in Eq. (19) leads to the uninfected steady state $\Pi_0 = (s_1^0, s_2^0, 0, 0, 0, 0, 0, 0)$. If $p \neq 0$, we have

$$\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i s_i}{a_i g (\alpha_i + \beta_i)(s_i + p)} - 1 = 0. \tag{21}$$

Eq. (14) implies that

$$s_i^{\pm} = \frac{1}{2}\left(\left(s_i^0 - \varphi_i p\right) \pm \sqrt{\left(\varphi_i p - s_i^0\right)^2 + 4 s_i^0 p} \right),$$

where, $s_i^0 = \frac{\rho_i}{\delta_i}, \varphi_i = \frac{\lambda_i}{\delta_i} + 1, i = 1, 2$. Clearly if $p > 0$ then $s_i^- < 0$ and $s_i^+ > 0$, then we choose $s_i = s_i^+$

$$s_i = \frac{1}{2}\left(\left(s_i^0 - \varphi_i p\right) + \sqrt{\left(\varphi_i p - s_i^0\right)^2 + 4 s_i^0 p} \right). \tag{22}$$

Substituting from Eqs. (14) and (19) into Eq. (17) we get

$$\sum_{i=1}^{2} \frac{k_i \beta_i}{d_{3i} (\alpha_i + \beta_i)} (\rho_i - \delta_i s_i) - gp = 0. \tag{23}$$

Since s_i is a function of p then from Eq. (23) we can define a function $H_1(p)$ as:

$$H_1(p) = \sum_{i=1}^{2} \frac{k_i \beta_i}{a_i (\alpha_i + \beta_i)} (\rho_i - \delta_i s_i(p)) - gp = 0. \tag{24}$$

We need to show that there exists a $p > 0$ such that $H_1(p) = 0$. It is clear that, if $p = 0$, then $s_i = s_i^0$ and $H_1(0) = 0$ and when $p = \hat{p} = \sum_{i=1}^{2} \frac{k_i \rho_i \beta_i}{a_i g(\alpha_i + \beta_i)} > 0$, we have $\hat{s}_i = s_i(\hat{p}) > 0$ and

$$H_1(\hat{p}) = -\sum_{i=1}^{2} \frac{k_i \beta_i \delta_i \hat{s}_i}{a_i g (\alpha_i + \beta_i)} < 0.$$

Since $H_1(p)$ is continuous for all $p \geq 0$, we obtain

$$H_1^{'}(0) = g \left(\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g (\alpha_i + \beta_i)} - 1 \right).$$

Therefore, $H_1^{'}(0) > 0$, if

$$\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g (\alpha_i + \beta_i)} > 1 \tag{25}$$

It means that if condition (25) is satisfied, then there exists $\tilde{p} \in (0, \hat{p})$ such that $H_1(\tilde{p}) = 0$. From Eqs. (19) and (22), we have $\tilde{s}_i, \tilde{w}_i, \tilde{u}_i, \tilde{p} > 0$. Thus, an infection steady state without antibodies $\Pi_1 = (\tilde{s}_1, \tilde{s}_2, \tilde{w}_1, \tilde{w}_2, \tilde{u}_1, \tilde{u}_2, \tilde{p}, 0)$ exists when $\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g(\alpha_i + \beta_i)} > 1$.Now we can define

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g (\alpha_i + \beta_i)},$$

Now if $z \neq 0$, then from Eqs. Eqs. (14)-(16),

$$\bar{s}_i = \frac{1}{2} \left[\left(s_i^0 - \varphi_i \frac{\zeta}{r} \right) + \sqrt{\left(\varphi_i \frac{\zeta}{r} - s_i^0 \right)^2 + \frac{4 \zeta s_i^0}{r}} \right], \qquad \bar{w}_i = \frac{\lambda_i \bar{s}_i \bar{p}}{(\alpha_i + \beta_i)(\bar{s}_i + \bar{p})},$$

$$\bar{u}_i = \frac{\lambda_i \beta_i \bar{s}_i \bar{p}}{a_i (\alpha_i + \beta_i)(\bar{s}_i + \bar{p})}, \qquad \bar{z} = \frac{g}{\mu} \left(\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i \bar{s}_i}{a_i g (\alpha_i + \beta_i)(\bar{s}_i + \bar{p})} - 1 \right),$$

Thus, $\bar{z} > 0$ when $\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i \bar{s}_i}{a_i g(\alpha_i + \beta_i)(\bar{s}_i + \bar{p})} > 1$. Let us define the parameter R_1 as:

$$R_1 = \sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i \bar{s}_i}{a_i g (\alpha_i + \beta_i)(\bar{s}_i + \bar{p})},$$

If $R_1 > 1$, then $\bar{z} = \frac{g}{\mu}(R_1 - 1) > 0$ and exists an infection steady state with antibodies $\Pi_2 = (\bar{s}_1, \bar{s}_2, \bar{w}_1, \bar{w}_2, \bar{u}_1, \bar{u}_2, \bar{p}, \bar{z})$ if $R_1 > 1$.

2.2 Global properties

We will use the following function throughout the paper, $F : (0, \infty) \longrightarrow [0, \infty)$ as $F(q) = q - 1 - \ln q$.

Theorem 1. The steady state Π_0 is globally asymptotically stable when $R_0 \leq 1$.

Proof. Define

$$W_{01} = \sum_{i=1}^{2} \gamma_i \left[w_i + \frac{(\alpha_i + \beta_i)}{\beta_i} u_i \right] + p + \frac{\zeta}{r} z,$$

where, $\gamma_i = \frac{k_i \beta_i}{a_i(\alpha_i + \beta_i)}$. We evaluate $\frac{dW_{01}}{dt}$ along the solutions of system (9)-(13) as:

$$\begin{aligned} \frac{dW_{01}}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\dot{w}_i + \frac{(\alpha_i + \beta_i)}{\beta_i} \dot{u}_i \right] + \dot{p} + \frac{\mu}{r} \dot{z} \\ &= \sum_{i=1}^{2} \gamma_i \left[\frac{\lambda_i s_i p}{s_i + p} - (\alpha_i + \beta_i) w_i + \frac{(\alpha_i + \beta_i)}{\beta_i} (\beta_i w_i - a_i u_i) \right] + \sum_{i=1}^{2} k_i u_i - gp - \mu p z + \frac{\mu}{r} (rpz - \zeta z). \end{aligned} \tag{26}$$

Eq. (26) can be simplified as

$$\begin{aligned} \frac{dW_{01}}{dt} &= \sum_{i=1}^{2} \gamma_i \frac{\lambda_i s_i p}{s_i + p} - gp - \frac{\mu \zeta}{r} z \leq \sum_{i=1}^{2} \gamma_i \lambda_i p - gp - \frac{\mu \zeta}{r} z \\ &= g \left(\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g (\alpha_i + \beta_i)} - 1 \right) p - \frac{\mu \zeta}{r} z = g (R_0 - 1) p - \frac{\mu \zeta}{r} z. \end{aligned}$$

If $R_0 \leq 1$, then $\frac{dW_{01}}{dt} \leq 0$ holds in Γ. Moreover, $\frac{dW_{01}}{dt} = 0$ when $p = 0$ and $z = 0$. Hence the largest compact invariant set in Γ is

$$\begin{aligned} Q_1 &= \left\{ (s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in \Gamma \mid \frac{dW_{01}}{dt} = 0 \right\} \\ &= \{ (s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in \Gamma \mid p = 0, z = 0 \}. \end{aligned}$$

LaSalle's invariance principle yields $\lim_{t \to +\infty} p(t) = 0$ and $\lim_{t \to +\infty} z(t) = 0$. One can get limit equations:

$$\dot{s}_i = \rho_i - \delta_i s_i, \tag{27}$$

$$\dot{w}_i = -(\alpha_i + \beta_i) w_i, \tag{28}$$

$$\dot{u}_i = \beta_i w_i - a_i u_i. \tag{29}$$

Define a function W_{02} by

$$W_{02} = \sum_{i=1}^{2} \gamma_i \left[s_i^0 F\left(\frac{s_i}{s_i^0} \right) + w_i + \frac{(\alpha_i + \beta_i)}{\beta_i} u_i \right].$$

Then

$$\begin{aligned} \frac{dW_{02}}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{s_i^0}{s_i} \right) \dot{s}_i + \dot{w}_i + \frac{(\alpha_i + \beta_i)}{\beta_i} \dot{u}_i \right] \\ &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{s_i^0}{s_i} \right) (\rho_i - \delta_i s_i) - (\alpha_i + \beta_i) w_i + \frac{(\alpha_i + \beta_i)}{\beta_i} (\beta_i w_i - a_i u_i) \right] \\ &= -\sum_{i=1}^{2} \gamma_i \delta_i \frac{(s_i - s_i^0)^2}{s_i} - \sum_{i=1}^{2} k_i u_i. \end{aligned}$$

Therefore, $\frac{dW_{02}}{dt} \leq 0$ holds in Q_1 and $\frac{dW_{02}}{dt} = 0$ if and only if $s_i = s_i^0$ and $u_i = 0$. There is the largest compact invariant set in Q_1:

$$\begin{aligned} Q_2 &= \left\{ (s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in Q_1 \mid \frac{dW_{02}}{dt} = 0 \right\} \\ &= \left\{ (s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in Q_1 \mid s_i = s_i^0, w_i \geq 0, u_i = 0 \right\}. \end{aligned}$$

In Q_2, from Eq. (29) we get $\beta_i w_i - a_i(0) = 0$, and then $w_i = 0$. So

$$\begin{aligned} Q_2 &= \left\{ (s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in Q_1 \mid \frac{dW_{02}}{dt} = 0 \right\} \\ &= \left\{ (s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in Q_1 \mid s_i = s_i^0, w_i = 0, u_i = 0 \right\} \\ &= \{\Pi_0\}. \end{aligned}$$

Hence, if $R_0 \le 1$, all solution trajectories in Γ approach the uninfected steady state Π_0.

Theorem 2. The steady state Π_1 is globally asymptotically stable when $R_1 \le 1 < R_0$.

Proof. We introduce

$$W_1 = \sum_{i=1}^{2} \gamma_i \left[s_i - \tilde{s}_i - \int_{\tilde{s}_i}^{s_i} \frac{(\alpha_i + \beta_i)\tilde{w}_i(\tau + \tilde{p})}{\lambda_i \tau \tilde{p}} d\tau + \tilde{w}_i F\left(\frac{w_i}{\tilde{w}_i}\right) + \frac{(\alpha_i + \beta_i)}{\beta_i} \tilde{u}_i F\left(\frac{u_i}{\tilde{u}_i}\right) \right] + \tilde{p} F\left(\frac{p}{\tilde{p}}\right) + \frac{\mu}{r} z.$$

Evaluating $\frac{dW_1}{dt}$ along the trajectories of system (9)-(13):

$$\begin{aligned} \frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{(\alpha_i + \beta_i)\tilde{w}_i(s_i + \tilde{p})}{\lambda_i s_i \tilde{p}}\right) \dot{s}_i + \left(1 - \frac{\tilde{w}_i}{w_i}\right) \dot{w}_i + \frac{(\alpha_i + \beta_i)}{\beta_i} \left(1 - \frac{\tilde{u}_i}{u_i}\right) \dot{u}_i \right] + \left(1 - \frac{\tilde{p}}{p}\right) \dot{p} + \frac{\mu}{r} \dot{z} \\ &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{(\alpha_i + \beta_i)\tilde{w}_i(s_i + \tilde{p})}{\lambda_i s_i \tilde{p}}\right) \left(\rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i + p}\right) + \left(1 - \frac{\tilde{w}_i}{w_i}\right) \left(\frac{\lambda_i s_i p}{s_i + p} - (\alpha_i + \beta_i) w_i\right) \right. \\ &\quad \left. + \frac{(\alpha_i + \beta_i)}{\beta_i} \left(1 - \frac{\tilde{u}_i}{u_i}\right) (\beta_i w_i - a_i u_i) \right] + \left(1 - \frac{\tilde{p}}{p}\right) \left(\sum_{i=1}^{2} k_i u_i - gp - \mu p z\right) + \frac{\mu}{r} (rpz - \zeta z). \end{aligned} \tag{30}$$

Simplify Eq. (30) as:

$$\begin{aligned} \frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\rho_i - \delta_i s_i - \frac{(\alpha_i + \beta_i)\tilde{w}_i(s_i + \tilde{p})}{\lambda_i s_i \tilde{p}} \left(\rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i + p}\right) - \frac{\lambda_i s_i p}{s_i + p} \frac{\tilde{w}_i}{w_i} + (\alpha_i + \beta_i) \tilde{w}_i \right. \\ &\quad \left. - (\alpha_i + \beta_i) w_i \frac{\tilde{u}_i}{u_i} + \frac{a_i(\alpha_i + \beta_i)}{\beta_i} \tilde{u}_i - \frac{a_i(\alpha_i + \beta_i)}{\beta_i} u_i \frac{\tilde{p}}{p} \right] - gp + g\tilde{p} + \mu \left(\tilde{p} - \frac{\zeta}{r}\right) z. \end{aligned} \tag{31}$$

From the conditions of Π_1, we obtain

$$\begin{aligned} &\rho_i = \delta_i \tilde{s}_i + (\alpha_i + \beta_i) \tilde{w}_i, \quad \frac{\lambda_i \tilde{s}_i \tilde{p}}{\tilde{s}_i + \tilde{p}} = (\alpha_i + \beta_i) \tilde{w}_i, \quad \frac{a_i(\alpha_i + \beta_i)}{\beta_i} \tilde{u}_i = (\alpha_i + \beta_i) \tilde{w}_i, \\ &g\tilde{p} = \sum_{i=1}^{2} k_i \tilde{u}_i, \quad \lambda_i = \frac{(\alpha_i + \beta_i)\tilde{w}_i(\tilde{s}_i + \tilde{p})}{\tilde{s}_i \tilde{p}}, \quad \frac{(\alpha_i + \beta_i)\tilde{w}_i(s_i + \tilde{p})}{\lambda_i s_i \tilde{p}} = \frac{\tilde{s}_i(s_i + \tilde{p})}{s_i(\tilde{s}_i + \tilde{p})}, \end{aligned}$$

then, we have

$$\begin{aligned} \frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\delta_i \tilde{s}_i \left(1 - \frac{s_i}{\tilde{s}_i} - \frac{\tilde{s}_i(s_i + \tilde{p})}{s_i(\tilde{s}_i + \tilde{p})} + \frac{s_i + \tilde{p}}{\tilde{s}_i + \tilde{p}}\right) + (\alpha_i + \beta_i) \tilde{w}_i \left(-1 - \frac{p}{\tilde{p}} + \frac{p(s_i + \tilde{p})}{\tilde{p}(s_i + p)} + \frac{s_i + p}{s_i + \tilde{p}}\right) \right. \\ &\quad \left. + (\alpha_i + \beta_i) \tilde{w}_i \left(5 - \frac{\tilde{s}_i(s_i + \tilde{p})}{s_i(\tilde{s}_i + \tilde{p})} - \frac{s_i \tilde{w}_i p(\tilde{s}_i + \tilde{p})}{\tilde{s}_i w_i \tilde{p}(s_i + p)} - \frac{w_i \tilde{u}_i}{\tilde{w}_i u_i} - \frac{u_i \tilde{p}}{\tilde{u}_i p} - \frac{s_i + p}{s_i + \tilde{p}}\right) \right] + \mu(\tilde{p} - \bar{p}) z. \end{aligned} \tag{32}$$

Eq. (32) becomes

$$\begin{aligned} \frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[-\frac{\delta_i \tilde{p}(s_i - \tilde{s}_i)^2}{s_i(s_i + p)} - (\alpha_i + \beta_i) \tilde{w}_i \frac{s_i(p - \tilde{p})^2}{\tilde{p}(s_i + p)(s_i + \tilde{p})} \right. \\ &\quad \left. + (\alpha_i + \beta_i) \tilde{w}_i \left(5 - \frac{\tilde{s}_i(s_i + \tilde{p})}{s_i(\tilde{s}_i + \tilde{p})} - \frac{s_i \tilde{w}_i p(\tilde{s}_i + \tilde{p})}{\tilde{s}_i w_i \tilde{p}(s_i + p)} - \frac{w_i \tilde{u}_i}{\tilde{w}_i u_i} - \frac{u_i \tilde{p}}{\tilde{u}_i p} - \frac{s_i + p}{s_i + \tilde{p}}\right) \right] + \mu(\tilde{p} - \bar{p}) z. \end{aligned}$$

Using the rule

$$\frac{1}{n}\sum_{i=1}^{n}a_i \geq \sqrt[n]{\prod_{i=1}^{n}a_i},$$

we obtain

$$\frac{\tilde{s}_i\left(s_i+\tilde{p}\right)}{s_i\left(\tilde{s}_i+\tilde{p}\right)}+\frac{s_i\tilde{w}_ip\left(\tilde{s}_i+\tilde{p}\right)}{\tilde{s}_iw_i\tilde{p}\left(s_i+p\right)}+\frac{w_i\tilde{u}_i}{\tilde{w}_iu_i}+\frac{u_i\tilde{p}}{\tilde{u}_ip}+\frac{s_i+p}{s_i+\tilde{p}}-5\geq 0.$$

Now we show that if $R_1 \leq 1$ then $\tilde{p} \leq \frac{\zeta}{r} = \bar{p}$. This can be shown if we prove that

$$sgn\left(\bar{s}_i-\tilde{s}_i\right)=sgn\left(\tilde{p}-\bar{p}\right)=sgn\left(R_1-1\right).$$

Suppose that, $sgn\left(\bar{p}-\tilde{p}\right)=sgn\left(\bar{s}_i-\tilde{s}_i\right)$.

$$\left(\rho_i-\delta_i\bar{s}_i\right)-\left(\rho_i-\delta_i\tilde{s}_i\right)=\frac{\lambda_i\bar{s}_i\bar{p}}{\bar{s}_i+\bar{p}}-\frac{\lambda_i\tilde{s}_i\tilde{p}}{\tilde{s}_i+\tilde{p}}=\lambda_i\left[\frac{\left(\bar{p}-\tilde{p}\right)s_i^2}{\left(\bar{s}_i+\bar{p}\right)\left(\bar{s}_i+\tilde{p}\right)}+\frac{\left(\bar{s}_i-\tilde{s}_i\right)\tilde{p}^2}{\left(\bar{s}_i+\tilde{p}\right)\left(\tilde{s}_i+\tilde{p}\right)}\right].$$

This yields, $sgn\left(\tilde{s}_i-\bar{s}_i\right)=sgn\left(\bar{s}_i-\tilde{s}_i\right)$, which leads to contradiction and then $sgn\left(\tilde{p}-\bar{p}\right)=sgn\left(\bar{s}_i-\tilde{s}_i\right)$. Using the condition for the steady state Π_1 we have $\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i\tilde{s}_i}{a_ig(\alpha_i+\beta_i)(\tilde{s}_i+\tilde{p})}=1$, then

$$\begin{aligned}R_1-1&=\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i\bar{s}_i}{a_ig\left(\alpha_i+\beta_i\right)\left(\bar{s}_i+\bar{p}\right)}-\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i\tilde{s}_i}{a_ig\left(\alpha_i+\beta_i\right)\left(\tilde{s}_i+\tilde{p}\right)}\\&=\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i}{a_ig\left(\alpha_i+\beta_i\right)}\left(\frac{\left(\bar{s}_i-\tilde{s}_i\right)\tilde{p}+\left(\tilde{p}-\bar{p}\right)\tilde{s}_i}{\left(\bar{s}_i+\bar{p}\right)\left(\tilde{s}_i+\tilde{p}\right)}\right).\end{aligned}\tag{33}$$

From (33) we get $sgn\left(R_1-1\right)=sgn\left(\tilde{p}-\bar{p}\right)$. So that, if $R_1\leq 1$ then $\tilde{p}\leq\frac{\zeta}{r}=\bar{p}$. So that, if $R_1\leq 1$ then $\frac{dW_1}{dt}\leq 0$ holds in Γ and $\frac{dW_1}{dt}=0$ when $s_i=\tilde{s}_i, w_i=\tilde{w}_i, u_i=\tilde{u}_i, p=\tilde{p}, z=0$. Hence the largest compact invariant subset in Γ is

$$\begin{aligned}Q_3&=\left\{\left(s_1,s_2,w_1,w_2,u_1,u_2,p,z\right)\in\Gamma \mid \frac{dW_1}{dt}=0\right\}\\&=\left\{\left(s_1,s_2,w_1,w_2,u_1,u_2,p,z\right)\in\Gamma \mid s_i=\tilde{s}_i,w_i=\tilde{w}_i,u_i=\tilde{u}_i,p=\tilde{p},z=0\right\}\\&=\left\{\Pi_1\right\}.\end{aligned}$$

It follows that, if $R_1\leq 1$ then Π_1 is GAS in Γ by LIP.

Theorem 3. The steady state Π_2 is globally asymptotically stable when $R_1>1$.

Proof. Define

$$W_2=\sum_{i=1}^{2}\gamma_i\left[s_i-\bar{s}_i-\int_{\bar{s}_i}^{s_i}\frac{\left(\alpha_i+\beta_i\right)\bar{w}_i\left(\tau+\bar{p}\right)}{\lambda_i\tau\bar{p}}d\tau+\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right)+\frac{\left(\alpha_i+\beta_i\right)}{\beta_i}\bar{u}_iF\left(\frac{u_i}{\bar{u}_i}\right)\right]+\bar{p}F\left(\frac{p}{\bar{p}}\right)+\frac{\mu}{r}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

Then $\frac{dW_2}{dt}$ is given as:

$$\begin{aligned}\frac{dW_2}{dt}&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\left(\alpha_i+\beta_i\right)\bar{w}_i\left(s_i+\bar{p}\right)}{\lambda_is_i\bar{p}}\right)\dot{s}_i+\left(1-\frac{\bar{w}_i}{w_i}\right)\dot{w}_i+\frac{\left(\alpha_i+\beta_i\right)}{\beta_i}\left(1-\frac{\bar{u}_i}{u_i}\right)\dot{u}_i\right]+\left(1-\frac{\bar{p}}{p}\right)\dot{p}+\frac{\mu}{r}\left(1-\frac{\bar{z}}{z}\right)\dot{z}\\&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\left(\alpha_i+\beta_i\right)\bar{w}_i\left(s_i+\bar{p}\right)}{\lambda_is_i\bar{p}}\right)\left(\rho_i-\delta_is_i-\frac{\lambda_is_ip}{s_i+p}\right)+\left(1-\frac{\bar{w}_i}{w_i}\right)\left(\frac{\lambda_is_ip}{s_i+p}-\left(\alpha_i+\beta_i\right)w_i\right)\right.\\&\quad\left.+\frac{\left(\alpha_i+\beta_i\right)}{\beta_i}\left(1-\frac{\bar{u}_i}{u_i}\right)\left(\beta_iw_i-a_iu_i\right)\right]+\left(1-\frac{\bar{p}}{p}\right)\left(\sum_{i=1}^{2}k_iu_i-gp-\mu pz\right)+\frac{\mu}{r}\left(1-\frac{\bar{z}}{z}\right)\left(rpz-\zeta z\right).\end{aligned}\tag{34}$$

Eq. (34) can be simplified as:

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\rho_i-\delta_i s_i-\frac{(\alpha_i+\beta_i)\bar{w}_i(s_i+\bar{p})}{\lambda_i s_i\bar{p}}\left(\rho_i-\delta_i s_i-\frac{\lambda_i s_i p}{s_i+p}\right)-\frac{\lambda_i s_i p}{s_i+p}\frac{\bar{w}_i}{w_i}+(\alpha_i+\beta_i)\bar{w}_i\right.$$
$$\left.-(\alpha_i+\beta_i)w_i\frac{\bar{u}_i}{u_i}+\frac{a_i(\alpha_i+\beta_i)}{\beta_i}\bar{u}_i-\frac{a_i(\alpha_i+\beta_i)}{\beta_i}u_i\frac{\bar{p}}{p}\right]-gp+g\bar{p}-\mu p\bar{z}+\frac{\mu\zeta}{r}\bar{z}.$$

Using conditions of Π_2 we get

$$\rho_i=\delta_i\bar{s}_i+(\alpha_i+\beta_i)\bar{w}_i,\quad \frac{a_i(\alpha_i+\beta_i)}{\beta_i}\bar{u}_i=(\alpha_i+\beta_i)\bar{w}_i,\quad \frac{(\alpha_i+\beta_i)\bar{w}_i(s_i+\bar{p})}{\lambda_i s_i\bar{p}}=\frac{\bar{s}_i(s_i+\bar{p})}{s_i(\bar{s}_i+\bar{p})},$$
$$\lambda_i=\frac{(\alpha_i+\beta_i)\bar{w}_i(\bar{s}_i+\bar{p})}{\bar{s}_i\bar{p}},\quad g\bar{p}=\sum_{i=1}^{2}k_i\bar{u}_i-\mu\bar{p}\bar{z},\quad gp=\frac{p}{\bar{p}}\sum_{i=1}^{2}k_i\bar{u}_i-\mu p\bar{z}$$

Then, we have

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\delta_i\bar{s}_i\left(1-\frac{s_i}{\bar{s}_i}-\frac{\bar{s}_i(s_i+\bar{p})}{s_i(\bar{s}_i+\bar{p})}+\frac{s_i+\bar{p}}{\bar{s}_i+\bar{p}}\right)+(\alpha_i+\beta_i)\bar{w}_i\left(-1-\frac{p}{\bar{p}}+\frac{p(s_i+\bar{p})}{\bar{p}(s_i+p)}+\frac{s_i+p}{s_i+\bar{p}}\right)\right.$$
$$\left.+(\alpha_i+\beta_i)\bar{w}_i\left(5-\frac{\bar{s}_i(s_i+\bar{p})}{s_i(\bar{s}_i+\bar{p})}-\frac{\bar{w}_i s_i p(\bar{s}_i+\bar{p})}{w_i\bar{s}_i\bar{p}(s_i+p)}-\frac{w_i\bar{u}_i}{\bar{w}_i u_i}-\frac{\bar{p}u_i}{p\bar{u}_i}-\frac{s_i+p}{s_i+\bar{p}}\right)\right].\tag{35}$$

Eq. (35) becomes

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[-\frac{\delta_i\bar{p}(s_i-\bar{s}_i)^2}{s_i(\bar{s}_i+\bar{p})}-(\alpha_i+\beta_i)\bar{w}_i\frac{s_i(p-\bar{p})^2}{\bar{p}(s_i+p)(s_i+\bar{p})}\right.$$
$$\left.+(\alpha_i+\beta_i)\bar{w}_i\left(5-\frac{\bar{s}_i(s_i+\bar{p})}{s_i(\bar{s}_i+\bar{p})}-\frac{s_i\bar{w}_i p(\bar{s}_i+\bar{p})}{\bar{s}_i w_i\bar{p}(s_i+p)}-\frac{w_i\bar{u}_i}{\bar{w}_i u_i}-\frac{u_i\bar{p}}{\bar{u}_i p}-\frac{s_i+p}{s_i+\bar{p}}\right)\right].$$

It follows that, $\frac{dW_2}{dt}\leq 0$ for all $s_i,w_i,u_i,p,z>0$ and $\frac{dW_2}{dt}=0$ when $s_i=\bar{s}_i,w_i=\bar{w}_i,u_i=\bar{u}_i,p=\bar{p},z=\bar{z}$. Hence

$$\begin{aligned}Q_4&=\left\{(s_1,s_2,w_1,w_2,u_1,u_2,p,z)\in\Gamma\mid\frac{dW_2}{dt}=0\right\}\\&=\{(s_1,s_2,w_1,w_2,u_1,u_2,p,z)\in\Gamma\mid s_i=\bar{s}_i,w_i=\bar{w}_i,u_i=\bar{u}_i,p=\bar{p},z=\bar{z}\}\\&=\{\Pi_2\}.\end{aligned}$$

It follows that, if $R_1>1$ then Π_2 is GAS in Γ by LIP.

3 Simulations

We support our results by numerical simulations using the values of the parameters given in Table 1.

For the parameters $\bar{\lambda}_1$, $\bar{\lambda}_2$ and r we have three cases to show its effect on the stability of the system. We assume that $\varepsilon_1=\varepsilon_2=0$ (there is no treatment). The initial condtions are considered to be: $s_1(0)=500, s_2(0)=20, w_1(0)=1, w_2(0)=0.3, u_1(0)=20, u_2(0)=0.2, p(0)=90, z(0)=40$.

Case (I) $R_0\leq 1$. We consider $\bar{\lambda}_1=0.002,\bar{\lambda}_2=0.00001$ and $r=0.0001$. Then, $R_0=0.2469<1$ and $R_1=0.1062<1$. This means that Π_0 is GAS. From Figures 1-8 we can see that the trajectory of the system converges the steady state $\Pi_0(830,24.6,0,0,0,0,0,0)$.

Case (II) $R_1\leq 1<R_0$. Choosing $\bar{\lambda}_1=0.02,\bar{\lambda}_2=0.0005$ and $r=0.0001$. In this case, $R_0=2.5694$ and $R_1=0.7141<1$ and Π_1 exists with $\Pi_1=(448.116,17.9,2.949,0.436,32.439,0.218,650.956,0)$. According to Theorem 2, Π_1 is GAS. Figures 1-8 show the validity of the theoretical results of Theorem 2.

Table 1: The values of parameters of the models.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
ρ_1	11.537	ρ_2	0.03198	k_1	10	k_2	5
δ_1	0.0139	δ_2	0.001	g	0.5	μ	0.01
α_1	0.57	α_2	0.5	ζ	0.05	f	0.5
a_1	0.1	a_2	0.02	h	0.5	λ_1	varied
β_1	1.1	β_2	0.01	λ_2	varied	r	varied

Case (III) $R_1 > 1$. We take $\bar{\lambda}_1 = 0.02, \bar{\lambda}_2 = 0.0005$ and $r = 0.002$. Then we get $R_0 = 2.5694 > 1$ and $R_1 = 2.3288 > 1$. Figures 1-8 show that, the steady state $\Pi_2(762.485, 19.254, 0.521, 0.348, 5.735, 0.174, 50, 66.438)$ is GAS which confirm the results of Theorem 3.

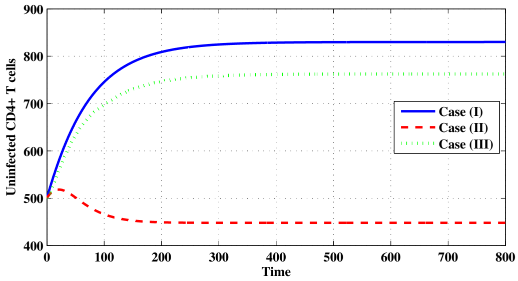

Figure 1: The concentration of uninfected CD4$^+$T cells.

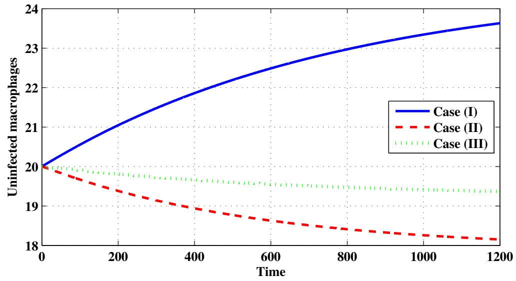

Figure 2: The concentration of uninfected macrophages.

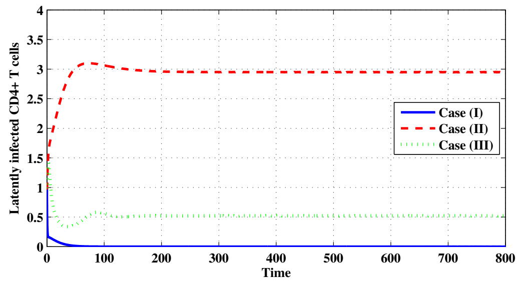

Figure 3: The concentration of latently infected CD4$^+$T cells.

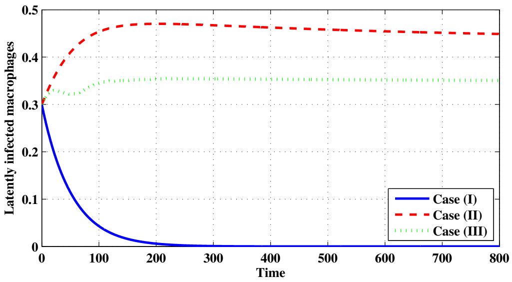

Figure 4: The concentration of latently infected macrophages.

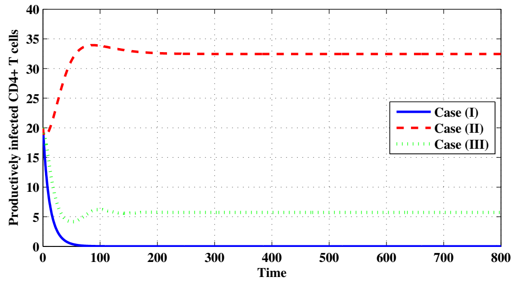

Figure 5: The concentration of productively infected CD4$^+$T cells.

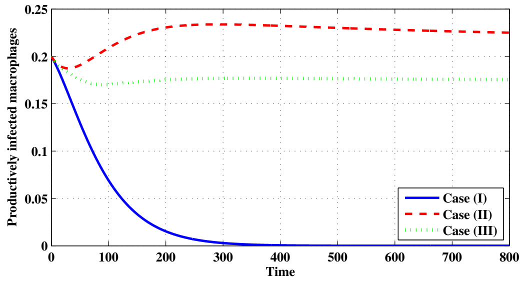

Figure 6: The concentration of productively infected macrophages.

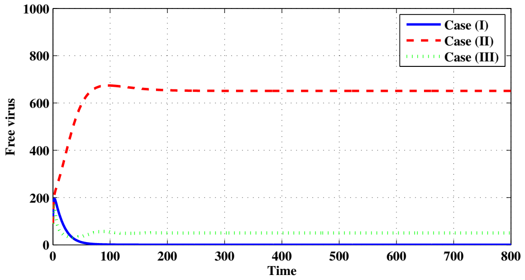

Figure 7: The concentration of HIV.

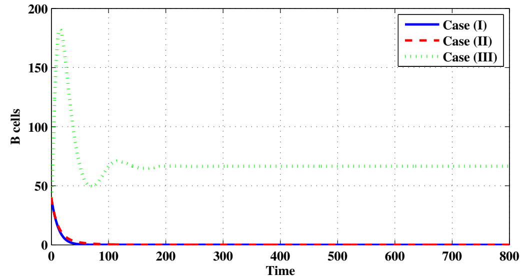

Figure 8: The concentration of B cells.

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FOURIER SERIES OF TWO VARIABLE HIGHER-ORDER FUBINI FUNCTIONS

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Abstract. In this paper, we consider the two variable higher-order Fubini functions and investigate their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions and obtain as a consequence the corresponding polynomial identities for the two variable higher-order Fubini polynomials.

1. Introduction

For each nonnegative integer r, the two variable Fubini polynomials $F_m^{(r)}(x;y)$ of order r are defined by

$$\frac{e^{xt}}{(1-y(e^t-1))^r}=\sum_{m=0}^{\infty}F_m^{(r)}(x;y)\frac{t^m}{m!},\quad \text{(see [4, 7])}. \tag{1.1}$$

However, in this paper y will be an arbitrary but fixed nonzero real number, and hence $F_m^{(r)}(x;y)$ are polynomials in x, for each $0\neq y\in\mathbb{R}$.

In the case of $r=1$, $F_m(x;y)=F_m^{(1)}(x;y)$ are called two variable Fubini polynomials and they were introduced by Kilar and Simsek in [4]. For $x=0$, $F_m^{(r)}(y)=F_m^{(r)}(0;y)$ are called Fubini polynomials of order r, and $F_m^{(r)}=F_m^{(r)}(1)=F_m^{(r)}(0;1)$ Fubini numbers of order r. Further, $F_m^{(r)}(x;1)$ are called ordered Bell polynomials of order r and they are denoted by $Ob_m^{(r)}(x)$; $F_m^{(r)}(1)=F_m^{(r)}(0;1)$ are also called ordered Bell numbers of order r and they are also denoted by $Ob_m^{(r)}$. Thus $Ob_m^{(r)}(x)$ and $Ob_m^{(r)}$ are respectively given by

$$\frac{e^{xt}}{(2-e^t)^r}=\sum_{m=0}^{\infty}Ob_m^{(r)}(x)\frac{t^m}{m!}, \tag{1.2}$$

$$\frac{1}{(2-e^t)^r}=\sum_{m=0}^{\infty}Ob_m^{(r)}\frac{t^m}{m!}, \tag{1.3}$$

(see [1, 3, 5]).

As we see from (1.1), $F_m^{(r)}(x;y)$ are Appell polynomials and hence

$$\frac{d}{dx}F_m^{(r)}(x;y)=mF_{m-1}^{(r)}(x;y),\qquad (m\geq 1). \tag{1.4}$$

Also, we have

$$yF_m^{(r)}(x+1;y)=(y+1)F_m^{(r)}(x;y)-F_m^{(r-1)}(x;y),\qquad (m\geq 0). \tag{1.5}$$

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Indeed,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(F_m^{(r)}(x+1;y)-F_m^{(r)}(x;y)\right)\frac{t^m}{m!}\\
&=\frac{e^{xt}(e^t-1)}{(1-y(e^t-1))^r}\\
&=\frac{1}{y}\left(\frac{e^{xt}}{(1-y(e^t-1))^r}-\frac{e^{xt}}{(1-y(e^t-1))^{r-1}}\right)\\
&=\frac{1}{y}\sum_{m=0}^{\infty}\left(F_m^{(r)}(x;y)-F_m^{(r-1)}(x;y)\right)\frac{t^m}{m!}.
\end{aligned}\tag{1.6}$$

The identity (1.5) follows from this. In turn, from (1.4) and (1.5), we obtain

$$F_m^{(r)}(1;y)-F_m^{(r)}(y)=\frac{1}{y}\left(F_m^{(r)}(y)-F_m^{(r-1)}(y)\right),\tag{1.7}$$

$$\begin{aligned}
\int_0^1 F_m^{(r)}(x;y)dx&=\frac{1}{m+1}\left(F_{m+1}^{(r)}(1;y)-F_{m+1}^{(r)}(y)\right)\\
&=\frac{1}{(m+1)y}\left(F_{m+1}^{(r)}(y)-F_{m+1}^{(r-1)}(y)\right).
\end{aligned}\tag{1.8}$$

As is well-known, the Bernoulli polynomials $B_m(x)$ are given by

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\quad \text{(see [2])}.\tag{1.9}$$

For any real number x, the fractional part of x is denoted by $< x >= x-[x]\in[0,1)$. We also need the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m\geq 2$,

$$B_m(< x >)=-m!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},\tag{1.10}$$

(b) for $m=1$,

$$-\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{2\pi in}=\begin{cases}B_1(< x >), & \text{for } x\in\mathbb{R}-\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}\tag{1.11}$$

In this paper, we will consider the two variable higher-order Fubini functions $F_m^{(r)}(< x >;y)$, for each $0\neq y\in\mathbb{R}$, and derive their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions and obtain as a consequence the corresponding polynomial identities for the two variable higher-order Fubini polynomials. For some related to Fourier series, we refer the reader to [5,6,8].

2. Main results

In this section, we assume that $m\geq 1$, $r\geq 1$, and $0\neq y\in\mathbb{R}$. For convenience, we set

$$\Delta_m^{(r)}(y)=F_m^{(r)}(1;y)-F_m^{(r)}(y)=\frac{1}{y}\left(F_m^{(r)}(y)-F_m^{(r-1)}(y)\right).\tag{2.1}$$

We note here that

$$\begin{aligned} F_m^{(r)}(1;y) = F_m^{(r)}(y) &\Leftrightarrow \Delta_m^{(r)}(y) = 0 \\ &\Leftrightarrow F_m^{(r)}(y) = F_m^{(r-1)}(y), \end{aligned} \tag{2.2}$$

and

$$\int_0^1 F_m^{(r)}(x;y)dx = \frac{1}{m+1}\Delta_{m+1}^{(r)}(y). \tag{2.3}$$

Before we move on our discussion for Fourier series expansions of $F_m^{(r)}(< x >; y)$, in passing we note the following:

$$\frac{1}{(1-y)^r}F_m^{(r)}(\frac{y}{1-y}) = \sum_{k=0}^{\infty}\binom{r+k-1}{k}k^m y^k, \tag{2.4}$$

from which, by letting $y = \frac{1}{2}$, we get

$$Ob_m^{(r)} = F_m^{(r)}(1) = \frac{1}{2^r}\sum_{k=0}^{\infty}\binom{r+k-1}{k}\frac{k^m}{2^k}. \tag{2.5}$$

Indeed, we may see (2.4) from

$$\begin{aligned} \frac{1}{(1-y)^r}\sum_{m=0}^{\infty}F_m^{(r)}\left(\frac{y}{1-y}\right)\frac{t^m}{m!} &= (1-ye^t)^{-r} \\ &= \sum_{k=0}^{\infty}\binom{r+k-1}{k}y^k e^{kt} \\ &= \sum_{k=0}^{\infty}\binom{r+k-1}{k}y^k\sum_{m=0}^{\infty}k^m\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}\binom{r+k-1}{k}k^m y^k\right)\frac{t^m}{m!}. \end{aligned} \tag{2.6}$$

$F_m^{(r)}(< x >; y)$ is a periodic function on $\mathbb{R}$ with period 1 and piecewise C^∞. Further, in view of (2.2), $F_m^{(r)}(< x >; y)$ is continuous from those (r, m) with $\Delta_m^{(r)}(y) = 0$ (or equivalently $F_m^{(r)}(y) = F_m^{(r-1)}(y)$), and is discontinuous with jump disconitinuities at integers for those (r, m) with $\Delta_m^{(r)}(y) \neq 0$ (or equivalently $F_m^{(r)}(y) \neq F_m^{(r-1)}(y)$).

The Fourier series of $F_m^{(r)}(< x >; y)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m,r,y)}e^{2\pi inx} \tag{2.7}$$

where

$$\begin{aligned} C_n^{(m)} = C_n^{(m,r,y)} &= \int_0^1 F_m^{(r)}(< x >; y)e^{-2\pi inx}dx \\ &= \int_0^1 F_m^{(r)}(x;y)e^{-2\pi inx}dx. \end{aligned} \tag{2.8}$$

Now, we would like to determine the Fourier coefficients $C_n^{(m)}$.

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Case 1 : $n \neq 0$.

$$\begin{aligned} C_n^{(m)} &= \int_0^1 F_m^{(r)}(x;y) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left[F_m^{(r)}(x;y)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \left(\frac{\partial}{\partial x} F_m^{(r)}(x;y)\right) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}(F_m^{(r)}(1;y) - F_m^{(r)}(y)) + \frac{m}{2\pi i n}\int_0^1 F_{m-1}^{(r)}(x;y) e^{-2\pi i n x} dx \\ &= \frac{m}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m^{(r)}(y). \end{aligned} \tag{2.9}$$

Thus we have shown that

$$C_n^{(m)} = \frac{m}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m^{(r)}(y), \tag{2.10}$$

from which by induction on m we get

$$C_n^{(m)} = -\frac{1}{m+1}\sum_{j=1}^{m} \frac{(m+1)_j}{(2\pi i n)^j}\Delta_{m-j+1}^{(r)}(y). \tag{2.11}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 F_m^{(r)}(x;y)dx = \frac{1}{m+1}\Delta_{m+1}^{(r)}(y). \tag{2.12}$$

Assume first that $\Delta_m^{(r)}(y) = 0$. Then $F_m^{(r)}(1;y) = F_m^{(r)}(y)$. As $F_m^{(r)}(< x >; y)$ is piecewise C^∞ and continuous, the Fourier series of $F_m^{(r)}(< x >; y)$ converges uniformly to $F_m^{(r)}(< x >; y)$, and

$$\begin{aligned} & F_m^{(r)}(< x >; y) \\ &= \frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi i n)^j}\Delta_{m-j+1}^{(r)}(y)\right)e^{2\pi i n x} \\ &= \frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\ &= \frac{1}{m+1}\sum_{j=0, j\neq 1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y) B_j(< x >) \\ &\quad + \Delta_m^{(r)}(y) \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

We are ready to state our first result.

Theorem 2.1. *For positive integers r, l, and $0 \neq y \in \mathbb{R}$, we let*

$$\Delta_l^{(r)}(y) = F_l^{(r)}(1;y) - F_l^{(r)}(y) = \frac{1}{y}\left(F_l^{(r)}(y) - F_l^{(r-1)}(y)\right). \tag{2.14}$$

Assume that $\Delta_m^{(r)}(y) = 0$. Then we have the following.

(a) $F_m^{(r)}(<x>;y)$ *has the Fourier series expansion*

$$F_m^{(r)}(<x>;y) = \frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi i n)^j}\Delta_{m-j+1}^{(r)}(y)\right)e^{2\pi i n x}, \tag{2.15}$$

for all $x \in \mathbb{R}$*, where the convergence is uniform.*

(b)

$$F_m^{(r)}(<x>;y) = \frac{1}{m+1}\sum_{j=0, j\neq 1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>), \tag{2.16}$$

for all $x \in \mathbb{R}$.

Assume next that $\Delta_m^{(r)}(y) \neq 0$. Then $F_m^{(r)}(1;y) \neq F_m^{(r)}(y)$. Hence $F_m^{(r)}(<x>;y)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $F_m^{(r)}(<x>;y)$ converges pointwise to $F_m^{(r)}(<x>;y)$, for $x \in \mathbb{R} - \mathbb{Z}$, and converges to

$$\frac{1}{2}(F_m^{(r)}(y) + F_m^{(r)}(1;y)) = F_m^{(r)}(y) + \frac{1}{2}\Delta_m^{(r)}(y), \tag{2.17}$$

for $x \in \mathbb{Z}$. We are now ready to state our second result.

Theorem 2.2. *For positive integers* r, l*, and* $0 \neq y \in \mathbb{R}$*, we let*

$$\Delta_l^{(r)}(y) = F_l^{(r)}(1;y) - F_l^{(r)}(y) = \frac{1}{y}\left(F_l^{(r)}(y) - F_l^{(r-1)}(y)\right). \tag{2.18}$$

Assume that $\Delta_m^{(r)}(y) \neq 0$*. Then we have the following.*

(a)

$$\begin{aligned}&\frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi i n)^j}\Delta_{m-j+1}^{(r)}(y)\right)e^{2\pi i n x}\\ &= \begin{cases} F_m^{(r)}(<x>;y), & \text{for} \quad x \in \mathbb{R} - \mathbb{Z}, \\ F_m^{(r)}(y) + \frac{1}{2}\Delta_m^{(r)}(y), & \text{for} \quad x \in \mathbb{Z}. \end{cases}\end{aligned} \tag{2.19}$$

(b)

$$\frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>) = F_m^{(r)}(<x>;y) \tag{2.20}$$

for all $x \in \mathbb{R} - \mathbb{Z}$*;*

$$\frac{1}{m+1}\sum_{j=0, j\neq 1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>) = F_m^{(r)}(y) + \frac{1}{2}\Delta_m^{(r)}(y) \tag{2.21}$$

for all $x \in \mathbb{Z}$.

We remark that the case of $y = 1$ had been treated in the previous paper [?]. From Theorems 2.1 and 2.2, we have

$$F_m^{(r)}(<x>;y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>), \tag{2.22}$$

for all $x \in \mathbb{R} - \mathbb{Z}$ and $0 \neq y \in \mathbb{R}$. We immediately obtain the following polynomial identities from this observation.

Corollary 2.3. *We have the following polynomial identities for two variable higher-order Fubini polynomials*

(a) $F_m^{(r)}(x;y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(F_{m-j+1}^{(r)}(1;y) - F_{m-j+1}^{(r)}(0;y)\right)B_j(x)$,

(b) $yF_m^{(r)}(x;y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(F_{m-j+1}^{(r)}(y) - F_{m-j+1}^{(r-1)}(y)\right)B_j(x)$.

For $x = 0$, we have the following identities for higher-order Fubini polynomials.

Corollary 2.4. *We have the following polynomial identities for higher-order Fubini polynomials*

(a) $F_m^{(r)}(y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j\left(F_{m-j+1}^{(r)}(1;y) - F_{m-j+1}^{(r)}(0;y)\right)$,

(b) $yF_m^{(r)}(y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j\left(F_{m-j+1}^{(r)}(y) - F_{m-j+1}^{(r-1)}(y)\right)$.

Finally, for $y = 1$, we get the following identities for higher-order ordered Bell polynomials.

Corollary 2.5. *We have the following polynomial identities for higher-order ordered Bell polynomials*

(a) $Ob_m^{(r)}(x) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(Ob_{m-j+1}^{(r)}(1) - Ob_{m-j+1}^{(r)}\right)B_j(x)$,

(b) $Ob_m^{(r)}(x) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(Ob_{m-j+1}^{(r)} - Ob_{m-j+1}^{(r-1)}\right)B_j(x)$,

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WEIGHTED COMPOSITION OPERATORS FROM DIRICHLET TYPE SPACES TO SOME WEIGHTED-TYPE SPACES

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Abstract. Let $K : [0, \infty) \to [0, \infty)$ be a right continuous increasing function and $\nu : \mathbb{D} \to (0, \infty)$ be any continuous function. In this paper by considering K and ν as weight functions, we characterize the boundedness and compactness of weighted composition operators from Dirichlet type spaces to some weighted-type spaces.

1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk and $\partial\mathbb{D}$ be its boundary in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denotes the class of all holomorphic functions on $\mathbb{D}$, $S(\mathbb{D})$ be the class of all holomorphic self-maps of $\mathbb{D}$ and H^∞ be the space of all bounded analytic functions on $\mathbb{D}$. Let $dA(z) = \frac{dxdy}{\pi} = r\frac{drd\theta}{\pi}$ be the normalized area measure on $\mathbb{D}$.
A continuous function $\nu : \mathbb{D} \longrightarrow (0, \infty)$ is called weight. For $\nu(z) = \nu(|z|)$, $z \in \mathbb{D}$, weight is radial and weight is a standard weight if $\lim_{|z|\to 1^-} \nu(z) = 0$.
For weight ν, the *Bers-type space* $\mathcal{A}_\nu$ is the collection of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty$$

and with the norm

$$\|f\|_{\mathcal{A}_\nu} = \sup_{z\in\mathbb{D}} \nu(z)|f(z)|,$$

it is a non-separable Banach space. The closure of the set of polynomials in $\mathcal{A}_\nu$ forms a separable Banach space. This set is denoted by $\mathcal{A}_{\nu,0}$ and contains exactly of those $f \in \mathcal{A}_\nu$ such that

$$\lim_{|z|\to 1^-} \nu(z)|f(z)| = 0.$$

The *Bloch-type space* $\mathcal{B}_\nu$ on $\mathbb{D}$ with the weight ν is the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty.$$

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The *little Bloch-type space* $\mathcal{B}_{\nu,0}$ is the closure of the set of polynomials in $\mathcal{B}_\nu$ and contains all those $f \in \mathcal{B}_\nu$ such that

$$\lim_{|z|\to 1} \nu(z)|f'(z)| = 0$$

and with the norm

$$\|f\|_{\mathcal{B}_\nu} = |f(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty,$$

both $\mathcal{B}_\nu$ and $\mathcal{B}_{\nu,0}$ form Banach spaces.
For more information about these spaces one may refer [20] and references therein.

Let $\varphi \in S(\mathbb{D})$ and ψ be an analytic map on $\mathbb{D}$. The operator C_φ so called as the *composition operator* and is defined as $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. The operator M_ψ which is called as the *multiplication operator* is defined by $M_\psi f = \psi \cdot f$, $f \in H(\mathbb{D})$. For $f \in H(\mathbb{D})$, the *weighted composition operator* on $H(\mathbb{D})$ is defined by

$$(W_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)),$$

where $\psi \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $z \in \mathbb{D}$.

It can be easily seen that for $\psi \equiv 1$, the operator reduced to C_φ. If $\varphi(z) = z$, operator get reduced to M_ψ. This operator is basically a linear transformation of $H(\mathbb{D})$ defined by $(W_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)) = (M_\psi C_\varphi f)(z)$, for f in $H(\mathbb{D})$ and z in $\mathbb{D}$. The basic problem is to give the function-theoretic characterization when between various function spaces ψ and φ induce bounded or compact weighted composition operator. Various holomorphic functions spaces on various domains have been studied for the the boundedness and compactness of weighted composition operators acting on them. Moreover, a number of papers have been studied on these operators acting on different spaces of holomorphic functions on various domains for more detail (see [1], [5], [7]-[11], [13], [15], [19]).

Consider a function $K : [0, \infty) \to [0, \infty)$ which is right continuous and increasing. The *Dirichlet type space* $\mathcal{D}_K$ consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{D}_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K\big(1 - |z|^2\big) dA(z) < \infty.$$

For more about the Dirichlet type spaces we refer ([2], [3], [4], [12], [14], [16]). In this paper we consider function K as a weight function satisfying the following two conditions:

(a) $K_1(t) = \int_0^t K(s)\frac{ds}{s} \approx K(t)$, $0 < t < 1$;

(b) $K_2(t) = t\int_t^\infty K(s)\frac{ds}{s^2} \approx K(t)$, $t > 0$.

From condition (b), we get that $K(2t) \approx K(t)$ for $0 < t < 1$. Also there exist $C > 0$ sufficiently small for which $t^{-C}K_1(t)$ is increasing and $K_2(t)t^{C-1}$ is decreasing (see [4],

[17], [18]).
This paper is entirely devoted to characterize the boundedness and compactness of operator $W_{\psi,\varphi}$ from Dirichlet type spaces to the Bers-type space and Bloch-type space.
Throughout this paper, C will represents a constant which may differ from one occurrence to another. The notation $A \lesssim B$ means that there exist $C > 0$ such that $A \leq CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.
The paper is organized in a systematic manner. Section 1 covers the introduction and literature part. Lemmas that are used to formulate our main theorems are kept in Section 2. Section 3 contains the boundedness and compactness of the oparator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$. Section 4 considers the boundedness and compactness of the oparator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$.

2. **Auxiliary Results**

To arrive at the main results we use some lemmas, as given below

Lemma 2.1. [4] *Let K be a weight function. Then for any $w \in \mathbb{D}$ and $\varepsilon > 0$, we have*

$$f_z(w) = \frac{(1-|z|^2)^{\varepsilon/2}}{\sqrt{K(1-|z|^2)}(1-w\bar{z})^{1+\varepsilon/2}}$$

is in $\mathcal{D}_K$. Moreover,

$$\sup_{z\in\mathbb{D}} \|f_z\|_{\mathcal{D}_K} \approx 1,$$

and f_z converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|z| \to 1^-$.

The following two lemmas can be proved easily by following the Lemma 2.1 and [4].

Lemma 2.2. [4] *Let K be a weight function. Then for every $f \in \mathcal{D}_K$ we have*

$$|f(z)| \leq C\frac{\|f\|_{\mathcal{D}_K}}{\sqrt{K(1-|z|^2)}(1-|z|^2)}, \quad z \in \mathbb{D}.$$

Lemma 2.3. [4] *Let K be a weight function and n be a positive integer. Then for every $f \in \mathcal{D}_K$ we have*

$$|f^{(n)}(z)| \leq C\frac{\|f\|_{\mathcal{D}_K}}{\sqrt{K(1-|z|^2)}(1-|z|^2)^{n+1}}, \quad z \in \mathbb{D}.$$

The following criterion characterize the compactness. It was given for the first time in [6]. Since the proof is standard, so we omit it.

Lemma 2.4. *Let ν be the standard weight and the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded. Then $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_K$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have*

$$\lim_{n\to\infty} \|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} = 0.$$

3. Boundedness and compactness of weighted composition operator from Dirichlet type space to Bloch-type space

Theorem 3.1. *Let ν and K be two weight functions, $\psi \in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Then the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded if and only if the following conditions are satisfied:*

$$(i)\ M_1 = \sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \infty;$$

$$(ii)\ M_2 = \sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)^2}(1-|\varphi(z)|^2)^2} < \infty.$$

Furthermore, if the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded, then

$$M_1 + M_2 \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K \to \mathcal{B}_\nu} \lesssim 1 + M_1 + M_2.$$

Proof. First suppose that condition (i) and (ii) hold. Using Lemma 2.2 we have,

$$\begin{aligned} \nu(z)|(W_{\psi,\varphi}f)'(z))| &\leq \nu(z)|\psi'(z)||f(\varphi(z))| + \nu(z)|\psi(z)\varphi'(z)||f'(\varphi(z))| \\ &\lesssim \Bigg(\frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} \\ &\quad + \frac{\nu(z)|\psi(z)\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} \Bigg) \|f\|_{\mathcal{D}_K}. \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} |(W_{\psi,\varphi}f)(0)| &= |\psi(0)||f(\varphi(0))| \\ &\lesssim \frac{|\psi(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)} \|f\|_{\mathcal{D}_K}. \end{aligned} \tag{3.2}$$

From conditions $(i), (ii)$ and equations (3.1) and (3.2), we get

$$\begin{aligned} \|W_{\psi,\varphi}f\|_{\mathcal{B}_\nu} &= |\psi(0)||f(\varphi(0))| + \sup_{z\in\mathbb{D}} \nu(z)|(W_{\psi,\varphi}f)'(z)| \\ &\lesssim \left(\frac{|\psi(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)} + M_1 + M_2 \right) \|f\|_{\mathcal{D}_K} \\ &\lesssim \big(1 + M_1 + M_2\big) \|f\|_{\mathcal{D}_K}. \end{aligned}$$

Therefore, $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded and

$$\|W_{\psi,\varphi}\|_{\mathcal{D}_K \to \mathcal{B}_\nu} \lesssim 1 + M_1 + M_2. \tag{3.3}$$

Conversely, suppose that $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded. Let $z = \varphi(\zeta)$, $\zeta \in \mathbb{D}$ and

$$g_z(w) = \tau_z(w) f_z(w), \tag{3.4}$$

where $f_z(w)$ is defined in Lemma 2.1 and $\tau_z(w)$ is defined as

$$\tau_z(w) = 1 - \frac{1-|z|^2}{1-\bar{z}w}. \tag{3.5}$$

Then $\tau_z \in H^\infty$ as

$$\sup_{w\in\mathbb{D}} |\tau_z(w)| \le \sup_{w\in\mathbb{D}} \left(1 + \frac{1-|z|^2}{1-|z||w|}\right) \le 3.$$

Therefore, $g_z \in \mathcal{D}_K$ and $\sup_{w\in\mathbb{D}} \|g_z\|_{\mathcal{D}_K} \lesssim 1$. From equation (3.5) we have,

$$\tau_z(z) = 0 \tag{3.6}$$

and

$$\tau_z'(w) = \frac{-\bar{z}(1-|z|^2)}{(1-\bar{z}w)^2}.$$

Thus,

$$\tau_z'(z) = \frac{-\bar{z}}{(1-|z|^2)}. \tag{3.7}$$

Therefore, $g_z(z) = 0$, using the value of $f_z(z)$ and from (3.7), we obtain

$$\begin{aligned} g_z'(z) &= \tau_z'(z) f_z(z) + \tau_z(z) f_z'(z) \\ &= \frac{-\bar{z}}{\sqrt{K(1-|z|^2)}(1-|z|^2)^2}. \end{aligned}$$

Using the above fact, we get

$$\begin{aligned} \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} &\gtrsim \|W_{\psi,\varphi} g_{\varphi(\zeta)}\|_{\mathcal{B}_\nu} \\ &\ge \nu(\zeta)|\psi'(\zeta) g_{\varphi(\zeta)}(\varphi(\zeta)) + \psi(\zeta)\varphi'(\zeta) g'_{\varphi(\zeta)}(\varphi(\zeta))| \\ &\ge \nu(\zeta)|\psi(\zeta)\varphi'(\zeta) g'_{\varphi(\zeta)}(\varphi(\zeta))| \\ &\ge \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||\varphi(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2}. \end{aligned}$$

When $\delta \in (0,1)$ is fixed, we have

$$\sup_{|\varphi(\zeta)|>\delta} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.8}$$

Taking $f_z(w) \equiv 1 \in \mathcal{D}_K$, implies that

$$\sup_{w\in\mathbb{D}} \nu(w)|\psi'(w)| = \|W_{\psi,\varphi}(1)\|_{\mathcal{B}_\nu} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.9}$$

Again taking $f(w) = w \in \mathcal{D}_K$, using the asymptotic estimate (3.9) and boundedness of φ, we get

$$\sup_{w\in\mathbb{D}} \nu(w)|\psi(w)\varphi'(w)| \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.10}$$

Using (3.10) and the compactness of φ, we easily get

$$\begin{aligned} &\sup_{|\varphi(\zeta)|\le\delta} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \\ &\lesssim \left(\frac{1}{\sqrt{K(1-\delta^2)}(1-\delta^2)^2}\right)\|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \end{aligned} \tag{3.11}$$

Further, from (3.8) and (3.11), we obtain

$$\sup_{\zeta\in\mathbb{D}} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.12}$$

Again, for f_z as defined in Lemma 2.1, we have

$$\begin{aligned}\|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} &\gtrsim \|W_{\psi,\varphi} f_{\varphi(\zeta)}\|_{\mathcal{B}_\nu}\\ &\geq \nu(\zeta)\big|\psi'(\zeta)f_{\varphi(\zeta)}(\varphi(\zeta)) + \psi(\zeta)\varphi'(\zeta)f'_{\varphi(\zeta)}(\varphi(\zeta))\big|\\ &\geq \frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)}\\ &\quad + \frac{(1+\epsilon/2)\nu(\zeta)|\psi(\zeta)||\varphi'(\zeta)||\varphi(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2}.\end{aligned}$$

By using the boundedness of φ, we get

$$\begin{aligned}&\frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)}\\ &\leq \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} + C\frac{\nu(\zeta)|\psi(\zeta)||\varphi'(\zeta)||\varphi(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2}.\end{aligned} \tag{3.13}$$

Taking the supremum over $\zeta\in\mathbb{D}$ in (3.13) and using (3.12), we get

$$\sup_{\zeta\in\mathbb{D}} \frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.14}$$

From (3.12) and (3.14), we obtain

$$M_1 + M_2 \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.15}$$

Hence, from (3.3) and (3.15), we get

$$M_1 + M_2 \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} \lesssim 1 + M_1 + M_2.$$

□

Theorem 3.2. *Let ν be a standard weight, $\psi\in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Let K be a weight function. Assume that $W_{\psi,\varphi}:\mathcal{D}_K\to\mathcal{B}_\nu$ is bounded. Then the operator $W_{\psi,\varphi}:\mathcal{D}_K\to\mathcal{B}_\nu$ is compact if and only if the following conditions are satisfied:*

$$(i)\ \lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} = 0;$$

$$(ii)\ \lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} = 0.$$

Proof. First suppose that (i) and (ii) hold. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $\mathcal{D}_K$ that converges to zero uniformly on compact subset of $\mathbb{D}$. To prove the compactness of $W_{\psi,\varphi}$, we have to show that $\|W_{\psi,\varphi}f_n\|_{\mathcal{B}_\nu}\to 0$ as $n\to\infty$.
Condition (i) and (ii) implies that for any $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$\frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \varepsilon \tag{3.16}$$

and

$$\frac{\nu(z)|\psi(z))\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} < \varepsilon, \tag{3.17}$$

whenever $\delta < |\varphi(z)| < 1$.
Let $A = \{z \in \mathbb{D} : |z| \le \delta\}$ be a compact subset of $\mathbb{D}$. We have

$$\begin{aligned}
\|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} &= |\psi(0)||f_n(\varphi(0))| + \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|(W_{\psi,\varphi} f_n)'(\zeta)| \\
&\le |\psi(0)||f_n(\varphi(0))| + \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|\psi'(\zeta)||f_n(\varphi(\zeta))| \\
&\quad + \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||f_n'(\varphi(\zeta))| \\
&\le |\psi(0)||f_n(\varphi(0))| + \sup_{\{\zeta\in\mathbb{D}:\varphi(\zeta)\in A\}} \nu(\zeta)|\psi'(\zeta)||f_n(\varphi(\zeta))| \\
&\quad + \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \nu(\zeta)|\psi'(\zeta)||f_n(\varphi(\zeta))| \\
&\quad + \sup_{\{\zeta\in\mathbb{D}:\varphi(\zeta)\in A\}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||f_n'(\varphi(\zeta))| \\
&\quad + \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||f_n'(\varphi(\zeta))| \\
&\le |\psi(0)||f_n(\varphi(0))| + \|\psi\|_{\mathcal{B}_\nu} \sup_{z\in A}|f_n(z)| + N \sup_{z\in A}|f_n'(z)| \\
&\quad + C \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)} \|f_n\|_{\mathcal{D}_K} \\
&\quad + C \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \|f_n\|_{\mathcal{D}_K},
\end{aligned} \tag{3.18}$$

where we have $|f_n(\varphi(0))| < \varepsilon$, $\sup_{z\in A}|f_n(z)| < \varepsilon$ and $\sup_{z\in A}|f_n'(z)| < \varepsilon$, for some $N_0 \in \mathbb{N}$ and for all $n \ge N_0$. Also we have used the fact that $\psi \in \mathcal{B}_\nu$ and $N = \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)| < \infty$. Using the above fact in (3.18) and along with (3.16) and (3.17), we get $\|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} < C\varepsilon$, for $n \ge N_0$. Since $\varepsilon > 0$ is arbitrary, so we have $\|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} \to 0$ as $n \to \infty$. Hence, $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact.

Conversely, suppose that $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact. Let $(\zeta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(\zeta_n)| \to 1$ as $n \to \infty$. Suppose such a sequence does not exist, then (i) & (ii) are vacuously satisfied. Let $g_n(w) = \tau_{\varphi(\zeta_n)}(w) f_{\varphi(\zeta_n)}(w)$, where f_z and τ_z are defined earlier in Lemma 2.1 and Theorem 3.1. Then, $\|\tau_{\varphi(\zeta_n)}\|_{\mathcal{D}_K} \lesssim 1$, $\|f_{\varphi(\zeta_n)}\|_{\mathcal{D}_K} \lesssim 1$ and $(f_{\varphi(\zeta_n)})_{n\in\mathbb{N}}$ converges to zero uniformly on compact subset of $\mathbb{D}$ as $n \to \infty$. So, $\|g_n\|_{\mathcal{D}_K} \lesssim 1$ and $(g_n)_{n\in\mathbb{N}}$ converges to zero uniformly on compact subset of $\mathbb{D}$ as $n \to \infty$.

Since $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact, so we have

$$\|W_{\psi,\varphi} g_n\|_{\mathcal{B}_\nu} \to 0 \text{ as } n \to \infty.$$

Also, we have (as in Theorem 3.1),

$$\|W_{\psi,\varphi} g_n\|_{\mathcal{B}_\nu} \ge \frac{\nu(\zeta_n)|\psi(\zeta_n)\varphi'(\zeta_n)||\varphi(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)^2}.$$

Using the above two facts, we get

$$\lim_{|\varphi(\zeta_n)|\to 1} \frac{\nu(\zeta_n)|\psi(\zeta_n)\varphi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)^2} = 0. \tag{3.19}$$

Using Lemma 2.1, we have $\sup_{n\in\mathbb{N}} \|f_{\varphi(\zeta_n)}\|_{\mathcal{D}_K} \lesssim 1$ and $f_{\varphi(\zeta_n)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Since $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact. Therefore,

$$\lim_{n\to\infty} \|W_{\psi,\varphi} f_{\varphi(\zeta_n)}\|_{\mathcal{B}_\nu} = 0. \tag{3.20}$$

From (3.13), we obtain

$$\frac{\nu(\zeta_n)|\psi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)} \leq \|W_{\psi,\varphi} f_{\varphi(\zeta_n)}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} + C\frac{\nu(\zeta_n)|\psi(\zeta_n)\varphi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)^2},$$

which on combining with (3.19) and (3.20) gives

$$\lim_{|\varphi(\zeta_n)|\to 1} \frac{\nu(\zeta_n)|\psi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)} = 0. \tag{3.21}$$

Hence, the result follows from (3.19) and (3.21). □

4. Boundedness and compactness of weighted composition operator from Dirichlet type space to Bers-type space

In this section, we consider the Bers-type spaces and characterize the boundedness and compactness of operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$. We omit the proofs as these are similar to Theorem 3.1 and 3.2 of Section 3.

Theorem 4.1. *Let ν be a weight and K be a weight function, $\psi \in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Then the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$ is bounded if and only if the following condition is satisfied:*

$$l_1 = \sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \infty.$$

Theorem 4.2. *Let ν be a standard weight, $\psi \in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Let K be a weight function. Assume that the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$ is bounded. Then $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$ is compact if and only if the following condition is satisfied:*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|\psi(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} = 0.$$

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OSCILLATION CRITERIA FOR DIFFERENTIAL EQUATIONS WITH SEVERAL NON-MONOTONE DEVIATING ARGUMENTS

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Abstract. Consider the first-order linear differential equation with several retarded arguments $x'(t)+\sum_{i=1}^{m} p_i(t)\,x(\tau_i(t))=0,\ t\geq t_0$, where the functions $p_i,\ \tau_i\in C\left([t_0,\infty),\ \mathbb{R}^+\right)$, for every $i=1,2,\ldots,m,\ \tau_i(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau_i(t)=\infty$. New oscillation criteria which essentially improve known results in the literature are established.An example illustrating the results is given.

1. Introduction

Consider the first-order linear differential equation with several non-monotone retarded arguments

$$x'(t)+\sum_{i=1}^{m} p_i(t)\,x(\tau_i(t))=0,\ t\geq t_0, \tag{1.1}$$

where the functions $p_i,\ \tau_i\in C\left([t_0,\infty),\mathbb{R}^+\right)$, for every $i=1,2,\ldots,m$, (here $\mathbb{R}^+=[0,\infty)$), $\tau_i(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau_i(t)=\infty$.

Let $T_0\in[t_0,+\infty)$, $\tau(t)=\min\{\tau_i(t):i=1,\ldots,m\}$ and $\tau_{-1}(t)=\sup\{s:\tau(s)\leq t\}$. By a solution of the equation (1.1) we understand a function $x\in C([T_0,+\infty),\mathbb{R})$, continuously differentiable on $[\tau_{-1}(T_0),+\infty]$ and that satisfies (1.1) for $t\geq\tau_{-1}(T_0)$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *non-oscillatory.*

In the special case where $m=1$ equation (1.1) reduces to the equation

$$x'(t)+p(t)\,x(\tau(t))=0,\ t\geq t_0, \tag{1.2}$$

where the functions $p,\ \tau\in C([t_0,\infty),\mathbb{R}^+)$, $\tau(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau(t)=\infty$.

For the general theory of these equations the reader is referred to [13,16,18,19,32].

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equations (1.1) and (1.2) has been the subject of many investigations. See, for example, [1-40] and the references cited therein.

In the case of monotone argumetns, a survey of the most interesting oscillation conditions for Eq.(1.2) can be found in [36]. While in the general case of non-monotone arguments we mention the following interesting sufficient oscillation conditions.

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In 1994, Koplatadze and Kvinikadze [26] established the following: Assume

$$\sigma(t) := \sup_{s\leq t} \tau(s), \qquad t \geq 0. \tag{1.3}$$

Clearly $\sigma(t)$ is non-decreasing and $\tau(t) \leq \sigma(t)$ for all $t \geq 0$. *Let* $k \in \mathbb{N}$ *exist such that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{ \int_{\sigma(s)}^{\sigma(t)} p(\xi)\psi_k(\xi)d\xi \right\} ds > 1 - c(\mathfrak{a}), \tag{1.4}$$

where $\mathfrak{a} := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e}$,

$$\psi_1(t) = 0,\ \psi_k(t) = \exp\left\{ \int_{\tau(t)}^{t} p(\xi)\psi_{k-1}(\xi)d\xi \right\},\ k = 2, 3, ... \text{ for } t \in \mathbb{R}^+. \tag{1.5}$$

and

$$c(\mathfrak{a}) = \begin{cases} 0 & \text{if } \mathfrak{a} > \frac{1}{e}, \\ \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) & \text{if } 0 < \mathfrak{a} \leq \frac{1}{e}\ . \end{cases} \tag{1.6}$$

Then all solutions of equation (1.2) oscillate.

In 2011 Braverman and Karpuz [6] derived the following sufficient oscillation condition for Eq.(1.2)

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{ \int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi \right\} ds > 1, \tag{1.7}$$

while in 2014 Stavroulakis [37] improved the above condition as follows:

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{ \int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi \right\} ds > 1 - \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) \tag{1.8}$$

In 2018 Chatzarakis, Purnaras and Stavroulakis [9] improved further these conditions as follows: Assume that for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\, du \right) ds > 1, \tag{1.9}$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\, du \right) ds > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}, \tag{1.10}$$

where $0 < \mathfrak{a} \leq \frac{1}{e}$, and

$$P_k(t) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\, d\xi \right) du \right) ds \right]$$

with $P_0(t) = p(t)$. Then all solutions of Eq. (1.2) oscillate.

Concerning the differential equation (1.1) with several non-monotone arguments the following related oscillation results have been recently published.

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Assume that there exist non-decreasing functions $\sigma_i \in C([t_0,\infty),\mathbb{R}^+)$ such that

$$\tau_i(t) \le \sigma_i(t) \le t, \ i = 1,2,\dots,m. \tag{1.11}$$

In 2015 Infante, Kopladatze and Stavroulakis [21] proved that if

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} \sum_{i=1}^{m} p_i(\xi) \exp\left(\int_{\tau_i(\xi)}^{\xi} \sum_{i=1}^{m} p_i(u)\,du \right) d\xi \right) ds \right]^{1/m} > \frac{1}{m^m}, \tag{1.12}$$

then all solutions of Eq. (1.1) oscillate.

Also in 2015 Kopladatze [27] improved the above condition as follows: Let there exist some $k \in \mathbb{N}$ such that

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(m \int_{\tau_i(s)}^{\sigma_i(t)} \left(\prod_{\ell=1}^{m} p_\ell(\xi) \right)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right) ds \right]^{\frac{1}{m}} > \frac{1}{m^m} \left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right], \tag{1.13}$$

where

$$\psi_1(t) = 0, \ \psi_i(t) = \exp\left(\sum_{j=1}^{m} \int_{\tau_j(t)}^{t} \left(\prod_{\ell=1}^{m} p_\ell(s) \right)^{\frac{1}{m}} \psi_{i-1}(s)\, ds \right), \ i = 2,3,\dots,$$

$$0 < \alpha_i := \liminf_{t\to\infty} \int_{\sigma_i(t)}^{t} p_i(s)\, ds < \frac{1}{e}, \ i = 1,2,\dots,m, \tag{1.14}$$

and

$$c_i(\alpha_i) = \frac{1 - \alpha_i - \sqrt{1 - 2\alpha_i - \alpha_i^2}}{2}, \ i = 1,2,\dots,m, \tag{1.15}$$

then all solutions of Eq. (1.1) oscillate.

In 2016 Braverman, Chatzarakis and Stavroulakis [7] obtained the following iterative sufficient oscillation conditions

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(t), \tau_i(u))\, du > 1, \tag{1.16}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(t), \tau_i(u))\, du > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{1.17}$$

or

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(t), \tau_i(u))\, du > \frac{1}{e}, \tag{1.18}$$

where

$$h(t) = \max_{1\le i\le m} h_i(t) \text{ and } h_i(t) = \sup_{t_0\le s\le t} \tau_i(s), \ i = 1, 2, \dots, m,$$

$$0 < \alpha := \liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s)\, ds \le \frac{1}{e} \tag{1.19}$$

and

$$a_1(t,s) = \exp\left(\int_s^t \sum_{i=1}^m p_i(u)\, du \right),$$

$$a_{r+1}(t,s) = \exp\left(\int_s^t \sum_{i=1}^m p_i(u)\, a_r(u, \tau_i(u))\, du \right), \ r \in \mathbb{N}.$$

Also, in 2016 Akca, Chatzarakis and Stavroulakis [1] improved that result replacing condition (1.8) by the iterative condition

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(u), \tau_i(u))\, du > \frac{1+\ln\lambda_0}{\lambda_0} \tag{1.20}$$

where λ_0 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$,

$$0 < \alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)\, ds \le \frac{1}{e}$$

and $\tau(t) = \max_{1\le i\le m} \{\tau_i(t)\}$.

In 2017 Chatzarakis [8] derived the following results: Assume that for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P_k(u)\, du \right) ds > 1, \tag{1.21}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P_k(u)\, du \right) ds > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.22}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_k(u) du \right) ds > \frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}, \tag{1.23}$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(s)} P_k(u) du \right) ds > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.24}$$

or

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(s)} P_k(u) du \right) ds > \frac{1}{e}, \tag{1.25}$$

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where $h(t), \tau(t), \alpha$ are defined as above, λ_1 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$, and

$$P_k(t) = P(t)\left[1 + \int_{\tau(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{t} P(u)\exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\,d\xi\right)du\right)ds\right]$$

with $P_0(t) = P(t) = \sum_{i=1}^{m} p_i(t)$. Then all solutions of Eq. (1.1) oscillate.

Recently Bereketoglu et al [4] improved the above conditions as follows:

Assume that there exist non-decreasing functions $\sigma_i \in C([t_0,\infty), \mathbb{R}^+)$ such that (1.11) is satisfied and for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m}, \tag{1.26}$$

or

$$\limsup_{t\to\infty} \prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m}\left[1 - \prod_{i=1}^{m} c_i(\alpha_i)\right], \tag{1.27}$$

where

$$P_k(t) = \sum_{j=1}^{m} p_j(t)\left\{1 + m\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{t} P_{k-1}(u)\,du\right)ds\right]^{1/m}\right\},$$

with

$$P_0(t) = m\left[\prod_{\ell=1}^{m} p_\ell(t)\right]^{1/m},$$

α_i is given by (1.14) and $c_i(\alpha_i)$ by (1.15). Then all solutions of Eq.(1.1) oscillate.

In 2018 Attia et al [3] established the following oscillation conditions.

Assume that

$$0 < \rho := \liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{k=1}^{n} p_k(s)ds \le \frac{1}{\mathrm{e}},$$

and

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^{t}\sum_{i=1}^{n} p_i(s)ds}\right) > 1,$$

where

$$Q(t) = \sum_{k=1}^{n}\sum_{i=1}^{n} p_i(t)\int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{g_k(t)}^{t}\sum_{i=1}^{n} p_i(s)ds + (\lambda(\rho)-\epsilon)\int_{\tau_k(s)}^{g_k(t)}\sum_{\ell=1}^{n} p_\ell(u)du}ds, \quad \epsilon \in (0, \lambda(\rho)),$$

or

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q_1(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^{t}\sum_{i=1}^{n} p_i(s)ds}\right) > 1,$$

where

$$Q_1(t) = \sum_{k=1}^{n}\sum_{i=1}^{n} p_i(t)\int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{g_k(t)}^{t}\sum_{i=1}^{n}p_i(s)ds+\int_{\tau_k(s)}^{g_k(t)}\sum_{\ell=1}^{n}(\lambda(q_\ell)-\epsilon_\ell)p_\ell(u)du}ds, \quad \epsilon_\ell\in(0,\,\lambda(q_\ell)),$$

and

$$q_\ell = \liminf_{t\to\infty}\int_{\tau_\ell(t)}^{t} p_\ell(s)ds, \quad \ell=1,2,...,m$$

or

$$\limsup_{t\to\infty}\left(\prod_{j=1}^{n}\left(\prod_{k=1}^{n}\int_{g_j(t)}^{t}R_k(s)ds\right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{n}c(\beta_k)}{n^n}\mathrm{e}^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n}p_\ell(s)ds}\right) > \frac{1}{n^n},$$

where

$$R_k(s) = \mathrm{e}^{\int_{g_k(s)}^{s}\sum_{i=1}^{n}p_i(u)du}\sum_{i=1}^{n}p_i(s)\int_{\tau_i(s)}^{s}p_k(u)\mathrm{e}^{(\lambda(\rho)-\epsilon)\int_{\tau_k(u)}^{g_k(s)}\sum_{\ell=1}^{n}p_\ell(v)dv}du, \quad \epsilon\in(0,\,\lambda(\rho)),$$

and

$$0<\beta_k := \liminf_{t\to\infty}\int_{\sigma_i(t)}^{t}p_i(s)ds \le \frac{1}{\mathrm{e}}.$$

Then Eq. (1.1) is oscillatory.

In this paper we further investigate the problem and derive oscillation conditions which essentially improve all the above mentioned conditions.

2. Main Results

Our main results are the following two theorems

Theorem 1. *Assume that there exist non-decreasing functions $\sigma_i \in C([t_0,\infty),\mathbb{R}^+)$ such that (1.11) is satisfied and for some $k\in\mathbb{N}$*

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)}P_k(u)\,du\right)ds\right]^{1/m} > \frac{1}{m^m}, \tag{2.1}$$

where

$$P_k(t) = P(t)\left[1+\int_{\sigma_i(t)}^{t}P(s)\exp\left(\int_{\tau_i(s)}^{t}P(u)\exp\left(\int_{\tau_i(u)}^{u}P_{k-1}(\xi)\,d\xi\right)du\right)ds\right] \tag{2.2}$$

with

$$P_0(t) = P(t) = \sum_{i=1}^{m}p_i(t).$$

Then all solutions of Eq.(1.1) oscillate.

Proof. Suppose for the sake of contradiction that Eq.(1.1) has a non-oscillatory solution $x(t)$. Since $-x(t)$ is also a solution to (1.1), we confine ourselves only to the case that $x(t)$ is an eventually positive solution of Eq.(1.1). Then there exists $t_1>t_0$ such that $x(t)>0,\ x(\tau_i(t))>0,\ x(\sigma_i(t))>0$. Thus, from Eq.(1.1) it

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follows that $x'(t) \leq 0$ for all $t \geq t_1$ and therefore $x(t)$ is non-increasing and taking into account that $\tau_i(t) \leq t$, it follows

$$x'(t) + \sum_{i=1}^{m} p_i(t) x(t) \leq 0, \ t \geq t_1. \tag{2.3}$$

Dividing the last inequality by $x(t)$ and integrating from $\tau_i(t)$ to t for sufficiently large t, we have

$$x(\tau_i(t)) \geq x(t) \exp\left(\int_{\tau_i(t)}^{t} \sum_{\ell=1}^{m} p_\ell(\xi)\, d\xi \right), \ i = 1, 2, ..., m. \tag{2.4}$$

Dividing (1.1) by $x(t)$ and integrating from $\tau_i(s)$ to t, $s \leq t$, we obtain

$$x(\tau_i(s)) = x(t) \exp\left(\int_{\tau_i(s)}^{t} \sum_{\ell=1}^{m} p_\ell(u) \frac{x(\tau_\ell(u))}{x(u)} du \right), \ i = 1, 2, ..., m. \tag{2.5}$$

Combining the last two relations, we obtain

$$x(\tau_i(s)) \geq x(t) \exp\left(\int_{\tau_i(s)}^{t} \sum_{\ell=1}^{m} p_\ell(u) \exp\left(\int_{\tau_i(u)}^{u} \sum_{\ell=1}^{m} p_\ell(\xi)\, d\xi \right) du \right). \tag{2.6}$$

Now, integrating (1.1) from $\tau_i(t)$ to t and using (2.6) for sufficiently large t, we have

$$x(\tau_i(t)) \geq x(t) \left[1 + \int_{\tau_i(t)}^{t} \sum_{\ell=1}^{m} p_\ell(s) \exp\left(\int_{\tau_i(s)}^{t} \sum_{\ell=1}^{m} p_\ell(u) \exp\left(\int_{\tau_i(u)}^{u} \sum_{\ell=1}^{m} p_\ell(\xi)\, d\xi \right) du \right) ds \right]. \tag{2.7}$$

Multiplying the last inequality by $p_i(t)$ [cf.10,3,4] and taking the sum over i $(i = 1, 2, ..., m)$, we have

$$x'(t) + P_1(t) x(t) \leq 0, \ t \geq t_1, \tag{2.8}$$

where

$$P_1(t) = P(t) \left[1 + \int_{\tau_i(t)}^{t} P(s) \exp\left(\int_{\tau_i(s)}^{t} P(u) \exp\left(\int_{\tau_i(u)}^{u} P_0(\xi) d\xi \right) du \right) ds \right].$$

Observe that (2.8) resembles with (2.3), where $\sum_{i=1}^{m} p_i(t)$ $[= P(t) = P_0(t)]$ is replaced by $P_1(t)$, and following the same steps as from (2.3) to (2.8), for sufficiently large t we find

$$x'(t) + P_2(t) x(t) \leq 0, \tag{2.9}$$

where

$$P_2(t) = P(t) \left[1 + \int_{\tau_i(t)}^{t} P(s) \exp\left(\int_{\tau_i(s)}^{t} P(u) \exp\left(\int_{\tau_i(u)}^{u} P_1(\xi) d\xi \right) du \right) ds \right].$$

Repeating the above procedure, it follows by induction, that for sufficiently large t

$$x'(t) + P_k(t)\,x(t) \le 0, \tag{2.10}$$

where $P_k(t)$ is given by

$$P_k(t) = P(t)\left[1 + \int_{\tau_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)d\xi\right)du\right)ds\right].$$

Dividing (2.10) by $x(t)$ and integrating from $\tau_i(s)$ to $\sigma_i(t)$, $s \le t$, for sufficiently large t, we get

$$x(\tau_i(s)) \ge x(\sigma_i(t))\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right). \tag{2.11}$$

On the other hand, integrating (1.1) from $\sigma_j(t)$ to t for sufficiently large t, we have

$$x(\sigma_j(t)) = x(t) + \sum_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\,x(\tau_i(s))\,ds. \tag{2.12}$$

Combining (2.12) with (2.11) and using the arithmetic mean-geometric mean inequality, we obtain

$$x(\sigma_j(t)) \ge m\left[\prod_{i=1}^{m} x(\sigma_i(t))\right]^{1/m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m}.$$

Now, taking the product on both sides of the last inequality, we find

$$\prod_{j=1}^{m} x(\sigma_j(t)) \ge m^m\left[\prod_{j=1}^{m} x(\sigma_j(t))\right]\left[\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right]^{1/m}\right].$$

Hence

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right]^{1/m} \le \frac{1}{m^m}$$

which contradicts (2.1). □

For the next theorem we need the following lemma (See [39,13,26,27,4]).

Lemma 1. *Let there exist non-decreasing functions $\sigma_i \in C([t_0,\infty), \mathbb{R}^+)$ such that condition (1.11) is fulfilled and equation (1.1) has an eventually positive solution $x : [t_0, +\infty) \to (0, +\infty)$. Then*

$$\liminf_{t\to\infty}\frac{x(t)}{x(\sigma_i(t))} \ge c_i(\alpha_i),\ \ i = 1, 2, \ldots, m,$$

where α_i and $c_i(\alpha_i)$ are given by (1.14) and (1.15).

Theorem 2. *Assume that there exist non-decreasing functions* $\sigma_i \in C([t_0,\infty),\mathbb{R}^+)$ *such that (1.11) is satisfied and for some* $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \left(\int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du \right) ds \right) \right]^{1/m} > \frac{1}{m^m} \left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right], \tag{2.13}$$

where $P_k(u)$ *is given by (2.2),* α_i *by (1.14) and* $c_i(\alpha_i)$ *by (1.15). Then all solutions of Eq.(1.1) oscillate.*

As in the proof of Theorem 1, we assume, for the sake of contradiction, that Eq.(1.1) has a non-oscillatory solution $x(t)$ and derive (2.11) and (2.12). Combining (2.12) with (2.11) and using the arithmetic mean-geometric mean inequality for sufficiently large t, we get

$$x(\sigma_j(t)) \geq x(t) + m \left[\prod_{i=1}^{m} x(\sigma_i(t)) \right]^{1/m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du \right) ds \right]^{1/m}.$$

Taking the product on both sides of the last inequalities and using Lemma 1, as in proof of [4, Theorem 2], we find

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du \right) ds \right]^{1/m} \leq$$

$$\leq \frac{1}{m^m} \left[1 - \liminf_{t\to\infty} \frac{x^m(t)}{\prod_{i=1}^{m} x(\sigma_i(t))} \right]$$

$$\leq \frac{1}{m^m} \left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right]$$

which contradicts (2.13).

Remark 1. It is clear that the left-hand sides of both conditions (2.1) and (2.13) are identically the same and also the right-hand side of (2.13) reduces to (2.1) when $c_i(\alpha_i) = 0$.Thus, it seems that Theorem 2 is exactly the same as Theorem 1, when $c_i(\alpha_i) = 0$. One may notice, however, that the condition (1.14) is required in Theorem 2 but not in Theorem 1.

In the case of monotone arguments we have the following theorem.

Theorem 3. *Let* τ_i *be non-decreasing functions and for some* $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\tau_j(t)}^{t} \left(p_i(s) \exp\left(\int_{\tau_i(s)}^{\tau_i(t)} P_k(u)\,du \right) ds \right) \right]^{1/m} > \begin{cases} \frac{1}{m^m} \\ or \\ \frac{1}{m^m}\left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right] \end{cases}$$

where

$$P_k(t) = P(t)\left[1 + \int_{\tau_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)d\xi\right)du\right)ds\right].$$

with

$$P_0(t) = P(t) = \sum_{j=1}^{m} p_j(t),$$

α_i *is given by (1.14), and* $c_i(\alpha_i)$ *by (1.15). Then all solutions of (1.1) osillate.*

3. Corollaries and Examples

In the case $m = 2$, Eq.(1.1) reduces to the equation

$$x'(t) + p_1(t)\,x(\tau_1(t)) + p_2(t)\,x(\tau_2(t)) = 0. \tag{3.1}$$

From Theorems 1 and 2 the following corollary is immediate

Corollary 1. *Assume that (1.11) holds and for for* $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{2}\left[\prod_{i=1}^{2}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right]^{1/2} > \begin{cases} \frac{1}{4} \\ or \\ \frac{1}{4}\left[1 - \prod_{i=1}^{2} c_i(\alpha_i)\right] \end{cases},$$

where,

$$P_k(t) = P(t)\left[1 + \int_{\sigma_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)d\xi\right)du\right)ds\right],$$

$$P_0(t) = 2\left(p_1(t)\,p_2(t)\right)^{1/2},$$

and for $i = 1, 2,$ α_i *is given by (1.14) and* $c_i(\alpha_i)$ *by (1.15). Then all solutions of Eq.(3.1) oscillate.*

Corollary 2. *Assume that there exist a non-decreasing function* $\sigma(t)$ *such that* $\tau(t) \le \sigma(t) \le t$ *and for some* $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\,du\right)ds > \begin{cases} 1 \\ or \\ 1 - c(\alpha) \end{cases} \tag{3.2}$$

where

$$P_k(t) = p(t)\left[1 + \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\,d\xi\right)du\right)ds\right],\ P_0(t) = p(t),$$

$$0 < \alpha := \liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\,ds \ \le \frac{1}{e}, \tag{3.3}$$

and

$$c(\alpha) = \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}$$

Then all solutions of Eq.(1.2) oscillate.

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The following example (cf.[6],[21],[4]) is given to illustrate our results. It is to be pointed out that in this example it is shown that our conditions essentially improve all the related known conditions in the literature.

Example 1. *(Cf. [6],[21],[4]) Consider the equation*

$$x'(t) + px(\tau(t)) = 0, \ t \geq 0, \ p > 0. \tag{3.4}$$

with the retarded argument

$$\tau(t) = \begin{cases} t-1, & t \in [3n, \ 3n+1], \\ -3t + (12n+3), & t \in [3n+1, \ 3n+2], \\ 5t - (12n+13), & t \in [3n+2, \ 3n+3]. \end{cases}$$

For this equation, as in [6,21,4], one may choose the funtion

$$\sigma(t) = \begin{cases} t-1, & t \in [3n, \ 3n+1], \\ -3n, & t \in [3n+1, \ 3n+2.6], \\ 5t - (12n+13), & t \in [3n+2.6, \ 3n+3]. \end{cases}$$

If we choose $t_n = 3n+3$, *(cf. [6, Example 1] and [21, Example 4.2]), then for* $k=1$, *the condition (2.1) of Theorem 1 (or the condition (3.2) of Corollary 2) reduces to*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\, du \right) ds \geq \lim_{n\to\infty} \int_{3n+2}^{3n+3} p \exp\left(\int_{5s-(12n+13)}^{3n+2} P_1(u)\, du \right) ds,$$

where

$$\begin{aligned} P_1(t) &= p\left[1 + \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{t} p \exp\left(\int_{\tau(u)}^{u} p d\xi \right) du \right) ds \right] \\ &\geq p\left[1 + \int_{3n+2}^{3n+3} p \exp\left(\int_{5s-(12n+13)}^{3n+3} p \exp(p) du \right) ds \right] \\ &= p\left[1 + \left(\frac{e^{6pe^p} - e^{pe^p}}{5} \right) e^{-p} \right] \end{aligned}$$

Therefore

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\, du \right) ds \geq \frac{p}{5P_1}\left(e^{5P_1} - 1 \right),$$

where $P_1 = p\left[1 + \left(\frac{e^{6pe^p} - e^{pe^p}}{5} \right) e^{-p} \right]$*.For* $p = 0.255$, $P_1 \approx 0.484721$, *and so*

$$\frac{p}{5P_1}\left(e^{5P_1} - 1 \right) \approx 1.082293 > 1.$$

Therefore all solutions of Eq.(3.4) *oscillate.*

Observe, however, that when we consider the conditions stated in [6], [37] [21], [27], [7], [1] and[4] for the above equation (3.4), we obtain the following:

1. Observe that, for $t_n = 3n+3$,

$$\int_{\sigma(3n+3)}^{3n+3} p \exp\left\{\int_{\tau(s)}^{\sigma(3n+3)} p d\xi\right\} ds = \int_{3n+2}^{3n+3} p \exp\left\{\int_{5s-(12n+13))}^{3n+2} p d\xi\right\} ds = \frac{e^{5p}-1}{5}$$

and condition (1.7) reduces to

$$\frac{e^{5p}-1}{5} > 1.$$

But, for $p = 0.255$

$$\frac{e^{5p}-1}{5} \approx 0.51574 < 1$$

therefore the condition (1.7) is not satisfied.

2. Similarly, in the condition (1.8),

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds = \lim_{n\to\infty} \int_{3n+2}^{3n+3} p ds = p$$

and

$$c(\mathfrak{a}) = c(p) = \frac{1-p-\sqrt{1-2p-p^2}}{2}.$$

and, as before, (1.8) reduces to

$$\frac{e^{5p}-1}{5} > 1 - \frac{1-p-\sqrt{1-2p-p^2}}{2}$$

Taking $p = 0.255$ the left-hand side of (1.8) is equal to 0.51574 while the right-hand side is 0.95345. Therefore this condition is not satisfied.

3. The condition (1.12) reduces to

$$\lim\sup_{t\to+\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} p \exp\left(\int_{\tau(\xi)}^{\xi} p du\right) d\xi\right) ds > 1, \tag{3.5}$$

and, as in [20,Example 4.2], the choice of $t_n = 3n+3$, leads to the inequality

$$\frac{\left(e^{5pe^p}-1\right)}{5e^p} > 1. \tag{3.6}$$

Observe, however, that for $p = 0.255$,

$$\frac{\left(e^{5pe^p}-1\right)}{5e^p} \approx 0.64849 < 1.$$

Therefore the condition (3.6) is not satisfied.

4. The condition (1.13), for $k = 2$, reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} p\ \psi_2(\xi)\, d\xi\right) ds > 1 - c(\alpha), \tag{3.7}$$

where $\psi_2(\xi) = 1$, and for $t_n = 3n + 3$, as before, it leads to

$$\frac{e^{5p} - 1}{5} > \frac{1 - p - \sqrt{1 - 2p - p^2}}{2}$$

For $p = 0.255$, we have

$$\frac{e^{5p} - 1}{5} \approx 0.51574,$$

while the right-hand side

$$1 - c(p) \approx 0.95345.$$

Therefore the condition (3.7) is not satisfied.

5. The condition (1.16) for $r = 1$ reduces to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} pa_1(h(t), \tau(s))\, ds > 1, \tag{3.8}$$

where

$$h(t) = \sigma(t) \text{ and } a_1(t, s) = \exp\left(\int_s^t pdu\right).$$

That is, to the condition

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right) ds > 1, \tag{3.9}$$

and, as before, for $t_n = 3n + 3$ and $p = 0.255$, we have

$$\frac{e^{5p} - 1}{5} \approx 0.51574 < 1. \tag{3.10}$$

Therefore the condition (3.8) is not satisfied.

6. Similarly, condition (1.20) for $r = 1$ reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right) ds > \frac{1 + \ln \lambda_0}{\lambda_0}, \tag{3.11}$$

where λ_0 is the smaller root of the equation $\lambda = e^{p\lambda}$. As before, for $t_n = 3n + 3$ and $p = 0.255$, we have

$$\frac{e^{5p} - 1}{5} \approx 0.51574,$$

while

$$\frac{1 + \ln \lambda_0}{\lambda_0} \approx 0.94664$$

Therefore the condition (3.11) is not satisfied.

7. For $k = 1$, condition (1.26) reduces to

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du\right)ds>1. \tag{3.12}$$

If we choose $t_n = 3n+3$,

$$\begin{aligned} P_1(t) &= p\left\{1+\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{t} pdu\right)ds\right\} = p\left\{1+\int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+3} pdu\right)ds\right\} \\ &= p\left(1+\frac{e^{6p}-e^{p}}{5}\right). \end{aligned}$$

and, as before, (3.12) reduces to

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)>1.$$

For $p = 0.255$ we find $P_1 \approx 0.424232$ and so

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)\approx 0.882491<1.$$

Therefore the condition (3,12) is not satisfied

We conclude, therefore, that for $p = 0.255$ no one of the conditions (1.7), (1.8), (1.12), (1.13) for $k = 2$, (1.16) and (1.20) for $r = 1$, and (1.26) is satisfied.

It should be also pointed out that not only for this value of $p = 0.255$ but for all values of $p > 0.255$, especially for all values of $p \in [0.255, 0.358]$, (cf. [21, Example 4.2]),

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)>1$$

and therefore all solutions of (3.4) oscillate. Observe, however, that for $p = 0.358$

$$\frac{e^{5p}-1}{5}\approx 0.99789<1,$$

also for $p = 0.3$

$$\frac{\left(e^{5pe^{p}}-1\right)}{5e^{p}}\approx 0.974101<1,$$

$$\frac{e^{5p}-1}{5}\approx 0.696337<0.912993\approx\frac{1+\ln\lambda_0}{\lambda_0},$$

and for $p = 0.263$, $P_1 \approx 0.44944$ and so

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)\approx 0.99024<1.$$

Therefore for all values of $p \in [0.255, 0.358]$ the conditions of Corollary 2 are satisfied and so all solutions to Eq.(3.4) oscillate, while no one of the above mentioned conditions is satisfied for these values of $p \in [0.255, 0.358]$.

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Hyers–Ulam stability of second-order nonhomogeneous linear difference equations with a constant stepsize

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Abstract

The present paper is concerned with Hyers–Ulam stability of the second-order linear difference equation $\Delta_h^2 x(t) + \alpha\Delta_h x(t) + \beta x(t) = f(t)$ on $h\mathbb{Z}$, where $\Delta_h x(t) = (x(t+h) - x(t))/h$ and $h\mathbb{Z} = \{hk|\, k \in \mathbb{Z}\}$ for the stepsize $h > 0$; α and β are real numbers; $f(t)$ is a real-valued function on $h\mathbb{Z}$. The purpose of this paper is to find an explicit HUS constant for the second-order linear difference equation whose characteristic equation has real roots. It is clarified that an HUS constant changes by the influence of the stepsize.

Keywords: Hyers–Ulam stability; HUS constant; second-order linear difference equation; stepsize.
2010 Mathematics Subject Classiffication: Primary 39B82; Secondary 39A06; 65Q10.

1 Introduction

Hyers–Ulam stability is originated from in the field of functional equations. In 1940, this problem was posed by Ulam [32, 33]. In the next year, it was solved by Hyers [9]. After that, there has been an increasing interest in studying Hyers–Ulam stability of functional equations, differential equations and difference equations (see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 34, 36]). In this paper, we will deal with Hyers–Ulam stability of the second-order nonhomogeneous linear difference equation

$$\Delta_h^2 x(t) + \alpha\Delta_h x(t) + \beta x(t) = f(t) \tag{1.1}$$

on $h\mathbb{Z}$, where

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \quad \text{and} \quad h\mathbb{Z} = \{hk|\, k \in \mathbb{Z}\}$$

for the stepsize $h > 0$; α and β are real numbers; $f(t)$ is a real-valued function on $h\mathbb{Z}$. If $1 - \alpha h + \beta h^2 = 0$ holds, then we no longer have a second-order difference equation. For this reason, we assume that

$$1 - \alpha h + \beta h^2 \neq 0. \tag{1.2}$$

It is well-known that the global existence and uniqueness of solutions of (1.1) are guaranteed for the initial-value problem. We say that (1.1) has "*Hyers–Ulam stability*" on $h\mathbb{Z}$ if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $\left|\Delta_h^2\phi(t) + \alpha\Delta_h\phi(t) + \beta\phi(t) - f(t)\right| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then there exists a solution $x : h\mathbb{Z} \to \mathbb{R}$ of (1.1) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in h\mathbb{Z}$. We call such K an "*HUS constant*" for (1.1) on $h\mathbb{Z}$. In addition, we call the minimum of HUS constants for (1.1) on $h\mathbb{Z}$ the "*best HUS constant*". Recently, the best HUS constant of various functional equations and linear operators has been discovered by Popa and Raşa (see [28, 29, 30] and the references cited therein). When $h \to 0$, (1.1) becomes the second-order linear differential equation

$$x'' + \alpha x' + \beta x = f(t), \tag{1.3}$$

that is, (1.1) is an approximation of the ordinary differential equation (1.3). In 2010, Li and Shen [18] proved that (1.3) has HUS on a finite interval I if characteristic equation has two different positive roots. In 2014, Xue [35] extended their results. Since the solution of the difference equation with small stepsize is a good approximate solution of the differential equation, studying Hyers–Ulam stability of difference equation (1.1) will contribute to computer science.

In 2018, the author [22] dealt with Hyers–Ulam stability of the first-order nonhomogeneous linear difference equation

$$\Delta_h x(t) - ax(t) = f(t) \tag{1.4}$$

on $h\mathbb{Z}$, where a is a real number and $f(t)$ is a real-valued function on $h\mathbb{Z}$. We say that (1.4) has "*Hyers–Ulam stability*" on $h\mathbb{Z}$ if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then there exists a solution $x : h\mathbb{Z} \to \mathbb{R}$ of (1.4) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in h\mathbb{Z}$. Noticing that if $f(t) \equiv 0$ with $a = 0$ or $a = -2/h$, then (1.4) does not have Hyers–Ulam stability on $h\mathbb{Z}$ (see [21]); if $a = -1/h$, then we no longer have a first-order difference equation. For this reason, we assume that

$$a \ne 0,\ -\frac{1}{h} \text{ and } -\frac{2}{h}.$$

In [22], the author proved that (1.4) has Hyers–Ulam stability on $h\mathbb{Z}$, and the best HUS constant for (1.4) on $h\mathbb{Z}$ is

$$B(a,h) = \begin{cases} \dfrac{1}{|a|}, & \text{if } a > 0 \text{ or } 0 < h < -\dfrac{1}{a}, \\ \dfrac{1}{|a+2/h|}, & \text{if } -\dfrac{1}{a} < h < -\dfrac{2}{a} \text{ or } -\dfrac{2}{a} < h. \end{cases}$$

This constant is rewritten as

$$B(a,h) = \frac{1}{||a+1/h| - 1/h|}. \tag{1.5}$$

Let $\Phi(t)$ be an antidifference of $\phi(t)$ on $h\mathbb{Z}$, that is, $\Delta_h\Phi(t) = \phi(t)$ holds on $h\mathbb{Z}$, and let C be an arbitrary real constant. We denote $\Phi(t) + C$ by $\Delta_h^{-1}\phi(t)$. We can obtain the above fact according to the following results.

Theorem A (see [22, Corollary 2.5]). *Suppose that $a > 0$ or $a < -2/h$. Then* (1.4) *has Hyers–Ulam stability with an HUS constant $B(a,h)$ on $h\mathbb{Z}$, where $B(a,h)$ is the constant given by* (1.5). *Furthermore, if a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then*

$$\lim_{t\to\infty} \left\{ \phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} \right\}$$

exists, and there exists a unique solution

$$x(t) = \Big[\Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} + \lim_{t\to\infty} \left\{ \phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} \right\} \Big] (ah+1)^{\frac{t}{h}}$$

of (1.4) *such that $|\phi(t) - x(t)| \le B(a,h)\varepsilon$ for all $t \in h\mathbb{Z}$.*

Theorem B (see [22, Corollary 2.6]). *Suppose that $-1/h < a < 0$ or $-2/h < a < -1/h$. Then* (1.4) *has Hyers–Ulam stability with an HUS constant $B(a,h)$ on $h\mathbb{Z}$, where $B(a,h)$ is the constant given by* (1.5). *Furthermore, if a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then*

$$\lim_{t\to-\infty} \left\{ \phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} \right\}$$

exists, and there exists a unique solution

$$x(t) = \Big[\Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} + \lim_{t\to-\infty} \Big\{\phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}}\Big\}\Big](ah+1)^{\frac{t}{h}}$$

of (1.4) *such that* $|\phi(t) - x(t)| \le B(a,h)\varepsilon$ *for all* $t \in h\mathbb{Z}$.

Remark 1.1. We can confirm that the best HUS constant for (1.4) on $h\mathbb{Z}$ is greater than or equal to $B(a,h)$ by the following example. Consider the first-order nonhomogeneous linear difference equation

$$\Delta_h \phi(t) - a\phi(t) - f(t) = \varepsilon(-1)^{\frac{mt}{h}} \tag{1.6}$$

on $h\mathbb{Z}$, where $\varepsilon > 0$ and $m \in \{1,2\}$. Let

$$\phi_0(t) = (ah+1)^{\frac{t}{h}} \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}},$$

$$\phi_m(t) = \frac{\varepsilon(-1)^{\frac{mt}{h}}}{\{(-1)^m - 1\}/h - a}$$

and $\phi(t) = \phi_0(t) + \phi_m(t)$ for all $t \in h\mathbb{Z}$. Then $\phi(t)$ is a solution of (1.6). Now we will check this fact. Since

$$\begin{aligned} f(t)(ah+1)^{-\frac{t+h}{h}} &= \Delta_h \phi_0(t)(ah+1)^{-\frac{t}{h}} \\ &= \frac{1}{h}\Big\{\phi_0(t+h)(ah+1)^{-\frac{t+h}{h}} - \phi_0(t)(ah+1)^{-\frac{t}{h}}\Big\} \\ &= \frac{\phi_0(t+h) - (ah+1)\phi_0(t)}{h}(ah+1)^{-\frac{t+h}{h}} \\ &= (\Delta_h\phi_0(t) - a\phi_0(t))(ah+1)^{-\frac{t+h}{h}} \end{aligned}$$

holds, $\phi_0(t)$ is a solution of (1.4). From

$$\Delta_h(-1)^{\frac{mt}{h}} = \frac{1}{h}\Big\{(-1)^{\frac{m(t+h)}{h}} - (-1)^{\frac{mt}{h}}\Big\} = \frac{(-1)^m - 1}{h}(-1)^{\frac{mt}{h}}, \tag{1.7}$$

we have

$$\Delta_h \phi_m(t) = \frac{\varepsilon\{(-1)^m - 1\}(-1)^{\frac{mt}{h}}}{\{(-1)^m - 1\} - ah} = \varepsilon(-1)^{\frac{mt}{h}} + a\phi_m(t).$$

That is, $\phi_m(t)$ is a solution of (1.6) with $f(t) \equiv 0$. Using the above facts, we obtain

$$\Delta_h\phi(t) - a\phi(t) = \Delta_h(\phi_0(t) + \phi_m(t)) - a(\phi_0(t) + \phi_m(t)) = f(t) + \varepsilon(-1)^{\frac{mt}{h}}.$$

This means that $\phi(t)$ is a solution of (1.6). Therefore,

$$|\Delta_h\phi(t) - a\phi(t) - f(t)| = \varepsilon$$

holds for all $t \in h\mathbb{Z}$. Since $\phi_0(t)$ is a solution of (1.4), and $(ah+1)^{t/h}$ is a solution of (1.4) with $f(t) \equiv 0$, the general solution of (1.4) is written as

$$x(t) = c(ah+1)^{\frac{t}{h}} + \phi_0(t)$$

for all $t \in h\mathbb{Z}$, where c is an arbitrary constant. Noticing that $c = 0$ holds if and only if $|\phi(t) - x(t)|$ is bounded on $h\mathbb{Z}$. When $c = 0$, we have

$$|\phi(t) - x(t)| = |\phi_m(t)| = \frac{\varepsilon}{|a + \{1 - (-1)^m\}/h|}$$

for all $t \in h\mathbb{Z}$ and $m \in \{1,2\}$. This means that the best HUS constant for (1.4) on $h\mathbb{Z}$ is greater than or equal to

$$\max\left\{\frac{1}{|a|}, \frac{1}{|a + 2/h|}\right\} = B(a,h).$$

Remark 1.2. Theorems A, B and Remark 1.1 imply that the best HUS constant for (1.4) on $h\mathbb{Z}$ is $B(a,h)$ given by (1.5).

The purpose of this paper is to find an HUS constant for (1.1) on $h\mathbb{Z}$. In addition, we will find an explicit solution $x(t)$ of (1.1) such that $|\phi(t)-x(t)|$ is less than or equal to HUS constant multiplied by ε on $h\mathbb{Z}$, where $\phi(t)$ is a function satisfying $\left|\Delta_h^2\phi(t)+\alpha\Delta_h\phi(t)+\beta\phi(t)-f(t)\right|\le\varepsilon$ on $h\mathbb{Z}$. In the next section, we will present main theorems and their proofs, and give an HUS constant for (1.1) on $h\mathbb{Z}$. In Section 3, we will classify HUS constants for (1.1) on $h\mathbb{Z}$ by coefficients α and β. For illustration of the obtained results, we will take an example.

2 HUS constant for the second-order linear difference equations

We can easily see that the quadratic equation

$$\lambda^2+\alpha\lambda+\beta=0 \tag{2.1}$$

is the characteristic equation for the second-order homogeneous linear difference equation

$$\Delta_h^2x(t)+\alpha\Delta_hx(t)+\beta x(t)=0 \tag{2.2}$$

on $h\mathbb{Z}$, where α and β are real numbers with (1.2). In fact, we consider the funtion $x(t)=(\lambda h+1)^{t/h}$ on $h\mathbb{Z}$, where λ is a root of (2.1). Notice that since (1.2), non of λ is equal to $-1/h$. On the other hand, if $\lambda\ne-1/h$ then (1.2) holds. Clearly, $\Delta_hx(t)=\lambda(\lambda h+1)^{t/h}$ and $\Delta_h^2x(t)=\lambda^2(\lambda h+1)^{t/h}$ hold on $h\mathbb{Z}$. Therefore, if (2.1) holds then $x(t)$ is a solution of (2.2). Conversely, (2.1) is satisfied whenever $x(t)$ is a solution of (2.2) on $h\mathbb{Z}$. Thus, $(\lambda h+1)^{t/h}$ is a solution of (2.2) on $h\mathbb{Z}$ if and only if (2.1) holds.

Throughout this paper, we define

$$\Lambda_1=\{\lambda\in\mathbb{R}|\,\lambda>0\},\quad \Lambda_2=\left\{\lambda\in\mathbb{R}\middle|\,-\frac{1}{h}<\lambda<0\right\},$$

and

$$\Lambda_3=\left\{\lambda\in\mathbb{R}\middle|\,-\frac{2}{h}<\lambda<-\frac{1}{h}\right\},\quad \Lambda_4=\left\{\lambda\in\mathbb{R}\middle|\,\lambda<-\frac{2}{h}\right\}.$$

First, the following simple result is obtained by using Theorems A and B.

Theorem 2.1. *Suppose that* (2.1) *has real roots* λ_1 *and* λ_2 *with* $\lambda_i\in\bigcup_{j=1}^4\Lambda_j$ *for* $i\in\{1,2\}$. *Then* (1.1) *has Hyers–Ulam stability with an HUS constant* $B(\lambda_1,h)B(\lambda_2,h)$ *on* $h\mathbb{Z}$, *where* $B(\cdot,h)$ *is the constant given by* (1.5).

Proof. Assume that a function $\phi:h\mathbb{Z}\to\mathbb{R}$ satisfies

$$\left|\Delta_h^2\phi(t)+\alpha\Delta_h\phi(t)+\beta\phi(t)-f(t)\right|\le\varepsilon$$

for all $t\in h\mathbb{Z}$. Let $\psi(t)=\Delta_h\phi(t)-\lambda_1\phi(t)$ for $t\in h\mathbb{Z}$. From $\lambda_1+\lambda_2=-\alpha$, $\lambda_1\lambda_2=\beta$ and the above assumption, we get the inequality

$$|\Delta_h\psi(t)-\lambda_2\psi(t)-f(t)|=\left|\Delta_h^2\phi(t)+\alpha\Delta_h\phi(t)+\beta\phi(t)-f(t)\right|\le\varepsilon \tag{2.3}$$

for all $t\in h\mathbb{Z}$. Using Theorems A and B, we can find a solution $u:h\mathbb{Z}\to\mathbb{R}$ of

$$\Delta_hu(t)-\lambda_2u(t)=f(t) \tag{2.4}$$

such that $|\psi(t)-u(t)|\le B(\lambda_2,h)\varepsilon$ for all $t\in h\mathbb{Z}$. Namely, we have

$$|\Delta_h\phi(t)-\lambda_1\phi(t)-u(t)|\le B(\lambda_2,h)\varepsilon \tag{2.5}$$

for all $t\in h\mathbb{Z}$. Using Theorems A and B again, there exists a solution $v:h\mathbb{Z}\to\mathbb{R}$ of

$$\Delta_hv(t)-\lambda_1v(t)=u(t) \tag{2.6}$$

such that $|\phi(t) - v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. Since $u(t)$ is a solution of (2.4), we have

$$\begin{aligned}\Delta_h^2 v(t) + \alpha\Delta_h v(t) + \beta v(t) &= \Delta_h^2 v(t) - (\lambda_1 + \lambda_2)\Delta_h v(t) + \lambda_1\lambda_2 v(t)\\ &= \Delta_h(\Delta_h v(t) - \lambda_1 v(t)) - \lambda_2(\Delta_h v(t) - \lambda_1 v(t))\\ &= \Delta_h u(t) - \lambda_2 u(t) = f(t)\end{aligned}$$

for all $t \in h\mathbb{Z}$. Therefore we can conclude that $v(t)$ is a solution of (1.1). □

More explicitly, we can obtain the following result.

Theorem 2.2. *Let $\varepsilon > 0$ be a given arbitrary constant, and let $B(\cdot, h)$ be the constant given by (1.5). Define*

$$F(t) = \Delta_h^{-1} f(t)(\lambda_2 h + 1)^{-\frac{t+h}{h}}$$

for $t \in h\mathbb{Z}$. Suppose that (2.1) has real roots λ_1 and λ_2 with $\lambda_i \in \bigcup_{j=1}^4 \Lambda_j$ for $i \in \{1, 2\}$. If a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies

$$\left|\Delta_h^2\phi(t) + \alpha\Delta_h\phi(t) + \beta\phi(t) - f(t)\right| \le \varepsilon$$

for all $t \in h\mathbb{Z}$, then one of the following holds:

(i) if $\lambda_1, \lambda_2 \in \Lambda_1 \cup \Lambda_4$, then the limiting values

$$c_1 = \lim_{t\to\infty}\left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\}$$

and

$$d_1 = \lim_{t\to\infty}\left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_1\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\}$$

exist, and there exists a unique solution

$$x(t) = \left\{\Delta_h^{-1}\left(F(t) + c_1\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_1\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$;

(ii) if $\lambda_1 \in \Lambda_1 \cup \Lambda_4$ and $\lambda_2 \in \Lambda_2 \cup \Lambda_3$, then the limiting values

$$c_2 = \lim_{t\to-\infty}\left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\}$$

and

$$d_2 = \lim_{t\to\infty}\left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\}$$

exist, and there exists a unique solution

$$x(t) = \left\{\Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_2\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$;

(iii) if $\lambda_1, \lambda_2 \in \Lambda_2 \cup \Lambda_3$, then the limiting values c_2 and

$$d_3 = \lim_{t\to-\infty}\left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\}$$

exist, and there exists a unique solution

$$x(t) = \left\{\Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_3\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$.

Proof. Assume that a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies

$$\left|\Delta_h^2\phi(t) + \alpha\Delta_h\phi(t) + \beta\phi(t) - f(t)\right| \le \varepsilon$$

on $h\mathbb{Z}$. Let $\psi(t) = \Delta_h\phi(t) - \lambda_1\phi(t)$ for $t \in h\mathbb{Z}$. Using the above assumption with $\lambda_1 + \lambda_2 = -\alpha$, $\lambda_1\lambda_2 = \beta$, we have (2.3) for $t \in h\mathbb{Z}$.

First we prove case (i). Using $\lambda_2 \in \Lambda_1 \cup \Lambda_4$ and Theorem A, we see that

$$\lim_{t\to\infty}\left\{\psi(t)(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\} = \lim_{t\to\infty}\left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\} = c_1$$

exists, and there exists a unique solution

$$u(t) = \left(F(t) + c_1\right)(\lambda_2 h + 1)^{\frac{t}{h}}$$

of (2.4) such that $|\psi(t) - u(t)| \le B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. That is, (2.5) holds on $h\mathbb{Z}$. Using $\lambda_1 \in \Lambda_1 \cup \Lambda_4$ and Theorem A again, we conclude that the limiting value

$$\lim_{t\to\infty}\left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_1\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\} = d_1$$

exists, and there exists a unique solution

$$v(t) = \left\{\Delta_h^{-1}\left(F(t) + c_1\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_1\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (2.6) such that $|\phi(t) - v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. Using the same argument as in the proof of Theorem 2.1, we see that $v(t)$ is a solution of (1.1). Noticing that $v(t)$ is a unique solution of (1.1) such that $|\phi(t) - v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$.

Next we prove case (ii). Using $\lambda_2 \in \Lambda_2 \cup \Lambda_3$ and Theorem B, we see that the limiting value

$$\lim_{t\to-\infty}\left\{\psi(t)(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\} = \lim_{t\to-\infty}\left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\} = c_2$$

exists, and there exists a unique solution

$$u(t) = \left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}$$

of (2.4) such that (2.5) holds for all $t \in h\mathbb{Z}$. Using $\lambda_1 \in \Lambda_1 \cup \Lambda_4$ and Theorem A, we can conclude that the limiting value

$$\lim_{t\to\infty}\left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\} = d_2$$

exists, and there exists a unique solution

$$v(t) = \left\{\Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_2\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (2.6) such that $|\phi(t) - v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. Repeating the same argument as in the proof of Theorem 2.1, $v(t)$ is a unique solution of (1.1) such that $|\phi(t) - v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$.

We prove case (iii). As in the same argument of the preceding paragraph, using $\lambda_2 \in \Lambda_2 \cup \Lambda_3$ and Theorem B, we see that c_2 exists, and there exists a unique solution

$$u(t) = \left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}$$

of (2.4) such that (2.5) holds on $h\mathbb{Z}$. Using $\lambda_1 \in \Lambda_2 \cup \Lambda_3$ and Theorem B again, we can find the limiting value

$$\lim_{t\to-\infty} \left\{\phi(t)(\lambda_1 h+1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t)+c_2\right)(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}}\right\} = d_3$$

and a unique solution

$$v(t) = \left\{\Delta_h^{-1}\left(F(t)+c_2\right)(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}} + d_3\right\}(\lambda_1 h+1)^{\frac{t}{h}}$$

of (2.6) such that $|\phi(t) - v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. By the same argument as in the proof of Theorem 2.1, $v(t)$ is a unique solution of (1.1) such that $|\phi(t)-v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. □

A natural question now arises. Is $B(\lambda_1, h)B(\lambda_2, h)$ the best HUS constant for (1.1) on $h\mathbb{Z}$? A partial answer to this question is as follows.

Theorem 2.3. *Suppose that* (2.1) *has real roots* λ_1 *and* λ_2 *with* $\lambda_i \in \bigcup_{j=1}^{4} \Lambda_j$ *for* $i \in \{1, 2\}$. *Then the best HUS constant for* (1.1) *on* $h\mathbb{Z}$ *is greater than or equal to*

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1+2/h)(\lambda_2+2/h)|}\right\}.$$

Before to prove this theorem, we will give a lemma.

Lemma 2.1. *Suppose that* (2.1) *has two roots* λ_1 *and* λ_2 *with* $\lambda_i \neq -1/h$ *for* $i \in \{1, 2\}$. *Define*

$$F(t) = \Delta_h^{-1} f(t)(\lambda_2 h+1)^{-\frac{t+h}{h}}$$

and

$$Y(t; \lambda_1, \lambda_2) = \left\{\Delta_h^{-1} F(t)(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}}\right\}(\lambda_1 h+1)^{\frac{t}{h}} \tag{2.7}$$

for $t \in h\mathbb{Z}$. *Then* $Y(t; \lambda_1, \lambda_2)$ *is a solution of* (1.1).

Proof. Since

$$\begin{aligned}
F(t)&(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}}\\
&= \Delta_h Y(t; \lambda_1, \lambda_2)(\lambda_1 h+1)^{-\frac{t}{h}}\\
&= \frac{1}{h}\left\{Y(t+h; \lambda_1, \lambda_2)(\lambda_1 h+1)^{-\frac{t+h}{h}} - Y(t; \lambda_1, \lambda_2)(\lambda_1 h+1)^{-\frac{t}{h}}\right\}\\
&= \frac{1}{h}\{Y(t+h; \lambda_1, \lambda_2) - (\lambda_1 h+1)Y(t; \lambda_1, \lambda_2)\}(\lambda_1 h+1)^{-\frac{t+h}{h}}\\
&= (\Delta_h Y(t; \lambda_1, \lambda_2) - \lambda_1 Y(t; \lambda_1, \lambda_2))(\lambda_1 h+1)^{-\frac{t+h}{h}}
\end{aligned}$$

holds, we have

$$\Delta_h Y(t; \lambda_1, \lambda_2) - \lambda_1 Y(t; \lambda_1, \lambda_2) = F(t)(\lambda_2 h+1)^{\frac{t}{h}}$$

for all $t \in h\mathbb{Z}$. Using this equality, we obtain

$$\begin{aligned}
\Delta_h^2 Y(t; \lambda_1, \lambda_2) &- \lambda_1 \Delta_h Y(t; \lambda_1, \lambda_2)\\
&= \Delta_h F(t)(\lambda_2 h+1)^{\frac{t}{h}}\\
&= \frac{1}{h}\left\{F(t+h)(\lambda_2 h+1)^{\frac{t+h}{h}} - F(t)(\lambda_2 h+1)^{\frac{t}{h}}\right\}\\
&= \frac{1}{h}\left(F(t+h) - \frac{1}{\lambda_2 h+1}F(t)\right)(\lambda_2 h+1)^{\frac{t+h}{h}}\\
&= \left(\Delta_h F(t) + \frac{\lambda_2}{\lambda_2 h+1}F(t)\right)(\lambda_2 h+1)^{\frac{t+h}{h}}\\
&= f(t) + \lambda_2 F(t)(\lambda_2 h+1)^{\frac{t}{h}}\\
&= f(t) + \lambda_2(\Delta_h Y(t; \lambda_1, \lambda_2) - \lambda_1 Y(t; \lambda_1, \lambda_2))
\end{aligned}$$

for all $t \in h\mathbb{Z}$. This means that $Y(t; \lambda_1, \lambda_2)$ is a solution of (1.1). □

Proof of Theorem 2.3. We have only to show that for a given $\varphi(t)$ satisfying

$$\left|\Delta_h^2\varphi(t) + \alpha\Delta_h\varphi(t) + \beta\varphi(t) - f(t)\right| \le \varepsilon$$

on $h\mathbb{Z}$, we find an explicit solution $x(t)$ of (1.1) such that

$$|\varphi(t) - x(t)| = \max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\}$$

for all $t \in h\mathbb{Z}$.

We now consider the second-order difference equation

$$\Delta_h^2\varphi(t) + \alpha\Delta_h\varphi(t) + \beta\varphi(t) - f(t) = \varepsilon(-1)^{\frac{mt}{h}} \tag{2.8}$$

on $h\mathbb{Z}$, where $\varepsilon > 0$ and $m \in \{1, 2\}$. Let

$$\varphi_m(t) = \frac{\varepsilon(-1)^{\frac{mt}{h}}}{\left\{\dfrac{(-1)^m - 1}{h} - \lambda_1\right\}\left\{\dfrac{(-1)^m - 1}{h} - \lambda_2\right\}}$$

and $\varphi(t) = Y(t;\lambda_1,\lambda_2) + \varphi_m(t)$ for all $t \in h\mathbb{Z}$, where $Y(t;\lambda_1,\lambda_2)$ is the function given by (2.7). Note here that $Y(t;\lambda_1,\lambda_2)$ is a solution of (1.1) from Lemma 2.1. Now, we will check that $\varphi(t)$ is a solution of (2.8). From (1.7), we have

$$\Delta_h^2(-1)^{\frac{mt}{h}} = \left\{\frac{(-1)^m - 1}{h}\right\}^2 (-1)^{\frac{mt}{h}}.$$

Using this, we get

$$\begin{aligned}
&\Delta_h^2\varphi_m(t) + \alpha\Delta_h\varphi_m(t) + \beta\varphi_m(t) \\
&= \left[\left\{\frac{(-1)^m - 1}{h}\right\}^2 + \alpha\frac{(-1)^m - 1}{h} + \beta\right] \frac{\varepsilon(-1)^{\frac{mt}{h}}}{\left\{\dfrac{(-1)^m - 1}{h} - \lambda_1\right\}\left\{\dfrac{(-1)^m - 1}{h} - \lambda_2\right\}} \\
&= \varepsilon(-1)^{\frac{mt}{h}}
\end{aligned}$$

for all $t \in h\mathbb{Z}$. That is, $\varphi_m(t)$ is a solution of (2.8) with $f(t) \equiv 0$. Using the above facts, we obtain

$$\begin{aligned}
&\Delta_h^2\varphi(t) + \alpha\Delta_h\varphi(t) + \beta\varphi(t) \\
&\quad= \Delta_h^2 Y(t;\lambda_1,\lambda_2) + \alpha\Delta_h Y(t;\lambda_1,\lambda_2) + \beta Y(t;\lambda_1,\lambda_2) \\
&\qquad + \Delta_h^2\varphi_m(t) + \alpha\Delta_h\varphi_m(t) + \beta\varphi_m(t) \\
&\quad= f(t) + \varepsilon(-1)^{\frac{mt}{h}}.
\end{aligned}$$

This means that $\varphi(t)$ is a solution of (2.8). Therefore,

$$\left|\Delta_h^2\varphi(t) + \alpha\Delta_h\varphi(t) + \beta\varphi(t) - f(t)\right| = \varepsilon$$

holds for all $t \in h\mathbb{Z}$. Let $x_0(t)$ be the general solution of (1.1) with $f(t) \equiv 0$. That is, $x_0(t)$ is written by

$$c_1(\lambda_1 h + 1)^{\frac{t}{h}} + c_2(\lambda_2 h + 1)^{\frac{t}{h}} \quad \text{or} \quad c_1(\lambda_1 h + 1)^{\frac{t}{h}} + c_2 t(\lambda_1 h + 1)^{\frac{t}{h}},$$

where c_1 and c_2 are arbitrary constants. Since $Y(t;\lambda_1,\lambda_2)$ is a solution of (1.1), the general solution of (1.1) is written as

$$x(t) = x_0(t) + Y(t;\lambda_1,\lambda_2)$$

for all $t \in h\mathbb{Z}$. Noticing that $c_1 = c_2 = 0$ holds if and only if $|\varphi(t) - x(t)|$ is bounded on $h\mathbb{Z}$. When $c_1 = c_2 = 0$, we have

$$|\varphi(t) - x(t)| = |\varphi_m(t)| = \frac{\varepsilon}{\left|\lambda_1 + \dfrac{1 - (-1)^m}{h}\right|\left|\lambda_2 + \dfrac{1 - (-1)^m}{h}\right|}$$

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for all $t \in h\mathbb{Z}$ and $m \in \{1, 2\}$. This means that the best HUS constant for (1.1) on $h\mathbb{Z}$ is greater than or equal to

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\}.$$

□

Theorems 2.1 and 2.3 imply the following result.

Corollary 2.4. *Suppose that* (2.1) *has real roots* λ_1 *and* λ_2. *If* $\lambda_1,\ \lambda_2 \in \Lambda_1 \bigcup \Lambda_2$ *or* $\lambda_1,\ \lambda_2 \in \Lambda_3 \bigcup \Lambda_4$, *then* (1.1) *has Hyers–Ulam stability with the best HUS constant* $B(\lambda_1, h)B(\lambda_2, h)$ *on* $h\mathbb{Z}$, *where* $B(\cdot, h)$ *is the constant given by* (1.5).

Proof. From Theorem 2.1, (1.1) has Hyers–Ulam stability with an HUS constant $B(\lambda_1, h)B(\lambda_2, h)$ on $h\mathbb{Z}$. Since

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\} = \frac{1}{|\lambda_1\lambda_2|}$$

if $\lambda_1,\ \lambda_2 \in \Lambda_1 \bigcup \Lambda_2$, and

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\} = \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}$$

if $\lambda_1,\ \lambda_2 \in \Lambda_3 \bigcup \Lambda_4$, we conclude that

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\} = B(\lambda_1, h)B(\lambda_2, h).$$

From Theorem 2.3 it follows that $B(\lambda_1, h)B(\lambda_2, h)$ is the best HUS constant. □

From Corollary 2.4, we obtain the following.

Corollary 2.5. *Suppose that* (2.1) *has exactly one real root* λ *with* $\lambda \in \bigcup_{j=1}^{4} \Lambda_j$. *Then* (1.1) *has Hyers–Ulam stability with the best HUS constant* $B^2(\lambda, h)$ *on* $h\mathbb{Z}$, *where* $B(\cdot, h)$ *is the constant given by* (1.5).

3 Classification of HUS constants by the coefficients

According to Theorem 2.1, we see that the following fact.

Remark 3.1. An HUS constant for (1.1) on $h\mathbb{Z}$ is rewritten as

$$B(\lambda_1, h)B(\lambda_2, h) = \begin{cases} \dfrac{1}{|\lambda_1\lambda_2|} & \text{if } \lambda_1, \lambda_2 \in \Lambda_1 \cup \Lambda_2, \\ \dfrac{1}{|\lambda_1(\lambda_2 + 2/h)|} & \text{if } \lambda_1 \in \Lambda_1 \cup \Lambda_2,\ \lambda_2 \in \Lambda_3 \cup \Lambda_4, \\ \dfrac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|} & \text{if } \lambda_1, \lambda_2 \in \Lambda_3 \cup \Lambda_4, \end{cases}$$

where λ_1 and λ_2 are real roots of (2.1) satisfying $\lambda_i \neq 0$, $-1/h$ and $-2/h$ for $i \in \{1, 2\}$.

Unfortunately, HUS constants in the right-hand side of the equation are implicit expressions. In this section, we will decide HUS constants more explicitly. To be specific, we will classify HUS constants for (1.1) on $h\mathbb{Z}$ by coefficients α and β. Let S be the set

$$S = \left\{(\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \beta \leq \frac{\alpha^2}{4},\ \beta \neq \frac{1}{h}\alpha - \frac{1}{h^2},\ \beta \neq \frac{2}{h}\alpha - \frac{4}{h^2},\ \beta \neq 0\right\}.$$

Since $\beta = \alpha/h - 1/h^2$ is the tangent line to the curve $\beta = \alpha^2/4$ at $(2/h, 1/h^2)$, S is divided into three sets as follows (see Figure 1):

$$S_1 = \left\{ (\alpha,\beta) \in \mathbb{R}^2 \,\middle|\, \frac{1}{h}\alpha - \frac{1}{h^2} < \beta \le \frac{\alpha^2}{4},\ \alpha < \frac{2}{h},\ \beta \neq 0 \right\},$$

$$S_2 = \left\{ (\alpha,\beta) \in \mathbb{R}^2 \,\middle|\, \beta < \frac{1}{h}\alpha - \frac{1}{h^2},\ \beta \neq \frac{2}{h}\alpha - \frac{4}{h^2},\ \beta \neq 0 \right\},$$

$$S_3 = \left\{ (\alpha,\beta) \in \mathbb{R}^2 \,\middle|\, \frac{1}{h}\alpha - \frac{1}{h^2} < \beta \le \frac{\alpha^2}{4},\ \alpha > \frac{2}{h},\ \beta \neq \frac{2}{h}\alpha - \frac{4}{h^2} \right\}.$$

Note that $\beta = 2\alpha/h - 4/h^2$ is the tangent line to the curve $\beta = \alpha^2/4$ at $(4/h, 4/h^2)$; $S_1 \cap S_2$, $S_2 \cap S_3$ and $S_3 \cap S_1$ are empty sets; $S = S_1 \cup S_2 \cup S_3$ holds. The above-mentioned sets are used without notice in this paper.

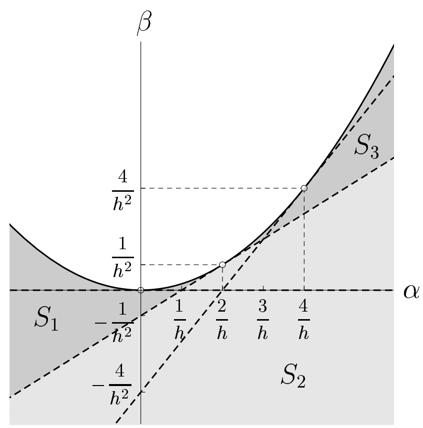

Figure 1: The sets S_1, S_2 and S_3 on the (α,β) plane.

The obtained result is as follows.

Corollary 3.1. *If $(\alpha,\beta) \in S$, then (1.1) has Hyers–Ulam stability with an HUS constant*

$$B\left(\frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, h\right) B\left(\frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}, h\right)$$

on $h\mathbb{Z}$, where $B(\cdot, h)$ is the constant given by (1.5). Furthermore, one of the following holds:

(i) if $(\alpha,\beta) \in S_1$, then the best HUS constant for (1.1) on $h\mathbb{Z}$ is $1/|\beta|$;

(ii) if $(\alpha,\beta) \in S_2$, then an HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{\left|\beta + \left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right)/h\right|};$$

(iii) if $(\alpha,\beta) \in S_3$, then the best HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|\beta - 2\alpha/h + 4/h^2|}.$$

Proof. Suppose that $(\alpha,\beta) \in S$. Let

$$\mu_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \quad \text{and} \quad \mu_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}.$$

Then μ_1 and μ_2 are real roots of (2.1) since $\beta \le \alpha^2/4$ holds. By $\beta \ne \alpha/h - 1/h^2$, (1.2) is satisfied, and therefore, we have $\mu_1 \ne -1/h \ne \mu_2$. From $\beta \ne 2\alpha/h - 4/h^2$ we see that $\mu_1 \ne -2/h \ne \mu_2$. In addition, by $\beta \ne 0$, non of μ_1 and μ_2 are equal to 0. Therefore, $\mu_1,\ \mu_2 \in \bigcup_{j=1}^{4} \Lambda_j$. Using Theorem 2.1, (1.1) has Hyers–Ulam stability with an HUS constant $B(\mu_1, h)\, B(\mu_2, h)$.

Next, we will show that the assertions (i)–(iii). We consider the case $(\alpha, \beta) \in S_1$. From

$$\frac{1}{h}\alpha - \frac{1}{h^2} < \beta \le \frac{\alpha^2}{4}$$

it follows that

$$0 \le \alpha^2 - 4\beta < \alpha^2 - \frac{4}{h}\alpha + \frac{4}{h^2} = \left(\alpha - \frac{2}{h}\right)^2.$$

That is, $0 \le \sqrt{\alpha^2 - 4\beta} < \sqrt{(\alpha - 2/h)^2} = |\alpha - 2/h|$ holds, and therefore, we have

$$\frac{-\alpha - |\alpha - 2/h|}{2} < \mu_2 \le \mu_1 < \frac{-\alpha + |\alpha - 2/h|}{2}. \tag{3.1}$$

By using $\alpha < 2/h$, we obtain $-1/h < \mu_2 \le \mu_1$. This means that $\mu_1,\ \mu_2 \in \Lambda_1 \cup \Lambda_2$. From Corollary 2.4 and Remark 3.1, the best HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|\mu_1\mu_2|} = \frac{1}{|\beta|}.$$

Next, we consider the case $(\alpha, \beta) \in S_2$. Since

$$\beta < \frac{1}{h}\alpha - \frac{1}{h^2}$$

holds, we have $\sqrt{\alpha^2 - 4\beta} > \sqrt{(\alpha - 2/h)^2} = |\alpha - 2/h|$. This means that

$$-\sqrt{\alpha^2 - 4\beta} < \alpha - \frac{2}{h} < \sqrt{\alpha^2 - 4\beta}.$$

Using this inequality we obtain

$$\mu_2 < -\frac{1}{h} < \mu_1.$$

That is, $\mu_1 \in \Lambda_1 \cup \Lambda_2$, $\mu_2 \in \Lambda_3 \cup \Lambda_4$. From Remark 3.1, an HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|\mu_1(\mu_2 + 2/h)|} = \frac{1}{\left|\beta + \left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right)/h\right|}.$$

Finally, we consider the case $(\alpha, \beta) \in S_3$. Using the same argument in the proof of the case $(\alpha, \beta) \in S_1$, we have (3.1). By using $\alpha > 2/h$, we obtain $\mu_2 \le \mu_1 < -1/h$. This and $\beta \ne 2\alpha/h - 4/h^2$ imply that μ_1, $\mu_2 \in \Lambda_3 \cup \Lambda_4$. From Corollary 2.4 and Remark 3.1, the best HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|(\mu_1 + 2/h)(\mu_2 + 2/h)|} = \frac{1}{|\mu_1\mu_2 + 2(\mu_1 + \mu_2)/h + 4/h^2|} = \frac{1}{|\beta - 2\alpha/h + 4/h^2|}.$$

This completes the proof of Corollary 3.1. □

For illustration of the obtained result, we will present an example.

Example. We consider the second-order linear difference equation

$$\Delta_h^2 x(t) + 3\Delta_h x(t) + x(t) = f(t) \tag{3.2}$$

on $h\mathbb{Z}$, where (1.2) and

$$1 \ne \frac{6}{h} - \frac{4}{h^2}$$

hold. Since $(3,1) \in S$, Corollary 3.1 implies that (3.2) has Hyers–Ulam stability. Moreover, fixing the stepsize gives an HUS constant. For example, if $h = 1/3$ then $(3,1) \in S_1$, and therefore, the best HUS constant for (3.2) is one. If $h = 1$ then $(3,1) \in S_2$. So, we get an HUS constant $1/\left(\sqrt{5}-2\right)$. If $h = 3$ then $(3,1) \in S_3$, and thus, the best HUS constant for (3.2) is 9/5.

Remark 3.2. Under the assumption that (α,β) is included in the first quadrant and S, if the stepsize is sufficiently small, then we can choose a h so that $(\alpha,\beta) \in S_1$. On the other hand, if the stepsize is sufficiently large, then we can choose a h so that $(\alpha,\beta) \in S_3$. From Corollary 3.1 and Example 3, we see that the best HUS constant for (1.1) on $h\mathbb{Z}$ is affected by the stepsize. In other words, it is concluded that the best HUS constant changes by the influence of the stepsize.

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Choquet-Iyengar type advanced inequalities

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Abstract

Here we extend advanced known Iyengar type inequalities to Choquet integrals setting with respect to distorted Lebesgue measures and for monotone functions.

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1 Background - I

In the year 1938, Iyengar [7] proved the following interesting inequality.

Theorem 1 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M_1$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leq \frac{M_1(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}. \quad (1)$$

In 2001, X.-L. Cheng [3] proved that

Theorem 2 *Let $f \in C^2([a,b])$ and $|f''(x)| \leq M_2$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \leq$$

$$\frac{M_2}{24}(b-a)^3 - \frac{(b-a)}{16M_2}\Delta_1^2, \quad (2)$$

where

$$\Delta_1 = f'(a) - \frac{2(f(b)-f(a))}{(b-a)} + f'(b).$$

In 2006, [6], the authors proved:

1

Theorem 3 *Let $f \in C^2([a,b])$ and $|f''(x)| \leq M$. Set*

$$I = \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a)). \quad (3)$$

Then

$$-\frac{M(b-a)^3}{24} + \frac{M}{3}\left(\lambda_a^3+\lambda_b^3\right) \leq I \leq$$

$$\frac{M(b-a)^3}{24} - \frac{M}{3}\left[\left(\frac{b-a}{2}-\lambda_a\right)^3 + \left(\frac{b-a}{2}-\lambda_b\right)^3\right], \quad (4)$$

where

$$\lambda_a = \frac{1}{2M}\left(f'\left(\frac{a+b}{2}\right) - f'(a)\right) + \frac{b-a}{4}, \quad (5)$$

$$\lambda_b = \frac{1}{2M}\left(f'(b) - f'\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{4}. \quad (6)$$

In 1996, Agarwal and Dragomir [1] obtained a generalization of (1):

Theorem 4 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > m$ we have $m \leq f'(x) \leq M$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leq$$

$$\frac{(f(b)-f(a)-m(b-a))(M(b-a)-f(b)+f(a))}{2(M-m)}. \quad (7)$$

In [9], Qi proved

Theorem 5 *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that for all $x \in [a,b]$ with $M > 0$ we have $|f''(x)| \leq M$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{(f(a)+f(b))}{2}(b-a) + \frac{\left(1+Q^2\right)}{8}(f'(b)-f'(a))(b-a)^2\right| \leq$$

$$\frac{M(b-a)^3}{24}\left(1-3Q^2\right), \quad (8)$$

where

$$Q^2 = \frac{\left(f'(a)+f'(b)-2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{M^2(b-a)^2 - (f'(b)-f'(a))^2}. \quad (9)$$

Finally in 2005, Zheng Liu, [8], proved the following:

2

Theorem 6 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a, b]$ and for all $x \in [a, b]$ with $M > m$ we have*

$$m \leq \frac{f'(x) - f'(a)}{x - a} \leq M \quad \text{and } m \leq \frac{f'(b) - f'(x)}{b - x} \leq M. \tag{10}$$

Then

$$\left| \int_a^b f(x)\,dx - \frac{(f(a) + f(b))}{2}(b - a) + \left(\frac{1 + P^2}{8}\right)(f'(b) - f'(a))(b - a)^2 \right.$$

$$\left. - \left(\frac{1 + 3P^2}{48}\right)(m + M)(b - a)^3 \right| \leq \frac{(M - m)(b - a)^3}{48}\left(1 - 3P^2\right), \tag{11}$$

where

$$P^2 = \frac{\left(f'(a) + f'(b) - 2\left(\frac{f(b) - f(a)}{b - a}\right)\right)^2}{\left(\frac{M - m}{2}\right)^2 (b - a)^2 - \left(f'(b) - f'(a) - \left(\frac{m + M}{2}\right)(b - a)\right)^2}. \tag{12}$$

In [2] we extended (1) for Choquet integrals. Motivated by these results we extend here Theorems 2-6 to the Choquet integrals setting.

2 Background - II

In the next assume that $(X, \mathcal{F})$ is a measurable space and $(\mathbb{R}^+)$ $\mathbb{R}$ is the set of all (nonnegative) real numbers.

We recall some concepts and some elementary results of capacity and the Choquet integral [4, 5].

Definition 7 *A set function $\mu : \mathcal{F} \to \mathbb{R}^+$ is called a non-additive measure (or capacity) if it satisfies*

(1) $\mu(\varnothing) = 0$;

(2) $\mu(A) \leq \mu(B)$ for any $A \subseteq B$ and $A, B \in \mathcal{F}$.

The non-additive measure μ is called concave if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \tag{13}$$

for all $A, B \in \mathcal{F}$. In the literature the concave non-additive measure is known as submodular or 2-alternating non-additive measure. If the above inequality is reverse, μ is called convex. Similarly, convexity is called supermodularity or 2-monotonicity, too.

First note that the Lebesgue measure λ for an interval $[a, b]$ is defined by $\lambda([a, b]) = b - a$, and that given a distortion function m, which is increasing (or non-decreasing) and such that $m(0) = 0$, the measure $\mu(A) = m(\lambda(A))$ is a distorted Lebesgue measure. We denote a Lebesgue measure with distortion m

by $\mu = \mu_m$. It is known that μ_m is concave (convex) if m is a concave (convex) function.

The family of all the nonnegative, measurable function $f : (X, \mathcal{F}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ is denoted as L_∞^+, where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field of $\mathbb{R}^+$. The concept of the integral with respect to a non-additive measure was introduced by Choquet [4].

Definition 8 *Let $f \in L_\infty^+$. The Choquet integral of f with respect to non-additive measure μ on $A \in \mathcal{F}$ is defined by*

$$(C)\int_A f d\mu := \int_0^\infty \mu(\{x : f(x) \geq t\} \cap A)\, dt, \tag{14}$$

where the integral on the right-hand side is a Riemann integral.

Instead of $(C)\int_X f d\mu$, we shall write $(C)\int f d\mu$. If $(C)\int f d\mu < \infty$, we say that f is Choquet integrable and we write

$$L_C^1(\mu) = \left\{ f : (C)\int f d\mu < \infty \right\}.$$

The next lemma summarizes the basic properties of Choquet integrals [5].

Lemma 9 *Assume that $f, g \in L_C^1(\mu)$.*

(1) $(C)\int 1_A d\mu = \mu(A)$, $A \in \mathcal{F}$.

(2) (Positive homogeneity) For all $\lambda \in \mathbb{R}^+$, we have $(C)\int \lambda f d\mu = \lambda \cdot (C)\int f d\mu$.

(3) (Translation invariance) For all $c \in \mathbb{R}$, we have $(C)\int_A (f + c)\, d\mu = (C)\int_A f d\mu + c\mu(A)$.

(4) (Monotonicity in the integrand) If $f \leq g$, then we have

$$(C)\int f d\mu \leq (C)\int g d\mu.$$

(Monotonicity in the set function) If $\mu \leq \nu$, then we have $(C)\int f d\mu \leq (C)\int f d\nu$.

(5) (Subadditivity) If μ is concave, then

$$(C)\int (f+g)\, d\mu \leq (C)\int f d\mu + (C)\int g d\mu.$$

(Superadditivity) If μ is convex, then

$$(C)\int (f+g)\, d\mu \geq (C)\int f d\mu + (C)\int g d\mu.$$

(6) (Comonotonic additivity) If f and g are comonotonic, then

$$(C)\int (f+g)\, d\mu = (C)\int f d\mu + (C)\int g d\mu,$$

where we say that f and g are comonotonic, if for any $x, x' \in X$, then

$$(f(x) - f(x'))(g(x) - g(x')) \geq 0.$$

4

We next mention the amazing result from [10], which permits us to compute the Choquet integral when the non-additive measure is a distorted Lebesgue measure.

Theorem 10 *Let f be a nonnegative and measurable function on $\mathbb{R}^+$ and $\mu = \mu_m$ be a distorted Lebesgue measure. Assume that $m(x)$ and $f(x)$ are both continuous and $m(x)$ is differentiable. When f is an increasing (non-decreasing) function on $\mathbb{R}^+$, the Choquet integral of f with respect to μ_m on $[0,t]$ is represented as*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(t-x) f(x) dx, \tag{15}$$

however, when f is a decreasing (non-increasing) function on $\mathbb{R}^+$, the Choquet integral of f is

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(x) f(x) dx. \tag{16}$$

Remark 11 *We denote by*

$$\gamma(t,x) := \begin{cases} m'(t-x), \text{ when } f \text{ is increasing (non-decreasing)}, \\ m'(x), \text{ when } f \text{ is decreasing (non-increasing)}. \end{cases} \tag{17}$$

So for f continuous and monotone we can combine (15) and (16) into

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t \gamma(t,x) f(x) dx. \tag{18}$$

3 Main Results

We present the following advanced Choquet-Iyengar type inequalities: The next is based on Theorem 2.

Theorem 12 *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone twice continuously differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and thrice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. Then*

i) if f is increasing and $\left|(m'(t-\cdot) f)''(x)\right| \leq M_2$, $\forall\ x \in [0,t]$, $M_2 > 0$, we have that

$$\Bigg|(C)\int_{[0,t]} f(x) d\mu_m(x) - \frac{t}{2}\left[m'(t) f(0) + m'(0) f(t)\right] +$$

$$\frac{t^2}{8}\left[(m'(0) f'(t) - m'(t) f'(0)) + (m''(t) f(0) - m''(0) f(t))\right]\Bigg| \leq$$

$$\frac{M_2}{24} t^3 - \frac{t}{16 M_2} \Delta_1^{*2}, \tag{19}$$

where

$$\Delta_1^* = (m'(t) f'(0) + m'(0) f'(t)) - \frac{2(m'(0) f(t) - m'(t) f(0))}{t} -$$
$$(m''(t) f(0) + m''(0) f(t)), \tag{20}$$

ii) if f is decreasing and $\left|(m'f)''(x)\right| \leq M_3$, $\forall\ x \in [0,t]$, $M_3 > 0$, we have that

$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}(m'(0) f(0) + m'(t) f(t)) + \right.$$
$$\left.\frac{t^2}{8}\left[(m''(t) f(t) - m''(0) f(0)) + (m'(t) f'(t) - m'(0) f'(0))\right]\right| \leq$$
$$\frac{M_3}{24}t^3 - \frac{t}{16M_3}\Delta_1^{**2}, \tag{21}$$

where

$$\Delta_1^{**} = [m''(t) f(t) + m''(0) f(0)] - \frac{2[m'(t) f(t) - m'(0) f(0)]}{t} +$$
$$[m'(t) f'(t) + m'(0) f'(0)]. \tag{22}$$

Proof. i) If f is increasing and $\left|(m'(t-\cdot) f)''(x)\right| \leq M_2$, $\forall\ x \in [0,t]$, $M_2 > 0$, we have that

$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}(m'(t) f(0) + m'(0) f(t)) + \right.$$
$$\left.\frac{t^2}{8}\Big((m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(0)\Big)\right| =$$
$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}[m'(t) f(0) + m'(0) f(t)] + \right.$$
$$\left.\frac{t^2}{8}\left[(m'(0) f'(t) - m'(t) f'(0)) + (m''(t) f(0) - m''(0) f(t))\right]\right| \overset{(\text{by } (2)\ \&\ (15))}{\leq}$$
$$\frac{M_2}{24}t^3 - \frac{t}{16M_2}\Delta_1^{*2}, \tag{23}$$

where

$$\Delta_1^* = (m'(t-\cdot) f)'(0) - \frac{2(m'(0) f(t) - m'(t) f(0))}{t} + (m'(t-\cdot) f)'(t) =$$
$$(m'(t) f'(0) + m'(0) f'(t)) - \frac{2(m'(0) f(t) - m'(t) f(0))}{t} -$$
$$(m''(t) f(0) + m''(0) f(t)). \tag{24}$$

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ii) If f is decreasing and $\left|(m'f)''(x)\right| \leq M_3$, $\forall\ x \in [0,t]$, $M_3 > 0$, we have that

$$\left|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left(m'(0)f(0) + m'(t)f(t)\right) + \frac{t^2}{8}\left((m'f)'(t) - (m'f)'(0)\right)\right| =$$

$$\left|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left(m'(0)f(0) + m'(t)f(t)\right) + \frac{t^2}{8}\left[\left(m''(t)f(t) - m''(0)f(0)\right) + \left(m'(t)f'(t) - m'(0)f'(0)\right)\right]\right| \overset{\text{(by (2) \& (16))}}{\leq}$$

$$\frac{M_3}{24}t^3 - \frac{t}{16M_3}\Delta_1^{**2}, \tag{25}$$

where

$$\Delta_1^{**} = \left[m''(t)f(t) + m''(0)f(0)\right] + \left[m'(t)f'(t) + m'(0)f'(0)\right] - \frac{2\left[(m'f)(t) - (m'f)(0)\right]}{t}. \tag{26}$$

The theorem is proved. ∎

The next result is based on Theorem 3.

Theorem 13 *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone twice continuously differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and thrice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. Then*

i) if f is increasing and $\left|(m'(t-\cdot)f)''(x)\right| \leq M_1$, $\forall\ x \in [0,t]$, $M_1 > 0$, we call:

$$I_1 = (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left[m'(t)f(0) + m'(0)f(t)\right] + \frac{t^2}{8}\left[\left(m'(0)f'(t) - m'(t)f'(0)\right) + \left(m''(t)f(0) - m''(0)f(t)\right)\right], \tag{27}$$

and

$$\lambda_0^{(1)} = \frac{1}{2M_1}\left[\left(-m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) + m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right) + \left(m''(t)f(0) - m'(t)f'(0)\right)\right] + \frac{t}{4}, \tag{28}$$

and

$$\lambda_t^{(1)} = \frac{1}{2M_1}\left[\left(-m''(0)f(t) + m'(0)f'(t)\right) + \left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) - m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right] + \frac{t}{4}, \tag{29}$$

and we obtain

$$-M_1\frac{t^3}{24}+\frac{M_1}{3}\left(\left(\lambda_0^{(1)}\right)^3+\left(\lambda_t^{(1)}\right)^3\right)\leq I_1\leq$$

$$\frac{M_1t^3}{24}-\frac{M_1}{3}\left[\left(\frac{t}{2}-\lambda_0^{(1)}\right)^3+\left(\frac{t}{2}-\lambda_t^{(1)}\right)^3\right], \tag{30}$$

ii) if f is decreasing and $\left|(m'f)''(x)\right|\leq M_2$, $\forall\ x\in[0,t]$, $M_2>0$, we call:

$$I_2=(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(0)f(0)+m'(t)f(t)\right)+$$

$$\frac{t^2}{8}\left[\left(m''(t)f(t)-m''(0)f(0)\right)+\left(m'(t)f'(t)-m'(0)f'(0)\right)\right] \tag{31}$$

and

$$\lambda_0^{(2)}=\frac{1}{2M_2}\left[\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)+m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right.$$

$$\left.-\left(m''(0)f(0)+m'(0)f'(0)\right)\right]+\frac{t}{4}, \tag{32}$$

and

$$\lambda_t^{(2)}=\frac{1}{2M_2}\left[\left(m''(t)f(t)+m'(t)f'(t)\right)\right.$$

$$\left.-\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)+m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right]+\frac{t}{4}, \tag{33}$$

and we obtain:

$$-\frac{M_2t^3}{24}+\frac{M_2}{3}\left(\left(\lambda_0^{(2)}\right)^3+\left(\lambda_t^{(2)}\right)^3\right)\leq I_2\leq$$

$$\frac{M_2t^3}{24}-\frac{M_2}{3}\left[\left(\frac{t}{2}-\lambda_0^{(2)}\right)^3+\left(\frac{t}{2}-\lambda_t^{(2)}\right)^3\right]. \tag{34}$$

Proof. i) Here f is increasing and $\left|(m'(t-\cdot)f)''(x)\right|\leq M_1$, $\forall\ x\in[0,t]$, $M_1>0$.

We call

$$I_1=(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(t)f(0)+m'(0)f(t)\right)+$$

$$\frac{t^2}{8}\left(\left(m'(t-\cdot)f\right)'(t)-\left(m'(t-\cdot)f\right)'(0)\right)=$$

$$(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left[m'(t)f(0)+m'(0)f(t)\right]+$$

$$\frac{t^2}{8}\left[\left(m'(0)f'(t)-m'(t)f'(0)\right)+\left(m''(t)f(0)-m''(0)f(t)\right)\right]. \tag{35}$$

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We set

$$\lambda_0^{(1)} = \frac{1}{2M_1}\left(\left(m'\left(t-\cdot\right)f\right)'\left(\frac{t}{2}\right) - \left(m'\left(t-\cdot\right)f\right)'\left(0\right)\right) + \frac{t}{4} = \tag{36}$$

$$\frac{1}{2M_1}\left[\left(-m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) + m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right.$$

$$\left. + \left(m''\left(t\right)f\left(0\right) - m'\left(t\right)f'\left(0\right)\right)\right] + \frac{t}{4}, \tag{37}$$

and

$$\lambda_t^{(1)} = \frac{1}{2M_1}\left(\left(m'\left(t-\cdot\right)f\right)'\left(t\right) - \left(m'\left(t-\cdot\right)f\right)'\left(\frac{t}{2}\right)\right) + \frac{t}{4} = \tag{38}$$

$$\frac{1}{2M_1}\left[\left(-m''\left(0\right)f\left(t\right) + m'\left(0\right)f'\left(t\right)\right)\right.$$

$$\left. + \left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) - m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right] + \frac{t}{4}.$$

By Theorem 3 and (15) we get

$$-M_1\frac{t^3}{24} + \frac{M_1}{3}\left(\left(\lambda_0^{(1)}\right)^3 + \left(\lambda_t^{(1)}\right)^3\right) \leq I_1 \leq$$

$$\frac{M_1 t^3}{24} - \frac{M_1}{3}\left[\left(\frac{t}{2} - \lambda_0^{(1)}\right)^3 + \left(\frac{t}{2} - \lambda_t^{(1)}\right)^3\right]. \tag{39}$$

ii) Next f is decreasing and $\left|\left(m'f\right)''\left(x\right)\right| \leq M_2$, $\forall\ x \in [0,t]$, $M_2 > 0$.
We call

$$I_2 = (C)\int_{[0,t]} f\left(x\right)d\mu_m\left(x\right) - \frac{t}{2}\left(m'\left(0\right)f\left(0\right) + m'\left(t\right)f\left(t\right)\right) +$$

$$\frac{t^2}{8}\left(\left(m'f\right)'\left(t\right) - \left(m'f\right)'\left(0\right)\right) =$$

$$(C)\int_{[0,t]} f\left(x\right)d\mu_m\left(x\right) - \frac{t}{2}\left(m'\left(0\right)f\left(0\right) + m'\left(t\right)f\left(t\right)\right) + \tag{40}$$

$$\frac{t^2}{8}\left[\left(m''\left(t\right)f\left(t\right) - m''\left(0\right)f\left(0\right)\right) + \left(m'\left(t\right)f'\left(t\right) - m'\left(0\right)f'\left(0\right)\right)\right].$$

We set

$$\lambda_0^{(2)} = \frac{1}{2M_2}\left[\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) + m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right. \tag{41}$$

$$\left. - \left(m''\left(0\right)f\left(0\right) + m'\left(0\right)f'\left(0\right)\right)\right] + \frac{t}{4},$$

and

$$\lambda_t^{(2)} = \frac{1}{2M_2}\left[\left(m''\left(t\right)f\left(t\right) + m'\left(t\right)f'\left(t\right)\right)\right. \tag{42}$$

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$$-\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)+m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\Bigg]+\frac{t}{4}.$$

By Theorem 3 and (16) we get

$$-\frac{M_2t^3}{24}+\frac{M_2}{3}\left(\left(\lambda_0^{(2)}\right)^3+\left(\lambda_t^{(2)}\right)^3\right)\leq I_2\leq$$

$$\frac{M_2t^3}{24}-\frac{M_2}{3}\left[\left(\frac{t}{2}-\lambda_0^{(2)}\right)^3+\left(\frac{t}{2}-\lambda_t^{(2)}\right)^3\right]. \tag{43}$$

The theorem is proved. ■

The next result is based on Theorem 4.

Theorem 14 *Here $f:\mathbb{R}^+\to\mathbb{R}^+$ is a monotone differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and twice differentiable on $\mathbb{R}^+$, $t\in\mathbb{R}^+$. Then*

i) if f is increasing, and $m_1\leq(m'(t-\cdot)f)'(x)\leq M_1$, $\forall\ x\in[0,t]$, where $M_1>m_1$, we obtain:

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}(m'(t)f(0)+m'(0)f(t))\right|\leq$$

$$\frac{(m'(0)f(t)-m'(t)f(0)-m_1t)(M_1t-m'(0)f(t)+m'(t)f(0))}{2(M_1-m_1)}. \tag{44}$$

ii) if f is decreasing, and $m_2\leq(m'f)'(x)\leq M_2$, $\forall\ x\in[0,t]$, where $M_2>m_2$, we obtain:

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}(m'(0)f(0)+m'(t)f(t))\right|\leq$$

$$\frac{(m'(t)f(t)-m'(0)f(0)-m_2t)(M_2t-m'(t)f(t)+m'(0)f(0))}{2(M_2-m_2)}. \tag{45}$$

Proof. i) Here f is increasing and $m_1\leq(m'(t-\cdot)f)'(x)\leq M_1$, $\forall\ x\in[0,t]$, where $M_1>m_1$. We get, by Theorem 4 and (15), that

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}(m'(t)f(0)+m'(0)f(t))\right|\leq$$

$$\frac{(m'(0)f(t)-m'(t)f(0)-m_1t)(M_1t-m'(0)f(t)+m'(t)f(0))}{2(M_1-m_1)}. \tag{46}$$

ii) Next f is decreasing and $m_2\leq(m'f)'(x)\leq M_2$, $\forall\ x\in[0,t]$, where $M_2>m_2$. We get, by Theorem 4 and (16), that

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}(m'(0)f(0)+m'(t)f(t))\right|\leq$$

$$\frac{\left(m'(t) f(t) - m'(0) f(0) - m_2 t\right)\left(M_2 t - m'(t) f(t) + m'(0) f(0)\right)}{2\left(M_2 - m_2\right)}. \qquad (47)$$

The theorem is proved. ■

The next result is based on Theorem 5.

Theorem 15 *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone twice differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and thrice differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. Then*

i) if f is increasing, and $\left|\left(m'(t - \cdot) f\right)''(x)\right| \leq M_1$, $\forall\, x \in [0, t]$, $M_1 > 0$, we call:

$$Q_1^2 =$$

$$\frac{\left[\left(-m''(t) f(0) + m'(t) f'(0)\right) + \left(-m''(0) f(t) + m'(0) f'(t)\right) - 2\left(\frac{m'(0) f(t) - m'(t) f(0)}{t}\right)\right]^2}{M_1^2 t^2 - \left(-m''(0) f(t) + m'(0) f'(t) + m''(t) f(0) - m'(t) f'(0)\right)^2}$$

and we obtain

$$\Bigg|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}\left[m'(t) f(0) + m'(0) f(t)\right] +$$

$$\left(\frac{\left(1 + Q_1^2\right) t^2}{8}\right)\left(-m''(0) f(t) + m'(0) f'(t) + m''(t) f(0) - m'(t) f'(0)\right)\Bigg| \leq$$

$$\frac{M_1 t^3}{24}\left(1 - 3Q_1^2\right), \qquad (49)$$

ii) if f is decreasing, and $\left|\left(m' f\right)''(x)\right| \leq M_2$, $\forall\, x \in [0, t]$, $M_2 > 0$, we call:

$$Q_2^2 =$$

$$\frac{\left[\left(m''(0) f(0) + m'(0) f'(0) + m''(t) f(t) + m'(t) f'(t)\right) - 2\left(\frac{m'(t) f(t) - m'(0) f(0)}{t}\right)\right]^2}{M_2^2 t^2 - \left[m''(t) f(t) + m'(t) f'(t) - m''(0) f(0) - m'(0) f'(0)\right]^2} \qquad (50)$$

and we obtain

$$\Bigg|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}\left[m'(0) f(0) + m'(t) f(t)\right] +$$

$$\left(\frac{\left(1 + Q_2^2\right) t^2}{8}\right)\left(m''(t) f(t) + m'(t) f'(t) - m''(0) f(0) - m'(0) f'(0)\right)\Bigg| \leq$$

$$\frac{M_2 t^3}{24}\left(1 - 3Q_2^2\right). \qquad (51)$$

Proof. i) If f is increasing, and $\left|(m'(t-\cdot)f)''(x)\right| \leq M_1$, $\forall\ x \in [0,t]$, $M_1 > 0$, we set:

$$Q_1^2 = \frac{\left((m'(t-\cdot)f)'(0) + (m'(t-\cdot)f)'(t) - 2\left(\frac{(m'(t-\cdot)f)(t) - (m'(t-\cdot)f)(0)}{t}\right)\right)^2}{M_1^2 t^2 - \left((m'(t-\cdot)f)'(t) - (m'(t-\cdot)f)'(0)\right)^2} =$$

$$\frac{\left((-m''(t)f(0) + m'(t)f'(0)) + (-m''(0)f(t) + m'(0)f'(t)) - 2\left(\frac{m'(0)f(t) - m'(t)f(0)}{t}\right)\right)^2}{M_1^2 t^2 - (-m''(0)f(t) + m'(0)f'(t) + m''(t)f(0) - m'(t)f'(0))^2}. \tag{52}$$

By Theorem 5 and (15) we derive

$$\Bigg|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left[m'(t)f(0) + m'(0)f(t)\right] +$$

$$\left(\frac{(1+Q_1^2)t^2}{8}\right)(-m''(0)f(t) + m'(0)f'(t) + m''(t)f(0) - m'(t)f'(0))\Bigg| \leq$$

$$\frac{M_1 t^3}{24}\left(1 - 3Q_1^2\right). \tag{53}$$

ii) If f is decreasing, and $\left|(m'f)''(x)\right| \leq M_2$, $\forall\ x \in [0,t]$, $M_2 > 0$, we set:

$$Q_2^2 = \frac{\left((m'f)'(0) + (m'f)'(t) - 2\left(\frac{(m'f)(t) - (m'f)(0)}{t}\right)\right)^2}{M_2^2 t^2 - \left((m'f)'(t) - (m'f)'(0)\right)^2} = \tag{54}$$

$$\frac{\left[(m''(0)f(0) + m'(0)f'(0) + m''(t)f(t) + m'(t)f'(t)) - 2\left(\frac{m'(t)f(t) - m'(0)f(0)}{t}\right)\right]^2}{M_2^2 t^2 - \left[m''(t)f(t) + m'(t)f'(t) - m''(0)f(0) - m'(0)f'(0)\right]^2}.$$

By Theorem 5 and (16) we derive

$$\Bigg|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left[m'(0)f(0) + m'(t)f(t)\right] +$$

$$\left(\frac{(1+Q_2^2)t^2}{8}\right)(m''(t)f(t) + m'(t)f'(t) - m''(0)f(0) - m'(0)f'(0))\Bigg| \leq$$

$$\frac{M_2 t^3}{24}\left(1 - 3Q_2^2\right). \tag{55}$$

The theorem is proved. ■

Finally we apply Theorem 6 to obtain:

Theorem 16 *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone and continuously differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and twice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. We have*

i) If f is increasing, and

$$m_1 \leq \frac{(m'(t-\cdot) f)'(x) - (m'(t-\cdot) f)'(0)}{x} \leq M_1, \tag{56}$$

and

$$m_1 \leq \frac{(m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(x)}{t-x} \leq M_1, \tag{57}$$

$\forall\ x \in [0,t]$, with $m_1 < M_1$, we set:

$$P_1^2 = \frac{\left((m'(t-\cdot) f)'(0) + (m'(t-\cdot) f)'(t) - 2\left(\frac{(m'(t-\cdot)f)(t) - (m'(t-\cdot)f)(0)}{t}\right)\right)^2}{\left(\frac{M_1-m_1}{2}\right)^2 t^2 - \left((m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(0) - \frac{(m_1+M_1)}{2}t\right)^2}. \tag{58}$$

Then

$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{((m'(t-\cdot) f)(0) + (m'(t-\cdot) f)(t))}{2}t + \right.$$

$$\left.\left(\frac{1+P_1^2}{8}\right)\left((m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(0)\right)t^2 - \left(\frac{1+3P_1^2}{48}\right)(m_1+M_1)\, t^3\right|$$

$$\leq \frac{(M_1-m_1)\, t^3}{48}\left(1-3P_1^2\right). \tag{59}$$

ii) If f is decreasing, and

$$m_2 \leq \frac{(m'f)'(x) - (m'f)'(0)}{x} \leq M_2, \tag{60}$$

and

$$m_2 \leq \frac{(m'f)'(t) - (m'f)'(x)}{t-x} \leq M_2, \tag{61}$$

$\forall\ x \in [0,t]$, with $m_2 < M_2$, we set:

$$P_2^2 = \frac{\left((m'f)'(0) + (m'f)'(t) - 2\left(\frac{(m'f)(t) - (m'f)(0)}{t}\right)\right)^2}{\left(\frac{M_2-m_2}{2}\right)^2 t^2 - \left((m'f)'(t) - (m'f)'(0) - \frac{(m_2+M_2)}{2}t\right)^2}. \tag{62}$$

Then

$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{((m'f)(0) + (m'f)(t))}{2}t + \right.$$

$$\left|\left(\frac{1+P_2^2}{8}\right)\left((m'f)'(t)-(m'f)'(0)\right)t^2-\left(\frac{1+3P_2^2}{48}\right)(m_2+M_2)\,t^3\right|\leq$$
$$\frac{(M_2-m_2)\,t^3}{48}\left(1-3P_2^2\right). \tag{63}$$

Example 17 *A well-known distortion function is* $m(t)=\frac{t}{1+t}$, $t\in\mathbb{R}^+$. *We have* $m(0)=0$, $m(t)\geq 0$, $m'(t)=\frac{1}{(1+t)^2}>0$, *that is* m *is strictly increasing. We have that* $m''(t)=-2(1+t)^{-3}$, $m^{(3)}(t)=6(1+t)^{-4}$, *and in general we get that* $m^{(n)}(t)=(-1)^{n+1}n!(1+t)^{-(n+1)}$, $\forall\ n\in\mathbb{N}$.

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SOME RESULTS ABOUT $\Delta\mathcal{I}-$STATISTICALLY PRE-CAUCHY SEQUENCES WITH AN ORLICZ FUNCTION

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Abstract. In this study, we define the concept of $\mathcal{I}-$statistically convergence for difference sequences and we use an Orlicz function to obtain more general results. We also show that an $\Delta\mathcal{I}-$statistically convergent sequence with an Orlicz function is $\Delta\mathcal{I}-$statistically pre-Cauchy .

1. Introduction

In this part, we give a short literature data about $\mathcal{I}-$statistical convergence, statistical pre-Cauchy sequences and difference sequence spaces. As is known, convergence is one of the basic notions of Mathematics and statistical convergence extends the notion. It is easy to see that any convergent sequence is statistically convergent but not conversely. Statistical convergence was given by Zygmund [35] in Warsaw in 1935 and then it was formally introduced by Fast [16] and Steinhaus [33], independently. Later it was reintroduced by Schoenberg [32] . Even now, this concept has very much applications in different areas such as number theory by Erdös and Tenenbaum [10] , measure theory by Miller [26] and summability theory by Freedman and Sember [17] . Statistical convergence is also applied to approximation theory by Gadjiev and Orhan [18], Anastassiou and Duman [1] and Sakaoğlu and Ünver [19]. If we want to briefly remember this concept by using the characteristic function, we should give the following definitions:

Definition 1.1. *Let E be a subset of $\mathbb{N}$, the set of all natural numbers. The natural density of E is defined by*

$$d(E) := \lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_E(j)$$

whenever the limit exists where $\chi(E)$ is characteristic function of E.

Definition 1.2. ([16]) *A number sequence (x_n) is statistically convergent to x provided that for every $\varepsilon > 0$,*

$$d\left\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\right\} = \lim_n \frac{1}{n} \left|\left\{k \leq n : |x_k - x| \geq \varepsilon\right\}\right| = 0$$

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or equivalently there exists a subset $K \subseteq \mathbb{N}$ *with* $d(E) = 1$ *and* $n_0(\varepsilon)$ *such that* $n > n_0(\varepsilon)$ *and* $n \in K$ *imply that* $|x_n - x| < \varepsilon$. *In this case we write* $st - \lim x_n = x$. *Statistical convergent sequences are generally denoted by* S.

$\mathcal{I}-$convergence has emerged as a kind of generalization form of many types of convergence. This means that, if we choose different ideals we will have different convergences such as usual convergence and statistical convergence as we will see from the examples below. In 2000, Koystro et. al. [24] introduced this concept in a metric space and then many concepts studied for statistical convergence have moved to ideal convergence. Before defining $\mathcal{I}-$convergence, the definitions of ideal and filter will be needed.

Definition 1.3. *A non-empty family of sets* $\mathcal{I} \subseteq 2^{\mathbb{N}}$ *is called an ideal if and only if i)* $\emptyset \in \mathcal{I}$, *ii) for each* $A, B \in \mathcal{I}$ *we have* $A \cup B \in \mathcal{I}$ *and iii) for each* $A \in \mathcal{I}$ *and each* $B \subseteq A$ *we have* $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 1.4. *A non-empty family of sets* $\mathcal{F} \subseteq 2^{\mathbb{N}}$ *is a filter in* $\mathbb{N}$ *if and only if i)* $\emptyset \notin \mathcal{F}$, *ii) for each* $A, B \in \mathcal{F}$ *we have* $A \cap B \in \mathcal{F}$ *and iii) for each* $A \in \mathcal{F}$ *and each* $B \supseteq A$ *we have* $B \in \mathcal{F}$.

If $\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter in $\mathbb{N}$.

Remark 1.1. *Generally we will use ideals in our proofs but if the notion is more familiar for filters, we will use the notion of filter.*

Definition 1.5. ([24]) *Let* $\mathcal{I} \subseteq 2^{\mathbb{N}}$ *be a proper ideal on* $\mathbb{N}$. *The real sequence* $x = (x_n)$ *is said to be* $\mathcal{I}-$*convergent to* $x \in \mathbb{R}$ *provided that for each* $\varepsilon > 0$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

The set of all $\mathcal{I}-$*convergent sequences usually denoted by* $c_{\mathcal{I}}$.

More investigations in this direction and more applications can be found in Kostyrko, Salát and Wilezyński's paper. We just want to give some well known examples which we mentioned before.

Example 1.1. *If* $\mathcal{I} = \mathcal{I}_f$ $=\{A \subseteq \mathbb{N} : A$ *is finite*$\}$ *then we have the usual convergence.*

Example 1.2. *If* $\mathcal{I} = \mathcal{I}_d$ $=\{A \subseteq \mathbb{N} : d(A\ ) = 0\}$ *then we have the statistical convergence where d is the asymptotic density of A.*

Following the statistical convergence and $\mathcal{I}-$convergence located an important role in this area, Das, Savaş and Ghosal [6] have introduced the concept of $\mathcal{I}-$statistical convergence as follows and they extend the important summability methods statistical convergence and $\mathcal{I}-$convergence using ideals.

Definition 1.6. ([6])*A sequence* $x = (x_n)$ *is said to be* $\mathcal{I}-$*statistically convergent to* L *for each* $\varepsilon > 0$ *and* $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

We will denote the set of all $\mathcal{I}-$statistically convergent sequences by $S_{\mathcal{I}}$.

Before giving information about the definitions and works of pre-Cauchy sequences, lets remember the definition of an Orlicz function. Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M satisfies the Δ_2-condition if there exits a constant $K > 0$ such that $M(2u) \leq KM(u)$ for all $u \geq 0$. We want to give a little note here that if convexity of Orlicz function M is replaced by $M(x+y) = M(x) + M(y)$ then we get the modulus function which is familiar to us.

Lindendstrauss and Tzafriri [25] used the idea of Orlicz function to define the following sequence space.

$$l_M := \left\{ x \in w : \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

which called an Orlicz sequence space. l_M is a Banach space with the norm

$$\|x\| := \inf\left\{ \rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) \leq 1 \right\}.$$

The notion of statistically pre-Cauchy for real sequences was introduced by Connor, Fridy and Kline [4] in 1994. They proved that statistically convergent sequences are statistically pre-Cauchy and any bounded statistical pre-Cauchy sequence with nowhere dense set of limit points is statistically convergent. Khan and Lohani [20] handled this concept in a different way with the Orlicz function. More works on statistically pre-Cauchy sequences are found in Dutta, Eşi and Tripathy [8] , Dutta and Tripathy [9] and Khan and Tabassum [21] .

As an expected result, in 2012, Khan, Ebedullah and Ahmad [22] defined pre-Cauchy sequences for $\mathcal{I}-$convergence and they introduced the concept of $\mathcal{I}-$pre-Cauchy sequence. They established the criterion for arbitrary sequence to be $\mathcal{I}-$pre-Cauchy and they also gave another criterion for $\mathcal{I}-$convergence.

Definition 1.7. ([22]) *Let $x = (x_n)$ be a sequence and let M be an Orlicz function then x is $\mathcal{I}-$pre-Cauchy if and only if*

$$\mathcal{I} - \lim_n \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|x_k - x_j|}{\rho}\right) = 0$$

for some $\rho > 0$.

Yamancı and Gürdal [34] , Ojha and Srivastava [27] and Saha et. al. [28] have some studies about this new definiton.

Definition 1.8. ([7]) *A sequence $x = (x_n)$ is said to be $\mathcal{I}-$statistically pre-Cauchy if for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n^2} \left|\{(j,k) : |x_k - x_j| \geq \varepsilon\},\ j,k \leq n\right| \geq \delta \right\} \in \mathcal{I}.$$

In another direction, in 1981, $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ difference sequence spaces defined by Kızmaz [23] where l_∞, c and c_0 are bounded, convergent and null sequence spaces, respectively. In this study the sequence $\Delta x = (\Delta x_n)$ defined by $(\Delta x_n) = (x_n - x_{n+1})$ for all $n \in \mathbb{N}$ and some relations between these spaces for example $c_0(\Delta) \subseteq c(\Delta) \subseteq l_\infty(\Delta)$ were obtained. In Et and Çolak's paper [11] Kızmaz's

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results generalized for Δ^m sequences such that,

$$\begin{array}{rcl} c_0(\Delta^m) & = & \{x = (x_n) : \Delta^m x \in c_0\} \\ c(\Delta^m) & = & \{x = (x_n) : \Delta^m x \in c\} \\ l_\infty(\Delta^m) & = & \{x = (x_n) : \Delta^m x \in l_\infty\} \end{array}$$

where $m \in \mathbb{N}$ and $\Delta^m x = (\Delta^m x_n) = (\Delta^{m-1} x_n - \Delta^{m-1} x_{n+1})$ i.e.
$\Delta^m x_n = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} x_{n+v}$. They proved that these spaces are Banach spaces with the norm

$$\|.\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty .$$

Following these definitions, Et [12], Et and Çolak [11], Et and Başarır [13], Aydın and Başar [2], Bektaş et. al. [3], Et and Eşi [14], Savaş [31] and many others searched various properties of this concept. Et and Nuray [15] have introduced the Δ^m-statistical convergence and the set of all Δ^m-statistical convergent sequences was denoted by $S(\Delta^m)$. Following this study, Gümüş and Nuray [19] have extended Δ^m-statistical convergence to Δ^m-ideal convergence.

Definition 1.9. ([19]) *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. The sequence $x = (x_n)$ of real numbers is said to be $\Delta\mathcal{I}-$convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set*

$$\{n \in \mathbb{N} : |\Delta x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

The space of all $\Delta\mathcal{I}-$convergent sequences is denoted by $c_{\mathcal{I}}(\Delta)$.

Before we get to the part where our main results are, we would like to give some expressions that have already been proved before about $\mathcal{I}-$convergence and $\Delta\mathcal{I}-$convergence, without moving away from our aim. At the same time it will be interesting to move these expressions to $\mathcal{I}-$statistical convergence.

Proposition 1.1. *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an ideal in $\mathbb{N}$ and (Δx_n) be a real sequence. Then*

$$c(\Delta) \subseteq c_{\mathcal{I}}(\Delta).$$

Note that the inverse of this proposition is not generally true as can be seen from the following example.

Example 1.3. *For the difference sequence $\Delta x = (\Delta x_n) = \begin{cases} 1, & n \text{ is square} \\ 0, & n \text{ is not square} \end{cases}$, $x \in c_{\mathcal{I}_d}(\Delta)$ but $x \notin c(\Delta)$.*

Definition 1.10. *Let $\mathcal{I}$ be an ideal in $\mathbb{N}$. If $\{n+1 : n \in \mathbb{N}\} \in \mathcal{I}$ for any $A \in \mathcal{I}$, then $\mathcal{I}$ is said to be a translation invariant ideal.*

Corollary 1.1. *If $\mathcal{I}$ is translation invariant and $(x_n) \in c_{\mathcal{I}}$ then $(x_{n+1}) \in c_{\mathcal{I}}$.*

Example 1.4. *$\mathcal{I}_d$ is a translation invariant ideal.*

Proposition 1.2. *If $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an admissible translation invariant ideal then $c_{\mathcal{I}} \subseteq c_{\mathcal{I}}(\Delta)$.*

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2. Main Results

In this section, we define $\Delta\mathcal{I}-$statistical convergent and $\Delta\mathcal{I}-$statistically pre-Cauchy sequences and we give some inclusion theorems.

Definition 2.1. *A sequence $x=(x_n)$ is said to be $\Delta\mathcal{I}-$statistically convergent to L provided that*

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\le n:|\Delta x_k-L|\ge\varepsilon\}\right|\ge\delta\right\}\in\mathcal{I}.$$

In our paper, the set of all $\Delta\mathcal{I}-$statistically convergent sequences will be denoted by $S_{\mathcal{I}}(\Delta)$.

Now, lets evaluate this new definition for the $\mathcal{I}_f$ ideal in the example mentioned above.

Example 2.1. *For the ideal $\mathcal{I}=\mathcal{I}_f$, $S_{\mathcal{I}_f}(\Delta)=S(\Delta)$.*

Definition 2.2. *A sequence $x=(x_n)$ is said to be $\Delta\mathcal{I}-$statistically pre-Cauchy if, for any $\varepsilon>0$ and $\delta>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n^2}\left|\{(j,k):|\Delta x_k-\Delta x_j|\ge\varepsilon,\quad j,k\le n\}\right|\ge\delta\right\}\in\mathcal{I}.$$

Theorem 2.1. *An $\Delta\mathcal{I}-$statistically convergent sequence is $\Delta\mathcal{I}-$statistically pre-Cauchy.*

Proof. Let $x=(x_n)$ be $\Delta\mathcal{I}-$statistically convergent to L. Let $\varepsilon>0$ and $\delta>0$ be given. We know that

$$A=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:|\Delta x_k-L|\ge\frac{\varepsilon}{2}\right\}\right|\ge\delta\right\}\in\mathcal{I}.$$

Then for all $n\in A^c$ where c stands for the complement,

$$\frac{1}{n}\left|\left\{k\le n:|\Delta x_k-L|\ge\frac{\varepsilon}{2}\right\}\right|<\delta \text{ i.e. } \frac{1}{n}\left|\left\{k\le n:|\Delta x_k-L|<\frac{\varepsilon}{2}\right\}\right|>1-\delta.$$

Writing $B_n=\left\{k\le n:|\Delta x_k-L|<\frac{\varepsilon}{2}\right\}$ we observe that for $j,k\in B_n$,

$$|\Delta x_k-\Delta x_j|\le|\Delta x_k-L|+|\Delta x_j-L|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Hence $B_n\times B_n\subset\{(j,k):|\Delta x_k-\Delta x_j|<\varepsilon,\quad j,k\le n\}$ which implies

$$\left[\frac{|B_n|}{n}\right]^2\le\frac{1}{n^2}\left|\{(j,k):|\Delta x_k-\Delta x_j|<\varepsilon,\quad j,k\le n\}\right|.$$

Thus for all $n\in A^c$,

$$\frac{1}{n^2}\left|\{(j,k):|\Delta x_k-\Delta x_j|<\varepsilon,\quad j,k\le n\}\right|\ge\left[\frac{|B_n|}{n}\right]^2>(1-\delta)^2$$

i.e.

$$\frac{1}{n^2}\left|\{(j,k):|\Delta x_k-\Delta x_j|\ge\varepsilon,\quad j,k\le n\}\right|<1-(1-\delta)^2.$$

Let $\delta_1>0$ be given. Choosing $\delta>0$ so that $1-(1-\delta)^2<\delta_1$ we see that $\forall n\in A^c$,

$$\frac{1}{n^2}\left|\{(j,k):|\Delta x_k-\Delta x_j|\ge\varepsilon,\quad j,k\le n\}\right|<\delta_1$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right| \geq \delta_1\right\} \subset A.$$

Since $A \in \mathcal{I}$, we have the proof. □

Theorem 2.2. *Let $x = (x_n)$ be a sequence and M be Orlicz function. Then x is $\Delta\mathcal{I}-$statistically pre-Cauchy if and only if*

$$\mathcal{I} - \lim_n \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) = 0 \text{ for some } \rho > 0.$$

Proof. First suppose that $\mathcal{I} - \lim_n \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) = 0$ for some $\rho > 0$. For each $\varepsilon > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) &= \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| < \varepsilon}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&+ \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \varepsilon}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\geq \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \varepsilon}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\geq M(\varepsilon) \left(\frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right|\right)
\end{aligned}$$

Then for any $\delta > 0$,

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \tfrac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right| \geq \delta\right\} \\
&\subset n \in \mathbb{N} : \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \geq \delta M(\varepsilon)
\end{aligned}$$

Thus x is $\Delta\mathcal{I}-$statistically pre-Cauchy.

Now conversely assume that x is $\Delta\mathcal{I}-$statistically pre-Cauchy and $\varepsilon > 0$ be given. Let $\eta > 0$ be such that $M(\eta) < \frac{\varepsilon}{2}$. Since Orlicz function is bounded, there exists an integer B such that $M(x) < \frac{B}{2}$ for all $x \geq 0$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) &= \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| < \eta}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&+ \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \eta}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\leq M(\eta) + \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \eta}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\leq \frac{\varepsilon}{2} + \frac{B}{2} \left(\frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \eta, \quad j,k \leq n\}\right|\right)
\end{aligned}$$

Since x is $\Delta\mathcal{I}-$statistically pre-Cauchy, for $\delta > 0$,

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n^2} \left| \{(j,k) : |\Delta x_k - \Delta x_j| \geq \eta, \quad j,k \leq n\} \right| \geq \delta \right\} \in \mathcal{I}.$$

Then for $n \in A^c$,

$$\frac{1}{n^2} \left| \{(j,k) : |\Delta x_k - \Delta x_j| \geq \eta, \quad j,k \leq n\} \right| < \delta$$

and so

$$\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho} \right) \leq \frac{\varepsilon}{2} + \frac{B}{2}\delta.$$

Let $\delta_1 > 0$ be given. Then choosing $\varepsilon, \delta > 0$ such that $\frac{\varepsilon}{2} + \frac{B}{2}\delta < \delta_1$ we see that for each $n \in A^c$,

$$\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho} \right) < \delta_1$$

i.e.

$$\left\{ n \in \mathbb{N} : \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho} \right) \geq \delta_1 \right\} \subset A \in \mathcal{I}.$$

□

Theorem 2.3. *Let $x = (x_n)$ be a sequence and M be Orlicz function. Then x is $\Delta\mathcal{I}-$statistically convergent to L if and only if*

$$\mathcal{I} - \lim_n \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) = 0 \quad \textit{for some } \rho > 0.$$

Proof. Suppose that $\mathcal{I} - \lim_n \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) = 0$ for some $\rho > 0$. We have,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) &= \frac{1}{n} \sum_{\substack{k=1 \\ |\Delta x_k - L| < \varepsilon}}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) + \frac{1}{n} \sum_{\substack{k=1 \\ |\Delta x_k - L| \geq \varepsilon}}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) \\ &\geq M(\varepsilon) \frac{1}{n} \left| \{k \leq n : |\Delta x_k - L| \geq \varepsilon\} \right|. \end{aligned}$$

Then for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{k \leq n : |\Delta x_k - L| \geq \varepsilon\} \right| \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) \geq M(\varepsilon).\delta \right\}$$

Due to the statement we accepted at the beginning of the theorem, right hand side belongs to the ideal. As we know from the second expression of ideal, left hand side is also in ideal and this proves the theorem.

Since the second part of the theory is very similar to the second part of the previous theorem, we can easily prove. □

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