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# Daubechies Wavelet Method for Second Kind Fredholm Integral Equations with Weakly Singular Kernel *

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## Abstract

In this paper, the weakly singular Fredholm integral equations of the second kind are solved by the periodized Daubechies wavelets method. In order to obtain a good degree of accuracy of the numerical solutions, the Sidi-Israeli quadrature formulae are used to construct the approximation of the singular kernel functions. By applying the asymptotically compact theory, we prove the convergence of approximate solutions. In addition, the sidi transformation can be used to degrade the singularities when the kernel function is non-periodic. At last, numerical examples show the method is efficient and errors of the numerical solutions possess high accuracy order $O\left(h^{3+\alpha}\right)$, where $h$ is the mesh size.

***Keyword***: Daubechies wavelets; weakly singular kernel; Fredholm integral equation of the second; linear and nonlinear integral equations; convergence rate.

# 1    Introduction

Many problems in science and engineering such as Lapalace's equation, problems in elasticity, conformal mapping, free surface flows and so on, result in Fredholm integral equations with singular or weakly singular (in general logarithmic) and periodic kernels [11]. Therefore, singular or weakly Singular Fredholm linear equations and its nonlinear counterparts are most frequently studied for decades.

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Generally, the weakly singular Fredholm integral equation of the second can be converted into the following form

$$u(x) - \int_0^1 k(x,y)g(u(y))\mathrm{d}y = f(x), \quad x \in [0,1], \tag{1.1}$$

where

$$k(x,y) = H_1(x,y)|x-y|^\alpha (\ln|x-y|)^\beta + H_2(x,y), \ \alpha > -1, \ \beta \geq 0, \tag{1.2}$$

$u(x)$ is an unknown function and $f \in L^2[0,1]$, and $H_j(x,y)$ $(j = 1,2)$ are continuous on $[0,1]$. The integral equation (1.1) is linear when $g(u(y)) = u(y)$, and when $g(u(y)) \neq u(y)$ the equation is nonlinear.

As is known, several different orthonormal basis functions, for example, Chebyshev polynomial [8], Fourier functions [2], and wavelets [3, 4, 5, 6, 7, 9, 10, 13, 14, 16, 17], can be used to approximate the solutions of integral equations. However, for large scale problems, the most attractive one among them may be the wavelet bases, in which the kernel can be transformed to a sparse matrix after discretization. This is mainly due to functions with fast oscillations, or even discontinuities, in localized regions may be approximated well by a linear combination of relatively few wavelets [3].

This paper is organized as follows: in Section 2, the periodized Daubechies wavelets is introduced for solving weakly singular Fredholm integral equations of the second in detail. In Section 3, the convergence and error analysis are investigated. In Section 4, numerical examples are provided to verify the theoretical results. Some useful conclusions are made in Section 5.

## 2 Periodized Daubechies wavelets method

### 2.1 Multiresolution analysis and function expansions

Wavelets are attractive for the numerical solution of integrations, because their vanishing moments property leads to operator compression. Especially, Daubechies wavelets [12, 15] have many good properties and can deal with some types of kernels arising from boundary integral formulation of elliptic PDEs, and the coefficient are often numerically sparse. In fact, there are only $O(n \log n)$ significant elements. Supposed that $\psi$ and $\phi$ be the the wavelet of genus $N$ and Daubechies scaling function respectively. Thus their support are $\mathrm{supp}(\phi) = \mathrm{supp}(\psi) = [0, N-1]$. For any $j, k \in Z$, we introduce the notations $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, then their periodic kin with period-1 can be described by

$$\tilde{\phi}_{j,k}(x) = \sum_{n \in Z} \phi_{j,k}(x+n), \quad \tilde{\psi}_{j,k}(x) = \sum_{n \in Z} \psi_{j,k}(x+n), \quad x \in R, \quad 0 \leq k < 2^j. \tag{2.1}$$

Here $\{\tilde{\phi}_{j,k}(x)\}_{k \in Z}$ and $\{\tilde{\psi}_{j,k}(x)\}_{k \in Z}$ are orthogonal [17]. Defining the periodic spaces $\tilde{V}_j = \mathrm{span}\{\tilde{\phi}_{j,k}\}_{k=0}^{2^j-1}$ and $\tilde{W}_j = \mathrm{span}\{\tilde{\psi}_{j,k}\}_{k=0}^{2^j-1}$. A chain of spaces $\tilde{V}_0 \subset \tilde{V}_1 \cdots \subset$

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$L^2[0,1]$ can be constructed, which subject to the following conditions: $(a)$ $\cup_{j\geq 0}\tilde{V}_j = L^2[0,1]$, $\cap_{j\in Z}\tilde{V}_j = \{0\}$; $(b)$ $h(x)\in\tilde{V}_j \Leftrightarrow h(2x)\in\tilde{V}_{j+1}$; $(c)$ $\tilde{V}_j\oplus\tilde{W}_j = \tilde{V}_{j+1}$, $\tilde{W}_j\perp\tilde{V}_j$. The Daubechies wavelets and scaling functions described above result in the wavelet theory (i.e., multiresolution analysis (MRA)) of $L^2[0,1]$.

Supposed that function $p(x)\in L^2[0,1]$ be approximated by scaling series at resolution $J$ as

$$p(x) = \sum_{k=0}^{2^J-1} c_{J,k}\tilde{\phi}_{J,k}(x) = \Phi^t(x)c, \quad x\in[0,1], \tag{2.2}$$

where

$$\Phi(x) = [\tilde{\phi}_{J,0}(x), \tilde{\phi}_{J,1}(x), \cdots, \tilde{\phi}_{J,2^J-1}(x)], \tag{2.3}$$

and

$$c = (c_{J,0}, c_{J,1}, \cdots, c_{J,2^J-1})^t, \quad c_{J,k} = \int_0^1 p(x)\tilde{\phi}_{J,k}(x)\mathrm{d}x. \tag{2.4}$$

First, we calculate the wavelet coefficient $c_{J,k}$ for nonsingular function $p(x)\in L^2[0,1]$. Let $x_i = i/2^J$, $i = 0, 1, \cdots, 2^{J-1}$. Substituting $x = x_i$ into Eq. (2.2), we have

$$p(x) = \sum_{k=0}^{2^J-1} c_{J,k}\tilde{\phi}_{J,k}(i/2^J) = 2^{J/2}\sum_{k=0}^{2^J-1} c_{J,k}\sum_{n\in Z}\phi_{j,k}(2^J n + i - k). \tag{2.5}$$

By using the relationship between $\mathrm{supp}(\phi)$ and $[0,1]$, we know when $J\geq J_0$ only finite terms of the inner summation in (2.5) contribute the following result

$$n = \begin{cases} 0 \text{ or } 1, & \text{if } 2^J - N + 2 \leq N - 1, \\ 0, & \text{if } 0 \leq k \leq 2^J s - N + 1. \end{cases}$$

Now we write (2.5) as the matrix form

$$p = Tc, \tag{2.6}$$

where $p = [p(0), p(1/2^J), \cdots, p((2^J - 1)/2^J)]^t$, $T$ is the nonsingular matrix which entries are the function values of $\phi(x)$ at integers (i.e., $\phi(0), \phi(1), \cdots, \phi(N-2)$) appear in it, and hence it satisfies

$$\sum_{i=1}^{2^J} T_{ij} = \sum_{j=1}^{2^J} T_{ij} = 2^{J/2}, \quad i, j = 1, 2, \cdots, 2^J. \tag{2.7}$$

Consequently, the function $k(x,y)\in L^2([0,1]\times[0,1])$ in Eq.(1.1) can be approximated at resolution $J$ as

$$k(x,y) = \Phi^t(x)Q\Phi(y), \tag{2.8}$$

where $Q$ is the $2^J\times 2^J$ coefficient matrix. Eq. (2.8) can be written as the following form

$$Q = T^{-1}KT^{-t}, \tag{2.9}$$

where $K$ is the $2^J\times 2^J$ kernel matrix with $K_{i,j} = k(i/2^J, j/2^J)$.

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Secondly, if the function $p(x)$ is singular on $[0, 1]$, some values of $p(x)$ at the dyadic points $x_i = i/2^J$, $i = 0, 1, \cdots, 2^J - 1$ may be unbounded and then $c$ can not be immediately solved from the Eq.(2.6). In order to avoid the Eq.(2.6) being invalid, we can use the method in the literature [17] to compute the values of $p(x)$. Without loss of generality, we assumed that function $p(x) \in L^2[0, 1]$ has only one singular point $x_i = i/2^J$, $i \in \{0, 1, \cdots, 2^J - 1\}$. Then the function value $p(x_i)$ in Eq.(2.6) can be computed via on the following ( see [17])

$$p(i/2^J) = 2^J \int_0^1 p(x)\mathrm{d}x - \sum_{j=0,j\neq i}^{2^J-1} p(j/2^J), \quad i \in \{0, 1, \cdots, 2^J - 1\}, \qquad (2.10)$$

where integration $\int_0^1 p(x)\mathrm{d}x$ can be calculated by Sidi-Israeli quadrature formulae [11].

## 2.2 Kernel function approximation and discretization of singular integral equation

Motivated by the Eq.(2.10) and by thinking $k(x, y)$ as a one-dimensional function of variable $x$ and $y$ respectively, we also have

$$k(x, m/2^J) = 2^J \int_0^1 k(x, y)\mathrm{d}y - \sum_{j=0,j\neq m}^{2^J-1} k(x, j/2^J), \quad m \in \{0, 1, \cdots, 2^J - 1\}. \quad (2.11)$$

The following Theorem 2.1 can be used to construct the kernel approximation of Eq.(1.1).

**Theorem 2.1** [11] Assume that the functions $H_1(x, y)$ and $H_2(x, y)$ are $2\ell$ times differentiable on $[a, b]$. Assume also that the functions $k(x, y)$ are periodic with period $T = b - a$, and that they are $2\ell$ times differentiable on $\widetilde{R} = (-\infty, \infty)\backslash\{x + jT\}_{j=-\infty}^{\infty}$. If $k(x, y) = H_1(x, y)|x - y|^{\alpha}(\ln|x - y|)^{\beta} + H_2(x, y)$, $s > -1$, $\beta = 0, 1$, then the quadrature rules of the following integral

$$I[k(x, y)] = \int_a^b k(x, y)\mathrm{d}y, \qquad (2.12)$$

are

$$I_n[k(x, y)] = h \sum_{j=1,y_j\neq x}^{n} k(x, y_j) + 2[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^{\beta}]H_1(x, x)h^{\alpha+1} + H_2(x, x)h,$$

$$(2.13)$$

and the quadrature errors are

$$E_n[k(x, y)] = 2\sum_{\mu=1}^{\ell-1}[\beta\zeta'(-\alpha - 2\mu) - \zeta(-\alpha - 2\mu)(\ln h)^{\beta}]\frac{H_1^{(2\mu)}(x, y_j)}{(2\mu)!}h^{2\mu+\alpha+1} + o\left(h^{2\ell}\right),$$

$$(2.14)$$

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where $E_n[k(x,y)] = I[k(x,y)] - I_n[k(x,y)]$, and the mesh size is $h = (b-a)/n$.

Let $n = 2^J$ and by (2.13), we can get the Nyström approximation for the kernel function $k(x,y)$

$$k_D(x_i, y_j) = \begin{cases} 2[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta]H_1(x_i, x_i)h^\alpha + H_2(x_i, x_i), & \text{if } i = j, \\ k(x_i, y_j), & \text{if } i \neq j. \end{cases}$$
$$(2.15)$$

Supposed that the kernel function $k(x,y)$, $u(x)$ and $f(x)$ be approximated at resolution $J$ as

$$k(x,y) = \Phi^t(x)Q\Phi(y), \quad f(x) = \Phi^t(x)b \text{ and } u(x) = \Phi^t(x)c, \qquad (2.16)$$

where $c = [c(0), c(1/2^J), \cdots, c((2^J - 1)/2^J)]^t$ is the expansion coefficient vector of $u(x)$. By the orthonormality of periodized wavelets, the integration of the product of the same two scaling function vectors is achieved as

$$\int_0^1 \Phi(x)\Phi^t(x)\mathrm{d}x = I, \qquad (2.17)$$

where $I$ is the $2^J$ by $2^J$ identity matrix. For the linear integral equation, we have

$$\Phi^t(x)c - \int_0^1 \Phi^t(x)Q\Phi(y)\Phi^t(y)c\mathrm{d}y = \Phi^t(x)b. \qquad (2.18)$$

Substituting (2.15), (2.16) and (2.17) into (2.18), and by invoking (2.9), we get a linear system

$$(I - K_D(T^tT)^{-1})u_D = f, \qquad (2.19)$$

where $f = [f(0), f(1/2^J), \cdots, f((2^J - 1)/2^J)]^t$ and $K_D = T^tQT$. Similarly, the nonlinear case for Eq. (1.1) can be transformed into the following by the wavelet method

$$u_D - K_D(TT^t)^{-1}g(u_D) = f, \qquad (2.20)$$

Eq. (2.20) is a system of nonlinear equations about $u$ and can be computed by Newton iteration method.

# 3  Convergence and error analysis

In this section, we mainly study the convergence and error for the linear case of (1.1) by wavelet method.

We write Eq. (1.1) as the operate form

$$(I - \tilde{K})u = f, \qquad (3.1)$$

where

$$(\tilde{K}u)(x) = \int_0^1 k(x,y)u(y)\mathrm{d}y, \qquad (3.2)$$

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with the kernel

$$k(x,y) = H_1(x,y)|x-y|^\alpha (\ln|x-y|)^\beta + H_2(x,y), \ \alpha > -1, \ \beta \geq 0, \qquad (3.3)$$

and the approximation of $\tilde{K}$ is defined by

$$(\tilde{K}_n u)(x) = h \sum_{j=1, y_j \neq x}^{n} k(x, y_j) u(y_j) + \omega_n(x) u(x), \qquad (3.4)$$

where the weight function

$$\omega(x) = 2[\beta \zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta] H_1(x,x) h^{\alpha+1} + H_2(x,x)h. \qquad (3.5)$$

Supposed that the approximation of (3.1) is

$$(I - \tilde{K}_n) u_n(x) = g. \qquad (3.6)$$

**Lemma 3.1** Supposed the the operator $\tilde{K}_n$ is defined by (3.4), then the operator sequence $\{\tilde{K}_n\}$ is asymptotically compactly convergent to $\tilde{K}$, *i.e.*,

$$\tilde{K}_n \overset{a.c}{\to} \tilde{K}, \qquad (3.7)$$

where $\overset{a.c}{\to}$ denotes the asymptotically compact convergence.

**Proof**. Let the continuous kernel approximation of $\tilde{K}$ be defined by

$$k_n^c(x,y) = \begin{cases} k(x,y), & \text{if } |x-y| \geq h, \\ H_1(x,x)h^\alpha (\ln h)^\beta + H_2(x,x), & \text{if } |x-y| < h, \end{cases} \qquad (3.8)$$

and the corresponding operator approximation be

$$(K_n^c u)(x) = h \sum_{j=1}^{n} k_n^c(x, y_j) u(y_j). \qquad (3.9)$$

For any $v \in C[0,1]$, we have

$$\begin{aligned}
\|(\tilde{K} - K_n^c)v\| &= \sup_{\|v\|_\infty \leq 1} \int_0^1 |(k(x,y) - k_n^c(x,y))v(y)|\mathrm{d}y \\
&\leq \int_0^1 |k(x,y) - k_n^c(x,y)|\mathrm{d}y\|v\|_\infty \\
&\leq \int_{|x-y|\leq h} |H_1(x,y)|||x-y|^\alpha(\ln|x-y|)^\beta - h^\alpha(\ln h)^\beta|\mathrm{d}y\|v\|_\infty \\
&\leq \max_{x,y\in C[0,1]} |H_1(x,y)| \int_{|x-y|\leq h} ||x-y|^\alpha(\ln|x-y|)^\beta - h^\alpha(\ln h)^\beta|\mathrm{d}y\|v\|_\infty \\
&= O((\ln h)^\beta h^\alpha)\|v\|_\infty,
\end{aligned}$$
$$(3.10)$$

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hence, we can obtain

$$\|\tilde{K} - K_n^c\| = O((\ln h)^\beta h^\alpha) \to 0, \quad \text{as } h \to 0. \tag{3.11}$$

On the other hand, we know $\omega(x) \to 0$ as $h \to 0$ by (3.5), then

$$\|K_n^c - \tilde{K}_n\| \to 0, \quad \text{as } h \to 0. \tag{3.12}$$

First, there exists a subsequence in $\{K_n^c y_n\}$ for any $y_n \subset C[0,1]$ by (3.11),. Without loss of generality, assumed that $K_n^c y_n \to z$ and by (3.12), then

$$\begin{aligned} \|\tilde{K}_n y_n - z\| &\le \|\tilde{K}_n y_n - K_n^c y_n\| + \|K_n^c y_n - z\| \\ &\le \|\tilde{K}_n - K_n^c\|\|y_n\| + \|K_n^c y_n - z\| \to 0, \end{aligned} \tag{3.13}$$

that is to say, the sequence $\{\tilde{K}_n\}$ is asymptotically compactly convergent. Secondly, for any $y \in C[0,1]$, we have

$$\|\tilde{K}_n y - \tilde{K} y\| \le \|\tilde{K}_n - K_n^c\|\|y\| + \|K_n^c y - \tilde{K} y\| \to 0. \tag{3.14}$$

The proof of Lemma 3.1 is completed. $\square$

**Corollary 3.2** The operator sequence $\{\tilde{K}_n(I - o(h)E)\}$ is asymptotically compactly convergent to $\tilde{K}$, *i.e.*,

$$\tilde{K}_n(I - o(h)E) \overset{a.c}{\to} \tilde{K}. \tag{3.15}$$

where $E$ is a matrix and every element in it is one.

**Proof**. By Lemma 3.1, we know

$$\tilde{K}_n \overset{a.c}{\to} \tilde{K}, \tag{3.16}$$

that is,

$$\|\tilde{K}_n - \tilde{K}\| \to 0. \tag{3.17}$$

Hence, we immediately have

$$\|\tilde{K}_n(I - o(h)E) - \tilde{K}\| \le \|\tilde{K}_n - \tilde{K}\| + \|\tilde{K}_n\|\|o(h)E\| \to 0. \tag{3.18}$$

The proof of Corollary 3.2 is completed. $\square$

Let $x = (i-1)h$, $i = 1, 2, \cdots, 2^J$, where $h = 1/2^J$. Using the trapezoidal rule to approximate Eq.(2.17), then we have $hTT^t = I + o(h)E$, where $E$ is a matrix and every element in it is one. By $(hTT^t)^{-1} = I + o(h)E$, (2.15) and (2.19), we get

$$(I - \tilde{K}_n(hT^{ht}T)^{-1})u_D = f, \tag{3.19}$$

which is equivalent to

$$(I - \tilde{K}_n(I - o(h)E)u_D = f. \tag{3.20}$$

Hence, by the Corollary 3.2 we get the following remark.

**Remark 1** According to the Corollary 3.2, the solutions $u_D$ of Eq.(3.20) by

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Daubechies wavelet method are convergent to the solutions $u_n$ of Eq.(3.6) when $h \to 0$.

**Theorem 3.3** The solutions of Eq.(3.6) have asymptotic expansions hold at nodes

$$u_n(x) = u(x) + \sigma_1(x)h^{\alpha+3} + \sigma_2(x)h^{\alpha+3}\ln h + o(h^{\alpha+5}\ln h), \qquad (3.21)$$

where $\sigma_j(x) \in C[0,1]$, $j = 1, 2$ are independent of $h$, and $\sigma_2 = 0$ when $\beta = 0$ and $\alpha > -1$, or $\beta = 1$ and $\alpha = 0$.

**Proof**. We construct the auxiliary equation

$$(I - \tilde{K})\sigma = P(x), \qquad (3.22)$$

where

$$P(x) = [\beta\zeta'(-\alpha - 2) - \zeta(-\alpha - 2)(\ln h)^\beta](H_1 u)^{(2)}h^{3+\alpha}. \qquad (3.23)$$

By invoking Eq.(2.14), we have

$$(\tilde{K}_n - \tilde{K})u(x) = -P(x) + o(h^{5+\alpha}\ln h). \qquad (3.24)$$

Using (3.22), we get

$$\begin{aligned}(I - \tilde{K}_n)(u_n - u - h^{3+\alpha}\sigma) &= f - u + \tilde{K}_n u + h^{3+\alpha}(I - \tilde{K}_n)\sigma \\ &= (\tilde{K}_n - \tilde{K})u + h^{3+\alpha}(I - \tilde{K})\sigma + h^{3+\alpha}(\tilde{K}_n - \tilde{K})\sigma \\ &= o(h^{\alpha+5}\ln h),\end{aligned} \qquad (3.25)$$

that is,

$$u_n - u - h^{3+\alpha}\sigma = o(h^{\alpha+5}\ln h). \qquad (3.26)$$

From (3.22), we obtain

$$\sigma = -\sigma_1 - \sigma_2(\ln h)^\beta, \qquad (3.27)$$

where

$$\sigma_1 = -\beta\zeta'(-\alpha-2)(I-\tilde{K})^{-1}(H_1 u)^{(2)}, \quad \text{and} \quad \sigma_2 = \zeta(-\alpha-2)(I-\tilde{K})^{-1}(H_1 u)^{(2)}. \quad (3.28)$$

Substituting (3.28) into (3.26), and by $\zeta(-2) = 0$ (see [1]), we know that (3.21) holds. The proof of Theorem 3.3 is completed. $\square$

**Remark 2** According to the Theorem 3.3 and Remark 1, the numerical solutions $u_D$ of Eq.(3.19) possess high accuracy order $O(h^{3+\alpha})$ as $h \to 0$.

# 4    Numerical experiments

In this section, two numerical examples about the Fredholm equations are computed by Daubechies wavelet method. Let $err_n^u(x) = |u(x) - u_n(x)|$ be the errors by Daubechies wavelet method using $n$ ($= 2^J$ $J = 3, \cdots, 8$) nodes, and let $EOC = \log(err_n/err_{2n})/\log 2$ be the estimated order of convergence.

If the kernel function $k(x, y)$ of Eq.(1.1) is not periodic, we can apply the Sidi transformation for Eq.(1.1) and make the kernel be periodic. The Sidi transformation is defined by (see [18])

$$\psi_\gamma(t) = \int_0^t (\sin\pi\tau)^\gamma d\tau \Big( \int_0^1 (\sin\pi\tau)^\gamma d\tau \Big)^{-1} : [0,1] \to [0,1], \quad \gamma \geq 1.$$

In the following three examples, the errors and error ratio of numerical solutions at the selected points $x_1 = 0$, $x_2 = 0.25$ and $x_3 = 0.5$ by Daubechies wavelet method using transformation $\psi_6(t)$ are listed in tables.

**Example 1.** Consider the linear Fredholm equation of the first kind

$$u(x) + \int_0^1 \ln|x - y| u(y) dy = g(x)$$

where $g(x) = x^2 \ln x/2 + (1 - x^2) \ln(1 - x)/2 + x/2 - 1/4$ and the exact solution is $u(x) = x$. We use periodic Daubechies wavelet of genus $D = 12$ as basis functions to compute the errors for Example 1 using different resolutions. The plots of computed errors are shown in Figure 1 and the errors and error ratio of numerical solutions are listed in Table 1. From the results in Table 1, we can see $EOC \approx 3$.

Table 1: The Errors of $u$.

| $J$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $err_n^u(x_1)$ | 2.058-02 | 2.111-03 | 3.214-04 | 3.935-05 | 4.896-06 | 6.116-07 |
| $EOC(x_1)$ | – | 3.2857 | 2.715 | 3.030 | 3.007 | 3.001 |
| $err_n^u(x_2)$ | 2.328-02 | 2.485-03 | 3.607-04 | 4.401-05 | 5.481-06 | 6.847-07 |
| $EOC(x_2)$ | – | 3.228 | 2.784 | 3.035 | 3.006 | 3.001 |
| $err_n^u(x_3)$ | 2.278-02 | 9.764-05 | 1.752-05 | 2.222-06 | 2.778-07 | 3.474-08 |
| $EOC(x_3)$ | – | 7.866 | 2.478 | 2.979 | 3.000 | 3.000 |
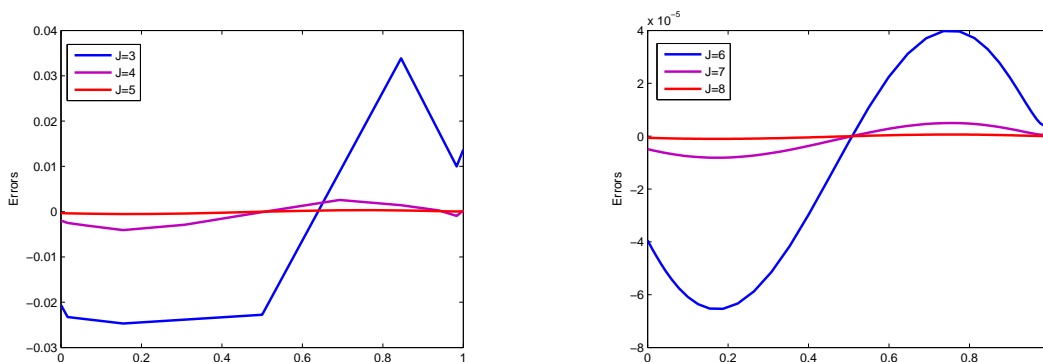


Figure 1: *The error distributions of Example 1 at different resolutions.*

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**Example 2.** Solving the following non-periodic second kind Fredholm integral equation with algebraic singular kernel

$$u(x) + \int_0^1 |x - y|^{-1/2} u(y) \mathrm{d}y = g(x),$$

where $g(x) = x + 2(x\sqrt{(x)} - x^{2/3}/3 + x\sqrt{(1-x)} + (1-x)^{2/3}/3)$ and the exact solution is $u(x) = x$. We also use periodic Daubechies wavelet of genus $D = 12$ as basis functions to compute the errors using different resolutions. The plots of computed errors are shown in Figure 2 and the errors and error ratio of numerical solutions are listed in Table 2. From the results in Table 2, we can see $EOC \approx 2.5$.

Table 2: The Errors of $u$.

| $J$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $err_n^u(x_1)$ | 1.713-03 | 3.964-05 | 1.638-05 | 2.627-06 | 4.505-07 | 7.898-08 |
| $EOC(x_1)$ | – | 5.433 | 1.275 | 2.640 | 2.544 | 2.512 |
| $err_n^u(x_2)$ | 8.526-03 | 3.899-04 | 3.847-05 | 4.843-06 | 8.529-07 | 1.508-07 |
| $EOC(x_2)$ | – | 4.451 | 3.341 | 2.990 | 2.505 | 2.500 |
| $err_n^u(x_3)$ | 4.698-03 | 2.975-04 | 6.878-05 | 1.216-05 | 2.150-06 | 3.800-07 |
| $EOC(x_3)$ | – | 3.981 | 2.113 | 2.499 | 2.500 | 2.500 |
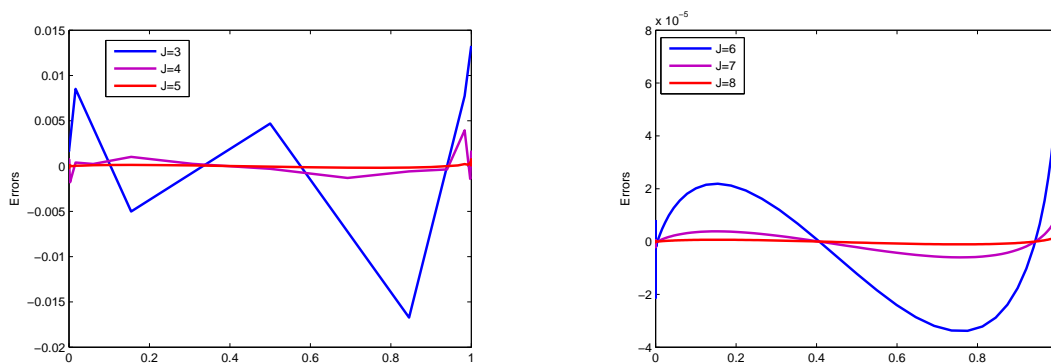


Figure 2: *The error distributions of Example 2 at different resolutions.*

**Example 3.** Solving the following nonlinear second kind Fredholm integral equation with weakly singular kernel

$$u(x) + \int_0^1 \ln|x - y| g(u(y)) \mathrm{d}y = f(x),$$

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where

$$f(x) = (x - 0.5)^{2/3} + \frac{1}{3}(x^2 - x + 1/3) - x(x^2/3 - x/2 + 0.25)\ln(\frac{x}{1-x}) - \frac{1}{12}\ln(1-x).$$

The exact solution is $u(x) = (x - 0.5)^{2/3}$. The periodic Daubechies wavelets of genus $D = 12$ as basis functions are used to compute the errors. The Newton iteration method is used for solve Example 3 and the initial vector of $u_0$ is given by $u_0 = (1, 1, \cdots, 1)^t_{2^J \times 1}$. After 4 iterations the errors are shown in Fig.3. The errors and error ratio of numerical solutions are listed in Table 3. From Table 3, we can see $EOC \approx 3$.

Table 3: The Errors of $u$.

| $J$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $err_n^u(x_1)$ | 4.900-04 | 4.755-05 | 4.371-06 | 5.481-07 | 6.865-08 | 8.582-09 |
| $EOC(x_1)$ | – | 3.365 | 3.443 | 2.996 | 2.997 | 3.000 |
| $err_n^u(x_2)$ | 1.659-03 | 1.565-04 | 1.714-05 | 2.141-06 | 2.673-07 | 3.340-08 |
| $EOC(x_2)$ | – | 3.407 | 3.190 | 3.001 | 3.002 | 3.001 |
| $err_n^u(x_3)$ | 6.442-04 | 2.102-04 | 6.570-05 | 8.145-06 | 1.004-06 | 1.252-07 |
| $EOC(x_3)$ | – | 1.616 | 1.678 | 3.012 | 3.019 | 3.003 |
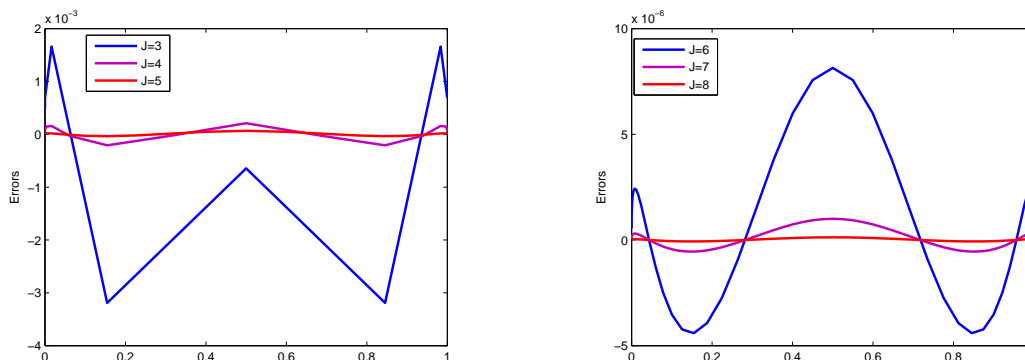


Figure 3: *The error distributions of Example 3 at different resolutions.*

# 5    Conclusions

In this paper, the Sidi-Israeli quadrature formula is used to construct the approximation of kernel functions and then the Daubechies wavelet method is used to solve Eq. (1.1). when the kernel functions are not periodic, we can apply the Sidi transformation for Eq.(1.1) and make the kernels be periodic. Because the wavelet integrations are completely avoided and the expansion coefficients obtained here are exact, which

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makes the wavelets method has a good degree of accuracy. In addition, the Daubechies wavelets method is used for linear Fredholm integration equation, the discrete matrix of the associated linear system can be transformed into a very sparse and symmetrical one. Accordingly, many preconditioners can be used to reduce the computational cost.

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# Some fixed point results in ordered complete dislocated quasi $G_d$ metric space

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*Abstract:* In this paper, we discuss the fixed points of mappings satisfying contractive type condition on a closed ball in an ordered complete dislocated quasi G metric space. The notion of dominated mappings is applied to approximate the unique solution of non linear functional equations. An example is given to show the validity of our work. Our results improve/generalize several well known recent and classical results.

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*Keywords and phrases:* fixed point; contractive dominated mappings; closed ball; ordered complete dislocated quasi metric spaces.

## 1    Introduction and Preliminaries

Let $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of $T$ if $x = Tx$. Let $x_0$ be an arbitrary chosen point in $X$. Define a sequence $\{x_n\}$ in $X$ by a simple iterative method given by $x_{n+1} = Tx_n$, where $n \in \{0, 1, 2, 3, ...\}$. Such a sequence is called a picard iterative sequence and its convergence plays a very important role in proving existence of fixed point of a mapping $T$. A self mapping $T$ on a metric space $X$ is said to be a Banach contraction mapping if,

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$ where $0 \leq k < 1$. Recently, many results appeared in literature related to fixed point results in complete metric spaces endowed with a partial ordering . Ran and Reurings [17] proved an analogue of Banach's fixed point theorem in metric space endowed with partial order and gave applications to matrix equations. Subsequently, Nieto et. al. [12] extended the results of [17] for non decreasing mappings and applied this results obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. On the other hand in 2005, Mustafa and Sims in [14] introduce the notion of a generalized metric space as generalization the usual metric space. Mustafa and others studied fixed point theorems for mappings satisfying different contractive conditions for further useful results can be seen in [3, 8, 9, 10, 15, 16, 21]. Recently, Arshad et. al. [4] proved a result concerning the existence of fixed points of a mapping satisfying a contractive condition on closed ball in a complete dislocated metric space. For further results on closed ball we refer the reader to [5, 6, 7, 13, 20] and references their in. The dominated mapping [2] which satisfies the condition $fx \preceq x$ occurs very naturally in several practical problems . For example $x$ denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$.

In this paper we have obtained fixed point results for dominated self- mappings in an ordered complete dislocated quasi $G_d$ metric space on a closed ball

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under contractive condition to generalize, extend and improve some classical fixed point results. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point. Our results do not exists even yet in metric spaces. An example shows how this result can be used when the corresponding results cannot.

**Definition 1** *Let $X$ be a nonempty set and let $G_d : X \times X \times X \to R^+$ be a function satisfying the*

following axioms:
(i) If $G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y) = 0$,then $x = y = z$,
(ii) $G_d(x, y, z) \leq G_d(x, a, a) + G_d(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then the pair $(X, G_d)$ is called the dislocated quasi $G_d$-metric space. It is clear that if
$G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y) = 0$ then from (i) $x = y = z$. But if $x = y = z$ then $G_d(x, y, z)$ may not be 0. It is observed that if $G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y)$ for all $x, y, z \in X$, then $(X, G_d)$ becomes a dislocated $G_d$-metric space.

**Example 2** *If $X = R^+ \cup \{0\}$ then $G_d(x, y, z) = x + \max\{x, y, z\}$ defines a dislocated quasi metric $G$ on $X$.*

**Definition 3** *Let $(X, G_d)$ be a $G_d$-metric space, and let $\{x_n\}$ be a sequence of points in $X$, a point $x$ in $X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n \to \infty} G_d(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is $G_d$-convergent to $x$. Thus, if $x_n \to x$ in a $G_d$-metric space $(X, G_d)$, then for any $\in > 0$, there exists $n, m \in \mathbb{N}$ such that $G_d(x, x_n, x_m) < \in$, for all $n, m \geq N$.*

**Definition 4** *Let $(X, G_d)$ be a $G_d$-metric space. A sequence $\{x_n\}$ is called $G_d$-Cauchy sequence if, for each $\in > 0$ there exists a positive integer $n^\star \in \mathbb{N}$ such that $G_d(x_n, x_m, x_l) < \in$ for all $n, l, m \geq n^\star$; i.e. if $G_d(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

**Definition 5** *$G_d$-metric space $(X, G_d)$ is said to be $G_d$-complete if every $G_d$-Cauchy sequence in $(X, G_d)$ is $G_d$-convergent in $X$.*

**Proposition 6** *Let $(X, G_d)$ be a $G_d$-metric space, then the following are equivalent:*

(1) $\{x_n\}$ is $G_d$ convergent to $x$.
(2) $G_d(x_n, x_n, x) \to 0$ as $n \to \infty$.
(3) $G_d(x_n, x, x) \to 0$ as $n \to \infty$.
(4) $G_d(x_n, x_m, x) \to 0$ as $m \, n \to \infty$.

**Definition 7** *Let $(X, G_d)$ be a $G_d$-metric space then for $x_0 \in X$, $r > 0$, the closed ball with centre $x_0$ and radius $r$ is,*

$$\overline{B(x_0, r)} = \{y \in X : G_d(x_0, y, y) \leq r\}.$$

2

**Definition 8** *[2] Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.*

**Definition 9** *[2] Let $(X, \preceq)$ be a partially ordered set. A self mapping $f$ on $X$ is called dominated if $fx \preceq x$ for each $x$ in $X$.*

**Example 10** *[2] Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore $f$ is a dominated map.*

## 2   Fixed Points of Contractive Mapping

**Theorem 11** *Let $(X, \preceq, G_d)$ be an ordered complete dislocated quasi $G_d$ metric space, and $T : X \to X$ be a dominated mapping. Suppose there exists $a, b$ such that $a + 3b < 1$ and for all comparable elements $x, y$ and $z$ in $\overline{B(x_0, r)}$, with $x_0 \in \overline{B(x_0, r)}$, $r > 0$,.*

$$
\begin{aligned}
G_d(Tx, Ty, Tz) &\leq \quad a \; G_d(x, y, z) + b \; [G_d(x, Tx, Tx) \\
&\qquad + G_d(y, Ty, Ty) + G_d(z, Tz, Tz)]
\end{aligned}
\tag{2.1}
$$

$$
\text{where } \lambda = \frac{a + b}{1 - 2b}
$$

$$
\text{and } G_d(x_0, Tx_0, Tx_0) \leq (1 - \lambda)r.
\tag{2.2}
$$

*If for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$ and*

$$
\begin{aligned}
&G(x_0, Tx_0, Tx_0) + G(v, Tv, Tv) + G(v, Tv, Tv) \\
&\leq \quad G(x_0, v, v) + G(Tx_0, Tv, Tv) + G(Tx_0, Tv, Tv)
\end{aligned}
\tag{2.3}
$$

*then there exists a point $x^\star$ in $\overline{B(x_0, r)}$ such that $G_d(x^\star, x^\star, x^\star) = 0$ and $x^\star = Tx^\star$.*

**Proof.** *Consider a picard sequence $x_{n+1} = Tx_n$ with initial guess $x_0$. As $x_{n+1} = Tx_n \preceq x_n$ for all $n \in \{0\} \cup \mathbb{N}$. By inequality (2.2), $G_d(x_0, x_1, x_1) \leq r$. It implies that $x_1 \in \overline{B(x_0, r)}$. Similarly $x_2 \ldots x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$.*

$$
\begin{aligned}
G_d(x_j, x_{j+1}, x_{j+1}) &= \quad G_d(Tx_{j-1}, Tx_j, Tx_j) \leq a \; G_d(x_{j-1}, x_j, x_j) \\
&\qquad + b[G_d(x_{j-1}, Tx_{j-1}, Tx_{j-1}) + G_d(x_j, x_{j+1}, x_{j+1}) \\
&\qquad + G_d(x_j, x_{j+1}, x_{j+1})] \\
(1 - 2b)G_d(x_j, x_{j+1}, x_{j+1}) &\leq \quad (a + b)G_d(x_{j-1}, x_j, x_j) \\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq \quad \frac{(a + b)}{(1 - 2b)}G_d(x_{j-1}, x_j, x_j) \\
&\quad \vdots \\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq \quad \lambda^j G_d(x_0, x_1, x_1).
\end{aligned}
\tag{2.4}
$$

3

*Now by using the inequality (2.2) and (2.4) we have*

$$
\begin{aligned}
G_d(x_j, x_{j+1}, x_{j+1}) &\leq G_d(x_0, x_1, x_1) + G_d(x_1, x_2, x_2) + \cdots + G_d(x_j, x_{j+1}, x_{j+1}) \\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq (1-\lambda)r + \lambda(1-\lambda)r + \cdots + \lambda^j(1-\lambda)r \\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq r(1-\lambda)[1 + \lambda + \lambda^2 + \cdots + \lambda^j] \\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq r(1-\lambda)\frac{(1-\lambda^{j+1})}{(1-\lambda)} \leq r \\
&\Rightarrow G_d(x_j, x_{j+1}, x_{j+1}) \leq r.
\end{aligned}
$$

*Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$. Now inequality (2.4) can be written as in the form of*

$$
G_d(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G_d(x_0, x_1, x_1) \text{ for all } n \in \mathbb{N}. \qquad (2.5)
$$

*By using inequality (2.5) we get*

$$
\begin{aligned}
G_d(x_n, x_{n+i}, x_{n+i}) &\leq G_d(x_n, x_{n+1}, x_{n+1}) + \cdots + G_d(x_{n+i-1}, x_{n+i}, x_{n+i}) \\
G_d(x_n, x_{n+i}, x_{n+i}) &\leq \frac{\lambda^n(1-\lambda^i)}{(1-\lambda)} G_d(x_0, x_1, x_1) \to 0 \text{ as } n \to \infty \qquad (2.6)
\end{aligned}
$$

*Notice that the sequence $\{x_n\}$ is Cauchy sequence in $\left(\overline{B(x_0, r)}, G_d\right)$. Therefore there exist a point $x^\star \in \overline{B(x_0, r)}$.*

$$
\begin{aligned}
\lim_{n\to\infty} G_d(x_n, x^\star, x^\star) &= \lim_{n\to\infty} G_d(x^\star, x^\star, x_n) = 0 \\
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + G_d(x_n, Tx^\star, Tx^\star)
\end{aligned}
$$

*By assumption $x^\star \preceq x_n \preceq x_{n-1}$, therefore,*

$$
\begin{aligned}
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + G_d(Tx_{n-1}, Tx^\star, Tx^\star) \\
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + a\, G_d(x_{n-1}, x^\star, x^\star) \\
&\quad + b[G_d(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G_d(x^\star, Tx^\star, Tx^\star) \\
&\quad\quad G_d(x^\star, Tx^\star, Tx^\star)] \\
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + a\, G_d(x_{n-1}, x^\star, x^\star) \\
&\quad + b[G_d(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + 2G_d(x^\star, Tx^\star, Tx^\star) \\
(1-2b)G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + a\, G_d(x_{n-1}, x^\star, x^\star) \\
&\quad + b\, G_d(x_{n-1}, x_n, x_n)
\end{aligned}
$$

*Taking $\lim_{n\to\infty}$ both sides and using (2.6) we have*

$$
\begin{aligned}
(1-2b)G_d(x^\star, Tx^\star, Tx^\star) &\leq 0 + a(0) + b(0) \\
&\Rightarrow G_d(x^\star, Tx^\star, Tx^\star) \leq 0 \\
&\Rightarrow x^\star = Tx^\star. \qquad (2.7)
\end{aligned}
$$

4

*Similarly* $G_d(Tx^\star, Tx^\star, x^\star) = 0$ *and* $G_d(Tx^\star, x^\star, Tx^\star) = 0$ *and hence* $x^\star = Tx^\star$. *Now*

$$
\begin{aligned}
G_d(x^\star, x^\star, x^\star) &= G_d(Tx^\star, Tx^\star, Tx^\star) \leq a\ G_d(x^\star, x^\star, x^\star) \\
&\quad +3bG_d(x^\star, Tx^\star, Tx^\star) \\
(1-a-3b)G_d(x^\star, x^\star, x^\star) &\leq 0 \\
&\Rightarrow G_d(x^\star, x^\star, x^\star) \leq 0.
\end{aligned}
$$

*This implies that* $G_d(x^\star, x^\star, x^\star) = 0$.

**Uniqueness:**

*Let* $y^\star$ *be another point in* $\overline{B(x_0, r)}$ *such that*

$$
\begin{aligned}
y^\star &= Ty^\star. &&(2.8) \\
G_d(y^\star, y^\star, y^\star) &= G_d(Ty^\star, Ty^\star, Ty^\star) \leq a\ G_d(y^\star, y^\star, y^\star) \\
&\quad +3b[G_d(y^\star, Ty^\star, Ty^\star)] \\
(1-a-3b)G_d(y^\star, y^\star, y^\star) &\leq 0 \\
&\Rightarrow G_d(y^\star, y^\star, y^\star) \leq 0. \\
&\Rightarrow G_d(y^\star, y^\star, y^\star) = 0.
\end{aligned}
$$

*If* $x^\star$ *and* $y^\star$ *are comparable then*

$$
\begin{aligned}
G_d(x^\star, y^\star, y^\star) &= G_d(Tx^\star, Ty^\star, Ty^\star) \leq a\ G_d(x^\star, y^\star, y^\star) \\
&\quad +b[G_d(x^\star, Tx^\star, Tx^\star) + 2G_d(y^\star, Ty^\star, Ty^\star)] \\
(1-a)G_d(x^\star, y^\star, y^\star) &\leq 0 \\
&\Rightarrow G_d(x^\star, y^\star, y^\star) = 0.
\end{aligned}
$$

*Similarly,* $G_d(y^\star, y^\star, x^\star) = 0$. *This shows that* $x^\star = y^\star$.

*If* $x^\star$ *and* $y^\star$ *are not comparable then there exist a point* $v \in \overline{B(x_0, r)}$ *which is a lower bound of both* $x^\star$ *and* $y^\star$. *Now we will to prove that* $T^n v \in \overline{B(x_0, r)}$. *Moreover by assumptions* $v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0$. *Now by using (2.1), we have,*

$$
G_d(Tx_0, Tv, Tv) \leq a\ G_d(x_0, v, v) + b\ [G_d(x_0, x_1, x_1) + 2G_d(v, Tv, Tv)].
$$

*By using (2.3), we have*

$$
\begin{aligned}
G_d(Tx_0, Tv, Tv) &\leq a\ G_d(x_0, v, v) + b\ [G_d(x_0, v, v) + 2G_d(x_1, Tv, Tv)] \\
(1-2b)G_d(Tx_0, Tv, Tv) &\leq (a+b)\ G_d(x_0, v, v) \\
G_d(Tx_0, Tv, Tv) &\leq \frac{(a+b)}{(1-2b)}\ G_d(x_0, v, v) \\
G_d(Tx_0, Tv, Tv) &\leq \lambda\ G_d(x_0, v, v). &&(2.9)
\end{aligned}
$$

*Now,*

$$
\begin{aligned}
G_d(x_0, Tv, Tv) &\leq G_d(x_0, x_1, x_1) + G_d(x_1, Tv, Tv) \\
G_d(x_0, Tv, Tv) &\leq G_d(x_0, x_1, x_1) + \lambda\ G_d(x_0, v, v)\ \text{by (2.9)} \\
G_d(x_0, Tv, Tv) &\leq (1-\lambda)r + \lambda r \\
G_d(x_0, Tv, Tv) &\leq r.
\end{aligned}
$$

5

*It follows that $Tv \in \overline{B(x_0, r)}$. Now we will prove that $T^n v \in \overline{B(x_0, r)}$. By using mathematical induction to apply inequality (2.1). Let $T^2 v,\ T^3 v, \cdots T^j v \in \overline{B(x_0, r)}$ for some $j \in N$. As*

$$T^j v \preceq T^{j-1} v \preceq \cdots \preceq v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0.$$

*Then,*

$$
\begin{aligned}
G_d(T^j v, T^{j+1} v, T^{j+1} v) &= G_d(T(T^{j-1} v), T(T^j v), T(T^j v)) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq a\, G_d(T^{j-1} v, T^j v, T^j v) + b\, [G_d(T^{j-1} v, T^j v, T^j v) \\
&\qquad + 2 G_d(T^j v, T^{j+1} v, T^{j+1} v)] \\
(1 - 2b) G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq (a + b) G_d(T^{j-1} v, T^j v, T^j v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda G_d(T^{j-1} v, T^j v, T^j v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^2 G_d(T^{j-2} v, T^{j-1} v, T^{j-1} v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^3 G_d(T^{j-3} v, T^{j-2} v, T^{j-2} v) \\
&\qquad\vdots \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^j G_d(T^{j-j} v, T^{j-(j-1)} v, T^{j-(j-1)} v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^j G_d(v, Tv, Tv) \qquad\qquad (2.10)
\end{aligned}
$$

*Now,*

$$
\begin{aligned}
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq G_d(Tx_j, T(T^j v), T(T^j v)) \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\, G_d(x_j, T^j v, T^j v) \\
&\qquad + b\, [G_d(x_j, Tx_j, Tx_j) + 2 G_d(T^j v, T^{j+1} v, T^{j+1} v)].
\end{aligned}
$$

*By using (2.4) and (2.10)*

$$
\begin{aligned}
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\lambda^j G_d(x_0, v, v) \\
&\qquad + b[\lambda^j G_d(x_0, x_1, x_1) + 2\lambda^j G_d(v, Tv, Tv)] \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\lambda^j G_d(x_0, v, v) \\
&\qquad + b\lambda^j [G_d(x_0, x_1, x_1) + 2 G_d(v, Tv, Tv)]
\end{aligned}
$$

*By using the condition (2.3)*

$$
\begin{aligned}
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\lambda^j G_d(x_0, v, v) \\
&\qquad + b\lambda^j [G_d(x_0, v, v) + 2\lambda G_d(x_0, v, v)] \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq \lambda^j (a + b + 2b\lambda) G_d(x_0, v, v) \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq \lambda^{j+1} G_d(x_0, v, v) \qquad\qquad (2.11)
\end{aligned}
$$

6

*Now ,*

$$
\begin{aligned}
G_d(x_0, T^{j+1}v, T^{j+1}v) &\leq G_d(x_0, x_{j+1}, x_{j+1}) + G_d(x_{j+1}, T^{j+1}v, T^{j+1}v) \\
G_d(x_0, T^{j+1}v, T^{j+1}v) &\leq G_d(x_0, x_1, x_1) + \cdots + G_d(x_j, x_{j+1}, x_{j+1}) \\
&\quad + G_d(x_{j+1}, T^{j+1}v, T^{j+1}v) \\
G_d(x_0, T^{j+1}v, T^{j+1}v) &\leq G_d(x_0, x_1, x_1) + \lambda G_d(x_0, x_1, x_1) \\
&\quad + \cdots + \lambda^{j+1} G_d(x_0, v, v) \ by \ (2.5) \ and \ (2.11) \\
G_d(x_0, T^{j+1}v, T^{j+1}v) &\leq G_d(x_0, x_1, x_1)[1 + \lambda + \lambda^2 + \cdots + \lambda^j] + \lambda^{j+1}r \ as \ v \in \overline{B(x_0, r)} \\
G_d(x_0, T^{j+1}v, T^{j+1}v) &\leq (1-\lambda)r \frac{(1-\lambda^{j+1})}{(1-\lambda)} + \lambda^{j+1}r = r \\
G_d(x_0, T^{j+1}v, T^{j+1}v) &\leq r.
\end{aligned}
$$

*It follows that* $T^{j+1}v \in \overline{B(x_0, r)}$ *and hence* $T^j v \in \overline{B(x_0, r)}$. *Now the inequality* (2.10) *can be written as*

$$
G_d(T^n v, T^{n+1}v, T^{n+1}v) \leq \lambda^n G_d(v, Tv, Tv) \to 0 \ as \ n \to \infty \qquad (2.12)
$$

*Now,*

$$
\begin{aligned}
G_d(x^\star, y^\star, y^\star) &= G_d(Tx^\star, Ty^\star, Ty^\star) \\
G_d(x^\star, y^\star, y^\star) &\leq G_d(Tx^\star, T^{n+1}v, T^{n+1}v) + G_d(T^{n+1}v, Ty^\star, Ty^\star) \\
G_d(x^\star, y^\star, y^\star) &\leq a\, G_d(x^\star, T^n v, T^n v) + b\, [G_d(x^\star, Tx^\star, Tx^\star) \\
&\quad + 2G_d(T^n v, T^{n+1}v, T^{n+1}v)] + a\, G_d(T^n v, Ty^\star, Ty^\star) \\
&\quad + b\, [G_d(T^n v, T^{n+1}v, T^{n+1}v) + 2G_d(y^\star, Ty^\star, Ty^\star)]
\end{aligned}
$$

*By using* (2.7), (2.8) *and* (2.12) *we have*

$$
\begin{aligned}
G_d(x^\star, y^\star, y^\star) &\leq a\, G_d(x^\star, T^n v, T^n v) + a\, G_d(T^n v, y^\star, y^\star) \\
G_d(x^\star, y^\star, y^\star) &\leq a\, [G_d(Tx^\star, T^n v, T^n v) + G_d(T^n v, Ty^\star, Ty^\star)] \\
G_d(x^\star, y^\star, y^\star) &\leq a\, [a\, G_d(x^\star, T^{n-1}v, T^{n-1}v) + b\, G_d(x^\star, Tx^\star, Tx^\star) \\
&\quad + 2b\, G_d(T^{n-1}v, T^n v, T^n v) + a\, G_d(T^{n-1}v, y^\star, y^\star) \\
&\quad + b\, G_d(T^{n-1}v, T^n v, T^n v) + 2b\, G_d(y^\star, Ty^\star, Ty^\star)].
\end{aligned}
$$

*By using* (2.7), (2.8) *and* (2.12) *we have*

$$
\begin{aligned}
G_d(x^\star, y^\star, y^\star) &\leq a^2\, [G_d(x^\star, T^{n-1}v, T^{n-1}v) + G_d(T^{n-1}v, y^\star, y^\star)] \\
G_d(x^\star, y^\star, y^\star) &\leq a^3\, [G_d(x^\star, T^{n-2}v, T^{n-2}v) + G_d(T^{n-2}v, y^\star, y^\star)] \\
&\quad \vdots \\
G_d(x^\star, y^\star, y^\star) &\leq a^n\, [G_d(x^\star, Tv, Tv) + G_d(Tv, y^\star, y^\star)] \\
G_d(x^\star, y^\star, y^\star) &\to 0 \ as \ n \to \infty \\
G_d(x^\star, y^\star, y^\star) &= 0 \\
x^\star &= y^\star.
\end{aligned}
$$

7

*This proves the uniqueness of the fixed point.* ∎

Now we give an example of an ordered complete dislocated quasi $G_d$-metric space in which the contraction does not hold on the whole space rather it holds on a closed ball only.

**Example 12** *Let $X = \mathbb{R}^+ \cup \{0\}$ be endowed with usual order and $G_d : X \times X \times X \to X$ be a complete dislocated quasi $G_d$ metric space defined by,*

$$G_d(x, y, z) = \left\{ \begin{array}{c} 0 \text{ if } x = y = z \\ \max \ \{2x, y, z\} \text{ otherwise.} \end{array} \right\}$$

Then $(X, G_d)$ is a $G_d$ complete $G$ dislocated quasi metric space.
Let $T : X \to X$ be defined by,

$$Tx = \left\{ \begin{array}{c} \frac{x}{5} \text{ if } x \in [0, \frac{3}{2}] \\ x - \frac{1}{3} \text{ if } x \in [\frac{3}{2}, \infty) \end{array} \right\}.$$

Clearly, $T$ is a dominated mappings. Take $x_0 = \frac{1}{3}$, $r = \frac{3}{2}$, $\overline{B(x_0, r)} = [0, \frac{3}{2}]$ and $\lambda = \frac{1}{4}$, $a + 3b < 1$, where $a = \frac{1}{10}$, and $b = \frac{1}{10}$.

$$
\begin{aligned}
G_d(x_0, Tx_0, Tx_0) &\leq (1 - \lambda)r \\
G_d(\frac{1}{3}, T\frac{1}{3}, T\frac{1}{3}) &= \max\{\frac{2}{3}, \frac{1}{15}, \frac{1}{15}\} = \frac{2}{3} \\
\text{Since } (1 - \lambda)r &= (1 - \frac{1}{4})\frac{3}{2} = \frac{9}{8} \\
&\Rightarrow \frac{2}{3} \leq \frac{9}{8} \\
&\Rightarrow 16 \leq 27
\end{aligned}
$$

Also if $x, y$ and $z \in [\frac{3}{2}, \infty)$. We assume that $x > y$, and $y > z$, then

$$
\begin{aligned}
\max\{2x - \frac{2}{3}, y - \frac{1}{3}, z - \frac{1}{3}\} &\geq \frac{1}{10} \max\{2x, y, z\} \\
&\frac{1}{10}[\max\{2x, x - \frac{1}{3}, x - \frac{1}{3}\} \\
&+ \max\{2y, y - \frac{1}{2}, y - \frac{1}{2}\} \\
&+ \max\{2z, z - \frac{1}{2}, z - \frac{1}{2}\}] \\
G_d(Tx, Ty, Tz) &\geq a \ G_d(x, y, z) + b \ [G_d(x, Tx, Tx) \\
&+ G_d(y, Ty, Ty) + G_d(z, Tz, Tz)]
\end{aligned}
$$

So the contractive conditions does not holds in $X$. Now if $x, y$ and $z \in \overline{B(x_0, r)}$

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then,

$$
\begin{aligned}
G_d(Tx, Ty, Tz) &= \max\{\frac{2x}{5}, \frac{y}{5}, \frac{z}{5}\} \leq \frac{1}{10}\{2x, y, z\} \\
&\quad + \frac{1}{10}[\max\{2x, \frac{x}{5}, \frac{x}{5}\} + \max\{2y, \frac{y}{5}, \frac{y}{5}\} \\
&\quad + \max\{z, \frac{z}{5}, \frac{z}{5}\}] \\
&\Rightarrow \quad G_d(Tx, Ty, Tz) \leq a \ G_d(x, y, z) + b \ [G_d(x, Tx, Tx) \\
&\quad + G_d(y, Ty, Ty) + G_d(z, Tz, Tz)].
\end{aligned}
$$

Hence it satisfies all the requirements of Theorem11. If we take $b = 0$ in inequality (2.1) then we obtain the following corollary.

**Corollary 13** *Let* $(X, \preceq, G)$ *be an ordered complete dislocated quasi* $G-$*metric space,* $T : X \to X$ *be a dominated mapping and* $x_0$ *be any arbitrary point in* $X$. *Suppose there exists* $a \in [0, 1)$ *with,*

$$
G(Tx, Ty, Tz) \leq a \ G(x, y, z), \ \ for \ all \ x, \ y \ and \ z \in Y = \overline{B(x_0, r)},
$$

*and*

$$
G(x_0, Tx_0, Tx_0) \leq (1 - a)r.
$$

*If for a nonincreasing sequence* $\{x_n\} \to u$ *implies that* $u \preceq x_n$. *Then there exists a point* $x^\star$ *in* $\overline{B(x_0, r)}$ *such that* $x^\star = Sx^\star$ *and* $G(x^\star, x^\star, x^\star) = 0$. *Moreover if for any three points* $x, y$ *and* $z$ *in* $\overline{B(x_0, r)}$ *such that there exists a point* $v \in \overline{B(x_0, r)}$ *such that* $v \preceq x, \ v \preceq y$ *and* $v \preceq z,$ *that is, every three of elements in* $\overline{B(x_0, r)}$ *has a lower bound, then the point* $x^\star$ *is unique.*
*Similarly if we take* $a = 0$ *in inequality (2.1) then we obtain the following corollary.*

**Corollary 14** *Let* $(X, \preceq, G)$ *be an ordered complete dislocated quasi* $G$-*metric space* $T : X \to R$ *be a mapping and* $x_0$ *be an arbitrary point in* $X$. *Suppose there exists* $b \in [0, \frac{1}{3})$ *with*

$$
G(Tx, Ty, Tz) \leq b \ (G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz))
$$

*for all comparable elements* $x, y, z \in \overline{B(x_0, r)}$ *and*

$$
G(x_0, Tx_0, Tx_0) \leq (1 - \lambda)r,
$$

*where* $\lambda = \frac{b}{1-2b}$. *If for non increasing sequence* $\{x_n\} \to u$ *implies that* $u \preceq x_n$. *Then there exists a point* $x^\star$ *in* $\overline{B(x_0, r)}$ *such that* $x^\star = Sx^\star$ *and* $G(x^\star, x^\star, x^\star) = 0$. *Moreover, if for any three points* $x, y, z \in \overline{B(x_o, r)}$, *there exists a point* $v$ *in* $\overline{B(x_0, r)}$ *such that* $v \preceq x$ *and* $v \preceq y, v \preceq z$.

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# MIZOGUCHI- TAKAHASHI'S FIXED POINT THEOREM IN $\nu$-GENERALIZED METRIC SPACES

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ABSTRACT. Our main work is to prove Mizogchi- Takahashi theorem in $\nu$-generalized metric space in the sense of Brancairi. In the same setting we prove two more theorems which are generalizations of the main one.

## 1. INTRODUCTION

A metric is defined as a mapping $d : X \times X \to [0, \infty)$, for any non-empty set $X$ which satisfying the following axioms, for any $x, y, z \in X$

(i) $d(x, y) = 0$ iff $x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

We said that the pair $(X, d)$ is a metric space. The theory of metric spaces form a basic environment for a lot of concepts in mathematics such as the fixed point theorems which have an important rules in various branches of mathematical analysis. One of the famous result of fixed point theorems is Banach Contraction Principle which state that,

**Theorem 1.1.** [11] *(Banach Contraction Principle)*
*Let $(X, d)$ be a complete metric space. Let $T : X \to X$ be a self map on $X$ such that*

$$d(Tx, Ty) \leq rd(x, y),$$

*hold for any $x, y \in X$, where $r \in [0, 1)$. Then $T$ has a unique fixed point.*

Many authors explored the importance of this theorem and extended it in different directions. For examples, we refer the reader to the following papers [2, 9, 8, 6], and the references therein. In (1969) Nadler extended theorem 1.1 for multi-valued mapping. Recall that the set of all non- empty, closed and bounded subsets of $X$ is denoted by $CB(X)$ and let $A, B$ be any sets in $CB(X)$. A Hausdorff metric is defined as

$$\mathcal{H}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

**Theorem 1.2.** [12] *(Nadler's theorem) Let $(X, d)$ be a complete metric space. Let $T : X \to CB(X)$ be a multi-valued map. Assume that*

$$\mathcal{H}(Tx, Ty) \leq rd(x, y),$$

*holds for each $x, y \in X$ and $r \in [0, 1)$. Then $T$ has a fixed point.*

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Many attempts have been done to generalize Nadler's theorem. One of these generalizations is Mizoguchi- Takahashi's theorem which stats that:

**Theorem 1.3.** [10] *Let $(X, d)$ be a complete metric space. Let $T : X \to CB(X)$ be a multi-valued mapping. Assume that*

$$\mathcal{H}(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

*hold for each $x, y \in X$, where $\beta : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{s \to t^+} \beta(s) < 1$. Then $T$ has a fixed point.*

**Remark 1.4.** *The function $\beta$ in theorem 1.3, which satisfies $\limsup_{s \to t^+} \beta(s) < 1$ is called Mizoguchi- Takahashi function (MT- function for short).*

Starting with Mizoguchi and Takahashi's paper, many generalizations of their theorem have been established see [3, 13]. Recently, Eldred et al [4], claimed that Nadler's and Mizoguchi- Takahashi's theorems are equivalent. However, in [14], Suzuki proved that their claim is not true and he shown that Mizoguchi- Takahashi's theorem (1.3) is a real extension of Nadler's theorem. This is why we are interesting in such theorem.

In another direction, in (2000) Branciari created a new concept of generalized metric spaces by modifying the triangle inequality to involve more points.

**Definition 1.5.** [1] *Let $X$ be a non- empty set and $d : X \times X \to [0, \infty)$. For $\nu \in \mathbb{N}$, a pair $(X, d)$ is called a $\nu$- generalized metric space if the following hold:*

(M1) $d(x, y) = 0$ *iff*    $x = y$
(M2) $d(x, y) = d(y, x)$
(M3) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + ... + d(u_\nu, y),$
     *for any $x, u_1, u_2, ... u_\nu, y \in X$, such that $x, u_1, u_2, ... u_\nu, y$ are all different.*

It is not difficult to show that the new space is not the same as the original one. Moreover, the new space is hard to deal with because it does not satisfy all topological properties that metric space has, see [15] for more details. Recently, in [16], Suzuki proved Nadler's theorem in $\nu$- generalized metric spaces. The main work of this paper is to prove Mizoguchi -Takahashi's theorem in $\nu$- generalized metric spaces. Firstly, we will list all the necessary definitions and some results that we will need. Then, we will be able to prove our main results.

## 2. Preliminary

**Definition 2.1.** *A point $x \in X$ is said to be a fixed point of multi-valued map $T$ if $x \in Tx$.*

**Definition 2.2.** [1] *Let $(X, d)$ be a $\nu$- generalized metric space. A sequence $\{x_n\}_{n \in \mathbb{N}} \in X$ is said to be Cauchy sequence if*

$$\lim_n \sup_{n > m} d(x_n, x_m) = 0$$

**Definition 2.3.** [16] *A sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be $(\sum, \neq)$- Cauchy sequence if all $x_n$'s are different and*

$$\sum_{j=1}^{\infty} d(x_j, x_{j+1}) < \infty$$

**Definition 2.4.** [16] *Let $(X, d)$ be a $\nu$- generalized metric space. We said that, $X$ is a $(\sum, \neq)$- complete if every $(\sum, \neq)$- Cauchy sequence converges.*

**Lemma 2.5.** [16, 5] *Let $(X, d)$ be a $\nu$- generalized metric space.*

- *Every converge $(\sum, \neq)$- Cauchy sequence is Cauchy.*
- *Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence converges to some $y \in X$ and $\{y_n\} \in X$ be a sequence such that $\lim_{n \to \infty} d(x_n, y_n) = 0$. Then, $\{y_n\}$ also converges to $y$.*

**Lemma 2.6.** [14] *Let $\beta : [0, \infty) \to [0, 1)$ is a MT-function. Then, for all $s \in [0, \infty)$, there exist $r_s \in [0, 1)$ and $\varepsilon_s > 0$ such that $\beta(t) \leq r_s$ for all $t \in [s, s + \varepsilon_s)$*

**Lemma 2.7.** [12] *Let $(X, d)$ be a metric space. For any $A, B \in CB(X)$ and $\varepsilon > 0$, there exist $a \in A$ and $b \in B$ such that $d(a, b) \leq \mathcal{H}(A, B) + \varepsilon$*

## 3. Main Result

In this section we prove Mizoguchi -Takahashi's theorem in $\nu$ -generalized metric spaces and some of its generalizations in the space.

**Theorem 3.1.** *Let $(X, d)$ be a $(\sum, \neq)$ complete, $\nu$- generalized metric space. and let $T$ be a multi-valued map defined from $X$ into $CB(X)$ satisfies the following:*

(i) *If $\{y_n\} \in Tx$ and $\{y_n\}$ converges to $y$ then $y \in Tx$.*
(ii) *For any $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$,*

*where $\alpha$ is MT-function. Then $T$ has a fixed point.*

*Proof.* Let define a function $\gamma : [0, \infty) \to [0, 1)$ as $\gamma(t) = \dfrac{1 + \alpha(t)}{2}$. It is not difficult to show that $\alpha(t) < \gamma(t)$, for any $t \in [0, \infty)$ and $\lim_{s \to t^+} \sup \gamma(s) < 1$. Moreover, for each $x, y \in X$ and $v \in Tx$, there exist $u \in Ty$ such that

$$d(v, u) \leq \gamma(d(x, y))d(x, y).$$

Putting $v = y$, we will get that

$$d(y, u) \leq \gamma(d(x, y))d(x, y)$$

Define $f(x) = \inf\{d(x, b) : b \in Tx\}$ and suppose that $T$ does not have a fixed point ( i.e., for all $x \in X, f(x) > 0$). Let $x_1 \in X$ be arbitrary and choose $x_2 \in Tx_1$ satisfying

(1) $$d(x_1, x_2) < \frac{1}{\gamma(d(x_1, x_2))} f(x_1).$$

Since $Tx_2 \neq \emptyset$, we can choose an arbitrary element $x_3 \in Tx_2$ such that

(2) $$f(x_2) \leq d(x_2, x_3) \leq \gamma(d(x_1, x_2))d(x_1, x_2).$$

Also, as in equation (1), we have

(3) $$d(x_2, x_3) < \frac{1}{\gamma(d(x_2, x_3))} f(x_2).$$

From (2) and (3), we have

$$d(x_2, x_3) \leq \min\{\gamma(d(x_1, x_2))d(x_1, x_2), \frac{1}{\gamma(d(x_2, x_3))} f(x_2)\}.$$

Thus

$$\gamma(d(x_2, x_3))d(x_2, x_3) < f(x_2) \leq \gamma(d(x_1, , x_2))d(x_1, x_2) < f(x_1).$$

Continuously, $\{x_n\}_{n\in\mathbb{N}} \in X$ is a sequence constructed such that $x_{n+1} \in Tx_n$ and satisfying

(4)
$$\gamma(d(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) < f(x_{n+1}) \leq \gamma(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < f(x_n),$$

and

(5)
$$d(x_{n+1}, x_{n+2}) \leq \gamma(d(x_{n+1}, x_n))d(x_{n+1}, x_n).$$

Since $\gamma(t) < 1$, we have $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$. Hence, from (4) and (5), the sequences $\{f(x_n)\}$ and $\{d(x_n, x_{n+1})\}$ are strictly decreasing. Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a $(\sum, \neq)$- Cauchy sequence in two steps:

**Step 1** we show that all terms different. Suppose not i.e suppose $x_n = x_m$ for some $n > m$, where $m, n \in \mathbb{N}$. Hence

$$f(x_m) = \inf\{d(x_m, b) : b \in Tx_m\}$$
$$= \inf\{d(x_n, b) : b \in Tx_n\}$$
$$= f(x_n),$$

which contradicts $\{f(x_n)\}$ being strictly decreasing.

**Step 2** We show that $\sum d(x_n, x_{n+1}) < \infty$. Since $\{d(x_n, x_{n+1})\}$ is an decreasing sequence in $R$ and bounded below, it converges to some positive real number (say $\delta$). Also, we have $\lim_{s \to t^+} \sup \gamma(s) < 1$, thus, there exist $r \in [0, 1)$ and $\varepsilon > 0$ such that $\gamma(s) \leq r$ for all $s \in [\delta, \delta + \varepsilon)$. For any $n \in \mathbb{N}$, we can choose $\mu \in \mathbb{N}$ satisfying $\delta \leq d(x_n, x_{n+1}) \leq \delta + \varepsilon$ with $n \geq \mu$. So,

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=\mu+1}^{\infty} d(x_n, x_{n+1})$$
$$\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n d(x_\mu, x_{\mu+1})$$
$$< \infty.$$

Thus $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence in $(\sum, \neq)$ complete $\nu$- generalize metric space. Then, it is converge to some $z \in X$ and by lemma (2.5), $\{x_n\}$ is a Cauchy sequence. From our assumption we choose $\{u_n\} \in Tz$ satisfy

$$d(x_{n+1}, u_n) \leq \mathcal{H}(Tx_n, Tz) \leq \gamma(d(x_n, z))d(x_n, z),$$

for any $n \in \mathbb{N}$. But $\{x_n\}$ converges to $z$, so $d(x_{n+1}, u_n) \to 0$ as $n \to \infty$. Thus we have $x_{n+1} \to z$ and $x_{n+1} \to u_n$. Therefore, by lemma(2.5) $d(u_n, z) = 0$ as $n \to \infty$. So $d(Tz, z) = 0$ implies $f(z) = 0$ which is a contradiction. Therefore, there exist $z \in X$ such that $f(z) = 0$ and hence $z \in Tz$ is a fixed point.                                                  $\square$

**Definition 3.2.** [7] *A multi- valued map $T$ from $X$ into $CB(X)$ is called $\alpha$- admissible if for any $x \in X$ and $y \in Tx$, $\alpha(x, y) \geq 1$ implies $\alpha(y, z) \geq 1$ for any $z \in Ty$, where $\alpha : X \times X \to [0, \infty)$.*

The up coming lemma proved in [18], for single-valued map here, we prove it for multi- valued map.

**Lemma 3.3.** *Let $(X, d)$ be a $\nu$- generalized metric space. Let $T$ be a multi-valued mapping from $X$ into $2^X$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $X$ defined by $x_{n+1} \in Tx_n$ such that $x_n \neq x_{n+1}$. Assume that*

(6) $$d(x_n, x_{n+1}) \leq \delta d(x_{n-1}, x_n)$$

*hold for any $\delta \in [0, 1)$. Then $x_n \neq x_m \ \forall n \neq m \in \mathbb{N}$.*

*Proof.* We prove that $x_{n+\ell} \neq x_n$ for all $n \in \mathbb{N}$ and $\ell \geq 1$. Suppose the contrary that is $x_{n+\ell} = x_n$ for some $n \in \mathbb{N}$ and $\ell \geq 1$. By assumption, we have that $x_{n+\ell+1} = x_{n+1}$. Then from (6) we get
(7)
$$d(x_n, x_{n+1}) = d(x_{n+\ell}, x_{n+\ell+1}) \leq \delta d(x_{n+\ell-1}, x_{n+\ell}) \leq \dots \leq \delta^\ell d(x_n, x_{n+1}) < d(x_n, x_{n+1})$$

which is contradiction. Thus, we get $x_m \neq x_n$ for all $m \neq n$ in $\mathbb{N}$. $\qquad\square$

Let $\Phi$ be the family of all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfying the following conditions:

(a) $\varphi(s) = 0$ iff $s = 0$.
(b) $\varphi$ is non-decreasing and lower semi-continuous

(c) $\lim_{s \to 0^+} \sup \dfrac{s}{\varphi(s)} < \infty.$

**Theorem 3.4.** *Let $(X, d)$ be a $(\sum, \neq)$ complete $\nu$- generalized metric space. Let $T : X \to CB(X)$ be an $\alpha$- admissible multi-valued mapping satisfying:*

(i) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*
(ii) *If $(y_n) \in Tx$ and $(y_n)$ converge to $y$ then $y \in Tx$*
(iii) *$\alpha(x, y)\mathcal{H}(Tx, Ty) \leq \phi(d(x, y))d(x, y)$ for any $x, y \in X$, and $\phi$ is MT-function.*

*Then $T$ has a fixed point.*

*Proof.* Let $\beta : [0, \infty) \to [0, 1)$ as $\beta(t) = \dfrac{1 + \phi(t)}{2}$ such that $\lim_{s \to t^+} \sup \beta(s) < 1$. Clearly $\phi(t) < \beta(t)$ for each $t \in [0, \infty)$. Let $x_0 \in X$ and choose $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume $x_0 \neq x_1$ so, $\dfrac{1 - \phi(d(x_0, x_1))}{2}d(x_0, x_1) > 0$. Since $Tx_1 \neq \emptyset$, choose $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \mathcal{H}(Tx_0, Tx_1) + \frac{1 - \phi(d(x_0, x_1))}{2}d(x_0, x_1)$$

$$\leq \alpha(x_0, x_1)\mathcal{H}(Tx_0, Tx_1) + \frac{1 - \phi(d(x_0, x_1))}{2}d(x_0, x_1)$$

$$\leq \phi(d(x_0, x_1))d(x_0, x_1) + \frac{1 - \phi(d(x_0, x_1))}{2}d(x_0, x_1)$$

$$\leq \beta(d(x_0, x_1))d(x_0, x_1).$$

Since $T$ is $\alpha$- admissible, $x_1 \in Tx_0$ and $\alpha(x_0, x_1) \geq 1$ then, $\alpha(Tx_0, Tx_1) \geq 1$ which implies $\alpha(x_1, x_2) \geq 1$. Similarly assume $x_1 \neq x_2$ we have $\dfrac{1 - \phi(d(x_1, x_2))}{2}d(x_1, x_2) >$

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0 and choose $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \mathcal{H}(Tx_1, Tx_2) + \frac{1 - \phi(d(x_1, x_2))}{2} d(x_1, x_2)$$

$$\leq \alpha(x_1, x_2)\mathcal{H}(Tx_1, Tx_2) + \frac{1 - \phi(d(x_1, x_2))}{2} d(x_1, x_2)$$

$$\leq \phi(d(x_1, x_2))d(x_1, x_2) + \frac{1 - \phi(d(x_1, x_2))}{2} d(x_1, x_2)$$

$$\leq \beta(d(x_1, x_2))d(x_1, x_2).$$

Similarly, using the same method of proving theorem (3.1), we have our result.

□

**Theorem 3.5.** *Let $(X, d)$ be a $(\sum, \neq)$ complete $\nu$- generalized metric space. Let $T : X \to CB(X)$ be a multi-valued map satisfying:*

$$\varphi(\mathcal{H}(Tx, Ty)) \leq \alpha(\varphi(d(x, y)))\varphi(d(x, y)),$$

*for each $x, y \in X$, where $\alpha$ is a MT- function and $\varphi \in \Phi$. Then $T$ has a fixed point.*

*Proof.* Let $\gamma : [0, \infty) \to [0, 1)$ defined by $\gamma(t) = \dfrac{1 + \alpha(t)}{2}$. Since $\varphi$ is non- decreasing function, then

(8)
$$\max \left\{ \sup_{v \in Tx} \varphi(d(v, Ty)), \sup_{u \in Ty} \varphi(d(u, Tx)) \right\}$$
$$= \max \left\{ \varphi(\sup_{v \in Tx} d(v, Ty)), \varphi(\sup_{u \in Ty} d(u, Tx)) \right\}$$
$$= \varphi(\mathcal{H}(Tx, Ty)) \leq \gamma(\varphi(d(x, y)))\varphi(d(x, y)).$$

There exist an element $z \in Ty$ such that

$$\varphi(d(y, z)) \leq \gamma(\varphi(d(x, y)))\varphi(d(x, y)),$$

for each $x \in X$ and $y \in Tx$. Thus, in the same way a sequence $\{x_n\}_{n \in \mathbb{N}} \in X$ defined as $x_{n+1} \in Tx_n$ is constructed such that

(9)        $$\varphi(d(x_n, x_{n+1}) \leq \gamma(\varphi(d(x_{n-1}, x_n))\varphi(d(x_{n-1}, x_n))$$

for all $n \in \mathbb{N}$. Since $\gamma(t) < 1$ for any $t \in [0, \infty)$, hence from (9) we get

(10)             $$\varphi(d(x_n, x_{n+1})) < \varphi(d(x_{n-1}, x_n)).$$

Clearly $\{\varphi(d(x_{n-1}, x_n))\}$ is decreasing sequence of positive real numbers. Hence it is converge to some non- negative real number, say $\epsilon$. By contradiction, it is easy to show that $\epsilon = 0$. Note that, $\varphi$ is a non- decreasing function which implies to $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$. Thus the sequence $\{d(x_n, x_{n+1})\}$ is also decreasing. Hence by lemma (3.3), the terms of the sequence all are different. Now, show that $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. Note that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded. Thus, it is converges to a positive real number (say $\delta$) which implies that $\varphi(\delta) \leq \varphi(d(x_n, x_{n+1}))$. Thus,

$$\varphi(\delta) \leq \lim_{n \to \infty} \varphi(d(x_n, x_{n+1})) = \epsilon = 0.$$

Since $\varphi(s) = 0$ if and only if $s = 0$ then, $\delta = 0$. By lemma (2.6), there exist $r \in [0,1)$ such that, $\varphi(d(x_n, x_{n+1})) \leq r\varphi(d(x_{n-1}, x_n))$. Therefore,

$$
\sum_{n=1}^{\infty} \varphi(d(x_n, x_{n+1})) \leq \sum_{n=1}^{\mu} \varphi(d(x_n, x_{n+1})) + \sum_{n=\mu+1}^{\infty} \varphi(d(x_n, x_{n+1}))
$$

$$
\leq \sum_{n=1}^{\mu} \varphi(d(x_n, x_{n+1})) + \sum_{n=1}^{\infty} r^n \varphi(d(x_\mu, x_{\mu+1}))
$$

$$
< \infty.
$$

By defintion of $\varphi$, we have

$$
\limsup_{n\to\infty} \frac{d(x_n, x_{n+1})}{\varphi(d(x_n, x_{n+1}))} \leq \lim_{s\to 0^+} \frac{s}{\varphi(s)} < \infty.
$$

Thus, the sequence $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence. Since $X$ is a $(\sum, \neq)$ complete $\nu$- generalized metric space and by lemma (2.5), it is Cauchy and then it is converge to some $z \in X$. From the definition of $\varphi$ and its increasing we conclude that,

$$
\varphi d(z, Tz)) \leq \liminf_{n\to\infty} \varphi(d(x_{n+1}, Tz)) \leq \liminf_{n\to\infty} \varphi(\mathcal{H}(Tx_n, Tz))
$$

$$
\leq \liminf_{n\to\infty} \gamma(\varphi(d(x_n, z)))\varphi(d(x_n, z)) \leq \liminf_{n\to\infty} \varphi(d(x_n, z))
$$

$$
= \lim_{s\to 0^+} \varphi(s) = \lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0.
$$

Therefore, $\varphi(d(z, Tz)) = 0$. Thus by the definition of $\varphi$ and since $Tz$ closed we have $z \in Tz$ is a fixed point. □

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# On Multiresolution Analyses Of Multiplicity $n$

Richard A. Zalik *

**Abstract**

This paper studies multiresolution analyses in $L^2(\mathbb{R}^d)$ that have more than one scaling function and are generated by an arbitrary dilation matrix. It provides a further analysis of a representation theorem obtained by the author for such MRA's.

## 1   Introduction

The concept of multiresolution analysis of multiplicity $n$ is due to Alpert [1, 2, 3] who introduced his now well known dyadic multiresolution analysis with an arbitrary number of filters in $L^2(\mathbb{R})$. Alpert's results motivated a number of papers, focused on the univariate case, such as Hervé [12, 13], Donovan, Geronimo and Hardin [6, 7], Geronimo and Marcellán [10], Goodman, Lee and Tang [9], Goodman and Lee [8], and Hardin, Kessler and Massopoust [11]. Multiresolution analyses of multiplicity 1 (i.e., with a single scaling function) with arbitrary expansive matrices in $L^2(\mathbb{R})$ were studied by Lemarié [15, 16] and Madych [17], among others, and we should also mention Wojtaszczyk's excellent textbook [25]. Properties of low pass filters and scaling functions in this context were studied by San Antolín [19, 20, 21, 22] and Cifuentes, Kazarian and San Antolín [5]. Saliani [18] extended these results to multiresolution analyses of multiplicity $n$ generated by an expansive matrix. These results were further extended by Soto–Bajo [24] to multiresolution analyses having an arbitrary (not necessarily finite) set of generator functions. In [4], Behera studied multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$ associated with multiresolution analyses of multiplicity $n$ generated by an expansive matrix. In [26] the author presented a representation theorem for such multiresolution analyses, in [27] he gave some simple examples for the case $n = 1$, and in [23] San Antolín and the author obtained representation theorems for vector valued wavelets. Some of the authors cited above have showed that by using more than one scaling function it is possible to

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construct wavelets that have a set of properties not available for wavelets associated with a multiresolution analysis having a single scaling function; for instance in [6] they constructed wavelets associated with more than two scaling functions having compact support, arbitrary regularity, orthogonality, and symmetry. These results would indicate that the further study of multiresolution analyses of multiplicity $n$ may lead to other interesting results.

In what follows, $\mathbb{Z}$ will denote the set of integers, $\mathbb{Z}_+$ the set of strictly positive integers and $\mathbb{R}$ the set of real numbers; $\mathbb{C}$ will denote the set of complex numbers, and $\mathbf{I}$ will stand for the identity matrix. Boldface lowcase letters will denote elements of $\mathbb{R}^d$; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; the vector norm $|| \cdot ||$ is defined by $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$. If $\mathbf{A}$ is a matrix $||\mathbf{A}||$ will denote the matrix norm induced by the vector norm $|| \cdot ||$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g \rangle$, their bracket product by $[f, g]$, and the norm of $f$ by $||f||$; thus,

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})}\, d\mathbf{t},$$

$$[f, g](\mathbf{t}) := \sum_{\mathbf{k} \in \mathbb{Z}^d} f(\mathbf{t} + \mathbf{k})\overline{g(\mathbf{t} + \mathbf{k})},$$

and

$$||f|| := \sqrt{\langle f, f \rangle}.$$

The Fourier transform of a function $f$ will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi \mathbf{x} \cdot \mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

Let $\mathbf{A} \in \mathbb{C}^{d \times d}$ and $|a| := \det(\mathbf{A})$. For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D_j^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D_j^{\mathbf{A}} f(\mathbf{t}) := |a|^{j/2} f(\mathbf{A}^j \mathbf{t})$$

and

$$T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t} + \mathbf{k})$$

respectively. Let $\mathbb{T} := [0, 1]$, and let $\mathbb{T}^d$ denote the $d$–fold cartesian product of $\mathbb{T}$. A function $f$ will be called $\mathbb{Z}^d$–*periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}} f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$.

Let $\mathbf{u} = \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$; then

$$T(\mathbf{u}) = T(u_1, \cdots, u_m) := \{T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}$$

and

$$S(\mathbf{u}) = S(u_1, \cdots, u_m) := \overline{\operatorname{span}}\, T(\mathbf{u}),$$

where the closure is in $L^2(\mathbb{R}^d)$. $S(\mathbf{u})$ is called a *finitely generated shift–invariant space* or FSI and the functions $u_\ell$ are called the *generators* of $S(\mathbf{u})$. In this case we will also use the symbols $T(u_1, \cdots, u_n)$ and $S(u_1, \cdots, u_n)$ to denote $S(\mathbf{u})$ and $T(\mathbf{u})$ respectively.

We also define

$$T(\mathbf{A}^j; \mathbf{u}) = T(\mathbf{A}^j; u_1, \cdots, u_m) := \{D_j^{\mathbf{A}} T_{\mathbf{k}} u_\ell; \ell = 1, \cdots m, \mathbf{k} \in \mathbb{Z}^d\},$$

and

$$S(\mathbf{A}^j; \mathbf{u}) = S_j(\mathbf{A}^j; u_1, \cdots, u_m) := \overline{\mathrm{span}}\, T(\mathbf{A}^j; \mathbf{u}).$$

Given a sequence of functions $\mathbf{u} := \{u_1, \cdots, u_m\}$ in $L^2(\mathbb{R}^d)$, by $G[u_1, \cdots, u_m](\mathbf{x})$, $G_{\mathbf{u}}(\mathbf{x})$ or $G(\mathbf{x})$ we will denote its *Gramian matrix*, viz.

$$G(\mathbf{x}) := \left( [\widehat{u_\ell}, \widehat{u_j}](\mathbf{x}) \right)_{\ell,j=1}^m.$$

Let $\Lambda \subset \mathbb{Z}$ and $\mathbf{u} = \{u_k; k \in \Lambda\} \subset S \subset L^2(\mathbb{R}^d)$. If $S$ is a shift–invariant space then $\mathbf{u}$ is called a *basis generator* of $S$, if for every $f \in S$ there are $\mathbb{Z}^d$–periodic functions $p_k$, uniquely determined by $f$ (up to a set of measure 0), such that

$$\widehat{f} = \sum_{k \in \Lambda} p_k \widehat{u_k}.$$

In what follows we will assume that $\mathbf{A}$ is a fixed matrix preserving the lattice $\mathbb{Z}^d$, i.e. $\mathbf{A}\mathbb{Z}^d \subset \mathbb{Z}^d$. We will also assume that $\mathbf{A}$ is expansive, that is, there exist constants $C > 0$ and $\delta > 1$ such that for every $j \in \mathbb{Z}_+$ and $\mathbf{x} \in \mathbb{R}^d$

$$||\mathbf{A}^j \mathbf{x}|| \geq C\delta^j ||\mathbf{x}||.$$

**Lemma 1.** $\mathbf{A}$ *is expansive if and only if all its eigenvalues have modulus larger that* 1.

*Proof.* Suppose first that all the eigenvalues of $\mathbf{A}$ have modulus larger that 1. If $\mathbf{A}$ is a Jordan block, then $\mathbf{A} = \lambda\mathbf{I} + \mathbf{N}$, where $\mathbf{N}$ has 1's on the superdiagonal and 0's elsewhere, and from e.g. [14, Lemma 3.1.4] we deduce that

$$\mathbf{A}^j \mathbf{x} = \sum_{k=0}^d \binom{j}{k} \lambda^k N^{j-k} \mathbf{x}$$

whence the assertion readily follows, and therefore it also follows when $\mathbf{A}$ is in Jordan canonical form. In general, if $\mathbf{Q}$ is the Jordan form of $\mathbf{A}$, then $\mathbf{Q} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ and we have

$$C\delta^j ||\mathbf{y}|| \leq ||\mathbf{Q}^j \mathbf{y}|| = ||\mathbf{B}^{-1}\mathbf{A}^j\mathbf{B}\mathbf{y}|| \leq ||\mathbf{B}^{-1}||\, ||\mathbf{A}^j\mathbf{B}\mathbf{y}||.$$

Setting $\mathbf{x} = \mathbf{B}\mathbf{y}$ the assertion readily follows.

Conversely, if $\mathbf{A}$ has an eigenvalue $\lambda$ with modulus less or equal to 1 and $\mathbf{v}$ is an eigenvector for $\lambda$ with $||\mathbf{v}|| = 1$, then $\mathbf{A}\mathbf{v} = \lambda\mathbf{I}$; hence $||\mathbf{A}^j\mathbf{v}|| = |\lambda|^j \leq |\lambda|$, and $||\mathbf{A}^j\mathbf{v}||$ remains bounded. So $\mathbf{A}$ is not expansive. $\square$

The previous proof was suggested by Wayne Lawton. An elementary proof may be found in San Antolín's thesis [19, Lema A.12].

A *multiresolution analysis* (MRA) of multiplicity $n$ in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}$, $f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{At}) \in V_{j+1}$.

(iii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) There are functions $\mathbf{u} := \{u_1, \cdots, u_n\}$ such that $T(\mathbf{u})$ is an orthonormal basis of $V_0$.

From [18, Lemma 17] we know that if $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis, then

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}. \tag{1}$$

This generalizes a result of Cifuentes, Kazarian and San Antolín, which was established for multiresolution analyses of multiplicity 1 (cf. [5, Lemma 4]).

It follows from the definition of multiresolution analysis that there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j} \in L^2(\mathbb{T}^d)$ such that the functions $u_\ell$ satisfy the *scaling identity*

$$\widehat{u}_\ell(\mathbf{A}^*\mathbf{x}) = \sum_{j=1}^{n} p_{\ell,j}(\mathbf{x})\widehat{u}_j(\mathbf{x}), \quad j, \ell = 1, \cdots, n \quad a.e.,$$

where $\mathbf{A}^*$ is the transpose of $\mathbf{A}$. The functions $u_\ell$ are called *scaling functions* for the multiresolution analysis, and the functions $p_{\ell,j}$ are called the *low pass filters* associated with $\mathbf{u}$.

A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ will be called an orthonormal wavelet system if the affine sequence

$$\{D_j^{\mathbf{A}} T_{\mathbf{k}} \psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be an orthogonal wavelet system in $L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$, let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis and let $W_j$ denote the orthogonal complement of $V_j$ in $V_{j+1}$. We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $T(\boldsymbol{\psi})$ is an orthonormal basis of $W_0$.

## 2    Representation of orthonormal wavelets.

For $k > 1$ let diag $\{-e^{i\omega}, 1, \cdots, 1\}_k$ denote the $k \times k$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries. With the convention that Arg $0 = 0$ we have

**Theorem 1.** *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity $n$ with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$, generated by a matrix $\mathbf{A}$ that preserves the lattice*

$\mathbb{Z}^d$. For $1 \leq \ell \leq n$, let $\{v_{\ell,1}, \cdots, v_{\ell,|a|}\}$ be an orthonormal basis generator of $S(\mathbf{A}, u_\ell)$, let $\mathbf{e} := (1, 0, \cdots, 0) \in \mathbb{R}^k$, and

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{|a|} b_{\ell,j}(\mathbf{x})\widehat{v}_{\ell,j}(\mathbf{x}), \tag{2}$$

$$\mathbf{b}_\ell(\mathbf{x}) := (b_{\ell,1}(\mathbf{x}), \cdots, b_{\ell,|a|}(\mathbf{x}))^T, \quad \delta_\ell(\mathbf{x}) := e^{iArg\, b_{\ell,1}(\mathbf{x})}, \quad \mathbf{q}_\ell(\mathbf{x}) := \mathbf{b}_\ell(\mathbf{x}) + \delta_\ell(\mathbf{x})\mathbf{e},$$

$$\widehat{\mathbf{v}}(\mathbf{x}) := (\widehat{v}_{1,1}(\mathbf{x}), \cdots, \widehat{v}_{1,|a|}(\mathbf{x}), \cdots, \widehat{v}_{n,1}(\mathbf{x}), \cdots, \widehat{v}_{n,|a|}(\mathbf{x}))^T,$$

and

$$\mathbf{Q}_\ell(\mathbf{x}) := \mathrm{diag}\{-\delta_\ell(\mathbf{x}), 1, \cdots, 1\}_{|a|} \left[\overline{\mathbf{I} - 2\mathbf{q}_\ell(\mathbf{x})\mathbf{q}_\ell(\mathbf{x})^*/\mathbf{q}_\ell(\mathbf{x})^*\mathbf{q}_\ell(\mathbf{x})}\right].$$

Let $a := \det \mathbf{A}$, $m := |a|n$, and let

$$\mathbf{Q}(\mathbf{x}) = \left(q_{\ell,k}(\mathbf{x})\right)_{\ell,k=1}^m$$

be the $m \times m$ block diagonal matrix

$$\mathbf{Q}_1(\mathbf{x}) \oplus \mathbf{Q}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{Q}_m(\mathbf{x}) = \begin{pmatrix} \mathbf{Q}_1(\mathbf{x}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_n(\mathbf{x}) \end{pmatrix}.$$

If

$$(\widehat{y}_1(\mathbf{x}), \cdots, \widehat{y}_m(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})\widehat{\mathbf{v}}(\mathbf{x}),$$

then

$$y_{(\ell-1)|a|+1} = u_\ell; \quad 1 \leq \ell \leq n, \tag{3}$$

and

$$\{y_{(\ell-1)|a|+k}; 1 \leq \ell \leq n, \ 2 \leq k \leq |a|\}$$

is an orthonormal wavelet system associated with $M$.

The preceding theorem was proved in [26, Theorem 9] but there was arguably a gap in the proof, which is bridged by the following

**Lemma 2.** *Let $m = n(|a| - 1)$, let $\{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis and assume that $\{u_\ell; \ell = 1, \cdots, n\}$ is an orthonormal basis generator of $V_0$. Then*

$$V_1 = S(\mathbf{A}, u_1) \oplus S(\mathbf{A}, u_2) \oplus \cdots \oplus S(\mathbf{A}, u_n).$$

*Proof.* From [26, Theorem 3] we know that there exist functions $v_{\ell,k}$, $\ell = 1, \cdots n$, such that $\{v_{\ell,1}, \cdots v_{\ell,|a|}\}$ is an orthonormal basis generator of $S(\mathbf{A}, u_\ell)$. Therefore $\{v_{\ell,k}; 1 \leq \ell \leq n, 1 \leq k \leq |a|\}$ is an orthogonal basis generator of $S(\mathbf{A}, u_1) \oplus S(\mathbf{A}, u_2) \oplus \cdots \oplus S(\mathbf{A}, u_n)$, which is a subspace of $V_1$. But [26, Theorem 3] also tells us that every Riesz generator of $V_1$ has $|a|n$ functions, and the assertion readily follows from [26, Theorem 1]. $\qquad\square$

We can actually say more: the following theorem elucidates the structure of a multiresolution analysis of multiplicity $n$:

**Theorem 2.** *Let $\{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis and assume that $\{u_\ell; \ell = 1, \cdots, n\}$ is an orthonormal basis generator of $V_0$. Then*
*(a) If $j > 0$,*
$$V_j = S(\mathbf{A}^j, u_1) \oplus S(\mathbf{A}^j, u_2) \oplus \cdots \oplus S(\mathbf{A}^j, u_n).$$

*(b)*

$$L^2(\mathbb{R}^d) = \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_1)} + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_2)} + \cdots \cdots + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_n)}$$

*Proof.* From [26, Theorem 4] we know that every orthonormal basis generator of $V_j, j > 0$, must have $n|a|^j$ functions, and an argument similar to the one employed in the proof of Lemma 2 yields (a).

To prove (b), let $f \in L^2(\mathbb{R}^d)$ and let $\varepsilon > 0$ be given; then there is a $j \in \mathbb{Z}_+$ and a $g \in V_j$ such that $||f - g|| < \varepsilon$. Since *a fortiori* $g$ belongs to the closed set

$$\overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_1)} + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_2)} + \cdots \cdots + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_n)}$$

and $\varepsilon$ is arbitrary, the assertion follows. $\square$

**Theorem 3.** *Let $m = n(|a| - 1)$, let the functions $y_k$, $k = 1 \cdots |a|n$ be constructed as in Theorem 1, and let $q_j = y_{j+\ell}$ for $\ell = 1, \cdots n$, $j = \ell|a|, \cdots, (\ell + 1)(|a| - 1)$. Then $\{w_1, \cdots w_r\}$ is an orthonormal wavelet system if and only if $r = m$ and there exists an orthogonal matrix $Q(x)$ such that*

$$(w_1, \cdots w_m)^T = Q(x)(q_1, \cdots q_m)^T.$$

*Proof.* From Theorem 1 we know that $\{q_1, \cdots q_m\}$ is an orthonormal basis system or, equivalently, that it is an orthonormal basis generator of $S(\mathbf{u})$. The assertion now readily follows from [26, Corollary 3 and Theorem 5]. $\square$

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# Generalized $g$–Fractional vector Representation Formula and integral Inequalities for Banach space valued functions

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### Abstract

Here we give a very general fractional Bochner integral representation formula for Banach space valued functions. We derive generalized left and righ fractional Opial type inequalities, fractional Ostrowski type inequalities and fractional Grüss type inequalities. All these inequalities are very general having in their background Bochner type integrals.

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## 1 Background

We need

**Definition 1** *([2]) Let $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a, b])$ and increasing, $f \in C([a, b], X)$, $\nu > 0$.*

*We define the left Riemann-Liouville generalized fractional Bochner integral operator*

$$\left(I_{a+;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_a^x \left(g(x) - g(z)\right)^{\nu-1} g'(z) f(z) \, dz, \qquad (1)$$

$\forall \ x \in [a, b]$, *where $\Gamma$ is the gamma function.*

*The last integral is of Bochner type. Since $f \in C([a, b], X)$, then $f \in L_\infty([a, b], X)$. By [2] we get that $I_{a+;g}^{\nu} f \in C([a, b], X)$. Above we set $I_{a+;g}^0 f := f$ and see that $\left(I_{a+;g}^{\nu} f\right)(a) = 0$.*

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When $g$ is the identity function $id$, we get that $I_{a+;id}^{\nu} = I_{a+}^{\nu}$, the ordinary left Riemann-Liouville fractional integral

$$\left(I_{a+}^{\nu}f\right)(x) = \frac{1}{\Gamma(\nu)}\int_{a}^{x}(x-t)^{\nu-1}f(t)\,dt, \qquad (2)$$

$\forall\ x \in [a,b]$, $\left(I_{a+}^{\nu}f\right)(a) = 0$.

We need

**Theorem 2** *([2]) Let $\mu, \nu > 0$ and $f \in C([a,b], X)$. Then*

$$I_{a+;g}^{\mu}I_{a+;g}^{\nu}f = I_{a+;g}^{\mu+\nu}f = I_{a+;g}^{\nu}I_{a+;g}^{\mu}f. \qquad (3)$$

We need

**Definition 3** *([2]) Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^{1}([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$.*

*We define the right Riemann-Liouville generalized fractional Bochner integral operator*

$$\left(I_{b-;g}^{\nu}f\right)(x) := \frac{1}{\Gamma(\nu)}\int_{x}^{b}(g(z)-g(x))^{\nu-1}g'(z)f(z)\,dz, \qquad (4)$$

$\forall\ x \in [a,b]$, where $\Gamma$ is the gamma function.

*The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_{\infty}([a,b], X)$. By [2] we get that $I_{b-;g}^{\nu}f \in C([a,b], X)$. Above we set $I_{b-;g}^{0}f := f$ and see that $\left(I_{b-;g}^{\nu}f\right)(b) = 0$.*

*When $g$ is the identity function $id$, we get that $I_{b-;id}^{\nu} = I_{b-}^{\nu}$, the ordinary right Riemann-Liouville fractional integral*

$$\left(I_{b-}^{\nu}f\right)(x) = \frac{1}{\Gamma(\nu)}\int_{x}^{b}(t-x)^{\nu-1}f(t)\,dt, \qquad (5)$$

$\forall\ x \in [a,b]$, with $\left(I_{b-}^{\nu}f\right)(b) = 0$.

We need

**Theorem 4** *([2]) Let $\mu, \nu > 0$ and $f \in C([a,b], X)$. Then*

$$I_{b-;g}^{\mu}I_{b-;g}^{\nu}f = I_{b-;g}^{\mu+\nu}f = I_{b-;g}^{\nu}I_{b-;g}^{\mu}f. \qquad (6)$$

We will use

**Definition 5** *([2]) Let $\alpha > 0$, $\lceil\alpha\rceil = n$, $\lceil\cdot\rceil$ the ceiling of the number. Let $f \in C^{n}([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^{1}([a,b])$, strictly increasing, such that $g^{-1} \in C^{n}([g(a), g(b)])$.*

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We define the left generalized $g$-fractional derivative $X$-valued of $f$ of order $\alpha$ as follows:

$$\left(D_{a+;g}^{\alpha}f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(g(x)-g(t)\right)^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t)) \, dt,$$
(7)

$\forall\ x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [2], we have that $\left(D_{a+;g}^{\alpha}f\right) \in C\left([a,b],X\right)$.

We see that

$$\left(I_{a+;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha}f\right)(x), \quad \forall\ x \in [a,b].$$
(8)

We set

$$D_{a+;g}^n f(x) := \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) \in C\left([a,b],X\right), \quad n \in \mathbb{N},$$
(9)

$$D_{a+;g}^0 f(x) = f(x), \quad \forall\ x \in [a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha}f = D_{a+;id}^{\alpha}f = D_{*a}^{\alpha}f,$$
(10)

the usual left $X$-valued Caputo fractional derivative, see [3].

We will use

**Definition 6** ([2]) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n\left([a,b],X\right)$, where $[a,b] \subset \mathbb{R}$, and $(X,\|\cdot\|)$ is a Banach space. Let $g \in C^1\left([a,b]\right)$, strictly increasing, such that $g^{-1} \in C^n\left([g(a),g(b)]\right)$.

We define the right generalized $g$-fractional derivative $X$-valued of $f$ of order $\alpha$ as follows:

$$\left(D_{b-;g}^{\alpha}f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \left(g(t)-g(x)\right)^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t)) \, dt,$$
(11)

$\forall\ x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [2], we have that $\left(D_{b-;g}^{\alpha}f\right) \in C\left([a,b],X\right)$.

We see that

$$I_{b-;g}^{n-\alpha}\left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \left(D_{b-;g}^{\alpha}f\right)(x), \quad a \le x \le b.$$
(12)

We set

$$D_{b-;g}^n f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^n \circ g\right)(x) \in C\left([a,b],X\right), \quad n \in \mathbb{N},$$
(13)

$$D_{b-;g}^0 f(x) := f(x), \quad \forall\ x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha}f(x) = D_{b-;id}^{\alpha}f(x) = D_{b-}^{\alpha}f,$$
(14)

the usual right $X$-valued Caputo fractional derivative, see [3].

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We make

**Remark 7** *All as in Definition 5. We have (by Theorem 2.5, p. 7, [5])*

$$\left\| \left(D^\alpha_{a+;g}f\right)(x)\right\| \leq \frac{1}{\Gamma(n-\alpha)}\int_a^x (g(x)-g(t))^{n-\alpha-1}\,g'(t)\left\|\left(f\circ g^{-1}\right)^{(n)}(g(t))\right\|dt$$

$$\leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)}\int_{g(a)}^{g(x)}(g(x)-g(t))^{n-\alpha-1}\,dg(t) =$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(x)-g(a))^{n-\alpha}. \tag{15}$$

*That is*

$$\left\|\left(D^\alpha_{a+;g}f\right)(x)\right\| \leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(x)-g(a))^{n-\alpha}, \tag{16}$$

$\forall\ x\in[a,b]$.

*If $\alpha\notin\mathbb{N}$, then $\left(D^\alpha_{a+;g}f\right)(a)=0$.*

*Similarly, by Definition 6 we derive*

$$\left\|\left(D^\alpha_{b-;g}f\right)(x)\right\| \leq \frac{1}{\Gamma(n-\alpha)}\int_x^b (g(t)-g(x))^{n-\alpha-1}\,g'(t)\left\|\left(f\circ g^{-1}\right)^{(n)}(g(t))\right\|dt$$

$$\leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)}\int_{g(x)}^{g(b)}(g(t)-g(x))^{n-\alpha-1}\,dg(t) =$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(b)-g(x))^{n-\alpha}. \tag{17}$$

*That is*

$$\left\|\left(D^\alpha_{b-;g}f\right)(x)\right\| \leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(b)-g(x))^{n-\alpha}, \tag{18}$$

$\forall\ x\in[a,b]$.

*If $\alpha\notin\mathbb{N}$, then $\left(D^\alpha_{b-;g}f\right)(b)=0$.*

**Notation 8** *We denote by*

$$D^{n\alpha}_{a+;g} := D^\alpha_{a+;g}D^\alpha_{a+;g}...D^\alpha_{a+;g}\quad (n\ times),\ n\in\mathbb{N}, \tag{19}$$

$$I^{n\alpha}_{a+;g} := I^\alpha_{a+;g}I^\alpha_{a+;g}...I^\alpha_{a+;g}, \tag{20}$$

$$D^{n\alpha}_{b-;g} := D^\alpha_{b-;g}D^\alpha_{b-;g}...D^\alpha_{b-;g}, \tag{21}$$

*and*

$$I^{n\alpha}_{b-;g} := I^\alpha_{b-;g}I^\alpha_{b-;g}...I^\alpha_{b-;g}, \tag{22}$$

*(n times), $n\in\mathbb{N}$.*

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We are motivated by the following generalized fractional Ostrowski type inequality:

**Theorem 9** *([2]) Let $g \in C^1([a,b])$ and strictly increasing, such that $g^{-1} \in C^1([g(a),g(b)])$, and $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b],X)$, where $(X,\|\cdot\|)$ is a Banach space. Let $x_0 \in [a,b]$ be fixed. Assume that $F_k^{x_0} := D_{x_0-;g}^{k\alpha}f$, for $k = 1,...,n$, fulfill $F_k^{x_0} \in C^1([a,x_0],X)$ and $\left(D_{x_0-;g}^{i\alpha}f\right)(x_0) = 0$, $i = 2,...,n$.*

*Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha}f$, for $k = 1,...,n$, fulfill $G_k^{x_0} \in C^1([x_0,b],X)$ and $\left(D_{x_0+;g}^{i\alpha}f\right)(x_0) = 0$, $i = 2,...,n$.*

*Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right\| \le \frac{1}{(b-a)\,\Gamma\left((n+1)\alpha + 1\right)} \cdot$$

$$\left\{ (g(b) - g(x_0))^{(n+1)\alpha}(b - x_0) \left\| D_{x_0+;g}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right.$$

$$\left. (g(x_0) - g(a))^{(n+1)\alpha}(x_0 - a) \left\| D_{x_0-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}. \tag{23}$$

In this work we will present several generalized fractional Bochner integral inequalities.

We mention the following $g$-left generalized $X$-valued Taylor's formula:

**Theorem 10** *([2]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $(X,\|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a),g(b)])$. Then*

$$f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(a)) +$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^\alpha f\right)(t)\,dt =$$

$$f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(a)) + \tag{24}$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^\alpha f\right) \circ g^{-1}\right)(z)\,dz, \quad \forall\, x \in [a,b].$$

We mention the following $g$-right generalized $X$-valued Taylor's formula:

**Theorem 11** *([2]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $(X,\|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a),g(b)])$. Then*

$$f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(b)) +$$

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$$\frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^\alpha f\right)(t) \, dt =$$

$$f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(b)) + \tag{25}$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^\alpha f\right) \circ g^{-1}\right)(z) \, dz, \quad \forall \, x \in [a,b].$$

For the Bochner integral excellent resources are [4], [6], [7] and [1], pp. 422-428.

## 2 Main Results

We give the following representation formula:

**Theorem 12** *All as in Theorem 10. Then*

$$f(y) = \frac{1}{b-a} \int_a^b f(x) \, dx - \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)} \int_a^b (g(x) - g(y))^k \, dx + R_1(y),$$
$$\tag{26}$$

*for any $y \in [a,b]$, where*

$$R_1(y) = -\frac{1}{\Gamma(\alpha)(b-a)}$$

$$\left[\int_a^b \chi_{[a,y)}(x) \left(\int_x^y |g(x) - g(t)|^{\alpha-1} g'(t) \left(D_{y-;g}^\alpha f\right)(t) \, dt\right) dx\right.$$

$$\left.\int_a^b \chi_{[y,b]}(x) \left(\int_y^x |g(x) - g(t)|^{\alpha-1} g'(t) \left(D_{y+;g}^\alpha f\right)(t) \, dt\right) dx\right]. \tag{27}$$

*here $\chi_A$ stands for the characteristic function set $A$, where $A$ is an arbitrary set.*

*One may write also that*

$$R_1(y) = -\frac{1}{\Gamma(\alpha)(b-a)} \left[\int_a^y \left(\int_x^y (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{y-;g}^\alpha f\right)(t) \, dt\right) dx\right.$$
$$\tag{28}$$

$$\left. + \int_y^b \left(\int_y^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{y+;g}^\alpha f\right)(t) \, dt\right) dx\right],$$

*for any $y \in [a,b]$.*

*Putting things together, one has*

$$f(y) = \frac{1}{b-a} \int_a^b f(x) \, dx - \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)} \int_a^b (g(x) - g(y))^k \, dx$$

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$$-\frac{1}{\Gamma\left(\alpha\right)\left(b-a\right)}\left[\int_{a}^{b}\chi_{[a,y)}\left(x\right)\left(\int_{x}^{y}\left|g\left(x\right)-g\left(t\right)\right|^{\alpha-1}g'\left(t\right)\left(D_{y-;g}^{\alpha}f\right)\left(t\right)dt\right)dx\right.$$

$$\left.\int_{a}^{b}\chi_{[y,b]}\left(x\right)\left(\int_{y}^{x}\left|g\left(x\right)-g\left(t\right)\right|^{\alpha-1}g'\left(t\right)\left(D_{y+;g}^{\alpha}f\right)\left(t\right)dt\right)dx\right]. \tag{29}$$

*In particular, one has*

$$f\left(y\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}\left(g\left(y\right)\right)}{k!\left(b-a\right)}\int_{a}^{b}\left(g\left(x\right)-g\left(y\right)\right)^{k}dx=R_{1}\left(y\right),$$
$$\tag{30}$$

*for any* $y\in[a,b]$.

**Proof.** Here $x,y\in[a,b]$. We keep $y$ as fixed.
By Theorem 10 we get:

$$f\left(x\right)=f\left(y\right)+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}\left(g\left(y\right)\right)}{k!}\left(g\left(x\right)-g\left(y\right)\right)^{k}+ \tag{31}$$

$$\frac{1}{\Gamma\left(\alpha\right)}\int_{y}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{\alpha-1}g'\left(t\right)\left(D_{y+;g}^{\alpha}f\right)\left(t\right)dt,\quad\text{for any }x\geq y.$$

By Theorem 11 we get:

$$f\left(x\right)=f\left(y\right)+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}\left(g\left(y\right)\right)}{k!}\left(g\left(x\right)-g\left(y\right)\right)^{k}+ \tag{32}$$

$$\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{y}\left(g\left(t\right)-g\left(x\right)\right)^{\alpha-1}g'\left(t\right)\left(D_{y-;g}^{\alpha}f\right)\left(t\right)dt,\quad\text{for any }x\leq y.$$

By (31), (32) we notice that

$$\int_{a}^{b}f\left(x\right)dx=\int_{a}^{y}f\left(x\right)dx+\int_{y}^{b}f\left(x\right)dx= \tag{33}$$

$$\int_{a}^{y}f\left(y\right)dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}\left(g\left(y\right)\right)}{k!}\int_{a}^{y}\left(g\left(x\right)-g\left(y\right)\right)^{k}dx+$$

$$\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{y}\left(\int_{x}^{y}\left(g\left(t\right)-g\left(x\right)\right)^{\alpha-1}g'\left(t\right)\left(D_{y-;g}^{\alpha}f\right)\left(t\right)dt\right)dx+$$

$$\int_{y}^{b}f\left(y\right)dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}\left(g\left(y\right)\right)}{k!}\int_{y}^{b}\left(g\left(x\right)-g\left(y\right)\right)^{k}dx+$$

$$\frac{1}{\Gamma\left(\alpha\right)}\int_{y}^{b}\left(\int_{y}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{\alpha-1}g'\left(t\right)\left(D_{y+;g}^{\alpha}f\right)\left(t\right)dt\right)dx.$$

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Hence it holds

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(y) + \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx +$$

$$\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y \left(\int_x^y |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y-;g}f\right)(t)\,dt\right)dx +\right.$$

$$\left.\int_y^b \left(\int_y^x |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y+;g}f\right)(t)\,dt\right)dx\right]. \tag{34}$$

Therefore we obtain

$$f(y) = \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx -$$

$$\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y \left(\int_x^y |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y-;g}f\right)(t)\,dt\right)dx +\right.$$

$$\left.\int_y^b \left(\int_y^x |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y+;g}f\right)(t)\,dt\right)dx\right]. \tag{35}$$

Hence the remainder

$$R_1(y) := -\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y \left(\int_x^y |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y-;g}f\right)(t)\,dt\right)dx\right.$$

$$\left. + \int_y^b \left(\int_y^x |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y+;g}f\right)(t)\,dt\right)dx\right] =$$

$$-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^b \chi_{[a,y)}(x)\left(\int_x^y |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y-;g}f\right)(t)\,dt\right)dx\right.$$

$$\left. + \int_a^b \chi_{[y,b]}(x)\left(\int_y^x |g(x)-g(t)|^{\alpha-1}\,g'(t)\left(D^\alpha_{y+;g}f\right)(t)\,dt\right)dx\right]. \tag{36}$$

The theorem is proved. ∎

Next we present a left fractional Opial type inequality:

**Theorem 13** *All as in Theorem 10. Additionally assume that $\alpha \geq 1$, $g \in C^1([a,b])$, and $\left(f\circ g^{-1}\right)^{(k)}(g(a)) = 0$, for $k = 0,1,...,n-1$. Let $p,q > 1$: $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_a^x \|f(w)\|\,\left\|\left(D^\alpha_{a+;g}f\right)(w)\right\|\,g'(w)\,dw \leq \frac{1}{\Gamma(\alpha)\,2^{\frac{1}{q}}}\cdot \tag{37}$$

$$\left(\int_a^x \left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}} \left(\int_a^x (g'(w))^q\,\left\|\left(D^\alpha_{a+;g}f\right)(w)\right\|^q\,dw\right)^{\frac{2}{q}},$$

$\forall\, x \in [a,b]\,.$

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**Proof.** By Theorem 10, we have that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^\alpha f\right)(t) dt, \quad \forall\, x \in [a,b]. \quad (38)$$

Then, by Hölder's inequality we obtain,

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x (g(x)-g(t))^{p(\alpha-1)} dt\right)^{\frac{1}{p}} \left(\int_a^x (g'(t))^q \left\|\left(D_{a+;g}^\alpha f\right)(t)\right\|^q dt\right)^{\frac{1}{q}}. \quad (39)$$

Call

$$z(x) := \int_a^x (g'(t))^q \left\|\left(D_{a+;g}^\alpha f\right)(t)\right\|^q dt, \quad (40)$$

$z(a) = 0$.

Thus

$$z'(x) = (g'(x))^q \left\|\left(D_{a+;g}^\alpha f\right)(x)\right\|^q \geq 0, \quad (41)$$

and

$$(z'(x))^{\frac{1}{q}} = g'(x) \left\|\left(D_{a+;g}^\alpha f\right)(x)\right\| \geq 0, \quad \forall\, x \in [a,b]. \quad (42)$$

Consequently, we get

$$\|f(w)\| g'(w) \left\|\left(D_{a+;g}^\alpha f\right)(w)\right\| \leq \quad (43)$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^w (g(w)-g(t))^{p(\alpha-1)} dt\right)^{\frac{1}{p}} (z(w) z'(w))^{\frac{1}{q}}, \quad \forall\, w \in [a,b].$$

Then

$$\int_a^x \|f(w)\| \left\|\left(D_{a+;g}^\alpha f\right)(w)\right\| g'(w) dw \leq \quad (44)$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x \left(\int_a^w (g(w)-g(t))^{p(\alpha-1)} dt\right)^{\frac{1}{p}} (z(w) z'(w))^{\frac{1}{q}} dw \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^x \left(\int_a^w (g(w)-g(t))^{p(\alpha-1)} dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x z(w) z'(w) dw\right)^{\frac{1}{q}} = \quad (45)$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^x \left(\int_a^w (g(w)-g(t))^{p(\alpha-1)} dt\right) dw\right)^{\frac{1}{p}} \left(\frac{z^2(x)}{2}\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^x \left(\int_a^w (g(w)-g(t))^{p(\alpha-1)} dt\right) dw\right)^{\frac{1}{p}}.$$

$$\left(\int_a^x (g'(t))^q \left\|\left(D_{a+;g}^\alpha f\right)(t)\right\|^q dt\right)^{\frac{2}{q}} \cdot 2^{-\frac{1}{q}}. \quad (46)$$

The theorem is proved. ∎

We also give a right fractional Opial type inequality:

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**Theorem 14** *All as in Theorem 11. Additionally assume that $\alpha \geq 1$, $g \in C^1([a, b])$, and $\left(f \circ g^{-1}\right)^{(k)}(g(b)) = 0$, $k = 0, 1, ..., n-1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_x^b \|f(w)\| \left\|\left(D_{b-;g}^\alpha f\right)(w)\right\| g'(w) \, dw \leq \frac{1}{2^{\frac{1}{q}} \Gamma(\alpha)} \cdot \quad (47)$$

$$\left(\int_x^b \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)} \, dt\right) dw\right)^{\frac{1}{p}} \left(\int_x^b (g'(w))^q \left\|\left(D_{b-;g}^\alpha f\right)(w)\right\|^q dw\right)^{\frac{2}{q}},$$

*all $a \leq x \leq b$.*

**Proof.** By Theorem 11, we have that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^\alpha f\right)(t) \, dt, \quad \text{all } a \leq x \leq b. \quad (48)$$

Then, by Hölder's inequality we obtain,

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (g(t) - g(x))^{p(\alpha-1)} \, dt\right)^{\frac{1}{p}} \left(\int_x^b (g'(t))^q \left\|\left(D_{b-;g}^\alpha f\right)(t)\right\|^q dt\right)^{\frac{1}{q}}. \quad (49)$$

Call

$$z(x) := \int_x^b (g'(t))^q \left\|\left(D_{b-;g}^\alpha f\right)(t)\right\|^q dt, \quad (50)$$

$z(b) = 0$.

Hence

$$z'(x) = -(g'(x))^q \left\|\left(D_{b-;g}^\alpha f\right)(x)\right\|^q \leq 0, \quad (51)$$

and

$$-z'(x) = (g'(x))^q \left\|\left(D_{b-;g}^\alpha f\right)(x)\right\|^q \geq 0, \quad (52)$$

and

$$(-z'(x))^{\frac{1}{q}} = g'(x) \left\|\left(D_{b-;g}^\alpha f\right)(x)\right\| \geq 0, \quad \forall\, x \in [a, b]. \quad (53)$$

Consequently, we get

$$\|f(w)\| \, g'(w) \left\|\left(D_{b-;g}^\alpha f\right)(w)\right\| \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)} \, dt\right)^{\frac{1}{p}} (z(w)(-z'(w)))^{\frac{1}{q}}, \quad \forall\, w \in [a, b]. \quad (54)$$

Then

$$\int_x^b \|f(w)\| \left\|\left(D_{b-;g}^\alpha f\right)(w)\right\| g'(w) \, dw \leq \quad (55)$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)} \, dt\right)^{\frac{1}{p}} (-z(w) z'(w))^{\frac{1}{q}} \, dw \leq$$

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$$\frac{1}{\Gamma(\alpha)} \left( \int_x^b \left( \int_w^b (g(t) - g(w))^{p(\alpha-1)} \, dt \right) dw \right)^{\frac{1}{p}} \left( - \int_x^b z(w) \, z'(w) \, dw \right)^{\frac{1}{q}} =$$

(56)

$$\frac{1}{\Gamma(\alpha)} \left( \int_x^b \left( \int_w^b (g(t) - g(w))^{p(\alpha-1)} \, dt \right) dw \right)^{\frac{1}{p}} \left( \frac{z^2(x)}{2} \right)^{\frac{1}{q}} =$$

$$\frac{1}{2^{\frac{1}{q}} \Gamma(\alpha)} \left( \int_x^b \left( \int_w^b (g(t) - g(w))^{p(\alpha-1)} \, dt \right) dw \right)^{\frac{1}{p}} \cdot$$

$$\left( \int_x^b (g'(t))^q \left\| (D_{b-;g}^{\alpha} f)(t) \right\|^q dt \right)^{\frac{2}{q}}.$$

(57)

The theorem is proved. ∎

Two extreme fractional Opial type inequalities follow (case $p = 1$, $q = \infty$).

**Theorem 15** *All as in Theorem 10. Assume that $\left( f \circ g^{-1} \right)^{(k)} (g(a)) = 0$, $k = 0, 1, ..., n - 1$. Then*

$$\int_a^x \|f(w)\| \left\| D_{a+;g}^{\alpha} f(w) \right\| dw \leq \frac{\left\| D_{a+;g}^{\alpha} f \right\|_{\infty}^2}{\Gamma(\alpha+1)} \left( \int_a^x (g(w) - g(a))^{\alpha} \, dw \right),$$

(58)

*all $a \leq x \leq b$.*

**Proof.** For any $w \in [a, b]$, we have that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^w (g(w) - g(t))^{\alpha-1} g'(t) \left( D_{a+;g}^{\alpha} f \right)(t) \, dt,$$

(59)

and

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^w (g(w) - g(t))^{\alpha-1} g'(t) \, dt \right) \left\| D_{a+;g}^{\alpha} f \right\|_{\infty}$$

$$= \frac{\left\| D_{a+;g}^{\alpha} f \right\|_{\infty}}{\Gamma(\alpha+1)} (g(w) - g(a))^{\alpha}.$$

(60)

Hence we obtain

$$\|f(w)\| \left\| D_{a+;g}^{\alpha} f(w) \right\| \leq \frac{\left\| D_{a+;g}^{\alpha} f \right\|_{\infty}^2}{\Gamma(\alpha+1)} (g(w) - g(a))^{\alpha}.$$

(61)

Integrating (61) over $[a, x]$ we derive (58). ∎

**Theorem 16** *All as in Theorem 11. Assume that $\left( f \circ g^{-1} \right)^{(k)} (g(b)) = 0$, $k = 0, 1, ..., n - 1$. Then*

$$\int_x^b \|f(w)\| \left\| D_{b-;g}^{\alpha} f(w) \right\| dw \leq \frac{\left\| D_{b-;g}^{\alpha} f \right\|_{\infty}^2}{\Gamma(\alpha+1)} \left( \int_x^b (g(b) - g(w))^{\alpha} \, dw \right),$$

(62)

*all $a \leq x \leq b$.*

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**Proof.** For any $w \in [a, b]$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_w^b (g(t) - g(w))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t) \, dt, \tag{63}$$

and

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \left( \int_w^b (g(t) - g(w))^{\alpha-1} g'(t) \, dt \right) \left\| D_{b-;g}^{\alpha} f \right\|_{\infty}$$

$$= \frac{\left\| D_{b-;g}^{\alpha} f \right\|_{\infty}}{\Gamma(\alpha+1)} (g(b) - g(w))^{\alpha}. \tag{64}$$

Hence we obtain

$$\|f(w)\| \left\| D_{b-;g}^{\alpha} f(w) \right\| \leq \frac{\left\| D_{b-;g}^{\alpha} f \right\|_{\infty}^2}{\Gamma(\alpha+1)} (g(b) - g(w))^{\alpha}. \tag{65}$$

Integrating (65) over $[x, b]$ we derive (62). ∎

Next we present three fractional Ostrowski type inequalities:

**Theorem 17** *All as in Theorem 10. Then*

$$\left\| f(y) - \frac{1}{b-a} \int_a^b f(x) \, dx + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!(b-a)} \int_a^b (g(x) - g(y))^k \, dx \right\| \leq \tag{66}$$

$$\frac{1}{\Gamma(\alpha+1)(b-a)}.$$

$$\left[ (g(y) - g(a))^{\alpha} (y-a) \left\| D_{y-;g}^{\alpha} f \right\|_{\infty} + (g(b) - g(y))^{\alpha} (b-y) \left\| D_{y+;g}^{\alpha} f \right\|_{\infty} \right],$$

$\forall \, y \in [a, b]$.

**Proof.** Define
$$\left(D_{y+;g}^{\alpha} f\right)(t) = 0, \quad \text{for } t < y,$$
$$\text{and} \tag{67}$$
$$\left(D_{y-;g}^{\alpha} f\right)(t) = 0, \quad \text{for } t > y.$$

Notice for $0 < \alpha \notin \mathbb{N}$ by Remark 7 we have

$$\left(D_{a+;g}^{\alpha} f\right)(a) = 0. \tag{68}$$

Similarly it holds $(0 < \alpha \notin \mathbb{N})$ by Remark 7 that

$$\left(D_{b-;g}^{\alpha} f\right)(b) = 0. \tag{69}$$

Thus
$$\left(D_{y+;g}^{\alpha} f\right)(y) = 0, \quad \left(D_{y-;g}^{\alpha} f\right)(y) = 0, \tag{70}$$

$0 < \alpha \notin \mathbb{N}$, any $y \in [a, b]$.

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We observe that

$$
\|R_1(y)\| \overset{(28)}{\leq} \frac{1}{\Gamma(\alpha)(b-a)} \left[ \left( \int_a^y \left( \int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\, dt \right) dx \right) \|D_{y-;g}^\alpha f\|_\infty \right.
$$

$$
\left. + \left( \int_y^b \left( \int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\, dt \right) dx \right) \|D_{y+;g}^\alpha f\|_\infty \right] =
\tag{71}
$$

$$
\frac{1}{\Gamma(\alpha)(b-a)} \left[ \left( \int_a^y \frac{(g(y)-g(x))^\alpha}{\alpha}\, dx \right) \|D_{y-;g}^\alpha f\|_\infty \right.
$$

$$
\left. + \left( \int_y^b \frac{(g(x)-g(y))^\alpha}{\alpha}\, dx \right) \|D_{y+;g}^\alpha f\|_\infty \right] \leq
$$

$$
\frac{1}{\Gamma(\alpha+1)(b-a)} \left[ (g(y)-g(a))^\alpha (y-a) \|D_{y-;g}^\alpha f\|_\infty + \right.
$$

$$
\left. (g(b)-g(y))^\alpha (b-y) \|D_{y+;g}^\alpha f\|_\infty \right].
\tag{72}
$$

We have proved that

$$
\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha+1)(b-a)} \left[ (g(y)-g(a))^\alpha (y-a) \|D_{y-;g}^\alpha f\|_\infty \right.
\tag{73}
$$

$$
\left. + (g(b)-g(y))^\alpha (b-y) \|D_{y+;g}^\alpha f\|_\infty \right],
$$

any $y \in [a,b]$.

We have established the theorem. ■

**Theorem 18** *All as in Theorem 10. Here we take $\alpha \geq 1$. Then*

$$
\left\| f(y) - \frac{1}{b-a} \int_a^b f(x)\, dx + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)} \int_a^b (g(x)-g(y))^k\, dx \right\| \leq
$$

$$
\frac{1}{\Gamma(\alpha)(b-a)} \left[ \left\| \left(D_{y-;g}^\alpha f\right) \circ g^{-1} \right\|_{1,[g(a),g(y)]} (y-a)(g(y)-g(a))^{\alpha-1} \right.
$$

$$
\left. + \left\| \left(D_{y+;g}^\alpha f\right) \circ g^{-1} \right\|_{1,[g(y),g(b)]} (b-y)(g(b)-g(y))^{\alpha-1} \right],
\tag{74}
$$

$\forall\ y \in [a,b]$.

**Proof.** We can rewrite

$$
R_1(y) = -\frac{1}{\Gamma(\alpha)(b-a)} \left[ \int_a^y \left( \int_{g(x)}^{g(y)} (z-g(x))^{\alpha-1} \left( \left(D_{y-;g}^\alpha f\right) \circ g^{-1} \right)(z)\, dz \right) dx \right.
$$

$$
\left. + \int_y^b \left( \int_{g(y)}^{g(x)} (g(x)-z)^{\alpha-1} \left( \left(D_{y+;g}^\alpha f\right) \circ g^{-1} \right)(z)\, dz \right) dx \right].
\tag{75}
$$

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We assumed $\alpha \geq 1$, then

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \cdot$$

$$\left[ \int_a^y \left( \int_{g(x)}^{g(y)} (z - g(x))^{\alpha-1} \left\| \left( \left( D_{y-;g}^\alpha f \right) \circ g^{-1} \right)(z) \right\| dz \right) dx \right. \tag{76}$$

$$\left. + \int_y^b \left( \int_{g(y)}^{g(x)} (g(x) - z)^{\alpha-1} \left\| \left( \left( D_{y+;g}^\alpha f \right) \circ g^{-1} \right)(z) \right\| dz \right) dx \right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[ \left( \int_a^y \left( \int_{g(x)}^{g(y)} \left\| \left( \left( D_{y-;g}^\alpha f \right) \circ g^{-1} \right)(z) \right\| dz \right) dx \right) (g(y) - g(a))^{\alpha-1} \right.$$

$$\left. + \left( \int_y^b \left( \int_{g(y)}^{g(x)} \left\| \left( \left( D_{y+;g}^\alpha f \right) \circ g^{-1} \right)(z) \right\| dz \right) dx \right) (g(b) - g(y))^{\alpha-1} \right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[ \left\| \left( D_{y-;g}^\alpha f \right) \circ g^{-1} \right\|_{1,[g(a),g(y)]} (y - a)(g(y) - g(a))^{\alpha-1} \right. \tag{77}$$

$$\left. + \left\| \left( D_{y+;g}^\alpha f \right) \circ g^{-1} \right\|_{1,[g(y),g(b)]} (b - y)(g(b) - g(y))^{\alpha-1} \right].$$

So when $\alpha \geq 1$, we obtained

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \left[ \left\| \left( D_{y-;g}^\alpha f \right) \circ g^{-1} \right\|_{1,[g(a),g(y)]} (y - a)(g(y) - g(a))^{\alpha-1} \right.$$

$$\tag{78}$$

$$\left. + \left\| \left( D_{y+;g}^\alpha f \right) \circ g^{-1} \right\|_{1,[g(y),g(b)]} (b - y)(g(b) - g(y))^{\alpha-1} \right].$$

Clearly here $g^{-1}$ is continuous, thus $\left( D_{y-;g}^\alpha f \right) \circ g^{-1} \in C\left( [g(a), g(y)], X \right)$, and $\left( D_{y+;g}^\alpha f \right) \circ g^{-1} \in C\left( [g(y), g(b)], X \right)$. Therefore

$$\left\| \left( D_{y-;g}^\alpha f \right) \circ g^{-1} \right\|_{1,[g(a),g(y)]}, \quad \left\| \left( D_{y+;g}^\alpha f \right) \circ g^{-1} \right\|_{1,[g(y),g(b)]} < \infty. \tag{79}$$

The proof of the theorem now is complete. ∎

**Theorem 19** *All as in Theorem 10. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$. Then*

$$\left\| f(y) - \frac{1}{b-a} \int_a^b f(x) \, dx + \sum_{k=1}^{n-1} \frac{\left( f \circ g^{-1} \right)^{(k)}(g(y))}{k!(b-a)} \int_a^b (g(x) - g(y))^k \, dx \right\|$$

$$\tag{80}$$

$$\leq \frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left[ (g(y) - g(a))^{\alpha-1+\frac{1}{p}} (y - a) \left\| \left( D_{y-;g}^\alpha f \right) \circ g^{-1} \right\|_{q,[g(a),g(y)]} \right.$$

$$\left. + (g(b) - g(y))^{\alpha-1+\frac{1}{p}} (b - y) \left\| \left( D_{y+;g}^\alpha f \right) \circ g^{-1} \right\|_{q,[g(y),g(b)]} \right],$$

$\forall \ y \in [a, b]$.

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**Proof.** Here we use (75).
We get that

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_{g(x)}^{g(y)}(z-g(x))^{p(\alpha-1)}dz\right)^{\frac{1}{p}}\cdot\right.$$

$$\left(\int_{g(x)}^{g(y)}\left\|\left(\left(D_{y-;g}^\alpha f\right)\circ g^{-1}\right)(z)\right\|^q dz\right)^{\frac{1}{q}}dx+\int_y^b\left(\int_{g(y)}^{g(x)}(g(x)-z)^{p(\alpha-1)}dz\right)^{\frac{1}{p}}\cdot$$

$$\left.\left(\int_{g(y)}^{g(x)}\left\|\left(\left(D_{y+;g}^\alpha f\right)\circ g^{-1}\right)(z)\right\|^q dz\right)^{\frac{1}{q}}dx\right]\leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)}\left[\left(\int_a^y\frac{(g(y)-g(x))^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}dx\right)\left\|\left(D_{y-;g}^\alpha f\right)\circ g^{-1}\right\|_{q,[g(a),g(y)]}\right.$$

$$\left.+\left(\int_a^y\frac{(g(x)-g(y))^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}dx\right)\left\|\left(D_{y+;g}^\alpha f\right)\circ g^{-1}\right\|_{q,[g(y),g(b)]}\right].$$
(81)

(here it is $\alpha-1+\frac{1}{p}>0$)
Hence it holds

$$\|R_1(y)\|\leq\frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}}\cdot$$
(82)

$$\left[(g(y)-g(a))^{\alpha-1+\frac{1}{p}}(y-a)\left\|\left(D_{y-;g}^\alpha f\right)\circ g^{-1}\right\|_{q,[g(a),g(y)]}+\right.$$

$$\left.(g(b)-g(y))^{\alpha-1+\frac{1}{p}}(b-y)\left\|\left(D_{y+;g}^\alpha f\right)\circ g^{-1}\right\|_{q,[g(y),g(b)]}\right].$$

Clearly here

$$\left\|\left(D_{y-;g}^\alpha f\right)\circ g^{-1}\right\|_{q,[g(a),g(y)]},\quad\left\|\left(D_{y+;g}^\alpha f\right)\circ g^{-1}\right\|_{q,[g(y),g(b)]}<\infty.$$

We have proved the theorem. ∎
Next we give some fractional Grüss type inequalities:

**Theorem 20** *Let $f,h$ as in Theorem 10. Here $R_1(y)$ will be renamed as $R_1(f,y)$, so we can consider $R_1(h,y)$. Then*
*1)*

$$\Delta_n(f,h):=\frac{1}{b-a}\int_a^b f(x)h(x)dx-\frac{\left(\int_a^b f(x)dx\right)\left(\int_a^b h(x)dx\right)}{(b-a)^2}+$$

$$\frac{1}{2(b-a)^2}\sum_{k=1}^{n-1}\frac{1}{k!}\left[\int_a^b\left(\int_a^b\left(h(y)\left(f\circ g^{-1}\right)^{(k)}(g(y))+\right.\right.\right.$$

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$$f\left(y\right)\left(h\circ g^{-1}\right)^{(k)}\left(g\left(y\right)\right)\right)\left(g\left(x\right)-g\left(y\right)\right)^{k}dx\right)dy\right]=$$

$$\frac{1}{2\left(b-a\right)}\left[\int_{a}^{b}\left(h\left(y\right)R_{1}\left(f,y\right)+f\left(y\right)R_{1}\left(h,y\right)\right)dy\right]=:K_{n}\left(f,h\right),\qquad(83)$$

*2) it holds*

$$\left\|\Delta_{n}\left(f,h\right)\right\|\leq\frac{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}{2\Gamma\left(\alpha+1\right)}\left[\left\|h\right\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|D_{y-;g}^{\alpha}f\right\|_{\infty}+\left\|D_{y+;g}^{\alpha}f\right\|_{\infty}\right)\right)\right.$$

$$\left.+\left\|f\right\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|D_{y-;g}^{\alpha}h\right\|_{\infty}+\left\|D_{y+;g}^{\alpha}h\right\|_{\infty}\right)\right)\right],\qquad(84)$$

*3) if $\alpha\geq1$, we get:*

$$\left\|\Delta_{n}\left(f,h\right)\right\|\leq\frac{1}{2\Gamma\left(\alpha\right)\left(b-a\right)}\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-1}\cdot\qquad(85)$$

$$\left\{\left\|h\right\|_{1}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}\right)\right)+\right.$$

$$\left.\left\|f\right\|_{1}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}h\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}h\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}\right)\right)\right\},$$

*4) if $p,q>1:\frac{1}{p}+\frac{1}{q}=1,\ \alpha>\frac{1}{q}$, we get:*

$$\left\|\Delta_{n}\left(f,h\right)\right\|\leq\frac{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-1+\frac{1}{p}}}{2\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\cdot\qquad(86)$$

$$\left\{\left\|h\right\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}\right)\right)+\right.$$

$$\left.\left\|f\right\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}h\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}h\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}\right)\right)\right\}.$$

*All right hand sides of (84)-(86) are finite.*

**Proof.** By Theorem 10 we have

$$h\left(y\right)f\left(y\right)=\frac{h\left(y\right)}{b-a}\int_{a}^{b}f\left(x\right)dx-$$

$$\sum_{k=1}^{n-1}\frac{h\left(y\right)\left(f\circ g^{-1}\right)^{(k)}\left(g\left(y\right)\right)}{k!\left(b-a\right)}\int_{a}^{b}\left(g\left(x\right)-g\left(y\right)\right)^{k}dx+h\left(y\right)R_{1}\left(f,y\right),\qquad(87)$$

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and

$$f(y) h(y) = \frac{f(y)}{b-a} \int_a^b h(x) \, dx -$$

$$\sum_{k=1}^{n-1} \frac{f(y) (h \circ g^{-1})^{(k)} (g(y))}{k! (b-a)} \int_a^b (g(x) - g(y))^k \, dx + f(y) R_1(h, y), \qquad (88)$$

$\forall \ y \in [a, b]$.

Then integrating (87) we find

$$\int_a^b h(y) f(y) \, dy = \frac{\left( \int_a^b h(y) \, dy \right)}{b-a} \left( \int_a^b f(x) \, dx \right) -$$

$$\sum_{k=1}^{n-1} \frac{1}{k! (b-a)} \int_a^b \int_a^b h(y) (f \circ g^{-1})^{(k)} (g(y)) (g(x) - g(y))^k \, dx dy$$

$$+ \int_a^b h(y) R_1(f, y) \, dy, \qquad (89)$$

and integrating (88) we obtain

$$\int_a^b f(y) h(y) \, dy = \frac{\left( \int_a^b f(y) \, dy \right) \left( \int_a^b h(x) \, dx \right)}{b-a} -$$

$$\sum_{k=1}^{n-1} \frac{1}{k! (b-a)} \int_a^b \int_a^b f(y) (h \circ g^{-1})^{(k)} (g(y)) (g(x) - g(y))^k \, dx dy$$

$$+ \int_a^b f(y) R_1(h, y) \, dy. \qquad (90)$$

Adding the last two equalities (89) and (90), we get:

$$2 \int_a^b f(x) h(x) \, dx = \frac{2 \left( \int_a^b f(x) \, dx \right) \left( \int_a^b h(x) \, dx \right)}{b-a} -$$

$$\sum_{k=1}^{n-1} \frac{1}{k! (b-a)} \left[ \left[ \int_a^b \int_a^b h(y) (f \circ g^{-1})^{(k)} (g(y)) + f(y) (h \circ g^{-1})^{(k)} (g(y)) \right] \cdot \right.$$

$$\left. (g(x) - g(y))^k \, dx dy \right] + \int_a^b (h(y) R_1(f, y) + f(y) R_1(h, y)) \, dy. \qquad (91)$$

Divide the last (91) by $2(b-a)$ to obtain (83).

Then, we upper bound $K_n(f, h)$ using Theorems 17, 18, 19, to obtain (84)-(86), respectively.

We use also that a norm is a continuous function. The theorem is proved.

∎

We make

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**Remark 21** *(in support of the proof of Theorem 20) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$. We have*

$$\left( D_{y+;g}^{\alpha} f \right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{y}^{x} \left( g(x) - g(t) \right)^{n-\alpha-1} g'(t) \left( f \circ g^{-1} \right)^{(n)} (g(t)) \, dt,$$

$$(92)$$

*$\forall \ x \in [y,b]$, and*

$$\left( D_{y-;g}^{\alpha} f \right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{y} \left( g(t) - g(x) \right)^{n-\alpha-1} g'(t) \left( f \circ g^{-1} \right)^{(n)} (g(t)) \, dt,$$

$$(93)$$

*$\forall \ x \in [a,y]$, both are Bochner type integrals.*

*By change of variables for Bochner integrals, see [6], Lemma B. 4.10 and [7], p. 158, we get:*

$$\left( D_{y+;g}^{\alpha} f \right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{g(y)}^{g(x)} \left( g(x) - z \right)^{n-\alpha-1} \left( f \circ g^{-1} \right)^{(n)} (z) \, dz =$$

$$\left( D_{g(y)+}^{\alpha} \left( f \circ g^{-1} \right) \right) (g(x)), \quad \forall \ x \in [y,b], \tag{94}$$

*and*

$$\left( D_{y-;g}^{\alpha} f \right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{g(x)}^{g(y)} \left( z - g(x) \right)^{n-\alpha-1} \left( f \circ g^{-1} \right)^{(n)} (z) \, dz =$$

$$\left( D_{g(y)-}^{\alpha} \left( f \circ g^{-1} \right) \right) (g(x)), \quad \forall \ x \in [a,y]. \tag{95}$$

*Here $D_{g(y)+}^{\alpha}, D_{g(y)-}^{\alpha}$ are the left and right $X$-valued Caputo fractional differentiation operators.*

*Fix $w : w \geq x_0 \geq y_0$; $w, x_0, y_0 \in [a,b]$, then $g(w) \geq g(x_0) \geq g(y_0)$. Hence*

$$\left\| \left( D_{y_0+;g}^{\alpha} f \right)(w) - \left( D_{x_0+;g}^{\alpha} f \right)(w) \right\| =$$

$$\left\| \left( D_{g(y_0)+}^{\alpha} \left( f \circ g^{-1} \right) \right) (g(w)) - \left( D_{g(x_0)+}^{\alpha} \left( f \circ g^{-1} \right) \right) (g(w)) \right\| =$$

$$\frac{1}{\Gamma(n-\alpha)} \left\| \int_{g(y_0)}^{g(x_0)} \left( g(w) - z \right)^{n-\alpha-1} \left( f \circ g^{-1} \right)^{(n)} (z) \, dz \right\| \leq \tag{96}$$

$$\frac{1}{\Gamma(n-\alpha)} \int_{g(y_0)}^{g(x_0)} \left( g(w) - z \right)^{n-\alpha-1} \left\| \left( f \circ g^{-1} \right)^{(n)} (z) \right\| dz \leq$$

$$\frac{\left\| \left( f \circ g^{-1} \right)^{(n)} \right\|_{\infty,[g(a),g(b)]}}{\Gamma(n-\alpha)} \int_{g(y_0)}^{g(x_0)} \left( g(w) - z \right)^{n-\alpha-1} dz =$$

$$\frac{\left\| \left( f \circ g^{-1} \right)^{(n)} \right\|_{\infty,[g(a),g(b)]}}{\Gamma(n-\alpha+1)} \left[ \left( g(y_0) - z \right)^{n-\alpha} - \left( g(x_0) - z \right)^{n-\alpha} \right] \to 0,$$

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*as $y_0 \to x_0$, then $g(y_0) \to g(x_0)$, proving continuity of $\left( D_{g(y)+}^{\alpha} \left( f \circ g^{-1} \right) \right)(g(x))$ with respect to $g(y)$, and of course continuity of $\left( D_{y+;g}^{\alpha} f \right)(x)$ in $y \in [a, b]$.*

    *Similarly, it is proved that $\left( D_{y-;g}^{\alpha} f \right)(x)$ is continuous in $y \in [a, b]$, the proof is omitted.*

**Remark 22** *Some examples for $g$ follow:*

$$g(x) = e^x, \quad x \in [a, b] \subset \mathbb{R},$$
$$g(x) = \sin x,$$
$$g(x) = \tan x,$$
$$where \; x \in \left[ -\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon \right], \; where \; \varepsilon > 0 \; small.$$

*Indeed, the above examples of $g$ are strictly increasing and continuous functions.*

    *One can apply all of our results here for the above specific choices of $g$. We choose to omit this job.*

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# On solutions of semilinear second-order impulsive functional differential equations

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## Abstract

This paper deals with the regularity for solutions of second-order semilinear impulsive differential equations contained the nonlinear convolution with cosine families, and obtain a variation of constant formula for solutions of the given equations.

*Keywords:*semilinear second-order equations, regularity for solutions, cosine family, sine family

*AMS Classification* Primary 35F25; Secondary 35K55

## 1    Introduction

In this paper we are concerned with the regularity of the following second-order semilinear impulsive differential system

$$
\begin{cases}
w''(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), & 0 < t \le T, \\
w(0) = x_0, \quad w'(0) = y_0, \\
\Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), & k = 1,2,...,m
\end{cases}
\tag{1.1}
$$

in a Banach space $X$. Here $k$ belongs to $L^2(0,T)$ and $g : [0,T] \times D(A) \to X$ is a nonlinear mapping such that $w \mapsto g(t,w)$ satisfies Lipschitz continuous. In (1.1),

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the principal operator $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$. The impulsive condition

$$\Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m$$

is combination of traditional evolution systems whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc.

In recent years the theory of impulsive differential systems has been emerging as an important area of investigation in applied sciences. The reason is that it is richer than the corresponding theory of classical differential equations and it is more adequate to represent some processes arising in various disciplines. The theory of impulsive systems provides a general framework for mathematical modeling of many real world phenomena(see [1, 2] and references therein). The theory of impulsive differential equations has seen considerable development. Impulsive differential systems have been studied in [3, 4, 5, 6], second-order impulsive integrodifferential systems in [7, 8], and Stochastic differential systems with impulsive conditions in [9, 10, 11].

In this paper, we allow implicit arguments about $L^2$-regularity results for semilinear hyperbolic equations with impulsive condition. These consequences are obtained by showing that results of the linear cases [12, 13] and semilinear case [14] on the $L^2$-regularity remain valid under the above formulation of (1.1). Earlier works prove existence of solution by using Azera Ascoli theorem. But we propose a different approach from that of earlier works to study mild, strong and classical solutions of Cauchy problems by using the properties of the linear equation in the hereditary part.

This paper is organized as follows. In Section 2, we give some definition, notation and the regularity for the corresponding linear equations. In Section 3, by using properties of the strict solutions of linear equations in dealt in Section 2, we will obtain the $L^2$-regularity of solutions of (1.1), and a variation of constant formula of solutions of (1.1). Finally, we also give an example to illustrate the applications of the abstract results..

## 2 Preliminaries

In this section, we give some definitions, notations, hypotheses and Lemmas. Let $X$ be a Banach space with norm denoted by $|| \cdot ||$.

**Definition 2.1.** *[15] A one parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in $X$ is called a strongly continuous cosine family if*

c(1)  $C(s+t) + C(s-t) = 2C(s)C(t), \quad$ *for all $s, \ t \in \mathbb{R}$,*

c(2)  $C(0) = I,$

c(3)  $C(t)x$  *is continuous in $t$  on  $\mathbb{R}$  for each fixed $x \in X$.*

If $C(t), \ t \in \mathbb{R}$ is a strongly continuous cosine family in $X$ , then $S(t), \ t \in \mathbb{R}$ is the one parameter family of operators in $X$ defined by

$$S(t)x = \int_0^t C(s)x \, ds, \ \ x \in X, \ \ t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator of a strongly continuous cosine family $C(t), \ t \in \mathbb{R}$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(0)x.$$

We endow with the domain $D(A) = \{x \in X : C(t)x$ is a twice continuously differentiable function of $t\}$ with norm

$$||x||_{D(A)} = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\} + ||Ax||.$$

We shall also make use of the set

$$E = \{x \in X : C(t)x \ \text{is a once continuously differentiable function of } t\}$$

with norm
$$||x||_E = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\}.$$

It is not difficult to show that $D(A)$ and $E$ with given norms are Banach spaces.
The following Lemma is from Proposition 2.1 and Proposition 2.2 of [1].

**Lemma 2.1.** *Let $C(t)(t \in \mathbb{R})$ be a strongly continuous cosine family in $X$. The following are true :*

c(4)  $C(t) = C(-t)$ *for all $t \in \mathbb{R}$,*

c(5)  $C(s), S(s), C(t)$ *and $S(t)$ commute for all $s, t \in \mathbb{R}$,*

c(6)  $S(t)x$ *is continuous in $t$ on $\mathbb{R}$ for each fixed $x \in X$,*

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*c(7)*   *there exist constants $K \geq 1$ and $\omega \geq 0$ such that*

$$||C(t)|| \leq Ke^{\omega|t|} \text{ for all } t \in \mathbb{R},$$

$$||S(t_1) - S(t_2)|| \leq K\left|\int_{t_2}^{t_1} e^{\omega|s|}ds\right| \text{ for all } t_1, t_2 \in \mathbb{R},$$

*c(8)*   *if $x \in E$, then $S(t)x \in D(A)$ and*

$$\frac{d}{dt}C(t)x = AS(t)x = S(t)Ax = \frac{d^2}{dt^2}S(t)x,$$

*c(9)*   *if $x \in D(A)$, then $C(t)x \in D(A)$ and*

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax,$$

*c(10)*   *if $x \in X$ and $r, s \in \mathbb{R}$, then*

$$\int_r^s S(\tau)x d\tau \in D(A) \quad \text{and} \quad A(\int_r^s S(\tau)x d\tau) = C(s)x - C(r)x,$$

*c(11)*   $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all $s, t \in \mathbb{R}$,*

*c(12)*   $S(s+t) = S(s)C(t) + S(t)C(s)$ *for all $s, t \in \mathbb{R}$,*

*c(13)*   $C(s+t) = C(t)C(s) - S(t)S(s)$ *for all $s, t \in \mathbb{R}$,*

*c(14)*   $C(s+t) - C(t-s) = 2AS(t)S(s)$ *for all $s, t \in \mathbb{R}$.*

The following Lemma is from Proposition 2.4 of [15].

**Lemma 2.2.** *Let $C(t)(t \in \mathbb{R})$ be a strongly continuous cosine family in $X$ with infinitesimal generator $A$. If $f : \mathbb{R} \to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$, and*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R},$$

*then $w(t) \in D(A)$ for $t \in \mathbb{R}$, $w$ is twice continuously differentiable, and $w$ satisfies*

$$w''(t) = Aw(t) + f(t), \ \ t \in R, \ w(0) = x_0, \ w'(0) = y_0. \tag{2.2}$$

*Conversely, if $f : \mathbb{R} \to X$ is continuous, $w(t) : \mathbb{R} \to X$ is twice continuously differentiable, $w(t) \in D(A)$ for $t \in \mathbb{R}$, and $w$ satisfies (2.2), then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R}.$$

**Proposition 2.1.** *Let $f : \mathbb{R} \to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$. Then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R}$$

*is a solution of (2.2) belonging to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$. Moreover, we have that there exists a positive constant $C_1$ such that for any $T > 0$,*

$$||w||_{L^2(0,T;D(A))} \leq C_1(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{2.3}$$

# 3 Nonlinear equations

This section is to investigate the regularity of solutions of a second-order nonlinear impulsive differential system

$$\begin{cases} w''(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), & 0 < t \leq T, \\ w(0) = x_0, \quad w'(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m \end{cases} \tag{3.1}$$

in a Banach space $X$.

**Assumption (G)** Let $g : [0,T] \times D(A) \to X$ be a nonlinear mapping such that $t \mapsto g(t,w)$ is measurable and

**(g1)** $||g(t,w_1) - g(t,w_2)||_{D(A)} \leq L||w_1 - w_2||,$

for a positive constant $L$.

**Assumption (I)** Let $I_k^1 : D(A) \to X$, $I_k^2 : E \to X$ be continuous and there exist positive constants $L(I_k^1)$, $L(I_k^2)$ such that

**(i1)** $||I_k^1(w_1) - I_k^1(w_2)|| \leq L(I_k^1)||w_1 - w_2||_{D(A)}$, for each $w_1, w_2 \in D(A)$
$||I_k^1(w)|| \leq L(I_k^1)$, for $w \in D(A)$

**(i2)** $||I_k^2(w_1') - I_k^2(w_2')|| \leq L(I_k^2)||w_1' - w_2'||_E$, for each $w_1', w_2' \in E$
$||I_k^2(w')|| \leq L(I_k^2)||$, for $w' \in E$.

For $w \in L^2(0,T : D(A))$, we set

$$F(t,w) = \int_0^t k(t-s)g(s,w(s))ds$$

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where $k$ belongs to $L^2(0,T)$. Then we will seek a mild solution of (3.1), that is, a solution of the integral equation

$$
\begin{aligned}
w(t) =& C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s,w) + f(s)\}ds \\
&+ \sum_{0<t_k<t} C(t-t_k)I_k^1(w(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w'(t_k^+)), \ \ t \in \mathbb{R}. \quad (3.2)
\end{aligned}
$$

**Remark 3.1.** *If $g : [0,T] \times X \to X$ is a nonlinear mapping satisfying*

$$
||g(t, w_1) - g(t, w_2)|| \le L||w_1 - w_2||
$$

*for a positive constant $L$, then our results can be obtained immediately.*

**Lemma 3.1.** *Let $w \in L^2(0,T;D(A))$, $T > 0$. Then $F(\cdot, w) \in L^2(0,T;X)$ and*

$$
||F(\cdot, w)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w||_{L^2(0,T;D(A))}.
$$

*Moreover if $w_1, \ w_2 \in L^2(0,T;D(A))$, then*

$$
||F(\cdot, w_1) - F(\cdot, w_2)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w_1 - w_2||_{L^2(0,T;D(A))}.
$$

**Lemma 3.2.** *If $k \in W^{1,2}(0,T)$, $T > 0$, then*

$$
A\int_0^t S(t-s)F(s,w)ds = -F(t,w) \quad (3.3)
$$
$$
+ \int_0^t (C(t-s) - I) \int_0^s \frac{d}{ds}k(s-\tau)g(\tau, w(\tau))d\tau \, ds
$$
$$
+ \int_0^t (C(t-s) - I)k(0)g(s, w(s))ds.
$$

**Theorem 3.1.** *Suppose that the Assumptions (G) and Assumption (I) are satisfied. If $f : \mathbb{R} \to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$, and $k \in W^{1,2}(0,T)$, $T > 0$, then there exists a time $T \ge T_0 > 0$ such that the functional differential equation (3.1) admits a unique solution $w$ in $L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)$.*

*Proof.* Let us fix $T_0 > 0$ so that

$$
\begin{aligned}
C_2 \equiv& \omega^{-1} K L T_0^{3/2}(e^{\omega T_0} - 1)||k||_{L^2(0,T_0)} \\
&+ \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} T_0^{3/2}/\sqrt{3} L ||Ke^{\omega T_0} + 1|| ||k||_{W^{1,2}(0,T_0)} \\
&+ \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} T_0/\sqrt{2} L ||Ke^{\omega T_0} + 1|| ||k(0)|| \\
&+ \{w^{-1} K(e^{wT_0} - 1) + 2\} \sum_{0<t_k<t} L(I_k^1) K e^{wT_0} \\
&+ \{2w^{-1} K(e^{wT_0} - 1) + 1\} \sum_{0<t_k<t} L(I_k^2) < 1
\end{aligned}
\tag{3.4}
$$

where $K$, $L$, $L(I_k^1)$ and $L(I_k^2)$ are constants in c(7), (g1) and Assumption (I) respectively. Invoking Proposition 2.1, for any $v \in L^2(0, T_0; D(A))$ we obtain the equation

$$
\begin{cases}
w''(t) = Aw(t) + F(t, v) + f(t), \quad 0 < t \le T_0, \\
w(0) = x_0, \quad w'(0) = y_0 \\
\Delta w(t_k) = I_k^1(v(t_k)), \quad \Delta w'(t_k) = I_k^2(v'(t_k^+)), \quad k = 1, 2, ..., m
\end{cases}
\tag{3.5}
$$

has a unique solution $w \in L^2(0, T_0; D(A)) \cap W^{1,2}(0, T_0; E)$. Let $w_1$, $w_2$ be the solutions of (3.5) with $v$ replaced by $v_1$, $v_2 \in L^2(0, T_0; D(A))$, respectively. Put

$$
\begin{aligned}
J(w)(t) = &C(t)x_0 + S(t)y_0 + \int_0^t S(t - s)\{F(s, v) + f(s)\}ds \\
&+ \sum_{0<t_k<t} C(t - t_k)I_k^1(v(t_k)) + \sum_{0<t_k<t} S(t - t_k)I_k^2(v'(t_k^+)).
\end{aligned}
$$

Then

$$
\begin{aligned}
J(w_1)(t) - J(w_2)(t) = &\int_0^t S(t - s)\{F(s, v_1) - F(s, v_2)\}ds \\
&+ \sum_{0<t_k<t} C(t - t_k)\{I_k^1(v_1(t_k)) - I_k^1(v_2(t_k))\} \\
&+ \sum_{0<t_k<t} S(t - t_k)\{I_k^2(v_1'(t_k^+)) - I_k^1(v_2'(t_k^+))\}, \\
=&I_1 + I_2 + I_3.
\end{aligned}
$$

So, from Lemmas 3.1, 3.2, it follows that for $0 \le t \le T_0$,

$$
\begin{aligned}
&||\int_0^t S(t - s)\{F(s, v_1) - F(s, v_2)\}ds|| \\
&\le \omega^{-1} K L T_0(e^{\omega T_0} - 1)||k||_{L^2(0,T_0)}||v_1 - v_2||_{L^2(0,T_0;D(A))},
\end{aligned}
$$

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$$||\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||$$

$$\leq ||AS(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||$$

$$= ||S(t)A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||,$$

and

$$||A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||$$

$$\leq ||\int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)(g(\tau,v_1(\tau))-g(\tau,v_2(\tau)))d\tau\, ds||$$

$$+ ||\int_0^t (C(t-s)-I)k(0)(g(s,v_1(s))-g(s,v_2(s)))ds||$$

$$\leq tL||Ke^{\omega t}+1||||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}$$

$$+ \sqrt{t}L||Ke^{\omega t}+1||||k(0)||||v_1-v_2||_{L^2(0,T_0;D(A))}.$$

Therefore, we have

$$||I_1||_{L^2(0,T_0;D(A))} \leq \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \qquad (3.6)$$

$$+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}$$

$$+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||_{L^2(0,T_0;D(A))}.$$

By Assumption (i1), we obtain

$$||\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1(t_k^-))-I_k^1(v_2(t_k^-))\}|| \leq \sum_{0<t_k<T_0}Ke^{wT_0}L(I_k^1)||v_1-v_2||_{D(A)},$$

$$||\frac{d}{dt}C(t)\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||$$

$$\leq ||AS(t)\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||$$

$$= ||S(t)A\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||,$$

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and

$$||A\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1(t_k^-))-I_k^1(v_2(t_k^-))\}|| = ||\sum_{0<t_k<t} C(t-t_k)A\{I_k^1(v_1)-I_k^1(v_2)\}||$$
$$\leq \sum_{0<t_k<t} Ke^{wt}||I_k^1(v_1)-I_k^1(v_2)||_{D(A)}$$
$$\leq \sum_{0<t_k<t} Ke^{wt}L(I_k^1)||v_1-v_2||_{D(A)}.$$

Therefore, we have

$$||I_2||_{L^2(0,T_0;D(A))} \leq \{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))}.$$
$$(3.7)$$

We also obtain from Assumption (i2),

$$||\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1'(t_k^+))-I_k^1(v_2'(t_k^+))\}|| \leq \sum_{0<t_k<T_0} Kw^{-1}(e^{wT_0}-1)L(I_k^2)||v_1-v_2||_{D(A)},$$

$$||\frac{d}{dt}C(t)\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||$$
$$\leq ||AS(t)\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||$$
$$= ||S(t)A\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||,$$

and

$$||A\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1'(t_k^+))-I_k^1(v_2'(t_k^+))\}|| = ||\sum_{0<t_k<t} \frac{d}{dt}C(t)\{I_k^2(v_1')-I_k^1(v_2')\}||$$
$$\leq \sum_{0<t_k<t} ||I_k^2(v_1')-I_k^1(v_2')||_E$$
$$\leq \sum_{0<t_k<t} L(I_k^2)||v_1'-v_2'||_E.$$

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Therefore, we have

$$||I_3||_{L^2(0,T_0;D(A))} \leq \{w^{-1}K(e^{wT_0} - 1) + 2\} \sum_{0 < t_k < t} L(I_k^1)Ke^{wT_0}||v_1 - v_2||_{L^2(0,T_0;D(A))}.$$

(3.8)

Thus, from (3.6),(3.7), and (3.8), we conclude that

$$||J(w_1) - J(w_2)||_{L^2(0,T_0;D(A))} \tag{3.9}$$
$$\leq \omega^{-1}KLT_0^{3/2}(e^{\omega T_0} - 1)||k||_{L^2(0,T_0)}||v_1 - v_2||_{L^2(0,T_0;D(A))}$$
$$+ \{\omega^{-1}K(e^{\omega T_0} - 1) + 1\}L||k||_{L^2(0,T_0)}\sqrt{T_0}||v_1 - v_2||_{L^2(0,T_0;D(A))}$$
$$+ \{\omega^{-1}K(e^{\omega T_0} - 1) + 1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0} + 1|| \, ||k||_{W^{1,2}(0,T_0)}||v_1 - v_2||_{L^2(0,T_0;D(A))}$$
$$+ \{\omega^{-1}K(e^{\omega T_0} - 1) + 1\}T_0/\sqrt{2}L||Ke^{\omega T_0} + 1|| \, ||k(0)|| \, ||v_1 - v_2||_{L^2(0,T_0;D(A))}$$
$$+ \{w^{-1}K(e^{wT_0} - 1) + 2\} \sum_{0 < t_k < t} L(I_k^1)Ke^{wT_0}||v_1 - v_2||_{L^2(0,T_0;D(A))}$$
$$+ \{2w^{-1}K(e^{wT_0} - 1) + 1\} \sum_{0 < t_k < t} L(I_k^2)||v_1 - v_2||_{W^{1,2}(0,T_0;D(A))}.$$

Moreover, it is easily seen that

$$||J(w_1) - J(w_2)||_{L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)} \leq C_2||v_1 - v_2||_{L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)}.$$

So by virtue of the condition (3.4) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $[0, T_0]$.  □

**Theorem 3.2.** *Suppose that the Assumptions (G) and (I) are satisfied. If $f : \mathbb{R} \to X$ is continuously differentiable, $x_0 \in D(A), y_0 \in E$, and $k \in W^{1,2}(0,T)$, $T > 0$, then the solution $w$ of (3.1) exists and is unique in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$, and there exists a constant $C_3$ depending on $T$ such that*

$$||w||_{L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)} \leq C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \quad (3.10)$$

*Proof.* Let $w(\cdot)$ be the solution of (3.1) in the interval $[0, T_0]$ where $T_0$ is a constant in (3.4) and $v(\cdot)$ be the solution of the following equation

$$v''(t) = Av(t) + f(t), \ \ 0 < t,$$
$$v(0) = x_0, \ \ v'(0) = y_0.$$

Then

$$(w-v)(t) = \int_0^t S(t-s)F(s,w)ds + \sum_{0<t_k<t} C(t-t_k)I_k^1(w(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w'(t_k^+)),$$

and in view of (3.9)

$$||w-v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \leq C_2||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}, \tag{3.11}$$

that is, combining (3.11) with Proposition 2.1 we have

$$||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \leq \frac{1}{1-C_2}||v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \tag{3.12}$$

$$\leq \frac{C_1}{1-C_2}(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T_0;X)}).$$

Now from

$$A\int_0^{T_0} S(T_0-s)\{F(s,w)+f(s)\}ds$$

$$= C(T_0)f(0) - f(T_0) + \int_0^{T_0}(C(T_0-s)-I)f'(s)ds$$

$$- F(T_0,w) + \int_0^{T_0}(C(T_0-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\,ds$$

$$+ \int_0^{T_0}(C(T_0-s)-I)k(0)g(s,w(s))ds,$$

$$||A\sum_{0<t_k<t} C(t-t_k)I_k^1(w_1)|| \leq Kw^{-1}(e^{wT_0-1})Ke^{wT_0}\sum_{0<t_k<t} L(I_k^1)||w(t_k)||_{D(A)},$$

$$||\sum_{0<t_k<t} S(t-t_k)I_k^2(v_1')|| \leq \sum_{0<t_k<t} L(I_k^2)||w'(t_k^+)||_E,$$

and since

$$\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,w)+f(s)\}ds = S(t)A\int_0^t S(t-s)\{F(s,w)+f(s)\}ds,$$

$$\frac{d}{dt}C(t)\sum_{0<t_k<t} C(t-t_k)I_k^1(w) \leq S(t)A\sum_{0<t_k<t} C(t-t_k)I_k^1(w).$$

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$$\frac{d}{dt}C(t)\sum_{0<t_k<t}S(t-t_k)I_k^2(w') \le S(t)A\sum_{0<t_k<t}S(t-t_k)I_k^2(w').$$

We have

$$||w(T_0)||_{D(A)} = ||C(T_0)x_0 + S(T_0)y_0 + \int_0^{T_0} S(T_0-s)\{F(s,w)+f(s)\}ds$$
$$+ \sum_{0<t_k<t}C(t-t_k)I_k^1(w) + \sum_{0<t_k<t}S(t-t_k)I_k^2(w')||_{D(A)}$$

$$\le (\omega^{-1}K(e^{\omega T_0}-1)+1)\{Ke^{\omega T_0}||x_0||_{D(A)} + ||y_0||_E + T_0L||k||_{L^2(0,T_0)}||w||_{L^2(0,T_0;D(A))}$$
$$+ ||Ke^{\omega T_0}f(0)|| + ||f(0)|| + ||K(e^{\omega T_0}+1)\sqrt{T_0}||f||_{W^{1,2}(0,T;X)}$$
$$+ tL||Ke^{\omega t}+1|| \, ||k||_{W^{1,2}(0,T_0)}||w||_{L^2(0,T_0;D(A))}$$
$$+ \sqrt{t}L||Ke^{\omega t}+1|| \, ||k(0)|| ||w||_{L^2(0,T_0;D(A))}\}$$
$$+ \{2 + Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t}Ke^{wT_0}L(I_k^1)$$
$$+ \{1 + 2Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t}L(I_k^2).$$

Hence, from (3.12), there exists a positive constant $C > 0$ such that

$$||w(T_0)||_{D(A)} \le C(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T_0;X)}).$$

Since the condition (3.4) is independent of initial values, the solution of (3.1) can be extended to the interval $[0, nT_0]$ for every natural number $n$. An analogous estimate to (3.12) holds for the solution in $[0, nT_0]$, and hence for the initial value $(w(nT_0), w'(nT_0)) \in D(A) \times E$ in the interval $[nT_0, (n+1)T_0]$. $\qquad\square$

**Example.** We consider the following partial differential equation

$$\begin{cases} w''(t,x) = Aw(t,x) + F(t,w) + f(t), & 0 < t, \quad 0 < x < \pi, \\ w(t,0) = w(t,\pi) = 0, & t \in \mathbb{R} \\ w(0,x) = x_0(x), \quad w'(0,x) = y_0(x), & 0 < x < \pi \\ \Delta w(t_k,x) = I_k^1(w(t_k)) = (\gamma_k||w''(t_k,x)|| + t_k), & 1 \le k \le m, \\ \Delta w'(t_k,x) = I_k^2(w'(t_k)) = \delta_k||w'(t_k,x)||, \end{cases} \quad (E)$$

where constants $\gamma_k$ and $\delta_k (k = 1, \cdots, m)$ are small.

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Let $X = L^2([0, \pi]; \mathbb{R})$, and let $e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$. Then $\{e_n : n = 1, \cdots\}$ is an orthonormal base for $X$. Let $A : X \to X$ be defined by

$$Aw(x) = w''(x),$$

where $D(A) = \{w \in X : w, w' \text{ are absolutely continuous}, w'' \in X, w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2(w, e_n)e_n, \quad w \in D(A),$$

and $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in $X$ given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, e_n)e_n, \quad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(w, e_n)e_n, \quad w \in X.$$

Let $g_1(t, x, w, p)$, $p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t, \delta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \leq \delta$ such that

(f1)  $g_1(t, x, 0, 0) = 0$,

(f2)  $|g_1(t, x, w, p) - g_1(t, x, w, q)| \leq \rho(t, |w|)|p - q|$,

(f3)  $|g_1(t, x, w_1, p) - g_1(t, x, w_2, p)| \leq \rho(t, |w_1| + |w_2|)|w_1 - w_2|$.

Let
$$g(t, w)x = g_1(t, x, w, Dw, D^2w).$$

Then noting that

$$\|g(t, w_1) - g(t, w_2)\|_{0,2}^2 \leq 2 \int_\Omega |g_1(t, x, w_1, p) - g_1(t, x, w_2, q)|^2 dx$$

$$+ 2 \int_\Omega |g_1(t, x, w_1, q) - g_1(t, x, w_2, q)|^2 dx$$

where $p = (Dw_1, D^2w_1)$ and $q = (Dw_2, D^2w_2)$, it follows from (f1), (f2) and (f3) that

$$\|g(t, w_1) - g(t, w_2)\|_{0,2}^2 \leq L(\|w_1\|_{D(A)}, \|w_2\|_{D(A)})\|w_1 - w_2\|_{D(A)}$$

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where $L(||w_1||_{D(A)}, ||w_2||_{D(A)})$ is a constant depending on $||w_1||_{D(A)}$ and $||w_2||_{D(A)}$. We set

$$F(t, w) = \int_0^t k(t-s)g(s, w(s))ds$$

where $k$ belongs to $L^2(0, T)$. Then, from the results in section 3, the solution $w$ of (E) exists and is unique in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$, and there exists a constant $C_3$ depending on $T$ such that

$$||w||_{L^2(0,T;D(A))} \leq C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}).$$

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# Caputo $\psi$-fractional Ostrowski and Grüss inequalities for several functions

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### Abstract

Very general univariate mixed Caputo $\psi$-fractional Ostrowski and Grüss type inequalities for several functions are presented. Estimates are with respect to $\left\Vert \cdot\right\Vert _{p}$, $1 \leq p \leq \infty$. We give also applications.

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**Keywords and Phrases:** Ostrowski and Grüss inequalities, right and left Caputo $\psi$-fractional derivatives.

## 1 Introduction

In 1938, A. Ostrowski [5] proved the following important inequality.

**Theorem 1** *Let* $f : [a,b] \rightarrow \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* $(a,b)$ *whose derivative* $f' : (a,b) \rightarrow \mathbb{R}$ *is bounded on* $(a,b)$, *i.e.,* $\left\Vert f'\right\Vert _{\infty} := \sup_{t\in(a,b)} \left\vert f'(t)\right\vert < +\infty$. *Then*

$$\left\vert \frac{1}{b-a} \int_{a}^{b} f(t)\, dt - f(x) \right\vert \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^{2}}{(b-a)^{2}} \right] \cdot (b-a) \left\Vert f'\right\Vert _{\infty}, \qquad (1)$$

*for any* $x \in [a,b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

In this article we are greatly motived and inspired by Theorem 1, see also [2]. Here we present various $\psi$-fractional Ostrowski and Grüss type inequalities for several functions and we give interesting applications.

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## 2 Background

Here we follow [1].

Let $\alpha > 0$, $[a, b] \subset \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ which is integrable and $\psi \in C^1([a, b])$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a, b]$. Consider $n = \lceil \alpha \rceil$, the ceiling of $\alpha$. The left and right fractional integrals are defined, respectively, as follows:

$$I_{a+}^{\alpha,\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) \, dt, \qquad (2)$$

and

$$I_{b-}^{\alpha,\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t) \, dt, \qquad (3)$$

for any $x \in [a, b]$, where $\Gamma$ is the gamma function.

The following semigroup property is valid for fractional integrals: if $\alpha, \beta > 0$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f(x) = I_{a+}^{\alpha+\beta,\psi} f(x), \quad \text{and} \quad I_{b-}^{\alpha,\psi} I_{b-}^{\beta,\psi} f(x) = I_{b-}^{\alpha+\beta,\psi} f(x).$$

We mention

**Definition 2** *([1]) Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$, $[a, b] \subset \mathbb{R}$ and $f, \psi \in C^n([a, b])$ with $\psi$ being increasing and $\psi'(x) \neq 0$, for all $x \in [a, b]$. The left $\psi$-Caputo fractional derivative of $f$ of order $\alpha$ is given by*

$$^C D_{a+}^{\alpha,\psi} f(x) := I_{a+}^{n-\alpha,\psi} \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x), \qquad (4)$$

*and the right $\psi$-Caputo fractional derivative of $f$ is given by*

$$^C D_{b-}^{\alpha,\psi} f(x) := I_{b-}^{n-\alpha,\psi} \left( -\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x). \qquad (5)$$

*To simplify notation, we will use the symbol*

$$f_\psi^{[n]}(x) := \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x), \qquad (6)$$

*with $f_\psi^{[0]}(x) = f(x)$.*

By the definition, whrn $\alpha = m \in \mathbb{N}$, we have

$$\begin{array}{c} ^C D_{a+}^{\alpha,\psi} f(x) = f_\psi^{[m]}(x) \\ \text{and} \\ ^C D_{b-}^{\alpha,\psi} f(x) = (-1)^m f_\psi^{[m]}(x). \end{array} \qquad (7)$$

If $\alpha \notin \mathbb{N}$, we have

$$^C D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f_\psi^{[n]}(t) \, dt, \qquad (8)$$

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and

$$^{C}D_{b-}^{\alpha,\psi}f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1} f_{\psi}^{[n]}(t)\,dt, \qquad (9)$$

$\forall\, x \in [a, b]$.

In particular, when $\alpha \in (0, 1)$, we have

$$
\begin{aligned}
&^{C}D_{a+}^{\alpha,\psi}f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} (\psi(x)-\psi(t))^{-\alpha} f'(t)\,dt,\\
&\text{and}\\
&^{C}D_{b-}^{\alpha,\psi}f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_{x}^{b} (\psi(t)-\psi(x))^{-\alpha} f'(t)\,dt
\end{aligned} \qquad (10)
$$

$\forall\, x \in [a, b]$.

Clearly the above is a generalization of left and right Caputo fractional derivatives.

For more see [1].

Still we need from [1] the following left and right fractional Taylor's formulae:

**Theorem 3** *([1]) Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$, $[a, b] \subset \mathbb{R}$ and $f, \psi \in C^{n}([a, b])$ with $\psi$ being increasing and $\psi'(x) \neq 0$, for all $x \in [a, b]$. Then, the left fractional Taylor formula follows,*

$$f(x) = \sum_{k=0}^{n-1} \frac{f_{\psi}^{[k]}(a)}{k!} (\psi(x)-\psi(a))^{k} + I_{a+}^{\alpha,\psi}\ ^{C}D_{a+}^{\alpha,\psi}f(x), \qquad (11)$$

*and the right fractional Taylor formula follows,*

$$f(x) = \sum_{k=0}^{n-1} (-1)^{k} \frac{f_{\psi}^{[k]}(b)}{k!} (\psi(b)-\psi(x))^{k} + I_{b-}^{\alpha,\psi}\ ^{C}D_{b-}^{\alpha,\psi}f(x), \qquad (12)$$

$\forall\, x \in [a, b]$.

*In particular, given $\alpha \in (0, 1)$, we have*

$$
\begin{aligned}
&f(x) = f(a) + I_{a+}^{\alpha,\psi}\ ^{C}D_{a+}^{\alpha,\psi}f(x),\\
&\text{and}\\
&f(x) = f(b) + I_{b-}^{\alpha,\psi}\ ^{C}D_{b-}^{\alpha,\psi}f(x),
\end{aligned} \qquad (13)
$$

$\forall\, x \in [a, b]$.

**Remark 4** *For convenience we can rewrite (11)-(13) as follows:*

$$f(x) = \sum_{k=0}^{n-1} \frac{f_{\psi}^{[k]}(a)}{k!} (\psi(x)-\psi(a))^{k} + \qquad (14)$$

$$\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ ^{C}D_{a+}^{\alpha,\psi}f(t)\,dt,$$

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*and*

$$f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k f_\psi^{[k]}(b)}{k!} (\psi(b) - \psi(x))^k + \tag{15}$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} {}^C D_{b-}^{\alpha,\psi} f(t) \, dt,$$

$\forall \ x \in [a, b]$.

When $\alpha \in (0, 1)$, we get:

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} {}^C D_{a+}^{\alpha,\psi} f(t) \, dt,$$
*and* $\tag{16}$
$$f(x) = f(b) + \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} {}^C D_{b-}^{\alpha,\psi} f(t) \, dt,$$

$\forall \ x \in [a, b]$.

Again from [1] we have the following:

Consider the norms $\|\cdot\|_\infty : C([a,b]) \to \mathbb{R}$ and $\|\cdot\|_{C_\psi^{[n]}} : C^n([a,b]) \to \mathbb{R}$, where

$\|f\|_{C_\psi^{[n]}} := \sum_{k=0}^n \left\| f_\psi^{[k]} \right\|_\infty$.

We have

**Theorem 5** *([1]) The $\psi$-Caputo fractional derivatives are bounded operators. For all $\alpha > 0 \ (n = \lceil \alpha \rceil)$*

$$\left\| {}^C D_{a+}^{\alpha,\psi} \right\|_\infty \le K \|f\|_{C_\psi^{[n]}} \tag{17}$$

*and*

$$\left\| {}^C D_{b-}^{\alpha,\psi} \right\|_\infty \le K \|f\|_{C_\psi^{[n]}}, \tag{18}$$

*where*

$$K = \frac{(\psi(b) - \psi(a))^{n-\alpha}}{\Gamma(n+1-\alpha)} > 0. \tag{19}$$

## 3  Main Results

At first we present the following $\psi$-fractional Ostrowski type inequalities for several functions:

**Theorem 6** *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f_i, \psi \in C^n([a,b])$, $i = 1, ..., r$; with $\psi$ being increasing and $\psi'(x) \ne 0$, for all $x \in [a,b]$. Let $x_0 \in [a,b]$ and assume that $f_{i\psi}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$; $i = 1, ..., r$. Set*

$$\Phi(f_1, ..., f_r)(x_0) := r \int_a^b \left( \prod_{k=1}^r f_k(x) \right) d\psi(x) - \tag{20}$$

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$$\sum_{i=1}^{r} \left[ f_i(x_0) \int_a^b \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) d\psi(x) \right].$$

*Then*

$$|\Phi(f_1, ..., f_r)(x_0)| \leq \sum_{i=1}^{r} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right. \quad (21)$$

$$+ \left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right].$$

*If $0 < \alpha \leq 1$, then (21) is valid without any initial conditions.*

**Proof.** By Theorem 3 we have that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} \, {}^C D_{x_0+}^{\alpha,\psi} f_i(t) \, dt, \quad (22)$$

$\forall \; x \in [x_0, b]$,
and

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\alpha)} \int_x^{x_0} \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} \, {}^C D_{x_0-}^{\alpha,\psi} f_i(t) \, dt, \quad (23)$$

$\forall \; x \in [a, x_0]$;
for all $i = 1, ..., r$.

Multiplying (22) and (23) by $\left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)$ we obtain, respectively,

$$\prod_{k=1}^{r} f_k(x) - \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

$$\frac{\left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)}{\Gamma(\alpha)} \int_{x_0}^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} \, {}^C D_{x_0+}^{\alpha,\psi} f_i(t) \, dt, \quad (24)$$

$\forall \; x \in [x_0, b]$,

$$\prod_{k=1}^{r} f_k(x) - \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

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$$\frac{\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right)}{\Gamma(\alpha)} \int_x^{x_0} \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} \; {}^C D_{x_0-}^{\alpha,\psi} f_i(t)\, dt, \qquad (25)$$

$\forall\; x \in [a, x_0]$;

for all $i = 1, ..., r$.

Adding (24) and (25), separately, we obtain

$$r\left(\prod_{k=1}^{r} f_k(x)\right) - \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right)\int_{x_0}^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} \; {}^C D_{x_0+}^{\alpha,\psi} f_i(t)\, dt\right], \quad (26)$$

$\forall\; x \in [x_0, b]$,

and

$$r\left(\prod_{k=1}^{r} f_k(x)\right) - \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right)\int_x^{x_0} \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} \; {}^C D_{x_0-}^{\alpha,\psi} f_i(t)\, dt\right], \quad (27)$$

$\forall\; x \in [a, x_0]$.

Next we integrate (26) and (27) with respect to $\psi(x)$, $x \in [a, b]$. We have

$$r\int_{x_0}^b\left(\prod_{k=1}^{r} f_k(x)\right) d\psi(x) - \sum_{i=1}^{r}\left[f_i(x_0)\int_{x_0}^b\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right) d\psi(x)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{x_0}^b\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right)[\int_{x_0}^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} \; {}^C D_{x_0+}^{\alpha,\psi} f_i(t)\, dt] d\psi(x)\right],$$
$$\qquad (28)$$

and

$$r\int_a^{x_0}\left(\prod_{k=1}^{r} f_k(x)\right) d\psi(x) - \sum_{i=1}^{r}\left[f_i(x_0)\int_a^{x_0}\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right) d\psi(x)\right] =$$

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$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \int_{a}^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left[ \int_{x}^{x_0} \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} \ ^C D_{x_0-}^{\alpha,\psi} f_i(t) \, dt \right] d\psi(x) \right].$$

(29)

Adding (28) and (29) we derive the identity:

$$\Phi(f_1, ..., f_r)(x_0) := r \int_{a}^{b} \left( \prod_{k=1}^{r} f_k(x) \right) d\psi(x) -$$

$$\sum_{i=1}^{r} \left[ f_i(x_0) \int_{a}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) d\psi(x) \right] =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \left[ \int_{a}^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left( \int_{x}^{x_0} \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} \ ^C D_{x_0-}^{\alpha,\psi} f_i(t) \, dt \right) d\psi(x) \right] \right.$$

(30)

$$\left. + \left[ \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left( \int_{x_0}^{x} \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \ ^C D_{x_0+}^{\alpha,\psi} f_i(t) \, dt \right) d\psi(x) \right] \right].$$

Hence it holds

$$|\Phi(f_1, ..., f_r)(x_0)| \leq$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \left[ \int_{a}^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) \left( \int_{x}^{x_0} \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} \left| ^C D_{x_0-}^{\alpha,\psi} f_i(t) \right| dt \right) d\psi(x) \right] \right.$$

$$\left. + \left[ \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) \left( \int_{x_0}^{x} \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \left| ^C D_{x_0+}^{\alpha,\psi} f_i(t) \right| dt \right) d\psi(x) \right] \right] =: (*).$$

(31)

We observe that

$$(*) \leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{r} \left[ \left[ \left\| ^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty, [a, x_0]} \int_{a}^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (\psi(x_0) - \psi(x))^{\alpha} \, d\psi(x) \right] \right.$$

$$\left. + \left[ \left\| ^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty, [x_0, b]} \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (\psi(x) - \psi(x_0))^{\alpha} \, d\psi(x) \right] \right] = \quad (32)$$

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$$\sum_{i=1}^{r} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] + \right.$$

$$\left. \left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right].$$

By Theorem 4.10, p. 98 of [3], we get that $I_{a+}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j| \right) \in C([a,b])$ and

so at any $x_0 \in [a,b]$ is finite, $i = 1, ..., r$. Similarly, by Theorem 4.11, p. 101 of

[3], we get that $I_{b-}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j| \right) \in C([a,b])$ and so at any $x_0 \in [a,b]$ is finite,

$i = 1, ..., r$. Arguing similarly, we get that ${}^C D_{a+}^{\alpha,\psi} f_i, {}^C D_{b-}^{\alpha,\psi} f_i \in C([a,b])$, for all
$i = 1, ..., r$.

The theorem is proved. ■

We continue with

**Theorem 7** *All as in Theorem 6 with $\alpha \geq 1$. Then*

$$|\Phi(f_1, ..., f_r)(x_0)| \leq \sum_{i=1}^{r} \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)} I_{a+}^{\alpha,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right.$$

$$\left. + \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)} I_{b-}^{\alpha,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right]. \tag{33}$$

**Proof.** From (31) we get

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)} \right. \right.$$

$$\left. \left( \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (\psi(x_0) - \psi(x))^{\alpha-1} d\psi(x) \right) \right] +$$

$$\left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)} \left( \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (\psi(x) - \psi(x_0))^{\alpha-1} d\psi(x) \right) \right] \right]$$
$$\tag{34}$$

$$= \sum_{i=1}^{r} \left[ \left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)} I_{a+}^{\alpha,\psi} \left( \prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) + \right.$$

$$\left. \left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)} I_{b-}^{\alpha,\psi} \left( \prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) \right], \qquad (35)$$

proving the theorem. ∎

We continue with

**Theorem 8** *All as in Theorem 6 with* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq 1$. *Then*

$$|\Phi(f_1, ..., f_r)(x_0)| \leq \frac{\Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r} \left[ \left[ \left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_q([a,x_0],\psi)} I_{a+}^{\alpha+\frac{1}{p},\psi} \left( \prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) \right] \qquad (36)$$

$$+ \left[ \left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_q([x_0,b],\psi)} I_{b-}^{\alpha+\frac{1}{p},\psi} \left( \prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) \right] \right].$$

**Proof.** From (31) we obtain

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \left[ \int_{a}^{x_0} \left( \prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x)| \right) \right. \right.$$

$$\left( \int_{x}^{x_0} (\psi(t) - \psi(x))^{p(\alpha-1)} d\psi(t) \right)^{\frac{1}{p}} \left( \int_{x}^{x_0} \left| {}^{C}D_{x_0-}^{\alpha,\psi} f_i(t) \right|^q d\psi(t) \right)^{\frac{1}{q}} d\psi(x) \right] +$$

$$\left[ \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x)| \right) \left( \int_{x_0}^{x} (\psi(x) - \psi(t))^{p(\alpha-1)} d\psi(t) \right)^{\frac{1}{p}} \qquad (37) \right.$$

$$\left. \left. \left( \int_{x_0}^{x} \left| {}^{C}D_{x_0+}^{\alpha,\psi} f_i(t) \right|^q d\psi(t) \right)^{\frac{1}{q}} d\psi(x) \right] \right] \leq \frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)}$$

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$$
\sum_{i=1}^{r} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_q([a,x_0],\psi)} \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (\psi(x_0) - \psi(x))^{\alpha-1+\frac{1}{p}} d\psi(x) \right] \right.
$$

$$
\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_q([x_0,b],\psi)} \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (\psi(x) - \psi(x_0))^{\alpha-1+\frac{1}{p}} d\psi(x) \right] \right] =
$$

(38)

$$
\frac{\Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \sum_{i=1}^{r} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_q([a,x_0],\psi)} I_{a+}^{\alpha+\frac{1}{p},\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right.
$$

$$
\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_q([x_0,b],\psi)} I_{b-}^{\alpha+\frac{1}{p},\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right], \tag{39}
$$

proving the theorem. ∎

We mention as motivation for Grüss type inequalities the following:

**Theorem 9** *(1882, Čebyšev [4]) Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a, b])$. Then*

$$
\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left( \frac{1}{b-a} \int_a^b f(x) \, dx \right) \left( \frac{1}{b-a} \int_a^b g(x) \, dx \right) \right|
$$

$$
\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{40}
$$

*The above integrals are assumed to exist.*

Next follow $\psi$-Caputo fractional Grüss type inequalities for several functions.

**Theorem 10** *Let $0 < \alpha \leq 1$, $[a, b] \subset \mathbb{R}$ and $f_i, \psi \in C^1([a, b])$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; with $\psi$ being increasing and $\psi'(x) \neq 0$, for all $x \in [a, b]$. Assume that $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty, [a,x_0]} < \infty$, and $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty, [x_0,b]} < \infty$, $i = 1, ..., r$. Set*

$$
\Delta^\psi(f_1, ..., f_r) := r(\psi(b) - \psi(a)) \left( \int_a^b \left( \prod_{k=1}^{r} f_k(x) \right) d\psi(x) \right) -
$$

$$
\sum_{i=1}^{r} \left[ \left( \int_a^b f_i(x) \, d\psi(x) \right) \left( \int_a^b \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) d\psi(x) \right) \right]. \tag{41}
$$

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*Then*

$$\left| \Delta^\psi \left( f_1, ..., f_r \right) \right| \le \left( \psi \left( b \right) - \psi \left( a \right) \right)$$

$$\left\{ \sum_{i=1}^{r} \left[ \left[ \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty, [a,x_0]} \sup_{x_0 \in [a,b]} I_{a+}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \ne i}}^{r} \left| f_j \left( x_0 \right) \right| \right) \right] \right. \right.$$

$$\left. \left. + \left[ \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty, [x_0,b]} \sup_{x_0 \in [a,b]} I_{b-}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \ne i}}^{r} \left| f_j \left( x_0 \right) \right| \right) \right] \right] \right\}. \qquad (42)$$

**Proof.** Here $0 < \alpha \le 1$, i.e. $n = 1$. Then (30) is valid without any initial conditions. Clearly $\Phi \left( f_1, ..., f_r \right) \in C \left( [a,b] \right)$. Thus, by integrating (30) against $\psi$ we obtain

$$\Delta^\psi \left( f_1, ..., f_r \right) = \int_a^b \Phi \left( f_1, ..., f_r \right) \left( x_0 \right) d\psi \left( x_0 \right) =$$

$$r \left( \psi \left( b \right) - \psi \left( a \right) \right) \left( \int_a^b \left( \prod_{k=1}^{r} f_k \left( x \right) \right) d\psi \left( x \right) \right) -$$

$$\sum_{i=1}^{r} \left[ \left( \int_a^b f_i \left( x \right) d\psi \left( x \right) \right) \left( \int_a^b \left( \prod_{\substack{j=1 \\ j \ne i}}^{r} f_j \left( x \right) \right) d\psi \left( x \right) \right) \right] =$$

$$\frac{1}{\Gamma \left( \alpha \right)} \int_a^b \left\{ \sum_{i=1}^{r} \left[ \left[ \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \ne i}}^{r} f_j \left( x \right) \right) \right. \right. \right.$$

$$\left. \left( \int_x^{x_0} \psi' \left( t \right) \left( \psi \left( t \right) - \psi \left( x \right) \right)^{\alpha-1} {}^C D_{x_0-}^{\alpha,\psi} f_i \left( t \right) dt \right) d\psi \left( x \right) \right] + \qquad (43)$$

$$\left[ \int_{x_0}^b \left( \prod_{\substack{j=1 \\ j \ne i}}^{r} f_j \left( x \right) \right) \left( \int_{x_0}^x \psi' \left( t \right) \left( \psi \left( x \right) - \psi \left( t \right) \right)^{\alpha-1} {}^C D_{x_0+}^{\alpha,\psi} f_i \left( t \right) dt \right) d\psi \left( x \right) \right] \right] \right\} d\psi \left( x_0 \right).$$

Hence it holds

$$\left| \Delta^\psi \left( f_1, ..., f_r \right) \right| \le \frac{1}{\Gamma \left( \alpha \right)} \int_a^b \left\{ \sum_{i=1}^{r} \left[ \left[ \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \ne i}}^{r} \left| f_j \left( x \right) \right| \right) \right. \right. \right.$$

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$$\left( \int_x^{x_0} \psi'(t) \left( \psi(t) - \psi(x) \right)^{\alpha-1} \left| {}^C D_{x_0-}^{\alpha,\psi} f_i(t) \right| dt \right) d\psi(x) \right] + \left[ \int_{x_0}^b \left( \prod_{\substack{j=1 \\ j \neq i}}^r |f_j(x)| \right) \right.$$

$$\left. \left( \int_{x_0}^x \psi'(t) \left( \psi(x) - \psi(t) \right)^{\alpha-1} \left| {}^C D_{x_0+}^{\alpha,\psi} f_i(t) \right| dt \right) d\psi(x) \right] \right] \right\} d\psi(x_0) =: (**).$$

(44)

Using (21) we derive

$$(**) \leq (\psi(b) - \psi(a))$$

$$\left\{ \sum_{i=1}^r \left[ \left[ \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} \sup_{x_0 \in [a,b]} I_{a+}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^r |f_j(x_0)| \right) \right] \right. \right.$$

$$\left. \left. + \left[ \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} \sup_{x_0 \in [a,b]} I_{b-}^{\alpha+1,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^r |f_j(x_0)| \right) \right] \right] \right\},$$

(45)

proving the theorem. ∎

We make

**Remark 11** *Let $\alpha > 0$, $[a,b] \subset \mathbb{R}$, $f \in C([a,b])$ and $\psi \in C^1([a,b])$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a,b]$. Let $x_0 \in [a,b]$. We observe the following*

$$\left| I_{a+}^{\alpha,\psi} f(x_0) \right| \overset{(2)}{\leq} \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \psi'(t) \left( \psi(x_0) - \psi(t) \right)^{\alpha-1} |f(t)| dt \leq$$

$$\frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha)} \int_a^{x_0} \left( \psi(x_0) - \psi(t) \right)^{\alpha-1} d\psi(t) = \frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \left( \psi(x_0) - \psi(a) \right)^\alpha.$$

(46)

*That is*

$$\left| I_{a+}^{\alpha,\psi} f(x_0) \right| \leq \frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \left( \psi(x_0) - \psi(a) \right)^\alpha.$$

(47)

*Similarly, we obtain*

$$\left| I_{b-}^{\alpha,\psi} f(x_0) \right| \overset{(3)}{\leq} \frac{1}{\Gamma(\alpha)} \int_{x_0}^b \psi'(t) \left( \psi(t) - \psi(x_0) \right)^{\alpha-1} |f(t)| dt \leq$$

$$\frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha)} \int_{x_0}^b \left( \psi(t) - \psi(x_0) \right)^{\alpha-1} d\psi(t) = \frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \left( \psi(b) - \psi(x_0) \right)^\alpha. \quad (48)$$

*That is*

$$\left| I_{b-}^{\alpha,\psi} f(x_0) \right| \leq \frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \left( \psi(b) - \psi(x_0) \right)^\alpha.$$

(49)

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We make

**Remark 12** *Let $\alpha \geq 1$, the rest as in Remark 11. We observe that*

$$\left| I_{a+}^{\alpha,\psi} f\left(x_0\right) \right| \overset{(2)}{\leq} \frac{1}{\Gamma\left(\alpha\right)} \int_a^{x_0} \psi'\left(t\right) \left(\psi\left(x_0\right) - \psi\left(t\right)\right)^{\alpha-1} \left|f\left(t\right)\right| dt \leq$$

$$\frac{\left(\psi\left(x_0\right) - \psi\left(a\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} \int_a^{x_0} \left|f\left(t\right)\right| d\psi\left(t\right) = \frac{\left(\psi\left(x_0\right) - \psi\left(a\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} \left\|f\right\|_{L_1\left(\left[a,x_0\right],\psi\right)}.$$

$$(50)$$

*That is*

$$\left| I_{a+}^{\alpha,\psi} f\left(x_0\right) \right| \leq \frac{\left(\psi\left(x_0\right) - \psi\left(a\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} \left\|f\right\|_{L_1\left(\left[a,x_0\right],\psi\right)}. \tag{51}$$

*Similarly, we get*

$$\left| I_{b-}^{\alpha,\psi} f\left(x_0\right) \right| \overset{(3)}{\leq} \frac{1}{\Gamma\left(\alpha\right)} \int_{x_0}^b \psi'\left(t\right) \left(\psi\left(t\right) - \psi\left(x_0\right)\right)^{\alpha-1} \left|f\left(t\right)\right| dt \leq$$

$$\frac{\left(\psi\left(b\right) - \psi\left(x_0\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} \int_{x_0}^b \left|f\left(t\right)\right| d\psi\left(t\right) = \frac{\left(\psi\left(b\right) - \psi\left(x_0\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} \left\|f\right\|_{L_1\left(\left[x_0,b\right],\psi\right)}.$$

$$(52)$$

*That is*

$$\left| I_{b-}^{\alpha,\psi} f\left(x_0\right) \right| \leq \frac{\left(\psi\left(b\right) - \psi\left(x_0\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} \left\|f\right\|_{L_1\left(\left[x_0,b\right],\psi\right)}. \tag{53}$$

Next, we simplify our main theorems:

**Proposition 13** *All as in Theorem 6. Then*

$$\left| \Phi\left(f_1, ..., f_r\right)\left(x_0\right) \right| \leq \frac{1}{\Gamma\left(\alpha + 2\right)}$$

$$\sum_{i=1}^r \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^r f_j \right\|_{\infty,[a,x_0]} \left(\psi\left(x_0\right) - \psi\left(a\right)\right)^{\alpha+1} \right] \right.$$

$$\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^r f_j \right\|_{\infty,[x_0,b]} \left(\psi\left(b\right) - \psi\left(x_0\right)\right)^{\alpha+1} \right] \right]. \tag{54}$$

*If $0 < \alpha \leq 1$, then (54) is valid without any initial conditions.*

**Proof.** By (21), (47) and (49). ∎
Next comes

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**Proposition 14** *All as in Theorem 6 with $\alpha \geq 1$. Then*

$$\left| \Phi \left( f_1, ..., f_r \right) \left( x_0 \right) \right| \leq \frac{1}{\Gamma \left( \alpha \right)}$$

$$\sum_{i=1}^{r} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{L_1([a,x_0],\psi)} \left( \psi \left( x_0 \right) - \psi \left( a \right) \right)^{\alpha-1} \right] \right.$$

$$\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{L_1([x_0,b],\psi)} \left( \psi \left( b \right) - \psi \left( x_0 \right) \right)^{\alpha-1} \right] \right]. \quad (55)$$

**Proof.** By (33), (51) and (53). ∎
Next follows

**Proposition 15** *All as in Theorem 6 with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq 1$. Then*

$$\left| \Phi \left( f_1, ..., f_r \right) \left( x_0 \right) \right| \leq \frac{1}{\Gamma \left( \alpha \right) \left( \alpha + \frac{1}{p} \right) \left( p \left( \alpha - 1 \right) + 1 \right)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_q([a,x_0],\psi)} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{\infty,[a,x_0]} \left( \psi \left( x_0 \right) - \psi \left( a \right) \right)^{\alpha+\frac{1}{p}} \right] \right.$$

$$\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_q([x_0,b],\psi)} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{\infty,[x_0,b]} \left( \psi \left( b \right) - \psi \left( x_0 \right) \right)^{\alpha+\frac{1}{p}} \right] \right]. \quad (56)$$

**Proof.** By (36), (47) and (49). ∎
We continue with

**Proposition 16** *All as in Theorem 10. Then*

$$\left| \Delta^{\psi} \left( f_1, ..., f_r \right) \right| \leq \frac{\left( \psi \left( b \right) - \psi \left( a \right) \right)^{\alpha+2}}{\Gamma \left( \alpha + 2 \right)}$$

$$\left\{ \sum_{i=1}^{r} \left[ \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} + \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} \right] \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{\infty,[a,b]} \right\}. \quad (57)$$

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**Proof.** By (42), (47) and (49). ∎

Next we make some applications of our main results.

We need

**Remark 17** *We have that $(r = 2)$*

$$\Phi\left(f_1, f_2\right)\left(x_0\right) = 2 \int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right) - \tag{58}$$

$$f_1\left(x_0\right) \int_a^b f_2\left(x\right) d\psi\left(x\right) - f_2\left(x_0\right) \int_a^b f_1\left(x\right) d\psi\left(x\right),$$

*and $(r = 3)$*

$$\Phi\left(f_1, f_2, f_3\right)\left(x_0\right) = 3 \int_a^b f_1\left(x\right) f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right) - f_1\left(x_0\right) \int_a^b f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right)$$

$$\tag{59}$$

$$- f_2\left(x_0\right) \int_a^b f_1\left(x\right) f_3\left(x\right) d\psi\left(x\right) - f_3\left(x_0\right) \int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right),$$

*etc.*

*Furthermore we derive $(r = 2)$*

$$\Delta^\psi\left(f_1, f_2\right) = 2 \left[\left(\psi\left(b\right) - \psi\left(a\right)\right) \left(\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right)\right) - \right.$$

$$\left. \left(\int_a^b f_1\left(x\right) d\psi\left(x\right)\right) \left(\int_a^b f_2\left(x\right) d\psi\left(x\right)\right) \right], \tag{60}$$

*and $(r = 3)$*

$$\Delta^\psi\left(f_1, f_2, f_3\right) = 3 \left(\psi\left(b\right) - \psi\left(a\right)\right) \left(\int_a^b f_1\left(x\right) f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_1\left(x\right) d\psi\left(x\right)\right) \left(\int_a^b f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_2\left(x\right) d\psi\left(x\right)\right) \left(\int_a^b f_1\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_3\left(x\right) d\psi\left(x\right)\right) \left(\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right)\right), \tag{61}$$

*etc.*

We give the special cases of fractional Ostrowski type inequalities.

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**Proposition 18** *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f_1, f_2 \in C^n([a,b])$. Let $x_0 \in [a,b]$ and assume that $f_{1e^x}^{[k]}(x_0) = f_{2e^x}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$. Then*

$$\left| 2 \int_a^b f_1(x) f_2(x) e^x dx - f_1(x_0) \int_a^b f_2(x) e^x dx - f_2(x_0) \int_a^b f_1(x) e^x dx \right| \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \sum_{i=1}^{2} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,e^x} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{2} f_j \right\|_{\infty,[a,x_0]} (e^{x_0} - e^a)^{\alpha+1} \right] \right.$$

$$\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,e^x} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{2} f_j \right\|_{\infty,[x_0,b]} (e^b - e^{x_0})^{\alpha+1} \right] \right]. \tag{62}$$

*If $0 < \alpha \leq 1$, then (62) is valid without any initial conditions.*

**Proof.** Case of $\psi(x) = e^x$, apply Proposition 13 for $r = 2$. ∎
We continue with

**Proposition 19** *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a,b] \subset (0, +\infty)$ and $f_1, f_2, f_3 \in C^n([a,b])$. Let $x_0 \in [a,b]$ and assume that $f_{i \ln x}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$; $i = 1, 2, 3$. Then*

$$\left| 3 \int_a^b \frac{f_1(x) f_2(x) f_3(x)}{x} dx - f_1(x_0) \int_a^b \frac{f_2(x) f_3(x)}{x} dx - \right.$$

$$\left. f_2(x_0) \int_a^b \frac{f_1(x) f_3(x)}{x} dx - f_3(x_0) \int_a^b \frac{f_1(x) f_2(x)}{x} dx \right| \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \sum_{i=1}^{3} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\ln x} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{3} f_j \right\|_{\infty,[a,x_0]} \left( \ln\left(\frac{x_0}{a}\right) \right)^{\alpha+1} \right] \right.$$

$$\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,\ln x} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{3} f_j \right\|_{\infty,[x_0,b]} \left( \ln\left(\frac{b}{x_0}\right) \right)^{\alpha+1} \right] \right]. \tag{63}$$

*If $0 < \alpha \leq 1$, then (63) is valid without any initial conditions.*

**Proof.** Case of $\psi(x) = \ln x$, apply Proposition 13 for $r = 3$. ∎
Next we present the special cases of fractional Grüss type inequality:

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**Proposition 20** *Let $0 < \alpha \leq 1$, $[a,b] \subset \mathbb{R}$ and $f_1, f_2 \in C^1([a,b])$. Assume that* $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,e^x} f_i \right\|_{\infty,[a,x_0]} < \infty$, *and* $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,e^x} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1, 2$.
*Then*

$$2 \left| \left( e^b - e^a \right) \left( \int_a^b f_1(x) f_2(x) e^x dx \right) - \left( \int_a^b f_1(x) e^x dx \right) \left( \int_a^b f_2(x) e^x dx \right) \right| \leq$$

$$\frac{\left( e^b - e^a \right)^{\alpha+2}}{\Gamma(\alpha+2)}$$

$$\left\{ \sum_{i=1}^2 \left[ \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,e^x} f_i \right\|_{\infty,[a,x_0]} + \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,e^x} f_i \right\|_{\infty,[x_0,b]} \right] \left\| \prod_{\substack{j=1 \\ j \neq i}}^2 f_j \right\|_{\infty,[a,b]} \right\}. \tag{64}$$

**Proof.** Apply Proposition 16, for $\psi(x) = e^x$, $r = 2$. ∎
We finish with another $\psi$-fractional Grüss type inequality:

**Proposition 21** *Let $0 < \alpha \leq 1$, $[a,b] \subset (0,\infty)$ and $f_i \in C^1([a,b])$, $i = 1, 2, 3$.
Assume that* $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,\ln x} f_i \right\|_{\infty,[a,x_0]} < \infty$, $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,\ln x} f_i \right\|_{\infty,[x_0,b]} < \infty$,
$i = 1, 2, 3$. *Then*

$$\left| \ln\left(\frac{b}{a}\right)^3 \left( \int_a^b \frac{f_1(x) f_2(x) f_3(x)}{x} dx \right) - \left( \int_a^b \frac{f_1(x)}{x} dx \right) \left( \int_a^b \frac{f_2(x) f_3(x)}{x} dx \right) - \right.$$

$$\left. \left( \int_a^b \frac{f_2(x)}{x} dx \right) \left( \int_a^b \frac{f_1(x) f_3(x)}{x} dx \right) - \left( \int_a^b \frac{f_3(x)}{x} dx \right) \left( \int_a^b \frac{f_1(x) f_2(x)}{x} dx \right) \right| \tag{65}$$

$$\leq \frac{\left( \ln\left(\frac{b}{a}\right) \right)^{\alpha+2}}{\Gamma(\alpha+2)} \left\{ \sum_{i=1}^3 \left[ \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha,\ln x} f_i \right\|_{\infty,[a,x_0]} + \right. \right.$$

$$\left. \left. \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha,\ln x} f_i \right\|_{\infty,[x_0,b]} \right] \left\| \prod_{\substack{j=1 \\ j \neq 1}}^3 f_j \right\|_{\infty,[a,b]} \right\}.$$

**Proof.** By Proposition 16, for $\psi(x) = \ln x$, $r = 3$. ∎

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# Results On Sequential Conformable Fractional Derivatives With Applications

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## Abstract

This paper investigates and states some properties of sequential conformable fractional derivative introduced by R. Khalil et. al. in [1]. Further some theorems of the classical power series are generalized for the fractional power series(CFPS), where the CFPS technique is used to find the solutions of conformable fractional deferential equation with variable coefficients.

## 1 Introduction

The correspondence between L'Höpital and Leibniz, in 1695, about what might be a derivative of order $1/2$, led to the introduction of a generalization of integral and derivative operators, known as Fractional Calculus. Since then, related to the definition of fractional derivatives have been many definitions. The most popular ones of these definitions are Riemann-Liouville and Caputo definitions see [6],[7].

Recently, R. Khalil et al. [1] give a new definition of fractional derivative and fractional integral. In their work they proved the product rule, the fractional Rolle's theorem and Mean Value Theorem utilizing the conformable fractional derivative definition. New construction of the generalized Taylor's power series is obtained by Abdeljawad in[2]. In recent years, many researchers have focused on the approximate analytical solutions of the system of fractional differential equations and some methods have been developed such as fractional power series method(FPS) in [5] . FPS method is a simple technique to find out the recurrence relation that determines the coefficients

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of the fractional power series, where this method is one of the most useful techniques to solve linear system and non-linear system of fractional differential equations with a fast convergence rate and small calculation error.

In this work, we state some properties of sequential conformable fractional derivative, then use the FPS technique to solve conformable fractional differential equation of order two.

# 2 Conformable fractional derivative

In this section, we present some definitions and some important properties of the conformable fractional derivative. The definition of the conformable fractional derivative is defined as follows;

**Definition 1** [1] *Given a function* $f : [0, \infty) \longrightarrow \mathbb{R}$. *For* $\alpha \in (0, 1)$, *the conformable fractional derivative (CFD) of* $f$ *of order* $\alpha$ *is defined by*

$$T_\alpha (f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1}$$

*for all* $t > 0$. *If* $f$ *is* $\alpha$*differentiable in some* $(0, a)$, $a > 0$, *and* $\lim_{t \to 0^+} f^{(\alpha)}(t)$ *exists, then define*

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

A function $f$ is called $\alpha$-differentiable at $t \geqslant 0$ if the above limits exists. For simplicity we sometimes use the notation $f^{(\alpha)}(t)$ instead of $T_\alpha (f)(t)$.

Consider the limit $\alpha \to 1^-$. In this case , for $t > 0$, we obtain the classical definition for the first derivative of a function $f^{(\alpha)}(t) = f'(t)$.

**Theorem 2** [1]*Let* $f : [0, \infty) \longrightarrow \mathbb{R}$ *be a differentiable function in the classical sense. Then* $f$ *is* $\alpha$*differentiable at* $t$ , $\alpha \in (0, 1)$ *and*

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t), \quad t > 0.$$

*Also, if* $f$ *is continuously differentiable at* $0$, *then* $f^{(\alpha)}(0) = 0$.

Note that the function could be $\alpha$-differentiable at a point $t_0$ but not differentiable at that point, as in the following example.

**Example 3** *For some fixed* $\alpha$, *with* $\alpha \in (0, 1)$, *let* $f(t) = \frac{t^\alpha}{\alpha}$, $t > 0$. *Note that* $f'(0)$ *does not exists but* $T_\alpha(f)(t) = 1$ *for* $t > 0$, *therefore* $f^{(\alpha)}(0) = \lim_{t \to 0^+} T_\alpha(f)(t) = 1$.

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**Theorem 4** $[1]$*If a function* $f : [0, \infty) \longrightarrow \mathbb{R}$ *is* $\alpha$-*differentiable at* $a > 0, \ \alpha \in (0, 1]$, *then* $f$ *is continuous at* $a$.

We list some Important properties of the operator $T_\alpha$ as follows.

**Theorem 5** $[1]$*Let* $\alpha \in (0, 1]$ *and* $f, g$ *be* $\alpha$*differentiable at a point* $t > 0$. *Then*

1. $T_\alpha(\lambda f + g) = \lambda T_\alpha(f) + T_\alpha(g)$, *for all* $\lambda \in \mathbb{R}$.

2. $T_\alpha(fg) = f T_\alpha(g) + g T_\alpha(f)$.

3. $T_\alpha(\frac{f}{g}) = \frac{g T_\alpha(f) - f T_\alpha(g)}{g^2}$.

Important examples of CFD are listed as follows:

**Example 6** $[1]$

1. $T_\alpha(t^p) = p t^{p-\alpha}$

2. $T_\alpha(e^{at}) = a t^{1-\alpha} e^{at}, \quad a \in \mathbb{R}$.

3. $T_\alpha(\sin(at)) = a t^{1-\alpha} \cos(at), \quad a \in \mathbb{R}$.

4. $T_\alpha(\cos(at)) = -a t^{1-\alpha} \sin(at), \quad a \in \mathbb{R}$.

5. $T_\alpha(e^{\lambda(\frac{1}{\alpha} t^\alpha)}) = \lambda e^{\frac{1}{\alpha} t^\alpha}$.

6. $T_\alpha(\sin(\frac{1}{\alpha} t^\alpha)) = \cos(\dfrac{1}{\alpha} t^\alpha)$.

7. $T_\alpha(\cos(\frac{1}{\alpha} t^\alpha)) = -\sin(\dfrac{1}{\alpha} t^\alpha)$.

8. $T_\alpha(\frac{1}{\alpha} t^\alpha) = 1$.

There are some properties that are not satisfied by operator $T_\alpha$ as follows:

**Theorem 7** $[2]$*For* $\alpha, \beta \in (0, 1]$ *and* $\alpha + \beta \in (1, 2]$, *and the function* $f : (0, \infty) \longrightarrow \mathbb{R}$ *is twice differentiable on* $(0, \infty)$. $T_\alpha$ *does not satisfy the Index Law;* $T_\alpha T_\beta = T_\beta T_\alpha$, *where*

$$T_\alpha T_\beta (f)(t) = t^{1-(\alpha+\beta)} \left( (1-\beta) f'(t) + t f''(t) \right),$$

1. *while,*

$$T_\beta T_\alpha (f)(t) = t^{1-(\alpha+\beta)} \left( (1-\alpha) f'(t) + t f''(t) \right).$$

**Proof.** Calculating $T_\alpha T_\beta, T_\beta T_\alpha$ gives that

$$T_\alpha T_\beta (f)(t) \neq T_\beta T_\alpha (f)(t),$$

for $\alpha \neq \beta$. ∎

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# 3 Sequential conformable fractional $\alpha$-derivatives

In this section the higher order of conformable fractional derivative will be defined and the relation between CFD and polynomials will be given.

**Definition 8** *The second order CFD operator $T_\alpha$, will be denoted by*

$$T_\alpha^2 = T_\alpha \left( T_\alpha \right).$$

*In general the $n^{th}$ order CFD operator $T_\alpha$ is defined as*

$$T_\alpha^n = T_\alpha \left( T_\alpha^{n-1} \right).$$

*Note that the operators commute for each positive integers $n, m$,*

$$T_\alpha^n T_\alpha^m = T_\alpha^m T_\alpha^n.$$

The calculation of the second and the third sequential orders of CF operator $T_\alpha$ can be found in [2].

**Theorem 9** *Given a function $f : (0, \infty) \longrightarrow \mathbb{R}$ . Then*

$$T_\alpha^2 \left( f \right) \left( t \right) = T_\alpha T_\alpha \left( f \right) \left( t \right) = t^{1-2\alpha} \left( \left( 1 - \alpha \right) f^{'} \left( t \right) + t \ f^{''} \left( t \right) \right),$$

*where $f$ is twice differentiable, and twice $\alpha$-differentiable at $t$ . Also*

$$T_\alpha^3 \left( f \right) \left( t \right) = t^{1-3\alpha} \left( \left( 1 - \alpha \right) \left( 1 - 2\alpha \right) f^{'} \left( t \right) + \left( 3 - 3\alpha \right) t \ f^{''} \left( t \right) + t^2 \ f^{'''} \left( t \right) \right),$$

*where $f$ is three times differentiable at $t$ and three times $\alpha$-differentiable at $t$ .*

**Lemma 10** *If $m$ and $n$ are any positive integers and $p$ a real number, then*

$$T_\alpha^m \left( t^p \right) = \Pi_{i=0}^{m-1} \left( p - i \ \alpha \right) t^{p-m\alpha}.$$

**Proof.**

$$\begin{aligned}
T_\alpha^2 \left( t^p \right) &= T_\alpha \left( T_\alpha t^p \right) \\
&= T_\alpha \left( p t^{p-\alpha} \right) \\
&= p \left( p - \alpha \right) t^{p-2\alpha}.
\end{aligned}$$

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Also,

$$T_\alpha^3 (t^p) = T_\alpha \left( p (p - \alpha) t^{p-2\alpha} \right)$$
$$= p (p - \alpha) (p - 2\alpha) t^{p-3\alpha}.$$

Then

$$T_\alpha^m (t^p) = p (p - \alpha) (p - 2\alpha) \dots (p - (m - 1) \alpha) t^{p-m\alpha}$$
$$= \Pi_{i=0}^{m-1} (p - i \ \alpha) t^{p-m\alpha}.$$

∎

**Corollary 11** *If $n$ is positive integer , then*

$$T_\alpha^n (t^n) = \Pi_{i=0}^{n-1} (n - i \ \alpha) t^{n(1-\alpha)}.$$

**Corollary 12** *If $m$ and $n$ are any positive integers and $0 < \alpha < 1$, then*

$$T_\alpha^m (t^{n\alpha}) = \Pi_{i=0}^{m-1} \alpha^m (n - i \ ) t^{\alpha(n-m)}.$$

Applying Lemma10 and using the linearity property of Theorem 5, we get

**Lemma 13** *If $P_n (t) = a_0 t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots + a_n$ is a polynomial in $t$ of degree $n$, then*

$$T_\alpha^m P_n (t) = \sum_{j=0}^{n} a_j \ \Pi_{i=0}^{m-1} (n - j - i \ \alpha) t^{n-j-m\alpha}.$$

Changing variable from $t \to t^\alpha$, we get the following result.

**Corollary 14** *If $f (t) = a_0 t^{\alpha n} + a_1 t^{\alpha(n-1)} + a_2 t^{\alpha(n-2)} + \dots + a_n$ , then*

$$T_\alpha^m f (t) = \sum_{j=0}^{n} a_j \ \Pi_{i=0}^{m-1} (\alpha (n - i \ ) - j) t^{\alpha(n-m)-j}.$$

The following theorem presents the $n^{th}$ sequential CF $\alpha$-derivative utilizing the limit definition as follows:

**Theorem 15** *Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then for $0 < \alpha < 1$ and $n$ a positive integer, the n th order of the $\alpha-$conformable fractional derivative of $f$ of is as follows*

$$T_\alpha^n f (t) = \lim_{\varepsilon \to 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f \left( (1 + \varepsilon t^{-\alpha})^j t \right)}{\varepsilon^n},$$

*where $t > 0$.*

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**Proof.** Since

$$T_\alpha f\,(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$
$$= \lim_{\varepsilon \to 0} \frac{f(t\,(1 + \varepsilon t^{-\alpha})) - f(t)}{\varepsilon}.$$

Let $\delta = 1 + \varepsilon t^{-\alpha}$, we get

$$T_\alpha f\,(t) = \lim_{\delta \to 1} \frac{f(\delta t) - f(t)}{(\delta - 1)\,t^\alpha}.$$

Thus

$$T_\alpha^2 f\,(t) = \lim_{\delta \to 1} \frac{T_\alpha f(\delta t) - T_\alpha f(t)}{(\delta - 1)\,t^\alpha}$$
$$= \lim_{\delta \to 1} \frac{f(\delta^2 t) - 2f(\delta t) + f(t)}{(\delta - 1)^2\,t^{2\alpha}}.$$

Again calculating the $3^{rd}$ order of the $\alpha-$conformable fractional derivative, we get

$$T_\alpha^3 f\,(t) = T_\alpha T_\alpha^2 f\,(t) = \lim_{\delta \to 1} \frac{T_\alpha f(\delta^2 t) - 2T_\alpha f(\delta t) + T_\alpha f(t)}{(\delta - 1)^2\,t^{2\alpha}}$$
$$= \lim_{\delta \to 1} \frac{f(\delta^3 t) - 3f(\delta^2 t) + 3f(\delta t) + f(t)}{(\delta - 1)^3\,t^{3\alpha}}$$
$$= \lim_{\delta \to 1} \frac{\sum_{j=0}^{3} \binom{3}{j} (-1)^{3-j} f\left(\delta^j t\right)}{(\delta - 1)^3\,t^{3\alpha}}.$$

Repeating this process $n$ times, we get

$$T_\alpha^n f\,(t) = \lim_{\delta \to 1} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f\left(\delta^j t\right)}{(\delta - 1)^n\,t^{n\alpha}}.$$

Substituting $\varepsilon = (\delta - 1)\,t^\alpha$, then

$$T_\alpha^n f\,(t) = \lim_{\varepsilon \to 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f\left((1 + \varepsilon t^{-\alpha})^j t\right)}{\varepsilon^n}.$$

∎

# 4   Conformable Fractional Power Series Representation

Power series is an important tool in the study of elementary functions. Using this power expansion gives us the ability to make an approximate study of many differential equations . In this section, we will recall some important definitions and theorems of fractional power series theory .

**Definition 16** *For $\alpha \in (0, 1)$ a conformable fractional power series of the form*

$$\sum_{n=0}^{\infty} c_n (t - t_0)^{n\alpha} = c_0 + c_1 (t - t_0)^{\alpha} + c_2 (t - t_0)^{2\alpha} + \cdots\cdots ,$$

*where $t > t_0 \geq 0$ tis called the conformable fractional power series (CFPS) about $t_0$,where $c_n$ denote the coefficients of the series, where $n \in \mathbb{N}$.*

Note that for CFPS, we have the value $c_0$ for$n = 0$,at $t = t_0$ , while $c_n = 0$ for $n \geq 1$at $t = t_0$ .Also note that for $t_0 = 0$, then the CFPS becomes

$$\sum_{n=0}^{\infty} c_n t^{n\alpha} = c_0 + c_1 t^{\alpha} + c_2 t^{2\alpha} + \cdots\cdots$$

**Proposition 17** $[4]$*If $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ converges absolutely for $t = t_0 > 0$, then we it converges absolutely for $t \in (0, t_0)$.*

**Theorem 18** *The series$\sum_{n=0}^{\infty} c_n t^{n\alpha}$ converges,$-\infty < t < \infty$ has radius of convergence $R$, if and only if the series $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \geq 0$ has radius of convergence $R^{\frac{1}{\alpha}}$, $R > 0$.*

**Theorem 19** $[2]$*Assume $f$ is an infinitely $\alpha$-differentiable function, for some $\alpha \in (0, 1]$ at a neighborhood of a point $t_0$. Then $f$ has the Taylor CFPS series expansion as follows*

$$f(t) = \sum_{k=0}^{\infty} \frac{T_{\alpha}^k f(t_0) (t - t_0)^{k\alpha}}{\alpha^k k!}, \ t \in \left( t_0, t_0 + R^{\frac{1}{\alpha}} \right) \ , \ R > 0.$$

The next following examples doesn't have the Taylor PS expansion about $t_0 \geq 0$ since there are not differentiable there. But they have Taylor CFPS expansion at $t_0$.

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**Example 20** $[2]$

1. $e^{\frac{1}{\alpha}(t-t_0)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(t-t_0)^{k\alpha}}{\alpha^k k!}$, *for* $t \in [t_0, \infty)$.

2. $\sin(\frac{1}{\alpha}(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{(t-t_0)^{(2k+1)\alpha}}{\alpha^{2k+1} 2k+1!}$, *for* $t \in [t_0, \infty)$.

3. $\cos(\frac{1}{\alpha}(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{(t-t_0)^{(2k)\alpha}}{\alpha^{2k} 2k!}$, *for* $t \in [t_0, \infty)$.

4. $\frac{1}{1-\frac{(t-t_0)^{\alpha}}{\alpha}} = \sum_{k=0}^{\infty} (t-t_0)^{k\alpha}$, *for* $t \in [t_0, t_0+1)$.

# 5   Solving CFD equation's using CFPS

**Example 21** *Consider the following conformable fractional differential equation*

$$T_{\alpha}^2 y(t) - t^{\alpha} y(t) = 0, \tag{2}$$

with initial conditions,

$$y(0) = 0, \ T_{\alpha} y(0) = y_0, \tag{3}$$

where $y_0$ is a real constant.
Now using CFPS technique, let

$$y(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}.$$

Then

$$T_{\alpha}^2 y(t) = \sum_{n=2}^{\infty \alpha} \alpha^2 (n)(n-1) c_n t^{(n-2)\alpha}$$

$$= \sum_{n=0}^{\infty} \alpha^2 (n+1)(n+2) c_{n+2} t^{n\alpha}$$

and

$$t^{\alpha} y(t) = \sum_{n=0}^{\infty} c_n t^{(n+1)\alpha}$$

$$= \sum_{n=1}^{\infty} c_{n-1} t^{n\alpha} \tag{4}$$

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Substituting $T_\alpha^2 y(t)$ and $t^\alpha y(t)$ in CFDE, we get

$$\sum_{n=0}^{\infty} \alpha^2 (n+1)(n+2) c_{n+2} t^{n\alpha} - \sum_{n=1}^{\infty} c_{n-1} t^{n\alpha} = 0.$$

Then from Formula  (), one can obtain

$$2\alpha^2 c_2 + \sum_{n=1}^{\infty} \left[ \alpha^2 (n+1)(n+2) c_{n+2} - c_{n-1} \right] t^{n\alpha} = 0.$$

Equating the coefficients of $t^{n\alpha}$ to zero in both sides gives the following;

$$c_2 = 0, \text{ and } c_{n+2} = \frac{c_{n-1}}{\alpha^2 (n+1)(n+2)}, \quad n = 1, 2, 3, \ldots . \tag{5}$$

Considering the initial conditions of the CFDE,

$$c_0 = 0, \text{ and } c_1 = \frac{1}{\alpha} y_0.$$

Based on Equation 5,the coefficients of $t^{n\alpha}$ can be divided into two categories: zero terms

$$c_2 = c_3 = c_5 = c_6 = c_8 = c_9 = c_{11} = c_{14} = \cdots = 0,$$

in general
$$c_{3n+2} = c_{3n+3}, \text{ for } n = 0, 1, 2, 3, \ldots$$

and non zero terms

$$c_{3n+1} \neq 0, \text{ for } n = 0, 1, 2, 3, \ldots,$$

that is $c_1, c_4, c_7, c_{10}, c_{13}, c_{16} \cdots$, where

$$c_4 = \frac{c_1}{\alpha^2 (3.4)} = \frac{y_0}{\alpha^3 (3.4)},$$

$$c_7 = \frac{c_4}{\alpha^2 (6.7)} = \frac{y_0}{\alpha^5 (3.4.6.7)},$$
$$c_{10} = \frac{c_7}{\alpha^2 (9.10)} = \frac{y_0}{\alpha^7 (3.4.6.7.9.10)},$$
$$c_{10} = \frac{c_{10}}{\alpha^2 (9.10)} = \frac{y_0}{\alpha^9 (3.4.6.7.9.10)},$$
$$\cdots \cdots \cdots$$

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Then, one can obtain the following CFPS as follows

$$y\left(t\right) = c_1 t^\alpha + c_4 t^{4\alpha} + c_7 t^{7\alpha} + c_{10} t^{10\alpha} + \cdots\cdots\cdots$$

$$= \frac{1}{\alpha} y_0 t^\alpha + \frac{y_0}{\alpha^3 \left(3.4\right)} t^{4\alpha} + \frac{y_0}{\alpha^5 \left(3.4.6.7\right)} t^{7\alpha} + \frac{y_0}{\alpha^9 \left(3.4.6.7.9.10\right)} t^{10\alpha} + \cdots\cdots$$

$$= \frac{1}{\alpha} y_0 \left( 1 + \sum_{k=1}^{\infty} \frac{1}{\alpha^{2k} \left( \prod_{i=0}^{k} \left(3i\right)\left(3i+1\right) \right)} t^{(3k+1)\alpha} \right).$$

**Conclusion 22** *The main aim of this work is to provide a reliable algorithm for the solutions to the systems of fractional differential equations by using the CFPS.*

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# Multivariate Ostrowski-Sugeno Fuzzy inequalities

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### Abstract

Here we present multivariate Ostrowski-Sugeno Fuzzy type inequalities. These are multivariate Ostrowski-like inequalities in the context of Sugeno fuzzy integral and its special properties. They give tight upper bounds to the deviation of a multivariate function from its Sugeno-fuzzy multivariate averages.

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**Keywords and phrases:** Sugeno fuzzy integral, multivariate function fuzzy average, deviation from fuzzy multivariate mean, Fuzzy multivariate Ostrowski inequality.

## 1   Introduction

The famous Ostrowski ([4]) inequality motivates this work and has as follows:

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \left( \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \left\| f' \right\|_\infty,$$

where $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality.

Another motivation comes from author's [2], pp. 507-508, see also [1]:

Let $f \in C^1\left( \prod_{i=1}^k [a_i, b_i] \right)$, where $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1,...,k$, and let $x_0 := (x_{01},...,x_{0k}) \in \prod_{i=1}^k [a_i, b_i]$ be fixed. Then

$$\left| \frac{1}{\prod_{i=1}^k (b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} ... \int_{a_k}^{b_k} f(z_1,...,z_k)\, dz_1...dz_k - f(x_0) \right| \leq$$

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$$\sum_{i=1}^{k} \left( \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2(b_i - a_i)} \right) \left\| \frac{\partial f}{\partial z_i} \right\|_{\infty}.$$

The last inequality is sharp, the optimal function is

$$f^* (z_1, ..., z_k) := \sum_{i=1}^{k} |z_i - x_{0i}|^{\alpha_i}, \ \ \alpha_i > 1.$$

Here first we give a survey about Sugeno fuzzy integral and its basic special properties. Then we derive a set of multivariate Ostrowski-like inequalities to all directions in the context of Sugeno integral whitin its basic important properties. We finish with an application to a special multivariate case.

## 2 Background

In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented. Also a preparation for the main results Section 3 is given.

**Definition 1** *(Fuzzy measure [6, 8]) Let $\Sigma$ be a $\sigma$-algebra of subsets of $X$, and let $\mu : \Sigma \to [0, +\infty]$ be a non-negative extended real-valued set function. We say that $\mu$ is a fuzzy measure iff:*
*(1) $\mu(\varnothing) = 0$,*
*(2) $E, F \in \Sigma : E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity),*
*(3) $E_n \in \Sigma \ (n \in \mathbb{N})$, $E_1 \subset E_2 \subset ...$, imply $\lim_{n \to \infty} \mu(E_n) = \mu(\cup_{n=1}^{\infty} E_n)$*
*(continuity from below);*
*(4) $E_n \in \Sigma \ (n \in \mathbb{N})$, $E_1 \supset E_2 \supset ...$, $\mu(E_1) < \infty$, imply $\lim_{n \to \infty} \mu(E_n) = \mu(\cap_{n=1}^{\infty} E_n)$ (continuity from above).*

Let $(X, \Sigma, \mu)$ be a fuzzy measure space and $f$ be a non-negative real-valued function on $X$. We denote by $\mathcal{F}_+$ the set of all non-negative real valued measurable functions, and by $L_\alpha f$ the set: $L_\alpha f := \{x \in X : f(x) \geq \alpha\}$, the $\alpha$-level of $f$ for $\alpha \geq 0$.

**Definition 2** *Let $(X, \Sigma, \mu)$ be a fuzzy measure space. If $f \in \mathcal{F}_+$ and $A \in \Sigma$, then the Sugeno integral (fuzzy integral) [7] of $f$ on $A$ with respect to the fuzzy measure $\mu$ is defined by*

$$(S) \int_A f d\mu := \vee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap L_\alpha f)), \tag{1}$$

*where $\vee$ and $\wedge$ denote the sup and inf on $[0, \infty]$, respectively.*

The basic properties of Sugeno integral follow:

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**Theorem 3** *([5, 8]) Let $(X, \Sigma, \mu)$ be a fuzzy measure space with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. Then*
*1) $(S) \int_A f d\mu \leq \mu(A)$;*
*2) $(S) \int_A k d\mu = k \wedge \mu(A)$ for a non-negative constant $k$;*
*3) if $f \leq g$ on $A$, then $(S) \int_A f d\mu \leq (S) \int_A g d\mu$;*
*4) if $A \subset B$, then $(S) \int_A f d\mu \leq (S) \int_B f d\mu$;*
*5) $\mu(A \cap L_\alpha f) \leq \alpha \Rightarrow (S) \int_A f d\mu \leq \alpha$;*
*6) if $\mu(A) < \infty$, then $\mu(A \cap L_\alpha f) \geq \alpha \Leftrightarrow (S) \int_A f d\mu \geq \alpha$;*
*7) when $A = X$, then $(S) \int_A f d\mu = \vee_{\alpha \geq 0} (\alpha \wedge \mu(L_\alpha f))$;*
*8) if $\alpha \leq \beta$, then $L_\beta f \subseteq L_\alpha f$;*
*9) $(S) \int_A f d\mu \geq 0$.*

**Theorem 4** *([8], p. 135) Here $f \in \mathcal{F}_+$, the class of all finite nonnegative measurable functions on $(X, \Sigma, \mu)$. Then*
*1) if $\mu(A) = 0$, then $(S) \int_A f d\mu = 0$, for any $f \in \mathcal{F}_+$;*
*2) if $(S) \int_A f d\mu = 0$, then $\mu(A \cap \{x | f(x) > 0\}) = 0$;*
*3) $(S) \int_A f d\mu = (S) \int_X f \cdot \chi_A d\mu$, where $\chi_A$ is the characteristic function of $A$;*
*4) $(S) \int_A (f + a) d\mu \leq (S) \int_A f d\mu + (S) \int_A a d\mu$, for any constant $a \in [0, \infty)$.*

**Corollary 5** *([8], p. 136) Here $f, f_1, f_2 \in \mathcal{F}_+$. Then*
*1) $(S) \int_A (f_1 \vee f_2) d\mu \geq (S) \int_A f_1 d\mu \vee (S) \int_A f_2 d\mu$;*
*2) $(S) \int_A (f_1 \wedge f_2) d\mu \leq (S) \int_A f_1 d\mu \wedge (S) \int_A f_2 d\mu$;*
*3) $(S) \int_{A \cup B} f d\mu \geq (S) \int_A f d\mu \vee (S) \int_B f d\mu$;*
*4) $(S) \int_{A \cap B} f d\mu \leq (S) \int_A f d\mu \wedge (S) \int_B f d\mu$.*

In general we have

$$(S) \int_A (f_1 + f_2) d\mu \neq (S) \int_A f_1 d\mu + (S) \int_A f_2 d\mu,$$

and

$$(S) \int_A a f d\mu \neq a (S) \int_A f d\mu, \text{ where } a \in \mathbb{R},$$

see [8], p. 137.

**Lemma 6** *([8], p. 138) $(S) \int_A f d\mu = \infty$ iff $\mu(A \cap L_\alpha f) = \infty$ for any $\alpha \in [0, \infty)$.*

We need

**Definition 7** *([3]) A fuzzy measure $\mu$ is subadditive iff $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for all $A, B \in \Sigma$.*

We mention

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**Theorem 8** *([3]) If $\mu$ is subadditive, then*

$$(S) \int_X (f + g) \, d\mu \leq (S) \int_X f d\mu + (S) \int_X g d\mu, \tag{2}$$

*for all measurable functions $f, g : X \to [0, \infty)$.*

*Moreover, if (2) holds for all measurable functions $f, g : X \to [0, \infty)$ and $\mu(X) < \infty$, then $\mu$ is subadditive.*

*Notice here in (1) we have that $\alpha \in [0, \infty)$.*

We have

**Corollary 9** *If $\mu$ is aubadditive, $n \in \mathbb{N}$, and $f : X \to [0, \infty)$ is a measurable function, then*

$$(S) \int_X n f d\mu \leq n (S) \int_X f d\mu, \tag{3}$$

*in particular it holds*

$$(S) \int_A n f d\mu \leq n (S) \int_A f d\mu, \tag{4}$$

*for any $A \in \Sigma$.*

**Proof.** By (2). ∎

A very important property of Sugeno integral follows.

**Theorem 10** *If $\mu$ is subadditive measure, and $f : X \to [0, \infty)$ is a measurable function, and $c > 0$, then*

$$(S) \int_A c f d\mu \leq (c + 1) (S) \int_A f d\mu, \tag{5}$$

*for any $A \in \Sigma$.*

**Proof.** Let the ceiling $\lceil c \rceil = m \in \mathbb{N}$, then by Theorem 3 (3) and (4) we get

$$(S) \int_A c f d\mu \leq (S) \int_A m f d\mu \leq m (S) \int_A f d\mu \leq (c + 1) (S) \int_A f d\mu,$$

proving (5). ∎

From now on in this article we work on the fuzzy measure space $(Q, \mathcal{B}, \mu)$, where $Q \subset \mathbb{R}^k$, $k \geq 1$ is a convex compact subset, $\mathcal{B}$ is the Borel $\sigma$-algebra on $Q$, and $\mu$ is a finite fuzzy measure on $\mathcal{B}$. Typically we take it to be subadditive.

The functions $f$ we deal with here are continuous from $Q$ into $\mathbb{R}_+$.

We make

**Remark 11** *Let $f \in C(Q, \mathbb{R}_+)$, and $\mu$ is a subadditive fuzzy measure such that $\mu(Q) > 0$, $x \in Q$. We will estimate*

$$E(x) := \left| (S) \int_Q f(t) \, d\mu(t) - \mu(Q) \wedge f(x) \right| \tag{6}$$

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*(by Theorem 3 (2))*

$$= \left| (S) \int_Q f(t) \, d\mu(t) - (S) \int_Q f(x) \, d\mu(t) \right|.$$

*We notice that*

$$f(t) = f(t) - f(x) + f(x) \le |f(t) - f(x)| + f(x),$$

*then (by Theorem 3 (3) and Theorem 4 (4))*

$$(S) \int_Q f(t) \, d\mu(t) \le (S) \int_Q |f(t) - f(x)| \, d\mu(t) + (S) \int_Q f(x) \, d\mu(t), \qquad (7)$$

*that is*

$$(S) \int_Q f(t) \, d\mu(t) - (S) \int_Q f(x) \, d\mu(t) \le (S) \int_Q |f(t) - f(x)| \, d\mu(t). \qquad (8)$$

*Similarly, we have*

$$f(x) = f(x) - f(t) + f(t) \le |f(t) - f(x)| + f(t),$$

*then (by Theorem 3 (3) and Theorem 8)*

$$(S) \int_Q f(x) \, d\mu(t) \le (S) \int_Q |f(t) - f(x)| \, d\mu(t) + (S) \int_Q f(t) \, d\mu(t),$$

*that is*

$$(S) \int_Q f(x) \, d\mu(t) - (S) \int_Q f(t) \, d\mu(t) \le (S) \int_Q |f(t) - f(x)| \, d\mu(t). \qquad (9)$$

*By (8) and (9) we derive that*

$$\left| (S) \int_Q f(t) \, d\mu(t) - (S) \int_Q f(x) \, d\mu(t) \right| \le (S) \int_Q |f(t) - f(x)| \, d\mu(t). \qquad (10)$$

*Consequently it holds*

$$E(x) \overset{(by\ (6),\ (10))}{\le} (S) \int_Q |f(t) - f(x)| \, d\mu(t), \qquad (11)$$

*where $t = (t_1, ..., t_k)$, $x = (x_1, ..., x_k)$.*

We will use (11).

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# 3   Main Results

We make

**Remark 12** *Here* $Q := \prod_{i=1}^{k} [a_i, b_i]$, *where* $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1, ..., k$; $x = (x_1, ..., x_k) \in \prod_{i=1}^{k} [a_i, b_i]$ *is fixed, and* $f \in C^1 \left( \prod_{i=1}^{k} [a_i, b_i], \mathbb{R}_+ \right)$. *Consider* $g_t(r) := f(x + r(t - x))$, $r \geq 0$. *Note that* $g_t(0) = f(x)$, $g_t(1) = f(t)$. *Thus*

$$f(t) - f(x) = g_t(1) - g_t(0) = g_t'(\xi)(1 - 0) = g_t'(\xi), \qquad (12)$$

*where* $\xi \in (0, 1)$.
   *I.e.*

$$f(t) - f(x) = \sum_{i=1}^{k} (t_i - x_i) \frac{\partial f}{\partial t_i} (x + \xi(t - x)). \qquad (13)$$

*Hence*

$$|f(t) - f(x)| \leq \sum_{i=1}^{k} |t_i - x_i| \left| \frac{\partial f}{\partial t_i} (x + \xi(t - x)) \right|$$

$$\leq \sum_{i=1}^{k} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_\infty. \qquad (14)$$

*By (11) we get*

$$\left| (S) \int_{\prod_{i=1}^{k} [a_i, b_i]} f(t) \, d\mu(t) - \mu \left( \prod_{i=1}^{k} [a_i, b_i] \right) \wedge f(x) \right| \leq$$

$$(S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |f(t) - f(x)| \, d\mu(t) \overset{(14)}{\leq}$$

$$(S) \int_{\prod_{i=1}^{k} [a_i, b_i]} \left( \sum_{i=1}^{k} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_\infty \right) d\mu(t) \overset{(2)}{\leq}$$

$$\sum_{i=1}^{k} (S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_\infty d\mu(t) \overset{(5)}{\leq}$$

$$\sum_{i=1}^{k} \left( \left\| \frac{\partial f}{\partial t_i} \right\|_\infty + 1 \right) \left( (S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |t_i - x_i| \, d\mu(t) \right). \qquad (15)$$

*Here* $\mu$ *is a fuzzy subadditive measure with* $\mu \left( \prod_{i=1}^{k} [a_i, b_i] \right) > 0$.

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*Therefore we get*

$$\left| \frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)} (S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right) \right| \overset{(15)}{\leq} \quad (16)$$

$$\sum_{i=1}^{k} \left( \frac{\left\| \frac{\partial f}{\partial t_i} \right\|_\infty + 1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)} \right) \left( (S)\int_{\prod_{i=1}^{k}[a_i,b_i]} |t_i - x_i|\,d\mu(t) \right).$$

*Notice here* $\left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right) \leq 1$, *and*

$$\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)} (S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) \overset{(by\ Thm.\ 3\ (1))}{\leq} 1,$$

*where* $(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) \geq 0$.

*If* $f : \prod_{i=1}^{k}[a_i,b_i] \to \mathbb{R}_+$ *is a Lipschitz function of order* $0 < \alpha \leq 1$, *i.e.*

$|f(x) - f(y)| \leq K \|x - y\|_{l_1}^\alpha$, $\forall\ x, y \in \prod_{i=1}^{k}[a_i,b_i]$, $K > 0$, *where* $\|x - y\|_{l_1} :=$

$\sum_{i=1}^{k} |x_i - y_i|$, *denoted by* $f \in Lip_{\alpha,K}\left(\prod_{i=1}^{k}[a_i,b_i], \mathbb{R}_+\right)$, *then by (11) we get*

$$\left| (S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \mu\left(\prod_{i=1}^{k}[a_i,b_i]\right) \wedge f(x) \right| \leq \quad (17)$$

$$(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} |f(t) - f(x)|\,d\mu(t) \leq$$

$$(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} K \|t - x\|_{l_1}^\alpha\,d\mu(t) \overset{(5)}{\leq}$$

$$(K+1)(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} \|t - x\|_{l_1}^\alpha\,d\mu(t).$$

*We have proved*

$$\left| \frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)} (S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right) \right| \leq \quad (18)$$

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$$\frac{(K+1)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}\|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We have established the following multivariate Ostrowski-Sugeno inequalities.

**Theorem 13** *Here $\mu$ is a fuzzy subadditive measure with $\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)>0$, $x\in\prod_{i=1}^{k}[a_i,b_i]$.*

*1) Let $f\in C^1\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, then*

$$\left|\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}f(t)\,d\mu(t)-\left(1\wedge\frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right|\leq \quad (19)$$

$$\sum_{i=1}^{k}\left(\frac{\left\|\frac{\partial f}{\partial t_i}\right\|_{\infty}+1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\left((S)\int_{\prod_{i=1}^{k}[a_i,b_i]}|t_i-x_i|\,d\mu(t)\right).$$

*2) Let $f\in Lip_{\alpha,K}\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, $0<\alpha\leq 1$, then*

$$\left|\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}f(t)\,d\mu(t)-\left(1\wedge\frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right|\leq \quad (20)$$

$$\frac{(K+1)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}\|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We make

**Remark 14** *Let $Q$ be a compact and convex subset of $\mathbb{R}^k$, $k\geq 1$. Let $f\in\left(C(Q,\mathbb{R}_+)\cap C^{n+1}(Q)\right)$, $n\in\mathbb{N}$ and $x\in Q$ is fixed such that all partial derivatives $f_{\alpha}:=\frac{\partial^{\alpha}f}{\partial t^{\alpha}}$, where $\alpha=(\alpha_1,...,\alpha_k)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,k$, $|\alpha|=\sum_{i=1}^{\kappa}\alpha_i=j$, $j=1,...,n$ fulfill $f_{\alpha}(x)=0$.*

*By [2], p. 513, we get that*

$$|f(t)-f(x)|\leq\frac{\left[\left(\sum_{i=1}^{k}|t_i-x_i|\left\|\frac{\partial}{\partial t_i}\right\|_{\infty}\right)^{n+1}f\right]}{(n+1)!},\quad\forall\,t\in Q. \quad (21)$$

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*Call*

$$D_{n+1}(f) := \max_{\alpha:|\alpha|=n+1} \|f_\alpha\|_\infty. \tag{22}$$

*For example, when $k = 2$ and $n = 1$, we get that*

$$\left[\left(\sum_{i=1}^{2} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^2 f\right] =$$

$$(t_1 - x_1)^2 \left\|\frac{\partial^2 f}{\partial t_1^2}\right\|_\infty + 2|t_1 - x_1||t_2 - x_2| \left\|\frac{\partial^2 f}{\partial t_1 \partial t_2}\right\|_\infty + (t_2 - x_2)^2 \left\|\frac{\partial^2 f}{\partial t_2^2}\right\|_\infty, \tag{23}$$

*and*

$$D_2(f) = \max_{\alpha:|\alpha|=2} \|f_\alpha\|_\infty. \tag{24}$$

*Clearly, it holds*

$$\left[\left(\sum_{i=1}^{2} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^2 f\right] \le D_2(f) \left(|t_1 - x_1| + |t_2 - x_2|\right)^2. \tag{25}$$

*Consequently, we derive that*

$$\left[\left(\sum_{i=1}^{k} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^{n+1} f\right] \le D_{n+1}(f) \|t - x\|_{l_1}^{n+1}, \quad \forall \ t \in Q. \tag{26}$$

*By (11) we get*

$$\left|(S)\int_Q f(t)\, d\mu(t) - \mu(Q) \wedge f(x)\right| \le$$

$$(S)\int_Q |f(t) - f(x)|\, d\mu(t) \overset{(21)}{\le} \tag{27}$$

$$(S)\int_Q \frac{\left[\left(\sum_{i=1}^{k} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^{n+1} f\right]}{(n+1)!}\, d\mu(t) \overset{(26)}{\le}$$

$$(S)\int_Q \frac{D_{n+1}(f) \|t - x\|_{l_1}^{n+1}}{(n+1)!}\, d\mu(t) \overset{(5)}{\le} \tag{28}$$

$$\left(\frac{D_{n+1}(f)}{(n+1)!} + 1\right)(S)\int_Q \|t - x\|_{l_1}^{n+1}\, d\mu(t).$$

*Here $\mu$ is a fuzzy subadditive measure with $\mu(Q) > 0$.*
*By (27) and (28) we obtain*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \le$$

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$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!} + 1\right)}{\mu(Q)} (S) \int_Q \|t - x\|_{l_1}^{n+1} d\mu(t).\tag{29}$$

We have established the following multivariate Ostrowski-Sugeno general inequality:

**Theorem 15** *Let $Q$ be a compact and convex subset of $\mathbb{R}^k$, $k \geq 1$. Let $f \in \left(C(Q, \mathbb{R}_+) \cap C^{n+1}(Q)\right)$, $n \in \mathbb{N}$, $x \in Q$ be fixed: $f_\alpha(x) = 0$, all $\alpha : |\alpha| = j$, $j = 1, ..., n$. Here $\mu$ is a fuzzy subadditive measure with $\mu(Q) > 0$. Then*

$$\left| \frac{1}{\mu(Q)} (S) \int_Q f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right) \right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!} + 1\right)}{\mu(Q)} (S) \int_Q \|t - x\|_{l_1}^{n+1} d\mu(t).\tag{30}$$

**Corollary 16** *All as in Theorem 15. Then*

$$\left| \frac{1}{\mu(Q)} (S) \int_Q f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right) \right| \leq$$

$$\frac{\left(1 + \frac{1}{(n+1)!}\right)}{\mu(Q)} (S) \int_Q \left[ \left(\sum_{i=1}^k |t_i - x_i| \left\|\frac{\partial}{\partial x_i}\right\|_\infty \right)^{n+1} f \right] d\mu(t).\tag{31}$$

Next we take again $Q := \prod_{i=1}^k [a_i, b_i]$, we set $a := (a_1, ..., a_k)$, $b := (b_1, ..., b_k)$, and $\frac{a+b}{2} = \left(\frac{a_1+b_1}{2}, ..., \frac{a_k+b_k}{2}\right) \in \prod_{i=1}^k [a_i, b_i]$.

**Corollary 17** *Let $f \in \left(C\left(\prod_{i=1}^k [a_i, b_i], \mathbb{R}_+\right) \cap C^{n+1}\left(\prod_{i=1}^k [a_i, b_i]\right)\right)$, $n \in \mathbb{N}$, such that $f_\alpha\left(\frac{a+b}{2}\right) = 0$, all $\alpha : |\alpha| = j$, $j = 1, ..., n$. Here $\mu$ is a fuzzy subadditive measure with $\mu\left(\prod_{i=1}^k [a_i, b_i]\right) > 0$. Then*

$$\left| \frac{1}{\mu\left(\prod_{i=1}^k [a_i, b_i]\right)} (S) \int_{\prod_{i=1}^k [a_i, b_i]} f(t) d\mu(t) - \left(1 \wedge \frac{f\left(\frac{a+b}{2}\right)}{\mu\left(\prod_{i=1}^k [a_i, b_i]\right)}\right) \right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!} + 1\right)}{\mu\left(\prod_{i=1}^k [a_i, b_i]\right)} (S) \int_{\prod_{i=1}^k [a_i, b_i]} \left\|t - \frac{a+b}{2}\right\|_{l_1}^{n+1} d\mu(t).\tag{32}$$

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**Proof.** By Theorem 15. ∎

We make

**Remark 18** *By multinomial theorem we have that*

$$\|t - x\|_{l_1}^{n+1} = \left( \sum_{i=1}^{k} |t_i - x_i| \right)^{n+1} =$$

$$\sum_{r_1+r_2+...+r_k=n+1} \binom{n+1}{r_1, r_2, ..., r_k} |t_1 - x_1|^{r_1} |t_2 - x_2|^{r_2} ... |t_k - x_k|^{r_k}, \qquad (33)$$

*where*

$$\binom{n+1}{r_1, r_2, ..., r_k} = \frac{(n+1)!}{r_1! r_2! ... r_k!}. \qquad (34)$$

*By (27), (28) we get*

$$\left| (S) \int_Q f(t) \, d\mu(t) - \mu(Q) \wedge f(x) \right| \le$$

$$(S) \int_Q \frac{D_{n+1}(f)}{(n+1)!} \|t - x\|_{l_1}^{n+1} \, d\mu(t) \overset{(by\ (33),\ (34))}{=}$$

$$(S) \int_Q \left[ \sum_{r_1+r_2+...+r_k=n+1} \left( \frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} \right) \left( \prod_{i=1}^{k} |t_i - x_i|^{r_i} \right) \right] d\mu(t) \overset{(2)}{\le}$$

$$\sum_{r_1+r_2+...+r_k=n+1} (S) \int_Q \left( \frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} \right) \left( \prod_{i=1}^{k} |t_i - x_i|^{r_i} \right) d\mu(t) \overset{(5)}{\le}$$

$$\sum_{r_1+r_2+...+r_k=n+1} \left( \frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} + 1 \right) (S) \int_Q \left( \prod_{i=1}^{k} |t_i - x_i|^{r_i} \right) d\mu(t). \qquad (35)$$

We have proved the following multivariate Ostrowski-Sugeno general inequality:

**Theorem 19** *Here all as in Theorem 15. Then*

$$\left| \frac{1}{\mu(Q)} (S) \int_Q f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(x)}{\mu(Q)} \right) \right| \le$$

$$\sum_{r_1+r_2+...+r_k=n+1} \left( \frac{\left( \frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} + 1 \right)}{\mu(Q)} \right) (S) \int_Q \left( \prod_{i=1}^{k} |t_i - x_i|^{r_i} \right) d\mu(t). \qquad (36)$$

We make

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**Remark 20** *In case $k = 2$, $n = 1$, by (27), (28) we get*

$$\left| (S) \int_Q f(t) \, d\mu(t) - \mu(Q) \wedge f(x) \right| \leq$$

$$(S) \int_Q \frac{D_2(f)}{2} \|t - x\|_{l_1}^2 \, d\mu(t) =$$

$$(S) \int_Q \frac{D_2(f)}{2} \left[ (t_1 - x_1)^2 + 2 |t_1 - x_1| |t_2 - x_2| + (t_2 - x_2)^2 \right] d\mu(t) \leq \quad (37)$$

$$(S) \int_Q \frac{D_2(f)}{2} (t_1 - x_1)^2 \, d\mu(t) + (S) \int_Q D_2(f) |t_1 - x_1| |t_2 - x_2| \, d\mu(t)$$

$$+ (S) \int_Q \frac{D_2(f)}{2} (t_2 - x_2)^2 \, d\mu(t) \leq$$

$$\left( 1 + \frac{D_2(f)}{2} \right) (S) \int_Q (t_1 - x_1)^2 \, d\mu(t) + (1 + D_2(f)) (S) \int_Q |t_1 - x_1| |t_2 - x_2| \, d\mu(t)$$

$$+ \left( 1 + \frac{D_2(f)}{2} \right) (S) \int_Q (t_2 - x_2)^2 \, d\mu(t).$$

We have proved

**Corollary 21** *Let $Q$ be a compact and convex subset of $\mathbb{R}^2$. Let $f \in (C(Q, \mathbb{R}_+)$ $\cap C^2(Q))$, $x = (x_1, x_2) \in Q$ be fixed: $\frac{\partial f}{\partial t_1}(x_1, x_2) = \frac{\partial f}{\partial t_2}(x_1, x_2) = 0$. Here $\mu$ is a fuzzy subadditive measure with $\mu(Q) > 0$. Then*

$$\left| \frac{1}{\mu(Q)} (S) \int_Q f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(x)}{\mu(Q)} \right) \right| \leq$$

$$\frac{\left( 1 + \frac{D_2(f)}{2} \right)}{\mu(Q)} (S) \int_Q (t_1 - x_1)^2 \, d\mu(t) + \frac{(1 + D_2(f))}{\mu(Q)} (S) \int_Q |t_1 - x_1| |t_2 - x_2| \, d\mu(t)$$

$$\quad (38)$$

$$+ \frac{\left( 1 + \frac{D_2(f)}{2} \right)}{\mu(Q)} (S) \int_Q (t_2 - x_2)^2 \, d\mu(t).$$

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# SOME APPLICATIONS OF INTERVAL-VALUED SUBSETHOOD MEASURES WHICH ARE DEFINED BY INTERVAL-VALUED CHOQUET INTEGRALS IN TRADE EXPORTS BETWEEN KOREA AND ITS TRADING PARTNERS

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ABSTRACT. In this paper, we consider subsethood measures introduced by Fan et al. [3] and the interval-valued Choquet integral with respect to a fuzzy measure of interval-valued fuzzy sets. Based on such a focus, we define three types of interval-valued subsethood measures and provide four interval-valued fuzzy sets to animal product exports between Korea and four selected trading partners.

In particular, we investigate a strong interval-valued subsethood measure defined by the interval-valued Choquet integral which represents the degree of trade surplus between Korea and three trading partners in terms of the model of trade transactions with the United States and Korea

## 1. INTRODUCTION

Zadeh[18] first developed fuzzy sets and Murofushi-Sugeno [11] have studied fuzzy measures and Choquet integrals. Subsequently, using set-valued analysis theory developed by Aumann[1], we studied interval-valued Choquet integrals and their related applications(see[5, 6, 7, 8, 9]). In particular, through the restudy of the interval-valued Choquet integral in 2004 by Zhang-Guo-Liu[21], this research has been developed in a much more systematical

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manner. Xuechang [16], Zeng-Li[19] have examined fuzzy entropy, distance and similarity measures, the likes of which form three key concepts of fuzzy set theory. Ruan-Kerre [12] also introduced various fuzzy implication operators and the Choquet integral were suggested for the first time by Choquet [2]. Further studied by Murofushi-Sugeno [9], Jang-Kwon [10], and Jang [12] provide some interesting interpretations of fuzzy measures and the Choquet integral. Subjective probability and Choquet expected utility were studied as an application of Choquet integral and form another pivotal component of fuzzy sets and information theories(see[13, 14, 15, 20]).

A subsethood measure refers to the degree to which a fuzzy set is a subset of another fuzzy set. Many researchers have contributed to the area of a fuzzy subsethood measure that is closely related to the various tools introduced above (see[6, 7, 8, 11]). Their efforts have considered axiomatizing the properties of a subsethoods measure.

In this paper, we consider subsethood measures introduced by Fan et al. [3] and the interval-valued Choquet integral with respect to a fuzzy measure of interval-valued fuzzy sets. Based on such a focus, we define three types of interval-valued subsethood measures and provide four interval-valued fuzzy sets to animal product exports between Korea and four selected trading partners. In configuring the four interval-valued fuzzy sets, the original data(see[4]) used had to be slightly modified to produce Table $A4$. In order to calculate the interval-valued Choquet integral, the rules (46) and (48) were introduced.

Furthermore, we also investigate a strong interval-valued subsethood measure defined by an interval-valued Choquet integral which represents the degree of trade surplus between Korea and 3 trading partners in terms of the model of trade transactions with the United States and Korea. The information for the above degree of surplus is of great significance in providing accurate comparative figures on the size of trade that exists between the four countries that trade with Korea.

## 2. Preliminaries and definitions

Throughout this paper, we write $X$ to denote a set,

$$F(X) = \{A | A = \{(x, m_A(x)) | x \in X\}, \ m_A : X \longrightarrow [0, 1] \text{ is a function}\} \tag{1}$$

stands for the set of fuzzy sets in $X$(see[18]). We note that $m_A$ expresses the membership of a fuzzy set $A$, $A^c$ is the complement of $A$, that is,

$$A^c = \{(x, m_{A^c}(x)) | \ m_{A^c}(x) = 1 - m_A(x), \ x \in X\}. \tag{2}$$

Recall that for $A, B \in F(X)$, $A \subset B$ if and only if $m_A(x) \leq m_B(x)$, for all $x \in X$, and for $A \in F(X)$, $[A] = \{x \in X | \ m_A(x) > 0\}$, $n(A)$ is the cardinal number of crisp set $[A]$, and $M(A)$ is the fuzzy cardinal of $A$, that is, $M(A) = \sum_{x \in X} m_A(x)$. Now, we introduce three types of subsethood measure in Fan et al. [3].

**Definition 2.1.** ([3]) Let $c : F(X) \times F(X) \longrightarrow [0, 1]$ be a function.
(1) $c$ is called a strong subsethood measure if $c$ has the following properties;

$$(S_1) \quad \text{if } A \subset B, \text{ then } c(A, B) = 1;$$
$$(S_2) \quad \text{if } A \neq \emptyset \text{ then } c(A, B)$$

$$(\text{S}_3) \quad \text{if } A \subset B \subset C, \text{ then } c(C, A) \leq c(B, A) \text{ and } c(C, A) \leq c(C, B). \tag{3}$$

(2) $c$ is called a subsethood measure if $c$ has the following properties;

$(\text{C}_1)$    if $A \subset B$, then $c(A, B) = 1$;
$(\text{C}_2)$    $c(X, \emptyset) = 0$
$(\text{C}_3)$    if $A \subset B \subset C$, then $c(C, A) \leq c(B, A)$ and $c(C, A) \leq c(C, B).$      (4)

(3) $c$ is called a weak subsethood measure if $c$ has the following properties;

$(\text{W}_1)$    $c(\emptyset, \emptyset) = 1,\ c(\emptyset, \emptyset) = 1,\ c(A, B) = 1;$ and $c(X, X) = 1$
$(\text{W}_2)$    if $A \neq \emptyset\ rmand\ A \cap B = \emptyset$, then $c(A, B) = 0$;
$(\text{W}_3)$    if $A \subset B \subset C$, then $c(C, A) \leq c(B, A)$ and $c(C, A) \leq c(C, B).$      (5)

We also list the set-theoretical arithmetic operators for the set of subintervals of an unit interval $[0, 1]$ in $\mathbb{R}$. We denote

$$I([0, 1]) = \{\overline{a} = [a^-, a^+] |\ a^-, a^+ \in [0, 1] \text{ and } a^- \leq a^+\}. \tag{6}$$

For any $a \in [0, 1]$, we define $a = [a, a]$.

**Definition 2.2.** ([5, 6, 7, 8, 9]) If $\overline{a} = [a^-, a^+], \overline{b} = [b^-, b^+] \in I([0, 1])$, and $k \in [0, 1]$, then the addition, scalar multiplication, minimum, maximum, inequality, subset, multiplication, and division as follows;

(1) $\overline{a} + \overline{b} = [a^- + b^-, a^+ + b^+]$,
(2) $k\overline{a} = [ka^-, ka^+]$,
(3) $\overline{a} \wedge \overline{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(4) $\overline{a} \vee \overline{b} = [a^- \vee b^-, a^+ \vee b^+]$,
(5) $\overline{a} \leq \overline{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(6) $\overline{a} < \overline{b}$ if and only if $\overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$,
(7) $\overline{a} \subset \overline{b}$ if and only if $b^- \leq a^-$ and $a^+ \leq b^+$,
(8) $\overline{a} \otimes \overline{b} = [a^- b^-, a^+ b^+]$, and
(9) $\overline{a} \oslash \overline{b} = [a^-/b^- \wedge a^+/b^+, a^-/b^- \vee a^+/b^+]$.

From Definition 2.1 (9), the following theorem can be easily obtained.

**Theorem 2.1.** (1) If $\overline{a} = [a^-, a^+] \in I([0, 1])$, then $\overline{a} \oslash \overline{a} = 1$.
(2) If $\overline{b} = [b^-, b^+] \in I([0, 1])$ and $b^- > 0$, then $1 \oslash \overline{b} = [1/b^+, 1/b^-]$.

**Definition 2.3.** ([5, 6, 8, 9, 21]) Let $(X, \Omega)$ be a measurable space. (1) A fuzzy measure on $X$ is a real-valued function $\mu : \Omega \longrightarrow [0, 1]$ satisfies

(i)    $\mu(\emptyset) = 0$
(ii)    $\mu(E_1) \leq \mu(E_2)$   whenever   $E_1, E_2 \in \Omega$   and   $E_1 \subset E_2.$      (7)

(2) A fuzzy measure $\mu$ is said to be continuous from below if for any sequence $\{E_n\} \subset \Omega$ and $E \in \Omega$, such that

$$\text{if } E_n \uparrow E, \quad \text{then} \quad \lim_{n \to \infty} \mu(E_n) = \mu(E). \tag{8}$$

(3) A fuzzy measure $\mu$ is said to be continuous from above if for any sequence $\{E_n\} \subset \Omega$ and $E \in \Omega$ such that

$$\text{if } \quad E_n \downarrow E, \quad \text{then} \quad \lim_{n \to \infty} \mu(E_n) = \mu(E). \tag{9}$$

(5) A fuzzy measure $\mu$ is said to be continuous if it is continuous from below and continuous from above.

**Definition 2.4.** ([5, 6, 8, 9, 21]) (1) Let $A \in F(X)$. The Choquet integrals with respect to a fuzzy measure $\mu$ of a fuzzy set $A$ on a set $E \in \Omega$ is defined by

$$C_{\mu,E}(A) = (C) \int_E m_A d\mu = \int_0^1 \mu_{E,m_A}(r) dr, \tag{10}$$

where $\mu_{E,m_A}(r) = \mu(\{x \in X | \ m_A(x) > r\} \cap E)$ and the integral on the right-hand side is an ordinary one.

(2) A measurable function is said to be integrable if $C_\mu(A) = C_{\mu,X}(A)$ exists.

It is well known that if $X$ is a finite set, that is, $X = \{x_1, x_2, \cdots, x_n\}$, and $A \in F(X)$, then we have

$$C_\mu(A) = \sum_{i=1}^n m_A(x_{(i)}) \left( \mu(E_{(i)}) - \mu(E_{(i+1)}) \right), \tag{11}$$

where $(\cdot)$ indicate a permutation on $\{1, 2, \cdots, n\}$ such that $m_A(x_{(1)}) \leq m_A(x_{(2)}) \leq \cdots \leq m_A(x_{(n)})$ and also $E_{(i)} = \{(i), (i+1), \cdots, (n)\}$ and $E_{(n+1)} = \emptyset$.

**Theorem 2.2.** *Let $A, B \in F(X)$.* (1) *If $A \leq B$, then $C_\mu(A) \leq C_\mu(B)$.*

(2) *If we define $(m_A \vee m_B)(x) = m_A(x) \vee m_B(x)$ for all $x \in X$, then $C_\mu(A \vee B) \geq C_\mu(A) \vee C_\mu(B)$.*

(3) *If we define $(m_A \wedge m_B)(x) = m_A(x) \wedge m_B(x)$ for all $x \in X$, then $C_\mu(A \wedge B) \geq C_\mu(A) \wedge C_\mu(B)$.*

## 3. Three types of interval-valued subsethood measures defined by interval-valued Choquet integral

In this section, we consider the interval-valued Choquet integral and list some properties of them.

**Definition 3.1.** ([5, 6, 8, 9, 21]) (1) The interval-valued Choquet integral of an interval-valued measurable function $\overline{f} = [f^-, f^+]$ on $E \in \Omega$ is defined by

$$\overline{C}_{\mu,E}(\overline{f}) = (C)\int_E \overline{f}d\mu = \left\{C_{\mu,E}(f) \mid f \in S(\overline{f})\right\},\tag{12}$$

where $S(\overline{f})$ is the family of measurable selection of $\overline{f}$.

(2) $\overline{f}$ is said to be integrable if $\overline{C}_\mu(\overline{f}) = \overline{C}_{\mu,X}(\overline{f}) \neq \emptyset$.

(3) $\overline{f}$ is said to be Choquet integrably bounded if there is an integrable function $g$ such that

$$||\overline{f}(x)|| = sup_{r \in \overline{f}(x)}|r| \leq g(x), \quad \text{for all } x \in X.\tag{13}$$

**Theorem 3.1.** ([5, 6, 21]) (1) If a closed set-valued measurable function $\overline{f}$ is integralble and if $E_1 \subset E_2$ and $E_1, E_2 \in \Omega$, then $\overline{C}_{\mu,E_1}(\overline{f}) \leq \overline{C}_{\mu,E_2}(\overline{f})$.

(2) If a fuzzy measure $\mu$ is continuous , and a closed set-valued measurable function $\overline{f}$ is Choquet integrably bounded, then $\overline{C}_\mu(\overline{f})$ is a closed set.

(3) If the fuzzy measure $\mu$ is continuous, and an interval-valued measurable function $\overline{f} = [f^-, f^+]$ is Choquet integrably bounded, then we have

$$\overline{C}_\mu(\overline{f}) = [C_\mu(f^-), C_\mu(f^+)].\tag{14}$$

Let $IF(X)$ be the set of all interval-valued fuzzy sets which are defined by
$$\overline{A} = \{(x, \overline{m}_{\overline{A}}) | \overline{m}_{\overline{A}} : X \longrightarrow I([0,1])\}.\tag{15}$$
By using Theorem 2.2 and Theorem 3.1(3), we easily obtain the following theorem.

**Theorem 3.2.** Let $\overline{A}, \overline{B} \in IF(X)$. (1) If $\overline{A} \leq \overline{B}$, then $\overline{C}_\mu(\overline{A}) \leq \overline{C}_\mu(\overline{B})$.

(2) If we define $(\overline{m}_{\overline{A}} \vee \overline{m}_{\overline{B}})(x) = \overline{m}_{\overline{A}}(x) \vee \overline{m}_{\overline{B}}(x)$ for all $x \in X$, then $\overline{C}_{\overline{\mu}}(\overline{A} \vee \overline{B}) \geq \overline{C}_\mu(\overline{A}) \vee \overline{C}_\mu(\overline{B})$.

(3) If we define $(\overline{m}_{\overline{A}} \wedge \overline{m}_{\overline{B}})(x) = \overline{m}_{\overline{A}}(x) \wedge \overline{m}_{\overline{B}}(x)$ for all $x \in X$, then $\overline{C}_\mu(\overline{A} \wedge \overline{B}) \geq \overline{C}_\mu(\overline{A}) \wedge \overline{C}_\mu(\overline{B})$.

We denote $\overline{m}_{\overline{A}} = [m_{A^-}, m_{A^+}]$ and define three types of interval-valued subsethood measures on $IF(X) \times IF(X)$ as follows:

**Definition 3.2.** Let $\overline{c} : IF(X) \times IF(X) \longrightarrow I([0,1])$ be a function.

(1) $\overline{c}$ is called a strong interval-valued subsethood measure if $\overline{c}$ has the following properties;

$(IS_1)$    if $\overline{A} \subset \overline{B}$, then $\overline{c}(\overline{A}, \overline{B}) = 1$;

$(IS_2)$    if $\overline{A} \neq \overline{\emptyset}$ then $\overline{c}(\overline{A}, \overline{B})$

$(IS_3)$    if $\overline{A} \subset \overline{B} \subset \overline{C}$, then $\overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{B}, \overline{A})$ and $\overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{C}, \overline{B})$.   (16)

(2) $c$ is called an interval-valued subsethood measure if $c$ has the following properties;

(IC$_1$)    if $\overline{A} \subset \overline{B}$, then $\overline{c}(\overline{A}, \overline{B}) = 1$;

(IC$_2$)    $\overline{c}(\overline{X}, \overline{\emptyset}) = 0$

(IC$_3$)    if $\overline{A} \subset \overline{B} \subset \overline{C}$, then $\overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{B}, \overline{A})$ and $\overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{C}, \overline{B})$.      (17)

(3) $c$ is called a weak interval-valued subsethood measure if $c$ has the following properties;

(IW$_1$)    $c(\emptyset, \emptyset) = 1$, $c(\emptyset, \emptyset) = 1$, $c(A, B) = 1$; and $c(X, X) = 1$

(IW$_2$)    if $A \neq \emptyset$ $rmand$ $A \cap B = \emptyset$, then $c(A, B) = 0$;

(IW$_3$)    if $A \subset B \subset C$, then $c(C, A) \leq c(B, A)$ and $c(C, A) \leq c(C, B)$.      (18)

Let $IF^*(X) = \{\overline{A} \in IF(X) | \ \overline{A}$ has the integrably bounded funstion $\overline{m_{\overline{A}}}\}$. Note that if $X$ is a finite set, then $IF(X) = IF^*(X)$. Finally, we give three types of interval-valued subsethood measures defined by the Choquet integral with respect to a fuzzy measure on $IF^*(X)$. By Theorem 3.1 (3), we note that for $\overline{A} = [A^-, A^+], \overline{B} = [B^-, B^+] \in IF^*(X)$, $\overline{C}_\mu(\overline{A}) = [C_\mu(A^-), C_\mu(A^+)]$, $\overline{C}_\mu(\overline{B}) = [C_\mu(B^-), C_\mu(B^+)]$, and $\overline{C}_\mu(\overline{A} \wedge \overline{B}) = [C_\mu(A^- \wedge B^-), C_\mu(A^+ \wedge B^+)]$.

**Theorem 3.3.** *Let $X$ be a set. If we define an interval-valued function $\overline{c}_1 : IF^*(X) \times IF^*(X) \longrightarrow I([0, 1])$,*

$$\overline{c}_1(\overline{A}, \overline{B}) = \begin{cases} 1, & \text{if } \overline{A} = \overline{B} = \overline{\emptyset}, \\ \frac{\overline{C}_\mu(\overline{A} \wedge \overline{B})}{\overline{C}_\mu(\overline{A})}, & \text{if} \quad \text{not,} \end{cases} \qquad (19)$$

*then $\overline{c}_1$ is a strong interval-valued subsethood measure on $IF^*(X)$.*

Proof. (IS$_1$) If $\overline{A} \leq \overline{B}$ and $\overline{B} = \emptyset$, that is, $\overline{A} = \overline{B} = \emptyset$, then by the definition of $\overline{c_1}$, we have $\overline{c_1}(\overline{AB}) = 1$. If $\overline{A} \leq \overline{B}$ and $\overline{B} \neq \emptyset$, then $\overline{m_{\overline{A}}} \leq \overline{m_{\overline{B}}}$. Thus, we have $m_{A^-} \leq m_{B^-}$ and $m_{A^+} \leq m_{B^+}$. Hence, we get

$$\begin{aligned} \overline{c}_1(\overline{A}, \overline{B}) &= \frac{\overline{C}_\mu(\overline{A} \wedge \overline{B})}{\overline{C}_\mu(\overline{A})} \\ &= \frac{[C_\mu(A^- \wedge B^-), C_\mu(A^+ \wedge B^+)]}{[C_\mu(A^-), C_\mu(A^+)]} \\ &= \frac{[C_\mu(A^-), C_\mu(A^+)]}{[C_\mu(A^-), C_\mu(A^+)]} = 1. \end{aligned} \qquad (20)$$

(IS$_2$) If $\overline{A} \neq \overline{\emptyset}$ and $\overline{A} \wedge \overline{B} = \overline{\overline{\emptyset}}$, then we get $0 = \overline{m_{\overline{\emptyset}}} = \overline{m_{\overline{A} \wedge \overline{B}}}$ and hence $\overline{c}_1(\overline{A}, \overline{B}) = \frac{\overline{C}_\mu(\overline{A} \wedge \overline{B})}{\overline{C}_\mu(\overline{A})} = 0$.

(IS$_3$) If $\overline{A} \leq \overline{B} \leq \overline{C}$, then we have

$$\overline{m_{\overline{A}}} \leq \overline{m_{\overline{B}}} \leq \overline{m_{\overline{C}}} \qquad (21)$$

and hence, by (21), we have

$$\overline{m_{\overline{C}}} \wedge \overline{m_{\overline{A}}} \leq \overline{m_{\overline{B}}} \wedge \overline{m_{\overline{A}}}. \qquad (22)$$

Thus, by (21) and (22), we get

$$\overline{C}_\mu(\overline{B}) \leq \overline{C}_\mu(\overline{C}), \text{ and } \overline{C}_\mu(\overline{C} \wedge \overline{A})) \leq \overline{C}_\mu(\overline{B} \wedge \overline{A}). \qquad (23)$$

Note that if $\overline{C} = \overline{\overline{\emptyset}}$, then $\overline{B} = \overline{\overline{\emptyset}}$. By using (23), we have

$$
\begin{aligned}
\overline{c}_1(\overline{C}, \overline{A}) &= \begin{cases} 1, & \text{if } \overline{C} = \overline{\overline{\emptyset}}, \\ \frac{\overline{C}_\mu(\overline{C} \wedge \overline{A})}{\overline{C}_\mu(\overline{C})}, & \text{if } \overline{C} \neq \overline{\overline{\emptyset}} \end{cases} \\
&= \begin{cases} 1, & \text{if } \overline{B} = \overline{\overline{\emptyset}}, \\ \frac{\overline{C}_\mu(\overline{B} \wedge \overline{A})}{\overline{C}_\mu(\overline{B})}, & \text{if } \overline{B} \neq \overline{\overline{\emptyset}} \end{cases} \\
&= \overline{c}_1(\overline{B}, \overline{A}).
\end{aligned}
\tag{24}
$$

From (21), we also get

$$
\overline{C}_\mu(\overline{C} \wedge \overline{A})) \leq \overline{C}_\mu(\overline{B} \wedge \overline{B}).
\tag{25}
$$

By using (25), we also have

$$
\begin{aligned}
\overline{c}_1(\overline{C}, \overline{A}) &= \begin{cases} 1, & \text{if } \overline{C} = \overline{\overline{\emptyset}}, \\ \frac{\overline{C}_\mu(\overline{C} \wedge \overline{A})}{\overline{C}_\mu(\overline{C})}, & \text{if } \overline{C} \neq \overline{\overline{\emptyset}} \end{cases} \\
&= \begin{cases} 1, & \text{if } \overline{B} = \overline{\overline{\emptyset}}, \\ \frac{\overline{C}_\mu(\overline{C} \wedge \overline{B})}{\overline{C}_\mu(\overline{C})}, & \text{if } \overline{B} \neq \overline{\overline{\emptyset}} \end{cases} \\
&= \overline{c}_1(\overline{C}, \overline{B}).
\end{aligned}
\tag{26}
$$

Therefore, $\overline{c}_1$ is a strong interval-valued subsethood measure.

**Theorem 3.4.** *Let $X$ be a set. If we define an interval-valued function $\overline{c}_2 : IF^*(X) \times IF^*(X) \longrightarrow I([0,1])$,*

$$
\overline{c}_2(\overline{A}, \overline{B}) = \begin{cases} 1, & \text{if } \overline{A} = \overline{B} = \overline{\overline{\emptyset}}, \\ \frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{A} \vee \wedge \overline{B})}, & \text{if } \text{ not,} \end{cases}
\tag{27}
$$

*then $\overline{c}_2$ is an interval-valued subsethood measure on $IF^*(X)$.*

Proof. (IC$_1$) If $\overline{A} = \overline{B} = \overline{\overline{\emptyset}}$, then $\overline{c}_2(\overline{A}, \overline{B}) = 1$. Since $\overline{A} \leq \overline{B}$, we have $\overline{m}_{\overline{A}} \leq \overline{m}_{\overline{B}}$. Thus, we get

$$
\overline{c}_2(\overline{A}, \overline{B}) = \frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{A} \vee \overline{B})} = 1.
\tag{28}
$$

(IC$_2$) By the definition of $\overline{c}_2$, we have

$$
\overline{c}_2(\overline{X}, \overline{\emptyset}) = \frac{\overline{C}_\mu(\overline{\emptyset})}{\overline{C}_\mu(\overline{X} \vee \overline{\emptyset})} = 0.
\tag{29}
$$

(IC$_3$) If $\overline{A} \leq \overline{B} \leq \overline{C}$, then we have

$$
\overline{m}_{\overline{A}} \leq \overline{m}_{\overline{B}} \leq \overline{m}_{\overline{C}}
\tag{30}
$$

and hence, by (30), we hsve

$$
\overline{C}_\mu(\overline{C}) \geq \overline{C}_\mu(\overline{B}), \ \ \overline{C}_\mu(\overline{C} \vee \overline{A}) = \overline{C}_\mu(\overline{C}), \ and \ \overline{C}_\mu(\overline{B} \vee \overline{A}) = \overline{C}_\mu(\overline{B}).
\tag{31}
$$

Therefore by using (31), we have

$$
\overline{c}_2(\overline{C}, \overline{A}) = \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C} \vee \overline{A})}
$$

$$
\begin{aligned}
&= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C})} \\
&\leq \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{B})} \\
&= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{B} \vee \overline{A})} = \bar{c}_2(\overline{B}, \overline{A}),
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
\bar{c}_2(\overline{C}, \overline{A}) &= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C} \vee \overline{A})} \\
&= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C})} \\
&\leq \frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{C})} \\
&= \frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{C} \vee \overline{B})} = \bar{c}_2(\overline{C}, \overline{B}).
\end{aligned}
\tag{33}
$$

Therefore, $\bar{c}_2$ is an interval-valued subsethood measure.

The following definition $\bar{c}_3$ has some problem because of the definition of a complement of interval-valued fuzzy set. So, we note that for interval-valued fuzzy sets $\overline{A} = [A^-, A^+]$, the modified complement $\overline{A}^{mc}$ of $\overline{A}$ is defined by

$$
\overline{m}_{\overline{A}^{mc}}(x) = [m_{A^+}(x), 1].
\tag{34}
$$

Through this definition $\overline{A}^{mc}$, we can take note of the followings:

(i)    $\overline{m}_{\overline{A}^{mc}} = [m_{A^+}, 1]$
(ii)   $\overline{m}_{\overline{A} \vee \overline{A}^{mc}} = [m_{A^-}, 1]$
(iii)  if $\overline{A} \leq \overline{B}$, then $\overline{B}^{mc} \leq \overline{A}^{mc}$.    (35)

**Theorem 3.5.** *Let $X$ be a set. If we define an interval-valued function $\bar{c}_3 : IF^*(X) \times IF^*(X) \longrightarrow I([0,1])$,*

$$
\bar{c}_3(\overline{A}, \overline{B}) = \frac{\overline{C}_\mu(\overline{A}^{mc}) \vee \overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{A} \vee \overline{A}^{mc} \vee \overline{B} \vee \overline{B}^{mc})}
\tag{36}
$$

*then $\bar{c}_3$ is an interval-valued subsethood measure on $IF^*(X)$.*

Proof. (IW$_1$) By the definition of $\bar{c}_3$, we get

$$
\begin{aligned}
\bar{c}_3(\overline{\emptyset}, \overline{\emptyset}) &= \frac{\overline{C}_\mu(\overline{\emptyset}) \vee \overline{C}_\mu(\emptyset)}{\overline{C}_\mu(\overline{\emptyset} \vee \overline{\emptyset}^{mc} \vee \overline{\emptyset} \vee \overline{\emptyset}^{mc})} \\
&= \frac{\overline{C}_\mu(\overline{X}) \vee \overline{C}_\mu(\overline{\emptyset})}{\overline{C}_\mu(\overline{\emptyset} \vee \overline{X} \vee \overline{\emptyset} \vee \overline{X})} \\
&= \frac{\overline{C}_\mu(\overline{X})}{\overline{C}_\mu(\overline{X})} = 1.
\end{aligned}
\tag{37}
$$

Similarly, we have $\bar{c}_3(\overline{\emptyset}, \overline{X}) = 1$ and $\bar{c}_3(\overline{X}, \overline{X}) = 1$.

(IW$_2$) By the definition of $\bar{c}_3$, we have

$$\begin{aligned} \bar{c}_3(\overline{X}, \bar{\emptyset}) &= \frac{\overline{C}_\mu(\overline{X}^{mc}) \vee \overline{C}_\mu(\bar{\emptyset})}{\overline{C}_\mu(\overline{X} \vee \overline{X}^{mc} \vee \bar{\emptyset} \vee \bar{\emptyset}^{mc})} \\ &= \frac{\overline{C}_\mu(\bar{\emptyset})}{\overline{C}_\mu(\overline{X})} = \frac{0}{1} = 0. \end{aligned} \tag{38}$$

(IW$_3$) If $\overline{A} \leq \overline{B} \leq \overline{C}$, then we have

$$\overline{m}_{\overline{A}} \leq \overline{m}_{\overline{B}} \leq \overline{m}_{\overline{C}}. \tag{39}$$

Thus, by (35)(i) and (39), we have

$$\overline{m}_{\overline{C}^{mc}} \leq \overline{m}_{\overline{B}^{mc}} \leq \overline{m}_{\overline{A}^{mc}}. \tag{40}$$

From (40), we get

$$\begin{aligned} \overline{C}_\mu(\overline{C} \vee \overline{C}^{mc} \vee \overline{A} \vee \overline{A}^{mc}) &= \overline{C}_\mu(\overline{C} \vee \overline{A}^{mc}) \\ &\geq \overline{C}_\mu(\overline{B} \vee \overline{A}^{mc}) \\ &= \overline{C}_\mu(\overline{B} \vee \overline{B}^{mc} \vee \overline{A} \vee \overline{A}^{mc}) \end{aligned} \tag{41}$$

Therefore by using (41), we have

$$\begin{aligned} \bar{c}_3(\overline{C}, \overline{A}) &= \frac{\overline{C}_\mu(\overline{C}^{mc}) \vee \overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C} \vee \overline{C}^{mc} \vee \overline{A} \vee \overline{A}^{cm})} \\ &\leq \frac{\overline{C}_\mu(\overline{B}^{mc}) \vee \overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{B}^{cm} \vee \overline{B}^{cm} \vee \overline{A} \vee \overline{A}^{cm})} \\ &= \bar{c}_3(\overline{B}, \overline{A}). \end{aligned} \tag{42}$$

Similarly, we have

$$\bar{c}_3(\overline{C}, \overline{A}) \leq \bar{c}_3(\overline{C}, \overline{B}). \tag{43}$$

Therefore, $\bar{c}_3$ is a weak interval-valued subsethood measure.


## 4. Applications


In this section, by using the of Harmonized system (HS) product code data for product categories $(s_1, \ldots, s_5)$ between Korea and its trading partners (that is, Korea-United States, Korea-New Zealand, Korea-Turkey, and Korea-Indea) over the 2010-2013 period, we construct four interval-valued fuzzy sets related with four countries and calculate a strong interval-valued subsethood measure $\bar{c}_1$.

Note that the product code definitions have been provided by the UN Comtrade's online data base(see[22]) and the relevant categories are defined as follows:

$s_1$. Live animals: animal products.

$s_2$. Meat and edible meat offal.

$s_3$. Fish and crustacreans, mollusks and other aguatic invertebrates.

$s_4$. Dairy produce: bird's eggs; natural honey; edible products of animal origin, not elsewhere specified or included.

$s_5$. Products of animal origin; not elsewhere specified or included.

Firstly, we denote that $s$ is year, $a(s)$ is trade value, and $u(a(s))$ is the utility of $a(s)$. By using the $\overline{u}(\overline{a}(s))$ for the trade values of animal product exports between Korea and selected trading partners for HS Product Codes $i = 1, 2, 3, 4, 5$ in Table A1 in [4], we can calculate the Choquet integral of an utility on the set of trade values (in USD) that represent Korea's trading relationship with a particular country for years $2010, 2012, 2012, 2013$. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ and $\overline{a}(s)$ be the interval-valued trade value of $s$ during four years and

$$\overline{u}(\overline{a}(s)) = \left[ \sqrt{\frac{a^-(s)}{100141401}}, \sqrt{\frac{a^+(s)}{100141401}} \right] \tag{44}$$

be an interval-valued utility of $\overline{a}(s)$. The following table $A1$ is used to create four interval-valued fuzzy sets required to draw a strong subsethood measure $\overline{c}_1$ defined by the interval-valued Choquet integral.

Table A1: The $\overline{u}(\overline{a}(s))$ for the trade value of animal product exports between Korea and selected trading partners for HS Product Codes $s_i$ for $i = 1, 2, 3, 4, 5.$

| TP | $s$ | $\overline{a}(s)$(USD) | $\overline{u}(\overline{a}(s))$ |
|---|---|---|---|
| USA | $s_1$ | $[144949, 364918] = \overline{a}(s^{(1)})$ | $[0.03542, 0.06037]$ |
| | $s_2$ | $[144949, 997539] = \overline{a}(s^{(3)})$ | $[0.03542, 0.09981]$ |
| | $s_3$ | $[74866073, 100141401] = \overline{a}(s^{(5)})$ | $[0.86464, 1.00000]$ |
| | $s_4$ | $[3722326, 5016833] = \overline{a}(s^{(4)})$ | $[0.19280, 0.22382]$ |
| | $s_5$ | $[1017895, 863858] = \overline{a}(s^{(2)})$ | $[0.09288, 0.10082]$ |
| NZ | $s_1$ | $[1589, 6650] = \overline{a}(s^{(2)})$ | $[0.00398, 0.00815]$ |
| | $s_2$ | $[0, 0] = \overline{a}(s^{(1)})$ | $[0.00000, 0.00000]$ |
| | $s_3$ | $[46632301, 91263506] = \overline{a}(s^{(5)})$ | $[0.68240, 0.95464]$ |
| | $s_4$ | $[113751, 277350] = \overline{a}(s^{(3)})$ | $[0.03370, 0.05263]$ |
| | $s_5$ | $[218022, 393025] = \overline{a}(s^{(4)})$ | $[0.04666, 0.06265]$ |
| TR | $s_1$ | $[150, 6900] = \overline{a}(s^{(4)})$ | $[0.00122, 0.00830]$ |
| | $s_2$ | $[0, 0] = \overline{a}(s^{(1)})$ | $[0.00000, 0.00000]$ |
| | $s_3$ | $[199874, 2532837] = \overline{a}(s^{(5)})$ | $[0.04468, 0.15904]$ |
| | $s_4$ | $[0, 0] = \overline{a}(s^{(2)})$ | $[0.00000, 0.00000]$ |
| | $s_4$ | $[0, 0] = \overline{a}(s^{(3)})$ | $[0.00000, 0.00000]$ |
| IND | $s_1$ | $[450, 1300] = \overline{a}(s^{(2)})$ | $[0.00212, 0.00360]$ |
| | $s_2$ | $[12135, 50630] = \overline{a}(s^{(5)})$ | $[0.00992, 0.05551]$ |
| | $s_3$ | $[1865, 8695] = \overline{a}(s^{(3)})$ | $[0.00432, 0.00932]$ |
| | $s_4$ | $[12135, 30938] = \overline{a}(s^{(4)})$ | $[0.00992, 0.02249]$ |
| | $s_5$ | $[0, 0] = \overline{a}(s^{(2)})$ | $[0.00000, 0.00000]$ |

We remark that in order to calculate the interval-valued Choquet integrals for four interval-valued fuzzy sets, we modified four interval-valued trading values for the United States and the India (see Table 5 in [4]) as follows;

$$[286892, 364918] = \overline{a}(s_1) \text{ and } [30005, 997539] = \overline{a}(s_2) \tag{45}$$

are changed by

$$\left[ \frac{286892 + 3005}{2}, 364918 \right] = \overline{a}(s_2) \text{ and } \left[ \frac{286892 + 3005}{2}, 997539 \right] = \overline{a}(s_2), \tag{46}$$

and

$$[2656, 50630] = \overline{a}(s_2) \text{ and } [21614, 30938] = \overline{a}(s_4) \tag{47}$$

are changed by

$$\left[\frac{2656 + 21614}{2}, 50630\right] = \bar{a}(s_2) \text{ and } \left[\frac{2656 + 21614}{2}, 30938\right] = \bar{a}(s_4), \quad (48)$$

From Table $A1$, we construct four interval-valued fuzzy sets from $S$ to $I([0,1])$, $\overline{U} = \overline{U}(USA), \overline{N} = \overline{N}(NZ), \overline{T} = \overline{T}(TR)$, and $\overline{I} = \overline{I}(ID)$ as follows;

$$
\begin{aligned}
\overline{U} \;=\; &\{(s_1, [0.03542, 0.06037], (s_2, [0.03542, 0.09981], (s_3, [0.86464, 1.00000]), \\
&(s_4, [0, 19280, 0.22382]), (s_5, [0.09288, 0.10082])\},
\end{aligned} \quad (49)
$$

$$
\begin{aligned}
\overline{N} \;=\; &\{(s_1, [0.00398, 0.00815], (s_2, [0.00000, 0.00000], (s_3, [0.68240, 0.95464]), \\
&(s_4, [0, 03370, 0.05263]), (s_5, [0.04666, 0.06265])\},
\end{aligned} \quad (50)
$$

$$
\begin{aligned}
\overline{T} \;=\; &\{(s_1, [0.00122, 0.00830], (s_2, [0.00000, 0.00000], (s_3, [0.04468, 0.15904]), \\
&(s_4, [0, 00000, 0.00000]), (s_5, [0.00000, 0.00000])\},
\end{aligned} \quad (51)
$$

and

$$
\begin{aligned}
\overline{I} \;=\; &\{(s_1, [0.00212, 0.00360], (s_2, [0.00992, 0.05551], (s_3, [0.00432, 0.00932]), \\
&(s_4, [0, 00992, 0.02249]), (s_5, [0.00000, 0.00000])\},
\end{aligned} \quad (52)
$$

In order to calculate the interval-valued Choquet integral, the four interval-valued fuzzy sets((49), (50), (51), (52)) were made to be increasing interval-valued fuzzy sets as follows:

$$
\begin{aligned}
\overline{U} \;=\; &\{(s_{(1)}, [0.03542, 0.06037]), (s_{(2)}, [0.03542, 0.09981]), (s_{(3)}, [0.09288, 0.10082]), \\
&(s_{(4)}, [0, 19280, 0.22382]), (s_{(5)}, [0.86464, 1.00000])\},
\end{aligned} \quad (53)
$$

$$
\begin{aligned}
\overline{N} \;=\; &\{(s_{(1)}, [0.00000, 0.00000]), (s_{(2)}, [0.00398, 0.00815]), (s_{(3)}, [0, 03370, 0.05263]), \\
&(s_{(4)}, [0.04666, 0.06265]), (s_{(5)}, [0.68240, 0.95464])\},
\end{aligned} \quad (54)
$$

$$
\begin{aligned}
\overline{T} \;=\; &\{(s_{(1)}, [0.00000, 0.00000], (s_{(2)}, [0, 00000, 0.00000], (s_{(3)}, [0.00000, 0.00000]), \\
&(s_{(4)}, [0.00122, 0.00830]), (s_{(5)}, [0.04468, 0.15904])\},
\end{aligned} \quad (55)
$$

and

$$
\begin{aligned}
\overline{I} \;=\; &\{(s_{(1)}, [0.00000, 0.00000], (s_{(2)}, [0.00212, 0.00360], (s_{(3)}, [0.00432, 0.00932]), \\
&(s_{(4)}, [0, 00992, 0.02249]), (s_{(5)}, [0.00992, 0.05551])\},
\end{aligned} \quad (56)
$$

Now, the more diversified export items, the higher fuzzy measure are defined as follows(see[4]):

$$
\begin{aligned}
&\mu(E_{(6)}) = \mu(\emptyset) = 0, \mu(E_{(5)}) = \mu_1(\{s_{(5)}\}) = 0.1, \;\; \mu(E_{(4)}) = \mu_1(\{s_{(4)}, s_{(5)}\}) = 0.2, \\
&\mu(E_{(3)}) = \mu_1(\{s_{(3)}, s_{(4)}, s_{(5)}\}) = 0.4, \;\; \mu(E_{(2)}) = \mu_1(\{s_{(2)}, s_{(3)}, s_{(4)}, s_{(5)}\}) = 0.7, \\
&\mu(E_{(1)}) = \mu_1(\{s_{(1)}, s_{(2)}, s_{(3)}, s_{(4)}, s_{(5)}\}) = 1.
\end{aligned} \quad (57)
$$

By using two interval-valued fuzzy sets (53), (54) and the above fuzzy measure (57), we can calculate a strong interval-valued subsethood measure for $\bar{c}_1(\overline{U}, \overline{N})$ as follows:

$$
\begin{aligned}
\overline{C}_\mu(\overline{U}) \;&=\; [C_\mu(U^-), C_\mu(U^+)] \\
\overline{C}_\mu(\overline{U} \wedge \overline{N}) \;&=\; [C_\mu(U^- \wedge N^-), C_\mu(U^+ \wedge N^+)],
\end{aligned} \quad (58)
$$

and

$$
\begin{aligned}
\bar{c}_1(\overline{U}, \overline{N}) \;&=\; \frac{\overline{C}_\mu(\overline{U} \wedge \overline{N})}{\overline{C}_\mu(\overline{U})} \\
&=\; \left[\frac{C_\mu(U^- \wedge N^-)}{C_\mu(U^-)} \wedge \frac{C_\mu(U^+ \wedge N^+)}{C_\mu(U^+)}, \frac{C_\mu(U^- \wedge N^-)}{C_\mu(U^-)} \vee \frac{C_\mu(U^+ \wedge N^+)}{C_\mu(U^+)}\right], (59)
\end{aligned}
$$

where

$$C_\mu(U^-) \;=\; m_{U^-}(s_{(1)})(\mu(E_{(1)}) - \mu(E_{(2)}))$$

$$+m_{U^-}(s_{(2)})(\mu(E_{(2)}) - \mu(E_{(3)}))$$
$$+m_{U^-}(s_{(3)})(\mu(E_{(3)}) - \mu(E_{(4)}))$$
$$+m_{U^-}(s_{(4)})(\mu(E_{(4)}) - \mu(E_{(5)}))$$
$$+m_{U^-}(s_{(5)})(\mu(E_{(5)})) - \mu(E_{(6)}), \tag{60}$$

$$
\begin{aligned}
C_\mu(U^+) =\ & m_{U^+}(s_{(1)})(\mu(E_{(1)}) - \mu(E_{(2)})) \\
& + m_{U^+}(s_{(2)})(\mu(E_{(2)}) - \mu(E_{(3)})) \\
& + m_{U^+}(s_{(3)})(\mu(E_{(3)}) - \mu(E_{(4)})) \\
& + m_{U^+}(s_{(4)})(\mu(E_{(4)}) - \mu(E_{(5)})) \\
& + m_{U^+}(s_{(5)})(\mu(E_{(5)}) - \mu(E_{(6)})),
\end{aligned}
\tag{61}
$$

and

$$
\begin{aligned}
C_\mu(U^- \wedge N^-) =\ & m_{U^- \wedge N^-}(s_{(1)})(\mu(E_{(1)}) - \mu(E_{(2)})) \\
& + m_{U^- \wedge N^-}(s_{(2)})(\mu(E_{(2)}) - \mu(E_{(3)})) \\
& + m_{U^- \wedge N^-}(s_{(3)})(\mu(E_{(3)}) - \mu(E_{(4)})) \\
& + m_{U^- \wedge N^-}(s_{(4)})(\mu(E_{(4)}) - \mu(E_{(5)})) \\
& + m_{U^- \wedge N^-}(s_{(5)})(\mu(E_{(5)}) - \mu(E_{(6)})) \\
=\ & (m_{U^-}(s_{(1)}) \wedge \mu_{N^-}(s_{(1)}))(\mu(E_{(1)}-)\mu(E_{(2)})) \\
& + (m_{U^-}(s_{(2)}) \wedge m_{N^-}(s_{(2)}))(\mu(E_{(2)}) - \mu(E_{(3)})) \\
& + (m_{U^-}(s_{(3)}) \wedge m_{N^-}(s_{(3)}))(\mu(E_{(3)}) - \mu(E_{(4)})) \\
& + (m_{U^-}(s_{(4)}) \wedge m_{N^-}(s_{(4)}))(\mu(E_{(4)}) - \mu(E_{(5)})) \\
& + (m_{U^-}(s_{(5)}) \wedge m_{N^-}(s_{(5)}))(\mu(E_{(5)}) - \mu(E_{(6)})),
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
C_\mu(U^+ \wedge N^+) =\ & m_{U^+ \wedge N^+}(s_{(1)})(\mu(E_{(1)}) - \mu(E_{(2)})) \\
& + m_{U^+ \wedge N^+}(s_{(2)})(\mu(E_{(2)}) - \mu(E_{(3)})) \\
& + m_{U^+ \wedge N^+}(s_{(3)})(\mu(E_{(3)}) - \mu(E_{(4)})) \\
& + m_{U^+ \wedge N^+}(s_{(4)})(\mu(E_{(4)}) - \mu(E_{(5)})) \\
& + m_{U^+ \wedge N^+}(s_{(5)})(\mu(E_{(5)}) - \mu(E_{(6)})) \\
=\ & (m_{U^+}(s_{(1)}) \wedge m_{N^+}(s_{(1)}))(\mu(E_{(1)}) - \mu(E_{(2)})) \\
& + (m_{U^+}(s_{(2)}) \wedge m_{N^+}(s_{(2)}))(\mu(E_{(2)}) - \mu(E_{(3)})) \\
& + (m_{U^+}(s_{(3)}) \wedge m_{N^+}(s_{(3)}))(\mu(E_{(3)}) - \mu(E_{(4)})) \\
& + (m_{U^+}(s_{(4)}) \wedge m_{N^+}(s_{(4)}))(\mu(E_{(4)}) - \mu(E_{(5)})) \\
& + (m_{U^+}(s_{(5)}) \wedge m_{N^+}(s_{(5)}))(\mu(E_{(5)}) - \mu(E_{(6)})).
\end{aligned}
\tag{63}
$$

Thus, by using (58) and (59), we get the following table A2 for the strong interval-valued subsethood measure between United States and New Zealand.

Table A2: The $\bar{c}_1(\overline{U}, \overline{N})$ between United States and New Zealand.

| $\overline{C}_\mu(\overline{U})$ | $\overline{C}_\mu(\overline{U}, \overline{N})$ | $\bar{c}_1(\overline{U}, \overline{N})$ |
|---|---|---|
| $[0.14557, 0.19060]$ | $[0.08084, 0.11470]$ | $[0.55534, 0.60178]$ |

Given that $\bar{c}_1(\overline{U}, \overline{V})$ represents the degree of trade surplus for the trading relationship for Korea and USA, and Korea and New Zealand.

Finally, we can calculate $\bar{c}_1(\overline{U}, \overline{T})$ and $\bar{c}_1(\overline{U}, \overline{I})$ in Table A1.

Table A3: The $\bar{c}_1(\overline{U}, \overline{T})$ between United States and Turkey.

| $\overline{C}_\mu(\overline{U})$ | $\overline{C}_\mu(\overline{U}, \overline{T})$ | $\bar{c}_1(\overline{U}, \overline{T})$ |
|---|---|---|
| $[0.14557, 0.19060]$ | $[0.00459, 0.01673]$ | $[0.03153, 0.08778]$ |

Table A4: The $\bar{c}_1(\overline{U}, \overline{I})$ between United States and India.

| $\overline{C}_\mu(\overline{U})$ | $\overline{C}_\mu(\overline{U}, \overline{I})$ | $\bar{c}_1(\overline{U}, \overline{I})$ |
|---|---|---|
| $[0.14557, 0.19060]$ | $[0.00348, 0.01074]$ | $[0.02393, 0.05635]$ |

Tables A2, A3, and A4, demonstrate the results $\bar{c}_1(\overline{U}, \overline{N}), \bar{c}_1(\overline{U}, \overline{T}), \bar{c}_1(\overline{U}, \overline{I})$. They highlight the degree of trade surplus that exists with the three trading partners in terms of the model of trade transactions with United States and Korea.

## 5. Conclusions

Using the concept of intervals, we defined three types of interval-valued subsethood measures in Definitions 3.2, 3.3 and 3.4. From these definitions, we proposed three types of interval-valued subsethood measures defined by the interval-valued Choquet integrals with respect to a continuous fuzzy measure in Theorems 3.2, 3.3, and 3.4. The fuzzy measure $\mu$ in (57) means that if set $E$ includes more categories between Korea and its trading partner, then $\mu(E)$ receives a higher score. Moreover, intervals are also a very useful tool to express the degree of trade surplus between Korea and its four trading partners analyzed over the 2010-2013 period.

In order to illustrate some applications of a strong interval-valued subsethood measure, we provided the four interval-valued fuzzy sets which were aggregated in (49), (50), (51), and (52) to animal product exports between Korea and four selected trading partners from 2010 to 2013. By using these interval-valued fuzzy sets, we obtained the strong interval-valued subsethood measure $\bar{c}_1(\overline{U}, \overline{N}), \bar{c}_1(\overline{U}, \overline{T}), \bar{c}_1(\overline{U}, \overline{I})$ which represent the degree of trade surplus between Korea and 3 trading partners in terms of the model of trade transactions with the United States and South Korea in Tables 2, 3, and 4. It was found that New Zealand was at least 0.55534 to 0.60178 times smaller than the United States, while Turkey and India were also smaller, with Turkey at least 0.03153 to 0.08778 times smaller and India being at least 0.2393 to 0.05635 times smaller than the United States between 2010 and 2013, in terms of the trade values of animal product exports that exists between Korea and selected trading partners.

**Data Availability:** All the authors solemnly declare that there is no data used to support the fndings of this study.

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# $M$-fractional integral inequalities

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### Abstract

Here we present $M$-fractional integral inequalities of Ostrowski and Polya types.

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## 1 Introduction

We are inspired by the following results:

**Theorem 1** *([2], p. 498, [1], [5]) (Ostrowski inequality)*
   *Let $f \in C^1([a,b])$, $x \in [a,b]$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(z)\, dz - f(x) \right| \leq \left( \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right) \| f' \|_\infty . \tag{1}$$

*Inequality (1) is sharp. In particular the optimal function is*

$$f^*(z) := |z-x|^\alpha (b-a), \quad \alpha > 1. \tag{2}$$

**Theorem 2** *([6], [7, p. 62], [8], [9, p. 83]) (Polya integral inequality)*
   *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then there exists at least one point $\xi \in [a,b]$ such that*

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x)\, dx. \tag{3}$$

In this short work we present inequalities of types (1) and (3) involving the left and right fractional local general $M$-derivatives, see [3], [4].

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## 2    Background

We need

**Definition 3** *([4]) Let $f : [a, \infty) \to \mathbb{R}$ and $t > a$, $a \in \mathbb{R}$. For $0 < \alpha \leq 1$ we define the left local general M-derivative of order $\alpha$ of function $f$, denoted by $D_{M,a}^{\alpha,\beta} f(t)$, by*

$$D_{M,a}^{\alpha,\beta} f(t) := \lim_{\varepsilon \to 0} \frac{f\left(t\mathbb{E}_\beta\left(\varepsilon (t-a)^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \tag{4}$$

*$\forall\ t > a$, where $\mathbb{E}_\beta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k+1)}$, $\beta > 0$, is the Mittag-Leffler function with one parameter.*

*If $D_{M,a}^{\alpha,\beta} f(t)$ exists over $(a, \gamma)$, $\gamma \in \mathbb{R}$ and $\lim_{t \to a+} D_{M,a}^{\alpha,\beta} f(t)$ exists, then*

$$D_{M,a}^{\alpha,\beta} f(a) = \lim_{t \to a+} D_{M,a}^{\alpha,\beta} f(t). \tag{5}$$

**Theorem 4** *([4]) If a function $f : [a, \infty) \to \mathbb{R}$ has the left local general M-derivative of order $\alpha \in (0, 1]$, $\beta > 0$, at $t_0 > a$, then $f$ is continuous at $t_0$.*

We need

**Theorem 5** *([4]) (Mean value theorem) Let $f : [\gamma, \delta] \to \mathbb{R}$ with $\gamma > a$, $0 \notin [\gamma, \delta]$, such that*
*(1) $f$ is continuous on $[\gamma, \delta]$,*
*(2) there exists $D_{M,a}^{\alpha,\beta} f$ on $(\gamma, \delta)$ for some $\alpha \in (0, 1]$.*
*Then, there exists $c \in (\gamma, \delta)$ such that*

$$f(\delta) - f(\gamma) = \left(D_{M,a}^{\alpha,\beta} f(c)\right) \frac{\Gamma(\beta+1)(c-a)^\alpha}{c} (\delta - \gamma). \tag{6}$$

We need

**Definition 6** *([3]) Let $f : (-\infty, b] \to \mathbb{R}$ and $t < b$, $b \in \mathbb{R}$. For $0 < \alpha \leq 1$ we define the right local general M-derivative of order $\alpha$ of function $f$, denoted as ${}_{M,b}^{\alpha,\beta}Df(t)$, by*

$${}_{M,b}^{\alpha,\beta}Df(t) := -\lim_{\varepsilon \to 0} \frac{f\left(t\mathbb{E}_\beta\left(\varepsilon (b-t)^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \tag{7}$$

*$\forall\ t < b$.*
*If ${}_{M,b}^{\alpha,\beta}Df(t)$ exists over $(\gamma, b)$, $\gamma \in \mathbb{R}$ and $\lim_{t \to b-} {}_{M,b}^{\alpha,\beta}Df(t)$ exists, then*

$${}_{M,b}^{\alpha,\beta}Df(b) = \lim_{t \to b-} {}_{M,b}^{\alpha,\beta}Df(t). \tag{8}$$

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**Theorem 7** *([3]) If a function $f : (-\infty, b] \to \mathbb{R}$ has the right local general M-derivative of order $\alpha \in (0, 1]$, $\beta > 0$, at $t_0 < b$, then $f$ is continuous at $t_0$.*

We also need

**Theorem 8** *([3]) (Mean value theorem) Let $f : [\gamma, \delta] \to \mathbb{R}$ with $\delta < b$, $0 \notin [\gamma, \delta]$, such that*
*(1) $f$ is continuous on $[\gamma, \delta]$,*
*(2) there exists $^{\alpha,\beta}_{M,b}Df$ on $(\gamma, \delta)$ for some $\alpha \in (0, 1]$.*
*Then, there exists $c \in (\gamma, \delta)$ such that*

$$f(\delta) - f(\gamma) = \left( -^{\alpha,\beta}_{M,b}Df(c) \right) \left( \frac{\Gamma(\beta + 1)(b - c)^\alpha}{c} \right) (\delta - \gamma).$$  (9)

Fractional derivatives $D^{\alpha,\beta}_{M,a}$ and $^{\alpha,\beta}_{M,b}D$ possess all basic properties of the ordinary derivatives and beyond, see [3], [4].

## 3  Main Results

We present the following $M$-fractional Ostrowski type inequality:

**Theorem 9** *Let $a < \gamma < \delta < b$, $0 \notin [\gamma, \delta]$, $f : [a, b] \to \mathbb{R}$, which is continuous over $[\gamma, \delta]$. We assume that $D^{\alpha,\beta}_{M,a}$, $^{\alpha,\beta}_{M,b}D$ exist and are continuous over $[\gamma, x_0]$ and $[x_0, \delta]$, respectively, where $x_0 \in [\gamma, \delta]$, for some $\alpha \in (0, 1]$. Then*

$$\left| \frac{1}{\delta - \gamma} \int_\gamma^\delta f(x) \, dx - f(x_0) \right| \leq \frac{\Gamma(\beta + 1)}{2(\delta - \gamma)}$$

$$\left[ \left\| \frac{D^{\alpha,\beta}_{M,a}f(x)}{x} \right\|_{\infty,[\gamma,x_0]} (x_0 - a)^\alpha (x_0 - \gamma)^2 + \left\| \frac{^{\alpha,\beta}_{M,b}Df(x)}{x} \right\|_{\infty,[x_0,\delta]} (b - x_0)^\alpha (\delta - x_0)^2 \right].$$  (10)

**Proof.** Let $x \in [\gamma, x_0]$, the by Theorem 5, there exists $c_1 \in (x, x_0)$, such that

$$f(x_0) - f(x) = \left( \frac{D^{\alpha,\beta}_{M,a}f(c_1)}{c_1} \right) \Gamma(\beta + 1)(c_1 - a)^\alpha (x_0 - x).$$  (11)

Thus

$$|f(x) - f(x_0)| = \left| \frac{D^{\alpha,\beta}_{M,a}f(c_1)}{c_1} \right| \Gamma(\beta + 1)(c_1 - a)^\alpha |x - x_0| \leq$$

$$\left\| \frac{D^{\alpha,\beta}_{M,a}f(x)}{x} \right\|_{\infty,[\gamma,x_0]} \Gamma(\beta + 1)(x_0 - a)^\alpha |x - x_0|,$$  (12)

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$\forall\ x \in [\gamma, x_0]$.

Let now $x \in [x_0, \delta]$, then by Theorem 8, there exists $c_2 \in (x_0, x)$, such that

$$f(x) - f(x_0) = - \left( \frac{{}^{\alpha,\beta}_{M,b} Df(c_2)}{c_2} \right) \Gamma(\beta+1)(b-c_2)^\alpha (x-x_0). \qquad (13)$$

Thus

$$|f(x) - f(x_0)| = \left| \frac{{}^{\alpha,\beta}_{M,b} Df(c_2)}{c_2} \right| \Gamma(\beta+1)(b-x_0)^\alpha |x-x_0| \leq$$

$$\left\| \frac{{}^{\alpha,\beta}_{M,b} Df(x)}{x} \right\|_{\infty, [x_0, \delta]} \Gamma(\beta+1)(b-x_0)^\alpha |x-x_0|, \qquad (14)$$

$\forall\ x \in [x_0, \delta]$.

We have that

$$\left| \frac{1}{\delta-\gamma} \int_\gamma^\delta f(x)\, dx - f(x_0) \right| = \frac{1}{\delta-\gamma} \left| \int_\gamma^\delta (f(x) - f(x_0))\, dx \right| \leq$$

$$\frac{1}{\delta-\gamma} \int_\gamma^\delta |f(x) - f(x_0)|\, dx = \qquad (15)$$

$$\frac{1}{\delta-\gamma} \left[ \int_\gamma^{x_0} |f(x) - f(x_0)|\, dx + \int_{x_0}^\delta |f(x) - f(x_0)|\, dx \right] \overset{\text{(by (12), (14))}}{\leq}$$

$$\frac{1}{\delta-\gamma} \left[ \left\| \frac{D^{\alpha,\beta}_{M,a} f(x)}{x} \right\|_{\infty, [\gamma, x_0]} \Gamma(\beta+1)(x_0-a)^\alpha \int_\gamma^{x_0} (x_0-x)\, dx \right.$$

$$\left. + \left\| \frac{{}^{\alpha,\beta}_{M,b} Df(x)}{x} \right\|_{\infty, [x_0, \delta]} \Gamma(\beta+1)(b-x_0)^\alpha \int_{x_0}^\delta (x-x_0)\, dx \right] =$$

$$\frac{\Gamma(\beta+1)}{2(\delta-\gamma)} \left[ \left\| \frac{D^{\alpha,\beta}_{M,a} f(x)}{x} \right\|_{\infty, [\gamma, x_0]} (x_0-a)^\alpha (x_0-\gamma)^2 + \qquad (16) \right.$$

$$\left. \left\| \frac{{}^{\alpha,\beta}_{M,b} Df(x)}{x} \right\|_{\infty, [x_0, \delta]} (b-x_0)^\alpha (\delta-x_0)^2 \right].$$

The theorem is proved. ∎

Next we give two $M$-fractional Polya type inequalities:

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**Theorem 10** *All as in Theorem 9 and* $f(x_0) = 0$. *Then*

$$\left| \int_\gamma^\delta f(x)\, dx \right| \le \int_\gamma^\delta |f(x)|\, dx \le \frac{\Gamma(\beta+1)}{2}$$

$$\left[ \left\| \frac{D_{M,a}^{\alpha,\beta} f(x)}{x} \right\|_{\infty,[\gamma,x_0]} (x_0-a)^\alpha (x_0-\gamma)^2 + \left\| \frac{{}_{M,b}^{\alpha,\beta} D f(x)}{x} \right\|_{\infty,[x_0,\delta]} (b-x_0)^\alpha (\delta-x_0)^2 \right].$$

$$\tag{17}$$

**Proof.** Same as in the proof of Theorem 9, by setting $f(x_0) = 0$. ∎

**Corollary 11** *(to Theorem 10, case of* $x_0 = \frac{\gamma+\delta}{2}$*)* *All as in Theorem 9 and* $f\left(\frac{\gamma+\delta}{2}\right) = 0$. *Then*

$$\int_\gamma^\delta |f(x)|\, dx \le \frac{\Gamma(\beta+1)(\delta-\gamma)^2}{8}$$

$$\left[ \left\| \frac{D_{M,a}^{\alpha,\beta} f(x)}{x} \right\|_{\infty,\left[\gamma,\frac{\gamma+\delta}{2}\right]} \left( \left(\frac{\gamma+\delta}{2}\right) - a \right)^\alpha + \left\| \frac{{}_{M,b}^{\alpha,\beta} D f(x)}{x} \right\|_{\infty,\left[\frac{\gamma+\delta}{2},\delta\right]} \left( b - \left(\frac{\gamma+\delta}{2}\right) \right)^\alpha \right].$$

$$\tag{18}$$

**Proof.** Apply (17) for $x_0 = \frac{\gamma+\delta}{2}$. ∎

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