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A study of a coupled system of nonlinear second-order ordinary differential equations with nonlocal integral multi-strip boundary conditions on an arbitrary domain

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Abstract

In this paper, we study a nonlinear system of second order ordinary differential equations with nonlocal integral multi-strip coupled boundary conditions. Leray-Schauder alternative criterion, Schauder fixed point theorem and Banach contraction mapping principle are employed to obtain the desired results. Examples are constructed for the illustration of the obtained results. We emphasize that our results are new and enhance the literature on boundary value problems of coupled systems of ordinary differential equations. Several new results appear as special cases of our work.

Keywords: System of ordinary differential equations; integral boundary condition; multi-strip; existence; fixed point.
MSC 2000: 34A34, 34B10, 34B15.

[1]Corresponding author

1 Introduction

This paper is concerned with the following coupled system of nonlinear second-order ordinary differential equations:

$$\begin{cases} u''(t) = f(t, u(t), v(t)), \ t \in [a, b], \\ v''(t) = g(t, u(t), v(t)), \ t \in [a, b], \end{cases} \tag{1.1}$$

supplemented with the nonlocal integral multi-strip coupled boundary conditions of the form:

$$\begin{cases} \displaystyle\int_a^b u(s)ds = \sum_{j=1}^{m} \gamma_j \int_{\xi_j}^{\eta_j} v(s)ds + \lambda_1, \int_a^b u'(s)ds = \sum_{j=1}^{m} \rho_j \int_{\xi_j}^{\eta_j} v'(s)ds + \lambda_2, \\ \displaystyle\int_a^b v(s)ds = \sum_{j=1}^{m} \sigma_j \int_{\xi_j}^{\eta_j} u(s)ds + \lambda_3, \int_a^b v'(s)ds = \sum_{j=1}^{m} \delta_j \int_{\xi_j}^{\eta_j} u'(s)ds + \lambda_4, \end{cases} \tag{1.2}$$

where $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $a < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < b$, and $\gamma_j,\ \rho_j,\ \sigma_j$ and $\delta_j\ \in \mathbb{R}^+$ $(j = 1, 2, \ldots, m)$, $\lambda_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$.

Mathematical modeling of several real world phenomena lead to the occurrence of nonlinear boundary value problems of differential equations. During the past few decades, the topic of boundary value problems has evolved as an important and interesting area of investigation in view of its extensive applications in diverse disciplines such as fluid mechanics, mathematical physics, etc. For application details, we refer the reader to the text [1], while some recent works on boundary value problems of ordinary differential equations can be found in the papers ([2]-[5]).

Much of the literature on boundary value problems involve classical boundary conditions. However, these conditions cannot cater the complexities of the physical and chemical processes occurring within the domain. In order to cope with this situation, the concept of nonlocal boundary conditions was introduced. Such conditions relate the boundary values of the unknown function to its values at some interior positions of the domain. For a detailed account of nonlocal nonlinear boundary value problems, for instance, see ([6]-[16]) and the references cited therein.

Computational fluid dynamics (CFD) technique are directly concerned with the boundary data [1]. However, the assumption of circular cross-section in the fluid flow problems is not justifiable in many situations. The concept of integral boundary conditions played a key role in resolving this issue as such conditions can be applied to arbitrary shaped structures. Integral boundary conditions are also found to be quite useful in the study of thermal and hydrodynamic problems. In fact, one can find numerous applications of integral boundary conditions in the fields like chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. ([17]-[20]). For some recent results on boundary value problems integral boundary conditions, we refer the reader to a series of articles ([21]-[32]) and the references cited therein.

Motivated by the importance of nonlocal and integral boundary conditions, we introduce a new kind of coupled integral boundary conditions (1.2) and solve a nonlinear coupled system of second-order ordinary differential equations (1.1) equipped with these conditions. Our main results rely on Leray-Schauder alternative and Banach contraction mapping principle.

The rest of the paper is organized as follows. In Section 2, we present an auxiliary lemma. The main results for the problem (1.1) and (1.2) are discussed in Section 3. We also construct examples illustrating the obtained results. The paper concludes with some interesting observations.

2 An auxiliary lemma

The following lemma plays a key role in defining the solution for the problem (1.1) – (1.2).

Lemma 2.1 *For $f_1, g_1 \in C([a,b],\mathbb{R})$, the solution of the linear system of differential equations*

$$\begin{aligned} u''(t) &= f_1(t),\ t\in[a,b], \\ v''(t) &= g_1(t),\ t\in[a,b], \end{aligned} \tag{2.1}$$

subject to the boundary conditions (1.2) *is equivalent to the system of integral equations*

$$\begin{aligned} u(t) &= \int_a^t (t-s)f_1(s)ds \\ &-\frac{1}{A_3}\Big\{\int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s)+L_1+(b-a)A_2(t-a)\Big](b-s)f_1(s)ds \\ &+\int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\ &\times(b-s)g_1(s)ds\Big\}+\frac{1}{A_3}\Big\{\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p)+\rho_jL_1 \\ &+\rho_j(b-a)A_2(t-a)\Big]g_1(p)dpds+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p) \\ &+\delta_jL_2+\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f_1(p)dpds\Big\}+\Omega_1(t), \end{aligned} \tag{2.2}$$

$$v(t) = \int_a^t (t-s)g_1(s)ds-\frac{1}{A_3}\Big\{\int_a^b\Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{2\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}(b-s)$$

$$
\begin{aligned}
&+L_3+A_2(t-a)\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big](b-s)f_1(s)ds\\
&+\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)+L_4+A_2(b-a)(t-a)\Big](b-s)g_1(s)ds\Big\}\\
&+\frac{1}{A_3}\Big\{\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[(s-p)\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}+\rho_jL_3\\
&+\delta_jA_2(t-a)\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big]g_1(p)dpds\\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4+\delta_jA_2(b-a)(t-a)\Big]f_1(p)dpds\Big\}\\
&+\Omega_2(t),
\end{aligned}
\tag{2.3}
$$

where

$$A_1 = (b-a)^2-\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)^2\Big), \tag{2.4}$$

$$A_2 = (b-a)^2-\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)^2\Big), \quad A_3=A_1A_2\neq 0, \tag{2.5}$$

$$
\begin{aligned}
L_1 &= (b-a)\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big(\sum_{j=1}^{m}\sigma_j+\sum_{j=1}^{m}\delta_j\Big)\\
&\quad-\frac{(b-a)^4}{2}-\frac{(b-a)^2}{2}\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)^2\Big),
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
L_2 &= \sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big(\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)^2\Big)+(b-a)^2\Big)\\
&\quad-\frac{(b-a)^3}{2}\sum_{j=1}^{m}(\eta_j-\xi_j)(\rho_j+\gamma_j),
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
L_3 &= \sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big((b-a)^2(\sigma_j+\delta_j)-A_2\delta_j\Big)\\
&\quad+\frac{(b-a)^3}{2}\Big[\frac{A_2-(b-a)^2}{\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)}-\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big],
\end{aligned}
\tag{2.8}
$$

$$L_4 = \frac{(b-a)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}\sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big[\sigma_j\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)^2$$

$$+(b-a)^2 - A_2\Big] - \frac{(b-a)^4}{2} + \frac{(b-a)^2}{2\sum_{j=1}^m \gamma_j}\sum_{j=1}^m \rho_j\Big(A_2 - (b-a)^2\Big), \tag{2.9}$$

$$\Omega_1(t) = \frac{1}{A_3}\Big\{A_1(b-a)\lambda_1 + \Big[L_1 + A_2(b-a)(t-a)\Big]\lambda_2 + A_1\sum_{j=1}^m \gamma_j(\eta_j - \xi_j)\lambda_3 + \Big[L_2 + A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big]\lambda_4\Big\}, \tag{2.10}$$

$$\Omega_2(t) = \frac{1}{A_3}\Big\{\frac{A_1\Big((b-a)^2 - A_2\Big)}{\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)}\lambda_1 + \Big[L_3 + A_2(t-a)\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\Big]\lambda_2 + A_1(b-a)\lambda_3 + \Big[L_4 + A_2(b-a)(t-a)\Big]\lambda_4\Big\}. \tag{2.11}$$

Proof. Integrating the linear system (2.1) twice from a to t, we get

$$u(t) = c_1 + c_2(t-a) + \int_a^t (t-s)f_1(s)ds, \tag{2.12}$$

$$v(t) = c_3 + c_4(t-a) + \int_a^t (t-s)g_1(s)ds, \tag{2.13}$$

where c_1, c_2, c_3 and c_4 are arbitrary real constants.
Using the boundary conditions (1.2) in (2.12) and (2.13), together with notations (2.4), we obtain

$$(b-a)c_1 + \frac{(b-a)^2}{2}c_2 - \sum_{j=1}^m \gamma_j(\eta_j-\xi_j)c_3 - \sum_{j=1}^m \gamma_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)c_4 = -\int_a^b \frac{(b-s)^2}{2}f_1(s)ds + \sum_{j=1}^m \gamma_j\int_{\xi_j}^{\eta_j}\int_a^s (s-p)g_1(p)dpds + \lambda_1, \tag{2.14}$$

$$(b-a)c_2 - \sum_{j=1}^m \rho_j(\eta_j-\xi_j)c_4 = -\int_a^b (b-s)f_1(s)ds + \sum_{j=1}^m \rho_j\int_{\xi_j}^{\eta_j}\int_a^s g_1(p)dpds + \lambda_2, \tag{2.15}$$

$$-\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)c_1 - \sum_{j=1}^m \sigma_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)c_2 + (b-a)c_3 + \frac{(b-a)^2}{2}c_4 = -\int_a^b \frac{(b-s)^2}{2}g_1(s)ds + \sum_{j=1}^m \sigma_j\int_{\xi_j}^{\eta_j}\int_a^s (s-p)f_1(p)dpds + \lambda_3, \tag{2.16}$$

$$-\sum_{j=1}^m \delta_j(\eta_j-\xi_j)c_2 + (b-a)c_4 = -\int_a^b (b-s)g_1(s)ds + \sum_{j=1}^m \delta_j\int_{\xi_j}^{\eta_j}\int_a^s f_1(p)dpds + \lambda_4. \tag{2.17}$$

Solving the equations (2.15) and (2.17) for c_2 and c_4, we find that

$$\begin{aligned} c_2 &= \frac{1}{A_1}\Big[-\int_a^b (b-s)\Big((b-a)f_1(s)+\sum_{j=1}^m \rho_j(\eta_j-\xi_j)g_1(s)\Big)ds \\ &+\sum_{j=1}^m \rho_j\int_{\xi_j}^{\eta_j}\int_a^s \Big((b-a)g_1(p)+\sum_{j=1}^m \delta_j(\eta_j-\xi_j)f_1(p)\Big)dpds \\ &\qquad +(b-a)\lambda_2+\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\lambda_4\Big], \end{aligned} \tag{2.18}$$

$$\begin{aligned} c_4 &= \frac{1}{A_1}\Big[-\int_a^b (b-s)\Big(\sum_{j=1}^m \delta_j(\eta_j-\xi_j)f_1(s)+(b-a)g_1(s)\Big)ds \\ &+\sum_{j=1}^m \delta_j\int_{\xi_j}^{\eta_j}\int_a^s \Big(\sum_{j=1}^m \rho_j(\eta_j-\xi_j)g_1(p)+(b-a)f_1(p)\Big)dpds \\ &\qquad +\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big]. \end{aligned} \tag{2.19}$$

Using (2.18) and (2.19) in (2.14) and (2.16) and then solving the resulting equations for c_1 and c_3, we obtain

$$\begin{aligned} c_1 &= \frac{1}{A_3}\Big\{-\int_a^b \Big[\frac{1}{2}(b-a)A_1(b-s)+L_1\Big](b-s)f_1(s)ds \\ &-\int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)+L_2\Big](b-s)g_1(s)ds \\ &+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s \Big[A_1\gamma_j(b-a)(s-p)+\rho_jL_1\Big]g_1(p)dpds \\ &+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s \Big[A_1\sum_{j=1}^m \gamma_j\sigma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\Big]f_1(p)dpds \\ &+A_1(b-a)\lambda_1+L_1\lambda_2+A_1\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)\lambda_3+L_2\lambda_4\Big\}, \end{aligned}$$

$$\begin{aligned} c_3 &= \frac{1}{A_3}\Big\{-\int_a^b \Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{2\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)}(b-s)+L_3\Big](-s)f_1(s)ds \\ &-\int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s)+L_4\Big](b-s)g_1(s)ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}\sum_{j=1}^m\gamma_j(s-p)+\rho_jL_3\Big]g_1(p)dpds\\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4\Big]f_1(p)dpds\\
&+\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}\lambda_1+L_3\lambda_2+A_1(b-a)\lambda_3+L_4\lambda_4\Big\}.
\end{aligned}
$$

Inserting the values of c_1, c_2, c_3 and c_4 in (2.12) and (2.13), we get the solutions (2.2) and (2.3). The converse follows by direct computation. This completes the proof. □

3 Main results

Let us introduce the space $\mathcal{X} = \{u(t)|u(t) \in C([a,b])\}$ equipped with norm $\|u\| = \sup\{|u(t)|, t \in [a,b]\}$. Obviously $(\mathcal{X}, \|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{X}\times\mathcal{X}, \|u,v\|)$ is a Banach space with norm $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in \mathcal{X}\times\mathcal{X}$.

By Lemma 2.1, we define an operator $\mathcal{T} : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ as

$$
\mathcal{T}(u,v)(t) := (\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)),
$$

where

$$
\begin{aligned}
\mathcal{T}_1(u,v)(t) \;=\;& \int_a^t (t-s)f(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)\\
&+L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds\\
&-\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]\\
&\times(b-s)g(s,u(s),v(s))ds+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p) \qquad (3.1)\\
&+\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds\\
&+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\\
&+\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t),
\end{aligned}
$$

$$
\mathcal{T}_2(u,v)(t) \;=\; \int_a^t (t-s)g(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{2\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}(b-s)
$$

$$
\begin{aligned}
&+L_3+A_2(t-a)\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big](b-s)f(s,u(s),v(s))ds\\
&-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)+L_4+A_2(b-a)(t-a)\Big](b-s)\\
&\times g(s,u(s),v(s))ds+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[(s-p)\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)} \qquad (3.2)\\
&+\rho_jL_3+\delta_jA_2(t-a)\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big]g(p,u(p),v(p))dpds\\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4+\delta_jA_2(b-a)(t-a)\Big]\\
&\times f(p,u(p),v(p))dpds\Big\}+\Omega_2(t).
\end{aligned}
$$

In order to prove our main results, we need the following assumptions.

(H_1) There exist real constants $m_i, n_i \geq 0,\ (i=1,2)$ and $m_0>0,\ n_0>0$ such that $\forall u,v\in\mathbb{R}$, we have

$$|f(t,u,v)|\leq m_0+m_1|u|+m_2|v|,$$
$$|g(t,u,v)|\leq n_0+n_1|u|+n_2|v|.$$

(H_2) There exist nonnegative functions $\alpha(t),\ \beta(t)\in L(0,1)$ and $u,v\in\mathbb{R}$, such that

$$|f(t,u,v)|\leq\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2},\ \epsilon_1,\epsilon_2>0,\ 0<p_1,p_2<1,$$

$$|g(t,u,v)|\leq\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2},\ d_1,d_2>0,\ 0<l_1,l_2<1.$$

(H_3) There exist ℓ_1 and ℓ_2 such that for all $t\in[a,b]$ and $u_i,v_i\in\mathbb{R},\ i=1,2$, we have

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\leq\ell_1(|u_1-u_2|+|v_1-v_2|),$$

$$|g(t,u_1,v_1)-g(t,u_2,v_2)|\leq\ell_2(|u_1-u_2|+|v_1-v_2|).$$

For the sake of convenience in the forthcoming analysis, we set

$$
\begin{aligned}
q_1 \;=\;& \frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_1|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^4}{2}\\
&+|A_1|\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+|A_2|(b-a)\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.3)\\
\bar{q}_1 \;=\;& \frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\\
&+|A_1|(b-a)\sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.4)\\
q_2 \;=\;& \frac{1}{|A_3|}\Big\{\Big|\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)}\Big|\frac{(b-a)^3}{6}+|L_3|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^3}{2}\\
&\times\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)+|A_1|(b-a)\sum_{j=1}^{m}\sigma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\delta_j|L_4|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^{m}\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.5)\\
\bar{q}_2 \;=\;& \frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_4|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^4}{2}\\
&+\Big|\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}\Big|\sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\rho_j|L_3|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)\Big(\sum_{j=1}^{m}\delta_j\Big)\Big(\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j))\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.6)
\end{aligned}
$$

$$\bar{\lambda}_1=\sup_{t\in[a,b]}|\Omega_1(t)|,\quad \bar{\lambda}_2=\sup_{t\in[a,b]}|\Omega_2(t)|. \qquad (3.7)$$

Moreover, we set

$$Q_1=q_1+q_2,\quad Q_2=\bar{q}_1+\bar{q}_2,\quad \bar{\lambda}=\bar{\lambda}_1+\bar{\lambda}_2, \qquad (3.8)$$

where q_i, $\bar{q}_i$ and $\bar{\lambda}_i$ (i=1,2) are given in the equations (3.3) – (3.7) and

$$Q_0=\min\{1-(Q_1m_1+Q_2n_1),1-(Q_1m_2+Q_2n_2)\},\ m_i,n_i\geq 0\ (i=1,2). \qquad (3.9)$$

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3.1 Existence of solutions

In this subsection, we discuss the existence of solutions for the problem (1.1)-(1.2) by using standard fixed poit theorems.

Lemma 3.1 *(Leray-Schauder alternative [33]). Let $T : K \to K$ be a completely continuous operator (i.e., a map that restricted to any bounded set in K is compact). Let $\omega(T) = \{x \in K : x = \varphi T(x)$ for some $0 < \varphi < 1\}$. Then either the set $\omega(T)$ is unbounded, or T has at least one fixed point.*

Theorem 3.2 *Assume that condition (H_1) holds. In addition it is assumed that*

$$Q_1 m_1 + Q_2 n_1 < 1 \quad \text{and} \quad Q_1 m_2 + Q_2 n_2 < 1, \tag{3.10}$$

where Q_1 and Q_2 are given by (3.8). Then there exist at least one solution for problem $(1.1)-(1.2)$ on $[a, b]$

Proof. First of all, we show that the operator $\mathcal{T} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is completely continuous. Notice that the operator $\mathcal{T}$ is continuous as the functions f and g are continuous. Let $\Upsilon \subset \mathcal{X} \times \mathcal{X}$ be bounded. Then there exist positive constants κ_f and κ_g such that $|f(t, u(t), v(t))| \le \kappa_f$, $|g(t, u(t), v(t))| \le \kappa_g$, $\forall (u, v) \in \Upsilon$. Then, for any $(u, v) \in \Upsilon$, we can obtain

$$\begin{aligned}
|\mathcal{T}_1(u,v)(t)| = \quad & \sup_{t\in[a,b]} \Big| \int_a^t (t-s) f(s,u(s),v(s))ds - \frac{1}{A_3}\Big\{ \int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s) \\
& + L_1 + (b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds \\
& + \int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j) + L_2 + A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times (b-s)g(s,u(s),v(s))ds\Big\} + \frac{1}{A_3}\Big\{\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j A_1(b-a)(s-p) \\
& + \rho_j L_1 + \rho_j(b-a)A_2(t-a)\Big] g(p,u(p),v(p))dpds \\
& + \sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j A_1 \sum_{j=1}^m \gamma_j(\eta_j-\xi_j)(s-p) + \delta_j L_2 \\
& + \delta_j A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] f(p,u(p),v(p))dpds\Big\} + \Omega_1(t)\Big| \\
& \le \kappa_f \Big\{ \frac{(b-a)^2}{2} + \frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^4}{6} + |L_1|\frac{(b-a)^2}{2} + |A_2|\frac{(b-a)^4}{2} \\
& + |A_1|\Big(\sum_{j=1}^m \gamma_j\Big)\Big(\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big]\Big\}\\
&+\kappa_g\Big\{\frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}\\
&+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\\
&+|A_1|(b-a)\sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\}+\bar{\lambda}_1\\
&\le \kappa_f q_1+\kappa_g\bar{q}_1+\bar{\lambda}_1,
\end{aligned}
$$

which implies that

$$\|\mathcal{T}_1(u,v)\|\le \kappa_f q_1+\kappa_g\bar{q}_1+\bar{\lambda}_1.$$

Similarly, it can be found that

$$\|\mathcal{T}_2(u,v)\|\le \kappa_f q_2+\kappa_g\bar{q}_2+\bar{\lambda}_2.$$

Consequently, we get $\|\mathcal{T}(u,v)(t)\|\le \kappa_f Q_1+\kappa_g Q_2+\bar{\lambda}$ (Q_1 , Q_2 and $\bar{\lambda}$ are given by (3.8)), which implies that the operator $\mathcal{T}$ is uniformly bounded. Next, we show that $\mathcal{T}$ is equicontinuous. For $t_1,t_2\in[a,b]$ with $t_1<t_2$, we have

$$
\begin{aligned}
&|\mathcal{T}_1(u,v)(t_2)-\mathcal{T}_1(u,v)(t_1)|\\
\le\ &\kappa_f\Big|\int_a^{t_1}\Big[(t_2-s)-(t_1-s)\Big]ds+\int_{t_1}^{t_2}(t_2-s)ds\Big|\\
&+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\int_a^b(b-a)(b-s)ds+\Big(\sum_{j=1}^{m}\delta_j\Big)\int_{\xi_j}^{\eta_j}\int_a^s\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)dpds\Big]\\
&+\kappa_g\Big[\int_a^b\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)(b-s)ds+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\rho_j(b-a)dpds\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\lambda_4\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \kappa_f\Big[(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\Big]+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\frac{(b-a)^3}{2}\\
&+\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\kappa_g\Big[\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\frac{(b-a)^2}{2}+(b-a)\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\lambda_4\Big\}\to 0 \text{ independent of } u \text{ and } v \text{ as } (t_2-t_1)\to 0.
\end{aligned}
$$

Similarly, one can obtain

$$
\begin{aligned}
&|\mathcal{T}_2(u,v)(t_2)-\mathcal{T}_2(u,v)(t_1)|\\
&\leq \kappa_g\Big[(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\Big]+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\frac{(b-a)^3}{6}\\
&+(b-a)\sum_{j=1}^m\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\kappa_g\Big[\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^m\delta_j\Big)\Big(\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big\}\to 0 \text{ independent of } u \text{ and } v \text{ as } (t_2-t_1)\to 0.
\end{aligned}
$$

Finally, we will verify that the set $\omega=\{(u,v)\in\mathcal{X}\times\mathcal{X}|(u,v)=\varphi\mathcal{T}(u,v),0<\varphi<1\}$ is bounded. Let $(u,v)\in\omega$. Then $(u,v)=\varphi\mathcal{T}(u,v)$ and for any $t\in[a,b]$, we have

$$u(t)=\varphi\mathcal{T}_1(u,v)(t),\quad v(t)=\varphi\mathcal{T}_2(u,v)(t).$$

Then

$$
\begin{aligned}
|u(t)| &\leq q_1(m_0+m_1\|u\|+m_2\|v\|)+\bar{q}_1(n_0+n_1\|u\|+n_2\|v\|)+\bar{\lambda}_1\\
&= q_1m_0+\bar{q}_1n_0+(q_1m_1+\bar{q}_1n_1)\|u\|+(q_1m_2+\bar{q}_1n_2)\|v\|+\bar{\lambda}_1,
\end{aligned}
$$

and

$$
\begin{aligned}
|v(t)| &\leq q_2(m_0+m_1\|u\|+m_2\|v\|)+\bar{q}_2(n_0+n_1\|u\|+n_2\|v\|)+\bar{\lambda}_2\\
&= q_2m_0+\bar{q}_2n_0+(q_2m_1+\bar{q}_2n_1)\|u\|+(q_2m_2+\bar{q}_2n_2)\|v\|+\bar{\lambda}_2.
\end{aligned}
$$

Hence, we have

$$\|u\|+\|v\| \leq (q_1+q_2)m_0+(\bar{q}_1+\bar{q}_2)n_0+[(q_1+q_2)m_1+(\bar{q}_1+\bar{q}_2)n_1]\|u\|$$

$$+[(q_1+q_2)m_2+(\bar{q}_1+\bar{q}_2)n_2]\|v\|+\bar{\lambda}_1+\bar{\lambda}_2,$$

which, in view of (3.9) and (3.10), yields

$$\|(u,v)\| \leq \frac{Q_1 m_0 + Q_2 n_0 + \bar{\lambda}}{Q_0},$$

for any $t \in [a,b]$, which proves that the set ω is bounded. Hence, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Therefore, the problem (1.1) − (1.2) has at least one solution on $[a,b]$. This completes the proof. □

Next, we apply Schauder fixed point theorem to prove the existence of solutions for the problem (1.1)-(1.2) by imposing the the sub-growth condition on the nonlinear functions involved in the problem.

Theorem 3.3 *Assume that* (H_2) *holds. Then, there exist at least one solution on* $[a,b]$ *for the problem* (1.1) − (1.2).

Proof. Define a set Y in the Banach space $\mathcal{X}\times\mathcal{X}$ by

$$Y = \{(u,v)\in\mathcal{X}\times\mathcal{X} : \|(u,v)\| \leq y\},$$

where

$$y \geqslant \max\{7\bar{\lambda}, 7Q_1\alpha(t), 7Q_2\beta(t), (7Q_1\epsilon_1)^{\frac{1}{1-p_1}}, (7Q_1\epsilon_2)^{\frac{1}{1-p_2}}, (7Q_2 d_1)^{\frac{1}{1-l_1}}, (7Q_2 d_2)^{\frac{1}{1-l_1}}\}.$$

In order to show that $\mathcal{T}: Y \to Y$. We have

$$\begin{aligned}
|\mathcal{T}_1(u,v)(t)| = \quad & \sup_{t\in[a,b]}\Big|\int_a^t (t-s)f(s,u(s),v(s))ds - \frac{1}{A_3}\Big\{\int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s) \\
& +L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds \\
& +\int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times(b-s)g(s,u(s),v(s))ds\Big\} + \frac{1}{A_3}\Big\{\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j A_1(b-a)(s-p) \\
& +\rho_j L_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds \\
& +\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j A_1\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)(s-p)+\delta_j L_2 \\
& +\delta_j A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\} + \Omega_1(t)\Big|
\end{aligned}$$

$$\leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_1+\bar{\lambda}_1,$$

which implies that

$$\|\mathcal{T}_1(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_1+\bar{\lambda}_1.$$

Analogously, we have

$$\|\mathcal{T}_2(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_2+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_2+\bar{\lambda}_2.$$

In consequence,

$$\|\mathcal{T}(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)Q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)Q_2+\bar{\lambda}\leq y,$$

where Q_1 , Q_2 and $\bar{\lambda}$ are given by (3.8). Therefore, we conclude that $\mathcal{T}: Y\to Y$, where $\mathcal{T}_1(u,v)(t)$ and $\mathcal{T}_2(u,v)(t)$ are continuous on $[a,b]$.

Now we prove that $\mathcal{T}$ is completely continuous operator by fixing that

$$G=\max_{t\in[a,b]}|f(t,u(t),v(t))|, \quad H=\max_{t\in[a,b]}|g(t,u(t),v(t))|.$$

Letting $t,\tau\in[a,b]$ with $a<t<\tau<b$ and $(u,v)\in Y$, we get

$$\begin{aligned}
&|\mathcal{T}_1(u,v)(\tau)-\mathcal{T}_1(u,v)(t)|\\
&\leq G\Big[(\tau-t)(t-a)+\frac{(\tau-t)^2}{2}\Big]+\frac{(\tau-t)}{|A_1|}\Big\{G\Big[\frac{(b-a)^3}{2}\\
&+\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+H\Big[\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\frac{(b-a)^2}{2}+(b-a)\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\lambda_4\Big\}\to 0 \text{ as } (\tau-t)\to 0.
\end{aligned}$$

In a similar manner, one can obtain

$$\begin{aligned}
&|\mathcal{T}_2(u,v)(\tau)-\mathcal{T}_2(u,v)(t)|\\
&\leq H\Big[(\tau-t)(t-a)+\frac{(\tau-t)^2}{2}\Big]+\frac{(\tau-t)}{|A_1|}\Big\{G\Big[\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\frac{(b-a)^3}{6}\\
&+(b-a)\sum_{j=1}^m\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]
\end{aligned}$$

$$+H\Big[\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^{m}\delta_j\Big)\Big(\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]$$
$$+\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big\}\to 0 \text{ as } (\tau-t)\to 0.$$

Thus the operator $\mathcal{T}Y\subset Y$ is equicontinuous and uniformaly bounded set. Hence $\mathcal{T}$ is a completely continuous operator. So, by Schauder fixed point theorem, there exist a solution to the problem (1.1) − (1.2). □

3.2 Uniqueness of solutions

Here we establish the uniqueness of solutions for the problem (1.1) − (1.2) by means of Banach's contraction mapping principle.

Theorem 3.4 *Assume that* (H_3) *holds and that*

$$Q_1\ell_1+Q_2\ell_2<1, \tag{3.11}$$

where Q_1 *and* Q_2 *are given by* (3.8). *Then the problem* (1.1)−(1.2) *has a unique solution on* $[a,b]$.

Proof. Define $\sup_{t\in[a,b]}|f(t,0,0)|=N_1, \sup_{t\in[a,b]}|g(t,0,0)|=N_2$ and

$$r\geq\frac{Q_1N_1+Q_2N_2+\bar{\lambda}}{1-(Q_1\ell_1+Q_2\ell_2)}.$$

Then we show that $\mathcal{T}B_r\subset B_r$, where $B_r=\{(u,v)\in\mathcal{X}\times\mathcal{X}:\|(u,v)\|\leq r\}$. For any $(u,v)\in B_r, t\in[a,b]$, we find that

$$\begin{aligned}|f(s,u(s),v(s))| &= |f(s,u(s),v(s))-f(s,0,0)+f(s,0,0)|\\ &\leq |f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|\\ &\leq \ell_1(\|u\|+\|v\|)+N_1\leq\ell_1\|(u,v)\|+N_1\leq\ell_1 r+N_1,\end{aligned}$$

and

$$\begin{aligned}|g(s,u(s),v(s))| &= |g(s,u(s),v(s))-g(s,0,0)+g(s,0,0)|\\ &\leq |g(s,u(s),v(s))-g(s,0,0)|+|g(s,0,0)|\\ &\leq \ell_2(\|u\|+\|v\|)+N_2\leq\ell_2\|(u,v)\|+N_2\leq\ell_2 r+N_2.\end{aligned}$$

Then, for $(u,v)\in B_r$, we obtain

$$|\mathcal{T}_1(u,v)(t)|\leq \sup_{t\in[a,b]}\Big|\int_a^t(t-s)f(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)$$

$$
\begin{aligned}
&+L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds\\
&-\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]\\
&\times(b-s)g(s,u(s),v(s))ds+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p)\\
&+\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds\\
&+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\\
&+\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t)\Big|\\
&\le[\ell_1r+N_1]\times\Big\{\frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_1|\frac{(b-a)^2}{2}\\
&+|A_2|\frac{(b-a)^4}{2}+\sum_{j=1}^m\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_1|\Big(\sum_{j=1}^m\gamma_j\Big)\Big(\sum_{j=1}^m\sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+|A_2|(b-a)\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}\\
&+[\ell_2r+N_2]\times\Big\{\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}\\
&+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^m\rho_j(\eta_j-\xi_j)+|A_1|(b-a)\\
&\times\sum_{j=1}^m\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)+\sum_{j=1}^m\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}+\bar\lambda_1\\
\le\;& q_1(\ell_1r+N_1)+\bar q_1(\ell_2r+N_2)+\bar\lambda_1.
\end{aligned}
$$

Hence

$$\|\mathcal{T}_1(u,v)\|\le q_1(\ell_1r+N_1)+\bar q_1(\ell_2r+N_2)+\bar\lambda_1.$$

Likewise, we find that

$$\|\mathcal{T}_2(u,v)\|\le q_2(\ell_1r+N_1)+\bar q_2(\ell_2r+N_2)+\bar\lambda_2.$$

From the above estimates, it follows that that $\|\mathcal{T}(u,v)\| \le r$.

Next we show that the operator $\mathcal{T}$ is a contraction. For $(u_1,v_1),\ (u_2,v_2) \in \mathcal{X}\times\mathcal{X}$, we have

$$
\begin{aligned}
& |\mathcal{T}_1(u_1,v_1)(t) - \mathcal{T}_1(u_2,v_2)(t)| \\
\le\ & \sup_{t\in[a,b]}\Big\{\int_a^t (t-s)\Big|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\Big|ds \\
& +\frac{1}{|A_3|}\Big\{\int_a^b \Big[\frac{1}{2}|A_1|(b-a)(b-s) + L_1 + (b-a)|A_2|(t-a)\Big](b-s) \\
& \times\Big|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\Big|ds \\
& +\int_a^b \Big[\frac{1}{2}|A_1|(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j) + L_2 + |A_2|(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times(b-s)\Big|g(s,u_1(s),v_1(s)) - g(s,u_2(s),v_2(s))\Big|ds\Big\} \\
& +\frac{1}{|A_3|}\Big\{\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j|A_1|(b-a)(s-p) + \rho_j L_1 + \rho_j(b-a)|A_2|(t-a)\Big] \\
& \times\Big|g(p,u_1(p),v_1(p)) - g(p,u_2(p),v_2(p))\Big|dpds \\
& +\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j|A_1|\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p) + \delta_j L_2 + \delta_j|A_2|(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big] \\
& \times\Big|f(p,u_1(p),v_1(p)) - f(p,u_2(p),v_2(p))\Big|dpds\Big\} \\
\le\ & \ell_1(|u_1-u_2| + |v_1-v_2|)\times\Big\{\frac{(b-a)^2}{2} + \frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^4}{6} + |L_1|\frac{(b-a)^2}{2} \\
& +|A_2|\frac{(b-a)^4}{2} + |A_1|\Big(\sum_{j=1}^m\gamma_j\Big)\Big(\sum_{j=1}^m\sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big) \\
& +\sum_{j=1}^m\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big) \\
& +|A_2|(b-a)\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\} \\
& +\ell_2(|u_1-u_2| + |v_1-v_2|)\times\Big\{\frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^m\gamma_j(\eta_j-\xi_j) + |L_2|\frac{(b-a)^2}{2} \\
& +|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^m\rho_j(\eta_j-\xi_j) + |A_1|(b-a)\sum_{j=1}^m\gamma_j\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\}\\
\le\ &(\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|),
\end{aligned}
$$

which yields

$$\|\mathcal{T}_1(u_1,v_1)-\mathcal{T}_1(u_2,v_2)\|\le(\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|).$$

Similarly,

$$\|\mathcal{T}_2(u_1,v_1)-\mathcal{T}_2(u_2,v_2)\|\le(\ell_1 q_2+\ell_2\bar{q}_2)(|u_1-u_2|+|v_1-v_2|).$$

So, it follows from the above inequalities that

$$\|\mathcal{T}(u_1,v_1)-\mathcal{T}(u_2,v_2)\|\le(Q_1\ell_1+Q_2\ell_2)(\|u_1-u_2\|+\|v_1-v_2\|),$$

where Q_1 and Q_2 are given by (3.8). By the given assumption (3.11), it follows that the operator $\mathcal{T}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{T}$ has a fixed point, which corresponds to a unique solution of the problem (1.1)-(1.2) on $[a,b]$. □

Example 3.5 *Consider the following second order system of ordinary differential equations*

$$
\begin{cases}
u''(t)=\dfrac{1}{10+t^2}\Big(\dfrac{|u|}{1+|u|}+v(t)\Big)+e^{-t}, & t\in[2,3],\\
v''(t)=\dfrac{1}{3\sqrt{32+t^2}}\Big(u(t)+\tan^{-1}v(t)\Big)+\cos(t-2), & t\in[2,3],
\end{cases}
\tag{3.12}
$$

subject to the boundary conditions

$$
\begin{cases}
\displaystyle\int_2^3 u(s)ds=\sum_{j=1}^{3}\gamma_j\int_{\xi_j}^{\eta_j}v(s)ds+2,\ \int_2^3 u'(s)ds=\sum_{j=1}^{3}\rho_j\int_{\xi_j}^{\eta_j}v'(s)ds+1,\\
\displaystyle\int_2^3 v(s)ds=\sum_{j=1}^{3}\sigma_j\int_{\xi_j}^{\eta_j}u(s)ds+\frac{3}{2},\ \int_2^3 v'(s)ds=\sum_{j=1}^{3}\delta_j\int_{\xi_j}^{\eta_j}u'(s)ds+\frac{1}{2},
\end{cases}
\tag{3.13}
$$

where $a=2, b=3, m=3, \lambda_1=2, \lambda_2=1, \lambda_3=3/2, \lambda_4=1/2, \gamma_1=2/5, \gamma_2=21/40, \gamma_3=13/20, \rho_1=1/3, \rho_2=1/2, \rho_3=2/3, \sigma_1=3/7, \sigma_2=5/7, \sigma_3=1, \delta_1=3/8, \delta_2=5/8, \delta_3=7/8, \xi_1=15/7, \eta_1=16/7, \xi_2=17/7, \eta_2=18/7, \xi_3=19/7, \eta_3=20/7$.

Using the given data, we find that $\ell_1=\frac{1}{7}, \ell_2=\frac{1}{9}$, $A_1\approx 0.827806\neq 0, A_2\approx 0.793367\neq 0, A_3\approx 0.656754, |L_1|=0.03337, |L_2|\approx 0.225389, |L_3|\approx 0.027121, |L_4|\approx 0.185097$, $q_1\approx 1.963984, q_2\approx 1.422591, \bar{q}_1\approx 1.290164$ and $\bar{q}_2\approx 1.851349$. Also $Q_1\ell_1+Q_2\ell_2\approx 0.832853<1$ (Q_1 and Q_2 are given by (3.8)). Thus, all the conditions of Theorem 3.4 are satisfied. Hence it follows by the conclusion of Theorem 3.4 that the problem $(3.12)-(3.13)$ has a unique solution on $[2,3]$.

4 Conclusions

The salient features of this work includes *(i)* considering a coupled system of nonlinear ordinary differential equations on an arbitrary domain *(ii)* a new kind of integral multi-strip coupled boundary conditions. The results obtained for the given problem are new and significantly contribute to the existing literature on the topic. As a special case, our results correspond to the uncoupled integral boundary conditions of the form:

$$\int_a^b u(s)ds = \lambda_1, \int_a^b u'(s)ds = \lambda_2; \int_a^b v(s)ds = \lambda_3, \int_a^b v'(s)ds = \lambda_4,$$

if we take all $\gamma_j = 0, \rho_j = 0, \sigma_j = 0, \delta_j = 0$ $(j = 1, \dots, m)$ in the results of this paper.

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Explicit identities involving truncated exponential polynomials and phenomenon of scattering of their zeros

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Abstract : In this paper, we study differential equations arising from the generating functions of truncated exponential polynomials. We give explicit identities for the truncated polynomials. Using numerical investigation, we observe the behavior of complex roots of the truncated polynomials $e_n(x)$. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the truncated polynomials $e_n(x)$.

Key words : Differential equations, complex roots, truncated polynomials.

AMS Mathematics Subject Classification : 05A19, 11B83, 34A30, 65L99.

1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, tangent numbers and polynomials, Genocchi numbers and polynomials, Laguerre polynomials, and Hermite polynomials. These numbers and polynomials possess many interesting properties and arising in many areas of mathematics, physics, and applied engineering(see [1-14]). By using software, many mathematicians can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concept. Numerical experiments of Euler polynomials, Bernoulli polynomials, tangent polynomials, Genocchi polynomials, Laguerre polynomials, and Hermite polynomials have been the subject of extensive study in recent year and much progress have been made both mathematically and computationally. Using computer, a realistic study for the zeros of truncated polynomials $e_n(x)$ is very interesting. The main purpose of this paper is to observe an interesting phenomenon of 'scattering' of the zeros of the truncated polynomials $e_n(x)$ in complex plane. Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. We first give the definitions of the truncated exponential polynomials. It should be mentioned that the definition of truncated exponential polynomials $e_n(x)$ can be found in [1, 3]. The truncated exponential polynomials $e_n(x)$ are defined by means of the generating function:

$$\left(\frac{1}{1-t}\right)e^{xt} = \sum_{n=0}^{\infty} e_n(x)t^n, \quad |t| < 1. \tag{1.1}$$

We recall that G. Dattoli and M. Migliorati(see [3]) studied some properties of truncated exponential polynomials $e_n(x)$. The truncated exponential polynomials $e_n(x)$ satisfy the following relations

$$\frac{d}{dx}e_n(x) = e_{n-1}(x),$$

$$e_{n+1}(x) = \left(1 + \frac{x}{n+1}\left(1 - \frac{d}{dx}\right)\right)e_n(x).$$

The Miller-Lee polynomials $G_n^{(k)}(x)$(see [1]), are defined by means of the following generating function

$$\left(\frac{1}{1-t}\right)^{k+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)t^n. \tag{1.2}$$

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials. In this paper, we study linear differential equations arising from the generating functions of truncated exponential polynomials $e_n(x)$. We give explicit identities for truncated exponential polynomials $e_n(x)$.

2. Differential equations associated with truncated exponential polynomials

In this section, we study linear differential equations arising from the generating functions of truncated exponential polynomials. Let

$$F = F(t,x) = \left(\frac{1}{1-t}\right) e^{xt}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,x) = \frac{d}{dt}\left(\frac{1}{1-t}\right) e^{xt} \\ &= \left(\frac{1}{1-t}\right)^2 e^{xt} + x\left(\frac{1}{1-t}\right) e^{xt} \\ &= \left(\frac{1}{1-t} + x\right) F(t,x), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^{(2)} &= \left(\frac{d}{dt}\right)^2 F(t,x) \\ &= \left(\frac{1}{1-t}\right)^2 e^{xt}F(t,x) + \left(\frac{1}{1-t} + x\right) F(t,x)F^{(1)} \\ &= \left(\left(\frac{2}{1-t}\right)^2 + \left(\frac{2x}{1-t} + x^2\right)\right) F(t,x). \end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right) F(t,x), \quad (N = 0,1,2,\ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we obtain

$$\begin{aligned}
F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\
&= \left(\sum_{i=0}^{N} ia_i(N,x)(1-t)^{-i-1}\right) F(t,x) + \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right) F^{(1)}(t,x) \\
&= \left(\sum_{i=0}^{N} ia_i(N,x)(1-t)^{-i-1}\right) F(t,x) \\
&\quad + \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right) \left((1-t)^{-1}+x\right) F(t,x) \\
&= \left(\sum_{i=0}^{N} (i+1)a_i(N,x)(1-t)^{-i-1}\right) F(t,x) + \left(\sum_{i=0}^{N} xa_i(N,x)(1-t)^{-i}\right) F(t,x) \\
&= \left(\sum_{i=0}^{N} xa_i(N,x)(1-t)^{-i}\right) F(t,x) + \left(\sum_{i=1}^{N+1} ia_{i-1}(N,x)(1-t)^{-i}\right) F(t,x).
\end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1,x)(1-t)^{-i}\right) F(t,x). \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}
&\left(\sum_{i=0}^{N} xa_i(N,x)(1-t)^{-i}\right) F(t,x) + \left(\sum_{i=1}^{N+1} ia_{i-1}(N,x)(1-t)^{-i}\right) F(t,x) \\
&= \left(\sum_{i=0}^{N+1} a_i(N+1,x)(1-t)^{-i}\right) F(t,x)..
\end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$\begin{aligned}
a_0(N+1,x) &= xa_0(N,x), \\
a_{N+1}(N+1,x) &= (N+1)a_N(N,x),
\end{aligned} \tag{2.8}$$

and

$$a_i(N+1,x) = xa_i(N,x) + ia_{i-1}(N,x), (1 \le i \le N). \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$F = F^{(0)} = a_0(0,x)F(t,x) = F(t,x). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0,x) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned}
&(1-t)^{-1}F(t,x) + xF(t,x) \\
&= \sum_{i=0}^{1} a_i(1,x)(1-t)^{-i}F(t,x) \\
&= a_0(1,x)F(t,x) + a_1(1,x)(1-t)^{-1}F(t,x).
\end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1,x) = x, \quad a_1(1,x) = 1. \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,x) = xa_0(N,x) = x^2 a_0(N-1,x) = \cdots = x^{N+1},$$

and

$$a_{N+1}(N+1,x) = (N+1)a_N(N,x) = \cdots = (N+1)!. \tag{2.14}$$

For $i = 1,2,3$ in (2.9), we get

$$\begin{aligned} a_1(N+1,x) &= \sum_{k=0}^{N} x^k a_0(N-k,x), \\ a_2(N+1,x) &= 2\sum_{k=0}^{N-1} x^k a_1(N-k,x), \text{ and} \\ a_3(N+1,x) &= 3\sum_{k=0}^{N-2} x^k a_2(N-k,x). \end{aligned}$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x) = i\sum_{k=0}^{N-i+1} x^k a_{i-1}(N-k,x). \tag{2.15}$$

Now, we give explicit expressions for $a_i(N+1,x)$. By (2.14) and (2.15), we get

$$a_1(N+1,x) = \sum_{k_1=0}^{N} x^{k_1} a_0(N-k_1,x) = x^N(N+1),$$

$$a_2(N+1,x) = 2\sum_{k_1=0}^{N-1} x^{k_1} a_1(N-k_1,x) = 2!\sum_{k_1=0}^{N-1} x^{N-1}(N-k_1),$$

and

$$\begin{aligned} a_3(N+1,x) &= 3\sum_{k_2=0}^{N-2} x^{k_2} a_2(N-k_2,x) \\ &= 3!\sum_{k_2=0}^{N-2}\sum_{k_1=0}^{N-k_2-2} x^{N-k_2-2}(N-k_2-k_1-1). \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} a_i(N+1,x) = i!\sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-k_{i-1}-i+1}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i+1} & x^{N-k_i-\cdots-k_2-i+1} \\ \times (N-k_{i-1}-k_{i-2}-\cdots-k_2-k_1-i+2). & \end{aligned} \tag{2.16}$$

Note that, here the matrix $a_i(j,x)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{N+1} \\ 0 & 1! & 2x & \cdot & \cdots & (N+1)x^N \\ 0 & 0 & 2! & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 3! & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N+1)! \end{pmatrix}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2, \dots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N, x) \left(\frac{1}{1-t} \right)^i \right) F$$

has a solution

$$F = F(t, x) = \left(\frac{1}{1-t} \right) e^{xt},$$

where

$$\begin{aligned} &a_0(N, x) = x^N, \\ &a_N(N, x) = N!, \\ &a_i(N, x) = i! \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-k_{i-1}-i} \cdots \sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} x^{N-k_{i-1}-\cdots-k_2-i} \\ &\qquad\qquad \times (N - k_{i-1} - k_{i-2} - \cdots - k_2 - k_1 - i + 1), \\ &(1 \le i \le N). \end{aligned}$$

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt} \right)^N F(t, x) = \sum_{k=0}^{\infty} \frac{(k+N)!}{k!} e_{k+N}(x) t^k. \tag{2.17}$$

From Theorem 1, (1.2), and (2.17), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(k+N)!}{k!} e_{k+N}(x) t^k &= \left(\sum_{i=0}^{N} a_i(N, x) \left(\frac{1}{1-t} \right)^i \right) F \\ &= \sum_{i=0}^{N} a_i(N, x) \left(\frac{1}{1-t} \right)^{i+1} e^{xt} \\ &= \sum_{i=0}^{N} a_i(N, x) \left(\sum_{k=0}^{\infty} G_k^{(i)}(x) t^k \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} a_i(N, x) G_k^{(i)}(x) \right) t^k. \end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2. For $k = 0, 1, \dots,$ and $N = 0, 1, 2, \dots$, we have

$$e_{k+N}(x) = \frac{k!}{(k+N)!} \sum_{i=0}^{N} a_i(N, x) G_k^{(i)}(x), \tag{2.19}$$

where

$$\begin{aligned} &a_0(N, x) = x^N, \\ &a_N(N, x) = N!, \\ &a_i(N, x) = i! \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-k_{i-1}-i} \cdots \sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} x^{N-k_{i-1}-\cdots-k_2-i} \\ &\qquad\qquad \times (N - k_{i-1} - k_{i-2} - \cdots - k_2 - k_1 - i + 1), \\ &(1 \le i \le N). \end{aligned}$$

Let us take $k = 0$ in (2.19). Then, we have the following corollary.

Corollary 3. For $N = 0, 1, 2, \dots$, we have

$$e_N(x) = \frac{1}{N!}\sum_{i=0}^{N} a_i(N,x)G_0^{(i)}(x).$$

For $N = 1, 2, \dots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N,x)\left(\frac{1}{1-t}\right)^i\right)F$$

has a solution

$$F = F(t,x) = \left(\frac{1}{1-t}\right)e^{xt}.$$

Here is a plot of the surface for this solution.

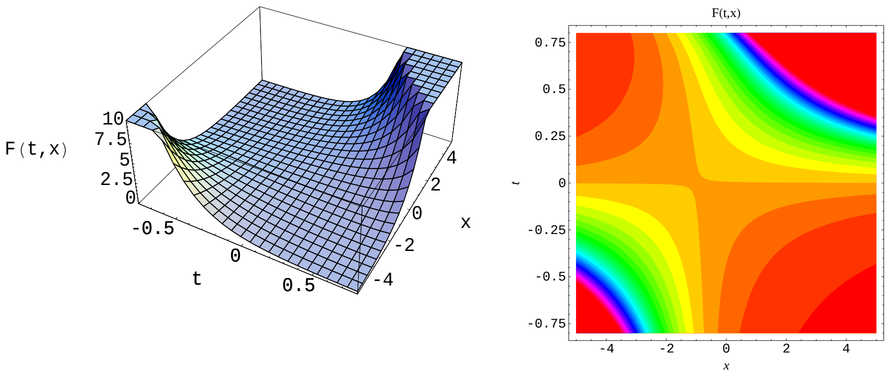

Figure 1: The surface for the solution $F(t,x)$

In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

3. Zeros of the truncated exponential polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the truncated exponential polynomials $e_n(x)$. By using computer, the truncated exponential polynomials $e_n(x)$ can be determined explicitly. We display the shapes of the truncated exponential polynomials $e_n(x)$ and investigate the zeros of the truncated exponential polynomials $e_n(x)$. We investigate the beautiful zeros of the truncated exponential polynomials $e_n(x)$ by using a computer. We plot the zeros of the $e_n(x)$ for $n = 20, 30, 40, 50$ and $x \in \mathbb{C}$(Figure 2). In Figure 2(top-left), we choose $n = 20$. In Figure 2(top-right), we choose $n = 30$. In Figure 2(bottom-left), we choose $n = 40$. In Figure 2(bottom-right), we choose $n = 50$.

Stacks of zeros of $e_n(x)$ for $1 \le n \le 40$, forming a 3D structure are presented(Figure 3). In Figure 3(top-left), we plot stacks of zeros of $e_n(x)$ for $1 \le n \le 40$. In Figure 3(top-right), we draw x and y axes but no z axis in three dimensions. In Figure 3(bottom-left), we draw y and z axes but no x axis in three dimensions. In Figure 3(bottom-right), we draw x and z axes but no y axis in three dimensions.

Our numerical results for approximate solutions of real zeros of the truncated exponential polynomials $e_n(x)$ are displayed(Tables 1, 2).

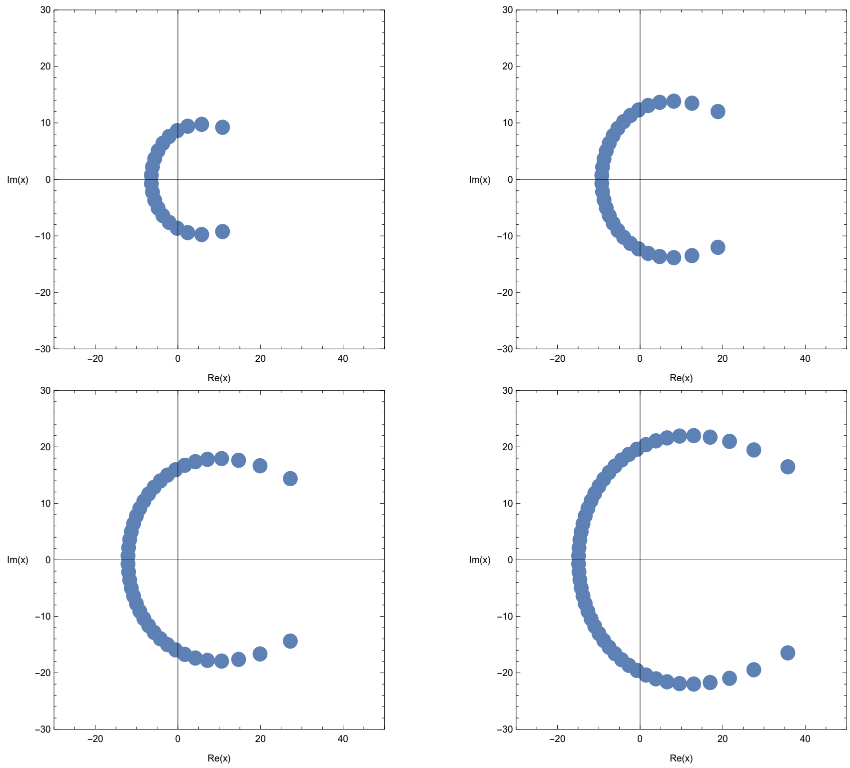

Figure 2: Zeros of $e_n(x)$

Table 1. Numbers of real and complex zeros of $e_n(x)$

degree n	real zeros	complex zeros
1	1	0
2	0	2
3	1	2
4	0	4
5	1	4
6	0	6
7	1	6
8	0	8
9	1	8
10	0	10
11	1	10
12	0	12
13	1	12
14	0	14

How many zeros does $e_n(x)$ have? We are not able to decide if $e_n(x)$ has n distinct solutions(see Table 1, Table 2). We would also like to know the number of complex zeros $C_{e_n(x)}$ of $e_n(x), Im(x) \neq 0$. Since n is the degree of the polynomial $e_n(x)$, the number of real zeros $R_{e_n(x)}$ lying on the real line $Im(x) = 0$ is then $R_{e_n(x)} = n - C_{e_n(x)}$, where $C_{e_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{e_n(x)}$ and $C_{e_n(x)}$.

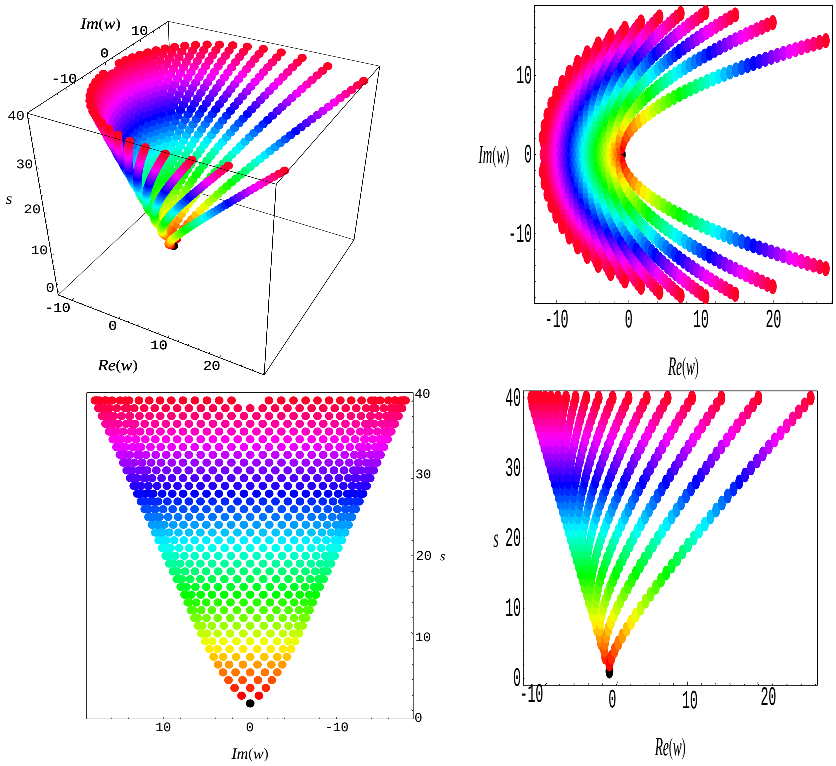

Figure 3: Stacks of zeros of $e_n(x)$ for $1 \leq n \leq 40$

Conjecture 5. Prove that $e_n(x) = 0$ has n distinct solutions.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n. Since n is the degree of the polynomial $e_n(x)$, the number of real zeros $R_{e_n(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{e_n(x)} = n - C_{e_n(x)}$, where $C_{e_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{e_n(x)}$ and $C_{e_n(x)}$.

Conjecture 6. Prove that the numbers of complex zeros $C_{e_n(x)}$ of $e_n(x), Im(x) \neq 0$ is

$$C_{e_n(x)} = 2\left[\frac{n}{2}\right],$$

where [ ] denotes taking the integer part.

Conjecture 7. For $n \in \mathbb{N}_0$, if $n \equiv 1 \pmod 2$, then $R_{e_n(x)} = 1$, if $n \equiv 0 \pmod 2$, then $R_{e_n(x)} = 0$.

The plot of real zeros of the truncated exponential polynomials $e_n(x)$ for $1 \leq n \leq 50$ structure are presented(Figure 4). It is expected that $e_n(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions (see Figure 2, Figure 3, Figure 4). For $a \in \mathbb{R}$, we expect that $e_n(x), x \in \mathbb{C}$, has not $Re(x) = a$ reflection symmetry analytic complex functions. We observe a remarkable regular structure of the complex roots of the truncated exponential polynomials $e_n(x)$. We also hope to verify a remarkable regular structure of the complex roots of the truncated exponential polynomials $e_n(x)$(Table 1). Next, we calculated an approximate solution satisfying $e_n(x) = 0, x \in \mathbb{C}$. The

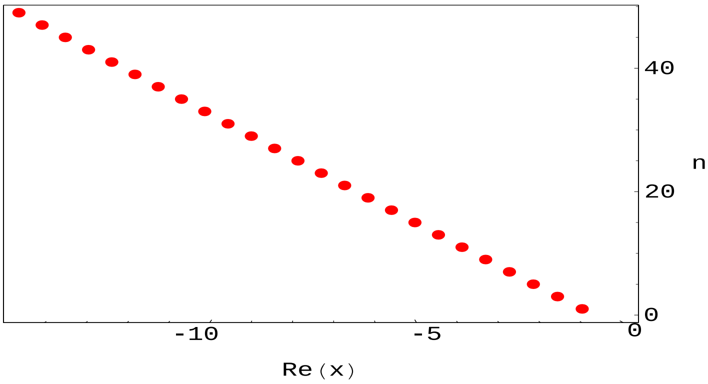

Figure 4: Real zeros of $e_n(x)$ for $1 \leq n \leq 50$

results are given in Table 2.

Table 2. Approximate solutions of $e_n(x) = 0, x \in \mathbb{C}$

degree n	x
1	-1.0000
2	$-1.0000 - 1.0000i$, $-1.0000 + 1.0000i$
3	-1.5961, $-0.7020 - 1.8073i$, $-0.7020 + 1.8073i$
4	$-1.7294 - 0.8890i$, $-1.7294 + 0.8890i$ $-0.2706 - 2.5048i$, $-0.2706 + 2.5048i$
5	-2.1806, $-1.6495 - 1.6939i$, $-1.6495 + 1.6939i$ $0.2398 - 3.1283i$, $0.2398 + 3.1283i$
6	$-2.3618 - 0.8384i$, $-2.3618 + 0.8384i$, $-1.4418 - 2.4345i$ $-1.4418 + 2.4345i$, $0.8036 - 3.6977i$, $0.8036 + 3.6977i$

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On generalized degenerate twisted (h, q)-tangent numbers and polynomials

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Abstract : We introduced the generalized twisted (h, q)-tangent numbers and polynomials. In this paper, our goal is to give generating functions of the generalized degenerate twisted (h, q)-tangent numbers and polynomials. We also obtain some explicit formulas for generalized degenerate twisted (h, q)-tangent numbers and polynomials.

Key words : Generalized tangent numbers and polynomials, degenerate generalized twisted (h, q)-tangent numbers and polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-16]). In [2], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.*[3] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. In this paper, we obtain some interesting properties for generalized degenerate tangent numbers and polynomials. Throughout this paper we use the following notations. Let p be a fixed odd prime number. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let r be a positive integer, and let ζ be rth root of 1. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. Then the generalized twisted (h, q)-tangent numbers associated with associated with χ, $T^{(h)}_{n,\chi,q,\zeta}$, are defined by the following generating function

$$\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}\frac{t^n}{n!}. \tag{1.1}$$

We now consider the generalized twisted (h, q)-tangent polynomials associated with χ, $T^{(h)}_{n,\chi,q,\zeta}(x)$, are also defined by

$$\left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}\right)e^{xt} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\frac{t^n}{n!}. \tag{1.2}$$

When $\chi = \chi^0$, above (1.1) and (1.2) will become the corresponding definitions of the twisted (h, q)-tangent numbers $T^{(h)}_{n,q,w}$ and polynomials $T^{(h)}_{n,q,w}(x)$. If $q \to 1$, above (1.1) and (1.2) will become the corresponding definitions of the generalized twisted tangent numbers $T_{n,\chi,w}$ and polynomials $T_{n,\chi,w}(x)$. We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.3}$$

We also have

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.5}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.6}$$

2. On the generalized degenerate twisted (h,q)-tangent polynomials

In this section, we define the generalized degenerate twisted (h,q)-tangent numbers and polynomials, and we obtain explicit formulas for them. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\text{mod } 2)$, and let ζ be rth root of 1. For $h \in \mathbb{Z}$, the generalized degenerate twisted (h,q)-tangent polynomials associated with associated with χ, $T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)$, are defined by the following generating function

$$\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} \tag{2.1}$$

and their values at $x=0$ are called the generalized degenerate twisted (h,q)-tangent numbers and denoted $T^{(h)}_{n,\chi,q,\zeta}(\lambda)$.

From (2.1) and (1.2), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty}\lim_{\lambda\to 0} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&= \left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}\right)e^{xt}\\
&= \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we get

$$\lim_{\lambda\to 0} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = T^{(h)}_{n,\chi,q,\zeta}(x), (n \geq 0).$$

From (2.1) and (1.6), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&= \left(\sum_{m=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l\frac{t^l}{l!}\right) \qquad (2.2)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} T^{(h)}_{l,\chi,q,\zeta}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing coefficients of $\dfrac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 1. For $n \geq 0$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} T^{(h)}_{l,\chi,q,\zeta}(\lambda)(x|\lambda)_{n-l}.$$

For $\chi = \chi^0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2}{\zeta q^h(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= \sum_{m=0}^{\infty} T^{(h)}_{n,q,\zeta}(x|\lambda)\frac{t^m}{m!}.\end{aligned} \tag{2.3}$$

Theorem 2. For $n \geq 0$ and $\chi = \chi^0$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = T^{(h)}_{n,q,\zeta}(x|\lambda).$$

For $d \in \mathbb{N}$ with $d \equiv 1(\text{mod}2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= \frac{2}{\zeta q^h(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^l\chi(l)(1+\lambda t)^{2l/\lambda} \\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}|\frac{\lambda}{d}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By comparing coefficients of $\dfrac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 3. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\text{mod } 2)$. Then we have

$$(1)\ T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}|\frac{\lambda}{d}\right),$$

$$(2)\ T^{(h)}_{n,\chi,q,\zeta}(\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}\right).$$

For $m \in \mathbb{Z}_+$, we obtain we can derive the following relation:

$$\begin{aligned}&\sum_{m=0}^{\infty}\zeta^d q^{hd}T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda)\frac{t^m}{m!} + \sum_{m=0}^{\infty}T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda)\frac{t^m}{m!} \\ &= 2\sum_{l=0}^{d-1}(-1)^l\chi(l)\zeta^l q^{hl}(1+\lambda t)^{2l/\lambda} \\ &= \sum_{m=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^{n-1-l}\chi(l)\zeta^l q^{hl}(2l|\lambda)_m\right)\frac{t^m}{m!}.\end{aligned} \tag{2.5}$$

By comparing of the coefficients $\frac{t^m}{m!}$ on the both sides of (2.5), we have the following theorem.

Theorem 4. For $m \in \mathbb{Z}_+$, we have

$$\zeta^d q^{hd} T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda) + T^{(h)}_{m,\chi,q,\zeta}(\lambda) = 2\sum_{l=0}^{d-1}(-1)^l \chi(l)\zeta^l q^{hl}(2l|\lambda)_m.$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x+y|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda} \\ &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{(2a+x)/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{y/\lambda} \\ &= \left(\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T^{(h)}_{l,\chi,q,\zeta}(x|\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.6), we have the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x+y|\lambda) = \sum_{k=0}^{n}\binom{n}{k}T^{(h)}_{k\chi,q,\zeta}(x|\lambda)(y|\lambda)_{n-k}.$$

From Theorem 5, we note that $T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)$ is a Sheffer sequence.

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (2.1), we obtain

$$\begin{aligned}\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}e^{xt} &= \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n\frac{1}{n!} \\ &= \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^m\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.7}$$

Thus, by (2.7) and (1.2), we have the following theorem.

Theorem 6. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x) = \sum_{n=0}^{m}\lambda^{m-n}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{(2a+x)/\lambda}}{\zeta^d q^{hd}(1+\lambda t)^{2d/\lambda}+1} \\ &= \sum_{m=0}^{\infty} T^{(h)}_{m,\chi,q,\zeta}(x|\lambda)\frac{t^m}{m!},\end{aligned} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}T^{(h)}_{n,\chi,q,\zeta}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}. \tag{2.9}$$

Thus, by (2.8) and (2.9), we have the following theorem.

Theorem 8. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x|\lambda) = \sum_{n=0}^{m} T^{(h)}_{n,\chi,q,\zeta}(x)\lambda^{m-n} S_1(m,n).$$

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New Oscillation Criteria of First Order Neutral Delay Difference Equations of Emden–Fowler Type

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Abstract

In this paper, we will establish some new sufficient condition for oscillation of solutions of a certain class of first-order neutral delay difference equations of the form

$$\Delta\left(x_n - p_n x_{n-1}\right) + q_n x_{n-\tau}^{\gamma} = 0,$$

where γ is a quotient of odd positive integers. We will consider the sublinear and super linear cases. The results will be obtained by using the oscillation theorems of second order delay difference equations.

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1 Introduction

In recent decades there has been much research activity concerning oscillation and nonoscillation of first and second order delay and neutral delay difference equations, we refer the reader to the papers [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and the references cited therein. In the following, we recall some results of first order neutral delay difference equations of sublinear and super linear types that motivate the contents of this paper. Xiaoyan Lin in [12] studied the oscillatory behavior of solutions of the neutral difference equations with nonlinear neutral term of the form

$$\Delta\left(x_n - p_n x_{n-\sigma}^{\alpha}\right) + q_n x_{n-\tau}^{\beta} = 0, \text{ for } n \in \mathbb{N}_{n_0}, \tag{1.1}$$

where α and β are quotient of odd positive integers, τ and σ are nonnegative integers and $\{p_n\}$ and $\{q_n\}$ are two sequences of nonnegative real numbers. The authors obtained necessary and sufficient conditions for existence of oscillatory solutions and studied the two cases when $0 < \alpha < 1$ and when $\alpha > 1$. As usual, a nontrivial solution x_n of (1.1) is called nonoscillatory if it eventually positive or eventually negative, otherwise it is called oscillatory and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\mathbb{N}_i = \{i+1, i+2, ...\}$. Lalli [11] established several sufficient conditions for oscillation of the equation

$$\Delta\left(x_n + p x_{n-\delta k}\right) + q_n f\left(x_{n-\tau}\right) = F_n, \; n \geq n_0, \tag{1.2}$$

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where $\delta = \pm 1$, p is a nonnegative real number, $k \in \mathbb{N} = \{1, 2, ...\}$, τ is a sequence of nonnegative integers with $\lim_{n\to\infty} \tau_n = \infty$, and $\{F_n\}$, $\{q_n\}$ are sequences of real numbers and f is a real valued function satisfying $xf(x) > 0$ for $x \neq 0$. El-Morshedy et al. [6] considered the equation

$$\Delta g\left(x_n + p_n x_{\sigma_n}\right) + f\left(n, x_{\tau_n}\right) = 0, \tag{1.3}$$

where $0 \leq p_n < p < 1$, σ_n and τ_n are sequences of integers such that $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} = \infty$ and $\sigma_{n+1} > \sigma_n$ for all $n \geq n_0$. They established several sufficient conditions for oscillation when the function f satisfies the condition

$$\frac{f(n,x)}{h(x)} \geq q_n, \ x \neq 0 \text{ and } n \geq n_0,$$

where $q_n \geq 0$ for $n \geq n_0$, $h \in C(\mathbb{R},\mathbb{R})$ and $xh(x) > 0$ for all $x \neq 0$. Recently Murugesan and Suganthi [13] discussed the oscillatory behavior of all solutions of the first order nonlinear neutral delay difference equation

$$\left[\Delta\left(r_n\left(a_n x_n - p_n x_{n-\tau}\right)\right)\right] + q_n x_{n-\sigma} = 0,$$

where r_n and a_n are sequences of positive real numbers p_n and q_n are sequences of non-negative real numbers, τ and σ are positive integers. Following this trend in this paper, we will consider the first order neutral delay difference equation

$$\Delta\left(x_n - p_n x_{n-1}\right) + q_n x_{n-\tau}^{\gamma} = 0, \text{ for } n \in \mathbb{N}_{n_0}, \tag{1.4}$$

Our aim in this paper is to establish some new sufficient conditions for oscillation of (1.4) by using a new technique when $0 < p_n \leq p \leq 1$ and we will consider the sublinear and the super linear cases. The new technique depends on the application of an invariant substitution which transforms the first nonlinear neutral difference equation to a second nonlinear difference equation. This allows us to obtain several sufficient conditions for oscillation of (1.4) by employing the oscillation conditions of second order delay difference equations by using the Riccati technique.

2 Main results

In this section, we prove the main results but before we do this, we apply an invariant substitution which transforms the first order neutral equation to a non-neutral second order difference equations. This substitution is given by

$$y_{n+1} = x_n \prod_{i=1}^{n} \frac{1}{p_i}, \quad \text{where} \quad \prod_{i=1}^{n} p_i = O(n), \tag{2.1}$$

This gives us that

$$x_n = y_{n+1} \prod_{i=1}^{n} p_i, \quad x_{n-1} = y_n \prod_{i=1}^{n-1} p_i, \quad \text{and } x_{n-\tau} = y_{n-\tau+1} \prod_{i=1}^{n-\tau} p_i. \tag{2.2}$$

From (2.2), we have

$$x_n - p_n x_{n-1} = \Delta y_n \prod_{i=1}^{n} p_i. \tag{2.3}$$

Substituting (2.3) into (1.4), we obtain

$$\Delta\left(\Delta y_n \prod_{i=1}^{n} p_i\right) + q_n \prod_{i=1}^{n-\tau} p_i y_{n-\tau+1} = 0. \tag{2.4}$$

Setting $d_n = \prod_{i=1}^{n} p_i$, and $Q_n = q_n d_{n-\tau}$ then (2.4) becomes

$$\Delta\left(d_n \Delta y_n\right) + Q_n y^{\gamma}_{n-(\tau-1)} = 0,\ n \in \mathbb{N}_0. \tag{2.5}$$

In this section, we intend to use the Riccati transformation technique for obtaining several new oscillation criteria for (1.4). First we state some fundamental lemmas for second order difference equations that will be used in the proofs of the main results (see [15]).

Lemma 2.1 *Assume that p_n is a real sequence with $0 < p_n \leq p < 1$ for all $n \in \mathbb{N}$. Furthermore assume that*

$$\sum_{n=1}^{\infty} \frac{1}{d_n} = \infty. \tag{2.6}$$

Let y be a positive solution of (2.5). Then
(I). $\Delta y(n) \geq 0$, $y(n) \geq n\Delta y(n)$ for $n \geq N$,
(II). y is nondecreasing, while $y(n)/n$ is nonincreasing for $n \geq N$.

Lemma 2.2 *Assume that p_n is a real sequence with $0 < p_n \leq p < 1$ for all $n \in \mathbb{N}$. Furthermore assume that (2.6) holds. If y_n be a nonoscillatory solution of (2.5) such that $y_n \geq 0$, $\Delta y_n \leq 0$, then $\lim_{n\to\infty} y_n = 0$ and hence*

$$\lim_{n\to\infty} \frac{x_n}{d_n} = 0, \tag{2.7}$$

where x_n is a solution of (1.4).

Throughout this paper, we will assume that the real sequences p_n, q_n are nonnegative, γ is a quotient of odd positive integers, τ is a nonnegative integer. Now, we state and prove the sufficient conditions which ensure that each solution of equation (1.4) is oscillatory or satisfies (2.7). We start with the case when $0 < \gamma \leq 1$.

Theorem 2.3 *Assume that (H_1) holds and $\Delta d_n \geq 0$. Furthermore, assume that there exists a positive sequence ρ_n such that,*

$$\lim_{n\to\infty} \sup \sum_{i=n_0}^{n} \left[\rho_i Q_i - \frac{d_{i-\tau+1}\beta^{1-\gamma}\left(i+2-\tau\right)^{1-\gamma}\left(\Delta\rho_i\right)^2}{\rho_i}\right] = \infty, \tag{2.8}$$

where $d_n = \prod_{i=1}^{n} p_i$ and $Q_n = q_n d_{n-\tau}$. Then every solution of (1.4) oscillates for all $0 < \gamma \leq 1$.

Proof. Assume to the contrary that x_n be a nonoscillatory solution of (1.4) such that x_{n-1}, $x_{n-\tau}$, $x_n > 0$ for all large $n \geq n_1 > n_0$ sufficiently large. We shall consider only this case, since the substitution $y_n = -x_n$ transforms equation (1.4) into an equation of the same form. From (2.1) we see that y_n is a positive solution of (2.5) such that $y_n > 0$ and $y_{n-\tau+1} > 0$ for $n > n_1 > n_0$ sufficiently large. From equation (2.5), we have

$$\Delta\left(d_n \Delta y_n\right) = -Q_n y^{\gamma}_{n-\tau+1} \leq 0,\ n \geq n_1, \tag{2.9}$$

and then $d_n\Delta y_n$ is an eventually nonincreasing sequence. We first show that $d_n\Delta y_n \geq 0$ for $n \geq n_0$. In fact, if there exists an integer $n_1 \geq n_0$ such that $d_{n_1}\Delta y_{n_1} = c < 0$ then (2.9) implies that $d_n\Delta y_n \leq c$ for $n \geq n_1$ that is $\Delta y_n \leq c/d_n$, and hence

$$y_n \leq y_{n_1} + c\sum_{i=n_1}^{n-1}\frac{1}{d_i} \to -\infty, \text{ as } n \to \infty, \tag{2.10}$$

which contradicts the fact that $y_n > 0$ for $n \geq n_0$ then $d_n\Delta y_n \geq 0$. Also since $\Delta d_n \geq 0$, we can prove that $\Delta^2 y_n > 0$ for $n \geq n_1$. Therefore we have

$$y_n > 0,\ \Delta y_n \geq 0,\ \text{ and } \Delta^2 y_n \leq 0,\ \text{for } n \geq n_1. \tag{2.11}$$

From (2.9) and (2.11)

$$d_{n-\tau+1}\Delta y_{n-\tau+1} \geq d_{n+1}\Delta\left(y_{n+1}\right) \text{ and } y_{n-\tau+1} \geq y_{n-\tau}. \tag{2.12}$$

Defining the sequence u_n by the Riccati substitution

$$u_n = \rho_n\frac{d_n\Delta y_n}{y^{\gamma}_{n-\tau+1}}, \quad \text{for} \quad n > n_1. \tag{2.13}$$

This implies that $u_n > 0$, and

$$\Delta u_n = d_{n+1}\Delta y_{n+1}\Delta\left[\frac{\rho_n}{y^{\gamma}_{n-\tau+1}}\right] + \rho_n\frac{\Delta\left(d_n\Delta y_n\right)}{y^{\gamma}_{n-\tau+1}}.$$

Hence

$$\Delta u_n = d_{n+1}\Delta y_{n+1}\left[\frac{\Delta\rho_n\left(y^{\gamma}_{n-\tau+1}\right) - \rho_n\left(\Delta y^{\gamma}_{n-\tau+1}\right)}{y^{\gamma}_{n-\tau+1}y^{\gamma}_{n-\tau+2}}\right] + \rho_n\frac{\Delta\left(d_n\Delta y_n\right)}{y^{\gamma}_{n-\tau+1}}. \tag{2.14}$$

From this, (2.5) and (2.14) we see that

$$\Delta u_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \left[\frac{d_{n+1}\Delta y_{n+1}\rho_n\Delta y^{\gamma}_{n-\tau+1}}{y^{\gamma}_{n-\tau+2}y^{\gamma}_{n-\tau+1}}\right] - \rho_n Q_n. \tag{2.15}$$

From (2.5) and (2.14), we have

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \frac{d_{n+1}\Delta y_{n+1}\rho_n\Delta y^{\gamma}_{n-\tau+1}}{y^{2\gamma}_{n-\tau+2}}. \tag{2.16}$$

By using the inequality (see [8]),

$$x^{\gamma} - y^{\gamma} \geq \gamma x^{\gamma-1}\left(x-y\right),\ \text{for all } x \neq y > 0 \text{ where } 0 < \gamma \leq 1, \tag{2.17}$$

we have

$$\begin{aligned} \Delta y^{\gamma}_{n-\tau+1} &= \left(y^{\gamma}_{n+2-\tau} - y^{\gamma}_{n+1-\tau}\right) \geq \gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(y_{n-\tau+2} - y_{n-\tau+1}\right) \\ &= \gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(\Delta y_{n-\tau+1}\right). \end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.16), we obtain that

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \rho_n d_{n+1}\frac{\gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(\Delta y_{n-\tau+1}\right)\Delta y_{n+1}}{y^{2\gamma}_{n-\tau+2}}. \tag{2.19}$$

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From (2.12) and (2.19), we have that

$$\Delta u_n \le -\rho_n Q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma \rho_n d_{n+1}^2 (\Delta y_{n+1})^2}{d_{n-\tau+1} (y_{n+2-\tau})^{1-\gamma} (y_{n-\tau+2}^{\gamma})^2}.$$

Hence,

$$\Delta u_n \le -\rho_n Q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma \rho_n}{(\rho_{n+1})^2 d_{n-\tau+1} (y_{n+2-\tau})^{1-\gamma}} (u_{n+1})^2 . \tag{2.20}$$

From (2.11), we conclude that

$$y_n \le y_{n_0} + \Delta y_{n_0} (n - n_0), \qquad n \ge n_1,$$

and consequently there exists a $n_2 \ge n_2$ and appropriate constant $\beta \ge 1$ such that

$$y_n \le \beta n, \quad \text{for} \quad n \ge n_2,$$

and this implies that

$$y_{n+2-\tau} \le \beta (n + 2 - \tau), \text{ for } \ n \ge n_3 = n_2 + \tau + 2,$$

and then

$$\frac{1}{(y_{n+2-\tau})^{1-\gamma}} \ge \frac{1}{(\beta (n + 2 - \tau))^{1-\gamma}}. \tag{2.21}$$

Substituting (2.21) into (2.20) we obtain

$$\Delta u_n \le -\rho_n Q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma \rho_n}{(\rho_{n+1})^2 d_{n-\tau+1} \beta^{1-\gamma} (n + 2 - \tau)^{1-\gamma}} (u_{n+1})^2 \tag{2.22}$$

Hence

$$\begin{aligned} \Delta u_n \quad &\le \quad -\rho_n Q_n + \frac{d_{n-\tau+1} \beta^{1-\gamma} (n + 2 - \tau)^{1-\gamma} (\Delta \rho_n)^2}{\rho_n} \\ &\quad - \left[\frac{\sqrt{\rho_n}}{\rho_{n+1} \sqrt{(\beta (n + 2 - \tau))^{1-\gamma}} d_{n-\tau+1}} u_{n+1} - \frac{\Delta \rho_n \sqrt{d_{n-\tau+1} \beta^{1-\gamma} (n + 2 - \tau)^{1-\gamma}}}{2 \rho_n} \right]^2 \end{aligned}$$

Then, we have

$$\Delta u_n \le - \left[\rho_n Q_n - \frac{d_{n-\tau+1} \beta^{1-\gamma} (n + 2 - \tau)^{1-\gamma} (\Delta \rho_n)^2}{\rho_n} \right] \tag{2.23}$$

Summing (2.23) from n_3 to n we obtain

$$-u_{n_3} < u_{n+1} - u_{n_3} \le - \sum_{i=n_3}^{n} \left[\rho_i Q_i - \frac{d_{i-\tau+1} \beta^{1-\gamma} (i + 2 - \tau)^{1-\gamma} (\Delta \rho_i)^2}{\rho_i} \right]$$

which yields

$$\sum_{i=n_3}^{n} \left[\rho_i Q_i - \frac{d_{i-\tau+1} \beta^{1-\gamma} (i + 2 - \tau)^{1-\gamma} (\Delta \rho_i)^2}{\rho_i} \right] < c_1,$$

for all large n, and this contrary to (2.8). The proof is complete. ■

From the Theorem 2.3, we can obtain different condition for oscillation of all solutions of (1.4) by different choices of ρ_n. For example if we take $\rho_n = n^{\lambda}$, $n \ge n_0$ and $\lambda > 1$ is a constant we have the following result.

Corollary 2.4 *Assume that all the assumptions of Theorem 2.3 hold, except that the condition (2.8) is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\left[s^{\lambda}Q_s-\frac{d_{s-\tau+1}\beta^{1-\gamma}\left(s+2-\tau\right)^{1-\gamma}\left(\Delta s^{\lambda}\right)^2}{s^{\lambda}}\right]=\infty.$$

Then every solution of (1.4) oscillates for all $0<\gamma\leq 1$.

Remark 2.5 *When* $\gamma=1$ *the equation (1.4) reduced to linear delay difference equation*

$$\Delta\left(x_n-p_nx_{n-1}\right)+q_nx_{n-\tau}=0,\ \textit{for } n\in\mathbb{N}_{n_0},$$

and the condition (2.8) in Theorem 2.3 reduced to

$$\lim_{n\to\infty}\sup\sum_{i=n_0}^{n}\left[\rho_iQ_i-\frac{d_{i-\tau+1}\left(\Delta\rho_i\right)^2}{\rho_i}\right]=\infty, \tag{2.24}$$

where $d_n=\prod_{i=1}^{n}p_i$ *and* $Q_n=q_nd_{n-\tau}$ *for all* $0<\gamma\leq 1$.

Now, we consider the case when $\gamma\geq 1$.

Theorem 2.6 *Assume that (2.6) holds. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=1}^{\infty}$ *such that for every positive constant* M,

$$\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left[\rho_lq_l-\frac{\left(d_{l-\sigma}\right)^{\gamma}\left(\Delta\rho_l\right)^2}{2^{3-\gamma}\left(M\right)^{\gamma-1}\left(d_{l+1}\right)^{2\gamma-2}\rho_l}\right]=\infty, \tag{2.25}$$

where $\sigma=\tau-1$. *Then every solution of (1.4) oscillates for all* $\gamma\geq 1$.

Proof. Suppose to the contrary that x_n is a nonoscillatory solution of (1.4). Without loss of generality, we may assume that x_n is an eventually positive solution of (1.4) such that $x_{n-1},\ x_{n-\tau},\ x_n>0$ for all large $n\geq n_1>n_0$ sufficiently large. We shall consider only this case, since the substitution $y_n=-x_n$ transforms equation (1.4) into an equation of the same form. As in the proof of Theorem 2.3, we have by (2.6) that

$$y_n>0,\ \Delta y_n\geq 0,\ \Delta(d_n\left(\Delta y_n\right))\leq 0,\ n\geq n_1. \tag{2.26}$$

Define the sequence u_n by

$$u_n:=\rho_n\frac{d_n\Delta y_n}{y_{n-\sigma}^{\gamma}}. \tag{2.27}$$

Then $u_n>0$, and

$$\Delta u_n=d_{n+1}\Delta y_{n+1}\Delta\left[\frac{\rho_n}{y_{n-\sigma}^{\gamma}}\right]+\frac{\rho_n\Delta(d_n\Delta y_n)}{y_{n-\sigma}^{\gamma}}. \tag{2.28}$$

In view of (2.5), (2.28), we have

$$\Delta u_n\leq-\rho_nq_n+\frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1}-\frac{\rho_nd_{n+1}\Delta y_{n+1}\Delta y_{n-\sigma}^{\gamma}}{y_{n+1-\sigma}^{\gamma}y_{n-\sigma}^{\gamma}}. \tag{2.29}$$

From (2.26), we see that

$$d_{n-\sigma}\Delta y_{n-\sigma}\geq d_{n+1}\Delta y_{n+1},\quad\text{and}\quad y_{n+1-\sigma}\geq y_{n-\sigma}. \tag{2.30}$$

Substituting (2.30) into (2.29), we have

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\rho_n d_{n+1} \Delta y_{n+1} \Delta y^{\gamma}_{n-\sigma}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2}. \tag{2.31}$$

Now, by using the inequality

$$x^{\gamma} - y^{\gamma} \geqslant 2^{1-\gamma}(x-y)^{\gamma}, \text{ for all } x \geqslant y > 0 \text{ and } \gamma \geqslant 1,$$

we find that

$$\Delta y^{\gamma}_{n-\sigma} = y^{\gamma}_{n+1-\sigma} - y^{\gamma}_{n-\sigma} \geqslant 2^{1-\gamma}(y_{n+1-\sigma} - y_{n-\sigma})^{\gamma} = 2^{1-\gamma}\left(\Delta y_{n-\sigma}\right)^{\gamma}. \tag{2.32}$$

Substituting (2.32) into (2.31), we have

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - 2^{1-\gamma} \rho_n d_{n+1} \frac{\Delta y_{n+1} \left(\Delta y_{n-\sigma}\right)^{\gamma}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2}. \tag{2.33}$$

From (2.30) and (2.33), we obtain

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - 2^{1-\gamma} \rho_n \frac{\left(d_{n+1}\right)^{\gamma+1}}{\left(d_{n-\sigma}\right)^{\gamma}} \frac{\left(\Delta y_{n+1}\right)^{\gamma+1}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2}. \tag{2.34}$$

Hence,

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\left(d_{n+1}\right)^{\gamma+1}}{\left(d_{n-\sigma}\right)^{\gamma}} \frac{2^{1-\gamma} \rho_n \left(\Delta y_{n+1}\right)^{2\gamma}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2 \left(\Delta y_{n+1}\right)^{\gamma-1}}. \tag{2.35}$$

From the definition of u_n, we get that

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{2^{1-\gamma} \rho_n}{\left(\rho_{n+1}\right)^2} \frac{\left(d_{n+1}\right)^{\gamma-1}}{\left(d_{n-\sigma}\right)^{\gamma}} \frac{u^2_{n+1}}{\left(\Delta y_{n+1}\right)^{\gamma-1}}. \tag{2.36}$$

Since $\{d_n\left(\Delta y_n\right)\}$ is a positive and nonincreasing sequence, there exists a $n_2 \geq n_1$ sufficiently large such that $d_n\left(\Delta y_n\right) \leq 1/M$ for some positive constant M and $n \geq n_1$, and hence by (2.26), we have

$$\frac{1}{\left(\Delta y_{n+1}\right)^{\gamma-1}} \geqslant \left(M d_{n+1}\right)^{\gamma-1}.$$

Substituting the last inequality into (2.36), we obtain

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \left(\frac{M}{2}\right)^{\gamma-1} \frac{\rho_n \left(d_{n+1}\right)^{2\gamma-2}}{\left(\rho_{n+1}\right)^2} \frac{1}{\left(d_{n-\sigma}\right)^{\gamma}} u^2_{n+1}, \tag{2.37}$$

so that

$$\begin{aligned}
\Delta u_n \quad &\leq \quad -\rho_n q_n + \frac{\left(d_{n-\sigma}\right)^{\gamma} (\Delta \rho_n)^2}{2^{3-\gamma}\left(M\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2} \rho_n} - \\
&\left[\frac{\sqrt{\left(\frac{M}{2}\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2} \rho_n}}{\rho_{n+1}\sqrt{\left(d_{n-\sigma}\right)^{\gamma}}} u_{n+1} - \frac{\sqrt{\left(d_{n-\sigma}\right)^{\gamma}} \Delta \rho_n}{2\sqrt{\left(\frac{M}{2}\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2} \rho_n}}\right]^2 \\
&< \quad -\left[\rho_n q_n - \frac{\left(d_{n-\sigma}\right)^{\gamma} (\Delta \rho_n)^2}{2^{3-\gamma}\left(M\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2} \rho_n}\right]
\end{aligned}$$

Then, we have

$$\Delta u_n < -\left[\rho_n q_n - \frac{(d_{n-\sigma})^{\gamma}(\Delta\rho_n)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}\right]. \tag{2.38}$$

Summing (2.38) from n_2 to n, we obtain

$$-u_{n_2} < u_{n+1} - u_{n_2} < -\sum_{l=n_2}^{n}\left[\rho_l q_l - \frac{(d_{l-\sigma})^{\gamma}(\Delta\rho_l)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{l+1})^{2\gamma-2}\rho_l}\right],$$

which yields

$$\sum_{l=n_2}^{n}\left[\rho_l q_l - \frac{(d_{l-\sigma})^{\gamma}(\Delta\rho_l)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{l+1})^{2\gamma-2}\rho_l}\right] < c_1,$$

for all large n. This contradicts (2.25). The proof is complete. ■

From Theorem 2.6, we can obtain different conditions for oscillation of all solutions of (1.4) when (2.6) holds by different choices of $\{\rho_n\}$. For example, let $\rho_n = n^{\lambda}$, $n \geq n_0$ and $\lambda > 1$ is a constant. From Theorem 2.6 we have the following result.

Corollary 2.7 *Assume that all the assumptions of Theorem 2.6 hold, except the condition (2.25) is replaced by*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[s^{\lambda}q_s - \frac{(d_{s-\sigma})^{\gamma}((s+1)^{\lambda}-s^{\lambda})^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{s+1})^{2\gamma-2}s^{\lambda}}\right] = \infty. \tag{2.39}$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

As a variant of the Riccati transformation technique used above, we will derive some oscillation criterion which can be considered as a discrete analogy of the Philos condition for oscillation of second order differential equation by introducing the following class of sequences that will be used in this chapter and later. Let

$$\pounds_0 = \{(m,n):\ m > n \geq n_0\}, \quad \pounds = \{(m,n):\ m \geq n \geq n_0\}.$$

The double sequence $H_{m,n} \in \Sigma$ if:

(I). $H(m,m) = 0$ on $\pounds$,

(II). $H(m,n) > 0$ on $\pounds_0$;

(III). $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *for* $m \geq n \geq 0$, and there exists a double sequence $h_{m,n}$ such that

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},\ \text{for}\ \ m > n \geq 0.$$

Theorem 2.8 *Assume that (2.6) hold. Let $\{\rho_n\}_{n=1}^{\infty}$ be a positive sequence and $H_{m,n} \in \Sigma$. If*

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - B_n\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] = \infty, \tag{2.40}$$

where

$$B_n := \frac{(d_{n-\sigma})^{\gamma}\rho_{n+1}^2}{2^{3-\gamma}M^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}.$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

Proof. We proceed as in the proof of Theorem 2.6, we may assume that (1.4) has a nonoscillatory solution x_n such that $x_n > 0$. As in the proof of Theorem 2.6 we get that (2.26) holds. Define $\{u_n\}$ by (2.27) as before, then we have $u_n > 0$ and there is some $M > 0$ such that (2.37) holds. For the sake of convenience, let us set

$$\bar{\rho}_n = \frac{2^{1-\gamma}(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}{(d_{n-\sigma})^{\gamma}}.$$

Then, we have from (2.37) that

$$\rho_n q_n \leq -\Delta u_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \frac{\bar{\rho}_n}{(\rho_{n+1})^2}u_{n+1}^2. \tag{2.41}$$

Therefore, we get

$$\sum_{n=n_1}^{m-1} H_{m,n}\rho_n q_n \leq -\sum_{n=n_1}^{m-1} H_{m,n}\Delta u_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\bar{\rho}_n u_{n+1}^2}{(\rho_{n+1})^2}. \tag{2.42}$$

The rest of the proof is similar to the proof of [15, Theorem 2.3.6]. ■

As an immediate consequence of Theorem 2.8, we get the following:

Corollary 2.9 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1} H_{m,n}\rho_n q_n = \infty,$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1}\frac{(d_{n-\sigma})^{\gamma}\rho_{n+1}^2}{(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2 < \infty.$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

By choosing the sequence $H_{m,n}$ in appropriate manners, we can derive several oscillation criteria for (1.4). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\left.\begin{aligned} H_{m,n} &= (m-n)^{\lambda}, \quad \lambda \geq 1, m \geq n \geq 0,\\ H_{m,n} &= \left(\log\tfrac{m+1}{n+1}\right)^{\lambda}, \lambda \geq 1, m \geq n \geq 0,\\ H_{m,n} &= (m-n)^{(\lambda)} \quad \lambda > 2,\ m \geq n \geq 0, \end{aligned}\right\} \tag{2.43}$$

where $(m-n)^{(\lambda)} = (m-n)(m-n+1)...(m-n+\lambda-1)$, and

$$\Delta_2(m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda(m-n)^{(\lambda-1)}.$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Hence we have the following result which gives new sufficient conditions for the oscillation of (1.4) of Kamenev type.

Corollary 2.10 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}\left[(m-n)^{\lambda}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n}V_{m,n}^2\right] = \infty, \tag{2.44}$$

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where

$$V_{m,n} := \left(\lambda(m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{(m-n)^{\lambda}}\right).$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

Corollary 2.11 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m-1}\left[\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n}^{2} R_{m,n}^2\right] = \infty, \tag{2.45}$$

where

$$R_{m,n} = \left(\frac{\lambda}{n+1}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{\left(\log\frac{m+1}{n+1}\right)^{\lambda}}\right).$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

Corollary 2.12 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{(\lambda)}}\sum_{n=0}^{m-1}(m-n)^{(\lambda)}\left[\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n}U_n^2\right] = \infty, \tag{2.46}$$

where

$$U_n := \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta\rho_n}{\rho_{n+1}}\right)^2.$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

In the following theorem, we consider the case when $0 < \gamma < 1$.

Theorem 2.13 *Assume that (2.6) holds and $\Delta d_n \geq 0$. If*

$$\sum_{n=n_0}^{\infty}\left(\frac{n-\sigma}{d_n}\right)^{\gamma} q_n = \infty. \tag{2.47}$$

Then every solution of (1.4) oscillates for all $0 < \gamma < 1$.

Proof. Proceeding as in Theorem *2.6*, we assume that (1.4) has a nonoscillatory solution, say $x_n > 0$ and $x_{n-\tau} > 0$ for all $n \geq n_0$. From the proof of Theorem 2.6 we know that $\Delta y_n > 0$, then y_n is nondecreasing sequence. Since $\Delta d_n \geq 0$ we obtain that $\Delta^2 y_n \leq 0$ and then Δy_n is a nonincreasing for all $n \geq n_1 \geq n_0$. Then, we have $y_n \geq (n-n_1)\Delta y_n$ which implies that $y_n \geq \frac{n}{2}\Delta y_n$ for $n \geq n_2 \geq 2n_1 + 1$. Then

$$y_{n-\sigma} \geq \frac{n-\sigma}{2}\Delta y_{n-\sigma} \geq \frac{n-\sigma}{2}\Delta y_{n+1}, \quad for\ n \geq N = n_2 + \sigma. \tag{2.48}$$

From (2.5) and (2.48) by dividing by $z_{n+1} = (d_n \Delta y_{n+1})^{\gamma} > 0$ and summing from $2N$ to k, we obtain

$$\sum_{n=2N}^{k}\left(\frac{n-\sigma}{2d_n}\right)^{\gamma} q_n \leq -\sum_{n=2N}^{k}\frac{\Delta(z_n)}{(z_{n+1})^{\gamma}}, \quad k \geq 2N. \tag{2.49}$$

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Since

$$y^{\gamma} - z^{\gamma} \leq \gamma y^{\gamma-1}(y - z) \text{ for } \gamma < 1 \text{ and } y > z > 0,$$

we see that

$$\Delta\left(z_n^{1-\gamma}\right) = \left(z_{n+1}^{1-\gamma}\right) - \left(z_n^{1-\gamma}\right) \leq (1-\gamma)(z(n+1))^{-\gamma}\Delta z(n).$$

Substituting in (2.49), we see that

$$\begin{aligned}\sum_{n=2N}^{k}\left(\frac{n-\sigma}{2d_n}\right)^{\gamma} q_n &\leq -\sum_{n=2N}^{k}\frac{\Delta\left(z_n^{1-\gamma}\right)}{(1-\gamma)} = -\frac{\left(z_{k-1}^{1-\gamma}\right)}{(1-\gamma)} + \frac{\left(z_{2N}^{1-\gamma}\right)}{(1-\gamma)} \\ &< \frac{\left(z_{2N}^{1-\gamma}\right)}{(1-\gamma)} < \infty, \text{ as } n \to \infty\end{aligned}$$

which contradicts (2.47). The proof is complete. ■

Now, we consider the case when

$$\sum_{n=0}^{\infty}\left(\frac{1}{d_n}\right) < \infty, \tag{2.50}$$

holds and establish some oscillation criteria for (1.4) in the sublinear and superlinear cases.

Theorem 2.14 *Assume that (2.50) holds and there exist positive sequences* $\{\rho_n\}_{n=1}^{\infty}$ *such that (2.25) holds for every positive constant* M*, and*

$$\sum_{n=0}^{\infty}\left(\frac{1}{d_n}\sum_{i=n_0}^{n-1} q_i\right) = \infty. \tag{2.51}$$

Then every solution of (1.4) oscillates or $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$*.*

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of (1.4). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. From (2.5), we have

$$\Delta(d_n \Delta y_n) \leq -q_n y_{n-\sigma}^{\gamma} \leq 0, \qquad n \geq n_0, \tag{2.52}$$

and so $\{d_n(\Delta y_n)\}$ is an eventually nonincreasing sequence. Since $\{q_n\}$ has a positive subsequence, either $\{\Delta y_n\}$ is eventually negative or eventually positive. If $\{\Delta y_n\}$ is eventually positive, we are then back to the case where (2.26) holds. Thus the proof of Theorem 2.6 goes through, and we may conclude that $\{y_n\}$ cannot be eventually positive, which is not possible. If $\{\Delta y_n\}$ is eventually negative, then $\lim_{n\to\infty} y_n = b \geq 0$. We assert that $b = 0$. If not then $y_{n-\sigma}^{\gamma} \to b^{\gamma} > 0$ as $n \to \infty$, and hence there exists $n_1 \geq n_0 > 0$ such that $y_{n-\sigma}^{\gamma} \geq b^{\gamma}$. Therefore from (2.52) we have

$$\Delta(d_n \Delta y_n) \leq -q_n b^{\gamma}.$$

The rest of the proof is similar to the proof of [15, Theorem 2.3.7] and hence is omitted. ■

By choosing $\{\rho_n\}_{n=1}^{\infty}$ in appropriate manners, we may obtain different oscillation criteria. For instance, let $\rho_n = n^{\lambda}$ for $n \geq 0$ and $\lambda > 1$. Then we have the following oscillation conditions of all solutions of (1.4) when (2.50) holds.

Corollary 2.15 *Assume that all assumptions of Theorem 2.14 hold, except that the condition (2.25) is replaced by (2.39). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Theorem 2.16 *Assume that (2.50) and (2.51) hold. Furthermore, assume that there exists a double sequence* $H_{m,n} \in \Sigma$ *such that (2.40) holds. Then every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1.4). Then as seen in the proof of Theorem *2.3*, either $\{\Delta x_n\}$ is eventually positive or is eventually negative. In the case when $\{\Delta y_n\}$ is eventually positive, we may follow the proof of Theorem *2.8* and obtain a contradiction. If $\{\Delta y_n\}$ is eventually negative, then we may follow the proof of Theorem *2.14* to show that $\{y_n\}$ converges to zero.

By choosing $H_{m,n}$ in appropriate manners, we can derive several oscillation criteria for (2.5) when (2.50) holds. For instance, let us consider the double sequence $H_{m,n}$ defined again by (2.43). Hence we have the following results.

Corollary 2.17 *Assume that all the assumptions of Theorem 2.16 hold, except that the condition (2.40) is replaced by (2.44). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Corollary 2.18 *Assume that all the assumptions of Theorem 2.16 hold, except that the condition (2.40) is replaced by(2.45) or (2.46). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Theorem 2.19 *Assume that (2.50) and (2.47) hold. Let* $\{\rho_n\}_{n=1}^{\infty}$ *such that (2.51) holds. Then every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $0 < \gamma < 1$.

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1.4). Then as seen in the proof of Theorem 2.6, either $\{\Delta y_n\}$ is eventually positive or is eventually negative. In the case when $\{\Delta y_n\}$ is eventually positive , we may follow the proof of Theorem 2.13 and obtain a contradiction. If $\{\Delta y_n\}$ is eventually negative, then we may follow the proof of Theorem 2.14 to show that $\{x_n/d_n\}$ converges to zero.

From Theorem 2.14 if $\rho_n = 1$, we see that the Riccati inequality associated with the equation (1.4) is given by

$$\Delta u_n + \rho_n q_n + \frac{1}{a_n} u_{n+1}^2 \leq 0, \tag{2.53}$$

where

$$A_n = \frac{2^{\gamma-1}\left(d_{n-\sigma}\right)^{\gamma}}{(M)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2}} > 0, \tag{2.54}$$

for every positive constant $M > 0$. Using the inequality (2.53) and proceeding as in the proof [15, Theorem 2.3.8], we can prove the following Hille and Nehari type results.

Theorem 2.20 *Assume that* (H_1) *holds and* $\Delta d_n \geq 0$. *Furthermore, assume that*

$$\liminf_{n\to\infty} \frac{n}{A_n} \sum_{n+1}^{\infty} q(s) > \frac{1}{4},$$

or

$$\liminf_{n\to\infty} \frac{n}{A_n} \sum_{n+1}^{\infty} q_s + \liminf_{n\to\infty} \frac{1}{n} \sum_{N}^{n-1} \frac{s^2}{A_n} q_s > 4.$$

Then every solution of (1.4) is oscillatory.

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RICCATI TECHNIQUE AND OSCILLATION OF SECOND ORDER NONLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS

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Abstract. In this paper, by using the Riccati technique which reduces the higher order dynamic equations to a Riccati dynamic inequality, we will establish some new sufficient conditions for oscillation of the second order nonlinear neutral dynamic equation

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))+v(t)x^{\beta}(\eta(t))=0,$$

on time scales where γ, α β are quotient of odd positive integers.

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1. Introduction

The theory of time scales has been introduced by Stefan Hilger in [14] in 1988 in his Ph.D thesis in order to unify continuous and discrete analysis. In the last decades the subject is going fast and simultaneously extending to the other areas of research and many researchers have contributed on different aspects of this new theory, see the survey paper by Agarwal et al. [1] and the references cited therein. In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of different classes of dynamic equations on a time scale $\mathbb{T}$ which may be an arbitrary closed subset of real numbers $\mathbb{R}$, and as special cases contains the continuous and the discrete results as well, we refer the reader to papers ([3],[6], [7], [21]) and the references cited therein.

Following this trend, in this paper, we are concerned with oscillation of a certain class of nonlinear neutral delay dynamic equations of the form

$$(1.1)\qquad (r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))+v(t)x^{\beta}(\eta(t))=0,\ \text{for } t\in[t_0,\infty)_{\mathbb{T}},$$

where γ, α, β are quotient of odd positive integers, $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and p, $q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ with $0\le p(t)<1$, $q(t)$, $v(t)\ge 0$ and τ, δ, $\eta\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and $\tau(t)\le t$, $\delta(t)\le t$, $\eta(t)\le t$ with $\lim_{t\to\infty}\tau(t)=\infty=\lim_{t\to\infty}\delta(t)=\infty=\lim_{t\to\infty}\eta(t)$. By a solution of (1.1), we mean a nontrivial real-valued function $x(t)\in C^1_{rd}([T_x,\infty),\mathbb{R})$, $T_x\ge t_0$ which has the properties that $r(z^{\Delta})^{\gamma})^{\Delta}\in C^1_{rd}([T_x,\infty),\mathbb{R})$ such that $x(t)$ satisfies (1.1) for all $[T_x,\infty)_{\mathbb{T}}$.

We mention here that the neutral delay differential equations appear in modelling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role, we refer the reader to the papers by Boe and Chang [4], Brayton and Willoughby [8] and to the books by Driver [9], Hale [13] and Popov [16] and reference cited therein.

For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [5], [6] which summarize and organize much of the time scale calculus. Throughout the paper, we will denote the time scale by the symbol $\mathbb{T}$. For example, the real numbers $\mathbb{R}$,

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the integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$ are time scales. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. A time-scale $\mathbb{T}$ equipped with the order topology is metrizable and is a K_σ −space; i.e. it is a union of at most countably many compact sets. The metric on $\mathbb{T}$ which generates the order topology is given by $d(r; s) := |\mu(r; s)|$, where $\mu(.) = \mu(.; \tau)$ for a fixed $\tau \in \mathbb{T}$ is defined as follows: The mapping $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called graininess.

When $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) \equiv 0$ for all $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t+1$ and $\mu(t) \equiv 1$ for all $t \in \mathbb{T}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The mapping : $\nu : \mathbb{T} \to \mathbb{R}_0^+$ such that $\nu(t) = t - \rho(t)$ is called the backward graininess. If $\sigma(t) > t$, we say that t is right-scattered , while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous ($rd-$continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, we define the derivative f^Δ as follows: Let $t \in \mathbb{T}$. If there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be differentiable at t, and we call α the delta derivative of f at t and denote it by $f^\Delta(t)$. For example, if $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f^{'}(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \text{ for all } t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{N}$, then $f^\Delta(t) = f(t + 1) - f(t)$ for all $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right–scattered. If t is not right–scattered then the derivative is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{t \to \infty} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right–dense continuous ($rd-$continuous) if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. The space of $rd-$continuous functions is denoted by $C_r(\mathbb{T}, \mathbb{R})$. A useful formula is

$$f^\sigma = f + \mu f^\Delta, \quad \text{where} f^\sigma := f \circ \sigma.$$

A time scale $\mathbb{T}$ is said to be regular if the following two conditions are satisfied simultaneously:
(a). For all $t \in \mathbb{T}$, $\sigma(\rho(t)) = t$,
(b). For all $t \in \mathbb{T}$, $\rho(\sigma(t)) = t$.

Remark 1.1. *If $\mathbb{T}$ is a regular time scale, then both operators and are invertible with $\sigma^{-1} = \rho$ and $\rho^{-1} = \sigma$.*

The following formulae give the product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable function f and g. Assume f; $g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}$, then

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \tag{1.2}$$

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{1.3}$$

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The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_{0}^{1} [hx^{\sigma} + (1-h)x]^{\gamma-1} \, dhx^{\Delta}(t), \tag{1.4}$$

which is a simple consequence of Keller's chain rule [5, Theorem 1.90]. Note that when $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = t, \; \mu(t) = 0, \quad f^{\Delta}(t) = f^{'}(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

When $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t+1, \; \mu(t) = 1, \; f^{\Delta}(t) = \Delta f(t), \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$,

$$f^{\Delta}(t) = \Delta_h f(t) = \frac{(f(t+h) - f(t))}{h}, \quad \int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h.$$

When $\mathbb{T} = \{t : t = q^k, \, k \in \mathbb{N}_0, \, q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$,

$$f^{\Delta}(t) = \Delta_q f(t) = \frac{(f(q\,t) - f(t))}{(q-1)\,t}, \quad \int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} f(q^k)\mu(q^k).$$

When $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}\}$, we have $\sigma(t) = (\sqrt{t}+1)^2$ and

$$\mu(t) = 1 + 2\sqrt{t}, \; f^{\Delta}(t) = \Delta_0 f(t) = (f((\sqrt{t}+1)^2) - f(t))/1 + 2\sqrt{t}.$$

When $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where (t_n) is the harmonic numbers that are defined by $t_0 = 0$ and $t_n = \sum_{k=1}^{n} \frac{1}{k}, n \in \mathbb{N}_0$, we have

$$\sigma(t_n) = t_{n+1}, \; \mu(t_n) = \frac{1}{n+1}, \; f^{\Delta}(t) = \Delta_1 f(t_n) = (n+1)f(t_n).$$

When $\mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}\}$, we have $\sigma(t) = \sqrt{t^2+1}$,

$$\mu(t) = \sqrt{t^2+1} - t, \; f^{\Delta}(t) = \Delta_2 f(t) = \frac{(f(\sqrt{t^2+1}) - f(t))}{\sqrt{t^2+1} - t}.$$

When $\mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}\}$, we have $\sigma(t) = \sqrt[3]{t^3+1}$ and

$$\mu(t) = \sqrt[3]{t^3+1} - t, \; f^{\Delta}(t) = \Delta_3 f(t) = \frac{(f(\sqrt[3]{t^3+1}) - f(t))}{\sqrt[3]{t^3+1} - t}.$$

Now, we pass to the antiderivative and the integration on time scales for detla differentiable functions. For $a, b \in \mathbb{T}$, and a delta differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = f(t)g(t)|_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t, \tag{1.5}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t.$$

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It is well known that $rd-$continuous functions possess antiderivative. If f is $rd-$continuous and $F^{\Delta}=f$, then

$$\int_t^{\sigma(t)} f(s)\Delta s = F(\sigma(t)) - F(t) = \mu(t)F^{\Delta}(t) = \mu(t)f(t).$$

Note that the integration formula on a discrete time scale is defined by

$$\int_a^b f(t)\Delta t = \sum_{t\in(a,b)} f(t)\mu(t).$$

We say that a solution x of (1.1) has a generalized zero at t if $x\,(t)=0$ and has a generalized zero in $(t,\sigma(t))$ in case $x\,(t)\,x^{\sigma}\,(t)<0$ and $\mu(t)>0$. To investigate the oscillation properties of (1.1) it is proper to use the notions such as conjugacy and disconjugacy of the equation (1.1). Equation (1.1) is disconjugate on the interval $[t_0,b]_{\mathbb{T}}$, if there is no nontrivial solution of (1.1) with two (or more) generalized zeros in $[t_0,b]_{\mathbb{T}}$.

Equation (1.1) is said to be nonoscillatory on $[t_0,\infty]_{\mathbb{T}}$ if there exists $c\in[t_0,\infty]_{\mathbb{T}}$ such that this equation is disconjugate on $[c,d]_{\mathbb{T}}$ for every $d>c$. In the opposite case (1.1) is said to be oscillatory on $[t_0,\infty]_{\mathbb{T}}$. A solution $x\,(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is oscillatory. We say that (1.1) is right disfocal (left disfocal) on $[a,b]_{\mathbb{T}}$ if the solutions of (1.1) such that $x^{\Delta}\,(a)=0$ ($x^{\Delta}\,(b)=0$) have no generalized zeros in $[a,b]_{\mathbb{T}}$.

In recent two decades some authors have been studied the oscillation of the second order nonlinear neutral delay dynamic equations on time scales and established several sufficient conditions for oscillation of some different types of equations by employing the Riccati transformation technique. For example, Saker [18] has studied the oscillation of second order neutral delay dynamic equations of Emden-fowler type of the form

$$[a(t)(y(t)+r(t)y(\tau(t))]^{\Delta}+p(t)|y(\delta(t))|^{\gamma}signy(\delta(t)))=0,$$

on time scale $\mathbb{T}$, where, $\gamma>1$, $a(t)$, $p(t)$, $r(t)$ and $\delta(t)$ are real-valued function defined on $\mathbb{T}$. Also Saker [19] studied the oscillation of the superlinear and sublinear neutral delay dynamic equations of the form

$$[a(t)([y(t)+p(t)y(\tau(t)))]^{\Delta})^{\gamma}]^{\Delta}+q(t)y^{\gamma}(\delta(t)))=0,$$

on time scale, where $\gamma>0$ is a quotient of odd positive integers. The main results has been obtained under the conditions $\tau(t):\mathbb{T}\to\mathbb{T}$, $\delta(t):\mathbb{T}\to\mathbb{T}$, $\tau(t)\le t$, $\delta(t)\le t$ for all $t\in\mathbb{T}$ and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$, $\int_{t_0}^{\infty}\frac{1}{a(t)}^{\frac{1}{\gamma}}\Delta t=\infty$, $a^{\Delta}(t)\ge t$ and $0\le p(t)<1$.

Thandapani et. al [24] studied the oscillation of second order nonlinear neutral dynamic equations on time scale of the form

$$(r(t)((y(t)+p(t)y(t-\tau))^{\Delta})^{\gamma})^{\Delta}+q(t)y^{\beta}(t-\delta)=0,\ t\in\mathbb{T},$$

where $\mathbb{T}$ is a time scales. They obtained their results under the conditions $\gamma\ge1$ and $\beta>0$ are quotients of odd positive integers, τ,δ are fixed nonnegative constants such that the delay function $\tau(t)=t-\tau<t$ and $\delta(t)=t-\delta<t$ satisfying $\tau:\mathbb{T}\to\mathbb{T}$ and $\delta:\mathbb{T}\to\mathbb{T}$ for all $t\in\mathbb{T}$, $q(t)$ and $\tau(t)$ real valued rd-continuous functions defined on $\mathbb{T}$, $p(t)$ is a positive and rd-continuous function $\mathbb{T}$ such that $0\le p(t)<1$.

Sun et al. [22] studied the oscillation of a second order quasiliniear neutral delay dynamic equation on time scales of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q_1(t)x^{\alpha}(\tau_1(t))+q_2(t)x^{\beta}(\tau_2(t))=0,$$

on time scale $\mathbb{T}$, where α,β,γ are quotients of odd positive integers, r, p, q_1, q_2 are rd-continuous function on $\mathbb{T}$ and r,q_1,q_2 are positive, $-1<-p_0\le p(t)<1$, $p_0>0$, the delay functions $\tau_i:\mathbb{T}\to\mathbb{T}$ satisfying $\tau_i(t)\le t$ for $t\in\mathbb{T}$ and $\tau_i(t)\to\infty$ as $t\to\infty$, for $i=0,1,2$ and there exists a function $\tau:\mathbb{T}\to\mathbb{T}$ which satisfying $\tau(t)\le\tau_1(t)$, $\tau(t)\le\tau_2(t)$, $\tau(t)\to\infty$ as $t\to\infty$.

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Gao et al. [12] established some oscillation theorems for second order neutral functional dynamic equations on time scale of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q_1(t)x^{\alpha}(\delta(t))+q_2(t)x^{\beta}(\eta(t))=0,$$

where γ,α,β are ratios of odd positive integers by using the comparison theorems for oscillation. Sethi [26] considered the second order sublinear neutral delay dynamic equations of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\gamma}(\alpha(t))+v(t)x^{\gamma}(\eta(t))=0,$$

under the assumptions:

(H_0) $\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=+\infty$,

(H_1). $\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t<\infty$,

where $0<\gamma\le 1$ is a quotient of odd positive integers, $q,\ v\to[0,\infty)$ and $p,\ q,\ v:\mathbb{T}\to\mathbb{T}$ are rd-continuous functions and $\tau,\ \sigma,\ \eta:\mathbb{T}\to\mathbb{T}$ are positive rd-continuous functions such that $\lim_{t\to\infty}\tau(t)=\infty=\lim\limits_{t\to\infty}\alpha(t)=\infty=\lim_{t\to\infty}\eta(t)$ and obtained some sufficient conditions for oscillation. Our aim in this paper is to establish some new sufficient conditions for oscillation of the equation (1.1) by employing the Riccati technique and some basic lemmas studied the behavior of nonoscillatory solutions. Our motivation of the present work has come under two ways. First is due to the work in [17] and [22] and second is due to the work in [10].

2. Main Results

In this section, we establish some sufficient conditions for oscillation of all solutions of (1.1) under the hypothesis (H_0). Throughout the paper, we use the notation

$$z(t)=x(t)+p(t)x(\tau(t)). \tag{2.1}$$

Lemma 2.1. [2] *Assume that (H_0) holds and $r(t)\in C^1_{rd}([(a,\infty),\mathbb{R}^+)$ such that $r^{\Delta}(t)\ge 0$. Let $x(t)$ be an eventually positive real valued function such that $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}\le 0$, for $t\ge t_1>t_0$. Then $x^{\Delta}(t)>0$ and $x^{\Delta\Delta}(t)<0$ for $t\ge t_1>t_0$.*

Lemma 2.2. [2] *Assume that the assumptions of Lemma 2.1 holds and let $\tau(t)$ be a positive continuous function such that $\tau(t)\le t$ and $\lim\limits_{t\to\infty}\tau(t)=\infty$. Then there exists $t_l>t_1$ such that for each $l\in(0,1)$*

$$\frac{x(\tau(t))}{x(\delta(t))}\ge l\frac{\tau(t)}{\delta(t)}.$$

Proof. Indeed, for $t\ge t_1$

$$u(\delta(t))-u(\tau(t))=\int_{\tau(t)}^{\delta(t)}u^{\Delta}(s)\Delta s\le(\delta(t)-\tau(t)))u^{\Delta}(\tau(t),$$

which implies that

$$\frac{u(\delta(t))}{u(\tau(t))}\le 1+(\delta(t)-\tau(t)))\frac{u^{\Delta}(\tau(t)}{u(\tau(t))}.$$

On the other hand, it follows that

$$u(\tau(t))-u(t_1))=\int_{t_1}^{\tau(t)}u^{\Delta}(s)\Delta s\ge(u(t)-t_1)u^{\Delta}(\tau(t)).$$

That is for each $l\in(0,1)$, there exists a $t_l>t_1$ such that

$$l(\tau(t))\le\frac{u(\tau(t))}{u^{\Delta}(\tau(t))},\ t\ge t_l.$$

Consequently,

$$\frac{u(\delta(t))}{u(\tau(t))}\le 1+(\delta(t)-\tau(t)))\frac{u^{\Delta}(\tau(t))}{u(\tau(t))}\le\frac{\delta(t)}{l\tau(t)}.$$

The proof is complete. □

In the following, for simplicity, we denote

$$a_1(t) := \int_t^\infty [q(s)(1-p(\delta(s))] \left(\frac{l\delta(s)}{\sigma(s)}\right)^\alpha \Delta s + \int_t^\infty [v(s)(1-p(\delta(s)))] \left(\frac{l\delta(s)}{\delta(s)}\right)^\alpha \Delta s,$$

and

$$A_1(t, K_1) := \left[a_1(t) + K_1 \int_t^\infty \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1^\delta(s))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}}, \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

where $K_1 > 0$ is an arbitrary constant.

Theorem 2.1. *Assume that* (H_0) *holds and let* $0 \le p(t) \le a < 1$, $r^\Delta(t) > 0$ *and* $\gamma < \alpha < \beta$, $\eta(t) \ge \delta(t)$ *and* $\delta^\Delta(t) \ge 1$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If*

(H_1). $\limsup_{t\to\infty} a_1(t) < \infty$,

(H_2). $\int_{t_0}^\infty (\frac{1}{r(s)})^{\frac{1}{\gamma}} A_1^\sigma(s, K_1)\Delta s = \infty$.

Then every solution of (1.1) oscillates on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \ge t_0$. Hence there exists $t \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ and $x(\eta(t)) > 0$ for $t \ge t_1$. Using (2.1), we see that (1.1) becomes

$$(r(t)(z^\Delta(t))^\gamma)^\Delta = -q(t)x^\alpha(\delta(t)) - v(t)x^\beta(\eta(t)) \le 0, \text{ for } t \ge t_2. \tag{2.2}$$

So $r(t)(z^\Delta(t))^\gamma$ is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$, that is, either $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$. By Lemma 2.1, it follows that $z^\Delta(t) > 0$ for $t \ge t_2$. Hence there exists $t_3 > t_2$ such that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &= x(t) + p(t)x(\tau(t)) - p(t)x(\tau(t)) \\ &\quad -p(t)p(\tau(t))p(\tau(\tau(t))) \\ &= x(t) - p(t)p(\tau(t))p(\tau(\tau(t))) \le x(t), \end{aligned}$$

which implies that

$$x(t) \ge (1-p(t))z(t), \text{ for } t \in [t_3, \infty)_{\mathbb{T}}.$$

Therefore (1.1) can be written as

$$(r(t)(z^\Delta(t))^\gamma)^\Delta + q(t)(1-p(\delta(t)))^\alpha z^\alpha(\delta(t)) + v(t)(1-p(\eta(t)))^\alpha z^\alpha(\eta(t)) \le 0,$$

where $\gamma < \alpha < \beta$. Define $w(t)$ by the Riccati transformation

$$w(t) = r(t)\frac{(z^\Delta(t))^\gamma}{z^\alpha(t)}, \quad \text{for} \quad t \in [t_3, \infty)_{\mathbb{T}}. \tag{2.3}$$

By using the product and quotient rules, we see that

$$w^\Delta(t) = \frac{(r(z^\Delta)^\gamma)^\Delta}{(z^\sigma)^\alpha} - \frac{(r(z^\Delta)^\gamma)^\sigma(z^\alpha)^\Delta}{z^\alpha(z^\sigma)^\alpha}, \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}. \tag{2.4}$$

Now, since $\eta(t) > \delta(t)$ and due to (2.3) and (2.4), we have

$$w^\Delta(t) \le -q(1-p^\delta)^\alpha - v(1-p^\delta)^\alpha \frac{(z^\delta)^\alpha}{(z^\sigma)^\alpha} - \frac{w^\sigma(z^\alpha)^\Delta}{z^\alpha}, \text{ for } t \in [t_3, \infty)_{\mathbb{T}},$$

Now, by using the chain rule [6], we get that

$$\begin{aligned} (z^\alpha(t))^\Delta &= \alpha \int_0^1 [(1-h)z(t) + hz(\sigma(t))]^{\alpha-1} dh z^\Delta(t) \\ &\ge \begin{cases} \alpha(z(t))]^{\alpha-1} z^\Delta(t), \ \alpha > 1, \\ \alpha(z(\sigma(t)))]^{\alpha-1} z^\Delta(t), \ 0 < \alpha \le 1. \end{cases} \end{aligned}$$

Since $z(t)$ is nondecreasing function on $[t_3,\infty)_{\mathbb{T}}$, then for $t \geq t_3$,

$$\frac{(z^{\alpha}(t))^{\Delta}}{z^{\alpha}(t)} \geq \begin{cases} \alpha\frac{z^{\Delta}(t)}{z(t)}, & \text{for} \quad \alpha > 1 \\ \alpha\frac{(z(\sigma(t)))^{\alpha-1}}{z^{\alpha}(t)}z^{\Delta}(t), & for\ 0<\alpha\leq 1. \end{cases}$$

Using the fact that $t \leq \sigma(t)$, we have

$$\frac{(z^{\alpha})^{\Delta}}{z^{\alpha}} \geq \alpha\frac{z^{\Delta}}{z^{\sigma}},\ \alpha>0 \quad \text{on} \quad [t_3,\infty)_{\mathbb{T}}.$$

Therefore (2.4) yields that

$$w^{\Delta} \leq -q(1-p^{\delta})^{\alpha} - v(1-p^{\delta})^{\alpha}\frac{(z^{\sigma})^{\alpha}}{(z^{\delta})^{\alpha}} - \alpha w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}},\ t \geq t_3. \tag{2.5}$$

Now, since $\left(r^{\frac{1}{\gamma}}z^{\Delta}\right)$ is nonincreasing on $[t_3,\infty)_{\mathbb{T}}$, then for $t \leq \sigma(t)$, we have that

$$z^{\Delta} \geq r^{-\frac{1}{\gamma}}(w^{\sigma})^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\alpha}{\gamma}},\ t \geq t_3. \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$w^{\Delta} \leq -q(1-p^{\delta})^{\alpha}\frac{(z^{\delta})^{\alpha}}{(z^{\sigma})^{\alpha}} - v(1-p^{\delta})^{\alpha}\frac{(z^{\delta})^{\alpha}}{(z^{\sigma})^{\alpha}} - \alpha r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}(z^{\sigma})\frac{\alpha}{\gamma}-1,\ t \geq t_3.$$

Since $z(t)$ is nondecreasing on $[t_3,\infty)_{\mathbb{T}}$, then there exists $t_4 > t_3$ and $C > 0$ such that

$$(z(\sigma(t)))^{\frac{\alpha}{\gamma}-1} \geq (z(t))^{\frac{\alpha}{\gamma}-1} \geq C, \quad \text{for} \quad t \geq t_4.$$

By using Lemma 2.2, it follows from the last inequality that

$$\begin{aligned} w^{\Delta}(t) \quad &\leq \quad -q(1-p(\delta(t)))^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} - v(1-p(\delta(t)))^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} \\ &-\alpha C r^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}},\ t \quad \geq \quad t_l > t_4. \end{aligned}$$

Integrating the above inequality from t to u $(t<u)$ for $t,\ u \in [t_4,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} -w(t) \quad &\leq \quad w(u)-w(t) \\ &\leq \quad -\int_t^u \left[q(1-p^{\delta})^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} + v(1-p^{\delta})^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} + \alpha C r^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}}\right]\Delta s, \end{aligned}$$

that is,

$$w(t) \geq a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s,\ t \geq t_1,$$

where $K_{1\alpha} = C\alpha$. Indeed, $w(t) > a_1(t)$ implies that

$$w(t) \geq a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(\sigma(s)))^{1+\frac{1}{\gamma}}\Delta s = A_1^{\gamma}(t,K_1).$$

Since $t \leq \sigma(t)$ we see

$$r(z^{\Delta})^{\gamma} \geq (r(z^{\Delta})^{\gamma})^{\sigma},$$

which implies that

$$\frac{r(z^{\Delta})^{\gamma}}{(z^{\sigma})^{\alpha}} \geq \frac{(r(z^{\Delta})^{\gamma})^{\sigma}}{(z^{\sigma})^{\alpha}} = w^{\sigma} \geq (A_1^{\gamma}(t,k_1))^{\sigma},$$

that is,

$$(z^{\sigma})^{\delta}z^{\Delta} \geq r^{-\frac{1}{\gamma}}(A_1^{\sigma}(t,k_1)),\ t \in [t_5,\infty]_{\mathbb{T}},$$

where $\delta = (\frac{\alpha}{\gamma}) > 1$. Using the chain rule, we have

$$\begin{aligned} (z^{1-\delta}(t))^{\Delta} &= (1-\delta)\int_0^1 [(1-h)z(t)+hz(\sigma(t))]^{\delta}dhz^{\Delta}(t) \\ &\leq (1-\delta)(z(\sigma(t)))^{-\delta}z^{\Delta}(t), \end{aligned}$$

that is,

$$\frac{(z^{1-\delta}(\sigma(t)))^{\Delta}}{1-\delta} \geq z(\sigma(t))^{-\delta} z^{\Delta}(\sigma(t)).$$

Hence

$$\frac{(z^{1-\delta}(t))^{\Delta}}{1-\delta} \geq (z(\sigma(t)))^{-\delta} z^{\Delta}(t),$$

and then due to (2.6), we see that

$$\frac{(z^{1-\delta}(t))^{\Delta}}{1-\delta} \geq r^{-\frac{1}{\gamma}}(t)(A_1^{\sigma}(t,k_1)),\ t\in[t_5,\infty)_{\mathbb{T}}.$$

Integrating above inequality from t_5 to t, we get

$$\int_{t_5}^{t} r(s)^{-\frac{1}{\gamma}}\Big(A_1^{\sigma}(s,K_1)\Big)^{\frac{1}{\gamma}}\Delta s<\infty,$$

which is a contradiction to (H_2). The proof is complete. □

Theorem 2.2. *Let $0 \leq p(t) \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\gamma=\alpha=\beta$, $\eta(t)\geq\sigma(t)$ and assume that (H_0), and (H_1) hold. Furthermore assume that*

(H_3). $\limsup\limits_{t\to\infty}\Big(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s)A_1(s,K_1)\Delta s\Big)>1.$

Then every solution of (1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.1, we have

$$w(t)\geq A_1^{\gamma}(t,K_1)\ for\ t\in[t_4,\infty)_{\mathbb{T}}.$$

Using the fact that $r^{\frac{1}{\gamma}}z^{\Delta}$ is nonincreasing on $[t_4,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} z(t)=z(t_4)+\int_{t_4}^{t} z^{\Delta}(s)\Delta s &= z(t_4)+\int_{t_4}^{t} r^{-\frac{1}{\gamma}}(s)\Big(r(s)^{-\frac{1}{\gamma}}z^{\Delta}(s)\Big)\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)r^{-\frac{1}{\gamma}}(s)\Delta s, \end{aligned}$$

that is,

$$\frac{r(t)^{\frac{1}{\gamma}}z^{\Delta}(t)}{z(t)}\leq\Big(\int_{t_4}^{t} r(s)^{-\frac{1}{\gamma}}\Delta s\Big)^{-1},\ t\geq t_4, \tag{2.7}$$

Consequently,

$$A_1(t,K_1)\leq w^{\frac{1}{\gamma}}(t)=\frac{r(t)^{\frac{1}{\gamma}}z'(t)}{z(t}\leq\Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)^{-1},$$

implies that

$$\Big(\int_{t_4}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)A_1(t,K_1)\leq 1,$$

which contradicts (H_3). Hence the theorem is proved. □

Theorem 2.3. *Let $0 \leq p(t) \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\gamma>\alpha>\beta$, $\eta(t)\geq\sigma(t)$ and assume that (H_0) and (H_2) hold. Furthermore assume that*

(H_4). $\limsup\limits_{t\to\infty}(a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}}\Big(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)\Big[a_1(t)+K_1\int_t^{\infty}\Big(\frac{1}{r(s)}\Big)^{\frac{1}{\gamma}}(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}}\Delta s\Big]^{\frac{1}{\gamma}}=\infty.$

Then every solution of (1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.1, we obtain (2.2) and (2.3) and hence $w(t) > a_1(t)$, for $t\in[t_4,\infty)$. Consequently, it follows from (2.3) that

$$r^{\frac{1}{\gamma}}z^{\Delta}>z^{\frac{\alpha}{\gamma}}a_1^{\frac{1}{\gamma}},\quad \text{for}\quad t\geq t_4.$$

We have $(rz^{\Delta})^{\gamma})^{\Delta} \leq 0$ implies that there exists a constant $C > 0$ and $t_5 > t_4$ such that $r^{\frac{1}{\gamma}}z^{\Delta} \leq C$, for $t \geq t_5$, that is $C \geq r^{\frac{1}{\gamma}}z^{\Delta} > z^{\frac{\alpha}{\gamma}}a_1^{\frac{1}{\gamma}}$ and hence

$$z(t) \leq C^{\frac{\gamma}{\alpha}}a_1(t)^{-\frac{1}{\alpha}}, \text{ for } t \in [t_5, \infty)_{\mathbb{T}}, \tag{2.8}$$

which implies that

$$(z^{\sigma})^{\frac{(\alpha-\gamma)}{\gamma}} \geq C^{\frac{(\alpha-\gamma)}{\alpha}}(a_1^{\sigma})^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \text{ for } t \in [t_5, \infty)_{\mathbb{T}}. \tag{2.9}$$

Due to (2.5), (2.6) and using Lemma 2.2, we have that

$$\begin{aligned} w^{\Delta}(t) \quad &\leq \quad -q(1-p(\delta(t)))^{\alpha}\Big(\frac{l\delta(t)}{\sigma(t)}\Big)^{\alpha} - v(1-p(\delta(t)))^{\alpha}\Big(\frac{l\delta(t)}{\sigma(t)}\Big)^{\alpha} \\ &\quad -\alpha C r^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}}(z^{\sigma}(t))^{\frac{(\alpha-\gamma)}{\alpha}}. \end{aligned}$$

Integrating the last inequality as in the proof of Theorem 2.1 and using (2.8), we obtain for $t \geq t_1 \geq t_5$ that

$$w(t) \geq a_1(t) + K_3\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(s))^{1+\frac{1}{\gamma}}\Delta s, \ for\ t \in [t_l, \infty)_{\mathbb{T}}, \tag{2.10}$$

where $K_1 = \alpha C^{\frac{(\alpha-\gamma)}{\gamma}}$. Substitute (2.9) into (2.3), it is easy to verify that

$$(z(t)^{\frac{(\alpha-\gamma)}{\gamma}}\frac{r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)}{z(t)} \geq \Big[a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}}\Delta s\Big]^{\frac{1}{\gamma}}. \tag{2.11}$$

Using (2.7) and (2.9) in (2.11), we can find

$$\begin{aligned} C^{\frac{\alpha-\gamma}{\alpha}}a_1(t)^{\frac{(\gamma-\alpha)}{\alpha\gamma}}\Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)^{-1} \quad &\geq \quad \Big[a_1(t) \\ +K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}}\Delta s\Big]^{\frac{1}{\gamma}}, \text{ for } t \quad &\in \quad [t_1, \infty)_{\mathbb{T}}. \end{aligned}$$

Therefore, for $t \geq t_1$ we have

$$(a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}}\Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)\Big[a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}}\Delta s\Big]^{\frac{1}{\gamma}} \leq C^{\frac{\alpha-\gamma}{\alpha}},$$

which contradicts (H_4). This completes the proof of theorem. □

Theorem 2.4. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\gamma < \beta < \alpha$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_2) and (H_3) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1. Hence the details are omitted. □

Theorem 2.5. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\alpha > \gamma > \beta$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_2) and (H_3) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1. □

Theorem 2.6. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\alpha < \beta < \gamma$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_1) and (H_4) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1 and Theorem 2.3. □

Theorem 2.7. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\alpha < \gamma < \beta$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_1) and (H_4) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1 and Theorem 2.3. □

In the following theorems we will denote

$$a_2(t) = \int_t^\infty \Big[\lambda Q(s)\left(\frac{l\delta(t)}{\delta(t)}\right)^\alpha + \mu V(s)\left(\frac{l\delta(t)}{\sigma(t)}\right)^\alpha\Big]\Delta s,\ t\in[t_0,\infty)_{\mathbb{T}},$$

and

$$A_2(t,K_2) = \left[\frac{\lambda}{1+a^\alpha}a_2(\tau^{-1}(t)) + \frac{\mu a_2(\tau^{-1}(t))}{1+a^\alpha} + K_2\int_{\tau^{-1}(t)}^\infty \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}((a_2(\tau^{-1\delta}(s)))^{1+\frac{1}{\gamma}}\Delta s\right]^{\frac{1}{\gamma}},$$

where K_2 is an arbitrary positive constant and $a>0$ $\lambda,\mu>0$ are positive constants, $Q(t) = min\{q(t), q(\tau(t))\}$, $V(t) = min\{v(t), v(\tau(t))\}$. From the definitions of τ, δ, η, we see that τ^{-1}, δ^{-1}, $\eta^{-1}:\mathbb{T}\to\mathbb{T}$ and τ^{-1}, δ^{-1}, η^{-1} are rd-continuous functions and $\tau^{-1}(t)\geq t, \delta^{-1}(t)\geq t$ and $\eta^{-1}(t)\geq t$.

Theorem 2.8. *Let $1\leq p(t)\leq p<\infty$, $r^\Delta(t)\geq 0$ $\tau(\delta(t)) = \delta(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$ and $\gamma<\alpha<\beta$, $\eta(t)\geq\delta(t)$ and If (H_0) holds and the following conditions hold:*
(H_5). $\limsup_{t\to\infty} a_2(t)<\infty$,
(H_6). $\int_{t_0}^\infty (\frac{1}{r(s)})^{\frac{1}{\gamma}} A_2^\sigma(s,K_2)\Delta s = \infty$,
Then every solution of (1.1) oscillates.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t)>0$ for $t\geq t_0$. Proceeding as in the proof of Theorem 2.1, we get (2.2) for $t\in[t_2,\infty)$, that is either $z^\Delta(t)>0$ or $z^\Delta(t)<0$. By lemma 2.1, it follows that $z^\Delta(t)>0$. From (1.1), it is easy to see for $t\geq t_1$, that

$$(r(t)(z^\Delta(t))^\gamma)^\Delta + p^\beta(r(\tau(t))(z^\Delta(\tau(t))^\gamma)^\Delta + q(t)x^\alpha(\delta(t)) + p^\beta q(\tau(t))x^\alpha(\delta(\tau(t)) + v(t)x^\beta(\eta(t)) + p^\beta v(\tau(t))x^\beta(\eta(\tau(t)) = 0. \tag{2.12}$$

By assuming that there exists $\lambda>0$ such that $u^\gamma(x)+u^\gamma(y)\geq\lambda u^\gamma(x+y)$, x, $y\in\mathbb{R}^+$, and there exists $\mu>0$ such that $u^\gamma(x)+u^\gamma(y)\geq\mu u^\gamma(x+y)$, x, $y\in\mathbb{R}^+$, we obtain (note that $\gamma<\alpha<\beta$) that

$$(r(t)(z^\Delta(t))^\gamma)^\Delta + p^\alpha(r(\tau(t))(z^\Delta(\tau(t))^\gamma)^\Delta + \lambda Q(t)z^\alpha(\delta(t)) + \mu V(t)z^\alpha(\eta(t))\leq 0.$$

for $t\in[t_2,\infty)_{\mathbb{T}}$, where $z(t)\leq x(t)+px(\tau(t))$. Define $w(t)$ as in (2.3). Upon using the fact that

$$w^\Delta(t) = \frac{(r(z^\Delta)^\gamma)^\Delta}{(z^\delta)^\alpha} - \frac{r(z^\Delta)^\gamma)^\delta(z^\alpha)^\Delta}{z^\alpha(z^\delta)^\alpha}$$

and

$$\frac{(z^\alpha)^\Delta}{(z^\sigma)^\alpha}\geq\alpha\frac{(z^\Delta)}{z^\sigma},\ \alpha>0\ for\ t\in[t_3,\infty)_{\mathbb{T}}.$$

By using the fact that $z(t)$ is nondecreasing and using (2.12) into (2.11) we obtain

$$w^\Delta\leq\frac{(r(z^\Delta)^\gamma)^\Delta}{(z^\sigma)^\alpha} - \alpha w^\sigma\frac{z^\Delta}{z^\rho},\ t\geq t_3.$$

Due to (2.6) and $(z(\sigma(t)))^{\frac{\alpha}{\gamma}}\geq C$, there exists $t_4>t_3$ such that, for $t\in[t_4,\infty)_{\mathbb{T}}$,

$$w^\Delta\leq\frac{(r(z^\Delta)^\gamma)^\Delta}{(z^\sigma)^\alpha} - \alpha Cr^{-\frac{1}{\gamma}}(w^\sigma)^{1+\frac{1}{\gamma}}. \tag{2.13}$$

From (2.13), we find

$$w^\Delta + a^\alpha w^{\tau\Delta}\leq\frac{(r(z^\Delta)^\gamma)^\Delta}{(z^\sigma)^\alpha} - \alpha Cr^{-\frac{1}{\gamma}}(w^\sigma)^{1+\frac{1}{\gamma}} + a^\alpha\frac{(r(z^\Delta)^\gamma)^{\tau\Delta}}{(z^{\sigma\Delta})^\alpha} - \alpha C(r^\tau)^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}},$$

that is,

$$w^{\Delta}+a^{\alpha}w^{\tau\Delta}\leq\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}+a^{\alpha}\frac{(r(z^{\Delta})^{\gamma})^{\tau\Delta}}{(z^{\sigma\Delta})^{\alpha}}$$
$$-\alpha C\left[r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}}\right].$$

Applying Lemma 2.2 on the above inequality, we get

$$\begin{aligned} w^{\Delta}+a^{\alpha}w^{\tau\Delta} &\leq -\lambda Q\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\mu V\left(\frac{l\delta}{\sigma}\right)^{\alpha} \\ &\quad -\alpha C\left[r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}}\right] \end{aligned}$$

for $t\in[t_1,\infty)_{\mathbb{T}}$, that is

$$w^{\Delta}+a^{\alpha}w^{\tau\Delta}\leq-\lambda Q(t)\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\mu V(t)\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\alpha Cr^{-\frac{1}{\gamma}}(1+a^{\alpha})(w^{\sigma})^{1+\frac{1}{\gamma}}, \tag{2.14}$$

where we used the fact that $r^{\Delta}(t)\geq 0$ and $w(t)$ is a decreasing function due to (2.6) and (2.14) on $[t_1,\infty)_{\mathbb{T}}$. Integrating (2.14) from t to v for t, $v\in[t_1,\infty)_{\mathbb{T}}$, it is easy to verify that

$$w^{\Delta}+a^{\alpha}w^{\tau(t)}\geq\int_t^{\infty}\lambda Q(s)\left(\frac{l\delta}{\sigma}\right)^{\alpha}\Delta s+\int_t^{\infty}\mu V(s)\left(\frac{l\delta}{\sigma}\right)^{\alpha}\Delta s$$
$$+\alpha C(1+a^{\alpha})\int_t^{\infty}\left[r(s)^{-\frac{1}{\gamma}}w(\sigma(s))^{1+\frac{1}{\gamma}}\right]\Delta s,$$

that is,

$$w^{\Delta}+a^{\alpha}w^{\tau(t)}=a_2(t)+\alpha C(1+a^{\alpha})\int_t^{\infty}\left[r(s)^{-\frac{1}{\gamma}}w(\sigma(s))^{1+\frac{1}{\gamma}}\right]\Delta s,$$

which implies that

$$(1+a^{\alpha})w(\tau(t))\geq a_2(t)+\alpha C(1+a^{\alpha})\int_t^{\infty}r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s. \tag{2.15}$$

Then (H_2) and (2.15) yield that

$$w(t)\geq\frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+\alpha C\int_{\tau^{-1}(t)}^{\infty}r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s.$$

Indeed

$$w(t)\geq\frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}.$$

Hence the last inequality becomes

$$\begin{aligned} w(t) &\geq \frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+\alpha C\int_{\tau^{-1}(t)}^{\infty}\left[r^{-\frac{1}{\gamma}}(s)\left(\frac{1}{1+a^{\alpha}}\right)^{1+\frac{1}{\gamma}}(a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right]\Delta s \\ &= \frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+K_2\int_{\tau^{-1}(t)}^{\infty}\left[r^{-\frac{1}{\gamma}}(s)(a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right]\Delta s \\ &= A_2^{\gamma}(t,K_2), K_2=\alpha C\left(\frac{1}{1+a^{\alpha}}\right)^{1+\frac{1}{\gamma}}. \end{aligned}$$

Proceeding as in the proof of theorem 2.1, we obtain

$$\int_{t_4}^{t}r^{-\frac{1}{\gamma}}(s)A_2^{\sigma}(s,K_2)\Delta s<\infty,$$

a contradiction due to (H_6). The proof is complete. □

Theorem 2.9. *Let* $1 \leq p(t) \leq p < \infty$, $r^{\Delta}(t) \geq 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $\tau(\delta(t)) = \delta(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$ *and* $\gamma = \alpha = \beta$, $\eta(t) \geq \delta(t)$. *If* (H_0), $(H_5) - (H_7)$ *and*

(H_7). $\limsup_{t\to\infty} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s) A_2(s, K_2)\Delta s \right) > 1.$

Then every solution of (1.1) oscillates.

Proof. The proof of the theorem follows from Theorem 2.2 and Theorem 2.8. Hence the details are omitted. □

Theorem 2.10. *Let* $1 \leq p(t) \leq a < \infty$, $r^{\Delta}(t) \geq 0$ $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\gamma > \alpha > \beta$, $\eta(t) \geq \delta(t)$. *If* (H_0), (H_2), $(H_5) - (H_7)$ *and*

(H_8). $\limsup_{t\to\infty} (a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s \right) \left[a_1(t) + K_3 \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s \right]^{\frac{1}{\gamma}} = \infty.$

Then every solution of (1.1) oscillates.

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Semilocal Convergence of a Newton-Secant Solver for Equations with a Decomposition of Operator

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Abstract. We provide the semilocal convergence analysis of the Newton-Secant solver with a decomposition of a nonlinear operator under classical Lipschitz conditions for the first order Fréchet derivative and divided differences. We have weakened the sufficient convergence criteria, and obtained tighter error estimates. We give numerical experiments that confirm theoretical results. The same technique without additional conditions can be used to extend the applicability of other iterative solvers using inverses of linear operators. The novelty of the paper is that the improved results are obtained using parameters which are special cases of the ones in earlier works. Therefore, no additional information is needed to establish these advantages.

Keywords: Newton-Secant solver; semilocal convergence analysis; Fréchet derivative; divided differences; decomposition of nonlinear operator

AMS Classification: 45B05, 47J05, 65J15, 65J22

1 Introduction

One of the important problems in Computational Mathematics including Mathematical Biology, Chemistry, Economic, Physics, Engineering and other disciplines is finding solutions of nonlinear equations and systems of nonlinear equations [1-14]. For most of these problems, to find the exact solution is difficult or impossible. Therefore, the development and research of numerical methods for solving nonlinear problems is an urgent task.

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A popular solver for dealing with nonlinear equations is Newton's [2, 3, 4]. But it is not applicable, if functions are nondifferentiable. In this case, we can apply solvers with divided differences [1, 2, 3, 7, 8, 10, 11]. If it is possible to decompose into differentiable and nondifferentiable parts, it is advisable to use combined methods [2, 3, 5, 6, 12, 13, 14].

Consider a nonlinear equation

$$F(x) + G(x) = 0, \tag{1}$$

where the operators F and G are defined on a open convex set D of a Banach space E_1 with values in a Banach space E_2 , F is a Fréchet differentiable operator, G is a continuous operator for which differentiability is not assumed. It is necessary to find an approximate solution $x_* \in D$ that satisfies equation (1).

In this paper, we consider the Newton-Secant solver

$$x_{n+1} = x_n - [F'(x_n) + G(x_{n-1}, x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, ... \tag{2}$$

This iterative process was proposed in [6] and studied in [2, 3, 13], and the convergence order $\frac{1+\sqrt{5}}{2}$ was established. It is shown that (2) converges faster than the Secant solver.

In this paper, we study solver (2) under the classical Lipschitz conditions for first-order Fréchet derivative and divided differences. Our technique allows to get the weaker convergence criteria, and tighter error estimates. This way, we extended the applicability of the results obtained in [13].

2 Convergence Analysis

Let $L(E_1, E_2)$ be a space of linear bounded operators from E_1 into E_2. Set $S(x, \tau) = \{y \in E_1 : \|y - x\| < \tau\}$ and let $\bar{S}(x, \tau)$ denote its closure.

Define quadratic polynomial φ by

$$\varphi(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3$$

and parameters r, and r_1 by

$$r = \frac{1 - (q_0 + \bar{q}_0)a}{p_0 + q_0 + 2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0},$$

$$r_1 = \frac{1 - \bar{q}_0 a}{2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0},$$

where

$$\alpha_1 = p_0 + q_0 + 2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0,$$

$$\alpha_2 = -[1 - (q_0 + \bar{q}_0)a + (2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0)c]$$

and

$$\alpha_3 = (1 - \bar{q}_0 a)c,$$

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where p_0, $\bar{p}_0$, q_0, $\bar{q}_0$, $\bar{\bar{q}}_0$, a and c are nonnegative numbers.

Suppose that $(q_0 + \bar{q}_0)a < 1$ and $\varphi\big(\frac{1}{2}r\big) \le 0$. Then, it is simple algebra to show, function φ has a unique root $\bar{r}_0 \in (0, \frac{r}{2}]$, and

$$r \le r_1,$$

$$\bar{\gamma} = \frac{p_0\bar{r}_0 + q_0(\bar{r}_0 + a)}{1 - \bar{q}_0 a - (2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0)\bar{r}_0} \in [0, 1)$$

and

$$\bar{r}_0 \ge \frac{c}{1 - \bar{\gamma}}.$$

Set $D_0 = D \cap S(x_0, r_1)$.

Definition 2.1. *We call an operator that acts from E_1 into E_2 and is denoted by $G(x, y)$ a first-order divided difference for the operator G by fixed points x and y $(x \ne y)$, if the equality*

$$G(x, y)(x - y) = G(x) - G(y)$$

is satisfied.

Theorem 2.2. *Suppose that:*

1) *F and G are nonlinear operators on an open convex set D of a Banach space E_1 into a Banach space E_2;*
2) *F is a Fréchet-differentiable operator, and let G is a continuous operator;*
3) *$G(\cdot, \cdot)$ is the first-order divided differences of the operator G defined on the set D;*
4) *the linear operator $A_0 = F'(x_0) + G(x_{-1}, x_0)$, where x_{-1}, $x_0 \in D$, is invertible;*
5) *the following conditions are satisfied for all $x, y, \in D$*

$$\|A_0^{-1}(F'(x_0) - F'(x))\| \le 2\bar{p}_0\|x_0 - x\|, \tag{3}$$

$$\|A_0^{-1}(G(x_{-1}, x_0) - G(x, x_0))\| \le \bar{q}_0\|x_{-1} - x\|, \tag{4}$$

$$\|A_0^{-1}(G(x, x_0) - G(x, y))\| \le \bar{\bar{q}}_0\|x_0 - y\|, \tag{5}$$

and for all $x, y, u \in D_0$

$$\|A_0^{-1}(F'(x) - F'(y))\| \le 2p_0\|x - y\|, \tag{6}$$

$$\|A_0^{-1}(G(x, y) - G(u, y))\| \le q_0\|x - u\|; \tag{7}$$

6) *a, c are nonnegative numbers such that*

$$\|x_0 - x_{-1}\| \le a,\ \|A_0^{-1}(F(x_0) + G(x_0))\| \le c,\ c > a, \tag{8}$$

$$(q_0 + \bar{q}_0)a < 1, \quad \varphi\Big(\frac{1}{2}r\Big) \le 0; \tag{9}$$

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7) $\bar{S}(x_0, \bar{r}_0) \subset D$.

Then, the solver (2) is well-defined and the sequence generated by it converges to the solution x_* *of equation (1), so that for each* $n \in \{-1, 0, 1, 2, ...\}$, *the following inequalities are satisfied*

$$\|x_n - x_{n+1}\| \leq t_n - t_{n+1}, \tag{10}$$

$$\|x_n - x_*\| \leq t_n - \bar{t}_*, \tag{11}$$

where sequence $\{t_n\}_{n\geq -1}$ *defined by the formulas*

$$t_{-1} = \bar{r}_0 + a,\ t_0 = \bar{r}_0,\ t_1 = \bar{r}_0 - c,$$

$$t_{n+1} - t_{n+2} = \bar{\gamma}_n(t_n - t_{n+1}),\ n \geq 0,$$

$$\bar{\gamma}_n = \frac{\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})},\ 0 \leq \bar{\gamma}_n < \bar{\gamma} \tag{12}$$

is decreasing, nonnegative, and converges to $\bar{t}_*$, *so that* $\bar{r}_0 - c/(1-\bar{\gamma}) \leq \bar{t}_* < t_0$, *where*

$$\tilde{p}_0 = \begin{cases} \bar{p}_0,\ n = 0 \\ p_0,\ n > 0 \end{cases},\quad \tilde{q}_0 = \begin{cases} \bar{q}_0,\ n = 0 \\ q_0,\ n > 0. \end{cases}$$

Proof. We use mathematical induction to show that, for each $k \geq 0$ the following inequalities are satisfied

$$t_{k+1} \geq t_{k+2} \geq \bar{r}_0 - \frac{1 - \bar{\gamma}^{k+2}}{1 - \bar{\gamma}} c \geq \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \geq 0, \tag{13}$$

$$t_{k+1} - t_{k+2} \leq \bar{\gamma}(t_k - t_{k+1}). \tag{14}$$

Setting $k = 0$ in (12), we get

$$t_1 - t_2 = \frac{\tilde{p}_0(t_0 - t_1) + \tilde{q}_0(t_{-1} - t_1)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_1) - \bar{\bar{q}}_0(t_0 - t_1)}(t_0 - t_1) \leq \bar{\gamma}(t_0 - t_1),$$

$$t_0 \geq t_1,\ t_1 \geq t_2 \geq t_1 - \bar{\gamma}(t_0 - t_1) \geq \bar{r}_0 - (1 + \bar{\gamma})c = \bar{r}_0 - \frac{(1 - \bar{\gamma}^2)c}{1 - \bar{\gamma}} \geq \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \geq 0.$$

Suppose that (13) and (14) are true for $k = 0, 1, ..., n-1$. Then, for $k = n$, we obtain

$$\begin{aligned} t_{n+1} - t_{n+2} &= \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\ &\leq \frac{\tilde{p}_0 t_n + \tilde{q}_0 t_{n-1}}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}(t_n - t_{n+1}) \leq \bar{\gamma}(t_n - t_{n+1}), \end{aligned}$$

$$t_{n+1} \geq t_{n+2} \geq t_{n+1} - \bar{\gamma}(t_n - t_{n+1}) \geq \bar{r}_0 - \frac{1 - \bar{\gamma}^{n+2}}{1 - \bar{\gamma}} c \geq \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \geq 0.$$

Thus, $\{t_n\}_{n\geq 0}$ is a decreasing nonnegative sequence, and converges to $\bar{t}_* \geq 0$.

Let us prove that the method (2) is well-defined, and for each $n \geq 0$ the inequality (10) is satisfied.

Since $t_{-1} - t_0 = a$, $t_0 - t_1 = c$ and conditions (8) are fulfilled then $x_1 \in S(x_0, \bar{r}_0)$ and (10) is satisfied for $n \in \{-1, 0\}$. Let conditions (8) be satisfied for $k = 0, 1, ..., n$. Let us prove that the method (2) is well-defined for $k = n + 1$.

Denote $A_n = F'(x_n) + G(x_{n-1}, x_n)$. Using the Lipschitz conditions (3) – (5), we have

$$\begin{aligned}
\|I - A_0^{-1}A_{n+1}\| &= \|A_0^{-1}(A_0 - A_{n+1})\| \leq \|A_0^{-1}(F'(x_0) - F'(x_{n+1}))\| \\
&+ \|A_0^{-1}(G(x_{-1}, x_0) - G(x_n, x_0) + G(x_n, x_0) - G(x_n, x_{n+1}))\| \\
&\leq 2\bar{p}_0\|x_0 - x_{n+1}\| + \bar{q}_0(\|x_{-1} - x_0\| + \|x_0 - x_n\|) + \bar{\bar{q}}_0\|x_0 - x_{n+1}\| \\
&\leq 2\bar{p}_0\|x_0 - x_{n+1}\| + \bar{q}_0 a + \bar{q}_0\|x_0 - x_n\| + \bar{\bar{q}}_0\|x_0 - x_{n+1}\| \\
&\leq \bar{q}_0 a + 2\bar{p}_0(t_0 - t_{n+1}) + \bar{q}_0(t_0 - t_n) + \bar{\bar{q}}_0(t_0 - t_{n+1}) \\
&\leq \bar{q}_0 a + 2\bar{p}_0\bar{r}_0 + \bar{q}_0\bar{r}_0 + \bar{\bar{q}}_0\bar{r}_0 < 1.
\end{aligned}$$

According to the Banach lemma on inverse operators [2] A_{n+1} is invertible, and

$$\|A_{n+1}^{-1}A_0\| \leq (1 - \bar{q}_0 a - 2\bar{p}_0\|x_0 - x_{n+1}\| - \bar{q}_0\|x_0 - x_n\| + \bar{\bar{q}}_0\|x_0 - x_{n+1}\|)^{-1}.$$

By the definition of the divided difference and conditions (6), (7), we obtain

$$\begin{aligned}
&\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\
&= \|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}) - F(x_n) - G(x_n) - A_n(x_n - x_{n+1}))\| \\
&\leq \|A_0^{-1}(\textstyle\int_0^1\{F'(x_{n+1} + t(x_n - x_{n+1})) - F'(x_n)\}dt)\|\|x_n - x_{n+1}\| \\
&+ \|A_0^{-1}(G(x_{n+1}, x_n) - G(x_{n-1}, x_n))\|\|x_n - x_{n+1}\| \\
&\leq (\tilde{p}_0\|x_n - x_{n+1}\| + \tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|))\|x_n - x_{n+1}\|.
\end{aligned}$$

In view of condition (10), we have

$$\begin{aligned}
\|x_{n+1} - x_{n+2}\| &= \|A_{n+1}^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\
&\leq \|A_{n+1}^{-1}A_0\|\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\
&\leq \frac{\tilde{p}_0\|x_n - x_{n+1}\| + \tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|)}{1 - \bar{q}_0 a - 2\bar{p}_0\|x_0 - x_{n+1}\| - \bar{q}_0\|x_0 - x_{n+1}\| + \bar{\bar{q}}_0\|x_0 - x_n\|}\|x_n - x_{n+1}\| \\
&\leq \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} = t_{n+1} - t_{n+2}.
\end{aligned}$$

Thus, the method (2) is well-defined for each $n \geq 0$. Hence it follows that

$$\|x_n - x_k\| \leq t_n - t_k, \quad -1 \leq n \leq k. \tag{15}$$

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Therefore, the sequence $\{x_n\}_{n\geq 0}$ is fundamental, so it converges to some $x_* \in \bar{S}(x_0, \bar{r}_0)$. Inequality (11) is obtained from (15) for $k \to \infty$. Let us show that x_* solves the equation $F(x) + G(x) = 0$. Indeed, we get in turn that

$$A_0^{-1}(F(x_{n+1}) + G(x_{n+1})) \leq \big(\tilde{p}_0\|x_n - x_{n+1}\|$$
$$+\tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|)\big)\|x_n - x_{n+1}\| \to 0, \quad n \to \infty.$$

Hence, $F(x_*) + G(x_*) = 0$. □

Remark 2.3. *The order of convergence of method (2) is equal to* $\dfrac{1+\sqrt{5}}{2}$.

Proof. In view of $t_n - t_{n+1} \leq \bar{\gamma}(t_{n-1} - t_n)$, and (12), we obtain

$$\begin{aligned} t_{n+1} - t_{n+2} &= \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_n - t_{n+1} + t_{n-1} - t_n)\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\ &\leq \frac{\tilde{p}_0\bar{\gamma}(t_{n-1} - t_n) + \tilde{q}_0(1+\bar{\gamma})(t_{n-1} - t_n)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})}(t_n - t_{n+1}) \\ &= \frac{\big(\bar{p}_0\bar{\gamma} + \bar{q}_0(1+\bar{\gamma})\big)(t_n - t_{n+1})(t_{n-1} - t_n)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\ &\leq \frac{\tilde{p}_0\bar{\gamma} + \tilde{q}_0(1+\bar{\gamma})}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}(t_n - t_{n+1})(t_{n-1} - t_n). \end{aligned}$$

Denote $\bar{C} = \dfrac{\bar{p}_0\bar{\gamma} + \bar{q}_0(1+\bar{\gamma})}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}$. Clearly,

$$t_{n+1} - t_{n+2} \leq \bar{C}(t_{n-1} - \bar{t}_*)(t_n - \bar{t}_*). \tag{16}$$

Since, for each $k > 2$, the estimate is satisfied

$$t_{n+k-1} - t_{n+k} \leq \bar{\gamma}^{k-2}(t_{n+1} - t_{n+2}),$$

we get

$$\begin{aligned} t_{n+1} - t_{n+k} &= t_{n+1} - t_{n+2} + t_{n+2} - t_{n+3} + \ldots + t_{n+k-1} - t_{n+k} \\ &\leq (1 + \bar{\gamma} + \ldots + \bar{\gamma}^{k-2})(t_{n+1} - t_{n+2}) \\ &= \frac{1 - \bar{\gamma}^{k-1}}{1 - \bar{\gamma}}(t_{n+1} - t_{n+2}) \leq \frac{1}{1 - \bar{\gamma}}(t_{n+1} - t_{n+2}). \end{aligned}$$

In view of (16), for $k \to \infty$, we have

$$t_{n+1} - \bar{t}_* \leq \frac{\bar{C}}{1 - \bar{\gamma}}(t_{n-1} - \bar{t}_*)(t_n - \bar{t}_*)$$

Hence, it follows that the order of convergence of the sequence $\{t_n\}_{n\geq 0}$ is equal to $\dfrac{1+\sqrt{5}}{2}$, and, according (11), the sequence $\{x_n\}_{n\geq 0}$ converges with the same order. □

Remark 2.4. *(a) The following conditions were used for each $x, y, u, v \in D$ in [13]*

$$\|A_0^{-1}(F'(y) - F'(x))\| \le 2P_0\|y - x\|, \tag{17}$$

$$\|A_0^{-1}(G(x,y) - G(u,v))\| \le Q_0(\|x - u\| + \|y - v\|), \tag{18}$$

$$\begin{aligned} r_0 &\ge \frac{c}{1-\gamma},\ Q_0 a + 2P_0 r_0 + 2Q_0 r_0 < 1, \\ \gamma &= \frac{P_0 r_0 + Q_0(r_0 + a)}{1 - Q_0 a - 2P_0 r_0 - 2Q_0 r_0},\ 0 \le \gamma < 1. \end{aligned} \tag{19}$$

But, then we have

$$\begin{aligned} \bar{p}_0 &\le P_0, \\ \bar{q}_0 &\le Q_0, \\ \bar{\bar{q}}_0 &\le Q_0, \end{aligned}$$

since $D_0 \subseteq D$, (3) and (4), (5), (7) are weaker than (17) and (18) respectively for $\bar{r}_0 \le r_0$. Notice that sufficient convergence criteria (9) imply (19) but not necessarily vice versa, unless if $\bar{p}_0 = P_0$, $\bar{q}_0 = \bar{\bar{q}}_0 = Q_0$ and $\bar{r}_0 = r_0$.

A simple inductive argument shows that

$$\bar{\gamma}_n \le \gamma_n, \tag{20}$$

$$t_n - t_{n+1} \le s_n - s_{n+1}, \tag{21}$$

where

$$s_{-1} = r_0 + a,\ s_0 = r_0,\ s_1 = r_0 - c,$$

$$s_{n+1} - s_{n+2} = \gamma_n(s_n - s_{n+1}),\ n \ge 0,$$

$$\gamma_n = \frac{P_0(s_n - s_{n+1}) + Q_0(s_{n-1} - s_{n+1})}{1 - Q_0 a - 2P_0(s_0 - s_{n+1}) - Q_0(2s_0 - s_n - s_{n+1})},\ 0 \le \gamma_n \le \gamma.$$

Notice that the corresponding quadratic polynomial φ_1 to φ is defined similarly by

$$\varphi_1(t) = b_1 t^2 + b_2 t + b_3$$

where

$$b_1 = 3P_0 + 3Q_0,$$

$$b_2 = -[1 - 2Q_0 a + (2P_0 + 2Q_0)c]$$

and

$$b_3 = (1 - Q_0 a)c.$$

We have by these definitions that

$$\alpha_1 < b_1,\ \alpha_2 < b_2,\ \text{but } \alpha_3 > b_3.$$

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Therefore, we cannot tell, if $r_0 < \bar{r}_0$ *or* $\bar{r}_0 < r_0$ *or* $r_0 = \bar{r}_0$. *But, we have*

$$\begin{aligned} \gamma \le \bar{\gamma} &\Rightarrow r_0 \le \bar{r}_0, \\ s_n &\le t_n, \\ s_* \le \bar{t}_* &= \lim_{n\to\infty} t_n \end{aligned} \tag{22}$$

and

$$\begin{aligned} \bar{\gamma} \le \gamma \Rightarrow \bar{r}_0 \le r_0 &\Rightarrow \bar{C} \le C, \\ t_n &\le s_n, \\ \bar{t}_* \le s_* &= \lim_{n\to\infty} s_n, \end{aligned} \tag{23}$$

It is simple algebra to show that $\varphi(r) \ge 0$, *and for* $r_{min} = -\dfrac{\alpha_2}{2\alpha_1}$ *(solving* $\varphi'(t) = 0$*),* $r_{min} \ge \dfrac{r}{2}$, $r_{min} \le \dfrac{r_1}{2}$. *Hence, one may replace the second inequation in (9) by* $\varphi(\lambda r) \le 0$ *for some* $\lambda \in (0, \frac{1}{2}]$ *to obtain a better information about the location of* $\bar{r}_0$, *if* $\lambda \ne \frac{1}{2}$, *especially in the case when we do not actually need to compute* $\bar{r}_0$.

(b) The Lipschitz parameters $\bar{p}_0$, $\bar{q}_0$, $\bar{\bar{q}}_0$ *can become even smaller, if we define the set* $D_1 = D \cap S(x_1, r_1 - c)$ *for* $r_1 > c$ *to replace* D_0 *in Theorem 2.2., since* $D_1 \subseteq D_0$.

3 Numerical experiments

Let us define function $F + G : R \to R$, where

$$F(x) = e^{x-0.5} + x^3 - 1.3,\ G(x) = 0.2x|x^2 - 2|.$$

The exact solution of $F(x) + G(x) = 0$ is $x_* = 0.5$. Let $D = (0, 1)$. Then

$$F'(x) = e^{x-0.5} + 3x^2,$$

$$G(x, y) = \frac{0.2x(2 - x^2) - 0.2y(2 - y^2)}{x - y} = 0.2(1 - x^2 - xy - y^2).$$

$$A_0 = e^{x_0-0.5} + 3x_0^2 + 0.2(1 - x_{-1}^2 - x_{-1}x_0 - x_0^2),$$

$$|A_0^{-1}(F'(x) - F'(y))| \le \frac{e^{0.5} + 3|x + y|}{|A_0|}|x - y|,$$

$$|A_0^{-1}(G(x, y) - G(u, v))| = \frac{0.2}{|A_0|}|(u + x + y)(u - x) + (v + y + u)(v - y)|.$$

Let $x_0 = 0.57$, $x_{-1} = 0.571$. Then, we have $a = 0.001$, $c \approx 0.0660157$, $\bar{p}_0 \approx 1.4118406$, $\bar{q}_0 \approx 0.1901483$, $\bar{\bar{q}}_0 \approx 0.2282491$, $r_1 \approx 0.3083854$,

$$D_0 = D \cap S(x_0, r_1) = (0.2616146, 0.8783854),$$

$p_0 \approx 1.5362481$, $q_0 \approx 0.2340358$, $P_0 \approx 1.6982621$, $Q_0 \approx 0.2664386$, and $r \approx 0.1994221$, $\varphi(\frac{1}{2}r) \approx -0.0051722 < 0$. So, $\bar{p}_0 < P_0$, $\bar{q}_0 < Q_0$, $\bar{\bar{q}}_0 < Q_0$.

By solving inequalities $\varphi(t) \leq 0$ and $\varphi_1(t) \leq 0$, we get

$$t \in [0.0824903,\ 0.1596319] \Rightarrow \bar{r}_0^{(1)} \approx 0.0824903,\ \bar{r}_0^{(2)} \approx 0.1596319,$$

$$t \in [0.0924062,\ 0.1211750] \Rightarrow r_0^{(1)} \approx 0.0924062,\ r_0^{(2)} \approx 0.1211750.$$

Then $\bar{r}_0 = \bar{r}_0^{(1)} \approx 0.0824903$, $r_0 = r_0^{(1)} \approx 0.0924062$, and

$$S(x_0, \bar{r}_0) = (0.4875097,\ 0.6524903),\ \bar{\gamma} \approx 0.1997151 < 1,\ \bar{C} \approx 0.8023108,$$

$$S(x_0, r_0) = (0.4775938,\ 0.6624062),\ \gamma \approx 0.2855916 < 1,\ C \approx 1.2998717.$$

In Table 1, there are results that confirm estimates (10), (11) and (21). Table 2 shows that sequences $\{t_n\}$ and $\{s_n\}$ converge to $\bar{t}_* \approx 0.0073550$ and $s_* \approx 0.0144209$, respectively, and confirms (20) and (23).

Table 1: Obtained results for $\varepsilon = 10^{-7}$

n	$\lvert x_{n-1} - x_n \rvert$	$t_{n-1} - t_n$	$s_{n-1} - s_n$	$\lvert x_n - x_* \rvert$	$t_n - \bar{t}_*$	$s_n - s_*$
1	0.0660157	0.0660157	0.0660157	0.0039843	0.0091195	0.0119695
2	0.0040123	0.0087609	0.0113203	0.0000281	0.0003586	0.0006492
3	0.0000281	0.0003573	0.0006452	1.761e-08	0.0000013	0.0000040
4	1.761e-08	0.0000040	0.0000040	7.438e-14	1.440e-10	1.033e-09

Table 2: Obtained results for $\varepsilon = 10^{-7}$

n	t_n	s_n	$\bar{\gamma}_{n-2}$	γ_{n-2}
-1	0.0834903	0.0934062		
0	0.0824903	0.0924062		
1	0.0164746	0.0263904		
2	0.0077136	0.0150701	0.1327096	0.1714793
3	0.0077136	0.0144249	0.0407873	0.0569927
4	0.0073550	0.0144209	0.0035475	0.0061771
5	0.0073550	0.0144209	0.0001136	0.0002592

4 Conclusions

We investigated the semilocal convergence of Newton-Secant solver under classical center and restricted Lipschitz conditions. This technique weakens the

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sufficient convergence criteria without adding more conditions and uses constants that are specializations of earlier ones. Moreover, tighter estimate errors are obtained. The theoretical results are confirmed by numerical experiments. Our technique can be used to extend the applicability of other iterative methods using inverses of linear operators [1-14] along the same lines.

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Global behavior of a nonlinear higher-order rational difference equation

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Abstract

In this paper, we investigate the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum_{i=1}^{k} x_{n-2m_i}^{p-1} \prod_{j=1}^{k} x_{n-2m_j}}, \quad n = 0, 1, 2, ...$$

with positive parameters and non-negative initial conditions.

Keywords: Recursive sequences; Global asymptotic stability; Oscillation; Period two solutions; Semicycles.

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1. INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

The study of these equations is quite challenging and rewarding and is still in its infancy.

We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore, that results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations.

El-Owaidy et al [1] investigated the global asymptotic behavior and the periodic character of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \qquad n = 0, 1, 2, ...$$

where the parameters $\alpha,\ \beta,\ \gamma$ and p are non-negative real numbers.

Other related results on rational difference equations can be found in refs. [2-15].

In this paper, we investigate the global asymptotic behavior and the periodic character of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum\limits_{i=1}^{k} x_{n-2m_i}^{p-1} \prod\limits_{j=1}^{k} x_{n-2m_j}}, \qquad n = 0, 1, 2, ... \tag{1.1}$$

where the parameters $\alpha,\ \beta,\ \gamma$ and p are positive real numbers, $k \in \{1, 2, ...\}$, $\{m_i\}_{i=1}^{k}$ be positive integers such that $m_i > m_{i-1}$; $i = 2, ...k$ and the initial conditions $x_{-2m_k}, x_{-2m_k+1}, ..., x_0$ are non-negative real numbers.

The results in this work are consistent with the results in [1] when $k = 1$ and $m_1 = 1$.

The results in this work are consistent with the results in [3] when $k = 2$, $m_1 = 1$ and $m_2 = 2$.

We need the following definitions.

Definition 1. *Let I be an interval of real numbers and let*

$$f : I^{k+1} \rightarrow I$$

be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

with $x_{-k}, x_{-k+1}, ..., x_0 \in I$. Let $\overline{x}$ be the equilibrium point of Eq.(1.2). The linearized equation of Eq.(1.2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + ... + c_{k+1} y_{n-k} \tag{1.3}$$

where

$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, ..., \overline{x})$, $c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, ..., \overline{x}), ..., c_{k+1} = \frac{\partial f}{\partial x_{n-k}}(\overline{x}, \overline{x}, ..., \overline{x})$.

The characteristic equation of Eq.(1.3) is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} c_i \lambda^{k-i+1} = 0. \tag{1.4}$$

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}$, $x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).
(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

Definition 2. *A positive semicycle of* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.2) consists of a 'string' of terms* $\{x_l, x_{l+1}, ..., x_m\}$ *, all greater than or equal to* $\overline{x}$*, with* $l \geq -k$ *and* $m < \infty$ *and such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} < \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} < \overline{x}$*.*

A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.2) consists of a 'string' of terms* $\{x_l, x_{l+1}, ..., x_m\}$ *, all less than* $\overline{x}$*, with* $l \geq -k$ *and* $m < \infty$ *and such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} \geq \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} \geq \overline{x}$*.*

Definition 3. *A solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.2) is called nonoscillatory if there exists* $N \geq -k$ *such that either*

$$x_n \geq \overline{x} \;\; \forall n \geq N \quad \text{or} \quad x_n < \overline{x} \;\; \forall n \geq N \;,$$

and it is called oscillatory if it is not nonoscillatory.

(a) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if

$$x_{n+p} = x_n \;\text{ for all } n \geq -k. \tag{1.5}$$

(b) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if it is periodic with period p and p is the least positive integer for which (1.5) holds.

We need the following theorem.

Theorem 1.1. *(i) If all roots of Eq.(1.4) have absolute value less than one, then the equilibrium point* $\overline{x}$ *of Eq.(1.2) is locally asymptotically stable.*

(ii) If at least one of the roots of Eq.(1.4) has absolute value greater than one, then $\overline{x}$ is unstable.

The equilibrium point $\overline{x}$ of Eq.(1.2) is called a saddle point if Eq.(1.4) has roots both inside and outside the unit disk.

2. Main results

In this section, we investigate the dynamics of Eq.(1.1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{p+k-1}} y_n$ reduces Eq.(1.1) to the difference equation

$$y_{n+1} = \frac{ry_{n-1}}{1+\sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}}, \qquad n = 0, 1, 2, ... \tag{2.1}$$

where $r = \frac{\alpha}{\beta} > 0$.

Note that $\overline{y_1} = 0$ is always an equilibrium point of Eq.(2.1). When $r > 1$, Eq.(2.1) also possesses the unique positive equilibrium $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$.

Theorem 2.1. *The following statements are true*

(i) If $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is locally asymptotically stable.

(ii) If $r > 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is a saddle point.

(iii) When $r > 1$, then the positive equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ of Eq.(2.1) is unstable.

Proof: The linearized equation of Eq.(2.1) about the equilibrium point $\overline{y_1} = 0$ is

$$z_{n+1} = rz_{n-1}, \qquad n = 0, 1, 2, ...$$

so, the characteristic equation of Eq.(2.1) about the equilibrium point $\overline{y_1} = 0$ is

$$\lambda^{2m_k+1} - r\lambda^{2m_k-1} = 0,$$

and hence, the proof of (i) and (ii) follows from Theorem A.

For (iii), we assume that $r > 1$, then the linearized equation of Eq.(2.1) about the equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ has the form

$$z_{n+1} = z_{n-1} - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_1} - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_2} - ... - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_k}, \quad n = 0, 1, 2, ...$$

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so, the characteristic equation of Eq.(2.1) about the equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ is

$$f(\lambda) = \lambda^{2m_k+1} - \lambda^{2m_k-1} + \frac{(r-1)(p+k-1)}{rk} \sum_{i=1}^{k} \lambda^{2m_k-2m_i} = 0,$$

It is clear that $f(\lambda)$ has a root in the interval $(-\infty, -1)$, and so, $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ is an unstable equilibrium point.

This completes the proof.

Theorem 2.2. *Assume that $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is globally asymptotically stable.*

Proof: We know by Theorem 2.1 that the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is locally asymptotically stable. So, let $\{y_n\}_{n=-2m_k}^{\infty}$ be a solution of Eq.(2.1). It suffices to show that $\lim_{n\to\infty} y_n = 0$. Since

$$0 \le y_{n+1} = \frac{r y_{n-1}}{1 + \sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}} \le r y_{n-1} < y_{n-1}.$$

So, the even terms of this solution decrease to a limit (say $L_1 \ge 0$), and the odd terms decrease to a limit (say $L_2 \ge 0$), which implies that

$$L_1 = \frac{rL_1}{1 + kL_2^{k+p-1}} \qquad \text{and} \qquad L_2 = \frac{rL_2}{1 + kL_1^{k+p-1}}.$$

If $L_1 \ne 0 \Rightarrow L_2^{k+p-1} = \frac{r-1}{k} < 0$, which is a contradiction, so $L_1 = 0$, which implies that $L_2 = 0$.

So, $\lim_{n\to\infty} y_n = 0$, which the proof is complete.

Theorem 2.3. *Assume that $r = 1$, then Eq.(2.1) possesses the prime period two solution*

$$..., \phi, 0, \phi, 0, ... \tag{2.2}$$

with $\phi > 0$. Furthermore, every solution of Eq.(2.1) converges to a period two solution (2.2) with $\phi \ge 0$.

Proof: Let

$$..., \phi, \psi, \phi, \psi, ...$$

be period two solutions of Eq.(2.1). Then

$$\phi = \frac{r\phi}{1 + k\psi^{k+p-1}}, \quad \text{and} \quad \psi = \frac{r\psi}{1 + k\phi^{k+p-1}},$$

so,

$$k\phi\psi = \frac{(r-1)(\phi - \psi)}{\psi^{k+p-2} - \phi^{k+p-2}} \geq 0,$$

If $k + p > 2$, then we have $r - 1 \leq 0$.

If $r < 1$, then this implies that $\phi < 0$ or $\psi < 0$, which is impossible, so $r = 1$.

If $k + p < 2$, then we have $r - 1 \geq 0$.

If $r > 1$, then we have either $\phi = \psi = 0$, which is impossible or $\phi = \psi = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$, which is impossible, so $r = 1$.

If $k + p = 2$, then we have$(r - 1)(\phi - \psi) = 0$, which implies that $r = 1$.

To complete the proof, assume that $r = 1$ and let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of Eq.(2.1), then

$$y_{n+1} - y_{n-1} = \frac{-y_{n-1}\left(\sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}\right)}{1 + \sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}} \leq 0, \qquad n = 0, 1, 2, ...$$

So, the even terms of this solution decrease to a limit (say $\Phi \geq 0$), and the odd terms decrease to a limit (say $\Psi \geq 0$). Thus,

$$\Phi = \frac{\Phi}{1 + k\Psi^{k+p-1}} \text{ and } \Psi = \frac{\Psi}{1 + k\Phi^{k+p-1}},$$

which implies that $k\Phi\Psi^{k+p-1} = 0$ and $k\Phi^{k+p-1}\Psi = 0$. Then the proof is complete.

Theorem 2.4. *Assume that $r > 1$, and let $\{y_n\}_{n=-2m_k}^{\infty}$ be a solution of Eq.(2.1) such that*

$$y_{-2m_k}, y_{-2m_k+2}, ..., y_0 \geq \overline{y_2} \text{ and } y_{-2m_k+1}, y_{-2m_k+3}, ..., y_{-1} < \overline{y_2}, \tag{2.3}$$

or

$$y_{-2m_k}, y_{-2m_k+2}, ..., y_0 < \overline{y_2} \text{ and } y_{-2m_k+1}, y_{-2m_k+3}, ..., y_{-1} \geq \overline{y_2}. \tag{2.4}$$

Then $\{y_n\}_{n=-2m_k}^{\infty}$oscillates about $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ with a semicycle of length one.

Proof: Assume that (2.3) holds. (The case where (2.4) holds is similar and will be omitted.) Then,

$$y_1 = \frac{ry_{-1}}{1+\sum_{i=1}^{k} y_{-2m_i}^{p-1} \prod_{j=1}^{k} y_{-2m_j}} < \frac{r\overline{y_2}}{1+k\overline{y_2}^{k+p-1}} = \overline{y_2}$$

and

$$y_2 = \frac{ry_0}{1+\sum_{i=1}^{k} y_{-2m_i+1}^{p-1} \prod_{j=1}^{k} y_{-2m_j+1}} > \frac{r\overline{y_2}}{1+k\overline{y_2}^{k+p-1}} = \overline{y_2}$$

and then the proof follows by induction.

Theorem 2.5. *Assume that $r > 1$, then Eq.(2.1) possesses an unbounded solution.*

Proof: From Theorem 2.4, we can assume without loss of generality that the solution $\{y_n\}_{n=-2k}^{\infty}$of Eq.(2.1) is such that

$$y_{2n-1} < \overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}} \quad \text{and} \quad y_{2n} > \overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}, \quad \text{for } n \geq -m_k+1.$$

Then

$$y_{2n+1} = \frac{ry_{2n-1}}{1+\sum_{i=1}^{k} y_{2n-2m_i}^{p-1} \prod_{j=1}^{k} y_{2n-2m_j}} < \frac{ry_{2n-1}}{1+k\overline{y_2}^{k+p-1}} = y_{2n-1}$$

and

$$y_{2n+2} = \frac{ry_{2n}}{1+\sum_{i=1}^{k} y_{2n-2m_i+1}^{p-1} \prod_{j=1}^{k} y_{2n-2m_j+1}} > \frac{ry_{2n}}{1+k\overline{y_2}^{k+p-1}} = y_{2n}$$

from which it follows that

$$\lim_{n\to\infty} y_{2n} = \infty \quad \text{and} \quad \lim_{n\to\infty} y_{2n+1} = 0.$$

Then, the proof is complete.

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Weighted composition operator acting between some classes of analytic function spaces

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Abstract

In this paper, we define some general classes of weighted analytic function spaces in the unit disc. For the new classes, we investigate boundedness and compactness of the weighted composition operator uC_ϕ under some mild conditions on the weighted functions of the classes.

1 Introduction

Let $\mathbb{H}(\mathbb{D})$ denote the class of analytic functions in the unit disk $\mathbb{D}$. As usual, two quantities L_f and M_f, both depending on analytic function f on the unit disk $\mathbb{D}$, are said to be equivalent, and written in the form $L_f \approx M_f$, if there exists a positive constant C such that

$$\frac{1}{C} M_f \le L_f \le C\, M_f.$$

The notation $A \lesssim B$ means that there exists a positive constant C_1 such that $A \le C_1 B$.
For $0 < \alpha < \infty$. The weighted type space H_α^∞ is the space of all $f \in \mathbb{H}(\mathbb{D})$ such that

$$\|f\|_{H_\alpha^\infty} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f(z)| < \infty.$$

and $H_{\alpha,\,0}^\infty$ denotes the closed subspace of H_α^∞ such that $f \in H_\alpha^\infty$ satisfies

$$(1-|z|^2)^\alpha |f(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

Let the Green's function $g(z,a) = \ln |\frac{1-\bar{a}z}{a-z}| = \ln \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{1-\bar{a}z}{a-z}$ stands for Möbius transformation. The following classes of weighted function spaces are defined in [7]:

Definition 1.1 *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and let f be an analytic function in $\mathbb{D}$ then $f \in \mathcal{N}_K$ if*

$$\|f\|_{\mathcal{N}_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))dA(z) < \infty,$$

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where $dA(z)$ defines the normalized area measure on $\mathbb{D}$, so that $A(\mathbb{D}) \equiv 1$.
Now, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))dA(z) = 0,$$

then f is said to belong to the class $\mathcal{N}_{K,0}$.

Clearly, if $K(t) = t^p$, then $\mathcal{N}_K = \mathcal{N}_p$ (see [19]), since $g(z,a) \approx (1 - |\varphi_a(z)|^2)$. For $K(t) = 1$ it gives the Bergman space $\mathcal{A}^2$ (see [17]).
It is easy to check that $\|\cdot\|_{\mathcal{N}_K}$ is a complete semi-norm on $\mathcal{N}_K$ and it is Möbius invariant in the sense that

$$\|f \circ \varphi_a\|_{\mathcal{N}_K} = \|f\|_{\mathcal{N}_K}, \ \ a \in \mathbb{D},$$

whenever $f \in \mathcal{N}_K$ and $\varphi_a \in \mathrm{Aut}(\mathbb{D})$ is the group of all Möbius maps of $\mathbb{D}$. If $\mathcal{N}_K$ consists of just the constant functions, we say that it is trivial.

We assume from now that all $K : [0,\infty) \to [0,\infty)$ to appear in this paper is right-continuous and nondecreasing function such that the integral

$$\int_0^{1/e} K(\log(1/\rho))\rho \, d\rho \ = \int_1^{\infty} K(t)e^{-2t} \, dt < \infty.$$

From a change of variables we see that the coordinate function z belongs to $\mathcal{N}_K$ space if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \ K\big(\log(1/|z|)\big)\, d\, A(z) < \infty.$$

Simplifying the above integral in polar coordinates, we conclude that $\mathcal{N}_K$ space is nontrivial if and only if

$$\sup_{t\in(0,1)} \int_0^1 \frac{(1-t)^2}{(1-tr^2)^3} K\big(\log(1/r)\big) r dr < \infty. \tag{1}$$

An important tool in the study of $\mathcal{N}_K$ space is the auxiliary function ϕ_K defined by

$$\phi_K(s) = \sup_{0<t<1} \frac{K(st)}{K(t)}, \ \ 0 < s < \infty.$$

The following condition has played a crucial role in the study of $\mathcal{N}_K$ space:

$$\int_1^{\infty} \phi_K(s) \frac{ds}{s^2} < \infty, \tag{2}$$

and

$$\int_0^1 \phi_K(s) \frac{ds}{s} < \infty. \tag{3}$$

The test function in $\mathcal{N}_K$ can be stated as follows (see [7]):

Lemma 1.1 *For $w \in \mathbb{D}$ we define*

$$h_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^2}.$$

Suppose that condition (1) is satisfied. Then $h_w \in \mathcal{N}_K$ and

$$\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_K} \leq 1.$$

2 Analytic $\mathcal{N}_{K,\,\omega}$ and $H^{\infty}_{\alpha,\,\omega}$-spaces

Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$ be any reasonable and right continuous nondecreasing function and $K : [0,\ \infty) \longrightarrow [0,\ \infty)$ be right continuous nondecreasing function too. Then, we give the following definitions.

Definition 2.1 *Let $0 < \alpha < \infty$. The weighted type space $H^{\infty}_{\alpha,\,\omega}$ is defined by*

$$H^{\infty}_{\alpha,\omega} := \{f \in \mathbb{H}(\mathbb{D})\ :\ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|^2)}|f(z)| < \infty\}.$$

If

$$\lim_{|z|\longrightarrow 1} \frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|^2)}|f(z)| = 0,$$

we say that f belongs to $H^{\infty}_{\alpha,\,\omega,0}$.

Definition 2.2 *The analytic $\mathcal{N}_{K,\omega}$ -space is defined by*

$$\mathcal{N}_{K,\omega} := \{f \in \mathbb{H}(\mathbb{D})\ :\ \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)} dA(z) < \infty\}.$$

If

$$\lim_{|a|\longrightarrow 1} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)} dA(z) = 0,$$

we say that f belongs to the class $\mathcal{N}_{K,\omega,0}$.

Clearly, if $K(t) = t^p$ and $\omega \equiv 1$, then $\mathcal{N}_{K,1} = \mathcal{N}_p$.
For $K(t) = 1$ and $\omega \equiv 1$, it gives the Bergman space $\mathcal{A}^2$.

In the study of the space $\mathcal{N}_{K,\omega}$, we assume the following condition holds:

$$\int_0^1 \frac{K(\log\frac{1}{r})}{\omega^2(1-r^2)}\, r\, dr < \infty. \tag{4}$$

Throughout this paper, we always assume that condition (4) is satisfied, so that the $\mathcal{N}_{K,\omega}$ space we study is not trivial.

Remark 2.1 *It should be remarked that the weight function $\omega(1-|z|)$ is used to define and study some general classes of function spaces, see [10, 15, 21] and others.*

For a point $a \in \mathbb{D}$ and $0 < r < 1$, let $D(a,r)$ denote an Euclidean disk with center $\frac{(1-r^2)a}{1-r^2|a|^2}$ and radius $\frac{(1-|a|^2)r}{1-r^2|a|^2}$ (see [20]). Suppose also that $E(a,r) = \{z \in \mathbb{D}\ :\ |z-a| < r(1-|a|)\ \}$.
Now, we will prove the following lemma:

Lemma 2.1 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$ be any reasonable and right continuous nondecreasing function and let $K : [0,\infty) \to [0,\infty)$ be right continuous nondecreasing function. Then*

$$\mathcal{N}_{K,\omega} \subset H^{\infty}_{1,\,\omega}.$$

Proof: Suppose that $f \in \mathcal{N}_{K,\,\omega}$, and let C be a constant such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)}\, dA(z) = C < \infty.$$

By the fact that K is nondecreasing, for all $r, 0 < r < 1$, we have

$$\begin{aligned} C &\geq \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \int_{D(a,r)} |f(z)|^2 \frac{K\big(\log \frac{1}{|\varphi_a(z)|}\big)}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \int_{E(a,r)} |f(z)|^2 \frac{K\big(\log \frac{1}{|\varphi_a(z)|}\big)}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} \int_{E(a,r)} |f(z)|^2\, dA(z). \end{aligned}$$

Since $|f(z)|^2$ is subharmonic it follows that,

$$\begin{aligned} C &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} \int_{E(a,r)} |f(z)|^2\, dA(z) \\ &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} 2r^2\pi(1-|a|^2)^2|f(a)|^2. \end{aligned}$$

For $r_0 \in (0,1)$, there exists a constant λ such that

$$\begin{aligned} \frac{|f(a)|^2(1-|a|^2)^2}{\omega^2(1-|a|^2)} &\leq \lambda \int_{D(a,r_0)} |f(z)|^2 dA(z) \\ &\leq \frac{C\,\lambda}{K(\log 1/r_0)}. \end{aligned}$$

Since r_0 is fixed, then

$$\sup_{a\in\mathbb{D}} \frac{|f(a)|(1-|a|^2)}{\omega(1-|a|^2)} \leq \sqrt{\frac{C\,\lambda}{K(\log 1/r_0)}}\ .$$

Thus $f \in H^{\infty}_{1,\,\omega}$ in $\mathbb{D}$. Hence, $\mathcal{N}_{K,\,\omega} \subset H^{\infty}_{1,\,\omega}$.

We will prove the following lemmas on $\mathcal{N}_{K,\,\omega}$- spaces:

Lemma 2.2 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ and $X, Y \in \{H^{\infty}_{\alpha,\,\omega}, \mathcal{N}_{K,\,\omega}\}$. Suppose that $uC_\phi(X) \subset Y$. Then $uC_\phi : X \to Y$ is compact if and only if for every bounded sequence $\{f_j\} \in X$ which converges to 0 uniformly on compact subset of $\mathbb{D}$, we have*

$$\lim_{j\to\infty} \|uC_\phi f_j\|_Y = 0.$$

Proof: This is an extension of a well-known result on the compactness of the composition operator on the Hardy spaces (see [9], Proposition 3.11). We see that any bounded sequence in $H^{\infty}_{\alpha,\,\omega}$ forms a normal family. Also by Lemma 2.1 we have the relation

$$\|f\|_{H^{\infty}_{1,\,\omega}} \leq \|f\|_{\mathcal{N}_{K,\,\omega}}$$

and the growth estimate for $f \in H^{\infty}_{1,\,\omega}$ imply that any bounded sequence in $\mathcal{N}_{K,\,\omega}$ forms a normal family. Hence a similar argument by using Montel's theorem also proves this lemma, and so we omit its proof.

Now, for $\alpha \in (0,\infty)$, $\theta \in [0, 2\pi)$ and $r \in (0,1]$, we put

$$f_{\theta,r}(z) := \sum_{k=0}^{\infty} 2^{\alpha k} (re^{i\theta})^{2^k} z^{2^k} \quad (z \in \mathbb{D}).$$

Using the function $f_{\theta,r}$, we have the following result:

Lemma 2.3 *The function $f_{\theta,r}(z)$ belongs to $H^{\infty}_{\alpha,\,\omega}$ and $\|f_{\theta,r}\|_{H^{\infty}_{\alpha,\,\omega}} \lesssim 1$ which is independent of θ and r. In particular, $f_{\theta,r} \in H^{\infty}_{\alpha,\,\omega.\,0}$ if $r \in (0,1)$.*

Proof: The proof is similar to the corresponding results in [27], with some simple modifications, so it will be omitted.

Lemma 2.4 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ with $\omega(kt) = k\omega(t)$, $k > 0$. Suppose that condition (4) is satisfied. For all $z, w \in \mathbb{D}$, we define the function $h_w(z)$ by*

$$h_w(z) = \frac{(1-|w|^2)(1-|z|)}{(1-\overline{w}z)^2}.$$

Then $h_w(z) \in \mathcal{N}_{K,\omega}$ and $\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_{K,\omega}} \lesssim 1$.
Proof: First, we have that

$$\|h_w\|_{\mathcal{N}_{K,\,\omega}} = \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{(1-|w|^2)}{(1-\overline{w}z)^2}\right|^2 \frac{(1-|z|)^2}{\omega^2(1-|z|^2)} K(g(z,w)dA(z).$$

Since, $1-|w| \le |1-\bar{w}z| \le 1+|w| < 2$ and $1-|z| \le |1-\bar{w}z| \le 1+|z| < 2$ where $z, w \in \mathbb{D}$, then

$$\|h_w\|_{\mathcal{N}_{K,\,\omega}} \le 4\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(g(z,w)}{\omega^2(1-|z|^2)} dA(z) = 4\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|\varphi_w(z)|})}{\omega^2(1-|z|^2)} dA(z).$$

Now, let $z = \varphi_w(z)$, then

$$\begin{aligned}
\|h_w\|_{\mathcal{N}_{K,\,\omega}} &\le 4\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{\omega^2(1-|\varphi_w(z)|^2)} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, dA(z) \\
&\le 16\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2\omega^2(1-|\varphi_w(z)|^2)}\, dA(z).
\end{aligned}$$

Since,

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}.$$

Then, we obtain that

$$\begin{aligned}
\|h_w\|_{\mathcal{N}_{K,\,\omega}} &\le 16\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2\,\omega^2\big(\frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}\big)}\, dA(z) \\
&= 16\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\,\omega^2\big(\frac{(1-|z|^2)}{|1-\overline{w}z|^2}\big)}\, dA(z) \\
&\le 16\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\,\omega^2\big(\frac{(1-|z|^2)}{(1-|w|)^2}\big)}\, dA(z) \\
&= 16\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|)^4 K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\,\omega^2(1-|z|^2)}\, dA(z) \\
&\le c\int_0^1 \frac{K(\log\frac{1}{r})}{\omega^2(1-r^2)}\, r\,dr < \infty,
\end{aligned}$$

where c is a positive constant. Then,

$$\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_{K,\omega}} \lesssim 1.$$

This completes the proof.

3 Weighted composition operator on $H^{\infty}_{\alpha,\,\omega}$ and $\mathcal{N}_{K,\,\omega}$ spaces

Let ϕ be an analytic self-map of the unit disk $\mathbb{D}$. For any $u \in \mathbb{H}(\mathbb{D})$ and $\phi : \mathbb{D} \to \mathbb{D}$, the weighted composition operator $uC_\phi : \mathbb{H}(\mathbb{D}) \to \mathbb{H}(\mathbb{D})$ is defined by $uC_\phi f = u.(f \circ \phi)$. This class of operators has been appeared in the studies of isometries of many holomorphic function spaces. In fact, many isometries of holomorphic function spaces are described as weighted composition operators. For more information and various studies on weighted composition operators, we refer to [8, 12, 13, 16, 18, 25, 27, 28, 29] and others.

In this section we study weighted composition operators acting on $\mathcal{N}_{K,\,\omega}$-space.
Let $\phi \in \mathbb{H}(\mathbb{D})$ to denoted a non-constant function satisfying $\phi(\mathbb{D}) \subset \mathbb{D}$. First, in the following result, we describe boundedness for the $\mathcal{N}_{K,\,\omega}$-class. The results in this section generalizing some results in [19].

Theorem 3.1 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2))} < \infty. \tag{5}$$

Proof: First assume that (5) holds. Then

$$\begin{aligned}
\|uC_\phi f\|_{H^{\infty}_{\alpha,\,\omega}} &= \sup_{z\in\mathbb{D}} |u(z)| f(\phi(z)) \left| \frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)} \right| \\
&\lesssim \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi_a(z)|^2)\omega(1-|z|^2)} \sup_{z\in\mathbb{D}} |f(\phi(z))| \frac{(1-|\phi_a(z)|^2)}{\omega(1-|\phi(z)|^2)} \\
&\lesssim \|f\|_{H^{\infty}_{1,\,\omega}} \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi_a(z)|^2)\omega(1-|\phi_a(z)|^2)} \\
&\leq \lambda \|f\|_{\mathcal{N}_{K,\,\omega}},
\end{aligned}$$

where λ is a positive constant.

Conversely, assume that $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is bounded, then

$$\|uC_\phi f\|_{H^{\infty}_{\alpha,\,\omega}} \lesssim \|f\|_{\mathcal{N}_{K,\,\omega}}.$$

Fix a $z_0 \in \mathbb{D}$, and let h_w be the test function in Lemma 2.4 with $w = \phi(z_0)$. Then

$$\begin{aligned}
1 \gtrsim \|h_w\|_{\mathcal{N}_{K,\,\omega}} &\geq \lambda_1 \|uC_\phi h_w\|_{H^{\infty}_{\alpha,\,\omega}} \\
&\geq \frac{|u(z_0)|(1-|w|^2)}{|1-\overline{w}\phi_a(z_0)|^2)\omega(1-|z_0|^2)} (1-|z_0|^2)^\alpha \\
&= \frac{|u(z_0)|(1-|z_0|^2)^\alpha}{(1-|\phi_a(z_0|^2))\omega(1-|z_0|^2)},
\end{aligned}$$

where λ_1 is a positive constant. The proof of Theorem 3.1 is therefore established.

Theorem 3.2 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then the weighted composition operator $uC_\phi : H^{\infty}_{\alpha,\,\omega} \to \mathcal{N}_{K,\,\omega}$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}(\omega^2(1-|z|^2))} K(g(z,a))dA(z) < \infty. \tag{6}$$

Proof: First we assume that condition (6) holds and let

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}(\omega^2(1-|z|^2))} K(g(z,a))dA(z) < C,$$

where C is a positive constant. If $f \in H^\infty_{\alpha,\,\omega}$, then for all $a \in \mathbb{D}$ we have

$$\begin{aligned}
\|uC_\phi f\|_{\mathcal{N}_{K,\,\omega}} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |u(z)|^2 |f(\phi(z)|^2 \frac{K(g(z,a)}{\omega^2(1-|z|^2)} dA(z)\\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2}{\omega^2(1-|z|^2)} \frac{(1-|\phi(z)|^2)^{2\alpha}|f(\phi(z)|^2}{\omega^2(1-|\phi(z)|^2)} . \frac{\omega^2(1-|\phi(z)|^2)K(g(z,a))}{(1-|\phi(z)|^2)^{2\alpha}} dA(z)\\
&\leq \|f\|^2_{H^\infty_{\alpha,\,\omega}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}\ \omega^2(1-|z|^2)} K(g(z,a))dA(z)\\
&\leq C\|f\|^2_{H^\infty_{\alpha,\,\omega}}.
\end{aligned}$$

Conversely, assume that $uC_\phi : H^\infty_{\alpha,\,\omega} \to \mathcal{N}_{K,\,\omega}$ is bounded, then

$$\|uC_\phi f\|^2_{\mathcal{N}_{K,\,\omega}} \lesssim \|f\|^2_{H^\infty_{\alpha,\,\omega}}.$$

fixing a point $z_0 \in \mathbb{D}$, with $w = \phi(z_0)$ then we set that

$$f_w(z) = \frac{\omega\big(1-\overline{w}\phi(z_0)\big)}{(1-\overline{w}z)^\alpha},$$

it is easy to check that $\|f_w\|_{H^\infty_{\alpha,\,\omega}} \lesssim 1$. Then,

$$\begin{aligned}
\|uC_\phi f_w\|^2_{\mathcal{N}_{K,\,\omega}} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z_0)|^2\omega^2(1-|\phi(z_0)|^2)K(g(z_0,a))}{\big(1-\overline{\phi(z_0)}\phi(z_0)\big)^{2\alpha}\omega^2(1-|z_0|^2)} dA(z_0)\\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z_0)|^2\omega^2(1-|\phi(z_0)|^2)K(g(z_0,a))}{\big(1-|\phi(z_0)|^2\big)^{2\alpha}\omega^2(1-|z_0|^2)} dA(z_0)\\
&\lesssim \|f_w\|^2_{H^\infty_{\alpha,\,\omega}}.
\end{aligned}$$

Theorem 3.3 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then, the operator $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^\infty_{\alpha,\,\omega}$ is compact if and only if*

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2)} = 0. \tag{7}$$

Proof: First assume that $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^\infty_{\alpha,\,\omega}$ is compact and suppose that there exists $\varepsilon_0 > 0$ a sequence $(z_n) \subset \mathbb{D}$ such that

$$\frac{|u(z_n)|(1-|z_n|^2)^\alpha}{(1-|\phi(z_n)|^2)\omega(1-|z_n|^2)} \geq \varepsilon_0 \quad \text{whenever} \quad |\phi(z_n)| > 1-\frac{1}{n}.$$

Clearly, we can assume that

$$w_n = \phi(z_n) \longrightarrow w_0 \in \partial\mathbb{D} \ \text{ as } \ n\to\infty.$$

Let $h_{w_n} = \dfrac{(1-|w_n|^2)}{(1-\overline{w_n}z)^2}$ be the test function in Lemma 2.4. Then $h_{w_n} \to h_{w_0}$ with respect to the compact open topology. Define $f_n = h_{w_n} - h_{w_0}$. Then $\|f_n\|_{\mathcal{N}_{K,\,\omega}} \leq 1$ (see Lemma 2.4) and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Thus, $uf_n \circ \phi \to 0$ in $H^\infty_{\alpha,\,\omega}$ by assumption. But, for n big enough, we obtain

$$\begin{aligned}
\|uC_\phi f_n\|_{H^\infty_{\alpha,\,\omega}} &\geq |u(z_n)|\big|h_{w_n}(\phi(z_n)) - h_{w_0}(\phi(z_n))\big| \frac{(1-|z_n|^2)^\alpha}{\omega(1-|z_n|^2)}\\
&\geq \underbrace{\frac{|u(z_n)|(1-|z_n|^2)^\alpha}{(1-|\phi(z_n)|^2)\omega(1-|z_n|^2)}}_{\geq\varepsilon_0} \underbrace{\left|1-\frac{(1-|w_n|^2)(1-|w_0|^2)}{\big|1-\overline{w_0}\,w_n\big|}\right|}_{=1},
\end{aligned}$$

which is a contradiction.

Conversely, assume that for all $\varepsilon > 0$ there exists $r \in (0,1)$ such that

$$\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2)} < \varepsilon \quad \text{whenever} \quad |\phi(z)| > r.$$

Let $(f_n)_n$ be a bounded sequence in $\mathcal{N}_{K,\,\omega}$ norm which converges to zero on compact subsets of $\mathbb{D}$. Clearly, we may assume that $|\phi(z)| > r$. Then

$$\begin{aligned} \|uC_\phi f_n\|_{H^\infty_{\alpha,\,\omega}} &= \sup_{z\in\mathbb{D}} |u(z)||f_n(\phi(z))| \frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)} \\ &= \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2)} \; \big|f_n(\phi(z))\big|(1-|\phi(z)|^2). \end{aligned}$$

It is not hard to show that

$$||.||_{H^\infty_{1,\,\omega}} \lesssim ||.||_{\mathcal{N}_{K,\omega}}.$$

Thus, we obtain that

$$\|uC_\phi f_n\|_{H^\infty_{\alpha,\,\omega}} \;\leq\; \varepsilon \; \|f_n\|_{H^\infty_{1,\,\omega}} \;\leq\; \varepsilon \; \|f_n\|_{\mathcal{N}_{K,\,\omega}} \;\leq\; \varepsilon.$$

It follows that uC_ϕ is a compact operator. This completes the proof of the theorem.

Remark 3.1 *It is still an open problem to extend the results of this paper in Clifford analysis, for several studies of function spaces in Clifford analysis, we refer to [1, 2, 3, 4, 5, 6] and others.*

Remark 3.2 *It is still an open problem to study properties for differences of weighted composition operators between $\mathcal{N}_{K,\,\omega}$ and $H^\infty_{\alpha,\,\omega}$ classes. For more information of studying differences of weighted composition operators, we refer to [14, 22, 23, 26] and others.*

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1

HERMITE-HADAMARD TYPE INEQUALITIES FOR THE ABK-FRACTIONAL INTEGRALS

ARTION KASHURI

ABSTRACT. The author introduced the new fractional integral operator called ABK-fractional integral and proved four identities for this type. By applying the established identities, some integral inequalities connected with the right hand side of the Hermite-Hadamard type inequalities for the ABK-fractional integrals are given. Various special cases have been identified. The ideas of this paper may stimulate further research in the field of integral inequalities.

1. Introduction

The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda e_1 + (1-\lambda)e_2) \leq \lambda f(e_1) + (1-\lambda) f(e_2) \tag{1.1}$$

holds for all $e_1, e_2 \in I$ and $\lambda \in [0,1]$. Also, we say that f is concave, if the inequality in (1.1) holds in the reverse direction.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $e_1, e_2 \in I$ with $e_1 < e_2$. Then the following inequality holds:*

$$f\left(\frac{e_1+e_2}{2}\right) \leq \frac{1}{e_2-e_1}\int_{e_1}^{e_2} f(x)dx \leq \frac{f(e_1)+f(e_2)}{2}. \tag{1.2}$$

This inequality (1.2) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [2],[4]-[20],[22]-[27].

In [8], Dragomir and Agarwal proved the following results connected with the right part of (1.2).

Lemma 1.3. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $e_1, e_2 \in I^\circ$ with $e_1 < e_2$. If $f' \in L[e_1, e_2]$, then the following equality holds:*

$$\frac{f(e_1)+f(e_2)}{2} - \frac{1}{e_2-e_1}\int_{e_1}^{e_2} f(x)dx = \frac{(e_2-e_1)}{2}\int_0^1 (1-2t) f'(te_1+(1-t)e_2)dt. \tag{1.3}$$

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Theorem 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $e_1, e_2 \in I^\circ$ with $e_1 < e_2$. If $|f'|$ is convex on $[e_1, e_2]$, then the following inequality holds:*

$$\left| \frac{f(e_1) + f(e_2)}{2} - \frac{1}{e_2 - e_1} \int_{e_1}^{e_2} f(x)dx \right| \leq \frac{(e_2 - e_1)}{8} \left(|f'(e_1)| + |f'(e_2)| \right). \tag{1.4}$$

Now, let us recall the following definitions.

Definition 1.5. $X_c^p(e_1, e_2)$ $(c \in \mathbb{R})$, $1 \leq p \leq \infty$ denotes the space of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where the norm $\|\cdot\|_{X_c^p}$ is defined by

$$\|f\|_{X_c^p} = \left(\int_{e_1}^{e_2} \left| t^c f(t) \right|^p \frac{dt}{t} \right)^{\frac{1}{p}} \quad (1 \leq p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{e_1 \leq t \leq e_2} \left| t^c f(t) \right|.$$

Recently, in [12], Katugampola introduced a new fractional integral operator which generalizes the Riemann-Liouville and Hadamard fractional integrals as follows:

Definition 1.6. Let $[e_1, e_2] \subset \mathbb{R}$ be a finite interval. Then, the left and right side Katugampola fractional integrals of order $\alpha\,(> 0)$ of $f \in X_c^p(e_1, e_2)$ are defined by

$${}^\rho I_{e_1^+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{e_1}^{x} \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt, \quad x > e_1 \tag{1.5}$$

and

$${}^\rho I_{e_2^-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{e_2} \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt, \quad x < e_2, \tag{1.6}$$

where $\rho > 0$, if the integrals exist.

In [3], Atangana and Baleanu produced two new fractional derivatives based on the Caputo and the Riemann-Liouville definitions of fractional order derivatives. They declared that their fractional derivative has a fractional integral as the antiderivative of their operators. The Atangana-Baleanu (AB) fractional order derivative is known to possess nonsingularity as well as nonlocality of the kernel, which adopts the generalized Mittag-Leffler function, see [15],[21].

Definition 1.7. The fractional AB-integral of the function $f \in H^*(e_1, e_2)$ is given by

$${}_{e_1}^{AB}I_t^\nu f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{t} (t-u)^{\nu-1} f(u)du, \quad t > e_1, \tag{1.7}$$

where $e_1 < e_2$, $0 < \nu < 1$ and $\mathbb{B}(\nu) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Similarly, we give the definition of the (1.7) opposite side is given by

$${}_{e_2}^{AB}I_t^\nu f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{t}^{e_2} (u-t)^{\nu-1} f(u)du, \quad t < e_2.$$

Here, $\Gamma(\nu)$ is the Gamma function. Since the normalization function $\mathbb{B}(\nu) > 0$ is positive, it immediately follows that the fractional AB-integral of a positive function is positive. It should be noted that, when the order $\nu \to 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\nu \to 0$.

Motivated by the above literatures, the main objective of this paper is to establish some new estimates for the right hand side of Hermite-Hadamard type integral inequalities for new fractional integral operator called the ABK-fractional integral operator. Various special cases will be identified. The ideas of this paper may stimulate further research in the field of integral inequalities.

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2. Hermite-Hadamard inequalities for ABK-fractional integrals

Now, we are in position to introduce the left and right side ABK-fractional integrals as follows.

Definition 2.1. Let $[e_1, e_2] \subset \mathbb{R}$ be a finite interval. Then, the left and right side ABK-fractional integrals of order $\nu \in (0,1)$ of $f \in X_c^p(e_1, e_2)$ are defined by

$$ {}^{ABK\,\rho}_{e_1^+}I_t^{\nu} f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\rho^{1-\nu}\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{t} \frac{u^{\rho-1}}{(t^\rho - u^\rho)^{1-\nu}} f(u)du, \quad t > e_1 \geq 0 \tag{2.1} $$

and

$$ {}^{ABK\,\rho}_{e_2^-}I_t^{\nu} f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\rho^{1-\nu}\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{t}^{e_2} \frac{u^{\rho-1}}{(u^\rho - t^\rho)^{1-\nu}} f(u)du, \quad t < e_2, \tag{2.2} $$

where $\rho > 0$ and $\mathbb{B}(\nu) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Remark 2.2. Since the normalization function $\mathbb{B}(\nu) > 0$ is positive, it immediately follows that the fractional ABK-integral of a positive function is positive. It should be noted that, when the $\rho \to 1$, we recover the AB-fractional integral. Also, using the same idea as in [12], the ABK-fractional integral operators are well-defined on $X_c^p(e_1, e_2)$. Finally, using the same idea as in [1], the interested reader can find new nonlocal fractional derivative of it with Mittag-Leffler nonsingular kernel, several formulae and many applications.

Let represent Hermite-Hadamard's inequalities in the ABK-fractional integral forms as follows:

Theorem 2.3. *Let $\nu \in (0,1)$ and $\rho > 0$. Let $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a function with $0 \leq e_1 < e_2$ and $f \in X_c^p(e_1^\rho, e_2^\rho)$. If f is a convex function on $[e_1^\rho, e_2^\rho]$, then the following inequalities for the ABK-fractional integrals hold:*

$$ \begin{aligned} & \frac{2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)\rho^{2-\nu}} f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) + \frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1^\rho) + f(e_2^\rho)\right] \\ \leq\ & \left[{}^{ABK\,\rho}_{e_1^+}I_{e_2^\rho}^{\nu} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I_{e_1^\rho}^{\nu} f(e_1^\rho)\right] \\ \leq\ & \left(\frac{(e_2^\rho - e_1^\rho)^\nu + \rho(1-\nu)\Gamma(\nu)}{\rho\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1^\rho) + f(e_2^\rho)\right]. \end{aligned} \tag{2.3} $$

Proof. Let $t \in [0,1]$. Consider x^ρ, $y^\rho \in [e_1^\rho, e_2^\rho]$, defined by $x^\rho = t^\rho e_1^\rho + (1-t^\rho)e_2^\rho$, $y^\rho = (1-t^\rho)e_1^\rho + t^\rho e_2^\rho$. Since f is a convex function on $[e_1^\rho, e_2^\rho]$, we have

$$ f\left(\frac{x^\rho + y^\rho}{2}\right) \leq \frac{f(x^\rho) + f(y^\rho)}{2}. $$

Then, we get

$$ 2f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) \leq f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) + f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right). \tag{2.4} $$

Multiplying both sides of (2.4) by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} t^{\rho\nu-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$ \begin{aligned} & \frac{2}{\rho\mathbb{B}(\nu)\Gamma(\nu)} f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) \\ \leq\ & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)dt + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)dt \\ =\ & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{e_2} \left(\frac{e_2^\rho - x^\rho}{e_2^\rho - e_1^\rho}\right)^{\nu-1} f(x^\rho) \frac{x^{\rho-1}}{e_2^\rho - e_1^\rho} dx \end{aligned} $$

$$+ \quad \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_{e_1}^{e_2}\left(\frac{y^\rho - e_1^\rho}{e_2^\rho - e_1^\rho}\right)^{\nu-1} f(y^\rho)\frac{y^{\rho-1}}{e_2^\rho - e_1^\rho}dy$$

Therefore, it follows that

$$\begin{aligned} &\frac{2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)\rho^{2-\nu}} f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) + \frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1^\rho) + f(e_2^\rho)\right] \\ \leq \quad & \left[{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right] \end{aligned}$$

and the left hand side inequality of (2.3) is proved.
For the proof of the right hand side inequality of (2.3) we first note that if f is a convex function, then

$$f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) \leq t^\rho f(e_1^\rho) + (1-t^\rho) f(e_2^\rho)$$

and

$$f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) \leq (1-t^\rho) f(e_1^\rho) + t^\rho f(e_2^\rho).$$

By adding these inequalities, we have

$$f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) + f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) \leq f(e_1^\rho) + f(e_2^\rho). \tag{2.5}$$

Then multiplying both sides of (2.5) by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}t^{\rho\nu-1}$ and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1} f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)dt + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1} f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)dt$$

$$\leq \quad \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\left[f(e_1^\rho) + f(e_2^\rho)\right]\int_0^1 t^{\rho\nu-1}dt$$

i.e.

$$\left[{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right] \quad \leq \quad \left(\frac{(e_2^\rho - e_1^\rho)^\nu + \rho(1-\nu)\Gamma(\nu)}{\rho\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1^\rho) + f(e_2^\rho)\right].$$

The proof of this theorem is complete. □

Corollary 2.4. *If we take $\rho \to 1$ in Theorem 2.3, then the following Hermite-Hadamard's inequalities for the AB-fractional integrals hold:*

$$\begin{aligned} &\frac{2\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)} f\left(\frac{e_1 + e_2}{2}\right) + \frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1) + f(e_2)\right] \\ \leq \quad & \left[{}^{AB}_{e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2}I^\nu_{e_1} f(e_1)\right] \\ \leq \quad & \left(\frac{(e_2 - e_1)^\nu + (1-\nu)\Gamma(\nu)}{\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1) + f(e_2)\right]. \end{aligned} \tag{2.6}$$

Remark 2.5. If in Corollary 2.4, we let $\nu \to 1$, then the inequalities (2.6) become the inequalities (1.2).

3. The ABK-fractional inequalities for convex functions

For establishing some new results regarding the right side of Hermite-Hadamard type inequalities for the ABK-fractional integrals we need to prove the following four lemmas.

Lemma 3.1. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1^\rho) + f(e_2^\rho)\right] - \left[{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right]$$

$$= \frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 \left[(1-t^\rho)^\nu - t^{\rho\nu}\right] t^{\rho-1} f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt. \tag{3.1}$$

Proof. Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\rho)^\nu t^{\rho-1} f'\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) dt \\ &= \frac{(1-t^\rho)^\nu}{\rho(e_1^\rho - e_2^\rho)} f\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right)\Big|_0^1 - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 (1-t^\rho)^{\nu-1} t^{\rho-1} f\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) dt \\ &= \frac{f(e_2^\rho)}{\rho(e_2^\rho - e_1^\rho)} - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 (1-t^\rho)^{\nu-1} t^{\rho-1} f\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) dt. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 t^{\rho(\nu+1)-1} f'\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) dt \\ &= \frac{t^{\rho(\nu+1)-1}}{\rho(e_1^\rho - e_2^\rho)} f\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right)\Big|_0^1 - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 t^{\rho(\nu+1)} f\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) dt \\ &= -\frac{f(e_1^\rho)}{\rho(e_2^\rho - e_1^\rho)} - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 t^{\rho(\nu+1)} f\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) dt. \end{aligned}$$

Thus, by multiplying I_1 and I_2 with $\frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}$, using definition of the ABK-fractional integrals and subtracting them, we get the result. □

Remark 3.2. If in Lemma 3.1, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned} &\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[\, {}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\ = &\frac{(e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 \left[(1-t)^\nu - t^\nu\right] f'\left(te_1 + (1-t)e_2\right) dt. \end{aligned} \tag{3.2}$$

Remark 3.3. If in Lemma 3.1, we let $\rho,\, \nu \to 1$, then we obtain the equality (1.3).

Lemma 3.4. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned} &\left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[\, {}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho)\right] \\ = &\frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho(\nu+1)-1}\left[f'\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) - f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\right] dt. \end{aligned} \tag{3.3}$$

Proof. The proof is similarly as Lemma 3.1, so we omit it. □

Remark 3.5. If in Lemma 3.4, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned} &\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[\, {}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\ = &\frac{(e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^\nu\left[f'\left((1-t)e_1 + te_2\right) - f'\left(te_1 + (1-t)e_2\right)\right] dt. \end{aligned} \tag{3.4}$$

Lemma 3.6. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned}
&\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^\rho I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^\rho I^\nu_{e_1^\rho} f(e_1^\rho)\right] \\
= \;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{ \int_0^1 \left[1 - t^{\rho(\nu+1)}\right] t^{\rho-1} f''\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) dt \right. \\
- \;& \left. \int_0^1 t^{\rho(\nu+2)-1} f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt \right\}.
\end{aligned}$$

Proof. By using twice integration by parts the proof is similarly as Lemma 3.1, so we omit it. □

Remark 3.7. If in Lemma 3.6, we let $\rho \to 1$, then we get the following equality for the *AB*-fractional integrals:

$$\begin{aligned}
&\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\
= \;& \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times \;& \left\{ \int_0^1 \left[1 - t^{\nu+1}\right] f''\left((1-t)e_1 + te_2\right) dt - \int_0^1 t^{\nu+1} f''\left(te_1 + (1-t)e_2\right) dt \right\}.
\end{aligned}$$

Lemma 3.8. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned}
&\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^\rho I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^\rho I^\nu_{e_1^\rho} f(e_1^\rho)\right] \\
= \;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu+2)} \int_0^1 \left[1 - (1-t^\rho)^{\nu+1} - t^{\rho(\nu+1)}\right] t^{\rho-1} f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt. \quad (3.5)
\end{aligned}$$

Proof. By using twice integration by parts and Lemma 3.1, we get the desired result. □

Remark 3.9. If in Lemma 3.8, we let $\rho \to 1$, then we get the following equality for the *AB*-fractional integrals:

$$\begin{aligned}
&\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\
= \;& \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \int_0^1 \left[1 - (1-t)^{\nu+1} - t^{\nu+1}\right] f''\left(te_1 + (1-t)e_2\right) dt. \quad (3.6)
\end{aligned}$$

Using Lemmas 3.1, 3.4, 3.6 and 3.8, we can obtain the following the *ABK*-fractional integral inequalities.

Theorem 3.10. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^\rho I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^\rho I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\
\leq \;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu+\frac{1}{q}}\mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[p]{D(p,\rho,\nu)} \sqrt[q]{\frac{\left|f'(e_1^\rho)\right|^q + \left|f'(e_2^\rho)\right|^q}{2}}, \quad (3.7)
\end{aligned}$$

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where

$$\begin{aligned} D(p,\rho,\nu) &:= \int_0^{\frac{1}{2}} \Big[(1-t^\rho)^{p\nu} - t^{p\rho\nu}\Big] t^{\rho-1}dt + \int_{\frac{1}{2}}^1 \Big[t^{p\rho\nu} - (1-t^\rho)^{p\nu}\Big] t^{\rho-1}dt \\ &= \frac{2}{\rho(p\nu+1)}\left\{1 - \left(1-\frac{1}{2^\rho}\right)^{p\nu+1} - \frac{1}{2^{\rho(p\nu+1)}}\right\}. \end{aligned}$$

Proof. Using Lemma 3.1, convexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \Big[\,{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\Big]\right| \\ \le\;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} \\ \times\;& \left(\int_0^1 \Big|(1-t^\rho)^\nu - t^{\rho\nu}\Big|^p t^{\rho-1}dt\right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1}\Big|f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\Big|^q dt\right)^{\frac{1}{q}} \\ \le\;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} \left(\int_0^{\frac{1}{2}} \Big[(1-t^\rho)^{p\nu} - t^{p\rho\nu}\Big] t^{\rho-1}dt + \int_{\frac{1}{2}}^1 \Big[t^{p\rho\nu} - (1-t^\rho)^{p\nu}\Big] t^{\rho-1}dt\right)^{\frac{1}{p}} \\ \times\;& \left(\int_0^1 t^{\rho-1}\left(t^\rho\big|f'(e_1^\rho)\big|^q + (1-t^\rho)\big|f''(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}} \\ =\;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu+\frac{1}{q}} \mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[p]{D(p,\rho,\nu)}\sqrt[q]{\frac{\big|f'(e_1^\rho)\big|^q + \big|f'(e_2^\rho)\big|^q}{2}}. \end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.11. *With the notations in Theorem 3.10, if we take $|f'| \le K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \Big[\,{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\Big]\right| \\ &\qquad \le \frac{\nu K (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu+\frac{1}{q}} \mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[p]{D(p,\rho,\nu)}. \end{aligned} \tag{3.8}$$

Corollary 3.12. *With the notations in Theorem 3.10, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)] - \Big[\,{}^{AB}_{e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2}I^\nu_{e_1} f(e_1)\Big]\right| \\ \le\;& \frac{\nu (e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[p]{D(p,1,\nu)}\sqrt[q]{\frac{\big|f'(e_1)\big|^q + \big|f'(e_2)\big|^q}{2}}. \end{aligned} \tag{3.9}$$

Theorem 3.13. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q \ge 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \Big[\,{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\Big]\right| \\ &\qquad \le \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} \left[D(1,\rho,\nu)\right]^{1-\frac{1}{q}} \end{aligned} \tag{3.10}$$

$$\times \left\{E(\rho,\nu)\left|f'(e_1^\rho)\right|^q + (F(\rho,\nu)-E(\rho,\nu))\left|f'(e_2^\rho)\right|^q\right.$$
$$\left. + \; G(\rho,\nu)\left|f'(e_1^\rho)\right|^q + (F(\rho,\nu)-G(\rho,\nu))\left|f'(e_2^\rho)\right|^q\right\}^{\frac{1}{q}},$$

where

$$E(\rho,\nu) := \int_0^{\frac{1}{2}} \Big[(1-t^\rho)^\nu - t^{\rho\nu}\Big] t^{2\rho-1}dt = \frac{1}{\rho}\left[\beta\left(\frac{1}{2^\rho};2,\nu+1\right) - \frac{1}{2^{\rho(\nu+2)}(\nu+2)}\right];$$

$$F(\rho,\nu) := \int_0^{\frac{1}{2}} \Big[(1-t^\rho)^\nu - t^{\rho\nu}\Big] t^{\rho-1}dt = \int_{\frac{1}{2}}^1 \Big[t^{\rho\nu} - (1-t^\rho)^\nu\Big] t^{\rho-1}dt$$
$$= \frac{1}{\rho(\nu+1)}\left[1 - \left(1-\frac{1}{2^\rho}\right)^{\nu+1} - \frac{1}{2^{\rho(\nu+1)}}\right];$$

$$G(\rho,\nu) := \int_{\frac{1}{2}}^1 \Big[t^{\rho\nu} - (1-t^\rho)^\nu\Big] t^{2\rho-1}dt = \frac{1}{\rho}\left[\frac{1-\frac{1}{2^{\rho(\nu+2)}}}{\nu+2} + \beta\left(\frac{1}{2^\rho};2,\nu+1\right) - \beta(2,\nu+1)\right],$$

where $\beta(\cdot\,;\cdot,\cdot)$, $\beta(\cdot,\cdot)$ *are respectively the incomplete and complete beta functions and* $D(1,\rho,\nu)$ *is defined as in Theorem 3.10 for value* $p=1$.

Proof. Using Lemma 3.1, convexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[\,{}^{ABK}_{e_1^+}{}^\rho I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^\rho I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right|$$
$$\le \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} \left(\int_0^1 \Big|(1-t^\rho)^\nu - t^{\rho\nu}\Big| t^{\rho-1}dt\right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \Big|(1-t^\rho)^\nu - t^{\rho\nu}\Big| t^{\rho-1}\Big|f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\Big|^q dt\right)^{\frac{1}{q}}$$
$$\le \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} [D(1,\rho,\nu)]^{1-\frac{1}{q}}$$
$$\times \left\{\int_0^{\frac{1}{2}} \Big[(1-t^\rho)^\nu - t^{\rho\nu}\Big] t^{\rho-1}\left(t^\rho\left|f'(e_1^\rho)\right|^q + (1-t^\rho)\left|f'(e_2^\rho)\right|^q\right)dt\right.$$
$$\left. + \int_{\frac{1}{2}}^1 \Big[t^{\rho\nu} - (1-t^\rho)^\nu\Big] t^{\rho-1}\left(t^\rho\left|f'(e_1^\rho)\right|^q + (1-t^\rho)\left|f'(e_2^\rho)\right|^q\right)dt\right\}^{\frac{1}{q}}$$
$$= \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} [D(1,\rho,\nu)]^{1-\frac{1}{q}}$$
$$\times \left\{E(\rho,\nu)\left|f'(e_1^\rho)\right|^q + (F(\rho,\nu)-E(\rho,\nu))\left|f'(e_2^\rho)\right|^q\right.$$
$$\left. + \; G(\rho,\nu)\left|f'(e_1^\rho)\right|^q + (F(\rho,\nu)-G(\rho,\nu))\left|f'(e_2^\rho)\right|^q\right\}^{\frac{1}{q}}.$$

The proof of this theorem is complete. □

Corollary 3.14. *With the notations in Theorem 3.13, if we take $|f'| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[\,{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\ \leq\ & \frac{\nu K (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}[D(1,\rho,\nu)]. \qquad (3.11)\end{aligned}$$

Corollary 3.15. *With the notations in Theorem 3.13, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)] - \left[\,{}^{AB}_{e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2}I^\nu_{e_1} f(e_1)\right]\right| \\ \leq\ & \frac{\nu (e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)}[D(1,1,\nu)]^{1-\frac{1}{q}} \\ \times\ & \Big\{E(1,\nu)\big|f'(e_1)\big|^q + (F(1,\nu)-E(1,\nu))\big|f'(e_2)\big|^q \qquad (3.12) \\ +\ & G(1,\nu)\big|f'(e_1)\big|^q + (F(1,\nu)-G(1,\nu))\big|f'(e_2)\big|^q\Big\}^{\frac{1}{q}}. \qquad (3.13)\end{aligned}$$

Theorem 3.16. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[\,{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\ &\leq \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}}\frac{1}{\sqrt[q]{\rho+1}} \qquad (3.14) \\ &\times\left\{\sqrt[q]{|f'(e_1^\rho)|^q + \rho|f'(e_2^\rho)|^q} + \sqrt[q]{\rho|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q}\right\}.\end{aligned}$$

Proof. Using Lemma 3.4, convexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[\,{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\ \leq\ & \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{p(\rho(\nu+1)-1)}dt\right)^{\frac{1}{p}} \\ \times\ & \left\{\left(\int_0^1 \left|f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|f'\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\ \leq\ & \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{p(\rho(\nu+1)-1)}dt\right)^{\frac{1}{p}} \\ \times\ & \left\{\left(\int_0^1 \left(t^\rho\big|f'(e_1^\rho)\big|^q + (1-t^\rho)\big|f'(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}}\right. \\ +\ & \left.\left(\int_0^1 \left((1-t^\rho)\big|f'(e_1^\rho)\big|^q + t^\rho\big|f'(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}}\right\}\end{aligned}$$

$$= \frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}} \frac{1}{\sqrt[q]{\rho+1}}$$
$$\times \left\{ \sqrt[q]{|f'(e_1^\rho)|^q + \rho|f'(e_2^\rho)|^q} + \sqrt[q]{\rho|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q} \right\}.$$

The proof of this theorem is complete. □

Corollary 3.17. *With the notations in Theorem 3.16, if we take $|f'| \le K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{2K\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}}. \tag{3.15}$$

Corollary 3.18. *With the notations in Theorem 3.16, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1) \right] \right|$$
$$\le \frac{2(e_2 - e_1)^{\nu+1}}{\sqrt[p]{p\nu+1}\mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[q]{\frac{|f'(e_1)|^q + |f'(e_2)|^q}{2}}. \tag{3.16}$$

Theorem 3.19. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q \ge 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{\nu\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \sqrt[q]{\nu+2}\mathbb{B}(\nu)\Gamma(\nu+2)} \tag{3.17}$$
$$\times \left\{ \sqrt[q]{|f'(e_1^\rho)|^q + (\nu+1)|f'(e_2^\rho)|^q} + \sqrt[q]{(\nu+1)|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q} \right\}.$$

Proof. Using Lemma 3.4, convexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{\rho(\nu+1)-1} dt \right)^{1-\frac{1}{q}}$$
$$\times \left\{ \left(\int_0^1 t^{\rho(\nu+1)-1} \left| f'\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}} \right.$$
$$+ \left. \left(\int_0^1 t^{\rho(\nu+1)-1} \left| f'\left((1-t^\rho) e_1^\rho + t^\rho e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}} \right\}$$
$$\le \frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{\rho(\nu+1)-1} dt \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left(\int_0^1 t^{\rho(\nu+1)-1} \left(t^\rho \left| f'(e_1^\rho) \right|^q + (1-t^\rho) \left| f'(e_2^\rho) \right|^q \right) dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 t^{\rho(\nu+1)-1} \left((1-t^\rho) \left| f'(e_1^\rho) \right|^q + t^\rho \left| f'(e_2^\rho) \right|^q \right) dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \sqrt[q]{\nu+2} \mathbb{B}(\nu) \Gamma(\nu+2)}$$

$$\times \left\{ \sqrt[q]{|f'(e_1^\rho)|^q + (\nu+1)|f'(e_2^\rho)|^q} + \sqrt[q]{(\nu+1)|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q} \right\}.$$

The proof of this theorem is complete. □

Corollary 3.20. *With the notations in Theorem 3.19, if we take $|f'| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{2\nu K \left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu+2)}. \tag{3.18}$$

Corollary 3.21. *With the notations in Theorem 3.19, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1) \right] \right|$$

$$\leq \frac{\nu (e_2 - e_1)^{\nu+1}}{\sqrt[q]{\nu+2} \mathbb{B}(\nu) \Gamma(\nu+2)} \tag{3.19}$$

$$\times \left\{ \sqrt[q]{|f'(e_1)|^q + (\nu+1)|f'(e_2)|^q} + \sqrt[q]{(\nu+1)|f'(e_1)|^q + |f'(e_2)|^q} \right\}.$$

Theorem 3.22. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu) \Gamma(\nu+2)} \times \left\{ \frac{1}{\rho} \sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + |f''(e_2^\rho)|^q}{2}} \right. \tag{3.20}$$

$$\left. + \frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + \rho|f''(e_2^\rho)|^q}{\rho+1}} \right\}.$$

Proof. Using Lemma 3.6, convexity of $|f''|^q$, Hölder inequality and properties of the modulus, we have

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu) \Gamma(\nu+2)}$$

$$\times \left\{ \left(\int_0^1 \left| 1 - t^{\rho(\nu+1)} \right|^p t^{\rho-1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1} \left| f''\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 t^{p(\rho(\nu+2)-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}} \right\}$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)}$$

$$\times \left\{ \left(\int_0^1 \left| 1 - t^{\rho(\nu+1)} \right|^p t^{\rho-1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1} \left((1-t^\rho)\left|f''(e_1^\rho)\right|^q + t^\rho \left|f''(e_2^\rho)\right|^q \right) dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 t^{p(\rho(\nu+2)-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left(t^\rho \left|f''(e_1^\rho)\right|^q + (1-t^\rho)\left|f''(e_2^\rho)\right|^q \right) dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{ \frac{1}{\rho} \sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + |f''(e_2^\rho)|^q}{2}} \right.$$

$$\left. + \frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + \rho |f''(e_2^\rho)|^q}{\rho+1}} \right\}.$$

The proof of this theorem is complete. □

Corollary 3.23. *With the notations in Theorem 3.22, if we take $|f''| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1^\rho) + f(e_2^\rho)\right] - \left[{}^{ABK\,\rho}_{\;e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu K \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{ \frac{1}{\rho} \sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}} + \frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}} \right\}. \tag{3.21}$$

Corollary 3.24. *With the notations in Theorem 3.22, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1) + f(e_2)\right] - \left[{}^{AB}_{\;e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{\;e_2}I^\nu_{e_1} f(e_1) \right] \right|$$

$$\leq \frac{\nu \left(e_2 - e_1\right)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \times \frac{\left[\sqrt[p]{p(\nu+1)} + 1\right]}{\sqrt[p]{p(\nu+1)+1}} \sqrt[q]{\frac{|f''(e_1)|^q + |f''(e_2)|^q}{2}} \tag{3.22}$$

Theorem 3.25. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q \geq 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1^\rho) + f(e_2^\rho)\right] - \left[{}^{ABK\,\rho}_{\;e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \tag{3.23}$$

$$\times \left\{ \left(\frac{\nu+1}{\rho(\nu+2)} \right)^{1-\frac{1}{q}} \sqrt[q]{\frac{(\nu+1)(\nu+4)}{2\rho(\nu+2)(\nu+3)} \left|f''(e_1^\rho)\right|^q + \frac{(\nu+1)}{2\rho(\nu+3)} \left|f''(e_2^\rho)\right|^q} \right.$$

$$+ \left(\frac{1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \sqrt[q]{\frac{1}{\rho(\nu+3)}\left|f''(e_1^\rho)\right|^q + \frac{1}{\rho(\nu+2)(\nu+3)}\left|f''(e_2^\rho)\right|^q}\Bigg\}.$$

Proof. Using Lemma 3.6, convexity of $|f''|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\
\leq\; & \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times\; & \left\{\left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}\left|f''\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
+\; & \left.\left(\int_0^1 t^{\rho(\nu+2)-1}dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\rho(\nu+2)-1}\left|f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\; & \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{\left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}}\right. \\
\times\; & \left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}\left((1-t^\rho)\left|f''(e_1^\rho)\right|^q + t^\rho\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}} \\
+\; & \left.\left(\int_0^1 t^{\rho(\nu+2)-1}dt\right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho(\nu+2)-1}\left(t^\rho\left|f''(e_1^\rho)\right|^q + (1-t^\rho)\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}}\right\} \\
=\; & \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times\; & \left\{\left(\frac{\nu+1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \sqrt[q]{\frac{(\nu+1)(\nu+4)}{2\rho(\nu+2)(\nu+3)}\left|f''(e_1^\rho)\right|^q + \frac{(\nu+1)}{2\rho(\nu+3)}\left|f''(e_2^\rho)\right|^q}\right. \\
+\; & \left.\left(\frac{1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \sqrt[q]{\frac{1}{\rho(\nu+3)}\left|f''(e_1^\rho)\right|^q + \frac{1}{\rho(\nu+2)(\nu+3)}\left|f''(e_2^\rho)\right|^q}\right\}.
\end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.26. *With the notations in Theorem 3.25, if we take $|f''| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\
\leq\; & \frac{\nu K(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu+2)}. \qquad (3.24)
\end{aligned}$$

Corollary 3.27. *With the notations in Theorem 3.25, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)] - \left[{}^{AB}_{e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2}I^\nu_{e_1} f(e_1)\right]\right| \\
\leq\; & \frac{\nu(e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)}
\end{aligned}$$

$$\times \quad \left\{ \left(\frac{\nu+1}{\nu+2}\right) \sqrt[q]{\frac{(\nu+4)\left|f''(e_1)\right|^q + (\nu+2)\left|f''(e_2)\right|^q}{2(\nu+3)}} + \frac{1}{(\nu+2)\sqrt[q]{\nu+3}} \sqrt[q]{(\nu+2)\left|f''(e_1)\right|^q + \left|f''(e_2)\right|^q} \right\}.$$

Theorem 3.28. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{e_1^+} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+1} \mathbb{B}(\nu) \Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\left|f''(e_1^\rho)\right|^q + \left|f''(e_2^\rho)\right|^q}{2}}. \tag{3.25}$$

Proof. Using Lemma 3.8, convexity of $|f''|^q$, Hölder inequality and properties of the modulus, we have

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{e_1^+} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu+2)}$$
$$\times \left(\int_0^1 \left| 1 - (1-t^\rho)^{\nu+1} - t^{\rho(\nu+1)} \right|^p t^{\rho-1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1} \left| f''\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}}$$
$$\le \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+\frac{1}{p}} \mathbb{B}(\nu) \Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \times \left(\int_0^1 t^{\rho-1} \left(t^\rho \left|f''(e_1^\rho)\right|^q + (1-t^\rho)\left|f''(e_2^\rho)\right|^q \right) dt \right)^{\frac{1}{q}}$$
$$= \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+1} \mathbb{B}(\nu) \Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\left|f''(e_1^\rho)\right|^q + \left|f''(e_2^\rho)\right|^q}{2}}.$$

The proof of this theorem is complete. □

Corollary 3.29. *With the notations in Theorem 3.28, if we take $|f''| \le K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{e_1^+} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{\nu K (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+1} \mathbb{B}(\nu) \Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}}. \tag{3.26}$$

Corollary 3.30. *With the notations in Theorem 3.28, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1) \right] \right|$$
$$\le \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu) \Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\left|f''(e_1)\right|^q + \left|f''(e_2)\right|^q}{2}}. \tag{3.27}$$

Theorem 3.31. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q \ge 1$, then the following inequality* *for the ABK-fractional integrals holds:*

$$\begin{aligned} & \left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho}f(e_2^\rho)+\,{}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]\right| \\ \leq\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \\ \times\ & \sqrt[q]{C(\rho,\nu)\left|f''(e_1^\rho)\right|^q+\left(\frac{\nu}{\rho(\nu+2)}-C(\rho,\nu)\right)\left|f''(e_2^\rho)\right|^q}, \end{aligned} \tag{3.28}$$

where

$$C(\rho,\nu):=\frac{1}{\rho}\left(\frac{\nu+1}{2(\nu+3)}-\beta(2,\nu+2)\right).$$

Proof. Using Lemma 3.8, convexity of $|f''|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned} & \left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho}f(e_2^\rho)+\,{}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]\right| \\ \leq\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu+2)} \\ \times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\ \times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}\left|f''\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}} \\ \leq\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \\ \times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}\left(t^\rho\left|f''(e_1^\rho)\right|^q+(1-t^\rho)\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}} \\ =\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\sqrt[q]{C(\rho,\nu)\left|f''(e_1^\rho)\right|^q+\left(\frac{\nu}{\rho(\nu+2)}-C(\rho,\nu)\right)\left|f''(e_2^\rho)\right|^q}. \end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.32. *With the notations in Theorem 3.31, if we take $|f''|\leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned} & \left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho}f(e_2^\rho)+\,{}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]\right| \\ \leq\ & \frac{\nu^2 K(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+3)}. \end{aligned} \tag{3.29}$$

Corollary 3.33. *With the notations in Theorem 3.31, if we take $\rho\to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned} & \left|\left(\frac{(e_2-e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)]-\left[\,{}^{AB}_{e_1}I^{\nu}_{e_2}f(e_2)+\,{}^{AB}_{e_2}I^{\nu}_{e_1}f(e_1)\right]\right| \\ \leq\ & \frac{\nu(e_2-e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\nu+2}\right)^{1-\frac{1}{q}} \end{aligned}$$

$$\times \quad \sqrt[q]{C(1,\nu)\left|f''(e_1)\right|^q + \left(\frac{\nu}{\nu+2} - C(1,\nu)\right)\left|f''(e_2)\right|^q}.$$

Theorem 3.34. *Let $\nu \in (0,1)$ and $\rho > 0$. Let f and g be real valued, nonnegative and convex functions on $[e_1^\rho, e_2^\rho]$, where $0 \le e_1 < e_2$. Then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}&\left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho} f(e_2^\rho)g(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho} f(e_1^\rho)g(e_1^\rho)\right]\\ \le\;& \left(\frac{1-\nu}{\mathbb{B}(\nu)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho\mathbb{B}(\nu)\Gamma(\nu+3)}\right)M(e_1^\rho, e_2^\rho) + \frac{2\nu^2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)}N(e_1^\rho, e_2^\rho), \quad (3.30)\end{aligned}$$

where

$$M(e_1^\rho, e_2^\rho) = f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)$$

and

$$N(e_1^\rho, e_2^\rho) = f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho).$$

Proof. Since f and g are convex on $[e_1^\rho, e_2^\rho]$, then

$$f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \le t^\rho f(e_1^\rho) + (1-t^\rho)f(e_2^\rho) \quad (3.31)$$

and

$$g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \le t^\rho g(e_1^\rho) + (1-t^\rho)g(e_2^\rho). \quad (3.32)$$

From (3.31) and (3.32), we get

$$\begin{aligned}f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \;\le\;& t^{2\rho}f(e_1^\rho)g(e_1^\rho) + (1-t^\rho)^2 f(e_2^\rho)g(e_2^\rho)\\ +\;& t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].\end{aligned}$$

Similarly,

$$\begin{aligned}f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho) \;\le\;& (1-t^\rho)^2 f(e_1^\rho)g(e_1^\rho) + t^{2\rho}f(e_2^\rho)g(e_2^\rho)\\ +\;& t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].\end{aligned}$$

By adding the above two inequalities, it follows that

$$f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) + f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)$$

$$\le \quad (2t^{2\rho} - 2t^\rho + 1)[f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)] + 2t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].$$

Multiplying both sides of above inequality by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}t^{\rho\nu-1}$ and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}&\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)dt\\ &+\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)dt\\ \le\;& \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}(2t^{2\rho} - 2t^\rho + 1)[f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)]dt\\ +\;& \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}2t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)]dt\\ =\;& \frac{\nu M(e_1^\rho, e_2^\rho)}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}(2t^{2\rho} - 2t^\rho + 1)dt + \frac{2\nu N(e_1^\rho, e_2^\rho)}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}t^\rho(1-t^\rho)dt\\ =\;& \frac{\nu(\nu^2+\nu+2)}{\rho\mathbb{B}(\nu)\Gamma(\nu+3)}M(e_1^\rho, e_2^\rho) + \frac{2\nu^2}{\mathbb{B}(\nu)\Gamma(\nu+3)}N(e_1^\rho, e_2^\rho).\end{aligned}$$

By the change of variables and with simple integral calculations, we get the desired result. □

Corollary 3.35. *With the notations in Theorem 3.34, if we choose $f = g$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}&\left[{}_{e_1^+}^{ABK\,\rho}I_{e_2^\rho}^{\nu} f^2(e_2^\rho) + {}_{e_2^-}^{ABK\,\rho}I_{e_1^\rho}^{\nu} f^2(e_1^\rho)\right]\\ \leq\ &\left(\frac{1-\nu}{\mathbb{B}(\nu)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho\mathbb{B}(\nu)\Gamma(\nu+3)}\right) M_1(e_1^\rho, e_2^\rho) + \frac{2\nu^2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)} N_1(e_1^\rho, e_2^\rho), \quad (3.33)\end{aligned}$$

where

$$M_1(e_1^\rho, e_2^\rho) = f^2(e_1^\rho) + f^2(e_2^\rho), \quad N_1(e_1^\rho, e_2^\rho) = 2f(e_1^\rho)f(e_2^\rho).$$

Corollary 3.36. *With the notations in Theorem 3.34, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}&\left[{}_{e_1}^{AB}I_{e_2}^{\nu} f(e_2)g(e_2) + {}_{e_2}^{AB}I_{e_1}^{\nu} f(e_1)g(e_1)\right]\\ \leq\ &\left(\frac{1-\nu}{\mathbb{B}(\nu)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)}\right) M(e_1, e_2) + \frac{2\nu^2\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)} N(e_1, e_2). \quad (3.34)\end{aligned}$$

Remark 3.37. With the notations in our theorems given in Section 3, if we take ρ, $\nu \to 1$, then we get some classical integral inequalities.

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A unified convergence analysis for single step-type methods for non-smooth operators

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Abstract

This paper is devoted to the approximation of solutions for nonlinear equations by using iterative methods. We present a unified convergence analysis for some Newton-type methods. We consider both semilocal and local analysis. In the first one, the hypotheses are imposed on the initial guess and in the second on the solution. The results can be applied for smooth and non-smooth operators. In the numerical section we study two applications, first one, it is devoted to a nonlinear integral equation of Hammerstein type and in second one, we approximate the solution of a nonlinear PDE related to image denoising.

1 Introduction

There are several situations in which the modeling of a problem leads us to calculate a solution of an equation

$$F(x) = 0. \tag{1}$$

This equation can represent a differential equation, ordinary or partial, an integral equation, an integro-differential equation or a simple system of equations. In general, mathematical methods that obtain exact solutions of (1) are not known, so that iterative methods are usually used to solve (1) [9, 10, 1, 2, 3, 4, 5, 7, 12]. For a greater generality,

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in this study, we consider $F : D \subset X \to Y$, where X, Y are Banach spaces and D is a nonempty, open and convex set. And we pay attention to F is continuous and Fréchet non-differentiable. In this case, to approximate a solution of (1), iterative methods using divided differences are usually applied instead of using derivatives [12]-[11]. It is common to approximate derivatives by divided differences for obtaining derivative free iterative schemes. So, given an operator $G : D \subset X \to Y$, let us denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X into Y, an operator $[x, y; G] \in \mathfrak{L}(X, Y)$ is called a first order divided difference for the operator G on the points x and y ($x \neq y$) in D if

$$[x, y; G](x - y) = G(x) - G(y). \tag{2}$$

Steffensen's method [13] is the most used iterative method using divided differences in the algorithm, which is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [x_n, x_n + F(x_n); F]^{-1}F(x_n), \quad n \geq 0. \end{cases} \tag{3}$$

As we can see in [14], Steffensen's method has a problem of accessibility that can be solved by using a procedure of decomposition ([15]) for operator F, the Fréchet differentiable part and the non-differentiable part. So, we consider

$$F(x) = F_1(x) + F_2(x) \tag{4}$$

where $F_1, F_2 : D \subset X \to Y$, F_1 is Fréchet differentiable and F_2 is continuous and Fréchet non-differentiable. Thus, in [14], we consider the method of Newton-Steffensen, given by the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - \left(F_1'(x_n) + [x_n, x_n + F(x_n); F_2]\right)^{-1} (F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{5}$$

with $X = Y$, which improves significantly the accessibility of method (3) and has quadratic convergence.

By using this procedure of decomposition for operator F, we see that we can also consider the application of iterative methods that use derivatives when F is non-differentiable. So, for example, we can consider the well-known Newton's method, which algorithm is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad n \geq 0, \end{cases} \tag{6}$$

Obviously, Newton's method is not applicable, under form (6), when F is not Fréchet differentiable. However, if we consider decomposition of F given in (4), we can use the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F_1'(x_n)]^{-1}(F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{7}$$

which is known as method of Zincenko [17].

The main aim of this paper consists of defining one-point iterative methods of Newton-type, as we can see previously, to obtain a general study for the convergence, local and semilocal, for these type of iterative methods. Moreover, in view of the last two considerations, with these one point iterative methods we can to improve the accessibility of one-point iterative methods that use divided differences and, in addition, to extend the application of iterative methods that use derivatives when F is Fréchet non-differentiable. For this aim, we consider the one-point iterative methods of Newton-type given by the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - L_n^{-1}(F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{8}$$

where $L_n := L(x_n)$ with $L(.) : D \to \mathfrak{L}(X, Y)$. Clearly, method (8) can be used to solve equations containing a nondifferentiable term.

There are a lot of iterative methods that can be written as algorithm (8), in addition to modifications of Steffensen and Newton given in (5) and (7), where $L(x) = F_1'(x) + [x, x + F_2(x); F_2]$ and $L(x) = F_1'(x)$, respectively. At the same time, we can also consider two interesting cases. Firstly, the generalized Steffensen methods [6], that are very used in the approximation of solutions of non-differentiable operators equations and the algorithm is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [x_n - aF(x_n), x_n + bF(x_n); F]^{-1}F(x_n), \quad n \geq 0. \end{cases}$$

Then, it is clear that we can define the generalized Newton-Steffensen method from 8) with $L(x) = F_1'(x) + [x - aF_2(x), x + bF_2(x); F_2]$, so we have the final iterative function given as:

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - (F_1'(x_n) + [x_n - aF_2(x_n), x_n + bF_2(x_n); F_2])^{-1}F(x_n), \quad n \geq 0. \end{cases} \tag{9}$$

where $a, b \in \mathbb{R}$.

In the same way as Newton's method, from Stirling method [16],

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F_1'(x_n - F(x_n))]^{-1}F(x_n), \quad n \geq 0, \end{cases} \tag{10}$$

we can define a modification of Newton-type, that can be applied to Fréchet non-differentiable operators. For this, just consider (8) with $L(x) = F_1'(x - F(x))$. In both cases, we choose $X = Y$. Obviously, we can include a lot of iterative methods in (8) if F is Fréchet differentiable.

So, in this paper, we study the convergence of algorithm (8). We analyze the semilocal and local convergences, so that we have a study of convergence of a lot of iterative methods that are usually used and can be written by algorithm (8).

Section 2 is devoted to the theoretical analysis about local and semilocal convergence for a very general single step Newton-like methods. In Section 3 we make a comparison for the behavior of some of these methods by solving a non-differentiable problem. In Section 4, we consider an application related to image denoising. Finally, in Section 5 we give some conclusions.

2 Convergence Analysis for single step Newton-like methods

In this section, we present both semilocal and local convergence analysis. In the first one, the hypotheses are imposed on the initial guess and in the second on the solution. The results can be applied for smooth and non-smooth operators.

2.1 Local Convergence Analysis

In this section, we first present the local followed by the semilocal convergence of method (8). Let $v_0 : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function with $v_0(0) = 0$. Suppose that the equation

$$v_0(t) = 1 \tag{11}$$

has at least one positive root r_0. Let also $v : [0, r_0) \to [0, +\infty)$ be a nondecreasing continuous function. Define function $\bar{v}$ on the interval $[0, r_0)$ by $\bar{v}(t) = \frac{v(t)}{1-v_0(t)} - 1$.

Suppose equation

$$\bar{v}(t) = 0 \tag{12}$$

has at least one positive root. Denote by r the smallest such root. It follows that for each $t \in [0, r)$

$$0 \leq v_0(t) < 1 \tag{13}$$

and

$$0 \leq \bar{v}(t) < 1. \tag{14}$$

The local convergence analysis of method (8) uses the conditions (A):

- (a_1) There exist a solution $x^* \in D$ of equation (4), and $B \in \mathfrak{L}(X, Y)$ such that $B^{-1} \in \mathfrak{L}(Y, X)$.

- (a_2) Condition (11) holds and for each $x \in D$

  $$\|B^{-1}(L(x) - B)\| \leq v_0(\|x - x^*\|),$$

  where v_0 is defined previously and r_0 is given in (11).

  Set $D_0 = D \cap \bar{U}(x*, r_0)$.

- (a_3) For $L : D_0 \to \mathfrak{L}(X, Y)$, any solution y of equation (4) and each $x \in D_0$
$$\|B^{-1}(F_1(x) + F_2(x) - L(x)(x - y))\| \leq v(\|x - y\|)\|x - y\|,$$
where v is defined previously.

- (a_4) $\bar{U}(x^*, r) \subset D$, where r is given in (12).

- (a_5) There exist $r^* \geq r$ such that
$$\xi := \frac{v(r^*)}{1 - v_0(r)} \in [0, 1).$$
Set $D_1 = D \cap \bar{U}(x*, r^*)$.

Remark 1

- *Condition (a_3) can be replaced by the stronger: for each $x, y, z \in D_0$*
$$\|B^{-1}(F_1(x) + F_2(x) - L(x)(x - y))\| \leq v_1(\|x - y\|)\|x - y\|,$$
where function v_1 is as v. But for each $t \geq 0$
$$v(t) \leq v_1(t).$$

- *Linear operator B does not necessarily depend on the solution x^*. It is used to determine the invertibility of linear operator $L(\cdot)$ appearing in the method. The invertibility of B can be assured by an additional condition of the form $||I - B|| < 1$ or some other way. A possible choice for B is $B = B(x^*)$ or $B = F_1'(x^*)$.*

- *It follows from the definition of r_0 and r that $r_0 \geq r$.*

We can present the local convergence analysis of method (8) based on the aforementioned conditions (A).

Theorem 2 *Suppose that the conditions (A) hold. Then, sequence x_k generated by method (8) for $x_0 \in U(x^*, r) - x^*$ is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ and converges to x^*. Moreover, the following estimates hold.*
$$\|x_{k+1} - x^*\| \leq \frac{v(\|x_k - x^*\|)}{1 - v_0(\|x_k - x^*\|)}\|x_k - x^*\| \leq \|x_k - x^*\| < r. \tag{15}$$
The vector x^* is the only solution of equation (4) in D_1, where D_1 is given in $(a5)$.

Proof We base the proof on k and mathematical induction. Let $x \in U(x^*, r)$. Using (8), $(a1)$ and $(a2)$, we have in turn that
$$\|B^{-1}(L(x) - B)\| \leq v_0(\|x - x^*\|) \leq v_0(r) < 1. \tag{16}$$

It follows by (16) and the Banach lemma on invertible operators [] that $L(x)^{-1} \in \mathfrak{L}(Y, X)$ and

$$\|L(x)^{-1}B\| \leq \frac{1}{1 - v_0(\|x - x^*\|)}. \tag{17}$$

In particular, estimate (17) holds for $x = x_0$, so x_1 is well defined by method (8) for $k = 0$. We also get by method (8) (for $k = 0$), $(a1)$, $(a3)$, (14) and (17) (for $k = 0$) that

$$\begin{aligned} \|x_1 - x^*\| &= \|x_0 - x^* - L(x_0)^{-1}(F_1(x_0) + F_2(x_0))\| \\ &= \|[-L(x_0)^{-1}B][B^{-1}(F_1(x_0) + F_2(x_0) - L(x_0)(x_0 - x^*))]\| \\ &\leq \|L(x_0)^{-1}B\|\|B^{-1}(F_1(x_0) + F_2(x_0) - L(x_0)(x_0 - x^*))\| \\ &\leq \frac{v(\|x_0 - x^*\|)}{1 - v_0(\|x_0 - x^*\|)}\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{18}$$

which shows estimate (15) for $k = 0$, and $x_1 \in U(x^*, r)$.

Simply, replace x_0, x_1 by x_i, x_{i+1} in the preceding estimates to complete the induction for estimate (15). Then, in view of the estimate

$$\|x_{i+1} - x^*\| \leq \xi\|x_i - x^*\| < r, \tag{19}$$

where

$$\xi = \frac{v(\|x_0 - x^*\|)}{1 - v_0(\|x_0 - x^*\|)} \in [0, 1),$$

we deduce that $\lim_{i \to +\infty} x_i = x^*$ and $x_{i+1} \in U(x^*, r)$. Moreover, to show the uniqueness part, let $y^* \in D_1$ with $F_1(y^*) + F_2(y^*) = 0$. Using $(a3)$, $(a5)$ and estimate (18), we obtain in turn that

$$\begin{aligned} \|x_{i+1} - y^*\| &\leq \|L(x_i)^{-1}B\|\|B^{-1}(F_1(x_i) + F_2(x_i) - L(x_i)(x_i - y^*))\| \\ &\leq \frac{v(\|x_i - y^*\|)}{1 - v_0(\|x_i - x^*\|)}\|x_i - y^*\| \\ &\leq \xi\|x_i - y^*\| < \xi^{i+1}\|x_0 - y^*\|, \end{aligned} \tag{20}$$

which shows $\lim_{i \to +\infty} x_i = y^*$. But, we showed $\lim_{i \to +\infty} x_i = x^*$. Hence, we conclude that $x^* = y^*$.

□

2.2 Semilocal Convergence Analysis

As in the local case it is convenient to define some functions and parameters for the semilocal analysis. Let $w_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function.

Suppose that equation

$$w_0(t) = 1. \tag{21}$$

has at least one positive root. Denote by ρ_0 the smallest such root. Let also $w : [0, \rho_0) \times [0, \rho_0) \to [0, +\infty)$ be a nondecreasing continuous function. Moreover, for $\eta \geq 0$, define parameters C_1 and C_2 by

$$\begin{aligned} C_1 &= \frac{w(\eta, 0)}{1 - w_0(\eta)}, \\ C_2 &= \frac{w(\frac{\eta}{1-C_1}, \eta)}{1 - w_0(\frac{\eta}{1-C_1})} \end{aligned}$$

and function $C : [0, \rho_0) \to [0, +\infty)$ by $C(t) = \frac{w(t,t)}{1-w_0(t)}$. Suppose that equation

$$(\frac{C_1 C_2}{1 - C(t)} + C_1 + 1)\eta - t = 0 \tag{22}$$

has as least one positive root. Denote by ρ the smallest such root.

Next, we show the semilocal convergence analysis of method (8) in an analogous way, under the conditions (H):

- $(h1)$ There exists $x_0 \in D$ and $B \in \mathfrak{L}(X, Y)$ such that $B^{-1} \in \mathfrak{L}(Y, X)$.
- $(h2)$ Condition (21) holds and for each $x \in D$

  $$\|B^{-1}(L(x) - B)\| \leq w_0(\|x - x_0\|),$$

  where w_0 is as defined previously, and ρ_0 is given in (21).

  Set $D_2 = D \bigcap \bar{U}(x_0, \rho_0)$.
- $(h3)$ For $L(\cdot) : D_2 \to \mathfrak{L}(X, Y)$, and each $x, y \in D_2$

  $$\begin{aligned} &\|B^{-1}(F_1(y) - F_1(x) + F_2(y) - F_2(x) - L(x)(y - x))\| \\ &\qquad \leq w(\|y - x_0\|, \|x - x_0\|)\|y - x\|, \end{aligned}$$

  where w is as defined previously.
- $(h4)$ $\bar{U}(x_0, \rho) \subseteq D$ and condition (22) holds for ρ, where $\|x_1 - x_0\| \leq \eta$.
- $(h5)$ There exists $\rho^* \geq \rho$ such that

  $$\xi_0 := \frac{w(\rho, \rho^*)}{1 - w_0(\rho)} \in [0, 1).$$

  Set $D_2 = D \bigcap \bar{U}(x^*, \rho^*)$.

Then, as in the local case but using the (H) instead of the (A) conditions, we have in turn the estimates:

$$\begin{aligned}
\|x_2-x_1\| &\le \frac{w(\|x_1-x_0\|,\|x_0-x_0\|)}{1-w_0(\|x_1-x_0\|)} = C_1\|x_1-x_0\|,\\
\|x_2-x_0\| &\le \|x_2-x_1\|+\|x_1-x_0\| \le (1+C_1)\|x_1-x_0\|\\
&= \frac{1-C_1^2}{1-C_1}\|x_1-x_0\|\\
&< \frac{\|x_1-x_0\|}{1-C_1}\eta < \rho,\\
\|x_3-x_2\| &\le \frac{w(\|x_2-x_0\|,\|x_1-x_0\|)}{1-w_0(\|x_2-x_0\|)}\|x_2-x_1\|\\
&\le \frac{w(\frac{\eta}{1-C_1},\eta)}{1-w_0(\frac{\eta}{1-C_1})}\|x_2-x_1\| = C_2\|x_2-x_1\|\\
\|x_3-x_0\| &\le \|x_3-x_2\|+\|x_2-x_1\|+\|x_1-x_0\|\\
&\le C_2\|x_2-x_1\|+C_1\|x_1-x_0\|+\|x_1-x_0\|\\
&\le (C_2C_1+C_1+1)\|x_1-x_0\|,\\
\|x_4-x_3\| &\le \frac{w(\|x_3-x_0\|,\|x_2-x_0\|)}{1-w_0(\|x_3-x_0\|)}\|x_3-x_2\|\\
&\le C(\rho)\|x_3-x_2\| \le C(\rho)C_2\|x_2-x_1\|\\
&\le C(\rho)C_2C_1\|x_1-x_0\|,\\
&\quad\ldots\\
\|x_{i+1}-x_i\| &\le C(\rho)\|x_i-x_{i-1}\| \le C(\rho)^{i-2}\|x_3-x_2\|\\
\|x_{i+1}-x_0\| &\le \|x_{i+1}-x_i\|+\ldots+\|x_4-x_3\|+\|x_3-x_0\|\\
&\le C(\rho)\|x_i-x_{i-1}\|+\ldots+C(\rho)\|x_3-x_2\|\\
&\quad +(C_2C_1+C_1+1)\|x_1-x_0\|\\
&\le C(\rho)^{i-2}\|x_3-x_2\|+\ldots+C(\rho)\|x_3-x_2\|\\
&\quad +(C_2C_1+C_1+1)\|x_1-x_0\|\\
&\le (\frac{1-C(\rho)^{i-1}}{1-C(\rho)}C_2C_1+C_1+1)\|x_1-x_0\|\\
&< (\frac{C_1C_2}{1-C(\rho)}+C_1+1)\eta \le \rho, \qquad (23)
\end{aligned}$$

$$\begin{aligned}\|x_{i+j}-x_i\| &\le \|x_{i+j}-x_{i+j-1}\|+\|x_{i+j-1}-x_{i+j-2}\|+...+\|x_{i+1}-x_i\|\\ &\le (C(\rho)^{i+j-3}+...+C(\rho)^{i-2})\|x_3-x_2\|\\ &\le C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}\|x_3-x_2\|\\ &\le C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}C_2C_1\|x_1-x_0\|\\ &\le C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}C_2C_1\eta. \end{aligned} \tag{24}$$

It follows from (23) that $x_i \in U(x_0,\rho)$ and from (24) that sequence x_i is complete in X and as such it converges to some $x^* \in \bar{U}(x_0,\rho)$. By letting $i \to +\infty$ in the estimate

$$\|B^{-1}(F_1(x_i)+F_2(x_i))\| = \|B^{-1}(F_1(x_i)+F_2(x_i)-F_1(x_{i-1})-F_2(x_{i-1})-B_{i-1}(x_i-x_{i-1}))\|$$

$$\le \frac{w(\|x_i-x_0\|,\|x_{i-1}-x_0\|)\|\|x_i-x_{i-1}\|}{1-w_0(\|x_i-x_0\|)} \le \frac{w(\rho,\rho)}{1-w_0(\rho)}\|x_i-x_{i-1}\|,$$

we obtain $F_1(x^*)+F_2(x^*)=0$. The uniqueness part is omitted as identical to the one in the local convergence case.

Hence, we arrived at the semilocal convergence result for method (8).

Theorem 3 *Suppose that the conditions (H) hold. Then, sequence x_k generated by method (8) for $x_0 \in D$ is well defined in $U(x_0,\rho)$ remains in $U(x_0,\rho)$ and converges $x^* \in \bar{U}(x_0,\rho)$ to a solution of equation (4). Moreover, the vector x^* is the only solution of equation (4) in D_3, where D_3 is defined previously.*

The same comments introduced in the previous remark are valid.

We emphasize the theoretical importance of this theorem because it presents a unified studied of the local and semilocal convergence of a big variety of Newton-Type methods and Steffensen type methods, so the study is applicable to differentiable an non differentiable equations.

3 Numerical Experiments

In this section, we consider a nonlinear integral equation of Hammerstein type, which can be used to describe applied problems in the fields of electro-magnetics, fluid dynamics, in the kinetic theory of gases and, in general, in the reformulation of boundary value problems. These equations are of the form:

$$x(s) = f(s) - \int_a^b K(s,t)\Phi(x(t))dt, \quad a \le s \le b, \tag{25}$$

where $x(s), f(s) \in C[a,b]$, with $-\infty < a < b < \infty$, and Φ is a polynomial function. One of the most used techniques to solve this kind of equations consists of expressing them as a nonlinear operator in a Banach space and solving the following operator equation:

$$F(x)(s) = x(s) - f(s) + \int_a^b K(s,t)\Phi(x(t))dt = 0, \tag{26}$$

where $F : D \subseteq C[a,b] \to C[a,b]$ with D a non-empty open convex subset of $C[a,b]$ with the max-norm $\|\nu\| = \max_{s\in[a,b]} |\nu(s)|$.

We consider (25), where K is the Green function in $[a,b] \times [a,b]$, and then use a discretization process to transform equation (26) into a finite dimensional problem by approximating the integral by an adequate quadrature formula

$$\int_a^b q(t)\,dt \simeq \sum_{i=1}^{p} w_i q(t_i),$$

where the nodes t_i and the weights w_i are known.

If we denote the approximations of $x(t_i)$ and $f(t_i)$ by x_i and f_i, respectively, with $i = 1, 2, \ldots, p$, then equation (26) is equivalent to the following system of nonlinear equations:

$$x_i = f_i + \sum_{j=1}^{p} a_{ij}\,\Phi(x_j), \quad j = 1, 2, \ldots, p, \tag{27}$$

where

$$a_{ij} = w_j K(t_i, t_j) = \begin{cases} w_j \frac{(b-t_i)(t_j-a)}{b-a}, & j \le i, \\ w_j \frac{(b-t_j)(t_i-a)}{b-a}, & j > i. \end{cases}$$

Now, system (27) can be written as

$$\mathbb{F}(\mathbf{x}) \equiv \mathbf{x} - \mathbf{f} - A\,\mathbf{z} = 0, \qquad \mathbb{F} : \Delta \subseteq \mathbb{R}^p \longrightarrow \mathbb{R}^p, \tag{28}$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_p)^T, \qquad \mathbf{f} = (f_1, f_2, \ldots, f_p)^T, \qquad A = (a_{ij})_{i,j=1}^p,$$
$$\mathbf{z} = (\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_p))^T.$$

After that, we choose $a = 0$, $b = 1$, $K(s,t)$ as the Green function in $[0,1] \times [0,1]$ and $\Phi(x(t)) = x(t)^3 + |x(t)|$ in (25). Then, the system of nonlinear equations given in (28) is of the form

$$\mathbb{F}(\mathbf{x}) = \mathbf{x} - \mathbf{f} - A\,(\mathbf{v_x} + \mathbf{w_x}) = 0, \qquad \mathbb{F} : \mathbb{R}^p \longrightarrow \mathbb{R}^p, \tag{29}$$

where

$$\mathbf{v_x} = (x_1^3, x_2^3, \ldots, x_p^3)^T, \qquad \mathbf{w_x} = (|x_1|, |x_2|, \ldots, |x_p|)^T.$$

It is obvious that the function $\mathbb{F}$ defined in (29) is nonlinear and non-differentiable. So, we consider $\mathbb{F}(\mathbf{x}) = \mathbb{F}_1(\mathbf{x}) + \mathbb{F}_2(\mathbf{x})$ where:

$$\mathbb{F}_1(\mathbf{x}) = \mathbf{x} - \mathbf{f} - A\mathbf{v_x} \quad \text{and} \quad \mathbb{F}_2(\mathbf{x}) = -\mathrm{A}\mathbf{w_x}.$$

As in $\mathbb{R}^p$ we can consider divided difference of first order that do not need that the function $\mathbb{F}$ is differentiable (see [16]), we use the divided difference of first order given by $[\mathbf{u},\mathbf{v};\mathbb{G}] = ([\mathbf{u},\mathbf{v};\mathbb{G}]_{ij})_{i,j=1}^{p} \in \mathcal{L}(\mathbb{R}^p,\mathbb{R}^p)$, where

$$[\mathbf{u},\mathbf{v};\mathbb{G}]_{ij} = \frac{1}{u_j - v_j}\left(\mathbb{G}_i(u_1,\ldots,u_j,v_{j+1},\ldots,v_p) - \mathbb{G}_i(u_1,\ldots,u_{j-1},v_j,\ldots,v_p)\right), \quad (30)$$

if $u_j \neq v_j$, in other case $[\mathbf{u},\mathbf{v};\mathbb{G}]_{ij} = 0$, for $\mathbf{u} = (u_1,u_2,\ldots,u_p)^T$ and $\mathbf{v} = (v_1,v_2,\ldots,v_p)^T$.

Now, to compare the behavior of different methods we consider the case $\mathbf{f} = \mathbf{0}$ in (29). Obviously, for this problem, $\mathbf{x}^* = \mathbf{0}$ is a solution of $\mathbb{F}(\mathbf{x}) = \mathbf{0}$. Then, the system of nonlinear equations given in (29) is of the form

$$\mathbb{F}(\mathbf{x}) = \mathbf{x} - A\,\mathbf{z}, \qquad z_j = x_j^3 + |x_j|,\ j = 1,\ldots,p. \quad (31)$$

The numerical results are obtained by using MATLAB 2018 and working with variable precision arithmetic with 100 digits. In Table 1 we can see the results obtained by using the methods mentioned in our study. First of all we take nodes and weights of Trapezoidal rule with $n = 10$ subintervals for approximatting the integral and starting guess $x_0(t) = 1/2\ \forall t \in [0,1]$. We compare the distance between consecutive iterates of the first 7 iterations of each method. In the case of the Newton-Steffensen General method (9), the parameters involved are $a = 0.5$ and $b = 1.5$.

	Stirling (10)	Zincenko (7)	Steffensen (3)	New-Steff. (5)	New-Steff. Gen.(9)
1	1.5887	1.1637	7.4375	2.9044	2.9044
2	$6.0578e-01$	$3.0210e-01$	$2.7350e-01$	1.3867	1.3867
3	$4.7941e-01$	$1.2065e-01$	$1.8235e-02$	$3.2041e-01$	$1.2942e-01$
4	$4.1942e-01$	$4.9511e-02$	$5.5411e-05$	$2.8725e-04$	$2.8725e-04$
5	$3.5456e-01$	$2.0403e-02$	$2.8134e-09$	$1.3552e-12$	$1.3552e-12$
6	$1.9024e-01$	$8.4133e-03$	$3.0173e-18$	$3.1538e-37$	$3.3246e-37$
7	$2.9676e-02$	$3.4697e-03$	$3.9490e-36$	$1.7796e-111$	$2.1782e-111$

Table 1: Results with different methods in the first iterations.

In Table 2 we work with same conditions, we obtain the iterations that each method needs to satisfy the stopping criterion $||x_{k+1} - x_k|| \leq 10^{-40}$. It should be noted that the first two methods never meet the required tolerance because they are not convergent and, therefore, the methods end when the required iterations are completed (in this case 15 iterations at most). Second, we observe a good approximation to the order of convergence of each method p in case the method converges. In the last two rows of Table 2 we compare

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	Stirling (10)	Zincenko (7)	Steffensen (3)	New-Steff. (5)	New-Steff. Gen.(9)
k	15	15	8	7	7
p	1.0000	1.0000	1.9994	3.0142	3.0148
$\|\|x_{k-1} - x_k\|\|$	$2.3258e-04$	$2.9041e-06$	$6.9382e-72$	$1.7796e-111$	$2.1782e-111$
$\|\|F(x_k)\|\|$	$9.5985e-05$	$1.1977e-06$	$1.2745e-107$	$7.8863e-219$	$6.8587e-219$

Table 2: Numerical results for comparing the proposed methods.

the difference between the last iterates of each method and we also see the norm of the function evaluted in the last iteration.

Now, we also want to use the Gauss-Legendre quadrature to approximate the integral of equation (25). Moreover, by using the Newton-Steffensen method we compare two different possibilities for implementing the divided differences given in (30), that is, in Tables 1 and 2 we obtain the divided difference like $[x_n, x_n + F_1(x_n) + F_2(x_n), F_2]$ but we want to compare with $[x_n, x_n + F_2(x_n), F_2]$. The results in Table 3 show that the use of first form used for obtaining the divided differences gives better residual errors, which was expected because $F_1(x_n) + F_2(x_n)$ tends to zero quicker than $F_2(x_n)$. Even in some different example the value $F_2(x_n)$ could not tend to zero, in this case only first form of obtaining the divided differences considered would work. In Table 3 we have also included the computational time, as can be observed in the last row, notice that the use of Gauss-Legedre quadrature needs much more time than the trapezoidal rule although in some cases reaches better accuracy.

	$\|\|x_n - x_{n-1}\|\|$			
$Iterations$	$Trapezoidal\quad rule$		$Gauss - Legendre$	
n	$[x, x + F_1 + F_2, F_2]$	$[x, x + F_2, F_2]$	$[x, x + F_1 + F_2, F_2]$	$[x, x + F_2, F_2]$
1	2.9044	2.9044	2.7204	2.7204
2	1.3867	1.3867	1.1355	1.1355
3	$3.2041e-01$	$1.2942e-01$	$6.6978e-02$	$6.6978e-02$
4	$2.8725e-04$	$2.8725e-04$	$3.4608e-05$	$3.4608e-05$
5	$1.3552e-12$	$1.3552e-12$	$2.1448e-15$	$2.1448e-15$
6	$3.1538e-37$	$3.3489e-28$	$1.124e-45$	$1.124e-45$
7	$1.7796e-111$	$1.3651e-43$	$8.0773e-137$	$7.8571e-137$

Table 3: Results with Trapezoidal rule and Gauss-Legendre method by using different form of obtaining the divided differences.

	$Trapezoidal \quad rule$		$Gauss - Legendre$	
n	$[x, x + F_1 + F_2, F_2]$	$[x, x + F_2, F_2]$	$[x, x + F_1 + F_2, F_2]$	$[x, x + F_2, F_2]$
k	7	8	7	7
p	3.0142	$unstable$	3.0099	3.0103
$\|\|x_{k-1} - x_k\|\|$	$1.7796e - 111$	$4.3463e - 59$	$8.0772e - 137$	$7.8571e - 137$
$\|\|F(x_k)\|\|$	$7.8863e - 219$	$1.0160e - 74$	$1.3057e - 243$	$1.5367e - 138$
$time$	17.796129	20.6134	282.5403	309.3090

Table 4: Numerical results and computational time for comparing the proposed methods.

4 Approximating the solution of a nonlinear PDE related to image denoising

In some steps of the manipulation of an image, some random noise is usually introduced. This noise makes the later steps of processing the image difficult and inaccurate.

In many applications like astrophysics, astronomy or meteorology we have to manipulate images contaminated by noise. The image processing becomes difficult and inaccurate. For these reasons, usually some image denoising strategies are developed. In this paper, we center our attention in the PDE framework.

Let $f : \Omega \to \mathbb{R}$ be a signal or image which we would like to denoise.

The usual PDE frameworks start with constrained optimization problems like

$$\begin{array}{l}\text{Minimize in } u\colon R(u)\\ \text{subject to } \|u - f\|^2_{L^2(\Omega)} = |\Omega|\sigma^2.\end{array}$$

where $n = u - f$ denotes the noise. If there is no good estimate of the variance of the noise, then we may consider the unconstrained optimization problem.

Different linear regularization functionals $R(u)$ can be consider, the most used is $\|\nabla u\|_{L^2}$. This type of functionals introduce diffusion near the edges of the images, this is their main limitation.

The TV norm does not penalize discontinuities in u, and thus allows us to improve the approximation near the edges.

$$\int_\Omega |\nabla u(x)| dx.$$

For the linear model its Euler–Lagrange equation, with Neumann's boundary conditions for u, is

$$-\triangle u + \lambda(u - f) = 0, \tag{32}$$

which comes from the corresponding unconstrained problem with the norm $\|\nabla u\|^2_{L^2(\Omega)}$ and where the positive parameter λ determines the relative importance of the smoothness of u and the quality of the approximation to the given signal f..

For the TV- model we have

$$-\nabla\cdot(\frac{\nabla u}{|\nabla u|}) + \lambda(u-f) = 0. \tag{33}$$

In practice, the term $|\nabla u|$ is replaced by $\sqrt{|\nabla u|^2+\epsilon}$, but even after this regularization, Newton's method does not work satisfactorily in the sense that its domain of convergence is very small. This is especially true if the regularizing parameter ϵ is small.

On the other hand, while the singularity and nondifferentiability of the term $w = \nabla u/|\nabla u|$ is the source of numerical problems, w itself is usually smooth because it is in fact the unit vector normal to the level sets of u. The numerical difficulties arise only because we linearize it the wrong way.

Thus we should introduce a new variable w; namely

$$w = \frac{\nabla u}{\sqrt{|\nabla u|^2}},$$

and replace (33) by the equivalent system of nonlinear PDEs:

$$\begin{aligned} -\nabla\cdot w + \lambda(u-f) &= 0, \\ w\sqrt{|\nabla u|^2} - \nabla u &= 0. \end{aligned}$$

Without the inclusion of the above regularization parameter ϵ, this system is nonlinear and nondifferentiable .

4.1 Discretization and numerical implementation

We present a comparison between the nonlinear model and the linear model using a simple finite difference discretization procedure.

For a regular mesh of size $h = 1/m$, $m \in \mathbb{N}$ ($x_i = i\cdot h$, $i = 0,\dots,m$), if in each iteration k we approximate the divergence and the gradient operators (these operators are the same in 1D) by

$$\nabla\cdot v(x_i) = \nabla v(x_i) \approx \frac{v_i - v_{i-1}}{h},$$

we obtain a nonlinear system for the unknowns w_i and u_i.

That is,

$$\begin{aligned} -\frac{w_i - w_{i-1}}{h} - \lambda(u_i - f_i) &= 0, \quad w_1 = w_m = 0, \\ w_i\cdot\sqrt{(\frac{u_i-u_{i-1}}{h})^2 - \frac{u_i-u_{i-1}}{h}} &= 0, \quad u_0 = f_0, u_m = f_m, \end{aligned}$$

for $i = 1, \ldots, m-1$.

We then consider the nonlinear and nondifferentiable operator

$$\begin{aligned} F_{2i-1}(u,w,\lambda_h) &= w_i - w_{i-1} + \lambda_h(u_i - f_i) = 0, \\ F_{2i}(u,w,\lambda_h) &= w_i\sqrt{(u_i - u_{i-1})^2 +} - (u_i - u_{i-1}) = 0, \quad 1 \le i \le m-1, \end{aligned}$$

with $\lambda_h = h\,\lambda$, $w_0 = w_m = 0$, $u_0 = f_0$ and $u_m = f_m$.

For the discretization of the linear model we can consider the system

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \lambda(u_i - f_i) = 0, \quad u_0 = f_0, u_m = f_m,$$

for $i = 1, \ldots, m-1$.

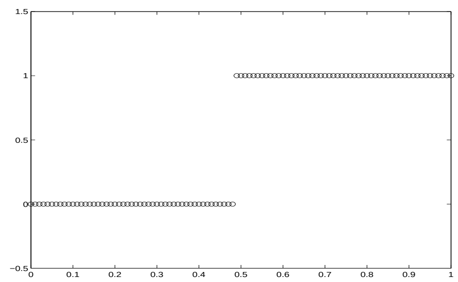

Figure 1: Original signal with a jump singularity.

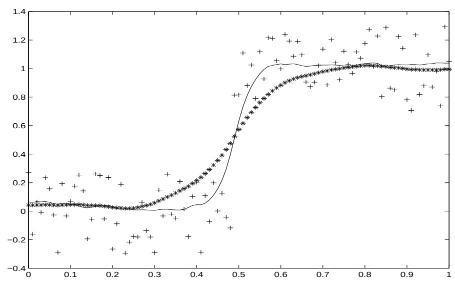

Figure 2: Solid lines = nonlinear model, starred lines = linear model and + lines = signal with noise. Noise level = 0.3, $\lambda = 10$.

In Figure 2, the solid lines are the function reconstructed by the nonlinear model approximated by the linearization based on a dual variable, solving the nonlinear system of equations by Steffensen's method 3 and the starred lines are given by the standard linear model, solving the associated linear system of equations by Gauss's method. The line with '+' is the noisy signal. The linear model introduces too much diffusion, giving a continuous function.

5 Conclusions

We have to point out the generalization of this study in which we have analyzed the local and semilocal convergence for Newton type methods and Steffensen like methods, so we can consider Newton-Steffensen's methods. The main idea it is to apply these kind of study to non-differentiable equations by taking in to account the advantages of consider the decomposition of the nonlinear equation into a sum of the differentiable part and the one non-differentiable.

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On the Localization of Factored Fourier Series

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Abstract

In the present paper, a theorem concerning local property of $|A, p_n|_k$ summability of factored Fourier series, which generalizes a result dealing with $|\bar{N}, p_n|_k$ summability of factored Fourier series, has been obtained. Also, some results have been given.
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1 Introduction

Let $\sum a_n$ be an infinite series with its partial sums (s_n) and (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

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The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [21])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

If we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability (see [2]). If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n (resp. $a_{nv} = \frac{p_v}{P_n}$ and $k = 1$), $|A, p_n|_k$ summability reduces to $|C, 1|_k$ summability (see [11]) (resp. $|\bar{N}, p_n|$) summability. Also, if we take $p_n = 1$ for all values of n, then $|A, p_n|_k$ summability reduces to $|A|_k$ summability (see [22]). Furthermore, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A|_k$ summability reduces to $|R, p_n|_k$ summability (see [4]).

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ (see [24]).

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) \quad = \quad \sum_{n=1}^{\infty} C_n(t),$$

where (a_n) and (b_n) denote the Fourier coefficients. It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function f (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f (see [23]).

2 Known Results

There are many different applications of Fourier series. Some of them can be find in [1], [5]-[10], [12]-[20]. Furthermore, Bor [3] has proved the following theorem.

Theorem 1 *Let $k \geq 1$ and (p_n) be a sequence such that*

$$P_n = O(np_n), \tag{1}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{2}$$

Then the summability $|\bar{N}, p_n|_k$ of the series $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ at a point can be ensured by local property, where (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.

3 Main Result

The purpose of this paper is to generalize Theorem 1 by using the definition of $|A, p_n|_k$ summability. Now, let us introduce some further notations. Let $A = (a_{nv})$ be a normal matrix, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{3}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{4}$$

and it is well known that

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{5}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{6}$$

Now, we will prove the following theorem.

Theorem 2 *Let $k \geq 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ n = 0, 1, ..., \tag{7}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{8}$$

3

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{9}$$

$$|\hat{a}_{n,v+1}| = O\left(v\,|\Delta_v \hat{a}_{nv}|\right), \tag{10}$$

where $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$. *Let the sequence* (p_n) *be such that the conditions (1) and (2) of Theorem 1 are satisfied. Then the summability* $|A, p_n|_k$ *of the series* $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ *at a point can be ensured by local property, where* (λ_n) *is as in Theorem 1.*

Here, if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1.

We should give the following lemmas for the proof of Theorem 2.

Lemma 3 ([13]) *If the sequence* (p_n) *is such that the conditions (1) and (2) of Theorem 1 are satisfied, then*

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{11}$$

Lemma 4 ([10]) *If* (λ_n) *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, then* (λ_n) *is non-negative and decreasing, and* $n\Delta\lambda_n \to 0 \quad as \quad n \to \infty$.

Lemma 5 *Let* $k \geq 1$ *and let the sequence* (p_n) *be such that the conditions (1) and (2) of Theorem 1 are satisfied. If* (s_n) *is bounded and the conditions (7)-(10) are satisfied, then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \lambda_n P_n}{np_n} \tag{12}$$

is summable $|A, p_n|_k$, *where* (λ_n) *is as in Theorem 1.*

Remark 6 *Since* (λ_n) *is a convex sequence, therefore* $(\lambda_n)^k$ *is also convex sequence and*

$$\sum \frac{1}{n}(\lambda_n)^k < \infty. \tag{13}$$

4 Proof of Lemma 5

Let (M_n) denotes the A-transform of the series $\sum \frac{a_n \lambda_n P_n}{n p_n}$. Then, we have

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{v p_v}$$

by (5) and (6).

Now, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= \frac{a_{nn} P_n \lambda_n}{n p_n} s_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{v p_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta \lambda_v P_v}{v p_v} s_v \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{v p_v} \right) s_v \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}
\end{aligned}$$

by applying Abel's transformation. For the proof of Lemma 5, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |M_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

First, we have

$$\begin{aligned}
\sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} |M_{n,1}|^k &= \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \right|^k \\
&= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{n^k} \left(\frac{P_n}{p_n} \right)^k (\lambda_n)^k |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \frac{1}{n} (\lambda_n)^k = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by (9), (1) and (13).

From Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v\Delta_v(\hat{a}_{nv})}{vp_v}s_v\right|^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)|\Delta_v(\hat{a}_{nv})|\,(\lambda_v)|s_v|\right\}^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k|s_v|^k\right\}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.\end{aligned}$$

By (4) and (3), we have that

$$\begin{aligned}\Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv}-\hat{a}_{n,v+1}\\ &= \bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}\\ &= a_{nv}-a_{n-1,v}. \qquad (14)\end{aligned}$$

Thus using (8), (3) and (7)

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv}) \leq a_{nn}. \qquad (15)$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k\right\}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|.\end{aligned}$$

Here, from (14) and (8), we obtain

$$\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| = \sum_{n=v+1}^{m+1}(a_{n-1,v}-a_{nv}) \leq a_{vv}.$$

Then,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k = O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k a_{vv}$$

$$\begin{aligned} &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}(\lambda_v)^k \\ &= O(1)\sum_{v=1}^{m} v^{k-1}\frac{1}{v^k}(\lambda_v)^k \\ &= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_v)^k = O(1) \quad as \quad m\to\infty, \end{aligned}$$

by (9), (1) and (13).

Now, by (1) and Hölder's inequality, we have

$$\begin{aligned} \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v P_v}{vp_v}s_v\right|^k \\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|s_v|\right\}^k \\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|s_v|^k\right\}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}^{k-1}. \end{aligned}$$

Now, (4), (3), (7) and (8) imply that

$$\begin{aligned} \hat{a}_{n,v+1} = \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1} &= \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i} \\ &= \sum_{i=0}^{n} a_{ni} - \sum_{i=0}^{v} a_{ni} - \sum_{i=0}^{n-1} a_{n-1,i} + \sum_{i=0}^{v} a_{n-1,i} \\ &= 1 - \sum_{i=0}^{v} a_{ni} - 1 + \sum_{i=0}^{v} a_{n-1,i} \\ &= \sum_{i=0}^{v}(a_{n-1,i} - a_{ni}) \geq 0 \qquad (16) \end{aligned}$$

and from this, using (4), (3) and (8), we have

$$\begin{aligned} |\hat{a}_{n,v+1}| &= \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1} \\ &= \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i} \\ &= a_{nn} + \sum_{i=v+1}^{n-1}(a_{ni} - a_{n-1,i}) \\ &\leq a_{nn}. \end{aligned}$$

Hence, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\left\{\sum_{v=1}^{n-1}\Delta\lambda_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}\\
&= O(1)\sum_{v=1}^{m}\Delta\lambda_v\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.
\end{aligned}$$

Now, by (16), (3) and (7), we find

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| \leq 1. \tag{17}$$

Thus,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k = O(1)\sum_{v=1}^{m}\Delta\lambda_v = O(1) \quad as \quad m\to\infty,$$

by Lemma 4.

Since $\Delta\left(\frac{P_v}{vp_v}\right) = O\left(\frac{1}{v}\right)$ by Lemma 3 and also by using (10), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\Delta\left(\frac{P_v}{vp_v}\right)s_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})|s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k|s_v|^k\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.
\end{aligned}$$

From (15) and (9),

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.
\end{aligned}$$

Again using (17),

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k = O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k = O(1) \quad as \quad m\to\infty,$$

by (13). Hence the proof of Lemma 5 is completed.

5 Proof of Theorem 2

The convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 2 is a consequence of Lemma 5.

6 Conclusions

For $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result concerning $|C, 1|_k$ summability factors of Fourier series. If we take $a_{nv} = \frac{p_v}{P_n}$ and $k = 1$, then we get a result concerning $|\bar{N}, p_n|$ summability factors of Fourier series (see [13]).

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Analysis of Solutions of Some Discrete Systems of Rational Difference Equations

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Abstract

The major objective of this article is to determine and formulate the analytical solutions of the following systems of rational recursive equations:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(\pm 1\mp x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(\mp 1\pm y_{n-1}x_{n-3}\right)},\quad n=0,1,...,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are required to be arbitrary non-zero real numbers. We also introduce some graphs describing these exact solutions under a suitable choice of some initial conditions.

Keywords: difference equations, system of recursive equations, periodicity, local stability, global stability.
Mathematics Subject Classification: 39A10.

1 Introduction

The global interest in exploring the qualitative behaviours of discrete systems of recursive equations has been recently emerged due to the significance of difference equations in modelling a considerable number of discrete phenomena. More specifically, recursive equations are utilized in describing some real life problems that originate in genetics in biology, queuing problems, enegineering, physics, etc. Some experts put effort to analyse dynamical systems of difference equations. Take, for instance, the following ones. Almatrafi et al. [1] studied the local stability, global attractivity, periodicity and solutions for a special case for the difference equation

$$x_{n+1}=ax_{n-1}-\frac{bx_{n-1}}{cx_{n-1}-dx_{n-3}}.$$

Clark and Kulenovic [6] investigated the global attractivity of the system

$$x_{n+1}=\frac{x_n}{a+cy_n},\quad y_{n+1}=\frac{y_n}{b+dx_n}.$$

The author in [8] explored the equilibrium points and the stability of a discrete Lotka-Volterra model shown as follows:

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \quad y_{n+1} = \frac{\delta y_n + \epsilon x_n y_n}{1 + \eta y_n}.$$

The positive solutions of the system

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}}.$$

were obtained in [14] by Gűműş and Őcalan. Moreover, Kurbanli et al. [18] solved the dynamical systems of recursive equations given by

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{x_n}{y_n z_{n-1}}.$$

In [19] Mansour et al. presented the analytical solutions of the system

$$x_{n+1} = \frac{x_{n-1}}{\alpha - x_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\beta + \gamma y_{n-1} x_n}.$$

Finally, the author in [23] demonstrated the dynamics of the system

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}.$$

To attain more information on the qualitative behaviours of dynamical difference equations, one can refer to refs [1–5, 7, 9–13, 15–17, 20–22]

In this paper, the rational solutions of the following discrete systems of difference equations will be discovered and given in four different theorems:

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}\left(\pm 1 \mp x_{n-1} y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}\left(\mp 1 \pm y_{n-1} x_{n-3}\right)}, \quad n = 0, 1, ...,$$

where the initial values are as described previously.

2 Main Results

2.1 First System $x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}(1 - x_{n-1} y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}(1 - y_{n-1} x_{n-3})}$

This subsection concentrates on obtaining the solutions of a dynamical system of fourth order difference equations given by the form:

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}\left(1 - x_{n-1} y_{n-3}\right)}, \; y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}\left(1 - y_{n-1} x_{n-3}\right)}, \; n = 0, \; 1, ... \; , \tag{1}$$

where the initial values are as shown previously. The following fundamental theorem presents the solutions of system (1).

Theorem 1 *Assume that* $\{x_n, y_n\}$ *is a solution to system (1) and let* $x_{-3} = \alpha,\ x_{-2} = \beta,\ x_{-1} = \gamma,\ x_0 = \delta,\ y_{-3} = \epsilon,\ y_{-2} = \eta,\ y_{-1} = \mu$ *and* $y_0 = \omega$. *Then, for* $n = 0,\ 1,\ \ldots$ *we have*

$$x_{4n-3} = \frac{\gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i)\,\alpha\mu - 1]}{\alpha^{n-1}\mu^n \prod_{i=0}^{n-1} [(2i+1)\,\gamma\epsilon - 1]}, \quad x_{4n-2} = \frac{\delta^n \eta^n \prod_{i=0}^{n-1} [(2i)\,\beta\omega - 1]}{\beta^{n-1}\omega^n \prod_{i=0}^{n-1} [(2i+1)\,\delta\eta - 1]},$$

$$x_{4n-1} = \frac{\gamma^{n+1} \epsilon^n \prod_{i=0}^{n-1} [(2i+1)\,\alpha\mu - 1]}{\alpha^{n}\mu^n \prod_{i=0}^{n-1} [(2i+2)\,\gamma\epsilon - 1]}, \quad x_{4n} = \frac{\delta^{n+1} \eta^n \prod_{i=0}^{n-1} [(2i+1)\,\beta\omega - 1]}{\beta^{n}\omega^n \prod_{i=0}^{n-1} [(2i+2)\,\delta\eta - 1]}.$$

And

$$y_{4n-3} = \frac{\alpha^n \mu^n \prod_{i=0}^{n-1} [(2i)\,\gamma\epsilon - 1]}{\gamma^{n}\epsilon^{n-1} \prod_{i=0}^{n-1} [(2i+1)\,\alpha\mu - 1]}, \quad y_{4n-2} = \frac{\beta^n \omega^n \prod_{i=0}^{n-1} [(2i)\,\delta\eta - 1]}{\delta^{n}\eta^{n-1} \prod_{i=0}^{n-1} [(2i+1)\,\beta\omega - 1]},$$

$$y_{4n-1} = \frac{\alpha^{n} \mu^{n+1} \prod_{i=0}^{n-1} [(2i+1)\,\gamma\epsilon - 1]}{\gamma^{n}\epsilon^n \prod_{i=0}^{n-1} [(2i+2)\,\alpha\mu - 1]}, \quad y_{4n} = \frac{\beta^{n} \omega^{n+1} \prod_{i=0}^{n-1} [(2i+1)\,\delta\eta - 1]}{\delta^{n}\eta^n \prod_{i=0}^{n-1} [(2i+2)\,\beta\omega - 1]}.$$

Proof. For $n = 0$, our results hold. Next, let $n > 1$ and suppose that the relations hold for $n - 1$. That is

$$x_{4n-7} = \frac{\gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i)\,\alpha\mu - 1]}{\alpha^{n-2}\mu^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\gamma\epsilon - 1]}, \quad x_{4n-6} = \frac{\delta^{n-1} \eta^{n-1} \prod_{i=0}^{n-2} [(2i)\,\beta\omega - 1]}{\beta^{n-2}\omega^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\delta\eta - 1]},$$

$$x_{4n-5} = \frac{\gamma^{n} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\alpha\mu - 1]}{\alpha^{n-1}\mu^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\gamma\epsilon - 1]}, \quad x_{4n-4} = \frac{\delta^{n} \eta^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\beta\omega - 1]}{\beta^{n-1}\omega^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\delta\eta - 1]}.$$

And

$$y_{4n-7} = \frac{\alpha^{n-1} \mu^{n-1} \prod_{i=0}^{n-2} [(2i)\,\gamma\epsilon - 1]}{\gamma^{n-1}\epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\,\alpha\mu - 1]}, \quad y_{4n-6} = \frac{\beta^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i)\,\delta\eta - 1]}{\delta^{n-1}\eta^{n-2} \prod_{i=0}^{n-2} [(2i+1)\,\beta\omega - 1]},$$

$$y_{4n-5} = \frac{\alpha^{n-1} \mu^{n} \prod_{i=0}^{n-2} [(2i+1)\,\gamma\epsilon - 1]}{\gamma^{n-1}\epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\alpha\mu - 1]}, \quad y_{4n} = \frac{\beta^{n-1} \omega^{n} \prod_{i=0}^{n-2} [(2i+1)\,\delta\eta - 1]}{\delta^{n-1}\eta^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\beta\omega - 1]}.$$

Now, it can be obviously observed from system (1) that

$$
\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(1-x_{4n-5}y_{4n-7}\right)} \\
&= \frac{\frac{\gamma^n\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}}{\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\left[1-\frac{\gamma^n\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}\right]} \\
&= \frac{\frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}}{\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\left[1-\frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\right]} \\
&= \frac{\gamma^n\epsilon^n\prod_{i=0}^{n-2}\left[(2i)\gamma\epsilon-1\right]\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}\left[(2i+1)\gamma\epsilon-1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]-\gamma\epsilon\prod_{i=0}^{n-2}\left[(2i)\gamma\epsilon-1\right]\right]} \\
&= -\frac{\gamma^n\epsilon^n\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}\left[(2i+1)\gamma\epsilon-1\right]} = \frac{\gamma^n\epsilon^n\prod_{i=0}^{n-1}\left[(2i)\alpha\mu-1\right]}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}\left[(2i+1)\gamma\epsilon-1\right]}.
\end{aligned}
$$

Now, system (1) gives us that

$$
\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left[1-y_{4n-5}x_{4n-7}\right]} \\
&= \frac{\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}}{\frac{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[1-\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}\right]} \\
&= \frac{\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}}{\frac{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[1-\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\right]} \\
&= \frac{\alpha^n\mu^n\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]\prod_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]}{\gamma^n\epsilon^{n-1}\prod_{i=0}^{n-2}\left[(2i+1)\alpha\mu-1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]-\alpha\mu\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]\right]}
\end{aligned}
$$

$$= -\frac{\alpha^n \mu^n \prod_{i=0}^{n-2} \left[(2i+2)\gamma\epsilon - 1\right]}{\gamma^n \epsilon^{n-1} \prod_{i=0}^{n-1} \left[(2i+1)\alpha\mu - 1\right]} = \frac{\alpha^n \mu^n \prod_{i=0}^{n-1} \left[(2i)\gamma\epsilon - 1\right]}{\gamma^n \epsilon^{n-1} \prod_{i=0}^{n-1} \left[(2i+1)\alpha\mu - 1\right]}.$$

Hence, the rest of the results can be similarly proved.

2.2 Second System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

Our leading duty in this subsection is to determine the solutions of the following discrete systems:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1 + x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1 + y_{n-1}x_{n-3}\right)}. \tag{2}$$

The initial values of this system are arbitrary real numbers.

Theorem 2 *Suppose that $\{x_n, y_n\}$ is a solution to system (2) and assume that $x_{-3} = \alpha,\ x_{-2} = \beta,\ x_{-1} = \gamma,\ x_0 = \delta,\ y_{-3} = \epsilon,\ y_{-2} = \eta,\ y_{-1} = \mu$ and $y_0 = \omega$. Then, for $n = 0,\ 1,\ ...$ we have*

$$\begin{aligned} x_{4n-3} &= \frac{\gamma^n \epsilon^n}{\alpha^{n-1}\mu^n \left(\gamma\epsilon - 1\right)^n}, \quad x_{4n-2} = \frac{\delta^n \eta^n}{\beta^{n-1}\omega^n \left(\delta\eta - 1\right)^n}, \\ x_{4n-1} &= \frac{\gamma^{n+1}\epsilon^n \left(\alpha\mu - 1\right)^n}{\alpha^n \mu^n}, \quad x_{4n} = \frac{\delta^{n+1}\eta^n \left(\beta\omega - 1\right)^n}{\beta^n \omega^n}. \end{aligned}$$

And

$$\begin{aligned} y_{4n-3} &= \frac{\alpha^n \mu^n}{\gamma^n \epsilon^{n-1} \left(\alpha\mu - 1\right)^n}, \quad y_{4n-2} = \frac{\beta^n \omega^n}{\delta^n \eta^{n-1} \left(\beta\omega - 1\right)^n}, \\ y_{4n-1} &= \frac{\alpha^n \mu^{n+1} \left(\gamma\epsilon - 1\right)^n}{\gamma^n \epsilon^n}, \quad y_{4n} = \frac{\beta^n \omega^{n+1} \left(\delta\eta - 1\right)^n}{\delta^n \eta^n}. \end{aligned}$$

Proof. It is obvious that all solutions are satisfied for $n = 0$. Next, we suppose that $n > 1$ and assume that the solutions hold for $n - 1$. That is

$$\begin{aligned} x_{4n-7} &= \frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\left(\gamma\epsilon - 1\right)^{n-1}}, \quad x_{4n-6} = \frac{\delta^{n-1}\eta^{n-1}}{\beta^{n-2}\omega^{n-1}\left(\delta\eta - 1\right)^{n-1}}, \\ x_{4n-5} &= \frac{\gamma^n \epsilon^{n-1}\left(\alpha\mu - 1\right)^{n-1}}{\alpha^{n-1}\mu^{n-1}}, \quad x_{4n-4} = \frac{\delta^n \eta^{n-1}\left(\beta\omega - 1\right)^{n-1}}{\beta^{n-1}\omega^{n-1}}. \end{aligned}$$

And

$$\begin{aligned} y_{4n-7} &= \frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}\left(\alpha\mu - 1\right)^{n-1}}, \quad y_{4n-6} = \frac{\beta^{n-1}\omega^{n-1}}{\delta^{n-1}\eta^{n-2}\left(\beta\omega - 1\right)^{n-1}}, \\ y_{4n-5} &= \frac{\alpha^{n-1}\mu^n\left(\gamma\epsilon - 1\right)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}, \quad y_{4n-4} = \frac{\beta^{n-1}\omega^n\left(\delta\eta - 1\right)^{n-1}}{\delta^{n-1}\eta^{n-1}}. \end{aligned}$$

We now turn to illustrate the first result. System (2) leads to

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(-1+x_{4n-5}y_{4n-7}\right)} \\ &= \frac{\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}}{\frac{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\left[-1+\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}\right]} \\ &= \frac{\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\left(\gamma\epsilon-1\right)^{n-1}\left[-1+\gamma\epsilon\right]} = \frac{\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\left(\gamma\epsilon-1\right)^n}. \end{aligned}$$

Similarly, it is easy to see from system (2) that

$$\begin{aligned} y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(-1+y_{4n-5}x_{4n-7}\right)} \\ &= \frac{\frac{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}}{\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\left[-1+\frac{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}\right]} \\ &= \frac{\alpha^n\mu^n}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^{n-1}\left[-1+\alpha\mu\right]} = \frac{\alpha^n\mu^n}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^n}. \end{aligned}$$

The remaining solutions of system (2) can be clearly justified in a similar technique. Thus, the proof is complete.

2.3 Third System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1-x_{n-1}y_{n-3})},\ y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

The central point of this subsection is to resolve a system of fourth order rational recursive equations given by the form:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1-x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1+y_{n-1}x_{n-3}\right)}, \tag{3}$$

where the initial values are as described previously.

Theorem 3 *Let $\{x_n, y_n\}$ be a solution to system (3) and suppose that $x_{-3} = \alpha,\ x_{-2} = \beta,\ x_{-1} = \gamma,\ x_0 = \delta,\ y_{-3} = \epsilon,\ y_{-2} = \eta,\ y_{-1} = \mu$ and $y_0 = \omega$. Then, for $n = 0,\ 1,\ ...$ we have*

$$\begin{aligned} x_{4n-3} &= \frac{\left(-1\right)^n\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}\left[\left(2i+1\right)\gamma\epsilon-1\right]}, \quad x_{4n-2} = \frac{\left(-1\right)^n\delta^n\eta^n}{\beta^{n-1}\omega^n\prod_{i=0}^{n-1}\left[\left(2i+1\right)\delta\eta-1\right]}, \\ x_{4n-1} &= \frac{\left(-1\right)^n\gamma^{n+1}\epsilon^n\left(\alpha\mu-1\right)^n}{\alpha^n\mu^n\prod_{i=0}^{n-1}\left[\left(2i+2\right)\gamma\epsilon-1\right]}, \quad x_{4n} = \frac{\left(-1\right)^n\delta^{n+1}\eta^n\left(\beta\omega-1\right)^n}{\beta^n\omega^n\prod_{i=0}^{n-1}\left[\left(2i+2\right)\delta\eta-1\right]}. \end{aligned}$$

And

$$y_{4n-3} = \frac{(-1)^n \alpha^n \mu^n \prod_{i=0}^{n-1} [(2i)\gamma\epsilon - 1]}{\gamma^n \epsilon^{n-1} (\alpha\mu - 1)^n}, \quad y_{4n-2} = \frac{(-1)^n \beta^n \omega^n \prod_{i=0}^{n-1} [(2i)\delta\eta - 1]}{\delta^n \eta^{n-1} (\beta\omega - 1)^n},$$
$$y_{4n-1} = \frac{(-1)^n \alpha^n \mu^{n+1} \prod_{i=0}^{n-1} [(2i+1)\gamma\epsilon - 1]}{\gamma^n \epsilon^n}, \quad y_{4n} = \frac{(-1)^n \beta^n \omega^{n+1} \prod_{i=0}^{n-1} [(2i+1)\delta\eta - 1]}{\delta^n \eta^n}.$$

Proof. The results are true for $n = 0$. Next, we suppose that $n > 1$ and assume that the relations hold for $n - 1$. That is

$$x_{4n-7} = \frac{(-1)^{n-1} \gamma^{n-1} \epsilon^{n-1}}{\alpha^{n-2} \mu^{n-1} \prod_{i=0}^{n-2} [(2i+1)\gamma\epsilon - 1]}, \quad x_{4n-6} = \frac{(-1)^{n-1} \delta^{n-1} \eta^{n-1}}{\beta^{n-2} \omega^{n-1} \prod_{i=0}^{n-2} [(2i+1)\delta\eta - 1]},$$
$$x_{4n-5} = \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} (\alpha\mu - 1)^{n-1}}{\alpha^{n-1} \mu^{n-1} \prod_{i=0}^{n-2} [(2i+2)\gamma\epsilon - 1]}, \quad x_{4n-4} = \frac{(-1)^{n-1} \delta^n \eta^{n-1} (\beta\omega - 1)^{n-1}}{\beta^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i+2)\delta\eta - 1]}.$$

And

$$y_{4n-7} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1} \prod_{i=0}^{n-2} [(2i)\gamma\epsilon - 1]}{\gamma^{n-1} \epsilon^{n-2} (\alpha\mu - 1)^{n-1}}, \quad y_{4n-6} = \frac{(-1)^{n-1} \beta^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i)\delta\eta - 1]}{\delta^{n-1} \eta^{n-2} (\beta\omega - 1)^{n-1}},$$
$$y_{4n-5} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^n \prod_{i=0}^{n-2} [(2i+1)\gamma\epsilon - 1]}{\gamma^{n-1} \epsilon^{n-1}}, \quad y_{4n-4} = \frac{(-1)^{n-1} \beta^{n-1} \omega^n \prod_{i=0}^{n-2} [(2i+1)\delta\eta - 1]}{\delta^{n-1} \eta^{n-1}}.$$

Now, we establish the proofs of two relations. Firstly, system (3) gives us that

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-5} y_{4n-7}}{y_{4n-5} (1 - x_{4n-5} y_{4n-7})} \\ &= \frac{\frac{(-1)^{n-1}\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]} \frac{(-1)^{n-1}\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}}{\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}} \left[1 - \frac{(-1)^{n-1}\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]} \frac{(-1)^{n-1}\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}\right]} \\ &= \frac{\frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}}{\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}} \left[1 - \frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\right]} \end{aligned}$$

$$= \frac{(-1)^{-n+1}\gamma^n\epsilon^n\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]\left[\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]-\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]\right]}$$

$$= \frac{-(-1)^{-n+1}\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}[(2i+1)\gamma\epsilon-1]} = \frac{(-1)^n\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}[(2i+1)\gamma\epsilon-1]}.$$

Next, it can be noticed from system (3) that

$$y_{4n-3} = \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}(-1+y_{4n-5}x_{4n-7})}$$

$$= \frac{\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}}{\frac{(-1)^{n-1}\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[-1+\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}\right]}$$

$$= \frac{(-1)^{-n+1}\alpha^n\mu^n\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}[-1+\alpha\mu]} = \frac{-(-1)^{n-1}\alpha^n\mu^n\prod_{i=0}^{n-1}[(2i)\gamma\epsilon-1]}{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^n}$$

$$= \frac{(-1)^n\alpha^n\mu^n\prod_{i=0}^{n-1}[(2i)\gamma\epsilon-1]}{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^n}.$$

The proofs of the remaining relations can be likewise achieved. Therefore, they are omitted.

2.4 Fourth System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})}$, $y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(1-y_{n-1}x_{n-3})}$

Our fundamental task in this subsection is to develop fractional solutions to the system of recursive equations given by the form:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(1-y_{n-1}x_{n-3})},\tag{4}$$

where the initial conditions are required to be non-zero real numbers.

Theorem 4 *Assume that $\{x_n, y_n\}$ is a solution to system (4) and suppose that $x_{-3}=\alpha$, $x_{-2}=\beta$, $x_{-1}=\gamma$, $x_0=\delta$, $y_{-3}=\epsilon$, $y_{-2}=\eta$, $y_{-1}=\mu$ and $y_0=\omega$. Then, for $n=0, 1, \ldots$ we have*

$$x_{4n-3} = \frac{(-1)^n\gamma^n\epsilon^n\prod_{i=0}^{n-1}[(2i)\alpha\mu-1]}{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^n},\quad x_{4n-2}=\frac{(-1)^n\delta^n\eta^n\prod_{i=0}^{n-1}[(2i)\beta\omega-1]}{\beta^{n-1}\omega^n(\delta\eta-1)^n},$$

$$x_{4n-1} = \frac{(-1)^n\gamma^{n+1}\epsilon^n\prod_{i=0}^{n-1}[(2i+1)\alpha\mu-1]}{\alpha^n\mu^n},\quad x_{4n}=\frac{(-1)^n\delta^{n+1}\eta^n\prod_{i=0}^{n-1}[(2i+1)\beta\omega-1]}{\beta^n\omega^n}.$$

And

$$y_{4n-3} = \frac{(-1)^n \alpha^n \mu^n}{\gamma^n \epsilon^{n-1} \prod_{i=0}^{n-1} [(2i+1)\alpha\mu - 1]}, \quad y_{4n-2} = \frac{(-1)^n \beta^n \omega^n}{\delta^n \eta^{n-1} \prod_{i=0}^{n-1} [(2i+1)\beta\omega - 1]},$$
$$y_{4n-1} = \frac{(-1)^n \alpha^n \mu^{n+1} (\gamma\epsilon - 1)^n}{\gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i+2)\alpha\mu - 1]}, \quad y_{4n} = \frac{(-1)^n \beta^n \omega^{n+1} (\delta\eta - 1)^n}{\delta^n \eta^n \prod_{i=0}^{n-1} [(2i+2)\beta\omega - 1]}.$$

Proof. The relations hold for $n = 0$. Next, we let $n > 1$ and assume that the formulas hold for $n-1$. That is

$$x_{4n-7} = \frac{(-1)^{n-1} \gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i)\alpha\mu - 1]}{\alpha^{n-2} \mu^{n-1} (\gamma\epsilon - 1)^{n-1}}, \quad x_{4n-6} = \frac{(-1)^{n-1} \delta^{n-1} \eta^{n-1} \prod_{i=0}^{n-2} [(2i)\beta\omega - 1]}{\beta^{n-2} \omega^{n-1} (\delta\eta - 1)^{n-1}},$$
$$x_{4n-5} = \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}}, \quad x_{4n-4} = \frac{(-1)^{n-1} \delta^n \eta^{n-1} \prod_{i=0}^{n-2} [(2i+1)\beta\omega - 1]}{\beta^{n-1} \omega^{n-1}}.$$

And

$$y_{4n-7} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}, \quad y_{4n-6} = \frac{(-1)^{n-1} \beta^{n-1} \omega^{n-1}}{\delta^{n-1} \eta^{n-2} \prod_{i=0}^{n-2} [(2i+1)\beta\omega - 1]},$$
$$y_{4n-5} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1}}{\gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+2)\alpha\mu - 1]}, \quad y_{4n-4} = \frac{(-1)^{n-1} \beta^{n-1} \omega^n (\delta\eta - 1)^{n-1}}{\delta^{n-1} \eta^{n-1} \prod_{i=0}^{n-2} [(2i+2)\beta\omega - 1]}.$$

We now turn to verify the proof of two relations. It can be obviously seen from system (4) that

$$\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5} y_{4n-7}}{y_{4n-5} (-1 + x_{4n-5} y_{4n-7})} \\
&= \frac{\frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}} \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}}{\frac{(-1)^{n-1} \alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1}}{\gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+2)\alpha\mu - 1]} \left[-1 + \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}} \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]} \right]} \\
&= \frac{(-1)^{-n+1} \gamma^n \epsilon^n \prod_{i=0}^{n-2} [(2i+2)\alpha\mu - 1]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1} [-1 + \gamma\epsilon]} = \frac{-(-1)^{n-1} \gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i)\alpha\mu - 1]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^n} \\
&= \frac{(-1)^n \gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i)\alpha\mu - 1]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^n}.
\end{aligned}$$

Further, it can be attained from system (4) that

$$
\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(1-y_{4n-5}x_{4n-7}\right)} \\
&= \frac{\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}}{\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}}\left[1-\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}\right]} \\
&= \frac{\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}}{\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}}\left[1-\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\right]} \\
&= \frac{(-1)^{-n+1}\alpha^{n}\mu^{n}\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]}{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}\left[(2i+1)\alpha\mu-1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]-\alpha\mu\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]\right]} \\
&= \frac{(-1)^{n}\alpha^{n}\mu^{n}}{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-1}\left[(2i+1)\alpha\mu-1\right]}.
\end{aligned}
$$

Other results can be proved in a similar way. Thus, the remaining proofs are omitted.

2.5 Numerical Examples

This subsection aims to present graphical confirmations to the whole solutions obtained in the previous subsections. Here, we plot the solutions (by using MATLAB software) under specific selections of some initial conditions.

Example 1. This example shows the paths of the solutions of system (1). The initial conditions of this example are given as follows: $x_{-3}=3,\ x_{-2}=1,\ x_{-1}=5,\ x_{0}=2,\ y_{-3}=1,\ y_{-2}=3,\ y_{-1}=5$ and $y_{0}=5$. See Figure 1.

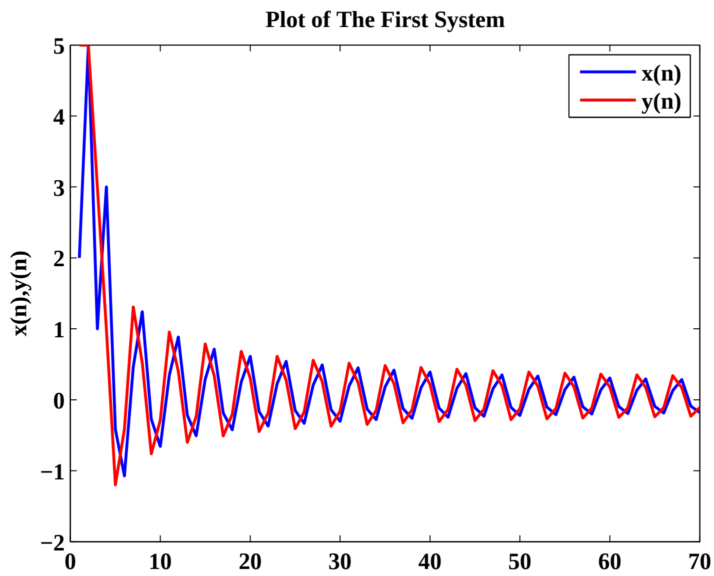

Figure 1: The behaviour of the solution of system (1).

Example 2. In Figure 2, we illustrate the behaviour of the solution of system (2) under the following selection of initial conditions: $x_{-3} = 3.4,\ x_{-2} = 0.7,\ x_{-1} = 2,\ x_0 = 3,\ y_{-3} = 1.5,\ y_{-2} = 1.5,\ y_{-1} = 0.5$ and $y_0 = 1.22$.

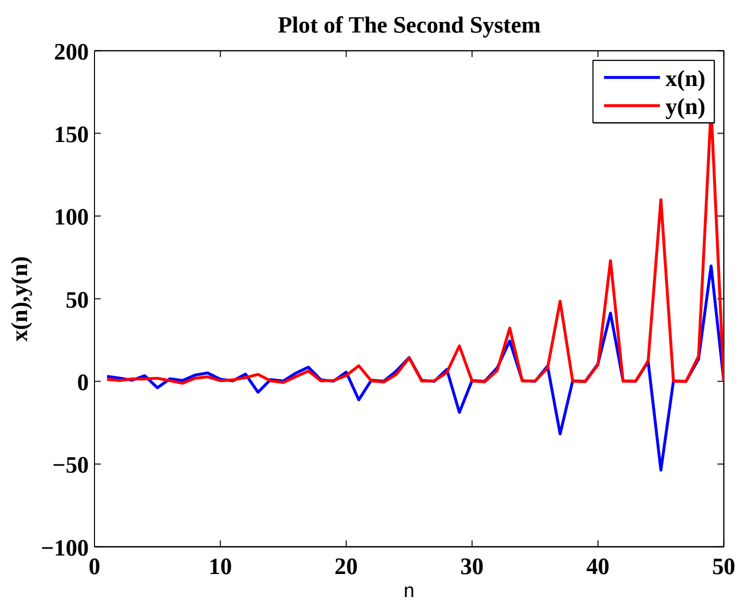

Figure 2: The behaviour of the solution of system (2).

Example 3. Figure 3 illustrates the curves of the solutions of system (3) when we assume that $x_{-3} = 0.7,\ x_{-2} = 2.1,\ x_{-1} = 1,\ x_0 = 0.5,\ y_{-3} = 0.1,\ y_{-2} = 0.2,\ y_{-1} = 2.2$ and $y_0 = 0.5$.

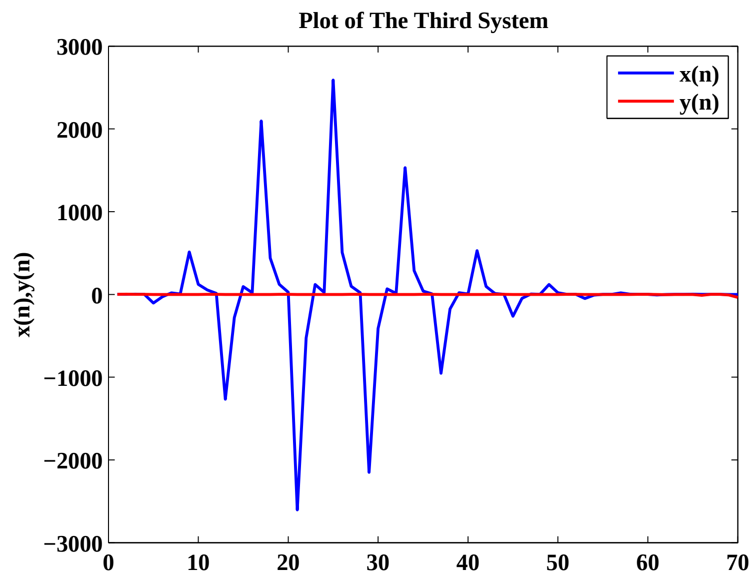

Figure 3: The behaviour of the solution of system (3).

Example 4. The solutions of system (4) are depicted in Figure 4 under the following initial data: $x_{-3} = 0.2,\ x_{-2} = 1,\ x_{-1} = 0.3,\ x_0 = 0.2,\ y_{-3} = 3,\ y_{-2} = 1,\ y_{-1} = 2$ and $y_0 = 0.3$.

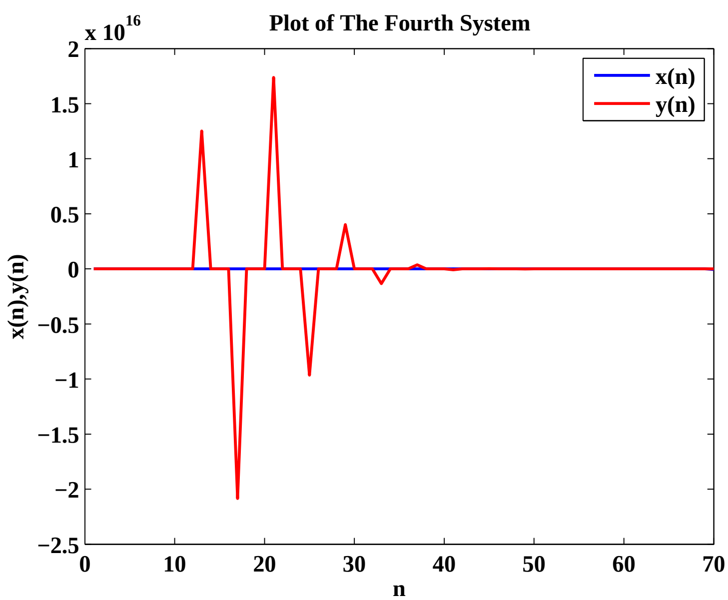

Figure 4: The behaviour of the solution of system (4).

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The ELECTRE multi-attribute group decision making method based on interval-valued intuitionistic fuzzy sets

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Abstract

In this paper, based on the ELECTRE method and new ranking for the interval-valued intuitionistic fuzzy set (IVIFS), the IVIF ELECTRE method to solve multi-attribute group decision-making problems with interval-valued intuitionistic fuzzy input data is proposed, it is extending the intuitionistic fuzzy set (IF) ELECTRE method. This method firstly use AHP (Analytic hierarchy process) to find the weights of attribute, and use new ranking method for IVIFS and similarity measure between IVIFS to determine the weights of decision makers (DMs), then give the concordance set, midrange concordance set, weak concordance set and cosponging discordance set, midrange discordance set, weak discordance set. From this, the concordance matrix, discordance index, concordance dominance matrix and discordance dominance matrix are proposed. Finally, the ranking order of all the alternatives $A_i (i = 1, 2, \ldots, n)$ and the best alternative are obtained. A numerical example is taken to illustrate the feasibility and practicability of the proposed method.

Keywords: Interval-valued intuitionistic fuzzy sets; ELECTRE method; Multi-attribute group decision making

1 Introduction

Since the multi-attribute decision making (MADM) was introduced in 1960's, it has been a hot topic in decision making and systems engineering, and been proven as a useful tool due to its broad applications in a number of practical problems. But in some real-life situations, a decision maker (DM) may not be able to accurately express his/her preferences for alternatives due to that (1) the DM may not possess a precise or sufficient level of knowledge of the problem; (2) the DM is unable to discriminate explicitly the degree to which one alternative are better than others. In order to handle inexact and imprecise data, in 1965 Zadeh [38] introduced fuzzy set (FS) theory. In 1983 Atanassov [1,2] generalized FS to intuitionistic fuzzy set (IFS) by using two characteristic functions to express the degree of membership and the degree of non-membership of elements of the universal set. Since IFS tackled the drawback of the single membership value in FS theory, IFS has been widely applied to the multi-attribute decision making (MADM) [4,7,8,10-14,20,22,23,28] and multi-attribute group decision making (MAGDM) [18,19,21].

In 1989 Atanassov and Gargov [3] further generalized the IFS in the spirit of the ordinary interval-valued fuzzy set (IVFS) and defined the concept of interval-valued intuitionistic fuzzy set (IVIFS), which enhances greatly the representation ability of uncertainty than IFS. Similar to the IFS, IVIFSs were also

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used in the problems of MADM [6,15-17,28,32] and MAGDM [29,31,33,34]. In these researches, some are extension of classic decision making methods in IVIFS environment. For example, Li [15] developed the closeness coefficient-based nonlinear-programming method for interval-valued intuitionistic fuzzy MAD-M with incomplete preference information, Li [16] proposed the TOPSIS-based nonlinear-programming methodology for MADM with IVIFSs, Li [17] proposed the linear-programming method for MADM with IVIFSs. These decision methods under interval-valued intuitionistic fuzzy environments also generalize the classic decision making methods, such as TOPSIS and LINMAP. In [32], Wang et al. proposed a expect to apply ELECTRE and PROMETHEE motheds to MADM and MAGDM with IVIFS.

In this paper, based on the new ranking method of interval in [27] and similarity measure of IVIFSs in [35, 37], the IF ELECTRE [30] method is applied to MAGDM with IVIFS, and obtain IVIFS ELECTRE method for solving MAGDM problems under IVIF environments.

This paper is organized as follows. Section 2 briefly reviews the analytic hierarchy process (AHP). Section 3 and Section 4 introduce the new ranking method of intervals and similarity measure between IVIFSs, respectively. Section 5 formulates an MAGDM problem in which the evaluation of alternatives in each attribute is expressed by IVIF sets, and also develops an extended ELECTRE method. Section 6 demonstrates the feasibility and applicability of the proposed method by applying it to the MAGDM problem of the air-condition. Finally, Section 7 presents the conclusions.

2 Analytic hierarchy process (AHP)

AHP was introduced for the first time in 1980 by Thomas L. Saaty [24]. For years, AHP has been used in various fields such as social sciences, health planning and management. Many researchers have preferred to use AHP to find the weights of attribute [25,26]. Due to the fact that attribute weights in the decision-making problems are various, it is not correct to assign all of them as equalled [5]. To solve the problem of indicating the weights, some methods like AHP, eigenvector, entropy analysis, and weighted least square methods were used. For the calculation of attribute weight in AHP the following steps are used:

(i) Arrange the attribute in $n \times n$ square matrix form as rows and columns.

(ii) Using pairwise comparisons, the relative importance of one attribute over another can be expressed as follow:

If two attribute have equal importance in pairwise comparison enter 1; if one of them is moderately more important than the other enter 3 and for the other enter 1/3; if one of them is strongly more important enter 5 and for the other enter 1/5; if one of them is very strongly more important enter 7 and for the other enter 1/7, and if one of them is extremely important enter 9 and for the other enter 1/9. 2, 4, 6 and 8 can be entered as intermediate values. Thus, pairwise comparison matrix is obtained as a result of the pairwise comparisons. Note that all elements in the comparison matrix are positive, in other words $a_{ij} > 0$ $(i, j = 1, 2, \dots, n)$.

(a) To find the maximum eigenvalue λ of the comparison matrix.

(b) Calculate consistency index $CI = \frac{\lambda-n}{n-1}$ and consistency ratio $CR = \frac{CI}{RI}$, where RI is the random consistency index given by Saaty.(Table 1)

(c) If $CR \geq 0.1$, then adjusts elements a_{ij} $(i, j = 1, 2, \dots, n)$ of the comparison matrix, (a) and (b) choices are done iteratively until $CR < 0.1$.

(d) Compute eigenvector of the maximum eigenvalue of the comparison matrix.

(e) Normalized eigenvector.

Table1:Random consistency index RI.

n	1	2	3	4	5	6	7	8	9	10	11
RI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51

3 Ranking method for intervals

Let $x = [a, b] \subseteq [0, 1]$ and $y = [c, d] \subseteq [0, 1]$ be two intervals. Since the location relations between $x = [a, b]$ and $y = [c, d]$ include the following six cases, Wan and Dong [27] calculated the occurrence probability for the fuzzy(or random) event $x \geq y$, denoted by $P(x \geq y)$, under different cases.

Case1: $a < b \leq c < d$,

$$P(x \geq y) = 0. \tag{1}$$

Case2: $a \leq c < b < d$ or $a < c < b \leq d$,

$$P(x \geq y) = \frac{(b-c)^2}{2(b-a)(d-c)}. \tag{2}$$

Case3: $a \leq c < d < b$ or $a < c < d \leq b$ or $a \leq c < d \leq b$,

$$P(x \geq y) = \frac{2b-d-c}{2(b-a)}. \tag{3}$$

Case4: $c \leq a < b < d$ or $c < a < b \leq d$,

$$P(x \geq y) = \frac{b+a-2c}{2(d-c)}. \tag{4}$$

Case5: $c \leq a < d < b$ or $c < a < d \leq b$,

$$P(x \geq y) = \frac{2bd+2ac-2bc-a^2-d^2}{2(b-a)(d-c)}. \tag{5}$$

Case6: $c \leq d \leq a < b$ or $c < a < b \leq d$,

$$P(x \geq y) = 1. \tag{6}$$

In order to rank intervals $\tilde{a}_i = [a_i, b_i]$ $(i = 1, 2, \ldots, n)$, Wang and Dong [27] construct the matrix of possibility degree as $P = (P_{ij})_{n\times n}$, where $P_{ij} = P(\tilde{a}_i \geq \tilde{a}_j)$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$. Then, the ranking vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is derived as follows:

$$\omega_i = (\sum_{j=1}^{n} P_{ij} + \frac{n}{2} - 1)/(n(n-1)) \ \ (i = 1, 2, \cdots, n). \tag{7}$$

The larger the value of ω_i, the bigger the corresponding intervals $\tilde{a}_i = [a_i, b_i]$. In other words, for the two intervals $\tilde{a}_i = [a_i, b_i]$ and $\tilde{a}_j = [a_j, b_j]$, if $\omega_i \geq \omega_j$, then $[a_i, b_i] \geq [a_j, b_j]$.

4 Similarity measure between IVIFSs

Definition 1.[3] An IVIFS A in the universe set of discourse X is defined as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\},$$

where $\mu_A(x) \subseteq [0, 1]$ and $\nu_A(x) \subseteq [0, 1]$ denote respectively the membership degree interval and the non-membership degree interval of x to A,with the condition:

$\sup\mu_A(x)+\sup\nu_A(x) \leq 1, \forall x \in X$.

Since IVIFS is composed of two ordered interval pairs, Xu [31,32] called them interval-valued intuitionistic fuzzy numbers(IVIFNs) and simply denoted by $G = ([a,b],[c,d])$, where $[a,b] \subseteq [0,1], [c,d] \subseteq [0,1]$ and $b + d \leq 1$.

Definition 2.[37] Let $G_i = ([a_i,b_i],[c_i,d_i])$ $(i = 1,2)$ be two IVIFNs, the normalized Hamming distance between G_1 and G_2 can be defined as:

$$d(G_1,G_2) = \frac{1}{4}(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2| + \left|\pi_1' - \pi_2'\right| + \left|\pi_1'' - \pi_2''\right|), \tag{8}$$

where $\pi_{G_i} = [\pi_i', \pi_i''] = [1 - b_i - d_i, 1 - a_i - c_i]$ $(i = 1,2)$ is called the degree of indeterminacy or called the degree of hesitancy of the IVIFN G_i.

Definition 3.[35, 37] Let $G_i = ([a_i,b_i],[c_i,d_i])$ $(i = 1,2)$ be two IVIFNs, then

$$s(G_1,G_2) = \begin{cases} 1, & if\ G_1 = G_2 = G_2^c, \\ \frac{d(G_1,G_2^c)}{d(G_1,G_2)+d(G_1,G_2^c)}, & otherwise \end{cases} \tag{9}$$

is called the degree of similarity between G_1 and G_2, where $G_2^c = ([c_2,d_2],[a_2,b_2])$ is denoted as the complement of G_2.

Definition 4.[37] Let A and B be two IVIFSs in X, then

$$s(A,B) = \frac{1}{n}\sum_{j=1}^{n} s(G_j^A, G_j^B) = \frac{1}{n}\sum_{j=1}^{n} \frac{d(G_j^A,(G_j^B)^c)}{d(G_j^A,G_j^B) + d(G_j^A,(G_j^B)^c)} \tag{10}$$

is called the degree of similarity between A and B , where G_j^A and G_j^B are j-th IVIFNs of A and B , respectively.

Definition 5.[6, 27] Let G_i $(i = 1,2,\ldots,n)$ be a collection of the IVIFNs, where $G_i = ([a_i,b_i],[c_i,d_i])$. If

$$Y_\omega(G_1,G_2,\cdots,G_n) = \frac{\sum\limits_{j=1}^{n} \omega_j G_j}{\sum\limits_{j=1}^{n} \omega_j}, \tag{11}$$

where $\omega = (\omega_1,\omega_2,...,\omega_n)^T$ is the weight vector, then the function Y_ω is called the weighted average operator for the IVIFNs. Particularly, if ω_j $(j = 1,2,\ldots,n)$ are crisp values, then the weighted average operator Y_ω is calculated as follows:

$$Y_\omega(G_1,G_2,\cdots,G_n) = \frac{\sum\limits_{j=1}^{n} \omega_j G_j}{\sum\limits_{j=1}^{n} \omega_j} = \left(\left[\frac{\sum\limits_{j=1}^{n} \omega_j a_j}{\sum\limits_{j=1}^{n} \omega_j}, \frac{\sum\limits_{j=1}^{n} \omega_j b_j}{\sum\limits_{j=1}^{n} \omega_j}\right], \left[\frac{\sum\limits_{j=1}^{n} \omega_j c_j}{\sum\limits_{j=1}^{n} \omega_j}, \frac{\sum\limits_{j=1}^{n} \omega_j d_j}{\sum\limits_{j=1}^{n} \omega_j}\right]\right). \tag{12}$$

5 MAGDM problems and ELECTRE method with IVIFSs

5.1 Problems description for MAGDM with IVIFSs

Assume that there are m alternatives $\{A_1,A_2,\ldots,A_m\}$ and k experts $\{p_1,p_2,\ldots,p_k\}$, each alternative A_i has n attributes $\{a_1,a_2,\ldots,a_n\}$. For each alternative A_i, each expert gives evaluation on different attribute.

The multi-attribute group decision making (MAGDM) is choose the best one from all alternatives according to these evaluations. Assume that $G^t_{Mij} = [a^t_{ij}, b^t_{ij}]$ and $G^t_{Nij} = [c^t_{ij}, d^t_{ij}]$ are respectively the membership degree and non-membership degree of alternative $A_i \in A$ on an attribute a_j given by DM p_t to the fuzzy concept "excellent". In other words, the evaluation of A_i on a_j given by p_t is an IVIFN as follows:

$$G^t_{ij} = (G^t_{Mij}, G^t_{Nij}), \tag{13}$$

where $[a^t_{ij}, b^t_{ij}] \subseteq [0,1], [c^t_{ij}, d^t_{ij}] \subseteq [0,1]$ and $b^t_{ij} + d^t_{ij} \leq 1$ $(1 \leq i \leq m, 1 \leq j \leq n, 1 \leq t \leq k)$.

5.2 Determination of the weights of DMs

Since the different DMs play different roles during the process of decision making, thus the importance of DMs should be taken into consideration. The weight vector of DMs is denoted by $z = (z_1, z_2, \ldots, z_k)^T$. In the following, an approach determined the weights of DMs is given.

Suppose that the evaluation of alternative A_i given by DM p_t on each attribute are respectively the IVIFNs $G^t_{i1}, G^t_{i2}, ..., G^t_{in}$. By Eq.(12), the individual overall attribute value of A_i given by p_t is obtained as follows:

$$E^t_i = ([a^t_i, b^t_i], [c^t_i, d^t_i]) = Y_\omega(G^t_{i1}, G^t_{i2}, \cdots G^t_{in}), \tag{14}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of attributes.

Let $E^t = (E^t_1, E^t_2, \ldots, E^t_m)$ and $E^u = (E^u_1, E^u_2, \ldots, E^u_m)$ are evaluation vectors of all alternatives given by DMs p_t and p_u, respectively. Using Eqs.(8-10), the similarity degree s_{tu} between E^t and E^u is obtained, and the similarity matrix S is constructed as follows:

$$S = (s_{tu})_{k\times k}. \tag{15}$$

Obviously, S is a non-negative symmetric matrix, by the Perron-Frobenius theorem [12], there exists the maximum module eigenvalue $\lambda > 0$, and the corresponding eigenvector $x = (x_1, x_2, \ldots, x_k)^T$ satisfies that $x_t > 0$ $(t = 1, 2, \ldots, k)$ and $\lambda x = Sx$.

Let $z = \lambda x = Sx$, then each component of z is the weight of corresponding expert. The normalized vector z, the weight z_t $(t = 1, 2, \ldots, k)$ of DM p_t is obtained as follows:

$$z_t = \frac{x_t}{(x_1 + x_2 + \cdots + x_k)} \quad (t = 1, 2, \cdots, k). \tag{16}$$

5.3 ELECTRE methods based on IVIFS

Based on the idea of ELECTRE method, a new approach, named as IVIF ELECTRE, is formulated to solve a MCDM problem under interval-valued intuitionistic fuzzy environment. For each pair of alternatives k and l $(k, l = 1, 2, \ldots, m$ and $k \neq l)$, each attribute in the different alternatives can be divided into two distinct subsets. The concordance set E_{kl} of A_k and A_l is composed of all attribute for which A_k is preferred to A_l. In other words, $E_{kl} = \{j | x_{kj} \geq x_{lj}\}$, where $J = \{j | j = 1, 2, \ldots, n\}$, x_{kj} and x_{lj} denoted the evaluation of DM in the jth attribute to alternative A_k and A_l, respectively. The complementary subset, which is the discordance set, is $F_{kl} = \{j | x_{kj} < x_{lj}\}$. In the proposed IVIF ELECTRE method, we can classify different types of concordance and discordance sets using the concepts of score function, accuracy function and hesitation degree, and use concordance and discordance sets to construct concordance and discordance matrices, respectively. The decision makers can choose the best alternative using the concepts of positive and negative ideal points.

Xu [31] and Xu and Chen [36] defined the score function $S(G)$ and accuracy function $H(G)$ for an IVIFN G=([a,b],[c,d]) as follows:

$$S(G) = \frac{1}{2}(a + b - c - d), \tag{17}$$

$$H(G) = \frac{1}{2}(a + b + c + d). \tag{18}$$

Here, we define the hesitation degree for an IVIFN G=([a,b],[c,d]) as follows:

$$\pi(G) = 1 - \frac{1}{2}(a + b + c + d). \tag{19}$$

From (18) and (19), easy to see that a higher accuracy degree $H(G)$ correlates with a lower hesitancy degree $\pi(G)$.

Considering the better alternative has the higher score degree or higher accuracy degree in cases where alternatives have the same score degree. A higher score degree refers to a larger membership degree or smaller non-membership degree, and a higher accuracy degree refers to a smaller hesitation degree. Based on this, using the above three functions to compare IVIF values of different alternatives. The concordance set can be classified as concordance set, midrange concordance set and weak concordance set. Similarly, The discordance sets can also be classified as the discordance set, midrange discordance set, and weak discordance set.

Next, the concordance set, midrange concordance set, weak concordance set, discordance set, midrange discordance set, weak discordance set are defined respectively as follows.

Let $G_{kj} = ([a_{kj}, b_{kj}], [c_{kj}, d_{kj}])$ and $G_{lj} = ([a_{lj}, b_{lj}], [c_{lj}, d_{lj}])$ denote the jth attribute value of alternative A_k and A_l, respectively. The concordance set C_{kl} is composed of all attribute for which A_k is preferred to A_l,i.e.,

$$C_{kl} = \{j|[a_{kj}, b_{kj}] \geq [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] < [c_{lj}, d_{lj}] \;\; and \; [\pi'_{kj}, \pi''_{kj}] < [\pi'_{lj}, \pi''_{lj}]\}, \tag{20}$$

where $J = \{j|j = 1, 2, \ldots, n\}$.

The midrange concordance set C'_{kl} is defined as

$$C'_{kl} = \{j|[a_{kj}, b_{kj}] \geq [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] < [c_{lj}, d_{lj}] \;\; and \; [\pi'_{kj}, \pi''_{kj}] \geq [\pi'_{lj}, \pi''_{lj}]\}. \tag{21}$$

The major difference between (20) and (21) is the hesitancy degree; the hesitancy degree at the kth alternative with respect to the jth attribute is higher than the lth alternative with respect to the jth attribute in the midrange concordance set. Thus, Eq. (20) is more concordant than (21).

The weak concordance set C''_{kl} is defined as

$$C''_{kl} = \{j|[a_{kj}, b_{kj}] \geq [a_{lj}, b_{lj}] \;\; and \;\; [c_{kj}, d_{kj}] \geq [c_{lj}, d_{lj}]\}. \tag{22}$$

The degree of non-membership at the kth alternative with respect to the jth attribute is higher than the lth alternative with respect to the jth attribute in the weak concordance set; thus, Eq. (21) is more concordant than (22).

The discordance set is composed of all attribute for which A_k is not preferred to A_l. The discordance set D_{kl} is formulated as follows:

$$D_{kl} = \{j|[a_{kj}, b_{kj}] < [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] \geq [c_{lj}, d_{lj}] \;\; and \; [\pi'_{kj}, \pi''_{kj}] \geq [\pi'_{lj}, \pi''_{lj}]\}, \tag{23}$$

The midrange discordance set D'_{kl} is defined as

$$D'_{kl} = \{j|[a_{kj}, b_{kj}] < [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] \geq [c_{lj}, d_{lj}] \;\; and \; [\pi'_{kj}, \pi''_{kj}] < [\pi'_{lj}, \pi''_{lj}]\}. \tag{24}$$

The weak discordance set D''_{kl} is defined as

$$D''_{kl} = \{j|[a_{kj}, b_{kj}] < [a_{lj}, b_{lj}] \ \ and \ \ [c_{kj}, d_{kj}] < [c_{lj}, d_{lj}]\}. \tag{25}$$

The IVIF ELECTRE method is an integrated IVIFS and ELECTRE method. The relative value of the concordance set of the IVIF ELECTRE method is measured through the concordance index. The concordance index e_{kl} between A_k and A_l is defined as:

$$e_{kl} = \min_{j \in C^*}\{w_{C^*} \times d(G_{kj}, G_{lj})\}, \tag{26}$$

where $d(G_{kj}, G_{lj})$ is defined in (8), denoted the distance between jth attribute of alternatives A_k and A_l, and w_{C^*} is equal to w_C, $w_{C'}$ or $w_{C''}$, which denoted the weight of the concordance, midrange concordance, and weak concordance sets, respectively.

The concordance matrix E is defined as follows:

$$E = \begin{bmatrix} - & e_{12} & \cdots & \cdots & e_{1m} \\ e_{21} & - & e_{23} & \cdots & e_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ e_{(m-1)1} & \cdots & \cdots & - & e_{(m-1)m} \\ e_{m1} & e_{m2} & \cdots & e_{m(m-1)} & - \end{bmatrix}, \tag{27}$$

where the maximum value of e_{kl} is denoted by e^*, which is the positive ideal point, and a higher value of e_{kl} indicates that A_k is preferred to A_l.

the discordance index is defined as follows:

$$h_{\mathrm{kl}} = \max_{j \in D^*}\{w_{D^*} \times d(G_{kj}, G_{lj})\}, \tag{28}$$

where $d(G_{kj}, G_{lj})$ is defined in (8), denoted the distance between jth attribute of alternatives A_k and A_l, and w_{D^*} is equal to w_D, $w_{D'}$ or $w_{D''}$, which denoted the weight of the discordance, midrange discordance, and weak discordance sets, respectively.

The discordance matrix H is defined as follows:

$$H = \begin{bmatrix} - & h_{12} & \cdots & \cdots & h_{1m} \\ h_{21} & - & h_{23} & \cdots & h_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ h_{(m-1)1} & \cdots & \cdots & - & h_{(m-1)m} \\ h_{m1} & h_{m2} & \cdots & h_{m(m-1)} & - \end{bmatrix}, \tag{29}$$

where the maximum value of h_{kl} is denoted by h^*, which is the negative ideal point, and a higher value of H_{kl} indicates that A_k is less favorable than A_l.

The concordance dominance matrix calculation process is based on the concept that the chosen alternative should have the shortest distance from the positive ideal solution, thus, the concordance dominance matrix K is defined as follows:

$$K = \begin{bmatrix} - & k_{12} & \cdots & \cdots & k_{1m} \\ k_{21} & - & k_{23} & \cdots & k_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ k_{(m-1)1} & \cdots & \cdots & - & k_{(m-1)m} \\ k_{m1} & k_{m2} & \cdots & k_{m(m-1)} & - \end{bmatrix}, \tag{30}$$

where $k_{kl} = e^* - e_{kl}$, which refers to the separation of each alternative from the positive ideal solution. A higher value of k_{kl} indicates that A_k is less favorable than A_l.

The discordance dominance matrix calculation process is based on the concept that the chosen alternative should have the farthest distance from the negative ideal solution, thus, the discordance dominance matrix L is defined as follows:

$$L = \begin{bmatrix} - & l_{12} & \cdots & \cdots & l_{1m} \\ l_{21} & - & l_{23} & \cdots & l_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ l_{(m-1)1} & \cdots & \cdots & - & l_{(m-1)m} \\ l_{m1} & l_{m2} & \cdots & l_{m(m-1)} & - \end{bmatrix}, \tag{31}$$

where $l_{kl} = h^* - h_{kl}$, which refers to the separation of each alternative from the negative ideal solution. A higher value of l_{kl} indicates that A_k is preferred to A_l.

In the aggregate dominance matrix determining process, the distance of each alternative to both positive and negative ideal points can be calculated to determine the ranking order of all alternatives. The aggregate dominance matrix R is defined as follows:

$$R = \begin{bmatrix} - & r_{12} & \cdots & \cdots & r_{1m} \\ r_{21} & - & r_{23} & \cdots & r_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ r_{(m-1)1} & \cdots & \cdots & - & r_{(m-1)m} \\ r_{m1} & r_{m2} & \cdots & r_{m(m-1)} & - \end{bmatrix}, \tag{32}$$

where

$$r_{kl} = \frac{l_{kl}}{k_{kl} + l_{kl}},$$

r_{kl} refers to the relative closeness to the ideal solution, with a range from 0 to 1. A higher value of r_{kl} indicates that the alternative A_k is simultaneously closer to the positive ideal point and farther from the negative ideal point than the alternative A_l, thus, it is a better alternative.

Let $\overline{T}_k = \frac{1}{m-1} \sum\limits_{l=1, l\neq k}^{m} r_{kl},\ k = 1, 2, \cdots, m,$ (33)

and $\overline{T}_k$ is the final value of evaluation. All alternatives can be ranked according to $\overline{T}_k$. The best alternative T^*, which is simultaneously the shortest distance to the positive ideal point and the farthest distance from the negative ideal point, can be generated and defined as follows:

$$T^* = \max_{1\leq k\leq m} \{\overline{T}_k\}, \tag{34}$$

where A^* is the best alternative.

5.4 Group decision making method

In the following we shall utilize the AHP and interval-valued intuitionistic fuzzy weighted average operator Y (i.e. Eq. (12)) to propose a new MAGDM method with IVIFN information. The detailed steps are summarized as follows:

Step 1. DMs use IVIFSs to represent the evaluation information in the each attribute of alternatives;
Step 2. Use AHP to calculate the weight of attribute;
Step 3. Calculate the individual overall attribute value of each alternative by Eq.(14);
Step 4. Obtain the similarity matrix of the DMs according to Eq.(10);

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Step 5. Derive the weight value of each DM from Eq.(16);
Step 6. Using the weight of DM to integrate the same attribute value of different DMs of each alternative in terms of Eq.(14);
Step 7. By the possibility degree ranking method for intervals in Section 3, calculate the ranking vector of the membership degree interval, the non-membership degree interval and the hesitancy degree interval of between the difference alternatives on each attribute, respectively.
Step 8. Obtain the concordance, midrange concordance, weak concordance, discordance, midrange discordance and weak discordance set according to Eqs.(20)-(25), respectively;
Step 9. Compute the concordance matrix, discordance matrix, concordance dominance matrix, discordance dominance matrix and aggregate dominance matrix in terms of Eqs.(26)-(32);
Step 10. Obtain the ranking order of all alternatives and the best alternative according to Eqs.(33)-(34).

6 Numerical example

In this section, we use the air-condition system selection problem given by [27] to verify the feasibility of the proposed method. The problem is described as follows:

Suppose there exist three air-condition systems $\{A_1, A_2, A_3\}$, four attributes a_1 (economical), a_2(function), a_3(being operative) and a_4(longevity) are taken into consideration in the selection problem. Three experts (DMs) $\{p_1, p_2, p_3\}$ participate in the decision making. The membership degrees and non-membership degrees for the alternative A_i on the attribute a_j given by expert p_t were listed in Tables $2-4$.

Table 2: IVIFNs given by the expert p_1.

Attribute	A_1	A_2	A_3
a_1	([0.4, 0.8], [0.0, 0.1])	([0.5, 0.7],[0.1, 0.2])	([0.5, 0.7],[0.2, 0.3])
a_2	([0.3, 0.6], [0.0, 0.2])	([0.3, 0.5],[0.2, 0.4])	([0.6, 0.8],[0.1, 0.2])
a_3	([0.2, 0.7], [0.2, 0.3])	([0.4, 0.7],[0.0, 0.2])	([0.4, 0.7],[0.1, 0.2])
a_4	([0.3, 0.4], [0.4, 0.5])	([0.1, 0.2],[0.7, 0.8])	([0.6, 0.8],[0.0, 0.2])

Table 3: IVIFNs given by the expert p_2.

Attribute	A_1	A_2	A_3
a_1	([0.5, 0.9], [0.0, 0.1])	([0.7, 0.8], [0.1, 0.2])	([0.5, 0.6], [0.1, 0.4])
a_2	([0.4, 0.5], [0.3, 0.5])	([0.5, 0.6], [0.2, 0.3])	([0.6, 0.7], [0.1, 0.2])
a_3	([0.5, 0.8], [0.0, 0.1])	([0.5, 0.8], [0.0, 0.2])	([0.4, 0.8], [0.1, 0.2])
a_4	([0.4, 0.7], [0.1, 0.2])	([0.5, 0.6], [0.3, 0.4])	([0.2, 0.6], [0.2, 0.3])

Table 4: IVIFNs given by the expert p_3.

Attribute	A_1	A_2	A_3
a_1	([0.3, 0.9], [0.0, 0.1])	([0.3, 0.8], [0.1, 0.2])	([0.2, 0.6], [0.1, 0.2])
a_2	([0.2, 0.5], [0.1, 0.4])	([0.5, 0.6], [0.1, 0.3])	([0.2, 0.6], [0.2, 0.3])
a_3	([0.4, 0.7], [0.1, 0.2])	([0.2, 0.8], [0.0, 0.2])	([0.3, 0.6], [0.1, 0.3])
a_4	([0.3, 0.6], [0.3, 0.4])	([0.3, 0.5], [0.2, 0.3])	([0.4, 0.7], [0.1, 0.2])

In the following, we will illustrate the decision making process.
(1) Calculation of weights of attributes

In order to find the weights of attributes, A commission, which is organized by sampling method, determined the importance of attribute by using AHP. A 4×4 size matrix is formed because 4 attribute are considered in this study. All the diagonal elements of the matrix will be 1, the elements of symmetrical position with respect to the diagonal are reciprocal, in other words, if a_{ij} is ith row and jth column element of matrix, then its symmetrical position is filled using $a_{ji} = 1/a_{ij}$ formula.

The comparison matrix W is obtained as follows:

$$W = \begin{pmatrix} 1 & 2 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{6} \\ 3 & 3 & 1 & \frac{1}{3} \\ 4 & 6 & 3 & 1 \end{pmatrix}.$$

By computing the eigenvalues and the eigenvectors of W, we obtained that the maximum eigenvalue of W was 4.0875, the corresponding eigenvector was $\omega = (0.1905, 0.1230, 0.4046, 0.8849)^T$, consistency index CI=0.0292 and consistency ratio $CR = 0.0324 < 0.1$.

Normalized eigenvectors, the four attributes weights are obtained as follows:
$\omega_1 = 0.1213$, $\omega_2 = 0.0765$, $\omega_3 = 0.2517$, $\omega_4 = 0.5505$.

(2) Calculate the individual overall attribute value of each alternative

By Eq.(14), the individual overall attribute value of each alternative can be obtained as in Table 5.

Table 5: The individual overall attribute values of the alternatives for weight vector of attributes.

E_i^t	A_1	A_2	A_3
p_1	([0.2870,0.5393],[0.2705,0.3782])	([0.2393,0.4095],[0.4128,0.5456])	([0.5375,0.7627],[0.0571,0.2121])
p_2	([0.4373,0.7341],[0.0780,0.1857])	([0.5242,0.6746],[0.1926,0.3178])	([0.3173,0.6580],[0.1551,0.2793])
p_3	([0.3175,0.6539],[0.1980,0.3133])	([0.2901,0.6196],[0.1299,0.2627])	([0.3353,0.6551],[0.1077,0.2328])

(3) Calculation of the similarity matrix and the weight vector of DMs

The similarity matrix for the DMs is constructed by Eq.(10) as follows:

$$S = \begin{pmatrix} 1 & 0.5415 & 0.6059 \\ 0.5415 & 1 & 0.7577 \\ 0.6059 & 0.7577 & 1 \end{pmatrix}.$$

Because the maximum eigenvalue of S is 2.2746, the corresponding eigenvector is $x = (0.5373, 0.5878, 0.6048)^T$, the expert′s weights are obtained from Eq.(16) as follows: $z_1 = 0.3106$, $z_2 = 0.3398$, $z_3 = 0.3496$.

(4) Integrate the attribute value of different DMs

By Eq.(14), the attribute value of different DMs are respectively integrated as in Table 6.

Table 6: The attribute value of different DMs in the different alternatives and different attributes.

	A_1	A_2	A_3
a_1	([0.3990,0.8689],[0,0.1])	([0.4980,0.7689],[0.1,0.2])	([0.3951,0.6311],[0.1311,0.2990])
a_2	([0.2990,0.5311],[0.1369,0.3719])	([0.4379,0.5689],[0.1650,0.3311])	([0.4602,0.6961],[0.1350,0.2350])
a_3	([0.3719,0.7340],[0.0971,0.1971])	([0.3641,0.7689],[0,0.2])	([0.3650,0.6990],[0.1,0.2350])
a_4	([0.3340,0.5719],[0.2631,0.3631])	([0.3058,0.4408],[0.3893,0.4893])	([0.3942,0.6971],[0.1029,0.2340])

(5) Calculate the ranking vector

The ranking vector of the membership degree interval, the non-membership degree interval and the hesitancy degree interval of between the difference alternatives on each attribute is calculated by Eqs.(1-7), respectively, as in Table 7.

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Table 7: The attribute value of different DMs in the different alternatives and different attributes.

	membership degree interval		non-membership degree interval		hesitancy degree interval	
a_1	A_1	A_2	A_1	A_2	A_1	A_2
	0.5006	0.4994	0.25	0.75	0.5873	0.4127
	A_1	A_3	A_1	A_3	A_1	A_3
	0.6286	0.3714	0.25	0.75	0.5388	0.4612
	A_2	A_3	A_2	A_3	A_2	A_3
	0.6808	0.3192	0.3207	0.6793	0.4341	0.5659
a_2	A_1	A_2	A_1	A_2	A_1	A_2
	0.3214	0.6786	0.5135	0.4865	0.5878	0.4122
	A_1	A_3	A_1	A_3	A_1	A_3
	0.27295	0.72705	0.6477	0.3523	0.59905	0.40095
	A_2	A_3	A_2	A_3	A_2	A_3
	0.34565	0.65435	0.6764	0.3236	0.5173	0.4827
a_3	A_1	A_2	A_1	A_2	A_1	A_2
	0.48325	0.51675	0.6177	0.3823	0.4723	0.5277
	A_1	A_3	A_1	A_3	A_1	A_3
	0.52875	0.47125	0.4246	0.5754	0.4995	0.5005
	A_2	A_3	A_2	A_3	A_2	A_3
	0.54255	0.45745	0.3426	0.6574	0.5273	0.4727
a_4	A_1	A_2	A_1	A_2	A_1	A_2
	0.66115	0.33885	0.25	0.75	0.56895	0.43105
	A_1	A_3	A_1	A_3	A_1	A_3
	0.35955	0.64045	0.75	0.25	0.44015	0.55985
	A_2	A_3	A_2	A_3	A_2	A_3
	0.2633	0.7367	0.75	0.25	0.365	0.635

(6) Determine the concordance, midrange concordance, weak concordance, discordance, midrange discordance and weak discordance set

Applying Eqs.(20-25) and Table 7, the concordance, midrange concordance, weak concordance, discordance, midrange discordance and weak discordance set is calculated, respectively, as follows:

$$C=\begin{pmatrix} - & - & 3\\ 2 & - & 1\\ 2 & 2 & - \end{pmatrix},\quad C'=\begin{pmatrix} - & 1,4 & 1\\ 3 & - & 3\\ 4 & 4 & - \end{pmatrix},\quad C''=\begin{pmatrix} - & - & -\\ - & - & -\\ - & - & - \end{pmatrix},$$
$$D=\begin{pmatrix} - & 2 & 2\\ - & - & 2\\ 3 & 1 & - \end{pmatrix},\quad D'=\begin{pmatrix} - & 3 & 4\\ 1,4 & - & 4\\ 1 & 3 & - \end{pmatrix},\quad D''=\begin{pmatrix} - & - & -\\ - & - & -\\ - & - & - \end{pmatrix}.$$

For example, $c_{13}=\{3\}$, which is in the 1st (horizontal) row and the 3rd (vertical) column of the concordance set, is "3." $c_{12}=\{-\}$, which is in the 1st row and 2nd column of the concordance set, is "empty," and so forth.

(7) Compute the concordance matrix, discordance matrix, concordance dominance matrix, discordance dominance matrix and aggregate dominance matrix

We give the relative weights as: $[\omega_C,\omega_{C'},\omega_{C''},\omega_D,\omega_{D'},\omega_{D''}]=[1,\frac{2}{3},\frac{1}{3},1,\frac{2}{3},\frac{1}{3}]$. By Eqs.(26)-(32), the concordance matrix, discordance matrix, concordance dominance matrix, discordance dominance matrix and aggregate dominance matrix are obtained, respectively, as follows:

$$E=\begin{pmatrix} - & 0.08575 & 0.02235\\ 0.04759 & - & 0.05697\\ 0.09643 & 0.07862 & - \end{pmatrix},\quad H=\begin{pmatrix} - & 0.1039 & 0.16309\\ 0.09967 & - & 0.18088\\ 0.12298 & 0.1204 & - \end{pmatrix},$$

$$K=\begin{pmatrix} - & 0.01068 & 0.07408\\ 0.04884 & - & 0.03946\\ 0 & 0.01781 & - \end{pmatrix},\quad L=\begin{pmatrix} - & 0.07698 & 0.01779\\ 0.08121 & - & 0\\ 0.0579 & 0.06048 & - \end{pmatrix}.$$

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$$R = \begin{pmatrix} - & 0.8782 & 0.1936 \\ 0.6246 & - & 0 \\ 1 & 0.7725 & - \end{pmatrix}$$

(8) Compute the ranking order of all alternatives and obtain the best alternative
Applying Eq.(33),

$$\overline{T} = \begin{pmatrix} 0.5359 \\ 0.3123 \\ 0.88625 \end{pmatrix}$$

The optimal ranking order of alternatives is given by $A_3 > A_1 > A_2$. The best alternative is A_3. The ranking order given by [27] is identical. The best air-condition system is A_3.

This example shows the effectiveness of the ranking method proposed in this paper.

7 Conclusion

Regarding the MAGDM problem, the IVIF theory provides a useful and convenient way to reflect the ambiguous nature of subjective judgments and assessments. In this paper, firstly, using the normalized Hamming distance between IVIFS to construct similarity matrix and obtain the wights of DMs. Then, using possibility degree of IVIF to calculate the ranking vector. Based on this, the concordance and discordance sets, concordance and discordance matrices etc. are obtained. Finally, by computing the ranking order of all alternatives, decision makers can choose the best alternative, the example verify the correctness of the method.

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The interior and closure of fuzzy topologies induced by the generalized fuzzy approximation spaces

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Abstract

With respect to the Alexandrov fuzzy topologies induced by the generalized fuzzy approximation spaces, Wang defined interior of fuzzy set. In this paper, we give the closure of fuzzy set and discuss some properties of the interior and closure.

Keywords: Alexandrov fuzzy topology; the generalized fuzzy approximation spaces; interior; closure; properties

1 Introduction

In his classical paper [36], Zadeh introduced the notation of fuzzy sets and fuzzy set operation. Subsequently, Chang [2] applied some basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. Pu etc.[18] defined a fuzzy point which took a crisp singleton, equivalently, an ordinary point, as a special case and gave the concepts of interior and closure operator w.r.t. fuzzy topology. Later, Lai and Zhang [11] modified the second axiom in Chang's definition of fuzzy topology to define an Alexandrov fuzzy topology.

The concept of Rough sets were introduced by Z. Pawlak [19] in 1982 as an powerful mathematical tool for uncertain data while modeling the problems in many fields [17,20,27]. Because the rough sets defined with equivalence relations limited the application of it. Thus many authors changed the equivalence relations into different binary relations to expand the application of it [23,35,37,38]. In recent years, the rough sets has been combined with some mathematical theories such as algebra and topology [1,5,6,8,10, 14, 16, 21, 25, 26, 28, 29]. With respect to different binary relations, the topological properties of rough sets were further investigated in [7,14,33,34].

In 1990, Dubois and Prade [3] combining fuzzy sets and rough sets proposed rough fuzzy sets and fuzzy rough sets. Afterward Morsi and Yakout [15] investigated fuzzy rough sets defined with left-continuous t-norms and R-implicators with respect to fuzzy similarity relations. Radzikowska and Kerre [24] defined a broad family of fuzzy rough sets based on t-norms and fuzzy implicators, which are called generalized fuzzy rough sets here. In recent years, the topological properties of fuzzy rough sets were further studied in many literatures [4,9,12,13,22]. Recently, with respect to the lower fuzzy rough approximation operator determined by a fuzzy implicator, Wang [30] studied various fuzzy topologies induced by different fuzzy relations and proved that I-lower fuzzy rough approximation operators were the interior operator w.r.t. some Alexandrov fuzzy topology.

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In this paper, we give closure operator w.r.t. some Alexandrov fuzzy topology given by Wang in [30]. Combined with the definition of Wang's interior, discuss some properties of the interior and closure of fuzzy set.

2 Preliminary

Definition 2.1.[36] A fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, A(x)) : x \in X\}$$

where $A(x) : X \to [0, 1]$ is a mapping and $A(x)$ states the grade of belongness of x in A. The family of all fuzzy sets in X is denoted by $\mathscr{F}(X)$.

Let $\alpha \in [0, 1]$, then a fuzzy set $A \in \mathscr{F}(X)$ is a constant, while $A(x) = \alpha$ for all $x \in X$, denoted as α_X.

Deflnition 2.2.[36] Let A, B be two fuzzy sets of $\mathscr{F}(X)$
(1) A is contained in B if and only if $A(x) \leq B(x)$ for every $x \in X$.
(2) The union of A and B is a fuzzy set C, denoted by $A \cup B = C$, whose membership function $C(x) = A(x) \vee B(x)$ for every $x \in X$.
(3) The intersection of A and B is a fuzzy set C, denoted by $A \cap B = C$, whose membership function $C(x) = A(x) \wedge B(x)$ for every $x \in X$.
(4) The complement of A is a fuzzy set, denoted by A^c, whose membership function $A^c(x) = 1 - A(x)$ for every $x \in X$.

A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ (resp. $S : [0, 1] \times [0, 1] \to [0, 1]$) is called a t-norm (resp. t-conorm) on [0, 1] if it is commutative, associative, increasing in each argument and has a unit element 1 (resp. 0).

A mapping $I : [0, 1] \times [0, 1] \to [0, 1]$ is called a fuzzy implicator on [0, 1] if it satisfies the boundary conditions according to the Boolean implicator, and is decreasing in the first argument and increasing in the second argument.

Definition 2.3.[30] A fuzzy implicator I is said to satisfy
(1) the left neutrality property ((NP), for short), if $I(1, b) = b$ for all $b \in [0, 1]$;
(2) the confinement principle ((CP), for short), if $I(a, b) = 1 \Leftrightarrow a \leq b$, for all $a, b \in [0, 1]$;
(3) the regular property ((RP), for short), if N_I is an involutive negation, where N_I is defined as $N_I(a) = I(a, 0)$ for all $a \in [0, 1]$.

Definition 2.4. [11] A subset $\tau \subseteq \mathscr{F}(X)$ is called an Alexandrov fuzzy topology if it satisfies:
(1) $\alpha_X \in \tau$ for all $\alpha \in [0, 1]$,
(2) $\cap_{i\in\Lambda} A_i \in \tau$ for all $\{A_i\}_{i\in\Lambda} \subseteq \tau$,
(3) $\cup_{i\in\Lambda} A_i \in \tau$ for all $\{A_i\}_{i\in\Lambda} \subseteq \tau$.

Every member of τ is called a τ-open fuzzy set. A fuzzy set is τ-closed if and only if its complement is τ-open. In the sequel, when no confusion is likely to arise, we shall call a τ-open (τ-closed) fuzzy set simply an open (closed) set.

Definition 2.5. [18,31]. Let $\tau \subseteq \mathscr{F}(X)$ be a fuzzy topology. Then the interior of $A \in \mathscr{F}(X)$ w.r.t. fuzzy topology τ denoted as A^o is defined as follows:

$$A^o = \cup\{B \in \tau | B \subseteq A\}.$$

The operator A^o is called an interior operator w.r.t. fuzzy topology τ.

According to definition of the fuzzy topology, obviously A^o is an open set.

Definition 2.6. [18]. Let $\tau \subseteq \mathscr{F}(X)$ be a fuzzy topology. Then the closure of $A \in \mathscr{F}(X)$ w.r.t. fuzzy topology τ denoted as $\overline{A}$ is defined as follows:

$$\overline{A} = \cap\{B|B \supseteq A, B^c \in \tau\}$$

The operator $\overline{A}$ is called a closure operator w.r.t. fuzzy topology τ.

According to De Morgan's Law and definition of the fuzzy topology, $\overline{A}$ is a closed set.

3 Fuzzy topologies induced by the generalized fuzzy approximation spaces

A fuzzy set $R \in \mathscr{F}(X \times Y)$ is called a fuzzy relation from X to Y. If $X = Y$, then R is a fuzzy relation on X. For every fuzzy relation R on X, a fuzzy relation R^{-1} is defined as $R^{-1}(x, y) = R(y, x)$ for all $x, y \in X$. Let R be a fuzzy relation from X to Y . Then the triple (X, Y, R) is called a fuzzy approximation space. When $X = Y$ and R is a fuzzy relation on X, we also call (X, R) a fuzzy approximation space.

Definition 3.1.[30]. Let R be a fuzzy relation on X. Then R is said to be
(1) reflexive if $R(x, x) = 1$ for all $x \in X$;
(2) symmetric if $R(x, y) = R(y, x)$ for all $x, y \in X$;
(3) T-transitive if $T(R(x, y), R(y, z)) \leq R(x, z)$ for all $x, y, z \in X$.

If $T = \wedge$, then T-transitive is said to be transitive for short. A fuzzy relation R is called a fuzzy tolerance if it is reflexive and symmetric, and a fuzzy T-preorder if it is reflexive and T-transitive. Similarly, a fuzzy relation R is called a fuzzy preorder if it is reflexive and transitive.

Definition 3.2.[24,30,32]. Let (X, Y, R) be a fuzzy approximation space. Then the following mappings $\underline{R}, \overline{R} : \mathscr{F}(Y) \to \mathscr{F}(X)$ are defined as follows: for all $A \in \mathscr{F}(Y)$ and $x \in X$,

$$\underline{R}(A)(x) = \bigwedge_{y\in Y} I(R(x, y), A(y)) \text{ and } \overline{R}(A)(x) = \bigvee_{y\in Y} T(R(x, y), A(y)).$$

The mappings $\underline{R}$ and $\overline{R}$ are called $I-lower$ and $T-upper$ fuzzy rough approximation operators, respectively. The pair $(\underline{R}(A), \overline{R}(A))$ is called a generalized fuzzy rough set of A w.r.t. (X, Y, R). Also known as generalized fuzzy approximation spaces.

Let R be a fuzzy relation on X. Then a fuzzy set $A \in \mathscr{F}(X)$ is said to be
(1) I-lower definable w.r.t. fuzzy relation R if $\underline{R}(A) = A$; the family of all $I - lower$ definable fuzzy sets w.r.t. R is denoted as $\mathscr{D}_I(R)$.
(2) T-upper definable w.r.t. fuzzy relation R if $\overline{R}(A) = A$; the family of all $T - upper$ definable fuzzy sets w.r.t. R is denoted as $\mathscr{D}_T(R)$.

Proposition 3.3.[30]. Let (X, R) be a fuzzy approximation space and R be reflexive. Then
(1)$\mathscr{D}_I(R)$ is an Alexandrov fuzzy topology, if I satisfies (NP).
(2)$\mathscr{D}_T(R)$ is an Alexandrov fuzzy topology.

Let (X, R) be a fuzzy approximation space. In [30] Wang defined

$$\mathscr{R}_I(R) = \{\underline{R}(A)|A \in \mathscr{F}(X)\} \text{ and } \mathscr{R}_T(R) = \{\overline{R}(A)|A \in \mathscr{F}(X)\}.$$

To discuss the properties of generalized fuzzy rough sets, Radzikowska and Kerre [19] introduced the following auxiliary conditions: for a fuzzy implicator I and a t-norm T,
(C1) $I(a, I(b, c)) = I(T(a, b), c)$ for all $a, b, c \in [0, 1]$,
(C2) $I(a, I(b, c)) \geq I(T(a, b), c)$ for all $a, b, c \in [0, 1]$,
(C3) $I(a, I(b, c)) \leq I(T(a, b), c)$ for all $a, b, c \in [0, 1]$.
If (C1) (resp. (C2), (C3)) holds for I and T , then we say that I satisfies (C1) (resp. (C2), (C3)) for T.

Proposition 3.4.[30]. Let (X, R) be a fuzzy approximation space and R be a fuzzy T-preorder. Then
(1) $\mathscr{R}_I(R)$ is an Alexandrov fuzzy topology and $\mathscr{R}_I(R) = \mathscr{D}_I(R)$, if I satisfies (NP) and (C2) for T.
(2) $\mathscr{R}_T(R)$ is an Alexandrov fuzzy topology and $\mathscr{R}_T(R) = \mathscr{D}_T(R)$.
The above $\mathscr{D}_I(R)$, $\mathscr{D}_T(R)$, $\mathscr{R}_I(R)$ and $\mathscr{R}_T(R)$ are called fuzzy topologies induced by the generalized fuzzy approximation spaces.

4 The interior and closure of fuzzy set

Proposition 4.1.[30]. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then $\underline{R}$ is the interior operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.

Proposition 4.2. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then A is an open set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ iff $\underline{R}(A) = A^o = A$.

Proof. Suppose A is an open set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$, again $A \subseteq A$, due to definition of A^o, $A \subseteq A^o$. On the other hand,

$$\forall x \in X, \ \ \underline{R}(A)(x) = \underset{y \in X}{\wedge} I(R(x, y), A(y)) \leq I(R(x, x), A(x)) = I(1, A(x)) = A(x).$$

This means $\underline{R}(A) = A^o \subseteq A$. Thus $\underline{R}(A) = A^o = A$.
Conversely, suppose $\underline{R}(A) = A^o = A$, A^o is an open set, thus A is an open set.

Proposition 4.3. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then for any $A \in \mathscr{F}_{(X)}$, $[\underline{R}(A^c)]^c$ is the closure operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
Proof. For any $A \in \mathscr{F}_{(X)}$, since $\underline{R}(A^c)$ is an open set, thus $(\underline{R}(A^c))^c$ is a closed set. Again

$$\forall x \in X, \ \ \underline{R}(A^c)(x) = \underset{y \in X}{\wedge} I(R(x, y), A^c(y)) \leq I(R(x, x), A^c(x)) = I(1, A^c(x)) = A^c(x),$$

this means $(\underline{R}(A^c))^c \supseteq A$.
On the other hand, for any $A \subseteq B \in \mathscr{F}_{(X)}$ and $B^c \in \mathscr{D}_I(R)$. By Proposition 4.2, $\underline{R}(B^c) = B^c$, and

$$\forall x \in X, \ \ \underline{R}(A^c)(x) = \underset{y \in X}{\wedge} I(R(x, y), A^c(y)) \geq \underset{y \in X}{\wedge} I(R(x, x), B^c(x)) = \underline{R}(B^c)(x).$$

We obtain $\underline{R}(A^c) \supseteq \underline{R}(B^c) = B^c$. This means $(\underline{R}(A^c))^c \subseteq B$. By Definition of the closure, for any $A \in \mathscr{F}_{(X)}$, $[\underline{R}(A^c)]^c$ is the closure operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ i.e. $[\underline{R}(A^c)]^c = \overline{A}$.

Proposition 4.4. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then A is a closed set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ iff $(\underline{R}(A^c))^c = \overline{A} = A$.

Proof. Suppose A is a closed set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$, then A^c is an open set. Therefore $\underline{R}(A^c) = A^c$, and then $\overline{A} = (\underline{R}(A^c))^c = A$.

4

Conversely, suppose $\overline{A} = (\underline{R}(A^c))^c = A$, $\overline{A}$ is a closed set, thus A is a closed set.

Proposition 4.5. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A, B \in \mathscr{F}_{(X)}$ the following formula hold w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $A \subseteq \overline{A}$;
(2) $\overline{\overline{A}} = \overline{A}$;
(3) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$;
(4) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proof. (1) For all $x \in X$,

$$\overline{A}(x) = (\underline{R}(A^c))^c(x) = 1 - \underline{R}(A^c)(x) = 1 - \bigwedge_{y\in X} I(R(x,y), A^c(y))$$
$$\geq 1 - I(R(x,x), A^c(x)) = 1 - I(1, A^c(x)) = 1 - A^c(x) = A(x),$$

thus $A \subseteq \overline{A}$.
(2) Since $\overline{A}$ is a closed set, By Proposition 4.4, $\overline{\overline{A}} = \overline{A}$.
(3) By $A \subseteq B$, we obtain $A^c \supseteq B^c$. According to Definition 3.2, obviously $\underline{R}(A^c) \supseteq \underline{R}(B^c)$, and then $\overline{A} = (\underline{R}(A^c))^c \subseteq (\underline{R}(B^c))^c = \overline{B}$.
(4) Since $A \subseteq A \cup B$, $B \subseteq A \cup B$, by (2)
$\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

On the other hand, by (1) $A \subseteq \overline{A}$, $B \subseteq \overline{B}$. Thus $A \cup B \subseteq \overline{A} \cup \overline{B}$. And then $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. Again $\overline{A} \cup \overline{B}$ is a closed set, according to Proposition 4.4 $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Thereby $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proposition 4.6. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A \in \mathscr{F}_{(X)}$, the following formula hold w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $\overline{A} = [(A^c)^o]^c$; (2) $A^o = [\overline{A^c}]^c$;
(3) $[\overline{A}]^c = [A^c]^o$; (4) $\overline{A^c} = [A^o]^c$.
Proof. (1) By Proposition 4.2, $(A^c)^o = \underline{R}(A^c)$, thus $[(A^c)^o]^c = [\underline{R}(A^c)]^c = \overline{A}$.

(2),(3),(4) can be proven in a similar way as for item (1).

Proposition 4.7. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A, B \in \mathscr{F}_{(X)}$ and $A \subseteq B$, the following holds w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $A^o \subseteq B^o$; (2) $A^{oo} = A^o$; (3) $(A \cap B)^o = A^o \cap B^o$.

Proof. (1) $\forall x \in X$, $\underline{R}(A)(x) = \bigwedge_{y\in X} I(R(x,y), A(y)) \leq \bigwedge_{y\in X} I(R(x,y), B(y)) = \underline{R}(B)(x)$. Thus $A^o \subseteq B^o$.
(2) Since A^o is a open set, by Proposition 4.2, $A^{oo} = A^o$.
(3) By Proposition 4.6 (2) and Proposition 4.5 (4),
$(A \cap B)^o = (\overline{(A \cap B)^c})^c = (\overline{A^c \cup B^c})^c = (\overline{A^c} \cup \overline{B^c})^c = (\overline{A^c})^c \cap (\overline{B^c})^c = A^o \cap B^o$.

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Weighted Lim's Geometric Mean of Positive Invertible Operators on a Hilbert Space

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Abstract

We generalize the weighted Lim's geometric mean of positive definite matrices to positive invertible operators on a Hilbert space. This mean is defined via a certain bijection map and parametrized over Hermitian unitary operators. We derive an explicit formula of the weighted Lim's geometric mean in terms of weighted metric/spectral geometric means. This kind of operator mean turns out to be a symmetric Lim-Pálfia weighted mean and satisfies the idempotency, the permutation invariance, the joint homogeneity, the self-duality, and the unitary invariance. Moreover, we obtain relations between weighted Lim geometric means and Tracy-Singh products via operator identities.

Keywords: positive invertible operator, metric geometric mean, spectral geometric mean, Lim's geometric mean, Tracy-Singh product
Mathematics Subject Classifications 2010: 47A64, 47A80.

1 Introduction

Recall that the Riccati equation for positive definite matrices A and B:

$$XA^{-1}X = B \tag{1}$$

has a unique positive solution

$$X = A\sharp B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}, \tag{2}$$

known as the metric geometric mean of A and B. This kind of mean was introduced by Ando [2] as the maximum (with respect to the Löwner partial

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order) of positive semidefinite matrices X satisfying

$$\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geqslant 0.$$

The above two definitions of the metric geometric mean are equivalent. See a nice discussion about the Riccati equation and the metric geometric mean of matrices in [4].

Fiedler and Pták [3] modified the notion of the metric geometric mean to the spectral geometric mean:

$$A \diamondsuit B \; = \; (A^{-1} \sharp B)^{\frac{1}{2}} A (A^{-1} \sharp B)^{\frac{1}{2}}. \tag{3}$$

One of the most important properties of the spectral geometric mean is the positive similarity between $(A\diamondsuit B)^2$ and AB. This shows that the eigenvalues of $A\diamondsuit B$ coincide with the positive square roots of the eigenvalues of AB.

Lee and Lim [5] introduced the notion of metric geometric means and spectral geometric means on symmetric cones of positive definite matrices and developed various properties of these means. Lim [6] provided a new geometric mean of positive definite matrices varying over Hermitian unitary matrices, including the metric geometric mean as a special case. The new mean has an explicit formula in terms of metric and spectral geometric means. He established basic properties of this mean including the idempotency, joint homogeneity, permutation invariance, non-expansiveness, self-duality, and a determinantal identity. He also gave this new geometric mean for the weighted case. Lim and Pálfia [7] presented a unified framework for weighted inductive means on the cone of positive definite matrices. The metric geometric mean, spectral geometric mean, and Lim geometric mean [6] are basic examples of the two-variable weighted mean.

In this paper, we extend the notion of weighted Lim's geometric mean [6] to the case of Hilbert-space operators via a certain bijection map (see Section 2). This operator mean is parametrized over Hermitian unitary operators. An explicit formula of the weighted Lim's geometric mean is given in term of weighted metric geometric means and spectral geometric means. This kind of operator mean serves the idempotency, the permutation invariance, the joint homogeneity, the self-duality, and the unitary invariance. Moreover, we establish certain operator identities involving Lim weighted geometric means and Tracy-Singh products (see Section 3). Our results include certain literature results involving weighted metric geometric means.

2 Lim's geometric mean of operators

In this section, we discuss the notion of Lim's geometric mean of positive invertible operators on any complex Hilbert space.

Throughout, let $\mathbb{H}$ be a complex Hilbert space. Denoted by $\mathfrak{B}(\mathbb{H})$ the Banach space of bounded linear operators on $\mathbb{H}$. The set of all positive invertible operators on $\mathbb{H}$ is denoted by $\mathbb{P}$.

First of all, we recall the notions of metric/spectral geometric means of operators. Recall that for any $t \in [0,1]$, the t-weighted metric geometric mean of $A, B \in \mathbb{P}$ is defined by

$$A \sharp_t B \;=\; A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}} \tag{4}$$

For briefly, we write $A \sharp B$ for $A \sharp_{1/2} B$. The spectral geometric mean of $A, B \in \mathbb{P}$ is defined by

$$A \diamondsuit B \;=\; (A^{-1} \sharp B)^{\frac{1}{2}} A (A^{-1} \sharp B)^{\frac{1}{2}}. \tag{5}$$

We list some basic properties of metric and spectral geometric means.

Lemma 1 (e.g. [1, 3, 4])**.** *Let $A, B \in \mathbb{P}$ and $t \in [0,1]$. Then*

(i) $A \sharp_t A = A$,

(ii) $(\alpha A) \sharp_t (\beta B) = \alpha^{1-t} \beta^t (A \sharp_t B)$,

(iii) $A \sharp_t B = B \sharp_{1-t} A$,

(iv) $(A \sharp_t B)^{-1} = A^{-1} \sharp_t B^{-1}$,

(v) *(Riccati Lemma) $A \sharp B$ is the unique positive invertible solution of $X A^{-1} X = B$,*

(vi) $(T^* A T) \sharp_t (T^* B T) = T^* (A \sharp_t B) T$ *for any invertible operator* $T \in \mathfrak{B}(\mathbb{H})$,

(vii) $(T^* A T) \diamondsuit (T^* B T) = T^* (A \diamondsuit B) T$ *for any unitary operator* $T \in \mathfrak{B}(\mathbb{H})$.

For a Hermitian unitary operator $U \in \mathfrak{B}(\mathbb{H})$, we set

$$\mathbb{P}_U^+ := \{X \in \mathbb{P} : UXU = X\}, \quad \mathbb{P}_U^- := \{X \in \mathbb{P} : UXU = X^{-1}\}$$

Lemma 2. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator. Then the map*

$$\Phi_U \;:\; \mathbb{P}_U^+ \times \mathbb{P}_U^- \;\to\; \mathbb{P}, \quad (A, B) \;\mapsto\; A^{\frac{1}{2}} B A^{\frac{1}{2}} \tag{6}$$

is bijective with the inverse map given by

$$X \;\mapsto\; (X \sharp (UXU), X \diamondsuit (UX^{-1}U)). \tag{7}$$

Proof. The proof is quite similar to [6, Theorem 2.6]. Let $A_1, A_2 \in \mathbb{P}_U^+$ and $B_1, B_2 \in \mathbb{P}_U^-$ such that $\Phi_U(A_1, B_1) = \Phi_U(A_2, B_2)$, i.e. $A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}} = A_2^{\frac{1}{2}} B_2 A_2^{\frac{1}{2}}$. Since $A_i \in \mathbb{P}_U^+$, we have

$$\begin{aligned} U A_i^{-1} U \;&=\; (U A_i U)^{-1} \;=\; A_i^{-1}, \\ U A_i^{\frac{1}{2}} U \;&=\; (U A_i U)^{\frac{1}{2}} \;=\; A_i^{\frac{1}{2}}. \end{aligned}$$

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and thus $A_i^{-1}, A_i^{\frac{1}{2}} \in \mathbb{P}_U^+$ for $i = 1, 2$. It follows that

$$\begin{aligned} B_1^{-1} &= UB_1U \\ &= U\Big(A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}B_2A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}}\Big)U \\ &= \big(UA_1^{-\frac{1}{2}}U\big)\big(UA_2^{\frac{1}{2}}U\big)\big(UB_2U\big)\big(UA_2^{\frac{1}{2}}U\big)\big(UA_1^{-\frac{1}{2}}U\big) \\ &= A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}B_2^{-1}A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}} \\ &= A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}\Big(A_2^{-\frac{1}{2}}A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}}A_2^{-\frac{1}{2}}\Big)^{-1}A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}} \\ &= \big(A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}\big)B_1^{-1}\big(A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}\big), \end{aligned}$$

i.e. $A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}$ is a solution of $XB_1^{-1}X = B_1^{-1}$. Since $B_1^{-1}\sharp B_1 = I$ is the unique solution of $XB_1^{-1}X = B_1^{-1}$ (Lemma 1 (v)), we conclude $A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}} = I$. This implies that $A_1 = A_2$ and then $B_1 = B_2$. Hence, Φ_U is injective. Next, let $X \in \mathbb{P}$. and consider $A = X\sharp(UXU)$ and $B = X\diamondsuit(UX^{-1}U) = A^{-\frac{1}{2}}XA^{-\frac{1}{2}}$. Consider

$$\begin{aligned} UAU &= U\big(X\sharp(UXU)\big)U \;=\; (UXU)\sharp(U^2XU^2) \\ &= (UXU)\sharp X \;=\; X\sharp(UXU) \;=\; A \end{aligned}$$

and

$$\begin{aligned} UBU &= U\big(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\big)U \;=\; \big(UA^{-\frac{1}{2}}U\big)(UXU)\big(UA^{-\frac{1}{2}}U\big) \\ &= A^{\frac{1}{2}}X^{-1}A^{\frac{1}{2}} \;=\; B^{-1}. \end{aligned}$$

This implies that $A \in \mathbb{P}_U^+$ and $B \in \mathbb{P}_U^-$. We have that there exist $A \in \mathbb{P}_U^+$ and $B \in \mathbb{P}_U^-$ such that $\Phi_U(A, B) = A^{\frac{1}{2}}BA^{\frac{1}{2}} = X$. Thus, Φ_U is surjective. Therefore Φ_U is bijective. □

By the bijectivity of Φ_U, we can define the t-weighted Lim geometric mean of operators as follows:

Definition 3. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0, 1]$. Let $X = \Phi_U(A_1, B_1), Y = \Phi_U(A_2, B_2) \in \mathbb{P}$. The t-weighted Lim geometric mean of X and Y is defined by*

$$\mathcal{G}_U(t; X, Y) \;=\; \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2). \tag{8}$$

We denote $\mathcal{G}_U(X, Y) = \mathcal{G}_U(1/2; X, Y)$ the Lim geometric mean.

The next theorem gives an explicit formula of $\mathcal{G}_U(X, Y)$.

Theorem 4. *Let U be a Hermitian unitary operator and $t \in [0, 1]$. Let $X, Y \in \mathbb{P}$. We have*

$$\mathcal{G}_U(t; X, Y) \;=\; (A_1\sharp_t A_2)^{\frac{1}{2}}(B_1\sharp_t B_2)(A_1\sharp_t A_2)^{\frac{1}{2}}, \tag{9}$$

where $A_1 = X\sharp(UXU), A_2 = Y\sharp(UYU), B_1 = X\diamondsuit(UX^{-1}U)$ and $B_2 = Y\diamondsuit(UY^{-1}U)$. In particular, $\mathcal{G}_I(X, Y) = X\sharp_t Y$.

Proof. Since f_U is surjective, there exist $A_1, A_2 \in \mathbb{P}_U^+$ and $B_1, B_2 \in \mathbb{P}_U^-$ such that $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. By using the inverse map (7), we have

$$\begin{aligned}(A_1, B_1) &= \Phi_U^{-1}(X) = (X\sharp(UXU), X\diamondsuit(UX^{-1}U)) \\ (A_2, B_2) &= \Phi_U^{-1}(Y) = (Y\sharp(UYU), Y\diamondsuit(UY^{-1}U)).\end{aligned}$$

For the case $U = I$, we have $\mathbb{P}_I^+ = \mathbb{P}$ and $\mathbb{P}_I^- = \{I\}$. It follows that $B_1 = B_2 = I$. By Lemma 1, we have $A_1 = X\sharp X = X$ and $A_2 = Y\sharp Y = Y$. Hence,

$$\mathcal{G}_I(t; X, Y) = (X\sharp_t Y)^{\frac{1}{2}}(I\sharp_t I)(X\sharp_t Y)^{\frac{1}{2}} = X\sharp_t Y. \qquad \square$$

Now, we give the definition of the Lim-Pálfia weighted mean [7] in the case of operators.

Definition 5. *The (two-variable) Lim-Pálfia weighted mean of positive invertible operators is the map* $\mathbb{M} : [0, 1] \times \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ *satisfying*

(i) $\mathbb{M}(0, X, Y) = X$,

(ii) $\mathbb{M}(1, X, Y) = Y$,

(iii) (Idempotency) $\mathbb{M}(t, X, X) = X$ *for all* $t \in [0, 1]$.

We say that $\mathbb{M}$ *is symmetric if*

(iv) (Permutation invariancy) $\mathbb{M}(t, X, Y) = \mathbb{M}(1 - t, Y, X)$ *for all* $t \in [0, 1]$.

Theorem 6. *The t-weighted Lim geometric mean of operators is a symmetric Lim-Pálfia weighted mean.*

Proof. Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0, 1]$. Let $X, Y \in \mathbb{P}$. Write $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. We have by Lemma 1 that

$$\begin{aligned}\mathcal{G}_U(0; X, Y) &= \Phi_U(A_1\sharp_0 A_2, B_1\sharp_0 B_2) = \Phi_U(A_1, B_1) = X, \\ \mathcal{G}_U(1; X, Y) &= \Phi_U(A_1\sharp_1 A_2, B_1\sharp_1 B_2) = \Phi_U(A_2, B_2) = Y, \\ \mathcal{G}_U(t; X, X) &= \Phi_U(A_1\sharp_t A_1, B_1\sharp_t B_1) = \Phi_U(A_1, B_1) = X.\end{aligned}$$

This implies that $\mathcal{G}_U$ is a Lim-Pálfia weighted mean. Using Lemma 1 again, we get

$$\begin{aligned}\mathcal{G}_U(t; X, Y) &= \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2) \\ &= \Phi_U(A_2\sharp_{1-t} A_1, B_2\sharp_{1-t} B_1) \\ &= \mathcal{G}_U(1 - t; Y, X).\end{aligned}$$

Thus, $\mathcal{G}_U$ is symmetric. $\square$

Corollary 7. *The t-weighted metric geometric mean of operators is a symmetric Lim-Pálfia weighted mean.*

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Theorem 8. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0,1]$. Let $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. We have*

(i) $\mathcal{G}_U(t; X, I) = \Phi_U(A_1^{1-t}, B_1^{1-t})$ *and* $\mathcal{G}_U(t; I, Y) = \Phi_U(A_2^t, B_2^t)$,

(ii) (Joint Homogeneity) $\mathcal{G}_U(t; \alpha X, \beta Y) = \alpha^{1-t}\beta^t \mathcal{G}_U(t; X, Y)$ *for any* $\alpha, \beta > 0$,

(iii) (Self-duality) $\mathcal{G}_U(t; X, Y)^{-1} = \mathcal{G}_U(t; X^{-1}, Y^{-1})$,

(iv) (Unitary invariance) $\mathcal{G}_U(t; T^*XT, T^*YT) = T^*\mathcal{G}_U(t; X, Y)T$ *where* $T \in \mathfrak{B}(\mathbb{H})$ *is a unitary operator such that* $UT = TU$,

(v) $\mathcal{G}_U(t; UXU, UYU) = U\mathcal{G}_U(t; X, Y)U$,

(vi) $\mathcal{G}_U(X, X^{-1}) = I$.

Proof. The first assertion is immediate from Definition 3. For the joint homogeneity, note that

$$\alpha X \;=\; \alpha \Phi_U(A_1, B_1) \;=\; \alpha\big(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\big) \;=\; A_1^{\frac{1}{2}}(\alpha B_1)A_1^{\frac{1}{2}} \;=\; \Phi_U(A_1, \alpha B_1).$$

Similarly, $\beta Y = \Phi_U(A_2, \beta B_2)$. Using Lemma 1, we obtain

$$\begin{aligned}\mathcal{G}_U(t; \alpha X, \beta Y) &= \Phi_U(A_1 \sharp_t A_2, (\alpha B_1)\sharp_t(\beta B_2)) \;=\; \Phi_U(A_1 \sharp_t A_2, \alpha^{1-t}\beta^t(B_1 \sharp_t B_2)) \\ &= \alpha^{1-t}\beta^t \Phi_U(A_1 \sharp_t A_2, B_1 \sharp_t B_2) \;=\; \alpha^{1-t}\beta^t \mathcal{G}_U(t; X, Y).\end{aligned}$$

For the self-duality, consider

$$X^{-1} = \Phi_U(A_1, B_1)^{-1} = \left(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\right)^{-1} = A_1^{-\frac{1}{2}} B_1^{-1} A_1^{-\frac{1}{2}} = \Phi_U(A_1^{-1}, B_1^{-1}).$$

Similarly, $Y^{-1} = \Phi_U(A_2^{-1}, B_2^{-1})$. Applying Lemma 1, we get

$$\begin{aligned}\mathcal{G}_U(t; X, Y)^{-1} &= \Phi_U(A_1 \sharp_t A_2, B_1 \sharp_t B_2)^{-1} \;=\; \Phi_U((A_1 \sharp_t A_2)^{-1}, (B_1 \sharp_t B_2)^{-1}) \\ &= \Phi_U(A_1^{-1} \sharp_t A_2^{-1}, B_1^{-1} \sharp_t B_2^{-1}) \;=\; \mathcal{G}_U(t; X^{-1}, Y^{-1})\end{aligned}$$

Now, let us prove the assertion (iv). Since T is unitary, we have by Lemma 1 that

$$\begin{aligned}(T^*XT)\sharp[U(T^*XT)U] &= (T^*XT)\sharp[T^*(UXU)T] \\ &= T^*[X\sharp(UXU)]T \\ &= T^*A_1T, \\ (T^*XT)\diamondsuit[U(T^*XT)^{-1}U] &= (T^*XT)\diamondsuit[UTX^{-1}T^*U] \\ &= (T^*XT)\diamondsuit[T^*(UX^{-1}U)T] \\ &= T^*[X\diamondsuit(UX^{-1}U)]T \\ &= T^*B_1T.\end{aligned}$$

Similarly,

$$(T^*YT)\sharp[U(T^*YT)U] \;=\; T^*A_2T \quad \text{and} \quad (T^*YT)\diamondsuit[U(T^*YT)^{-1}U] \;=\; T^*B_2T$$

Then $T^*XT = \Phi_U(T^*A_1T, T^*B_1T)$ and $T^*YT = \Phi_U(T^*A_2T, T^*B_2T)$. Thus

$$\begin{aligned}
&\mathcal{G}_U(t; T^*XT, T^*YT) \\
&\quad = [(T^*A_1T)\sharp_t(T^*A_2T)]^{\frac{1}{2}}[(T^*B_1T)\sharp_t(T^*B_2T)][(T^*A_1T)\sharp_t(T^*A_2T)]^{\frac{1}{2}} \\
&\quad = [T^*(A_1\sharp_tA_2)T]^{\frac{1}{2}}[T^*(B_1\sharp_tB_2)T][T^*(A_1\sharp_tA_2)T]^{\frac{1}{2}} \\
&\quad = [T^*(A_1\sharp_tA_2)^{\frac{1}{2}}T][T^*(B_1\sharp_tB_2)T][T^*(A_1\sharp_tA_2)^{\frac{1}{2}}T] \\
&\quad = T^*(A_1\sharp_tA_2)^{\frac{1}{2}}(B_1\sharp_tB_2)(A_1\sharp_tA_2)^{\frac{1}{2}}T \\
&\quad = T^*\mathcal{G}_U(t; X, Y)T.
\end{aligned}$$

Setting $T = U$, we get the result in the assertion (v). For the last assertion, since $X^{-1} = \Phi_U(A_1^{-1}, B_1^{-1})$, we have

$$\mathcal{G}_U(X, X^{-1}) \;=\; \Phi_U(A_1\sharp A_1^{-1}, B_1\sharp B_1^{-1}) \;=\; \Phi_U(I, I) \;=\; I. \qquad \square$$

3 Weighted Lim geometric means and Tracy-Singh products

In this section, we investigate relations between Weighted Lim geometric means and Tracy-Singh products of operators. Let us recalling the notion of Tracy-Singh product.

3.1 Preliminaries on the Tracy-Singh product of operators

The projection theorem for Hilbert space allows us to decompose

$$\mathbb{H} \;=\; \bigoplus_{i=1}^{n} \mathbb{H}_i \tag{10}$$

where all $\mathbb{H}_i$ are Hilbert spaces. For each $i = 1, \dots, n$, let P_i be the natural projection map from $\mathbb{H}$ onto $\mathbb{H}_i$. Each operator $A \in \mathfrak{B}(\mathbb{H})$ can be uniquely determined by an operator matrix

$$A \;=\; [A_{ij}]_{i,j=1}^{n,n}$$

where $A_{ij} : \mathbb{H}_j \to \mathbb{H}_i$ is defined by $A_{ij} = P_iAP_j^*$ for each $i, j = 1 \dots, n$.

Recall that the tensor product of $A, B \in \mathfrak{B}(\mathbb{H})$ is the operator $A \otimes B \in \mathfrak{B}(\mathbb{H} \otimes \mathbb{H})$ such that for all $x, y \in \mathbb{H}$,

$$(A \otimes B)(x \otimes y) \;=\; (Ax) \otimes (By).$$

Definition 9. *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ and $B = [B_{ij}]_{i,j=1}^{n,n}$ be operators in $\mathfrak{B}(\mathbb{H})$. The Tracy-Singh product of A and B is defined to be*

$$A \boxtimes B \;=\; \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij} \tag{11}$$

which is a bounded linear operator from $\bigoplus_{i,j=1}^{n,n} \mathbb{H}_i \otimes \mathbb{H}_j$ into itself.

Lemma 10 ([9, 10, 11])**.** *Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})$.*

(i) $(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.

(ii) If $A, B \in \mathbb{P}$, then $A \boxtimes B \in \mathbb{P}$.

(iii) If $A, B \in \mathbb{P}$, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any $\alpha \in \mathbb{R}$.

(iv) If A and B are Hermitian, then $A \boxtimes B$ is also.

(v) If A and B are unitary, then $A \boxtimes B$ is also.

Lemma 11 ([8])**.** *Let $A, B, C, D \in \mathbb{P}$ and $t \in [0, 1]$. Then*

$$(A \boxtimes B)\sharp_t(C \boxtimes D) \;=\; (A\sharp_t C) \boxtimes (B\sharp_t D).$$

For each $i = 1, \ldots, k$, let $\mathbb{H}_i$ be a Hilbert space and decompose

$$\mathbb{H}_i = \bigoplus_{r=1}^{n_i} \mathbb{H}_{i,r}$$

where all $\mathbb{H}_{i,r}$ are Hilbert spaces. We set $\boxtimes_{i=1}^{1} A_i = A_1$. For $k \in \mathbb{N} - \{1\}$ and $A_i \in \mathfrak{B}(\mathbb{H}_i)$ $(i = 1, \ldots, k)$, we use the notation

$$\mathop{\boxtimes}_{i=1}^{k} A_i \;=\; ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{k-1}) \boxtimes A_k.$$

3.2 The compatibility between weighted Lim geometric means and Tracy-Singh products

The following theorem provides an operator identity involving t-weighted Lim geometric means and Tracy-Singh products.

Theorem 12. *Let U, V be Hermitian unitary operators, $X_1, X_2, Y_1, Y_2 \in \mathbb{P}$ and $t \in [0, 1]$.*

$$\mathcal{G}_U(t; X_1, Y_1) \boxtimes \mathcal{G}_V(t; X_2, Y_2) \;=\; \mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2). \tag{12}$$

Proof. Write

$$X_1 = \Phi_U(A_1, B_1), \quad Y_1 = \Phi_U(C_1, D_1), \quad X_2 = \Phi_V(A_2, B_2), \quad Y_2 = \Phi_V(C_2, D_2),$$

where $A_1, C_1 \in \mathbb{P}_U^+,\ B_1, D_1 \in \mathbb{P}_U^-,\ A_2, C_2 \in \mathbb{P}_V^+,\ B_2, D_2 \in \mathbb{P}_V^-$. Since U and V are Hermitian unitary operators, we have by Lemma 10 that $U \boxtimes V$ is also a Hermitian unitary operator. Thus $\mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2)$ is well-defined. By Lemma 10, we get

$$(U \boxtimes V)(A_1 \boxtimes A_2)(U \boxtimes V) \;=\; (UA_1U) \boxtimes (VA_2V) \;=\; A_1 \boxtimes A_2$$

and

$$\begin{aligned}(U \boxtimes V)(B_1 \boxtimes B_2)(U \boxtimes V) \;&=\; (UB_1U) \boxtimes (VB_2V) \;=\; B_1^{-1} \boxtimes B_2^{-1} \\ &=\; (B_1 \boxtimes B_2)^{-1}.\end{aligned}$$

Thus $A_1 \boxtimes A_2 \in \mathbb{P}_{U\boxtimes V}^+$ and $B_1 \boxtimes B_2 \in \mathbb{P}_{U\boxtimes V}^-$. Similarly, we have $C_1 \boxtimes C_2 \in \mathbb{P}_{U\boxtimes V}^+$ and $D_1 \boxtimes D_2 \in \mathbb{P}_{U\boxtimes V}^-$. Using Lemma 10, we get

$$\begin{aligned}X_1 \boxtimes X_2 \;&=\; \Phi_U(A_1, B_1) \boxtimes \Phi_V(A_2, B_2) \\ &=\; \big(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\big) \boxtimes \big(A_2^{\frac{1}{2}} B_2 A_2^{\frac{1}{2}}\big) \\ &=\; (A_1^{\frac{1}{2}} \boxtimes A_2^{\frac{1}{2}})(B_1 \boxtimes B-2)(A_1^{\frac{1}{2}} \boxtimes A_2^{\frac{1}{2}}) \\ &=\; (A_1 \boxtimes A_2)^{\frac{1}{2}}(B_1 \boxtimes B_2)(A_1 \boxtimes A_2)^{\frac{1}{2}} \\ &=\; \Phi_{U\boxtimes V}(A_1 \boxtimes A_2, B_1 \boxtimes B_2).\end{aligned}$$

Similarly, $Y_1 \boxtimes Y_2 = \Phi_{U\boxtimes V}(C_1 \boxtimes C_2, D_1 \boxtimes D_2)$. Then

$$\mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \;=\; \Phi_{U\boxtimes V}\big((A_1 \boxtimes A_2)\sharp_t(C_1 \boxtimes C_2), (B_1 \boxtimes B_2)\sharp_t(D_1 \boxtimes D_2)\big).$$

We have by applying Lemmas 10 and 11 that

$$\begin{aligned}&\mathcal{G}_U(t;X_1, Y_1) \boxtimes \mathcal{G}_V(t; X_2, Y_2) \\ &\quad=\; \Phi_U(A_1\sharp_t C_1, B_1\sharp_t D_1) \boxtimes \Phi_V(A_1\sharp_t C_2, B_2\sharp_t D_2) \\ &\quad=\; \big[(A_1\sharp_t C_1)^{\frac{1}{2}}(B_1\sharp_t D_1)(A_1\sharp_t C_1)^{\frac{1}{2}}\big] \boxtimes \big[(A_2\sharp_t C_2)^{\frac{1}{2}}(B_2\sharp_t D_2)(A_2\sharp_t C_2)^{\frac{1}{2}}\big] \\ &\quad=\; \big[(A_1\sharp_t C_1)^{\frac{1}{2}} \boxtimes (A_2\sharp_t C_2)^{\frac{1}{2}}\big]\big[(B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big]\big[(A_1\sharp_t C_1)^{\frac{1}{2}} \boxtimes (A_2\sharp_t C_2)^{\frac{1}{2}}\big] \\ &\quad=\; \big[(A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2)\big]^{\frac{1}{2}}\big[(B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big]\big[(A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2)\big]^{\frac{1}{2}} \\ &\quad=\; \Phi_{U\boxtimes V}\big((A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2), (B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big) \\ &\quad=\; \Phi_{U\boxtimes V}\big((A_1 \boxtimes A_2)\sharp_t(C_1 \boxtimes C_2), (B_1 \boxtimes B_2)\sharp_t(D_1 \boxtimes D_2)\big) \\ &\quad=\; \mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2). \qquad \square\end{aligned}$$

Corollary 13. *Let $k \in \mathbb{N}$ and $t \in [0,1]$. For each $1 \leqslant i \leqslant k$, let $U_i \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $X_i, Y_i \in \mathbb{P}$. Then*

$$\mathop{\boxtimes}_{i=1}^{k} \mathcal{G}_{U_i}(t; X_i, Y_i) \;=\; \mathcal{G}_U\Big(t; \mathop{\boxtimes}_{i=1}^{k} X_i, \mathop{\boxtimes}_{i=1}^{k} Y_i\Big), \tag{13}$$

where $U = \boxtimes_{i=1}^{k} U_i$.

Proof. Since U_i is a Hermitian unitary operator for all $i = 1, \dots, k$, we have by Lemma 10 that $\boxtimes_{i=1}^{k} U_i$ is also. Using the positivity of the Tracy-Singh product, we get $\boxtimes_{i=1}^{k} X_i, \boxtimes_{i=1}^{k} Y_i \in \mathbb{P}$. Hence, the right hand side of (13) is well-defined. We reach the result by applying Thoerem 12 and induction on k. □

From Corollary 13, setting $U_i = I$ for all $i = 1, \dots, k$, we have

$$\boxtimes_{i=1}^{k}(X_i \sharp_t Y_i) \ = \ \left(\boxtimes_{i=1}^{k} X_i\right) \sharp_t \left(\boxtimes_{i=1}^{k} Y_i\right).$$

This equality were proved already in [8, Corollary 1].

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