Reduced Differential Transform Method for Fractional order partial differential Equations

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Received: 19.04.2024 Revised: 24.05.2024 Accepted: 27.05.2024

ABSTRACT
In this paper, we discussed Reduced Differential Transform Method (RDTM) for solving fractional order partial differential equations. In this study, we find analytic approximate solutions of initialvalue problems of one dimensional homogeneous time fractional Cahn-Hilliard equation by reduced differential transform method. The result of same fractional order partial differential equations are calculated in the form of convergent power series with easily computable components. The results show that the proposed technique, without linearization or small perturbation, is very effective and convenient.

Keywords: Reduced Differential Transform Method, Fractional Order Partial Differential Equations, Approximate Solutions

INTRODUCTION
Many applied problems can be described by mathematical models that involve partial differential equations. A mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus the investigation of the exact or approximation solution helps us to understand the means of these mathematical models. Several numerical methods were developed for solving partial differential equations with variable coefficients such us He's Polynomials[1], the homotopy perturbation method[2], homotopy analysis method [3] and the modified variational iteration method [4]. The main goal of this paper is to apply the reduced differential transform method (RDTM)[5-9] to obtain the exact solution fractional Cahn-Hilliard equation [10] Keskin introduced a reduced form of differential transform method (DTM) as reduced differential transform method (RDTM) and applied to approximate some PDEs and fractional PDEs . Abazari and Ganji extended RDTM to study the partial differential equation with proportional delay in t and shrinking in x, and shown that as a special advantage of RDTM rather than DTM. the classical Cahn–Hilliard equation (C-Hequation) introduced by American scientists JW Cahn and J Hilliard is one of the most studied models of mathematicalphysics. The equation is related to a number of physical phenomena like the spinodal decomposition, phase separation and phase ordering dynamics. This equation of mathematical physics describes the process of phase separation by which the two components of a binary fluidare spontaneously separated.

Analysis of the method
The basic definition and theorem for RDTM areintroduced as fallows
If the function u (x,t) is analytic and continuous differentiable function with respective to time tand space x then ,
Let,

$$U_K(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right] \qquad (1)$$

Where the t-dimensional spectrum function $U_k(x)$ is the transformed function u (x, t) is originfunction and the differential inverse transform of U_k (x) is defined as,

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \qquad (2)$$

By combining equations (1) and (2) we write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} t^k \qquad \textbf{(3)}$$

which gives the power series solution

Theorem1] If u (x , t) is original function then it's transformed form is

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=t_0}$$

Theorem2] If Z(x, t) = u(x, t) + v(x ,t) then $Z_k(x) = U(x) + V_k(x)$

Theorem3] If Z(x, t) = p u(x,t) then $Z_k(x) = p\, U_k(x)$, p is constant

Theorem4] If $Z(x,t) = x^m t^n$ then $Z_k(x) = x^m\delta(k-m)$ where δ = 1, if k = 0 and δ = 0 if k≠ 0

Theorem 5] Z(x,t) = u(x,t) . v(x,t) then

$$Z_k(x)=\sum_{l=0}^{k} V_l U_{k-l}(x)=\sum_{l=0}^{k} U_l V_{k-l}(x)$$

Theorem7]If $Z(x,t)=\frac{\partial}{\partial t}u(x,t)$ *then* $Z_k(x)=\frac{\partial}{\partial t}U_k(x)$

Theorem8]If $Z(x,t)=\frac{\partial^r}{\partial t^r}u(x,t)$ *then* $Z_k(x)=\frac{\partial^n}{\partial x^n}u_k(x)$

Theorem 9 *If* $Z(x,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$ *then* $Z_k(x)=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}u_{k+N}(x)$

Reduced Differential Transform Method for homogeneous time fractional Cahn-Hilliard equations

Consider the one dimensional Reduced Differential Transform Method for homogeneous time fractional Cahn-Hilliard partial differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}(x,t)-\frac{\partial^2}{\partial x^2}(x,t)-u(x,t)+u^3(x,t)=0, 0<a\le 1, t>0 \qquad (4)$$

subjected to the initial condition $u(x,0)=g(x), x\epsilon R$ (5)

Where α is a parameter that describes the order of time derivative by fractional derivative in thesense of Caputo fractional derivatives. By applying Reduced Differential Transform on (4) and (5) we get

$$R_D\left[\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)\right]=\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(x)$$

$$R_D\left[\frac{\partial^2}{\partial x^2}(x,t)\right]=\frac{\partial^2}{\partial x^2}(x)$$

$$u(x,t)=u_k(x)$$

$$[U^3(x,t)]=\sum_{i=0}^{k}\sum_{j=0}^{i}u_{i-j}(x)u_j(x)u_{k-i}(x)=F_k(x)$$

Where (x) is transformed value of $U^3(x,t)$,hence we get

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(x)=\frac{\partial^2}{\partial x^2}u_k(x)+u_k(x)-F_k(x), 0<\alpha\le 1, t>0 \text{ and } u_0(x)=(x), x\in R$$

For $k = 0,1,2 \ldots$ we get

$$u_1(x)=\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_0(x)+u_0(x)-u_0^3(x)\right]$$

$$u_2(x)=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_1(x)+u_1(x)-3u_0^2(x)u_1(x)\right]$$

And so on
Therefore using (2) the solution become

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x)t^{k\alpha}=u_0(x)+\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_0(x)+u_0(x)-u_0^3(x)\right]+\ldots$$

Example 1. Consider the one dimensional Reduced Differential Transform Method for homogeneoustime fractional Cahn-Hilliard partial differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)-\frac{\partial^2}{\partial x^2}(x,t)-(x,t)+u^3(x,t)=0, 0<a\le 1, t>0$$

subjected to the initial condition

$$(x,0)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}$$

By applying Reduced Differential Transform

$$RD\left[\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)\right]=\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(x), R_D\left[\frac{\partial^2}{\partial x^2}u(x,t)\right]=\frac{\partial^2}{\partial x^2}u_k(x)$$

$$[u(x,t)]=RD\left[\frac{1}{1+e^{e^{\frac{x}{\sqrt{2}}}}}\right], RD[u(x,t)]=u_k(x)$$

$$[u^3(x,t)]=F_k(x)$$

For $k=0, u_0(x)=\frac{1}{1+e^{e^{\frac{x}{\sqrt{2}}}}}$

For $k=0, u_1(x)=\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left[\frac{3e^{e^{\frac{x}{\sqrt{2}}}}}{2(1+e^{\frac{x}{\sqrt{2}}})^2}\right]$

$$\text{For } k = 1, u_2(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[\frac{9e^{\frac{x}{\sqrt{3}}}\,(e^{\frac{2x}{\sqrt{2}}}-1)}{4(1+e^{\frac{x}{\sqrt{2}}})^4}\right] \text{ and so on} \ldots$$

By applying inverse Differential Transform

$$u(x,t) = \frac{\Gamma(1)}{\Gamma(\alpha+1)}\frac{3e^{e^{\frac{x}{\sqrt{2}}}}}{\left[2\,(1+e^{\frac{x}{\sqrt{2}}})^2\right]} + \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[\frac{9e^{\frac{x}{\sqrt{3}}}\,(e^{\frac{2x}{\sqrt{2}}}-1)^1}{4\,(1+e^{\frac{x}{\sqrt{2}}})^4}\right] + \cdots$$

Which is approximate solution of example 1

Example 2. Consider the one dimensional Reduced Differential Transform Method for homogeneoustime fractional Cahn-Hilliard partial differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - \frac{\partial^2}{\partial x^2}(x,t) - (x,t) + u^3(x,t) = 0, 0 < a \leq 1, t > 0$$

subjected to the condition
$(x,0) = e^x$

By applying Reduced Differential Transform

$$RD\left[\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)\right] = \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(x), R_D\left[\frac{\partial^2}{\partial x^2}u(x,t)\right] = \frac{\partial^2}{\partial^2}u_k(x)$$

$$[u(x,t)] = RD\left[\frac{1}{1+e^{e^{\frac{x}{\sqrt{2}}}}}\right], RD[u(x,t)] = u_k(x)$$

$$[u^3(x,t)] = F_k(x)$$

$$RD[u(x,t)] = u_k(x) = e^x$$

$$u_{k+1}(x) = \frac{\Gamma(k\alpha+1}{\Gamma(k\alpha+\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_k(x) + (x) - F_k(x)\right], 0 < \alpha \leq 1, t > 0$$

and $u_0(x) = g(x), x \in R$

$$\text{For } k = 0, u_1(x) = \frac{2e^x - 3e^x}{\Gamma(\alpha+1)}$$

$$\text{For } k = 1, u_2(x) = \frac{4e^x - 16e^{3x} + 3e^{5x}}{\Gamma(2\alpha+1)} \text{ and so on} \ldots$$

By applying inverse Differential Transform we get

$$u(x,t) = e^x + \frac{2e^x - 3e^x}{\Gamma(\alpha+1)} + \cdots.$$

CONCLUSION

The reduced differential transform method is proposed to solve one dimensional homogeneous time fractional Cahn Hilliard equation in sense of Caputo fractional derivatives. One dimensional homogeneous time fractional CahnHilliard equation subject to the initial condition by using definitions of reduced and inverse reduced differential transformed function are used to find the approximate solutions .The solution are obtained in infinite power series. The proposed method is very effective, less computable , simple and can be applied to other non -linear partial differential equations models

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