Differential Transform Method for Fractional Order Differential Equation

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ABSTRACT
In this paper we study fractional order differential Equations and Differential Transform Method. We solved some numerical of fractional order differential equation in the (Reimann- Liouville) and (up to sence) by using Differential Transformed Method. And the results obtain by Differential Transformed Method use compared with exact solutions.

Keywords: Differential Transformed Method, Fractional order Differential Equations Reimann-liouville Integral, (up to) Integral

INTRODUCTION
A variety of methods, exact, approximate and purely numerical are available for the solution of differential equations. Most of these methods are computationally intensive because they are trial-and error in nature, or need complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations. The concept of differential transformation was first proposed by Zhou [19] in 1986 [2-5] and (Arikhoglui and Ozkol, Ayaz, Chens and Ho, 1996, 1999; Hassan and Abdel- Halims 2008, Duan, Khaled Batihas)[6-11] it was applied to solve linear and non-linear initial value problems in Electric circuit analysis. This method constructs a semi - analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equations.[The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The Differential transformation method is very effective and powerful for solving various kinds of Differential equation

1. Consider system of fractional differential equations

$$D_*^{\propto_1} x_1(t) = f_1(t_1 x_1 x_2 \dots x_n)$$
$$D_*^{\propto_2} x_2(t) = f_2(t_1 x_1 x_2 \dots x_n)$$
$$\vdots$$
$$D_*^{\propto_n} x_n(t) = f_n(t_1 x_1 x_2 \dots x_n)$$

Where $D_*^{\propto_i}$ is the derivative of x_i of order $\propto_i$ in the sence caputo $0 \leq \propto_i \leq 1$subjected to
Definition- Riemann- Liouville Fractional Integration of order $\propto$ is defined as,

$$J^{\propto}{}_{x_0} f(x) = \frac{1}{\surd(\propto)} \int_{x_0}^{x} (x-t)^{\propto -1} f(t) dt, \qquad \propto, x, 0$$

Riemann- Liouville Fractional Derivative:

$$D_{x_0}{}^{x} f(x) = \frac{D^m}{dx^m} [\, J^{m-\propto} f(x)\,]$$

Caputo Fractional Derivative:

$$D_{x_0}{}^{\propto} f(x) = J^{m-\propto} \left[\frac{D^m}{dx^m} f(x) \right]$$

Where m-1$\leq \propto \leq m \quad , m \in N$

2. Fractional Differential Transform Method :

The generalized Differential Transform of k^{th} derivative of the function $f(x)$ in one variable is given by,

$$F_\beta(k) = \frac{1}{\sqrt{(\beta k + 1)}}\left[(D_{x_0}{}^{\beta})^k f(x)\right]_{x=x_0}$$

Where $0 \le \beta \le 1$,

$(D_{x_0}{}^{\beta})^k = D_{x_0}{}^{\beta}, D_{x_0}{}^{\beta} \dots D_{x_0}{}^{\beta} (k\ times)$

And $F_\beta(k)$ is transform function.

Definition: The inverse generalized transform of $F_\beta(k)$is defined by,

$$f(x) = \sum_{k=0}^{\infty} F_\beta\ (k)(x - x_0)^{\beta k}$$

We use the following fractional Differential Transform theorem to solve the numerical

Theorem1: If $f(x) = g(x) \pm h(x) then\ F(k) = G(k) \pm H(k)$

Theorem 2: If $f(x) = g(x) \cdot h(x) then\ F(k) = \sum_{l=0}^{k} \mathrm{G(l)H(k-l)}$

Theorem 3: If $f(x) = g_1(x) \cdot g_2(x) \dots g_{n-1}(x) \cdot g_n$ then

$$F(k) = \sum_{k_{n-1=0}}^{k} \sum_{k_{n-2=0}}^{k_{n-1}} \dots \sum_{k!=0}^{k_2} G_1(k_1)G_2(k_2 - k_1) \dots G_n(k - k_{n-1})$$

Theorem 4: If $f(x) = (x - x_0)^P then\ F(k) = \partial(k - \alpha p)$

Where $\partial(k) = \begin{cases} 1\ if k = 0 \\ 0\ if\ \ \neq 0 \end{cases}$

Theorem 5: If $f(x) = D_{x_0}{}^{q}[g(x)]\ then$

$$F(k) = \frac{\sqrt{(q + 1 + k/\alpha)}}{\sqrt{1 + k/\alpha}} G(k + 2q)$$

Theorem 6: If $f(x) = \frac{d^{q1}}{dx^{q1}}[g_1(x)] \cdot \frac{d^{q2}}{dx^{q2}}[g_2(x)] \dots \frac{d^{qn}}{dx^{qn}}[g_1(x)]$

$$F(k) = \sum_{k_{n-1=0}}^{k} \sum_{k_{n-2=0}}^{k_{n-1}} \dots \sum_{k!=0}^{k_2} \frac{\sqrt{(q + 1 + k_1/\alpha)}}{\sqrt{1 + k_1/\alpha}} \cdot \frac{\sqrt{(q_2 + 1 + k_2 - k_1/\alpha)}}{\sqrt{1 + k_2 - k_1/\alpha}}$$

$$\dots \frac{\sqrt{(q_n + 1 + (k - k_{n-1}/\alpha)}}{\sqrt{1 + k - (k_{n-1}/\alpha)}} G_1(k_1 + 2q_1) \times G_2(k_2 - k_1 + \alpha q_2) \dots G_{n-1}(k_{n-1} - k_{n-2} + \alpha q_{n-1}) \times G_n(k - k_{n-1} + \alpha q_n)$$

Where $\alpha q_i \in Z^+ for\ i = 1,2 \dots, n$

Theorem 7: If$\left(D_a{}^{\beta}\right)^k f(x) \in [a, b] \neq k = 0,1,2 \dots n + 1$ where $0 < \alpha \le 1\ then$

$$f(x) \sum_{r=0}^{n} \frac{\left(x - a\right)^{p\beta}}{\sqrt{(r\beta + 1)}}$$

Example 1: $y_x{}^{\alpha} = y^2 x f(x) + f(x) where\ f(x) = x^2$ $\alpha = 1, \beta = 1, y(0) = 1$ by using theorem of Differential Transform Method,

$$y_1 = \frac{\sqrt{(\alpha k + 1)}}{\sqrt{\alpha k + \beta + 1}} \left[\sum_{r=0}^{k} y_1(r) \cdot y_1(k - r) + r\delta r - 3y_1(k) - r + \delta k - 2\right]$$

Put $k = 0, y_1(1) = 1$

$k = 1, y_1(2) = 1$

$k = 2, y_1(3) = 1$

...

Series solution becomes

$$y(x) = \sum_{k=0}^{\infty} y_1(k) x^k$$

$$= 1 + x + x^2 + \frac{4}{3}x^3 + \cdots$$

Exact solution of example is

$$y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \cdots$$

Example 2: Consider the system of two linear fractional order differential equations.

$D_*{}^{\beta} u(t) = u(t) + v(t)$

$D_*{}^{\gamma} u(t) = -u(t) + v(t)$

Subjected to condition $u(0) = 0, v(0) = 0$ using theorems on fractional order Differential equations we get,

$$U(K + \beta\alpha_1) = \frac{\sqrt{(1 + k/\alpha_1)}}{\sqrt{(\beta + 1 + k/\alpha_1)}}[U(K) + V(K)]$$

$$V(K + \gamma\alpha_2) = \frac{\sqrt{(1 + k/\alpha_2)}}{\sqrt{(\gamma + 1 + k/\alpha_2)}}[-U(X) + V(X)]$$

Where α_1, α_2 are known values of the fractional for $\beta = 1, \gamma = 1$

Put $k = 0, U(0) = 0$

$$k = 1, U(1) = 0$$

…

$$= \beta_{\alpha_1} - 1 , U(\beta_{\alpha_1} - 1) = 0$$

For $k = 1, \dots \gamma_{\alpha_2} - 1, V(k) = 0, V(0) = 1$

For $\beta = 1, \gamma = 1,$

$$u(t) = t + t^2 + \frac{t^3}{3} - \frac{t^5}{30} - \frac{t^6}{90} - \frac{t^7}{630} + \frac{t^9}{22680} + \frac{t^{10}}{113400} + \cdots$$

$$v(t) = 1 + t - \frac{t^3}{3} - \frac{t^4}{6} - \frac{t^5}{30} - \frac{t^7}{630} + \frac{t^8}{2520} + \frac{t^9}{22680} + \cdots$$

Exact solution for examples $u(t) = e^t sint\ v(t) = e^t . cost$

CONCLUSION

In this paper the analytic solution of fractional order differential equation with boundary conditions in one dimensional is constructed and we apply Differential Transform Method on the boundary value problem to get series solution. With some examples we find out the convergent series solution of system of fractional order differential equations in the series of Riemann-Liouville equations in the sense of Riemann-Liouville fractional order derivative.

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