Solution of the Fractional Order Airy's Partial Differential Equations by using Reduced Differential Transform Method

Mayur Vijay Solanki[1] , Jayshree Patil[2]

[1] Research Schloar, Department of mathematics, BAMU University, mCh. Sambhajinagar, Maharashtra, India, Email:mayursolankims1010@gmail.com
[2]Professor,VasantraoNaik College Ch. Sambhajinagar, Maharashtra, India, Email: jv.patil29@gmail.com

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ABSTRACT
In this paper, we use Reduced Differential Transform Method (RDTM) to compute analytical and semi analytical approximate solutions of fractional order Airy's ordinary differential equations and fractional order Airy's and Airy's type partial differential equations subjected to certain initial boundary conditions. We compare the solutions obtain by RDTM with exact solutions . we find out convergent series solutions by taking different fractional values. RDTM gives series solutions easily by dropping calculus work and computed mathematical results in order to easily solve fractional order Airy's type differential equations.

Keywords: Reduced Differential Transform(RDTM) Method, Airy's fractional order Partial differential equations, Series solutions

INTRODUCTION

Reduced Differential Transform Method is an alternative approach to overcome the demerit of complex calculation of Differential Transform Method, capable of reducing the size of the calculation. As a special advantage of (RDTM) rather than DTM, the reduced differential transform recursive equations produce exactly all the Poisson series coefficients of solutions, whereas the differential transform recursive equations produce exactly all the Taylor series coefficients of solutions. We notice that the Reduced Differential Transform Method technique[10,11] gives accurate and converge solutions [1,2,7,10]

METHODOLOGY

Consider a function of two variables $f(s,t)$ and suppose that it can be represented as product of two single variables, i.e. $f(s,t)=f_1(t)f_2(t)$ on the basis of properties of differential transform the function $f(s,t)$ can be represented as

$$f(s,t)=\sum_{i=0}^{\infty}F_1(i)\,s^i\sum_{j=0}^{\infty}F_2(j)t^j=\sum_{k=0}^{\infty}U_k(x)t^k$$

The basic definitions of Reduced Differential Transform Method are introduced as follows.

Definition: If the function u(x, t) is analytic and differential continuously with respect to time t and space x in the domain of interest then let,

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} \qquad (1)$$

Where the t – dimensional spectrum function $U_k(x)$ is the transformed function. $U(x,t)$ represent transformed function.

Definition: The differential inverse transform of $U_k(x)$ is defined as ,

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)\,t^k \qquad (2)$$

Then combining equations (1) and (2) we write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0}t^k \qquad (3)$$

From the above definition, it can be found that the concept of Reduced Differential Transform Method is derived from the power series expansion. The fundamental operation performed by Reduced Differential Transform Method are given table1

Table 1

function	**DTM form of function**
$u(x,t)$	$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0}$
$w(x,t)=u(x,t)\pm v(x,t)$	$W_k(x)=U_k(x)\pm V_k(x)$
$w(x,t)=\alpha\, u(x,t)$	$W_k(x)=\alpha\, U_k(x), \alpha$ is constant
$w(x,t)=x^m\ t^m$	$W_k(x)=x^m\delta(k-n), \delta(k)=1, k=0$ $=0, k\neq 0$
$w(x,t)=u(x,t)\ v(x,t)$	$W_k(x)=\sum_{r=0}^{k} U_r(x)\, V_{k-r}(x)$
$w(x,t)=\frac{\partial^r u(x,t)}{\partial t^r}$	$W_k(x)=(k+1)...(k+r)\, U_{k+r}(x)$
$w(x,t)=\frac{\partial u(x,t)}{\partial x}$	$W_k(x)=\frac{\partial U_k(x)}{\partial x}$
$w(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}$	$W_k(x)=\frac{\partial^2 U_k(x)}{\partial x^2}$
$f(s,t)=\sin(a\,s+b\,t),$	$F_k(x)=\frac{a^k}{k!}\sin\left(\frac{k\,\pi}{2}+a\,s\right)$
$f(s,t)=\cos(a\,s+b\,t),$	$F_k(x)=\frac{a^n}{k!}\cos\left(\frac{k\,\pi}{2}+a\,s\right)$
$f(s,t)=e^{a\,x+b\,t}$	$F_k(x)=\frac{b^k}{k!}\,e^{a\,x}$

Time Fractional Airy's Equations: one dimensional fractional Airy's partial differential equation in Caputo sense

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)=\beta\frac{\partial^3}{\partial x^3}u(x,t), x\in R, t>0,\ \ 0<\alpha<1 \qquad (4)$$

Where $\beta=\pm 1$ with conditions $u(x,0)=\emptyset(x),\ x\in R$ (5)

Applying RDTM on (4),(5) we get

$$R_D\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)\right]=R_D\left[\beta\frac{\partial^3}{\partial x^3}u(x,t)\right] \qquad (6)$$

and

$$R_D[u(x,0)]=R_D[\emptyset(x)] \qquad (7)$$

By applying table(1) we get,

$$U_{k+1}(x)=\beta\ \frac{\Gamma(k\alpha+1)}{\Gamma(k+1)(\alpha+1)}\left[\frac{\partial^3}{\partial x^3}U_k(x)\right], x\in R,\ k=0,1,2... \qquad (8)$$

$$U_0(x)=\phi(x), x\in R \qquad (9)$$

Putting (9) in (8) we get

For $k=0, U_1(x)=\beta\ \frac{1}{\Gamma(\alpha+1)}\left[\frac{\partial^3}{\partial x^3}\phi(x)\right]$

For $k=1, U_2(x)=\beta^2\ \frac{1}{\Gamma(2\alpha+1)}\left[\frac{\partial^6}{\partial x^6}\phi(x)\right],...$

Putting these equations in (2), we get

$$U(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{k\alpha}, t\succ 0$$

Example1:Consider one dimensional time fractional Airy partial differential equation for $\beta=1$

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^3 u}{\partial x^3}\, x\in R, t\succ 0, 0\le\alpha\le 1 \qquad (10)$$

subjected to the condition

$$u(x,0)=\cos\pi x+e^{\pi x} \qquad (11)$$

Applying RDTM on equations (10,11) we get

$$U_{k+1}(x)=\beta\,\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\frac{\partial^3}{\partial x^3}U_k(x)\right], x\in R,\ k=0,1,2\ldots \qquad (12)$$

Using RDTM on (11)

$$u(x,0)=u_0(x)=\cos\pi x+e^{\pi x}, x\in R \qquad (13)$$

Using equations (12) and (13)we get $U_k(x)$

$$U_1(x)=\frac{\pi^3\left(\sin\pi x+e^{\pi x}\right)}{\Gamma(\alpha+1)}$$

$$U_2(x)=\frac{-\pi^6\left(\cos\pi x-e^{\pi x}\right)}{\Gamma(2\alpha+1)}$$

$$U_3(x)=\frac{-\pi^9\left(\sin\pi x-e^{\pi x}\right)}{\Gamma(3\alpha+1)},\ldots$$

Thus the fractional differential inverse transform of $U_k(x)$ gives

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{k\alpha}$$

$$u(x,t)=\left(\cos\pi x+e^{\pi x}\right)+\frac{\pi^3\left(\sin\pi x+e^{\pi x}\right)}{1!}t^1-\frac{\pi^6\left(\cos\pi x-e^{\pi x}\right)}{2!}t^2-\frac{\pi^9\left(\sin\pi x+e^{\pi x}\right)}{3!}t^3+\ldots$$

For $\alpha=\frac{1}{2}$

$$u(x,t)=\left(\cos\pi x+e^{\pi x}\right)+\frac{\pi^3\left(\sin\pi x+e^{\pi x}\right)}{1!}t^{\frac{1}{2}}-\frac{\pi^6\left(\cos\pi x-e^{\pi x}\right)}{2!}t^1-\frac{\pi^9\left(\sin\pi x+e^{\pi x}\right)}{3!}t^{\frac{3}{2}}+\ldots$$

Example 2 Consider one dimensional time fractional Airy partial differential equation for $\beta=1$

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^3 u}{\partial x^3}\, x\in R, t\ge 0, 0\le\alpha\le 1 \qquad (14)$$

subjected to the condition $u(x,0)=\frac{1}{6}x^3$ (15)

Applying RDTM on equations (14,15) we get

$$U_{k+1}(x)=\beta\,\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\frac{\partial^3}{\partial x^3}U_k(x)\right], x\in R,\ k=0,1,2\ldots \qquad (16)$$

$$u_0(x)=\frac{1}{6}x^3 \qquad (17)$$

Using equations (16) and (17)we get $U_k(x)$

$$U_1(x)=\frac{1}{\Gamma(\alpha+1)}$$

For k = 1, $U_2(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[\frac{\partial^3}{\partial x^3}U_1(x)\right]$

For k = 2, $U_3(x) = 0, \ldots \quad , U_k(x) = 0$

Thus the fractional differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = \frac{1}{6}x^3 + \frac{1}{\Gamma(\alpha+1)}t^{\alpha}$$

For $\alpha = 1$, $u(x,t) = \frac{1}{6}x^3 + t$

CONCLUSION

In this study we have discussed the reduced differential transform method to find the solution of one dimensional time fractional Airy's and Airy's type partial differential equation based on the basic Caputo's definition of fractional derivatives. The answers show that the RDTM technique gives the approximate series solutions. The accuracy of the solutions can be improve by increasing number of terms in series solutions. The techniques used in this work can also be applied to solve linear and non-linear time fractional partial differential equation and multi-dimensional physical problems emerging in various fields of engineering and applied sciences.

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