The Edge-to-Edge Steiner Number of a Graph

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ABSTRACT
For a non-empty set Wof edges in a connected graphG,the edge-to-vertex Steiner distance $d_{ev}(W)$of Wis the minimum size of a tree containing V(W) and is called an edge-to-vertex Steiner tree with respect to W or a Steiner W_{ev} - tree.A set $W \subseteq E$ is called an edge-to-edge Steiner set if every edge of G lies on a Steiner W_{ev} - tree of G. The edge-to-edge Steiner number $s_{ee}(G)$of Gis the minimum cardinality of its edge-to-edge Steiner sets and any edge-to-edge Steiner sets of cardinality$s_{ee}(G)$ is called a minimum edge-to-edge Steiner set of G or a s_{ee}-set of G. The edge-to-edge Steiner number of certain classes of graphs are determined. We characterize a connected graph of size $m \geq 3$with an edge-to-edge Steiner numberm or $m - 1$.

Keywords: Steiner distance, Steiner number, edge-to-vertex Steiner distance, edge-to-vertex Steiner set, edge-to-edge Steiner set.

1. INTRODUCTION

Let $G = (V, E)$ be a graph having a vertex set V(G) and an edge setE(G)(V(G) andE(G) correspondingly). In addition, we state that a graph G has size $m = |E(G)|$ and order $n = |V(G)|$. We refer to [1] for the fundamental terms used in graph theory. A vertex v is adjacent to another vertex u if and only if there exists an edge $e = uv \in E(G)$.If $uv \in E(G)$, then u is a neighbor of v, and the set of neighbors of vis denoted by $N_G(v)$. The degree of a vertex$v \in V$ is $deg_G(v) = |N_G(v)|$. If $deg_G(v) = n - 1$, thenv is said to be a universal vertex. A vertex v is called an extreme vertex if the subgraph induced by v is complete.

The distance $d(u, v)$between two verticesuand v in a connected graph Gis the length of a shortest $u - v$path in G.For a nonempty set W of vertices in a connected graph G,the Steiner distance d(W) of W is the smallest size of a connected subgraph of G containing W.The Steiner distance for a graph Gisstudied in [3,8,11,13]. Let S(W) stand for the collection of all vertices on Steiner W-trees. If $S(W) = V(G)$, then a set $W \subseteq V(G)$ is referred to as a Steiner set of G.A Steiner set of minimum cardinality is a minimum Steiner set or simply as-set and its cardinality is the Steiner number s(G) of G. The Steiner number was introduce in [3] and further studied in [8-13,15].

Definition 1.1.[15] Consider a connected graph $G = (V, E)$ with at least three vertices. For a non-empty set Wof edges in a connected graphG,the edge-to-vertex Steiner distance $d_{ev}(W)$ of W is the minimum size of a tree containing V(W) and is called an edge-to-vertex Steiner tree with respect to W or a Steiner W_{ev} -tree. There might be more than one Steiner W_{ev}-tree in G for a specific set $W \subseteq E(G)$. However, $V(W) \subseteq V(T_1) \cap V(T_2)$ despite the possibility that T_1 and $T_2$are Steiner W_{ev}-trees.

Example 1.2. Let G be a graph shown in figure 1.1.Let $W = \{v_1v_6, v_2v_5, v_3v_4\}$. Figures 1.1(a) and 1.1(b) show the two Steiner W_{ev}-trees.

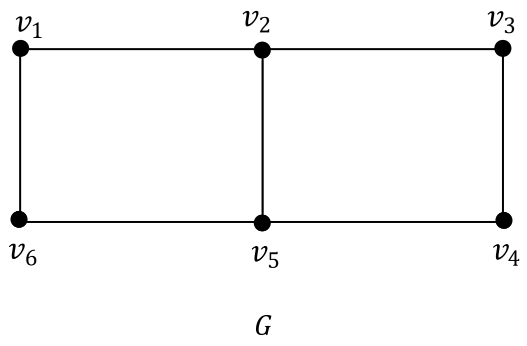

Figure 1.1

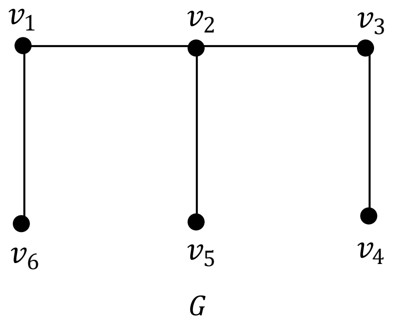

Figure 1.1(a)

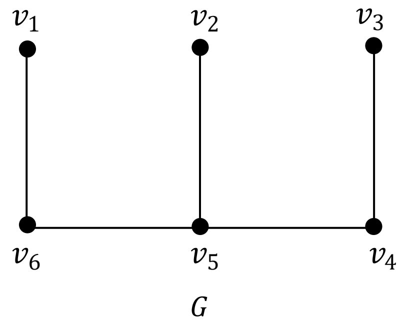

Figure 1.1(b)

Remark 1.3.[11] If the edges $e = uv$ and $f = vw$ are adjacent in G, then the Steiner W_{ev}-tree is a path between u, v and w.
Theorem 1.4.[14]Each end edge of the connected graph G belongs to every edge-to-vertex Steiner (edge-to-edge geodetic) set of G.
Theorem 1.5.[11]For the star $G = K_{1,m}$ $(m \geq 2)$, $s_{ev}(G) = m$.

2.The Edge-to-Edge Steiner Number of a Graph
Definition 2.1.A connected graph $G = (V, E)$with at least three vertices. A set $W \subseteq E$ is called an edge-to-edge Steiner set if every edge of G lies on a Steiner W_{ev}-tree of G. The edge-to-edge Steiner number$s_{ee}(G)$ is the minimum cardinality of its edge-to-edge Steiner sets.Any edge-to-edge Steiner set with cardinality $s_{ee}(G)$ is a minimum edge-to-edge Steiner set of G or as_{ee}-set of G.
Example 2.2. Let G be a graph shown in figure 1.1.Let $W_1 = \{v_1v_6, v_2v_5, v_3v_4\}$. For the graph G in Figure 1.1,W_1 is an edge-to-edge Steiner set of G since each edge of G is contained in one of the two Steiner W_{ev}-trees, and as a result, $s_{ee}(G) \leq 3$. No edge-to-edge Steiner set of G is a two elements subset of E, hence $s_{ee}(G) = 3$.
Remark 2.3. For the graph G given in Figure 1.1,$W_1 = \{v_1v_6, v_3v_4\}$ is a s_{ev}-set of G. Consequently, there is a difference between the edge-to-edge Steiner number and the edge-to-vertex Steiner number.

Theorem 2.4. For a connected graphG of size $m \geq 3$, $2 \leq s_{ee}(G) \leq m$.
Proof. A s_{ee}-set requires a minimum of two edges, therefore $s_{ee}(G) \geq 2$. Let $W = E$. Let uvbe any edge of G. Let T be aW_{ev}-tree of G. Then T is a spanning tree of G. If $uv \in E(T)$, thenW is an edge-to-edge Steiner set of G. If $uv \notin E(T)$, let$u = u_0, u_1, u_2, \dots\dots, u_n = v$ be the unique path in T. Thenu and vlie on different components of $T - uu_1$. LetT' be the tree obtained from$T - uu_1$ by joining the edge uv. Then $T'$is a spanning tree of G such that $|T| = |T'|$. Since $uv \in E(T')$,W is an edge-to-edge Steiner set of Gand so$s_{ee}(G) \leq m$. Therefore, $2 \leq s_{ee}(G) \leq m$.
Remark 2.5. The bounds in Theorem 2.4 are sharp. For $G = C_{2k}$ $(k \geq 2)$, $s_{ee}(G) = 2$ and for $G = K_{1,m}$, $s_{ee}(G) = m$. Also the bounds in Theorem 2.4 can be strict. For the graph G is given in Figure 1.1, $s_{ee}(G) = 3$ and $m = 7$. Thus $2 < s_{ee}(G) < m$.

Theorem 2.6. Every edge-to-edge Steiner set contains at least one extreme edge that is incident with v if v is an extreme vertex of a connected graph G.

Proof. Assume that v in G is an extreme vertex and that $deg_G(v) = k$. Let $N(v) = \{v_1, v_2, \ldots \ldots, v_k\}$ represent the area around v in G. Let W represent a edge-to-edge Steiner set of G. Assume that for any i, $(1 \le i \le k)$, $vv_i \notin W$. vv_i is located on a Steiner W_{ev}-tree of G for $1 \le i \le k$ since W is an edge-to-edge Steiner set of G. Assume that T is a Steiner W_{ev}- tree of W where $vv_i \in T$. Suppose $deg(v) = l$. For $l = 1$, the result is obviously trivial, therefore let $l = 2$. Let $N_T(v) = \{u_1, u_2, \ldots \ldots, u_l\}$ represent the area around v in T. Given that v is an extreme vertex of G, it follows that for any i, j with $1 \le i; j \le k-1$ and $i \ne j$, $u_i u_j \in E(G)$. Let T' be a tree in G that was created by removing the vertex v from the originaltree T and adding the edges $u_i u_{i-1}$ $(1 \le i \le k-1)$ to it. Then, $V(W) \subseteq V(T'')$ and $|V(T'')| = |V(T)| - 1$, which is a contradiction to W and a Steiner set of G that extends from edge-to-edge. Therefore there is at least one extreme edge that is incident with v in every edge-to-edge Steiner set.

Corollary 2.7. Each end vertex of G belongs to every edge-to-edge Steiner set of G.

Proof. This follows from Theorem 2.6.

Theorem 2.8. Let G be a connected graph with k end-vertices and a size of $m \ge 3$, then $\max\{2, k\} \le s_{ee}(G) \le m$.

Proof. This follows from Theorem 2.4 and Corollary 2.7.

Corollary 2.9. Let W be the edge-to-edge Steiner set of a connected graph G, which has cut edges. Then each of the two components of $G - e$ contains an element of W for each cut edge e of G that is not an end edge.

Proof. Let the cut edge of G be $e = uv$. Suppose G_1 and G_2 are the two parts of $G - e$ that have the relationship $u \in V(G_1)$ and $v \in V(G_2)$. Let W represent an edge-to-edge Steiner set of G. Assume that W doesn't have any G_1 edges. Every edge of G resides on a Steiner W_{ev}-tree since W is an edge-to-edge Steiner set of G. A Steiner W_{ev}-tree with $V(G_1) \in V(T)$ is called T. Then, in T, u occurs twice. As a result, T must be present in the cycle. T is not a tree, which is a contradiction. Therefore there is at least one edge each from G_1 and G_2 in every edge-to-edge Steiner set of G.

Corollary 2.10. Let G be a connected graph with a cut edge e. Then e lies in every Steiner W_{ev}-tree.

Theorem 2.11. Let W be a s_{ee}-set of G and G be a connected graph. Then, none of the cut edges of G that are not end edges belong to W.

Proof. Let W be any s_{ee}-set of G. Assume that the cut edge, $e = uv$, which is not G is an end edge and that $e \in W$. Assume that G_1 and G_2 two parts are $G - e$. Suppose $W' = W - \{uv\}$. We assert that W' is a Steiner set of G that spans its entire edge. According to Corollary 2.10, each Steiner W_{ev}-tree contains e because e is a cut edge of G. Since each Steiner W_{ev}-tree is a Steiner W_{ev}-tree of G, it follows that $e \notin W'$. Thus W' is an edge-to-edge Steiner set of G such that $|W'| < |W|$. Which is a contradiction to W an edge-to-edge Steiner set of G. Thus, the theorem is implied.

Corollary 2.12. The set of all end-edges of any non-trivial tree T is the unique minimum edge-to-edge Steiner set of T, and for any non-trivial tree T with k end-vertices, $s_{ee}(T) = k$.

Proof. This follows from Corollary 2.7 and Theorem 2.11.

Observation 2.13 (i) For the complete graph $G = K_n$ $(n \ge 3)$, $s_{ee}(G) = \begin{cases} n & \text{if } n = 3 \\ \frac{n}{2} & \text{if n is even} \\ \frac{n+1}{2} & \text{if n is odd} \end{cases}$.

(ii) For the cycle $G = C_n$ $(n \ge 4)$, $s_{ee}(G) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd} \end{cases}$.

(iii) For the complete bipartite graph $G = K_{s,s}$ $(s \ge 2)$, $s_{ee}(G) = s$.

Theorem 2.14. If G is a connected graph with $m \ge 4$ such that G is not a star, then $s_{ee}(G) \le m - 1$.

Proof. Assume that e is an edge that is not an end-edge of G. Let $W = E(G) - \{e\}$. Then $V(W) = V$. Since $s_{ee}(G) \le m - 1$, then W is a edge-to-edge Steiner set of G.

Theorem 2.15. For any connected graph G of size $m \ge 2$, $s_{ee}(G) = m$ if and only if G is either K_3 or the star $K_{1,m}$.

Proof. Let $s_{ee}(G) = m$. Assume that $m = 2$. Therefore $= P_3 \cong K_{1,2}$. Suppose $m = 3$. Then G is either K_3 or P_4. If $G = K_3$ then $s_{ee}(G) = 3 = m$. If $G = P_4$, then $s_{ee}(G) = 2 = m - 1$, which is not the case. Let $m \ge 4$. If G is not a star. Then by Theorem 2.14, $s_{ee}(G) \leq m-1$, which is a contradiction. Therefore G is the star, $K_{1,m}$. The converse is obvious.

Theorem 2.16. Let G be a connected graph of size $m \geq 4$which is not a tree. Then $s_{ee}(G) \leq m-2$.
Proof. By Observation 2.13(ii), the graph G is a cycle C_n $(n \geq 4)$. Therefore $s_{ee}(G) \leq m-2$. Let $C: v_1, v_2, v_3, \ldots\ldots, v_k, v_1$ $(k \geq 3)$ be the smallest cycle in the graph G.If it is not a cycle, let v be a vertex such that v is not on C and v is next to v_1 (say). Therefore$s_{ee}(G) \leq m-2$, $W = E(G) - \{v_1v_2, v_1v_k\}$ is an edge-to-edge Steiner set.
Remark 2.17. The bound in Theorem 2.16 is sharp. For the graph G given in Figure 2.1,$W = \{v_1v_2, v_3v_4\}$ is a s_{ee}-set of G so that $s_{ee}(G) = 2 = m-2$.

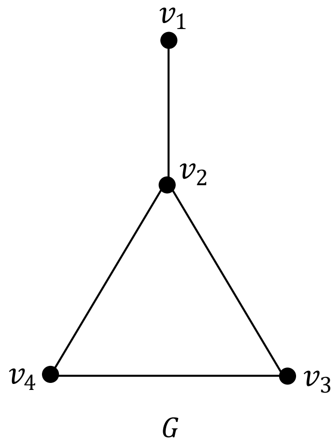

Figure 2.1

Theorem 2.18. For any connected graph G with $m \geq 4$, $s_{ee}(G) = m-1$ if and onlyif G isa double star.
Proof. Let $s_{ee}(G) = m-1$. Let $m = 3$. If G is a tree, then $G = P_4$ or $K_{1,3}$. By Corollary 2.12, if $G = K_{1,3}$, then $s_{ee}(G) = 3 = m$, which is a contradiction. If $G = P_4$ then it is a double star and meets the criteria of this theorem. If G is not a tree then $G = K_3$.Therefore by Theeorem 2.15$s_{ee}(G) = m$, which is a contradiction. Let $m \geq 4$. If G is not a tree then by Theorem 2.16, $s_{ee}(G) \leq m-2$ contradicts itself. G is therefore a tree. We assert that $d \leq 3$. Assume $d > 4$. Theorem 2.11 states that since G is a tree and has at least two internal edges, the statement $s_{ee}(G) \leq m-2$ is false.As a result, $d \leq 3$. If $d = 2$, then G is the star $K_{1,m}$. By Corollary 2.12, $s_{ee}(G) = m$, which contradicts the hypothesis. If $d = 3$ then G is a double star and complies with the theorem's conditions. ThereforeG is eitherP_4 or a double star. The converse is obvious.

3. CONCLUSIONS

In this article we studied the edge-to-edge Steiner number of a graph. We extend this concept to other distance related parameters in graphs.

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