The Geodetic Certified Domination Number of Graphs

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ABSTRACT
A set $D \subseteq E(G)$ is called a geodetic dominating set of G if D is both ageodetic dominating set and a certified dominating set of G. The geodetic certified domination number $\gamma_{gcer}(G)$ is the minimum cardinality of its geodetic certified dominating sets. The geodetic certified domination number of some standard graphs are determined.

Keywords: domination number, geodetic set, certified domination number, geodetic certified domination set, geodetic certified domination number.

1. INTRODUCTION

Let $G = (V, E)$ be a finite, undirected graph without loops and multiple edges. The graph $G = (V, E)$ has $n = |D|$ vertices and $m = |E|$ edges. The essential terms and definitions are comprehensively covered in Harary's work [8]. Two vertices u and v are said to be adjacent if uv is an edge of G. The open neighbourhood of a vertex v in a graph G is defined as the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. While the closed neighbourhood of a vertex v in a graph G is defined as the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. While the closed neighbourhood of a vertex v in a graph G is defined as the set $N(G) = N_G(v) \cup \{v\}$.
For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. The eccentricity $e(v)$ of a vertex v in G is the minimum eccentricity among the vertices of G is the radius, rad G or $r(G)$ and the maximum eccentricity is its diameter, diam G of G. Let $x, y \in V$ and let $I[x, y]$ be the set of all vertices that lies in $x - y$ geodetic including x and y. Let $D \subseteq V(G)$ and $I[D] = \cup I[x, y]$. Then D is said to be a geodetic set of G, if $I[D] = v$. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a g- set of G.
A set $D \subseteq V(G)$ is called a dominating set of G if for every $v \in V \backslash D$ is adjacent to atleast one vertex in D. A dominating set D is said to be minimal if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the domination number of G and is denoted by $\gamma(G)$ [9]. Any dominating set of cardinality$\gamma(G)$is a γ-set of G. A dominating set D of $G = (V, E)$ is a certified dominating set, if every vertex in D has either zero or atleast two neighbours in $V \backslash D$.The certified domination number $\gamma_{cer}(G)$ of G is the minimum cardinality of certified dominating set[7]. A set $D \subseteq V$ is said to be a geodetic certified dominating set of G, if D is both geodetic set and certified dominating set of G. The geodetic certified dominating number of G is the minimum cardinality among all geodetic certified dominating sets in G and denoted by$\gamma_{gcer}(G)$[10]. A geodetic certified domination set of minimum cardinality is called the $\gamma_{gcer}(G)$-set of G. In this article, the geodetic certified domination number of a graph is introduced and studied.

2. The Geodetic Certified Domination Number of Graphs

Definition 2.1.
A set $D \subseteq V(G)$ is said to be geodetic certified dominating set of G if D is bothgeodetic set and certified dominating set of G.The geodetic certified dominating number of G is the minimum cardinality among all the geodetic certified dominating sets in G and denoted by $\gamma_{gcer}(G)$. A geodetic certified dominating set of minimum cardinality is said to be a $\gamma_{gcer}(G)$-set of G.

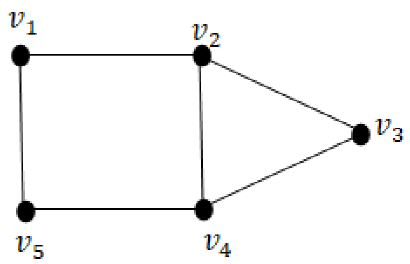

Figure 1

For the graph G given in Figure 1,$D_1 = \{v_1, v_3\}$ is a minimum geodetic certified dominating set. Therefore $\gamma_{gcer}(G) = 2$.

Theorem 2.2. For the path $G = P_n (n \geq 2)$, $\gamma_{gcer}(G) = n$

Proof: Let $V(G) = \{v_1, v_2, \dots, v_n\}$be the vertex set of Gand $E(G) = \{e_1, e_2, \dots, e_n\}$ be the edge set of G, where $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$.Let $D = \{v_1, v_2, v_3, \dots, v_{n-1}\}$. Then D is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) \leq n$.Here, it is to be claimed that $\gamma_{gcer}(G) = n$.Suppose that $\gamma_{gcer}(G) \leq n-1$. Then there exists a geodetic certified dominating set D' of G such that $|D'| \leq n-1$. First assume that $D' \subset D$. Let's consider a vertexx suchthat $x \in D$ and $x \notin D'$. Without loss of generality, let us assume that $x = v_n$. Thenx is not dominated by any element of D',which gives a contradiction. Suppose if$D' \not\subset D$. Let's consider a vertex x suchthat $x \notin D$ and $x \in D'$. Without loss of generality, let us assume that $x = v_{n+1}$. Then x is not dominated by any element of D', which gives a contradiction. Therefore $\gamma_{gcer}(G) = n$.

Theorem 2.3. For the cycle $G = C_n (n \geq 5)$,

$$\gamma_{gcer}(G) = \begin{cases} 2 & \text{if } n = 4 \\ 3 & \text{if } n = 5 \text{ or } 7 \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 0 (\text{mod } 3), n \equiv 1 (\text{mod } 3), n \equiv 2 (\text{mod } 3) \end{cases}$$

Proof: Let $V(G) = \{v_1, v_2, \dots, v_n, v_1\}$ and $E(G) = \{e_1, e_2, \dots, e_n\}$.

Case(i) $n = 4$. Then$D = \{v_1, v_3\}$ is the unique γ_{gcer} - set of G so that $\gamma_{gcer}(G) = 2$.

Case(ii) $n = 5$. Then $D = \{v_1, v_3, v_4\}$ is the unique γ_{gcer} - set of G so that $\gamma_{gcer}(G) = 3$.

Case(iii) $n = 7$. Then $D = \{v_1, v_3, v_5\}$ is the unique γ_{gcer} - set of G so that $\gamma_{gcer}(G) = 3$.

Case(iv) $n \equiv 0(\text{mod } 3)$.Let$D = \{v_1, v_4, v_7, \dots, v_{n-5}, v_{n-2}\}$. Then D is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) \leq \left\lceil \frac{n}{3} \right\rceil$. Establish that $\gamma_{gcer}(G) = \left\lceil \frac{n}{3} \right\rceil$. On the contrary, assume that $\gamma_{gcer}(G) \leq \left\lceil \frac{n}{3} \right\rceil - 1$. Then there exists a geodetic certified dominating set D' of G such that $|D'| \leq \left\lceil \frac{n}{3} \right\rceil - 1$. Assume that $D' \subset D$. Now consider a vertex x suchthat $x \in D$ and $x \notin D'$. Without loss of generality, let us assume that $x = v_{n-2}$. Thenx is not dominated by any element of D',which is a contradiction. Suppose if$D' \not\subset D$. Now consider a vertex x suchthat $x \notin D$ and $x \in D'$. Without loss of generality, let us assume that $x = v_n$. Thenx is not dominated by any element of D', which gives a contradiction to our assumption. Therefore $\gamma_{gcer}(G) = \left\lceil \frac{n}{3} \right\rceil$.

Case(v) $n \equiv 1(\text{mod } 3)$.Let$D = \{v_1, v_4, v_7, \dots, v_{n-3}, v_{n-1}\}$. Then D is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) \leq \left\lceil \frac{n}{3} \right\rceil$. Establish that $\gamma_{gcer}(G) = \left\lceil \frac{n}{3} \right\rceil$. On the contrary, assume that $\gamma_{gcer}(G) \leq \left\lceil \frac{n}{3} \right\rceil - 1$. Then there exists a geodetic certified dominating set D' of G such that $|D'| \leq \left\lceil \frac{n}{3} \right\rceil - 1$. Let $D' \subset D$. Let's consider a vertexx suchthat $x \in D$ and $x \notin D'$. Without loss of generality, let us assume that $x = v_{n-1}$. Thenx is not dominated by any element of D',which is a contradiction. Now assume that $D' \not\subset D$. Let x be a vertex such that $x \notin D$ and $x \in D'$. Without loss of generality, let us assume that $x = v_n$. Thenx is not dominated by any element of D', which gives a contradiction to our assumption. Therefore $\gamma_{gcer}(G) = \left\lceil \frac{n}{3} \right\rceil$.

Case(vi) $n \equiv 2(\text{mod } 3)$.Let$D = \{v_1, v_4, v_7, \dots, v_{n-4}, v_{n-1}\}$. Then D is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) \leq \left\lceil \frac{n}{3} \right\rceil$. Establish that $\gamma_{gcer}(G) = \left\lceil \frac{n}{3} \right\rceil$. On the contrary, assume that $\gamma_{gcer}(G) \leq \left\lceil \frac{n}{3} \right\rceil - 1$. Then there exists a geodetic certified dominating set D' of G such that $|D'| \leq \left\lceil \frac{n}{3} \right\rceil - 1$. Let$D' \subset D$. Let x be a vertex suchthat $x \in D$ and $x \notin D'$. Without loss of generality, let us assume that $x = v_{n-1}$. Thenx is not dominated by any element of D',which is a contradiction. Now assume that $D' \not\subset D$. Let x be a vertex suchthat $x \notin D$ and $x \in D'$.

Without loss of generality, let us assume that $x = v_n$. Then x is not dominated by any element of D', which gives a contradiction to our assumption. Therefore $\gamma_{gcer}(G) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem 2.4. For the wheel graph $G = K_1 + C_{n-1}$, $\gamma_{gcer}(G) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil & \text{if } n \text{ is odd} \end{cases}$

Proof: Letx be the central vertex of G and C_{n-1} be$v_1, v_2, \dots, v_{n-1}$.
Case(i) Let n be even. Let $D = \{v_1, v_3, \dots, v_{n-3}, v_{n-1}\}$. Then D is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) \le \frac{n}{2}$. It is to be claimed $\gamma_{gcer}(G) = \frac{n}{2}$.Suppose $\gamma_{gcer}(G) \le \frac{n}{2} - 1$, then there exists a geodetic certified dominating set D' of G such that $|D'| \le \frac{n}{2} - 1$. Assume that $D' \subset D$. Now consider a vertex x suchthat $x \in D$ and $x \notin D'$. Without loss of generality, let us assume that $x = v_{n-1}$. Then x is not dominated by any element of D',which is a contradiction. Suppose if$D' \not\subset D$. Let's consider a vertex x suchthat $x \notin D$and $x \in D'$. Without loss of generality, let us assume that $x = v_n$. Then x is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{gcer}(G) = \frac{n}{2}$.
Case(ii) Let n be odd. Let $D = \{v_1, v_3, \dots, v_{n-4}, v_{n-2}\}$. Then D is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) \le \left\lceil \frac{n}{2} \right\rceil$. It is to be claimed $\gamma_{gcer}(G) = \left\lceil \frac{n}{2} \right\rceil$.Suppose that $\gamma_{gcer}(G) \le \left\lceil \frac{n}{2} \right\rceil - 1$. Then there exists a geodetic certified dominating set D' of G such that $|D'| \le \left\lceil \frac{n}{2} \right\rceil - 1$. Assume that $D' \subset D$. Now consider a vertex x suchthat $x \in D$ and $x \notin D'$. Without loss of generality, let us assume that $x = v_{n-1}$. Then x is not dominated by any element of D',which is a contradiction. Suppose if$D' \not\subset D$. Now consider a vertex xsuchthat $x \notin D$ and $x \in D'$. Without loss of generality, let us assume that $x = v_n$. Then x is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{gcer}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.5. For the star graph $G = K_{1,n-1}$,$\gamma_{gcer}(G) = n$.
Proof. Let $v_1, v_2, \dots, v_{n-1}$ are the end vertices and x is the full vertex of the star $K_{1,n-1}$. Then every subset of V(G) is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) = n$.

Theorem 2.6. For the fan graph $G = K_1 + P_{n-1}$, $\gamma_{gcer}(G) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} \end{cases}$
Proof: Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$.
Case(i) Let n be oddwhere$n = 2k + 1 (k \ge 1)$. Let $D = \{v_1, v_3, \dots, v_{n-2}, v_n\}$. Then D is the geodetic certified dominating set of G so that $\gamma_{gcer}(G) \le \left\lceil \frac{n}{2} \right\rceil$. Here, it is to be established that $\gamma_{gcer}(G) = \left\lceil \frac{n}{2} \right\rceil$. On the contrary, suppose that $\gamma_{gcer}(G) \le \left\lceil \frac{n}{2} \right\rceil - 1$. Then there exists a geodetic certified dominating set D' of G such that $|D'| \le \left\lceil \frac{n}{2} \right\rceil - 1$. Assume that $D' \subset D$. Let's consider a vertexx suchthat $x \in D$ and $x \notin D'$. Without loss of generality,assume that $x = v_n$. Then x is not dominated by any element of D',which is a contradiction. Suppose if $D' \not\subset D$. Let's consider a vertex x suchthat $x \notin D$ and $x \in D'$. Without loss of generality, let us assume that $x = v_{n-1}$. Then x is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{geecer}(G) = \left\lceil \frac{n}{2} \right\rceil$.
Case(ii) Let n be even where $n = 2k + 2 (k \ge 1)$. Let $D = \{x, v_1, v_3, \dots, v_n\}$. Then D is the geodeticcertifieddominating set of G so that$\gamma_{gcer}(G) \le n + 1$. Here, it is to be established that $\gamma_{gcer}(G) = n + 1$ .On the contrary, suppose that $\gamma_{gcer}(G) \le n$.Then there exists a geodetic certified dominating set D' of G such that $|D'| \le n$. Assume that $D' \subset D$. Let's consider a vertex x suchthat $x \in D$ and $x \notin D'$. Without loss of generality, assume that $x = v_n$. Then x is not dominated by any element of D',which is a contradiction. Suppose if$D' \not\subset D$. Let's consider a vertex x suchthat $x \notin D$ and $x \in D'$. Without loss of generality, let us assume that $x = v_n$. Then x is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{gcer}(G) = n + 1$.

Observations

(i) For the complete graph $G = K_n (n \ge 3)$, $\gamma_{gcer}(G) = n$.
(ii) For the helm graph $G = H_n (n \ge 3)$, $\gamma_{gcer}(G) = n$.
(iii) For the bistar graph $G = B_{m,n} (n \ge 3)$, $\gamma_{gcer}(G) = n$.

CONCLUSION

In this article, the geodetic certified domination number of graphs has been introduced and some theorems are derived and analyzed by applying the concepts to various graphs. This study can be extended to other certified domination parameters.

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