Stability Analysis of $SIR\langle^{I}_{S}$ Epidemic Model with Time Delay In the Interaction of Susceptible and Infected Individuals

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ABSTRACT
In this paper, athree-compartment $SIR\langle^{I}_{S}$epidemic model with time delay introduced in the interaction of susceptibleand infected individuals is considered and discussed local, global stability at its equilibrium points. Hopf bifurcation is used to identify the point at which system becomes unstable to stable. Numerical simulation is carried out to support the results using MATLAB. Numerical examples are presented in support of the increase in the transmission rate at a particular critical time delay parameter the system becomes unstable. Further, observed that due to increase in the additional transaction rate subject to fixed time delay, the system becoming unstable.

Keywords: Re-infection rate, Additional transaction rate, Disease free equilibrium point, Endemic equilibrium point,Time Delay, Hopf bifurcation.

1. INTRODUCTION

A collection of methods known as "epidemic modelling" involve the use of computational, statistical, and mathematical tools to investigate the transmission of infectious diseases within host populations. It makes use of information and theories to explain how diseases are transmitted, how populations change over time, and how they affect people's health [7]. Daniel Bernoulli began researching mathematical models of epidemics in 1766. Eventually, in 1927, Kermack and Mc Kendrick [8] used a system of differential equations to explain the dynamics of disease transmission. In epidemiology, mathematical modeling provides a broad understanding of the factors influencing the spread of a disease. Differential equations play a vital role in dealing the models and are useful tool for stability analysis. The population under consideration in an epidemiological model can be classified into many classes that vary over time (t).These fall into three categories: susceptible (S(t)), infective (I(t)), and removable (R(t)).Those within the population who actively spread the sickness to others are considered infectious class. Populations that are susceptible are those who have not yet contracted the illness, while the removable class consists of those who have either recovered, were placed in isolation, or have passed away. The average number of new infections generated by each infected person is called the basic reproductive number and it is denoted by R_0. The infection does not spread if $R_0 < 1$ and if $R_0 > 1$, the disease will spread.

The most effective and thoroughly researched epidemiological models are the SIS and SIR models, among many others. Appa Rao D, Kalesha Vali S, et al. [1, 2, 3, 4, 5, 6] has been thoroughly examined the stability of these SIR models. Later, Divya kumari G, Kalesha Vali S et al.[13] has been studied the Stability Analysis of SIR Epidemic model under Vaccination Coverage on newborns with time delay in the interaction of Susceptible and Infected Individuals. More complexity can be added to the SIR model in several ways, including byage structure, regional heterogeneity, or several infection stages. One such extension is the SIRI model. These partial immunity models, also referred to as (Susceptible - infected - recovered - infected) SIRI models, are explored in the book "Population Biology and Criticality" [9,10]. In this model the recovered individuals Tudor was the one who initially developed the SIRI epidemiological model in a community [12]. Subsequently, an enormous number of mathematicians and scientists began working on SIRI epidemic models in an effort to comprehend the idea of disease transmission. Time delays are incorporated to provide for multiple factors that impact the disease's course within the population. The stability study of the SIRI epidemic model with reintroduction to susceptibles were discussed by Kanaka Maha Lakshmi E, Kalesha Vali S, et al. [11] under the assumption of the population is constant using analytical techniques. Also, discussed delay dynamics of the same model by incorporating delay on susceptible individuals and presented that the increase in the transmission rate the system becomes unstable [14].

In this paper, a study is made to examine the stability of $SIR\left\langle{}^{I}_{S}\right.$ epidemic model when time delay is introduced in the susceptible and infective populations. It is observed that the system is asymptotically stable at the endemic equilibrium point identified. Hopf bifurcation is used to identify the point at which system becomes unstable to stable. In support of the results numerical simulation is carried outusing MATLAB.

2. Basic Equations Of The Model

A three-compartment SIRI model with re-introduced susceptibles i.e., $SIR\left\langle{}^{I}_{S}\right.$ model [11] is considered inwith populations categorizes into three compartments: Susceptible (S), Infective (I),and Recovered (R). The susceptible compartment includes individuals who have not yet contacted the disease but are at risk of infection. These individuals transition to the infective compartment upon exposure to the pathogen. The infective compartment comprises individuals who have been infected and are capable of transmitting the disease to susceptible individuals. The size of this compartment varies based on the number of new infections and the rate at which infected individuals recover. The recovered compartment consists of individuals who have either recovered from the infection and gained immunity or poor/partial immunity. The individuals from the removable class or infective class may go to the susceptible class because of getting reinfection of poor immunity.

The following assumptions were made to develop the model $SIR\left\langle{}^{I}_{S}\right.$.

(a) The total human population is divided into three compartments susceptible (S), infectious (I), and removable / recovered (R).
(b) The total population N is constant, that is, the sum of the individuals in the three compartments does not change.
(c) Susceptible humans if interacted with infected humans will become infected and will go to infected class.
(d) Some exposed humans having sufficient natural immunity will recover from the infection naturally and will go to recovered class.
(e) The individuals from the removable class may go to the susceptible class because of getting reinfection The individuals from the removable class may go to the infected class because of poor immunity or not getting recovered fully from the infection.
(f) Transmission rate from susceptible (S) to infectious (I) and re-infection rate from recovered (R) to infective (I) are same.

Based on the model assumptions, the model flowchart is drawn as shown in Figure 1. This flow chart describes the flow of humans among the model compartments.

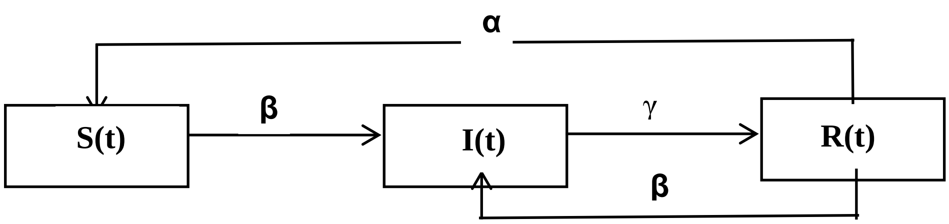


Fig 1. Flowchart showing flow of humans among the model compartments

The model variables and parameters description is detailed in the tables 1 and 2.

Table 1. Description of model variables

Variable	Description
S(t)	Susceptible population at time t
I(t)	Infected population at time t
R(t)	Recovered population at time t

Table 2. Description of model parameters

Parameter	**Description**
β	Rate of transmission of infection / infection rate or Reinfection rate
γ	Rate of the infected individual recover or recovery rate
α	Additional transition rate from recovered (R) to susceptible (S)
τ	Time delay parameter
τ_0	Critical time delay parameter or Bifurcation point

Based on the model assumptions, model flow chart and description of model variables and parameters, the system of model equations is constructed and presented in (1). The system is a group of three nonlinear ordinary differential equations

$$\frac{dS}{dt} = \alpha R - \beta S(t-\tau)I(t-\tau)$$
$$\frac{dI}{dt} = \beta S(t-\tau)I(t-\tau) - \gamma I + \beta RI$$
$$\frac{dR}{dt} = \gamma I - \alpha R - \beta RI \qquad (1)$$

3. Equilibrium points

In this section, the equilibrium points, namely a disease-free equilibrium point, and endemic equilibrium points are determined by solving the equations in (1) after equating them individually to zero

i.e., $\frac{dS}{dt} = 0$, $\frac{dI}{dt} = 0$ and $\frac{dR}{dt} = 0$.

The equilibrium points are

E_1 : Disease free equilibrium point is $E_1(S^*, I^*, R^*) = (N, 0, 0)$ (2)

E_2 : Endemic equilibrium point is $E_2(S^*, I^*, R^*) = \left(\frac{\alpha(\beta\gamma N - \gamma^2)}{\beta(\alpha+\beta N-\gamma)(\beta N-\gamma)}, \frac{\beta N-\gamma}{\beta}, \frac{\beta N\gamma-\gamma^2}{\beta(\alpha+\beta N-\gamma)}\right)$(3)

Here, the basic reproduction rate of the system (1) is identified as $R_0 = \frac{\beta N}{\gamma}$. Also it is observed that if $R_0 > 1$, the endemic equilibrium point exist and if $R_0 < 1$ the system has no feasible solution.

4. Stability at equilibrium points

In this section, we prove that the system (1) is asymptotically stable at the endemic equilibrium point E_2 subject to some constraints. Stability is studied even when time delay is introduced in the interaction of susceptible and infective individuals,identified the point at which the system becomes unstable to stable using Half bifurcation.

Theorem 4.1: The system (1) is asymptotically stable at endemic equilibrium point $E_2(S^*, I^*, R^*)$

Proof : The Jacobean matrix of the system (1) at endemic equilibrium point $E_2(S^*, I^*, R^*)$ is

$$J = \begin{bmatrix} -\beta e^{-\lambda\tau} I^* & -\beta S^* e^{-\lambda\tau} & \alpha \\ \beta e^{-\lambda\tau} I^* & \beta S^* e^{-\lambda\tau} - \gamma + \beta R^* & \beta I^* \\ 0 & \gamma - \beta R^* & -\alpha - \beta I^* \end{bmatrix} (4)$$

The characteristic equation of (4) is given by $|J - \lambda I| = 0$,where λ is a parameter is,

$$\lambda^3 + \lambda^2(\gamma + \alpha + \beta I^* - \beta R^*) + \lambda(\alpha\gamma - \alpha\beta R^*) + e^{-\lambda\tau}[\lambda^2(\beta I^* - \beta S^*) + \lambda(\beta\gamma I^* - \beta^2 R^* I^* + \alpha\beta I^* + \beta^2 I^{*2} - \alpha\beta S^* - \beta^2 S^* I^*)] = 0$$

i.e., $$\varphi(\lambda,\tau) = \lambda^3 + u_1\lambda^2 + u_2\lambda + e^{-\lambda\tau}(v_1\lambda^2 + v_2\lambda) = 0 \qquad (5)$$

Where

$$u_1 = \gamma + \alpha + \beta I^* - \beta R^*$$
$$u_2 = \alpha\gamma - \alpha\beta R^*$$

$v_1 = \beta I^* - \beta S^*$

$v_2 = \beta\gamma I^* - \beta^2 R^* I^* + \alpha\beta I^* + \beta^2 I^{*2} - \alpha\beta S^* - \beta^2 S^* I^*$

To find the condition for existence of negative real roots of (5), we get

Case (i): For $\tau = 0$,the equation (5) becomes

$$\varphi(\lambda, 0) = \lambda\Big[\lambda^2 + \lambda(\gamma + \alpha + \beta I^* + \beta(I^* - (S^* + R^*))) + \Big(\alpha\gamma + \beta\gamma I^* + \alpha\beta\big(I^* - (S^* + R^*)\big) + \beta^2 I^*\big(I^* - (S^* + R^*)\big)\Big)\Big] = 0 \qquad (6)$$

Which gives that

$\lambda = 0$ or

$$\lambda^2 + \lambda(\gamma + \alpha + \beta I^* + \beta(I^* - (S^* + R^*))) + \Big(\alpha\gamma + \beta\gamma I^* + \alpha\beta\big(I^* - (S^* + R^*)\big) + \beta^2 I^*\big(I^* - (S^* + R^*)\big)\Big) = 0(7)$$

Thus, one of the roots of (6) is zero and the system is stable if equation (7) possesses negative real roots.
Here $\frac{-(\gamma+\alpha+\beta I^*+\beta(I^*-(S^*+R^*)))}{1} < 0$ if $2I^* > N$ and $\frac{(\alpha\gamma+\beta\gamma I^*+\alpha\beta(I^*-(S^*+R^*))+\beta^2 I^*(I^*-(S^*+R^*)))}{1} > 0$ if $2I^* > N$.
Therefore, the system (1) is locally asymptotically stable at the endemic equilibrium point $E_2(S^*, I^*, R^*)$, if $2I^* > N$.
Case (ii): For $\tau > 0$, there exists a positiveτ_0 such that the equation (5) has pair of purely imaginary roots, and
can be taken as $\pm i\omega$, $\omega > 0$
Since equation (5) is a transcendental equation, Routh – Hurwitz criterion cannot be applied to find the roots of the equation. So, by Rouche's theorem, the transcendental equation shall have positive real part only when the equation has purely imaginary roots.
Let $\lambda = \pm i\omega$ be a purely imaginary roots of the equation (5)
Then (5) becomes$(i\omega)^3 + u_1(i\omega)^2 + u_2(i\omega) + e^{-i\omega\tau}(v_1(i\omega)^2 + v_2(i\omega)) = 0$
Which imply $-\omega^2 u_1 - v_1\omega^2 cos\omega\tau + v_2\omega sin\omega\tau + i(-\omega^3 + \omega v_2 + v_2\omega cos\omega\tau + v_1\omega^2 sin\omega\tau) = 0$
On collecting real and imaginary parts, we get

$$v_2\omega sin\omega\tau - v_1\omega^2 cos\omega\tau = \omega^2 u_1 \quad (8)$$

$$v_2\omega cos\omega\tau + v_1\omega^2 sin\omega\tau = \omega^3 - \omega u_2 \quad (9)$$

On adding, the above two equations after squaring, we get

$$(v_1\omega^2)^2 + (v_2\omega)^2 = (\omega^2 u_1)^2 + (\omega^3 - \omega u_2)^2$$

(10)
i.e., $\omega^6 + \omega^4({u_1}^2 - 2u_2 - {v_1}^2) + \omega^2({u_2}^2 - {v_2}^2) = 0$
Let $\psi(t) = t^3 + t^2 M_1 + tM_2 = 0$, Where $M_1 = {u_1}^2 - 2u_2 - {v_1}^2, M_2 = {u_2}^2 - {v_2}^2, t = \omega^2$ (11)
Thus, $\psi(t) = 0$
If $M_1 > 0, M_2 > 0$ then equation (11) will have no positive real roots
Therefore, the equation (11) admits negative real roots. Hence, we can derive the conditions for existences of stability at endemic equilibrium point.

Theorem 4.2: The system is locally asymptotically stable for allτ, at endemic equilibrium E_2 if following conditions hold.
(i). $R_0 > 1$
(ii). $(u_1 + v_1) > 0, (u_2 + v_2) > 0$
(iii). $M_1 > 0, M_2 > 0$
Proof: Let either of$M_1, M_2$is negative then equation (11) shall have a positive root ω_0
From equation (10), we have

$$cos\omega\tau = \frac{\begin{vmatrix} \omega^2 u_1 & v_2\omega \\ \omega^3 - \omega u_2 & v_1\omega^2 \end{vmatrix}}{\begin{vmatrix} -v_1\omega^2 & v_2\omega \\ v_2\omega & v_1\omega^2 \end{vmatrix}} \text{ (by cramer's rule)}$$

i.e., $cos\omega\tau = \frac{\omega^4(v_2 - u_1 v_1) - \omega^2 u_2 v_2}{{v_1}^2\omega^4 + {v_2}^2\omega^2}$
or $\tau_k = \frac{1}{\omega_0} cos^{-1}\left[\frac{{\omega_0}^2(v_2 - u_1 v_1) - u_2 v_2}{{v_1}^2{\omega_0}^2 + {v_2}^2}\right] + \frac{2k\pi}{\omega_0}$. Where k=0, 1, 2, 3 ……..

5. Hopf bifurcation

A critical point where the system stability switches, and periodic solution arise is called Hopf bifurcation. Generally, the system losses its stability due to unbounded periodic oscillations arise at this critical point. Hopf bifurcation takes place when a pair of complex conjugate eigen values cross the imaginary axis around the equilibrium points. Hopf bifurcation exists for the system when it is characterized by the system of ordinary differential equations.The following theorem establishes the existence of Hopf bifurcation for the three-dimensional system with time delay.

Theorem 5.1: If $R_0 > 1$ there exist a positive τ_0 such that the following results hold
(i) If $0 < \tau < \tau_0$, the system of equations (1) is locally asymptotically stable at endemic equilibrium point E_2
(ii) The system (1) exhibits Hopf bifurcation if $\tau > \tau_0$
Proof: To obtain Hopf bifurcation, we need to check the transversal condition for the existence of complex eigen value at $\tau = \tau_0$
i.e., the real part of $\lambda(\tau)$ become positive when $\tau > \tau_0$. At this stage, the stable state become unstable. That is the system exhibits Hopf bifurcation when τcrosses the critical valueτ_0.

Differentiating (5) w .r t. τ,

$$3\lambda^2\frac{d\lambda}{d\tau}+2u_1\lambda\frac{d\lambda}{d\tau}+u_2\frac{d\lambda}{d\tau}+e^{-\lambda\tau}\left(2v_1\lambda\frac{d\lambda}{d\tau}+v_2\frac{d\lambda}{d\tau}\right)+(v_1\lambda^2+v_2\lambda)\left(-\lambda-\tau\frac{d\lambda}{d\tau}\right)e^{-\lambda\tau}=0$$

i.e., $\frac{d\lambda}{d\tau}\left[3\lambda^2+2u_1\lambda+u_2+e^{-\lambda\tau}(2v_1\lambda+v_2)-(v_1\lambda^2+v_2\lambda)\tau e^{-\lambda\tau}\right]=(v_1\lambda^2+v_2\lambda)\lambda e^{-\lambda\tau}$

$$or\ \left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{\left[3\lambda^2+2u_1\lambda+u_2+e^{-\lambda\tau}(2v_1\lambda+v_2)-(v_1\lambda^2+v_2\lambda)\tau e^{-\lambda\tau}\right]}{(v_1\lambda^2+v_2\lambda)\lambda e^{-\lambda\tau}}$$

At $\lambda=i\omega_0$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{1}{\omega_0}\left[\frac{-3\omega_0{}^2+2iu_1\omega_0+u_2}{(-\omega_0{}^3+u_2\omega_0+i(u_1\omega_0{}^2)}+\frac{(2iv_1\omega_0+v_2)}{-v_2\omega_0+i(-v_1\omega_0{}^2)}+i\tau\right]$$

Now, real part of $\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{1}{\omega_0}\left[\frac{(-3\omega_0{}^2+u_2)(-\omega_0{}^3+u_2\omega_0)+2u_1\omega_0(u_1\omega_0{}^2)}{(-\omega_0{}^3+u_2\omega_0)^2+(u_1\omega_0{}^2)^2}+\frac{\left(-v_2{}^2\omega_0+2v_1\omega_0(-v_1\omega_0{}^2)\right)}{(v_2\omega_0)^2+(v_1\omega_0{}^2)^2}\right]$

$$i.e., (v_1\omega_0{}^2)^2+(v_2\omega_0)^2=(\omega_0{}^2u_1)^2+(-\omega_0{}^3+\omega_0u_2)^2$$

Which imply real part of $\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{1}{\omega_0}\left[\frac{3\omega_0{}^5+\omega_0{}^3(2u_1{}^2-u_2-3u_2-2v_1{}^2)+(u_2{}^2-v_2{}^2)\omega_0}{(v_2\omega_0)^2+(v_1\omega_0{}^2)^2}\right]$

$$or\ \left[\frac{d}{d\tau}Re(\lambda)\right]=\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0}=\left[\frac{3\omega_0{}^4+\omega_0{}^2(2u_1{}^2-4u_2-2v_1{}^2)+(u_2{}^2-v_2{}^2)}{(v_2\omega_0)^2+(v_1\omega_0{}^2)^2}\right]$$

If$u_1{}^2-2u_2-v_1{}^2>0\ and\ u_2{}^2-v_2{}^2>0$,

then$\left[\frac{d}{d\tau}Re(\lambda)\right]_{\lambda=i\omega_0}>0$

Therefore, the transversal condition holds and Hopfbifurcation exists at$\tau=\tau_0$ i.e., the system switches unstable to stable at τ_0. Thisτ_0 is called critical time delay parameter.

6. Numerical simulation

In this section, numerical simulation for different sets of parametric values is carried out using MATLAB. Twenty-four examples (6.1 to 6.24) are considered to analyse and ascertain our results. Each example consists of two types of graphical representations: Time series responses (Figure A) and Phase Portraits (Figure B). These plots show how the populations susceptible, infective, and recovered individuals change over time and phase portraits provide a phase-space representation of the dynamics, illustrating the trajectories of the system in the (S, I) or (I, R) planes. Time series responses and phase portraits help to visualize the stability and convergence behaviour of the model. For all the examples, S, I, R values are fixed and considered N=60, $S=25, I=31, R=4$.

The following examples (6.1-6.12) illustrates the existence of the critical time delay parameter τ_0(bifurcation point) for three sets of parametric values in which the transmission rate β, recovery rate γ varies separately or both at a time subject to the assumption, all the parameters remaining are fixed constant.

Example 6.1: For the parametric values$\beta=0.02, \gamma=0.5, \alpha=0.7, \tau=3.97>\tau_0=3.3$.

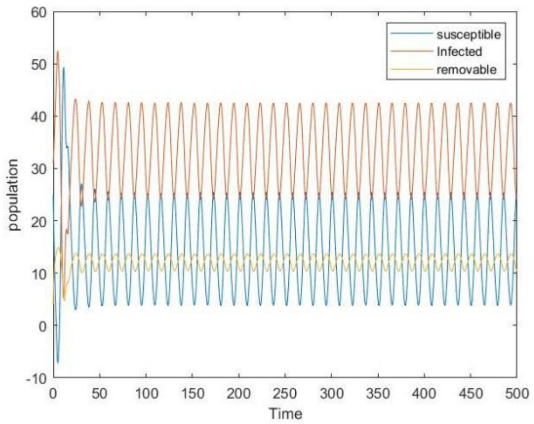


Fig.6.1-A

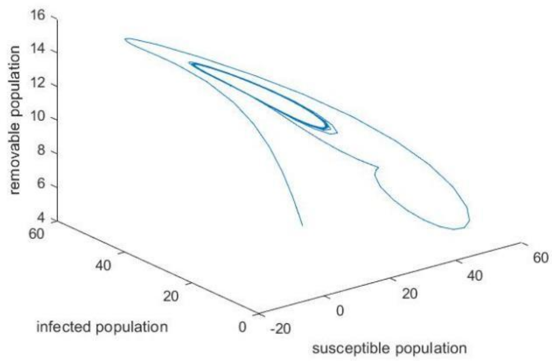


Fig.6.1-B

The system is unstable when $\tau>\tau_0$.

Example 6.2: For the parametric values, $\beta = 0.02, \gamma = 0.5, \alpha = 0.7, \tau = 3.38 > \tau_0 = 3.3$.

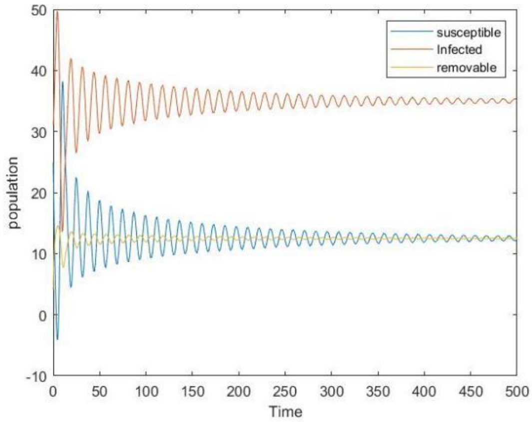


Fig.6.2-A

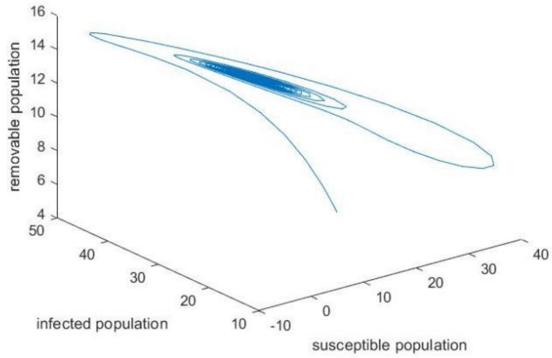


Fig.6.2-B

The system is unstable when $\tau > \tau_0$.

Example 6.3: For the parametric values, $\beta = 0.02, \gamma = 0.5, \alpha = 0.7$.

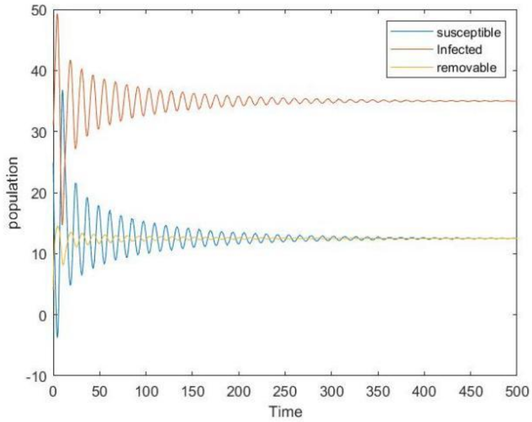


Fig.6.3-A

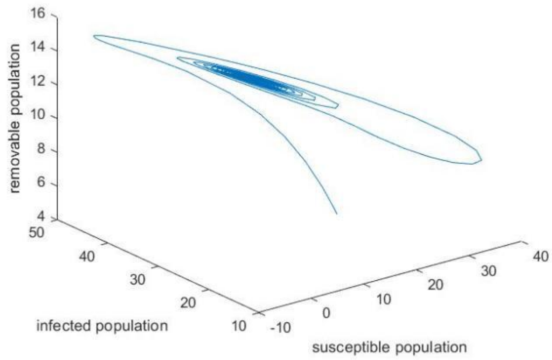


Fig.6.3-B

The critical time delay parameter τ_0 is 3.3

Example 6.4: For the parametric values, $\beta = 0.02, \gamma = 0.5, \alpha = 0.7, \tau = 3.24 < \tau_0 = 3.3$.

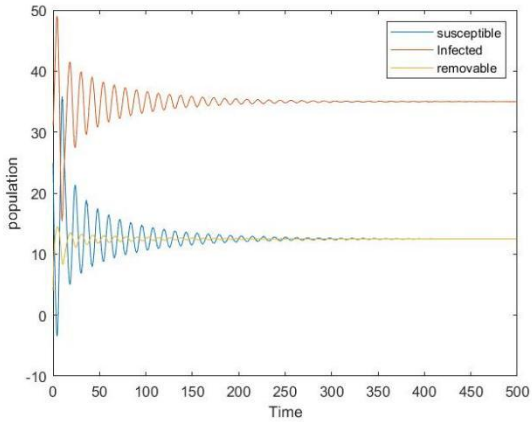


Fig.6.4-A

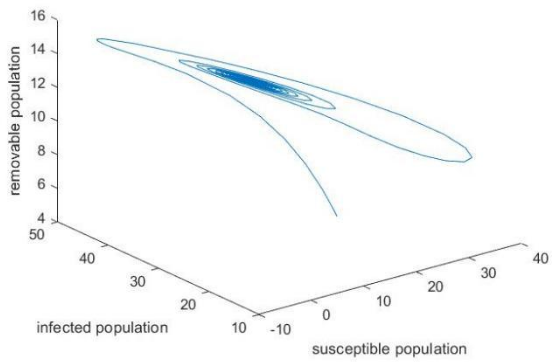


Fig.6.4-B

Here, the system is asymptotically stable and converges to the fixed equilibrium point $E(12,35,13)$ at $\tau = 3.24 < \tau_0 = 3.3$. Thus, the examples 6.1-6.4, illustrates the existence of the critical time delay parameter τ_0 and it is identified as3.3.

Example 6.5: For the parametric values$\beta = 0.018, \gamma = 0.3, \alpha = 0.7, \tau = 2.9 > \tau_0 = 2.39$.

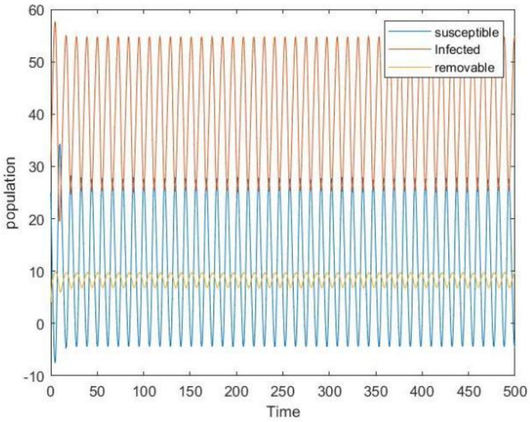


Fig.6.5-A

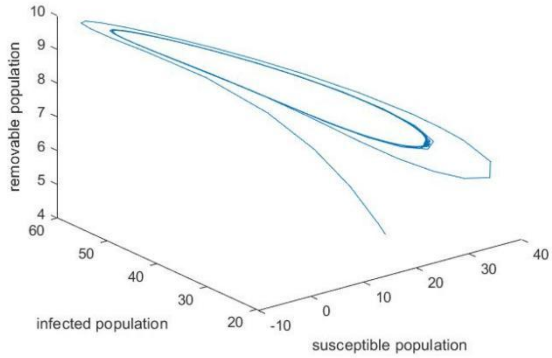


Fig.6.5-B

The system is unstable when$\tau > \tau_0$.

Example 6.6: For the parametric values$\beta = 0.018, \gamma = 0.3, \alpha = 0.7, \tau = 2.65 > \tau_0 = 2.39$.

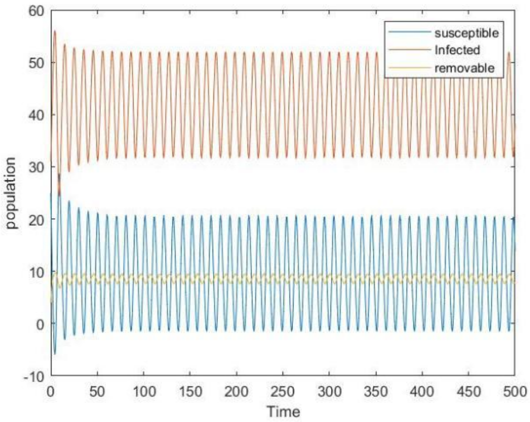


Fig.6.6-A

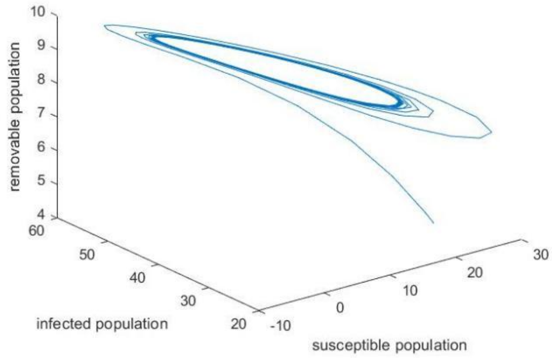


Fig.6.6-B

The system is unstable when $\tau > \tau_0$.

Example 6.7: For the parametric values$\beta = 0.018, \gamma = 0.3, \alpha = 0.7$.

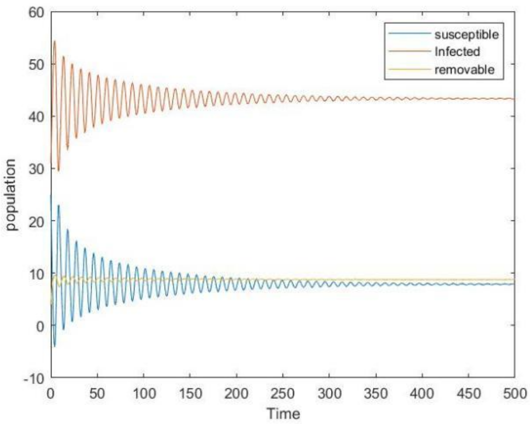


Fig.6.7-A

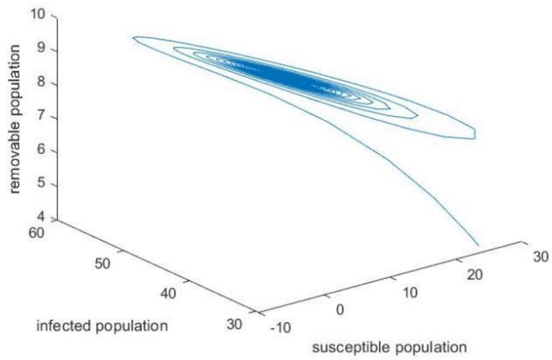


Fig.6.7-B

The critical time delay parameter τ_0 is 2.39.

Example 6.8: For the parametric values, $\beta = 0.018, \gamma = 0.3, \alpha = 0.7, \tau = 2.36 < \tau_0 = 2.39$.

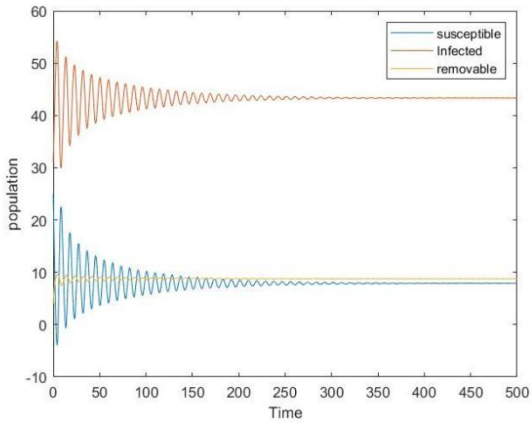


Fig.6.8-A

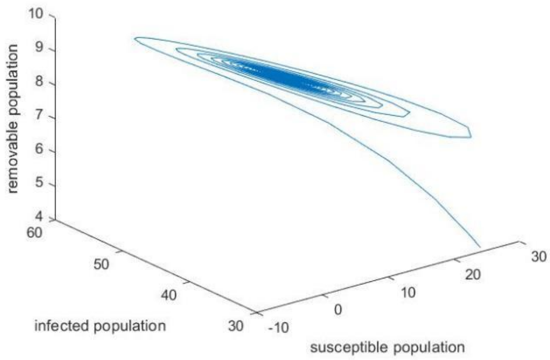


Fig.6.8-B

The system is asymptotically stable and converges to the fixed equilibrium point $E(8,43,9)$ at $\tau = 2.36 < \tau_0 = 2.39$. Thus, the examples 6.5- 6.8, illustrates the existence of the critical time delay parameter τ_0 and it is identified as 2.39.

Example 6.9: For the parametric values $\beta = 0.02, \gamma = 0.3, \alpha = 0.7, \tau = 2.5 > \tau_0 = 1.99$.

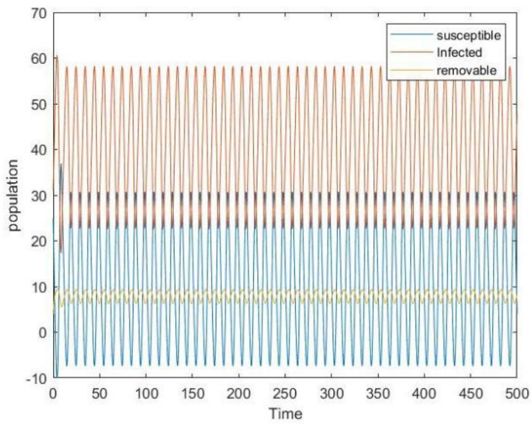


Fig.6.9-A

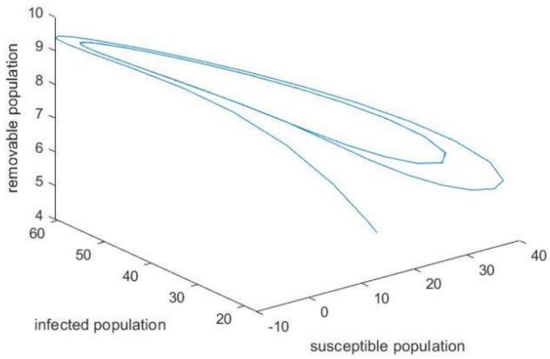


Fig.6.9-B

The system is unstable when $\tau > \tau_0$.

Example 6.10: For the parametric values $\beta = 0.02, \gamma = 0.3, \alpha = 0.7, \tau = 2.1 > \tau_0 = 1.99$.

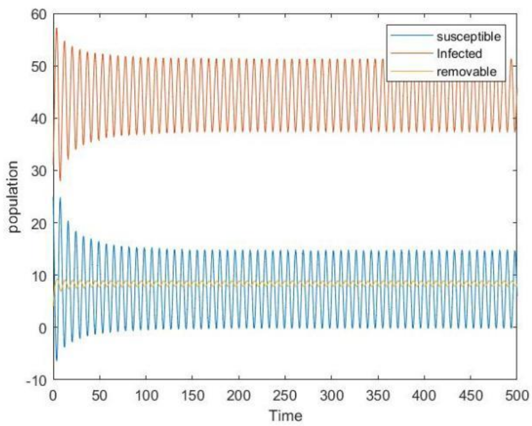


Fig.6.10-A

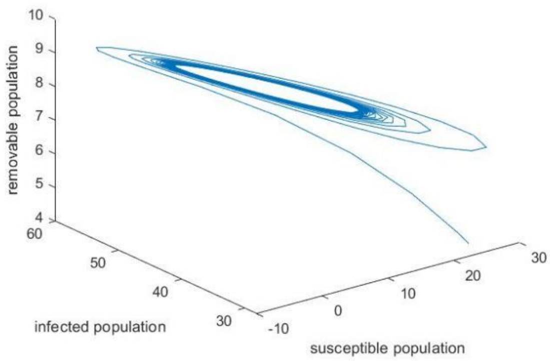


Fig.6.10-B

The system is unstable when $\tau > \tau_0$.

Example 6.11: For the parametric values $\beta = 0.02, \gamma = 0.3, \alpha = 0.7$.

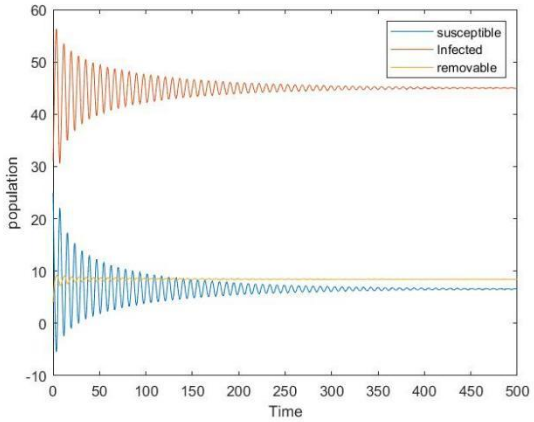


Fig.6.11-A

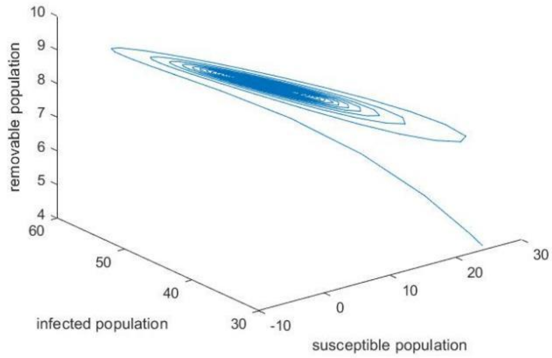


Fig.6.11-B

The critical time delay parameter τ_0 is 1.99.

Example 6.12: For the parametric values, $\beta = 0.02, \gamma = 0.3, \alpha = 0.7, \tau = 1.9 < \tau_0 = 1.99$.

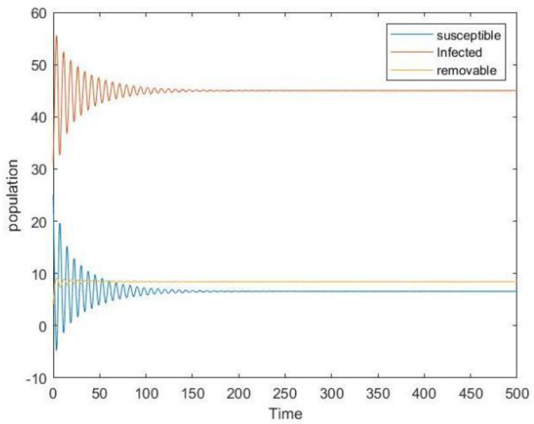


Fig.6.12-A

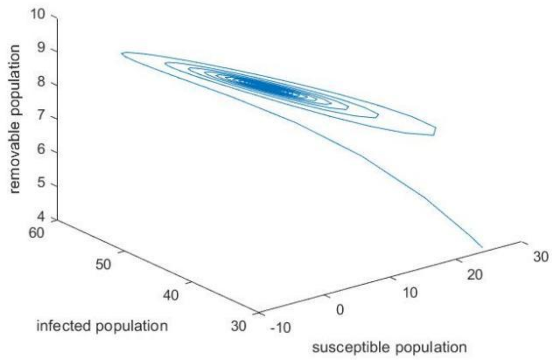


Fig.6.12-B

The system is asymptotically stable and converges to the fixed equilibrium point $E(7,45,8)$ at $\tau = 1.9 < \tau_0 = 1.99$. Thus, the examples 6.9-6.12, illustrates the existence of the critical time delay parameter τ_0 and is identified as 1.99.

The parametric values and the corresponding critical time delay parameters τ_0 (bifurcation points) are tabulated in the table 3.

Table 3. critical time delay parameters for different set of parametric values

S. No	Example	Parametric values	Critical Bifurcation Parameter (τ_0)
1	6.1-6.4	$N = 60, S = 25, I = 31, R = 4, \beta = 0.02, \gamma = 0.5, \alpha = 0.7$	$\tau_0 = 3.3$
2	6.5–6.8	$N = 60, S = 25, I = 31, R = 4, \beta = 0.018, \gamma = 0.3, \alpha = 0.7$	$\tau_0 = 2.39$
3	6.9-6.12	$N = 60, S = 25, I = 31, R = 4, \beta = 0.02, \gamma = 0.3, \alpha = 0.7$	$\tau_0 = 1.99$

For the above three sets of parametric values, the critical time delay parameter τ_0 evaluated theoretically and validated with the simulated values and observed that they are very closure. Validation of theoretical findings through and numerical simulation of critical time delay parameter $\tau_0$is shownin the table 4.

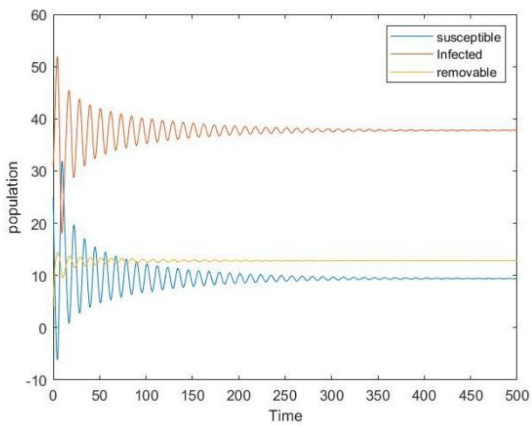
Table 4. Validation of theoretical finding through and numerical simulation of critical time delay parameter

Parameter values	$N = 60, S = 12, I = 35, R = 13, \beta = 0.02, \gamma = 0.5, \alpha = 0.7$	$N = 60, S = 8, I = 43, R = 9, \beta = 0.018, \gamma = 0.3, \alpha = 0.7$	$N = 60, S = 7, I = 45, R = 8, \beta = 0.02, \gamma = 0.3, \alpha = 0.7$
u_1	1.64	1.612	1.74
u_2	0.168	0.0966	0.098
v_1	0.46	0.63	0.76
v_2	0.812	1.035432	1.342
N_1	2.142	2.008444	2.254
N_2	-0.63112	-1.062788	-1.79136
ω_0	0.513	0.6595	0.7891
τ_0 (Theoretical)	3.39	2.49	2.06
τ_0 (Graphical)	3.3	2.39	1.99

τ_0

In the following examples (6.13-6.18), considered two sets of parametric values in which the transmission rate β is varied to ascertain the unstability of the system at fixed critical time delay parameter τ_0.

Example 6.13: For the parametric values $\beta = 0.018, \gamma = 0.4, \alpha = 0.5$.



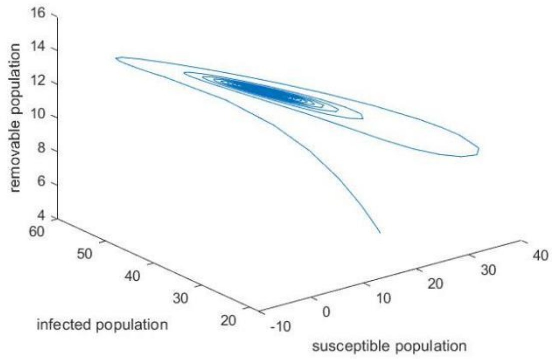
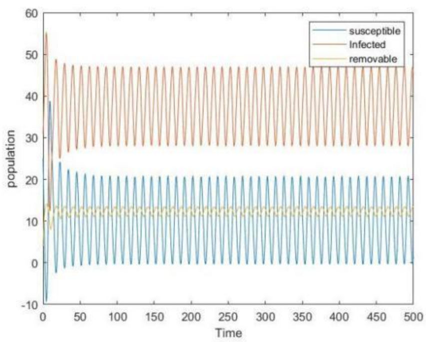


Fig.6.13-A Fig.6.13-B

The critical time delay parameter is $\tau_0 = 2.9$.

Example 6.14: For the parametric values $\beta = 0.019, \gamma = 0.4, \alpha = 0.5, \tau_0 = 2.9$.



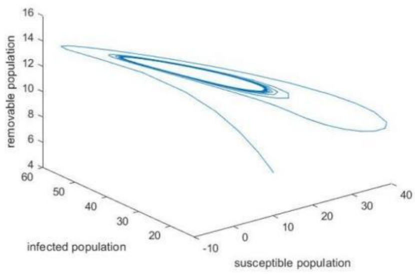


Fig.6.14-A Fig.6.14-B

The system becomes unstable.

Example 6.15: For the parametric values $\beta = 0.02, \gamma = 0.4, \alpha = 0.5, \tau_0 = 2.9$.

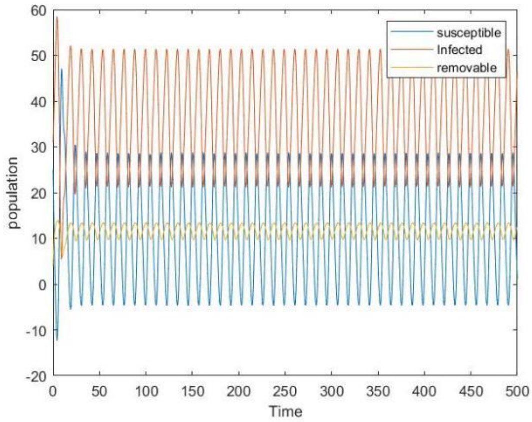


Fig.6.15-A

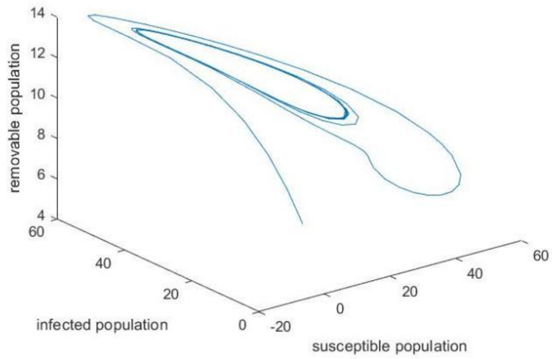


Fig.6.15-B

The system becomes unstable.
From the examples (6.13-6.15), it is understood that due to the increase in transmission rate (β) at critical time delay parameter$\tau_0 = 2.9$, the system becomes unstable.

Example 6.16: For the parametric values $\beta = 0.015, \gamma = 0.3, \alpha = 0.7$.

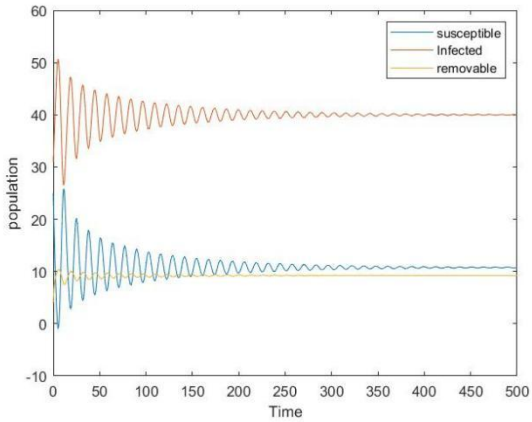


Fig.6.16-A

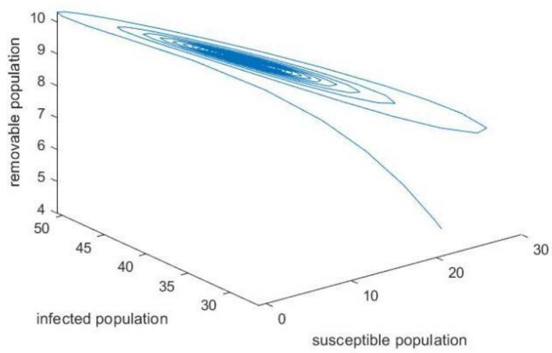


Fig.6.16-B

The critical time delay parameter is τ_0 =3.46

Example 6.17: For the parametric values $\beta = 0.016, \gamma = 0.3, \alpha = 0.7, \tau_0$ =3.46.

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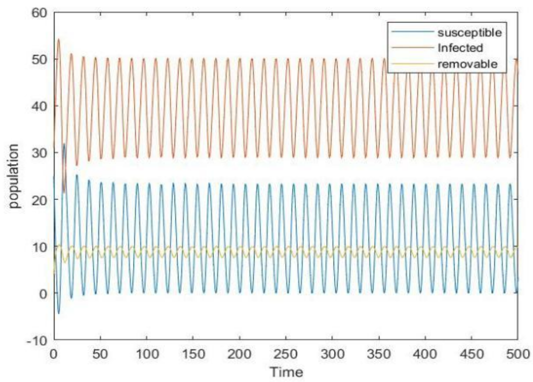

Fig.6.17-A

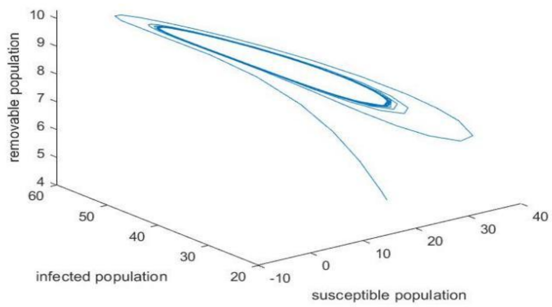

Fig.6.17-B

The system becomes unstable .

Example 6.18: For the parametric values $\beta = 0.017, \gamma = 0.3, \alpha = 0.7, \tau_0$ =3.46.

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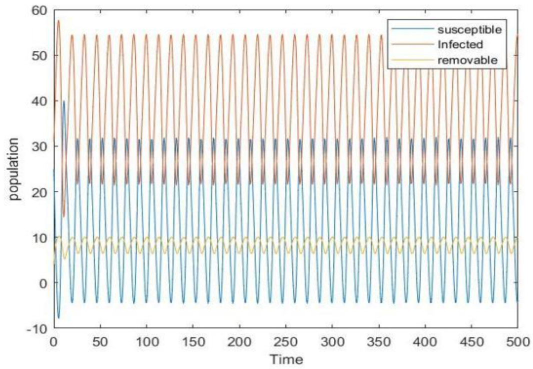

Fig.6.18-A

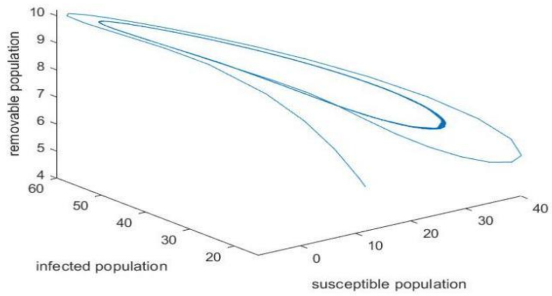

Fig.6.18-B

The system becomes unstable.
From the examples (6.16-6.18), it is understood that due to increase in the transmission rate (β) at critical time delay parameter$\tau_0 = 3.46$, the system becomes unstable.
The effects of volatile transmission rate on delay for different sets of parametric values are presented in the table 5.

Table 5: Effect of transmission rate on delay

S. No	Example	Parametric values	β	Observation
1	6.13	$N = 60, S = 25, I = 31, R = 4, \gamma = 0.4, \alpha = 0.5, \tau_0$ =2.9	0.018	$\tau_0 = 2.9$
2	6.14	$N = 60, S = 25, I = 31, R = 4, \gamma = 0.4, \alpha = 0.5, \tau_0$ =2.9	0.019	The system becomes unstable
3	6.15	$N = 60, S = 25, I = 31, R = 4, \gamma = 0.4, \alpha = 0.5, \tau_0$ =2.9	0.02	The system becomes unstable.
4	6.16	$N = 60, S = 25, I = 31, R = 4, \gamma = 0.3, \alpha = 0.7, \tau_0$ =3.46.	0.015	$\tau_0 = 3.46$
5	6.17	$N = 60, S = 25, I = 31, R = 4, \gamma = 0.3, \alpha = 0.7, \tau_0$ =3.46.	0.016	The system becomes unstable.
6	6.18	$N = 60, S = 25, I = 31, R = 4, \gamma = 0.3, \alpha = 0.7, \tau_0$ =3.46.	0.017	The system becomes unstable.

Two sets of parametric values are considered in which the additional transaction rate α is varied to ascertain the unstability of the system at fixed critical time delay parameter τ_0 in the examples (6.21-6.26).

Example 6.19: For the parametric values $\beta = 0.02, \gamma = 0.5, \alpha = 0.3$.

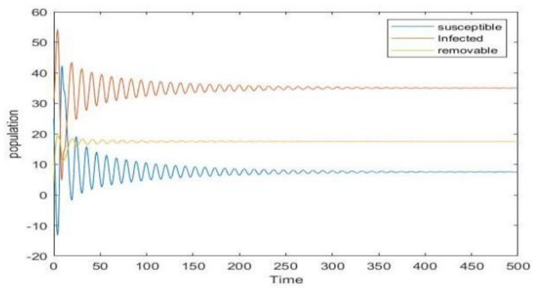

Fig.6.19-A

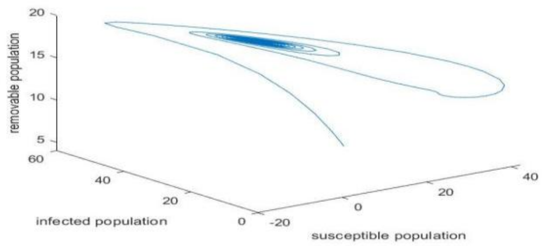

Fig.6.19-B

The critical time delay parameter is $\tau_0 = 2.68$.

Example 6.20: For the parametric values$\beta = 0.02, \gamma = 0.5, \alpha = 0.5, \tau_0 = 2.68$.

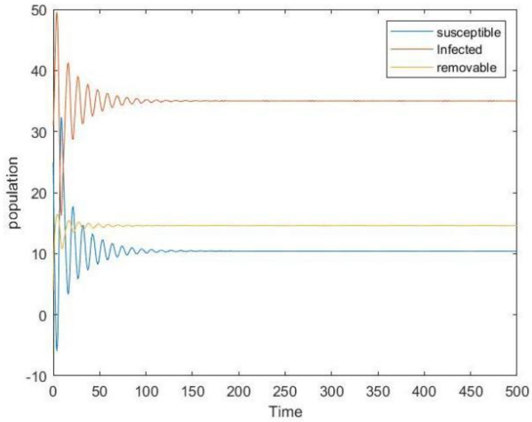

Fig.6.20-A

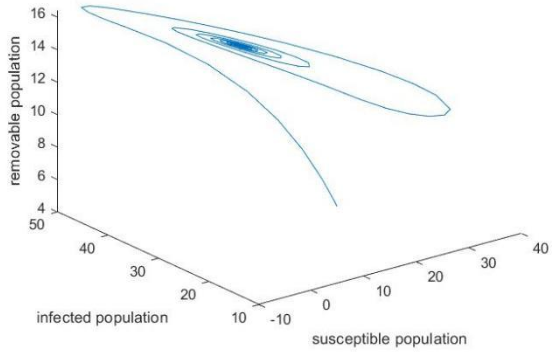

Fig.6.20-B

The system converges to the fixed equilibrium point$E(10,35,15)$.

Example 6.21: For the parametric values $\beta = 0.02, \gamma = 0.5, \alpha = 0.7, \tau_0 = 2.68$.

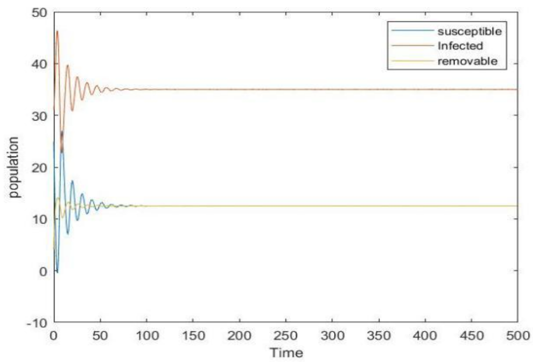

Fig.6.21-A

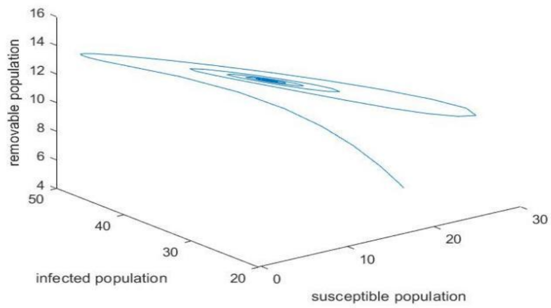

Fig.6.21-B

The system converges to the fixed equilibrium point $E(13,35,12)$.
From the examples (6.19-6.21), it is observed that due to increase in the additional transaction rate (α)at critical time delay parameter$\tau_0 = 2.68$, the system is asymptotically stable and there is significant growth in susceptible population.

Example 6.22: For the parametric values $\beta = 0.018, \gamma = 0.3, \alpha = 0.05$.

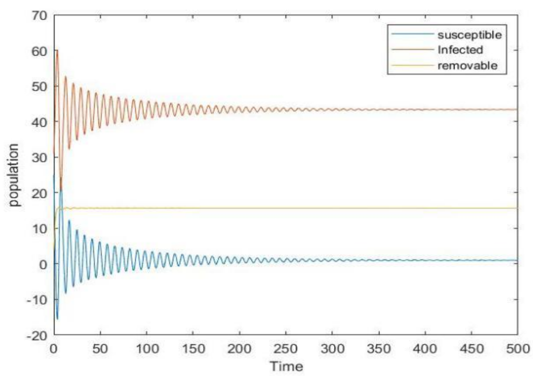

Fig.6.22-A

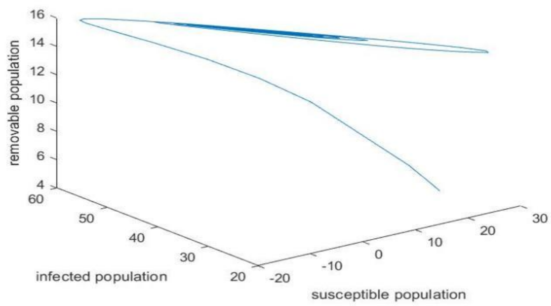

Fig.6.22-B

The critical time delay parameter is $\tau_0 = 2$

Example 6.23: For the parametric values $\beta = 0.018, \gamma = 0.3, \alpha = 0.1, \tau_0 = 2$.

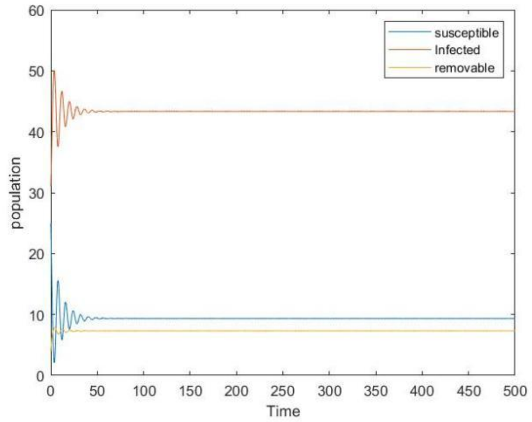

Fig.6.23-A

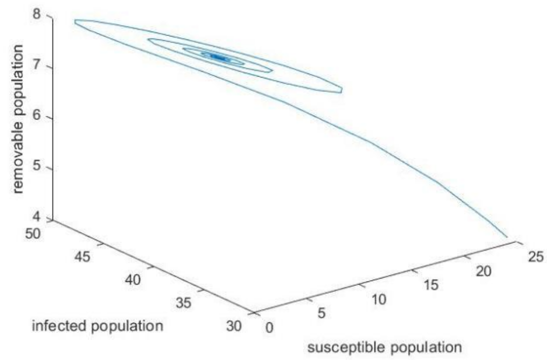

Fig.6.23-B

The system converges to the fixed equilibrium point $E(2,43,15)$.

Example 6.24: For the parametric values $\beta = 0.018, \gamma = 0.3, \alpha = 0.15, \tau_0 = 2$.

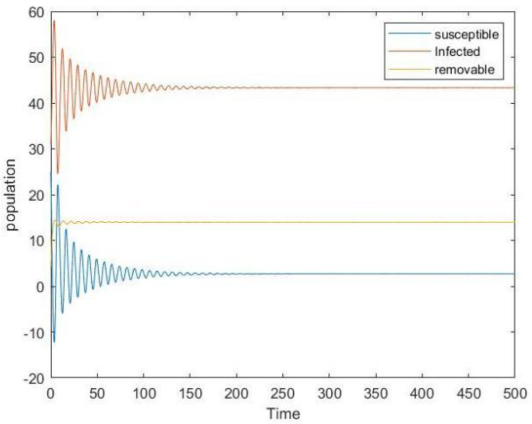

Fig.6.24-A

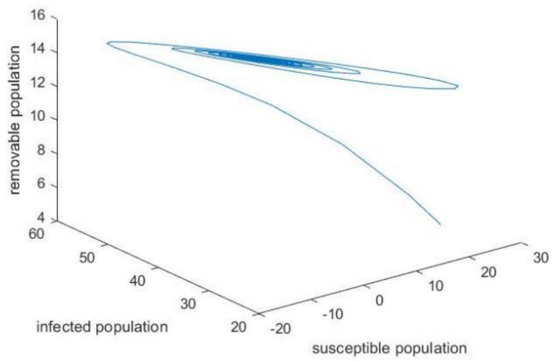

Fig.6.24-B

The system converges to the fixed equilibrium point $E(3,43,14)$.

From the examples (6.22-6.24), it is observed that due to increase in the additional transaction rate (α)at critical time delay parameter$\tau_0 = 2$, the system is asymptotically stable and there is significant growth in susceptible population.

The effects of volatile additional transaction rate (α) on delay for different sets of parametric values and are presented in the table 6.

Table 6. Effect of additional transaction rate (α) on delay

S. No	Example	Parametric values	α	Observation
1	6.18	$N = 60, S = 25, I = 31, R = 4, \beta = 0.02, \gamma = 0.5, \tau_0 = 2.68$	0.3	$\tau_0 = 2.68$
2	6.19	$N = 60, S = 25, I = 31, R = 4, \beta = 0.02, \gamma = 0.5, \tau_0 = 2.68$	0.5	The system converges to the fixed equilibrium point E(10,35,15).
3	6.21	$\mathrm{N} = 60, \mathrm{S} = 25, \mathrm{I} = 31, \mathrm{R} = 4, \beta = 0.02, \gamma = 0.5, \tau_0 = 2.68$	0.7	The system converges to the fixed equilibrium point E(13,35,12).
4	6.22	$\mathrm{N} = 60, \mathrm{S} = 25, \mathrm{I} = 31, \mathrm{R} = 4, \beta = 0.018, \gamma = 0.3, \tau_0 = 2$	0.05	$\tau_0 = 2$

5	6.23	$N = 60, S = 25, I = 31, R = 4, \beta = 0.018, \gamma = 0.3, \tau_0 = 2$	0.1	The system converges to the fixed equilibrium pointE(2,43,15).
6	6.24	$N = 60, S = 25, I = 31, R = 4, \beta = 0.018, \gamma = 0.3, \tau_0 = 2$	0.15	The system converges to the fixed equilibrium pointE(3,43,14).

7. RESULTS AND CONCLUSIONS

A three compartment $SIR\langle^{I}_{S}$ epidemic model with time delay in the interaction of susceptible and infective individuals is considered. It is observed that the model admits a disease free equilibrium point if the basic reproductive rate is less than one and the system is locally asymptotically stable if $\gamma > \beta N$. Also, the system admits an endemic equilibrium point, if the basic reproductive rate is greater than one and the system is locally asymptotically stable if $2I^* > N$. Numerical simulation is carried out to support the results using MATLAB. Obtained bifurcation points for three different sets of parametric values subject to variation in the transmission rate $\beta$and recovery rate γ independently subject to the condition that all the parameters fixed constant. Also considered two sets of parametric values to observe the effect of transmission rate β on delay and observed that due to change in the transmission rate β is varied the system is becoming unstable at fixed critical time delay parameter τ_0.Further, considered two sets of parametric values to observe the effect of additional transaction rate $\alpha$and observed that due to increase in the additional transaction rate α the system is becoming stable at fixed critical time delay parameter τ_0 and there is significant growth in the susceptible individuals. Further, validated the theoretical values of critical time delay parameter τ_0 through numerical simulation.

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