Optimal Replacement Model for a Deteriorating System under Partial Sum Process

I. Ashwini[1], Zahiruddeen[2], U. Rizwan[3*]

[1,2,3]Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi, Affiliated to Thiruvalluvar University, Vellore
Email: dr.u.rizwan@gmail.com
*Corresponding Author

Received: 17.04.2024 Revised: 18.05.2024 Accepted: 28.05.2024

ABSTRACT
In this paper, the maintenance model for a deteriorating system under partial sum process is studied. Whenever a failure arrives, the system operating time is reduced. Assuming that successive operating times after repairs form a decreasing Partial Sum Process and also consecutive repair times of the system after failures form an increasing Geometric Process, a replacement policy T, by which we replace the system whenever the working age of the system reaches T, is adopted. An explicit expression for the long run average cost rate per unit per unit time is derived. Optimality conditions are deduced. Numerical illustration is included to strengthen the theoretical results devolved.

Keyeords: Partial Sum Process (PSP), Geometric Process (GP), Renewal Process (RP),Replacement policy.

1. INTRODUCTION

At initial stage the common assumption made after repair, in modelling the operating and repair times is that the system is as good as new, but this is not always true for a deteriorating system in a real situation due to accumulated wear and ageing effect. Barlow and Proschan [1975] introduced the minimal repair model in which failed item after repair will have the same failure rate and same effective age at the time of the failure. Lam Y., [1988a] presented a geometric process repair model to model a deteriorating system under some univariate replacement policy.

Definition 1. Given two random variables X and Y, X is said to be stochastically smaller than Y (or Y is stochastically greater than X), if $P(X > a) \le P(Y > a)$, for real a.

Definition 2. A stochastic process $\{X_n,\ n = 1,2,3,\ldots\}$ is said to be stochastically decreasing(or increasing) $X_n \ge_{st} (\le_{st}) X_{n+1}$.

Definition 3. Given a sequence of non-negative random variables $\{X_n,\ n = 1,2,3,\ldots\}$, they are independent and the cdf of X_n is given by $F\left(a^{k-1}x\right)$ for $k = 1,2,3,\ldots,$ where a is a positive constant, then $\{X_n,\ n = 1,2,3,\ldots\}$ is called a geometric process (GP).

Definition 4. Let $\{X_n,\ n = 1,2,3,\ldots\}$ be a sequence of independent non negative random variables and let $F(x)$ be the distribution of X_1. Then $\{X_n,\ n = 1,2,3,\ldots\}$ is called partial sum process, if the distribution of X_{i+1} is $F(\beta_i x)\ (i = 1,2,3,\ldots)$, where β_i >0 are constants with $\beta_i = \beta_0 + \beta_1 + \beta_2 + \ldots \beta_{i-1}$ and $\beta_0 = \beta > 0$.

Lemma 1. The partial sum process $\{X_n,\ n = 1,2,3,\ldots\}$ is stochastically decreasing and hence it is a monotone process.

Lemma 2. Let $E[X_i] = \mu,$ then for $i = 1,2,3,\ldots$ $E[X_{i+1}] = \dfrac{\mu}{2^{i-1}\beta}$ and $\operatorname{var}[X_{i+1}] = \left(\dfrac{\sigma}{2^{i-1}\beta}\right)^2$.

2. Model Assumptions

Under the replacement policy T, the problem is to determine an optimal replacement policy T^* such that long run average cost is minimized. We make the following assumptions for the maintenance model of a deteriorating system.

A1 : Initially a new system is installed. Whenever the system fails, it is either repairedor replaced by an identical new one.

A2 : Let X_1 be the operating time before the first failure and let $F(x)$ be the distribution function of X_1. $E[X_1]=\mu>0$. Let X_{i+1} be the operating time after the $i-$th failure, for $i=1,2,3,\ldots$. Then the distribution function of X_{i+1} is $F\left(2^{i-1}\beta x\right)$, where $\beta>0$ and $E[X_{i+1}]=\dfrac{\mu}{2^{i-1}\beta}$ for $i=1,2,3,\ldots$. The successive operating times after repairs $\{X_n,\ n=1,2,3,\ldots\}$ follow partial sum process.

A3 : After the first failure, let Y_1 be the repair time and let $G(y)$ be the distribution of Y_1. Assume that $E[Y_1]=\xi>0$. Let Y_n be the repair time and let the distribution function of Y_n $G\left(a^{n-1}x\right)$, where $0\le a\le 1$ is constant, that is, the successive repair times $\{Y_n,\ n=1,2,3,\ldots\}$ form an increasing geometric process. By Lam Y., (1988a), $E[Y_i]=\dfrac{\xi}{a^{i-1}}$, $i=1,2,3,\ldots$.

A4 : The working age T of a system at time t is the cumulative life time given by

$$T(t)=\ T_n=\begin{cases} t-V_n & U_n+V_n\le t\le U_{n+1}+V_n \\ U_{n+1} & U_{n+1}+V_n\le t\le U_{n+1}+V_{n+1}, \end{cases}$$

where, $U_n=\sum_{k=1}^{n}X_k$ and $V_n=\sum_{k=1}^{n}Y_k$.

A5 : The working times, repair times and replacement time are independent.

A6 : The repair cost is c, the reward rate r and replacement cost is R.

A7 : The replacement time is a random variable $E[Z]=\tau$.

A8 : A cycle is completed, if the replacement is done.

3. The Replacement Policy T

A cycle is the time between two consecutive replacements. The successive cycles forms a renewal process. By the renewal reward theorem, the long run average cost per unit time under the replacement policy T is given by

$$C(T)=\frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}$$

$$=\frac{E\left[\sum_{k=1}^{n-1}cY_k\right]-E\left[\sum_{k=1}^{n}rX_k\right]+R}{E\left[\sum_{k=1}^{n-1}X_k\right]+E\left[\sum_{k=1}^{n-1}Y_k\right]+E[Z]}, \tag{1}$$

where η is a random variable denoting the number of failures before the working age of the system reaches T.

Consider

$$E\left[\sum_{k=1}^{\eta} X_k\right] = E\left[E\left[\sum_{k=1}^{\eta} X_k \mid \eta = n\right]\right]$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\eta} EX_k\right) P(\eta = n)$$

$$= \sum_{n=1}^{\infty}\left(E[X_1] + \sum_{k=2}^{\eta} E[X_k]\right) P(\eta = n)$$

$$= E[X_1]\sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=1}^{\infty}\sum_{k=2}^{n} E[X_k] P(\eta = n)$$

$$= \mu\sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} E[X_{k+1}] P(\eta = n)$$

$$= \mu P(\eta = 1) + \mu\sum_{n=2}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} E[X_{k+1}] P(\eta = n)$$

$$= \mu P(\eta = 1) + \mu\sum_{n=2}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} \frac{\mu}{2^{k-1}} P(\eta = n)$$

$$= \mu\left(F_1(T) - F_2(T)\right) + \mu\sum_{n=2}^{\infty}\left(1 + \sum_{n=2}^{\infty}\frac{1}{2^{k-1}}\right) P(\eta = n)$$

$$= \mu\left(F_1(T) - F_2(T)\right) + \mu\sum_{n=2}^{\infty}\frac{1}{2^{n-1}} F_{n+1}(T) + \mu F_2(T)$$

$$= \mu\left(F_1(T)\right) + \mu\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} F_{n+1}(T) \tag{2}$$

Consider

$$E\left[\sum_{k=1}^{n-1} Y_k\right] = E\left[E\left[\sum_{k=1}^{n-1} Y_k \mid \eta = n-1\right]\right]$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1} E[Y_k]\right) P(\eta = n-1)$$

$$= \sum_{n=1}^{\infty}\left(E[Y_1] + \sum_{k=1}^{n-1} E[Y_k]\right) P(\eta = n-1)$$

$$= E[Y_1]\sum_{n=1}^{\infty} P(\eta = n-1) + \sum_{n=1}^{\infty}\sum_{k=1}^{n-1} E[Y_k] P(\eta = n-1)$$

$$= \xi P(\eta = 1) + \xi\sum_{n=2}^{\infty} P(\eta = n-1) + \sum_{n=1}^{\infty}\sum_{k=1}^{n-1} E[Y_k]\frac{\xi}{a^{k-1}} P(\eta = n-1)$$

$$= \xi\left(G_1(T) - G_2(T)\right) + \xi\sum_{n=2}^{\infty}\left(1 + \sum_{k=1}^{n-1}\frac{1}{a^{k-1}}\right) P(\eta = n-1)$$

$$= \xi\left(G_1(T) - G_2(T)\right) + \xi\sum_{n=2}^{\infty}\left(1 + \sum_{k=1}^{n-1}\frac{1}{a^{k-1}}\right)\left(G_n(T) - G_{n+1}(T)\right)$$

$$= \xi\left(G_1(T)\right) + \xi\sum_{n=1}^{\infty}\frac{1}{a^{n-1}} G_{n+1}(T).. \tag{3}$$

Substituting the equations (2) and (3) in equation (1), we obtain

$$C(T)=\frac{E\left[\sum_{k=1}^{n-1}cY_k\right]-E\left[\sum_{k=1}^{n}rX_k\right]+R}{E\left[\sum_{k=1}^{n}X_k\right]+E\left[\sum_{k=1}^{n-1}Y_k\right]+E[Z]}$$

$$=\frac{\left[\begin{array}{c} c\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right) \\ -r\ \mu\left(\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+R\right.\end{array}\right]}{\left[\begin{array}{c}\mu\left(F_1(T)\right)+\mu\left(\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right) \\ +\xi\left(G_1(T)\right)+\xi\left(\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+\ \tau\end{array}\right]}$$

$$=\frac{\left[\begin{array}{c} c\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right) \\ -r\ \mu\left(\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+R\right.\end{array}\right]}{\left[\begin{array}{c}\mu\left(F_1(T)\right)+\mu\left(\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right) \\ +\xi\left(G_1(T)\right)+\xi\left(\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+\ \tau\end{array}\right]}+r-r$$

$C(T)=C_1(T)-r,$

where

$$C_1(T)=\frac{\left[\begin{array}{c} c\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right) \\ -r\ \mu\left(\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+R\right. \\ r\left(\mu\left(F_1(T)\right)+\mu\left(\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)\right. \\ \left.+\xi\left(G_1(T)\right)+\xi\left(\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+\ \tau\right)\end{array}\right]}{\left[\begin{array}{c}\mu\left(F_1(T)\right)+\mu\left(\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right) \\ +\xi\left(G_1(T)\right)+\xi\left(\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+\ \tau\end{array}\right]}$$

$$= \frac{(c+r)\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+(R+r\tau)}{\left[\begin{array}{c}\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)\\ +\xi(G_1(T))+\xi\left(\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+\ \tau\end{array}\right]}. \tag{4}$$

On differentiating equation (4), we obtain on simplification

$$C_1^{'}(T) = \frac{\left[\begin{array}{l}\left[\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+\tau\right]\\ \left[(c+r)\xi\left(G_1^{'}(T)+\ \ \sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}{'}(T)\right)\right]\\ -\left[(c+r)\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+(R+r\tau)\right]\\ \left[\mu\left(F_1{'}(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}{'}(T)\right)\right]\end{array}\right]}{\left[\begin{array}{c}\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)\\ +\xi(G_1(T))+\xi\left(\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+\ \tau\end{array}\right]^2}. \tag{5}$$

If $C_1^{'}(T)=0$, then

$$\left[\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+\tau\right]\left[(c+r)\xi\left(G_1^{'}(T)+\ \ \sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}{'}(T)\right)\right]$$
$$-\left[(c+r)\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+(R+r\tau)\right]\left[\mu\left(F_1{'}(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}{'}(T)\right)\right]=0$$

which implies

$$\begin{array}{l}\left[\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+\tau\right]\left[(c+r)\xi\left(G_1^{'}(T)+\ \ \sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}{'}(T)\right)\right]\\ =\left[(c+r)\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+(R+r\tau)\right]\left[\mu\left(F_1{'}(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}{'}(T)\right)\right]\end{array} \tag{6}$$

Again differentiating equation (5) and using the equation (6), we obtain on simplification that

$$C_1^{''}(T) = \frac{\left[\begin{array}{l}\left[\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+\tau\right]\\ \left[(c+r)\xi\left(G_1^{''}(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}^{''}(T)\right)\right]\\ -\left[(c+r)\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+(R+r\tau)\right]\\ \left[\mu\left(F_1^{''}(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}^{''}(T)\right)\right]\end{array}\right]}{\left[\begin{array}{c}\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)\\ +\xi(G_1(T))+\xi\left(\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+\ \tau\end{array}\right]^2}. \tag{7}$$

If $C_1{''}(T)>0$, then

$$\left[\mu\left(F_1(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}(T)\right)+\tau\right]\left[(c+r)\xi\left(G_1^{"}(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}^{"}(T)\right)\right]$$

$$>\left[(c+r)\xi\left(G_1(T)+\sum_{n=1}^{\infty}\frac{1}{a^{n-1}}G_{n+1}(T)\right)+(R+r\tau)\right]\left[\mu\left(\mathrm{F}_1^{"}(T)+\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}F_{n+1}^{"}(T)\right)\right]..$$

For if $C_1^{'}(T)=0$, $C_1"(T)>0$, then $C_1(T)$ attains its minimum .

4. Numerical Example

In this section, we give an example to strengthen the theoretical results. Assume that $(W_i,\ i=1,2,...)$ is a sequence of independent random variable and each W_i, has an exponential distribution $\exp(\lambda_i)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the probability density function of $\sum_{i=1}^{n} W_i$ is given by

$$f_n(t)=\begin{cases}(-1)^{n-1}\lambda_1\lambda_2\lambda_3\cdots\lambda_n\sum_{i=1}^{n}\dfrac{\exp(-\lambda_i t)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(\lambda_{i-1}-\lambda_{j-1})}, & \text{for } x\geq 0\\ 0, & \text{otherwise.}\end{cases}$$

Let $\lambda_i = a^{i-1}\lambda$. The density function of $\sum_{i=1}^{n} W_i$ then becomes

$$f_n(t)=\begin{cases}(-1)^{n-1}\lambda a^{\frac{n(n-1)}{2}}\sum_{i=1}^{n}\dfrac{\exp(-a^{i-1}\lambda t)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(a^{i-1}-a^{j-1})}, & \text{for } x\geq 0\\ 0, & \text{otherwise.}\end{cases}$$

The distribution function of $\sum_{i=1}^{n} W_i$ is

$$F_n(t)=(-1)^{n-1}\lambda a^{\frac{n(n-1)}{2}}\sum_{i=1}^{n}\frac{1-\exp(-a^{i-1}\lambda T)}{a^{i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(a^{i-1}-a^{j-1})}.$$

The distribution function of $\sum_{i=1}^{n} Y_i$ is

$$G_n(t)=(-1)^{n-1}\left(\frac{\mu}{\beta}\right)^{n}\left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}}\sum_{i=1}^{n}\frac{1-\exp\left(-\frac{\mu}{\beta 2^{i-1}}T\right)}{\prod_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{\mu}{\beta 2^{i-1}}-\frac{\mu}{\beta 2^{i-1}}\right)}.$$

$$G_n(t)=(-1)^{n-1}\left(\frac{\mu}{\beta}\right)^{n}\left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}}\sum_{i-1}^{n}\frac{1-exp\left(-\frac{\mu}{\beta 2^{i-1}}T\right)}{\prod_{j=1\, j\neq i}^{n}\left(\frac{\mu}{\beta 2^{i-1}}-\frac{\mu}{\beta 2^{i-1}}\right)}$$

Let the parameter values be $c=15$, $a=0.956$, $\mu=25$, $r=50$, $\beta=1.054$, $\xi=12$, $\tau=10$,

$R = 50000$.

On substituting these values in equation (7) and passing over numerical calculations, we arrive at $T^* = 410$, so that $C(T)$ is minimum at T^* and the long run average cost $C(T) = C(410) = -1452.746$ monetary units. The value of $C(T)$ for T ranging from 110 to 500 in steps of 10 are given in Table 1. Further the tabulated values are plotted in Figure 1

Table 1. Values of $C(T)$ against T

T	$C(T)$	T	$C(T)$	T	$C(T)$	T	$C(T)$
110	–1436.987	210	–1145.783	310	–1147.801	**410**	
120	–1437.254	220	–1145.846	320	–1147.939	420	–1451.023
130	–1438.723	230	–1145.902	330	–1147.972	430	–1450.974
140	–1439.085	240	–1147.143	340	–1148.548	440	–1450.257
150	–1439.156	250	–1147.235	350	–1148.697	450	–1446.938
160	–1442.358	260	–1147.357	360	–1148.913	460	–1445.845
170	–1442.456	270	–1147.489	370	–1149.216	470	–1443.789
180	–1442.569	280	–1147.504	380	–1149.864	480	–1442.971
190	–1442.631	290	–1147.671	390	–1149.265	490	–1439.582
200	–1443.705	300	–1147.738	400	–1149.327	500	–1435.768

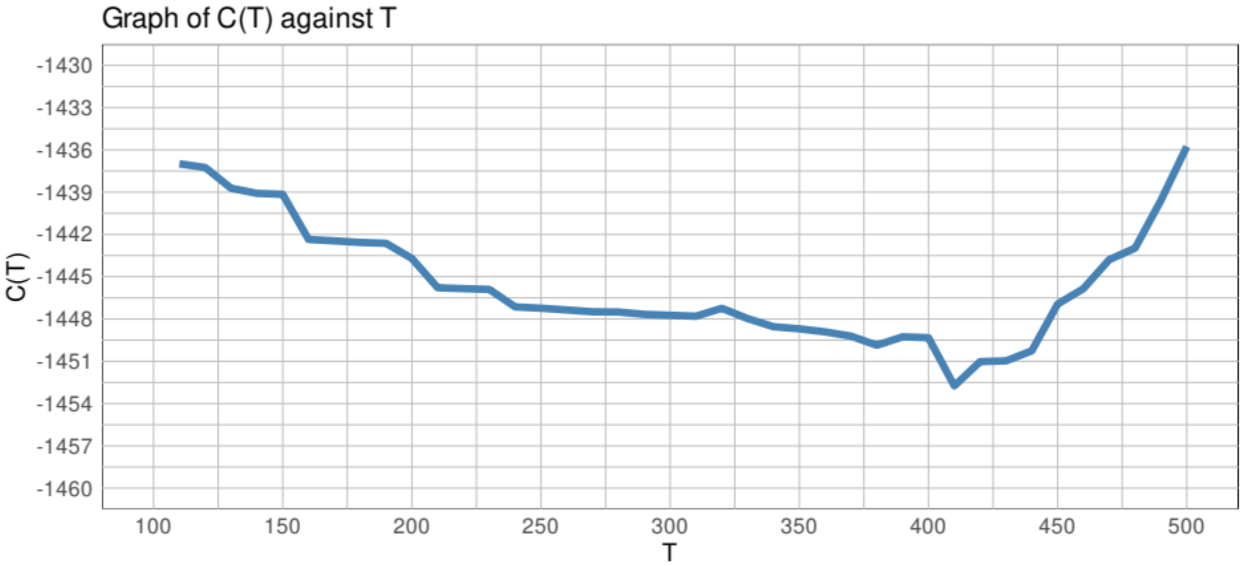

Figure 1. Graph of $C(T)$ against T

5. CONCLUSION

In this paper, we have studied maintenance model for a deteriorating system under partial sum process. An explicit expression for the long run average cost per unit time under the replacement policy T is derived. Optimality conditions are determined analytically. Numerical example is given for verifying the theoretical result.

REFERENCES

[1] Abdel−Hameed, H., (1986), Optimal Replacement of a System Subject to Shocks, Journal of Applied Probability, 23(1), pp.107−114.
[2] Babu, D., Govindaraju, P., and Rizwan, U., (2019), A Maintenance Model for a Deteriorating System with Imperfect Delayed Repair Under Partial Product Process, Journal of Applied Sciences And Computations, 6(4), pp.795−800.
[3] Babu, D., Manigandan, C., and Parvatham, R., (2020), A Deteriorating System Maintenance Model

with Imperfect Delay Repair Under Partial Sum Process, Stochastic Modeling and Applications, 24(2), pp.111−119.

[4] Barlow, R.E., and Proschan, F., (1975), Statistical Theory of Reliability and Life Testing, John Wiley, New York.

[5] Barlow, R.E., And Proschan, F., (1983), Imperfect Repair, Journal of Applied Probability, 20(4), pp.851−859.

[6] Lam, Y., (1988a), A Note on Optimal Replacement Problem, Advances in Applied Probability, 20(2), pp.479−482.

[7] Lam, Y., (1988b), Geometric Processes and Replacement Problem, Acta Mathematicae Applicatae Sinica, 4, pp.366−377.

[8] Lam, Y., (2007), The Geometric Processes and Its Applications, World Scientific, Singapore.

[9] Raajpandiyan, T.R., Syed Tahir Hussainy, and Rizwan, U., (2022), Optimal Replacement Model Under Partial Product Process, Stochastic Modeling and Applications, 26(3), pp.170−176.

[10] Ross, S.M., (1983), Stochastic Process, Wiley, New York.

[11] Thangaraj, V., And Rizwan, U., (2001), Optimal Replacement Policies In Burn-In Process for An Alternative Repair Model, International Journal of Information And Management Sciences, 12(3), pp.43−56.

[12] Wang, H., (2002), A Survey of Maintenance Policies of Deteriorating Systems, European Journal of Operational Research, 139, pp.469−489.