Thermodynamics of Anisotropic Universe with Hubble Parameterization and Observational Constraints

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ABSTRACT
In this paper, we investigate the thermodynamic properties of anisotropic universe with Hubble parameterization and estimates the model parameters by bounding our model with 57 H(z) observational data and ColFI methods. We estimate the parameters using the Artificial Neural Networks (ANN), Mixture Density Network (MDN) and Mixture Neural Network (MNN). It has been observed that the entropy of the universe is increasing function of time while the entropy density decreases with time. We also found that the cosmological constant $\Lambda(t)$ is decreasing function of time, which is supported by results from SNe Ia observations and this cosmological constant Λ (t) affects the entropy. Some physical and geometric behaviour of the universe also are discussed.

Keywords: Hubble's parameters, ColFI method, Accelerating universe, Entropy, Cosmological constant

1. INTRODUCTION

The Bianchi types models and anisotropic geometry of the universe create more interest due to its applicability to permit small anisotropy with evolution process of the universe and become isotropic at late times. Therefore, these models are the simplest generalizations of FRW models [1 -6]. In the literature, Bianchi type I, II, III, V, VI and IX models are investigated by various authors in different physical contexts. Among these models, the Bianchi type V universe is the natural generalization of the open FRW model, which are eventually become isotropic [7 - 16]. Following the work of Saha [17], Singh and Chaubey [18-19] have investigated a quadrature form of metric functions for Bianchi type V model with perfect fluid and viscous fluid. Moreover, in recent times, the solution of Einstein's field equation for Bianchi type models have been studied by several authors such as Hajj-Boutros [20,21], Mazumdar [22] using various aspect of cosmology. Berman [23] have proposed the simplest solution in power law form by using the law of variation for Hubble's parameter which leads the constant value of deceleration parameter. The cosmological models with constant deceleration parameter have been studied by Maharaj and Naidoo [24], Johri and Desikan[25], Singh and Desikan [26] in isotropic and anisotropic space-time. In Ref. [27], the authors have extended power law in Bianchi type II space-time and described the late time acceleration of the universe with particular choice of the parameters.

The cosmological models with a relic cosmological constant Λ have received considerable attention r among the researchers for various reasons [28- 32]. In Ref. [33], the author has investigated that the time varying cosmological constant $\Lambda(t)$ plays the role of dark energy component and contribute the late time acceleration of the universe. In general, $\Lambda(t) > 0$ which lead the current accelerating phase of the universe but the possibility of negative cosmological constant i. e. $\Lambda(t) < 0$, can not completely denied in our previous research [33] and today the sign switching $\Lambda(t)$ from initially negative to its positive values on later time becomes a promising candidate to solve the tensions and anomalies found in cosmological parameters due to the different observational approaches [34]. So the idea of negative cosmological constant was initially initiated in Ref. [33] which adds attractive gravity, therefore universe with negative cosmological constant are invariably doomed to re-collapse.

In recent times the deep learning, which utilizes a computer models known as artificial neural networks (ANNs), has emerged as the fastest-growing subfield in machine learning. An assortment of cosmological tasks have made use of artificial neural networks, including N-body simulations [35, 36] and statistical approaches [37–39]. Using a mixture model, the literature recommends using the mixture density network (MDN) to describe the conditional probability density $P(\Theta|d)$ [40]. It is worthwhile to note that the conditional probability density $P(\Theta|d)$ is learned using samples generated by models and utilized an artificial neural network (ANN) to estimate parameters [41].

In this paper, we investigate the interaction matter or radiation with dark energy term by taking into account a special law of variation of Hubble parameter in Bianchi V space – time. We also probes the model parameters by applying ColFI methods [40, 41, 42] and analyzed its thermodynamic properties. In section 2, the field equations and its solutions are given while in section 3, we describe the thermodynamic properties of the derived model of the universe. In section 4, we explore the method of observational constraints and its numerical results. Finally the conclusions are given in section 5.

2. The Field Equations and its solution

The energy momentum tensor for dark energy is read as

$$T_{ij}^{(vac)} = -\Lambda(t) g_{ij}\,, \tag{1}$$

where Λ(t) denotes cosmological term - a suitable candidate of dark energy and g_{ij} is the metric tensor. Therefore, the effective energy momentum tensor is defined as

$$T_{ij} = (p+\rho)u_i u_j - p g_{ij} - \Lambda(t) g_{ij}\,, \tag{2}$$

Where p and ρ are the pressure and energy density of perfect fluid respectively and u_i is the fluid four velocity with condition $u^i u_i = 1$.

The spatially homogeneous and anisotropic Bianchi type V space time is read as

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{2\eta x}\left[B^2(t)dy^2 + C^2(t)dz^2\right] \tag{3}$$

where η is constant.

For the energy momentum tensor Eq. (2) and Bianchi type V space time Eq. (3), the Einstien's field equations

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} \tag{4}$$

yielding the following equations

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{\eta^2}{A^2} = -8\pi(p+\Lambda)\,, \tag{5}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{A_4 C_4} - \frac{\eta^2}{A^2} = -8\pi(p+\Lambda)\,, \tag{6}$$

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} - \frac{\eta^2}{A^2} = -8\pi(p+\Lambda)\,, \tag{7}$$

$$\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} + \frac{B_4 C_4}{BC} - \frac{3\eta^2}{A^2} = -8\pi(\rho-\Lambda)\,, \tag{8}$$

$$2\frac{A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C} = 0\,, \tag{9}$$

Note that suffix 4 by the symbol A, B and C represent differentiation with respect to t.

Eqs. (5) – (8), yield the conservation equation as follows

$$\rho_4 + (\rho + p)\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) = 0 \tag{10}$$

2.1. Model and law of variation for Hubble's parameter

The law of variation of the Hubble parameter in the case of a spatially homogeneous and anisotropic for Bianchi type V space time gives a constant value of deceleration parameter (q). If q < 0 then the model of the universe evolves with acceleration in its expansion while q > 0 leads the decelerating phase of the universe model. In Refs. [43, 44], the authors have investigated that the universe is evolving with acceleration at present epoch.

The average scale factor (a) of model (3) is read as

$$a = (ABC)^{1/3} \tag{11}$$

The spatial volume V is given by

$$V = a^3 = ABC \tag{12}$$

The generalized mean Hubble parameter (H) is obtained as

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right) \qquad (13)$$

where, $H_1 = \frac{A_4}{A}$, $H_2 = \frac{B_4}{B}$ and $H_3 = \frac{C_4}{C}$.

From equation (11) - (13), we obtain the following relation

$$H = \frac{1}{3}\frac{V_4}{V} = \frac{a_4}{a} = \frac{1}{3}\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) \qquad (14)$$

Since the model (3) is completely characterized by Hubble parameter H, therefore, the mean Hubble parameter H is defined in term of average scale factor as following

$$H = ka^{-\alpha} = k(ABC)^{-\alpha/3} \qquad (18)$$

Note that k and α is a positive constant.

The deceleration parameter (q) is defined as

$$q = \frac{-a_{44}a}{a_4^{\,2}} \qquad (19)$$

From equation (14) and (18), we get

$$a_4 = ka^{-\alpha+1} \qquad (20)$$

$$a_{44} = -k^2(n-1)a^{-2\alpha+1} \qquad (21)$$

From equation (19), (20) and (21), we get

$$q = \alpha - 1 \qquad (22)$$

From Eq. (22), we observe that the deceleration parameter q is constant. The sign of q indicates whether the model accelerates or not. The positive sign of q ($\alpha > 1$) corresponds to standard decelerating model whereas the negative sign $-1 \le q \le 0$ implies that $(0 \le \alpha \le 1)$ and the universe is evolving with acceleration. It is worthwhile to note that the current observations of SN Ia [43, 44] favors an accelerated expansion of the universe at present epoch.

Solving Eq. (20), we obtain

$$a = (\alpha kt + c_1)^{1/\alpha}, \qquad (23)$$

where $\alpha \neq 0$ and c_1 is a constants of integration.

2.2. Solution of field equations

The field equations (5) – (8) and (10) are expressed in terms of H, σ^2 and q as

$$8\pi(p + \Lambda) = H^2(2q-1) - \sigma^2 + \frac{\eta^2}{A^2}, \qquad (24)$$

$$8\pi(\rho - \Lambda) = 3H^2 - \sigma^2 - \frac{3\eta^2}{A^2}, \qquad (25)$$

$$\rho_4 + 3(\rho + p)H = 0, \qquad (26)$$

From Eq. (9), we obtain

$$A^2 = BC \qquad (27)$$

Now, Subtracting (7) from (6), and taking the second integration, one can find the following relation between A and B

$$\frac{A}{B} = d_1 \exp\left(k_1 \int \frac{dt}{a^3}\right) \qquad (28)$$

Similarly, we obtain the other relations

$$\frac{B}{C} = d_2 \exp\left(k_2 \int \frac{dt}{a^3}\right), \qquad (29)$$

$$\frac{C}{A} = d_3 \exp\left(k_3 \int \frac{dt}{a^3}\right), \tag{30}$$

where d_1, d_2, d_3, and k_1, k_2, k_3, are constants of integration, obeying

$$d_1 d_3 = d_2^{-1}, \quad k_1 + k_2 + k_3 = 0 \tag{31}$$

In view of equation (31), we obtain the metric functions from (28)–(30) as following

$$A(t) = (l_1)^{1/3} a \exp\left(\frac{K_1}{3} \int \frac{dt}{a^3}\right), \tag{32}$$

$$B(t) = (l_2)^{1/3} a \exp\left(\frac{K_2}{3} \int \frac{dt}{a^3}\right), \tag{33}$$

$$C(t) = (l_3)^{1/3} a \exp\left(\frac{K_3}{3} \int \frac{dt}{a^3}\right), \tag{34}$$

Where, $K_1 = k_1 - k_3$, $K_2 = -2k_1 - k_3$, $K_3 = k_1 + 2k_3$,

$$l_1 = \sqrt[3]{d_1/d_3}, \quad l_2 = \sqrt[3]{1/d_1^2 d_3}, \quad l_3 = \sqrt[3]{d_1 d_3^2}.$$

Also, we define the following relation among the constant from equation (27) and (32) - (34)

$K_1 = 0$, $K_2 = -K_3 = K$, $l_1 = 1$, $l_2 = l_3^{-1} = M^3$,

where, K and M are constants.

Thus, we obtain

$$A(t) = a, \tag{35}$$

$$B(t) = Ma \exp\left(\frac{K}{3} \int \frac{dt}{a^3}\right), \tag{36}$$

$$C(t) = M^{-1} a \exp\left(-\frac{K}{3} \int \frac{dt}{a^3}\right), \tag{37}$$

Therefore, inserting Eq. (23) into (35) - (37), the expressions of A(t), B(t) and C(t) are read as

$$A(t) = (\alpha kt + c_1)^{\frac{1}{\alpha}}, \tag{38}$$

$$B(t) = M(\alpha kt + c_1)^{\frac{1}{\alpha}} \times \exp\left[\frac{K}{3k(\alpha - 3)} (\alpha lt + c_1)^{(\alpha-3)/\alpha}\right], \tag{39}$$

$$C(t) = M^{-1}(\alpha kt + c_1)^{\frac{1}{\alpha}} \times \exp\left[-\frac{K}{3k(\alpha - 3)} (\alpha lt + c_1)^{(\alpha-3)/\alpha}\right], \tag{40}$$

This solution exists for $\alpha \neq 3$.

The pressure and density are obtained as

$$8\pi(p + \Lambda) = (2\alpha - 3)k^2 (\alpha kt + c_1)^{-2} - \frac{K^2}{9} (\alpha kt + c_1)^{-\frac{6}{\alpha}} + \eta^2 (\alpha kt + c_1)^{-\frac{2}{\alpha}} \tag{41}$$

$$8\pi(\rho - \Lambda) = 3k^2 (\alpha kt + c_1)^{-2} - \frac{K^2}{9} (\alpha kt + c_1)^{-\frac{6}{\alpha}} - 6\eta^2 (\alpha kt + c_1)^{-\frac{2}{\alpha}} \tag{42}$$

For complete determinacy of the system, we consider a perfect fluid equation of state

$$p = \gamma \rho, \quad 0 \leq \gamma \leq 1. \tag{43}$$

Eq. (41), with the use of (43) and (42), leads to

$$8\pi(1 + \gamma)\rho = 2\alpha k^2 (\alpha kt + c_1)^{-2} - \frac{2K^2}{9} (\alpha kt + c_1)^{-\frac{6}{\alpha}} - 2\eta^2 (\alpha kt + c_1)^{-\frac{2}{\alpha}} \tag{44}$$

Therefore, the cosmological constant Λ (t) is obtained as

$$8\pi(1+\gamma)\Lambda = (2\alpha - 1 - \gamma)k^2(\alpha kt + c_1)^{-2} - (1-\gamma)\frac{K^2}{9}(\alpha kt + c_1)^{-\frac{6}{\alpha}} + (3 + 3\gamma - 2)\eta^2(\alpha kt + c_1)^{-\frac{2}{\alpha}}$$

(45)

3. The thermodynamic properties of the model

It is possible to discuss entropy (s) of our universe as we have obtained the expressions for energy density, pressure and volume.

In thermodynamics the expression for entropy is given by

$$Tds = d(\rho V) + PdV \qquad (45)$$

Where, V is the spatial volume and T denotes the temperature of the universe.

To solve the entropy problem of the standard model, it is mandatory to consider ds > 0, because for a stable system, change in entropy is always positive. Here, our aim is to investigate a physically viable model of the current universe.

Therefore, Eq. (45) leads to

$$Tds = \rho_4 + (\rho + P)\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) > 0 \qquad (46)$$

The energy conservation equation $T_{ij}^{\;j} = 0$; for metric (3) is read as

$$\rho_4 + (\rho + P)\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{3}{2}\Lambda\Lambda_4 + \frac{3}{2}\Lambda^2\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) = 0 \qquad (47)$$

Eq. (46) and (47) leads to

$$\Lambda_4 + \Lambda\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) < 0 \qquad (48)$$

From equation (14) and (48), we have

$$\Lambda_4 + 3\Lambda H < 0 \qquad (49)$$

Hence, for α ≠ 0, we obtain

$$\Lambda < \frac{\alpha_0}{(\alpha kt + c_1)^{\frac{3}{\alpha}}} \;, \qquad (50)$$

where, α_0 is any arbitrary constant.

Moreover, from Eq. (45), one may obtain entropy in term of temperature as

$$s = \frac{(\rho + p)V}{T} = \frac{(1+\gamma)\rho V}{T} \qquad (51)$$

Therefore, the entropy density (S) is obtained as

$$S = \frac{(1+\gamma)\rho}{T} = (1+\gamma)\rho^{\frac{1}{1+\gamma}} \qquad (52)$$

where, $T = \rho^{\frac{\gamma}{1+\gamma}}$.

The Hubble parameter H, expansion scalar and anisotropic parameter are obtained as

$$H = k(\alpha kt + c_1)^{-1} \qquad (54)$$

$$\theta = 3k(\alpha kt + c_1)^{-1}, \qquad (55)$$

$$A_m = \frac{2}{27}\frac{K^2}{k^2}(\alpha kt + c_1)^{\frac{2\alpha - 6}{\alpha}} \qquad (56)$$

4. Observational confrontation and Physical Parameters

The scale factor in terms of z is defined as a = 1/(1+z), and using the transformation equation dz/dt = -(1+z)H and Eq. (23), we obtain the following expression for H in term of z

$$H = H_0(1+z)^{\alpha} \tag{57}$$

where, H_0 is the present value of Hubble constant.

In this paper, the model parameter H_0 and α are estimated by bounding our model with 57 H(z) observational data. These 57 H(z) data points are obtained from Cosmic Chronometric (CC) and BAO. The CC technique yielded 31 H(z) and BAO yielded 26 H(z) data points in the range $0 \le z \le 2.36$. Table II of Ref. [45] includes all 57 H(z) data points.

The main idea of parameter estimation using the Artificial Neural Networks (ANN), Mixture Density Network (MDN) and Mixture Neural Network (MNN) are described in Refs. [40, 41, 42]. In this paper, we are using the methods and approaches given in Ref. [41] and a publicly available code called Cosmological Likelihood free Inference (CoLFI) - https://github.com/Guo-Jian-Wang/colfi for estimating the model parameters.

Figs. 1, 2 and 3 depicts 1σ and 2σ contours contours in one- and two-dimensional marginalized distributions, constrained from 57 point H(z) data trained and simulated using the ANN model, MDn model and MNN model respectively. Fig. 4 and 5 exhibit the behavior of energy density ρ and cosmological constant Λ (t) with respect to cosmic time and from these Figures, we observe that the ρ and Λ decreases with evolution of the universe. The behavior of pressure is shown in Fig. 6 and it shows similar behavior as energy density. The evolution of entropy density is shown in Fig. 7 which shows that energy density of the universe in derived model decreases with expansion of the universe while the entropy of the universe is increasing function of time (see Fig. 8). Therefore, the entropy of the universe increases with its age.

When a model is chosen for the medium then by restricting the pressure and density values, the energy conditions are used to check the model either it is stable or not. The Energy condition equations are named as:

Weak energy condition (WEC) $\rho \ge 0$

Null energy condition (NEC) $\rho - p \ge 0$

Dominant energy condition (DEC) $\rho + p \ge 0$

Strong energy conditions (SEC) $\rho + 3p \ge 0$

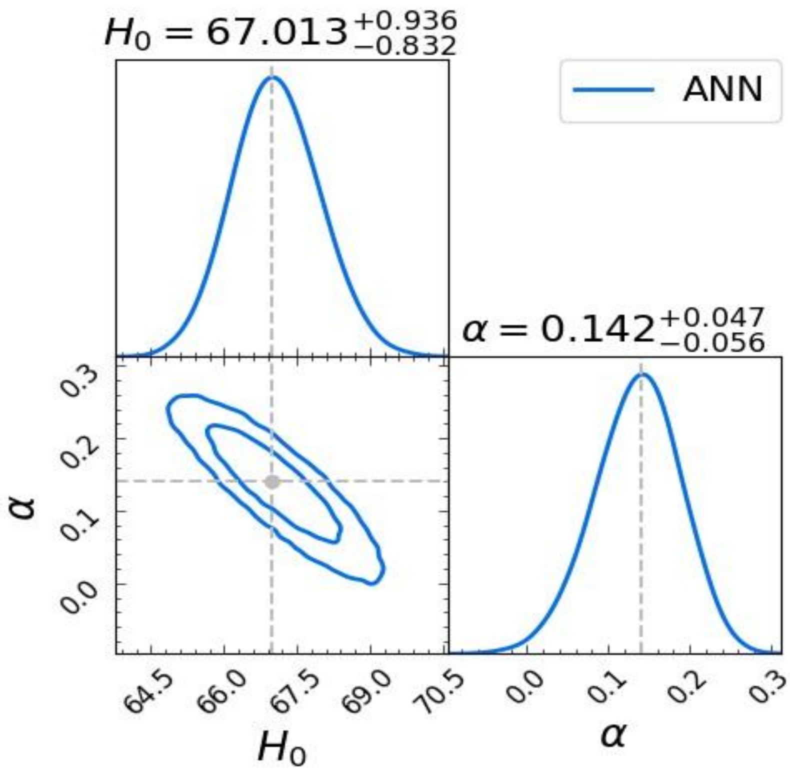

Fig 1. 1σ and 2σ contours contours in one- and two-dimensional marginalized distributions, constrained from 57 point H(z) data trained and simulated using the ANN model.

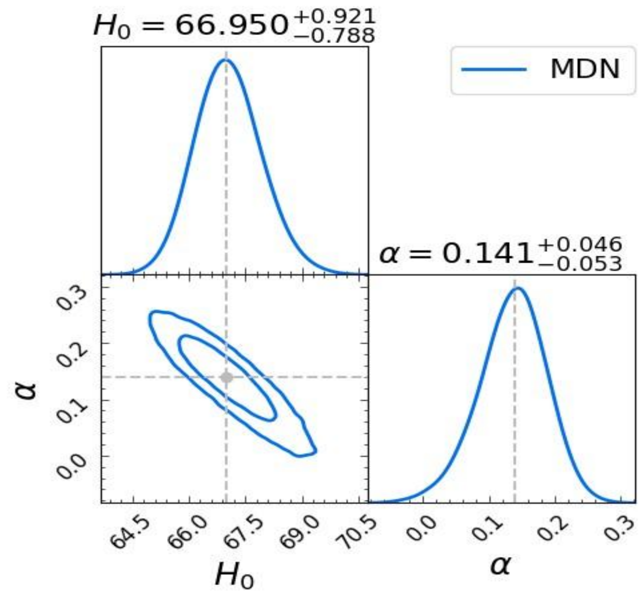

Fig 2. 1σ and 2σ contours contours in one- and two-dimensional marginalized distributions, constrained from 57 point H(z) data trained and simulated using the MDN model.

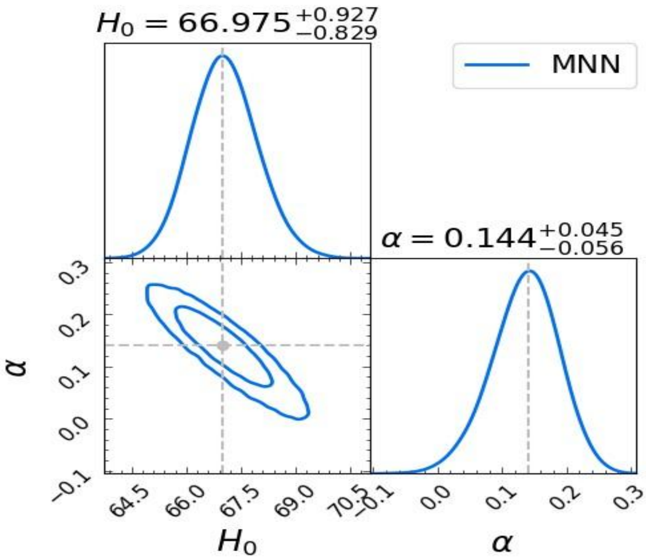

Fig 3. 1σ and 2σ contours contours in one- and two-dimensional marginalized distributions, constrained from 57 point H(z) data trained and simulated using the MNN model.

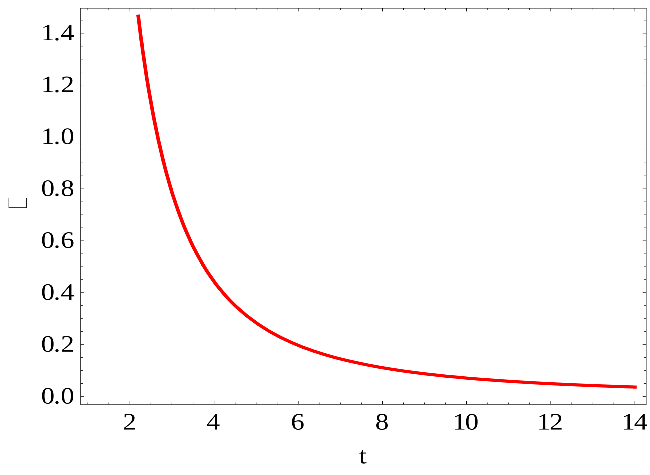

Fig 4. Plot of energy density versus time for α = 0.144

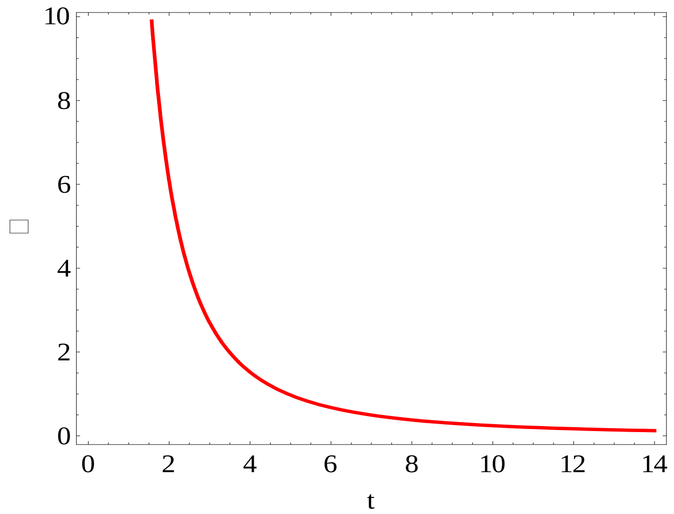

Fig 5. Plot of cosmological constant versus time for α = 0.144

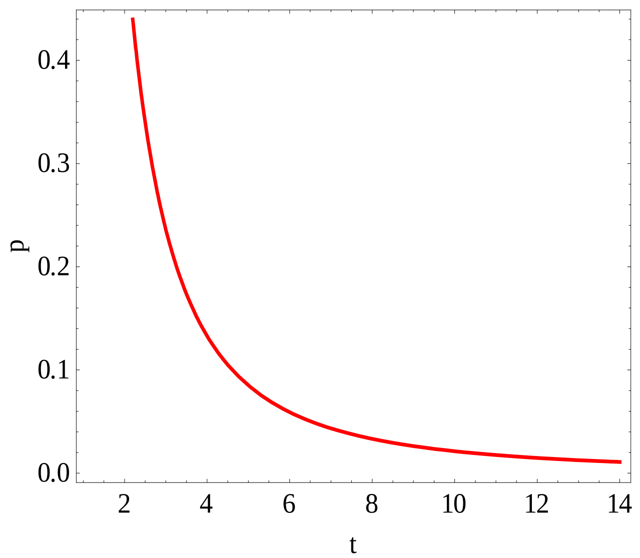

Fig 6. Plot of pressure versus time for α = 0.144

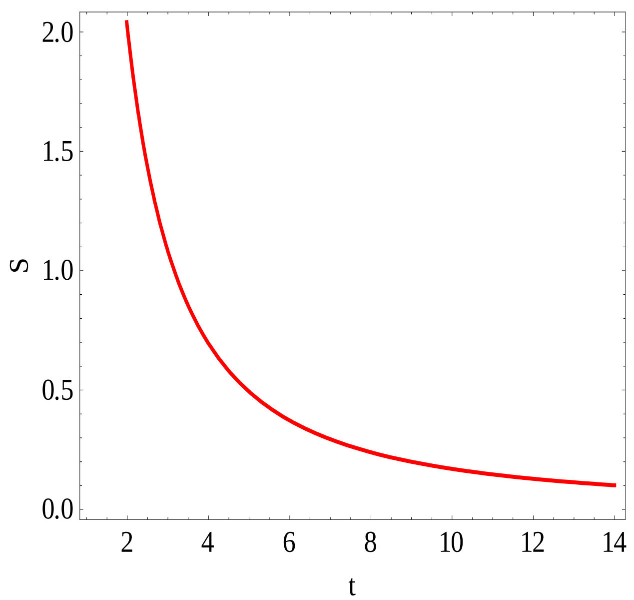

Fig 7. Plot of entropy density versus time for α = 0.144

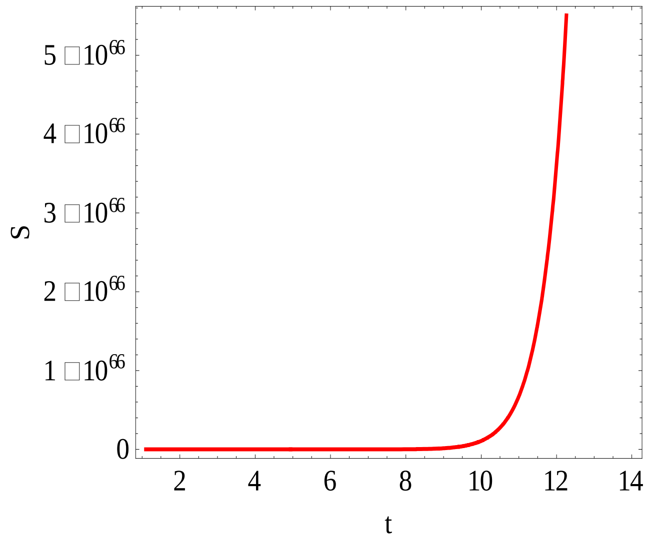

Fig 8. Plot of entropy versus time for α = 0.144

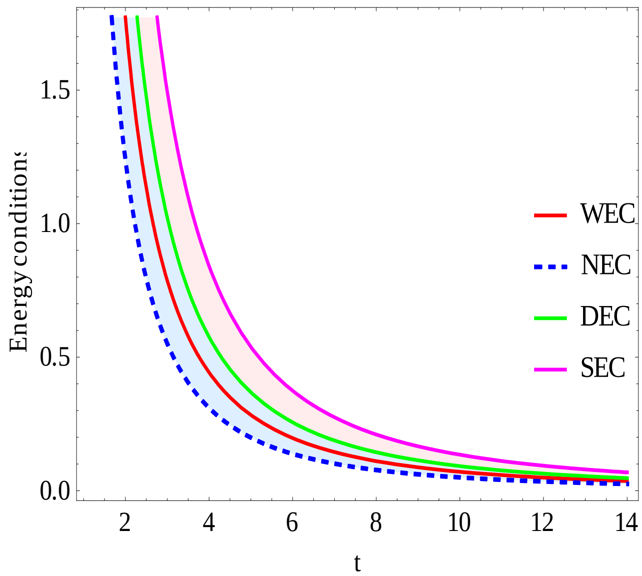

Fig 9. Validation of energy conditions for α = 0.144

Fig. 9 depicts the validation of WEC, NEC, DEC and SEC in the derived model of the universe. Therefore, the analysis of energy condition predicts that this model of the universe is physically viable. Moreover, we also note that the cosmological constant Λ(t) is decreasing function of time and obtained a diminish value at present epoch. This behavior of Λ(t) is supported by SNe Ia observations. It is worthwhile to note that the cosmological constant Λ (t) affects the entropy.

5. CONCLUSIONS

In this paper we have investigated the thermodynamic properties of anisotropic universe by taking into account Hubble parameterization and we also have estimated model parameters H_0 and α restricting our model with 57 H(z) observational data by applying CoLFI methods [41]. The following are some aspects of this research that were noted:

- CoLFI: The cosmological Likelihood-free Inference with Neural network estimates cosmological parameter in good pattern. The whole process is easier in comparision to Bayesian analysis to probing the cosmological parameters and also predicted parameter accuracy. Comparative in�vestigation of our neural network techniques versus classic MCMC methodology showed that MNN produced practically comparable results, proving its fit and reliability.

- The cosmological constant $\Lambda(t)$ is found to be decreasing function of time and it approaches a small positive value at late time which are supported by the results from supernovae observations obtained by the High-z Supernova Team and Supernova Cosmological project.
- The energy density of derived model stays positive throughout the evolution of the universe, and it approaches zero at late time. Pressure drops as time increases, indicating that the cosmos is expanding. Furthermore, we observe that this model validates the energy conditions WEC, NEC, DEC and SEC.
- We estimate the model parameters as $H_0 = 67.013$ and $\alpha = 0.142$ (ANN), $H_0 = 66.950$ and $\alpha = 0.141$ (MDN) and $H_0 = 66.975$ and $\alpha = 0.144$ (MNN). These estimated values of H_0 are very close to its value given in Planck result [46]. Thus, the solution presented in this paper also minimized recently observed H_0 tension.

Finally the model of the universe investigated in this paper is accelerating and approaches isotropic at later time of its expansion. The more realistic models may be analyzed using this technique, which may lead to interesting and describes various physical phenomenon and fate of the universe.

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