J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 3, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

On asymptotic behavior of a quadratic functional equation

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December 9, 2022

Abstract

The main goal of this paper is to investigate the stability problems for the following quadratic functional equation

$$f(x+y+z)+f(x+y-z)+f(x-y+z)+f(-x+y+z) = 4f(x)+4f(y)+4f(z)$$

on an unbounded restricted domain. As a consequence, we can apply the obtained results to obtain some asymptotic behaviors of that equation in normed spaces. Moreover, we introduce a new inequality that characterizes the inner product spaces.

1 Introduction

In 1940, Stanisław Marcin Ulam proposed the following problem [25]:

> Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given a real number $\varepsilon > 0$, does there exist a real number $\delta > 0$ such that if a mapping $h: G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) \leq \delta$ for all $x, y \in G_1$, then there is a homomorphism $H: G_1 \to G_2$ with $d(h(x), H(x)) \leq \varepsilon$ for all $x \in G_1$?

This problem gave rise to what we now call Ulam's stability of functional equations.

In a later year, an affirmative answer to the Ulam stability problem was given by D. H. Hyers for Banach spaces (see [13]). Several generalizations of this result are discussed. T. Aoki [5] for additive maps and by T.M. Rassias

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[19] for linear maps considering an unbounded Cauchy difference. P. Găvruţă [12] provided a further generalization of the Rassias' theorem by using a general control function. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (see, [1, 19, 20, 21, 23]).

Throughout the paper, let $(G,+)$ be an Abelian group and Y be a linear space on the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Let us note that a mapping $q\colon G \to Y$ is called quadratic if q satisfies the well-known quadratic functional equation

$$q(x+y)+q(x-y)=2q(x)+2q(y), \quad x,y \in G. \tag{1.1}$$

Quadratic functional equation (1.1) was used by Jordan and von Neumann [14] to characterize inner product spaces. Several other functional equations are used for this characterization. Maurice Fréchet in [11] obtained a characterization of the inner product spaces among normed linear spaces by using the following functional equation

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(x+z).$$

Theorem 1.1. *[11] Let $(X,\|.\|)$ be a normed linear space. Then X is an inner product space with respect to $\|.\|$ if and only if*

$$\|x+y+z\|^2+\|x\|^2+\|y\|^2+\|z\|^2=\|x+y\|^2+\|x+z\|^2+\|y+z\|^2, \quad x,y,z \in X.$$

Motivated by this idea, we deal with the following functional equation:

$$\begin{aligned} f(x+y+z)+f(x+y-z)+f(x-y+z)+f(-x+y+z) \\ =4f(x)+4f(y)+4f(z). \end{aligned} \tag{1.2}$$

This equation was first introduced and solved by S. Jung [15]. In fact, he proved the following theorem

Theorem 1.2 ([15], Theorem 2.1.)**.** *Let X and Y be vector spaces over fields of characteristic different from 2, respectively. If $f\colon X \to Y$ satisfies the functional equations*

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.3}$$

$$f(x-y-z)+f(x)+f(y)+f(z)=f(x-y)+f(y+z)+f(z-x) \tag{1.4}$$

and

$$\begin{aligned} f(x+y+z)+f(x+y-z)+f(x-y+z)+f(-x+y+z) \\ =4f(x)+4f(y)+4f(z), \end{aligned} \tag{1.5}$$

then each of the equations (1.3), (1.4), *and* (1.5) *is equivalent to the other.*

Recently, EL-Fassi *et al.* [9] treated the Ulam-type stability of (1.2) in the class of functions from an Abelian group into a Banach space. The method used in [9, Theorem 4] based on a fixed point theorem [7, Theorem 1] and the argument presented is all on the whole set. This aspect of the domain is very important. However if we consider a subset of the domain which does not present all the possibilities when we have on the whole set, is the stability of the equation still valid? Many others studies this question about the stability on restricted domains of some functional equations (see [24, 22]).

Inspired by the works of Hyers [13], Brzdęk [8] and Park [17], and by a direct method, we investigate the stability of (1.2) on a restricted unbounded domain. Then, using these results, we study an asymptotic behavior of this functional equation. We also obtain a new criterion on characterization of inner product spaces by involving our functional equation. Before we state it, let us recall the definitions of quasi-norm and quasi-normed Abelian group.

We recall some basic facts concerning quasi-norm and quasi-normed Abelian group.

Definition 1.3. [4] Let $(G,+)$ be an Abelian group. A function $\rho\colon G \to \mathbb{R}$ is called a quasi-norm on G if:

1. $0 \leq \rho(x) \leq +\infty$ for all $x \in G$ (positive definite);
2. $\rho(x) = \rho(-x)$ for all $x \in G$ (even);
3. $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in G$ (subadditivity);
4. $\rho(0) = 0$.

If $\rho(x) < +\infty$ for all $x \in G$ we say that ρ is a finite quasi-norm. The pair (G, ρ) is called quasi-normed Abelian group if ρ is a quasi-norm on G.

A triplet $(G, +, \delta)$ is called metric Abelian group if $(G,+)$ is an Abelian group and δ is a translation invariant metric on G. This metric can be turned into a quasi-norm $\|.\|_\delta\colon G \to \mathbb{R}$, via $\|x\|_\delta = \delta(x, 0)$ and the pair $(G, \|.\|_\delta)$ is a quasi-normed Abelian group. For a more detailed definition of such terminology, one can refer to [10, 18].

In this paper, assume that $(G,+)$ be an Abelian group, Y be a linear space on the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a given mapping $f\colon G \longrightarrow Y$, we define the function $\Delta_f\colon G \times G \times G \longrightarrow Y$ by

$$\begin{aligned}\Delta_f(x,y,z) := {} & f(x+y+z) + f(x+y-z) + f(x-y+z) \\ & + f(-x+y+z) - 4f(x) - 4f(y) - 4f(z), \quad x, y, z \in G.\end{aligned}$$

2 Stability in restricted domains

In this section we study the stability problem of the functional equation (1.2).

Theorem 2.1. *Let $(G,+,\delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y,\|.\|)$ be a Banach space. Let $\varepsilon \geq 0$, $d>0$ be arbitrary real numbers. Suppose that $f: G\to Y$ is a function satisfies*

$$\|\Delta_f(x,y,z)\| \leq \varepsilon, \quad \|x+y+z\|_\delta \geq d. \tag{2.1}$$

Then, there exists a unique quadratic function $Q: G\to Y$ such that

$$\|Q(x)-f(x)\| \leq \frac{5\varepsilon}{8}, \qquad x\in G. \tag{2.2}$$

Proof. Let $f: G\to Y$ be a function fulfilling (2.1). Taking $x=y=z$ in (2.1), we get

$$\|f(3x)-9f(x)\| \leq \varepsilon, \quad \|3x\|_\delta \geq d,$$

witch implies

$$\|f(3x)-9f(x)\| \leq \varepsilon, \quad \|x\|_\delta \geq d.$$

Therefore,

$$\left\|\frac{f(3^{n+1}x)}{9^{n+1}} - \frac{f(3^m x)}{9^m}\right\| \leq \sum_{k=m}^{k=n} \frac{\varepsilon}{9^{k+1}}, \tag{2.3}$$

for all natural numbers ; $n\geq m$, and $\|x\|_\delta \geq d$. Therefore, $\left\{\frac{f(3^n x)}{9^n}\right\}_{n=0}^{\infty}$ is a Cauchy sequence for each $x\in G$ with $\|x\|_\delta \geq d$. It is easily to infer that the sequence $\left\{\frac{f(3^n x)}{9^n}\right\}_n$ is Cauchy in the whole G. As Y is Banach space, this Cauchy sequence is convergent. We define $Q: G\to Y$ by

$$Q(x) := \lim_{n\to+\infty} \frac{f(3^n x)}{9^n}, \quad x\in G. \tag{2.4}$$

For $x\in G\setminus\{0\}$, we choose $N\in\mathbb{N}$ so large that for all $n\geq N$, $\|3^n x\|_\delta \geq d$. By (2.4), we see that

$$\begin{aligned}\|\Delta_Q(x,y,z)\| = \lim_{n\to+\infty}\frac{1}{9^n}\Big\| & f(3^n x+3^n y+3^n z) + f(3^n x+3^n y-3^n z) \\ & + f(3^n x-3^n y+3^n z) + f(-3^n x+3^n y+3^n z) \\ & - 4[f(3^n x)+f(3^n y)+f(3^n z)]\Big\| \\ \leq & \lim_{n\to+\infty}\frac{\varepsilon}{9^n} = 0.\end{aligned}$$

Hence, Q fulfills equation (1.2) for all $x\in G\setminus\{0\}$.
Since

$$Q(0) = \lim_{n\to+\infty}\frac{f(0)}{9^n} = 0,$$

the function Q fulfills equation (1.2) for all $x\in G$. Since Q is a solution of (1.2), we infer that Q is a quadratic function in G.

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Taking the limit as $n \to +\infty$ and putting $m = 0$, we get from (2.3)

$$\|Q(x) - f(x)\| \leq \frac{\varepsilon}{8}, \quad \|x\|_\delta \geq d. \tag{2.5}$$

Next, we extend (2.5) to the whole G. Let $z \in G$ and choose $\|x\|_\delta \geq \|z\|_\delta + d$ such that $\|y\|_\delta \geq \|z\|_\delta + d$ and $\|x+y\|_\delta \geq \|z\|_\delta + d$. Clearly, $\|x+y\|_\delta \geq d$, $\|x+z\|_\delta \geq d$ and $\|x+y+z\|_\delta \geq d$.Then by (2.5), we get

$$\begin{aligned}
&\|Q(x+y+z) - f(x+y+z)\| \leq \frac{\varepsilon}{8};\\
&\|Q(x+y-z) - f(x+y-z)\| \leq \frac{\varepsilon}{8};\\
&\|Q(x-y+z) - f(x-y+z)\| \leq \frac{\varepsilon}{8};\\
&\|Q(-x+y+z) - f(-x+y+z)\| \leq \frac{\varepsilon}{8};\\
&\|-4Q(x) + 4f(x)\| \leq \frac{4\varepsilon}{8};\\
&\|-4Q(y) + 4f(y)\| \leq \frac{4\varepsilon}{8}.
\end{aligned}$$

Adding these inequalities and applying (2.5) and (2.1), we get

$$\|4Q(z) - 4f(z)\| \leq \varepsilon + \frac{3\varepsilon}{2}.$$

Therefore

$$\|Q(z) - f(z)\| \leq \frac{5\varepsilon}{8},$$

for $z \in G$.

It remains to prove the uniqueness of Q. Assume that $Q' : G \to Y$ is another quadratic function that satisfies inequality (2.2). Then we have

$$\begin{aligned}
\|Q(x) - Q'(x)\| &\leq \|Q(x) - f(x)\| + \|Q'(x) - f(x)\|\\
&\leq \frac{5\varepsilon}{4}, \quad x \in G,
\end{aligned}$$

Since Q and Q' are quadratic, the last inequality implies that

$$\begin{aligned}
\|Q(x) - Q'(x)\| &= \frac{1}{9^n}\|Q(3^n x) - Q'(3^n x)\|\\
&\leq \frac{1}{9^n} \times \frac{5\varepsilon}{4}, \qquad x \in G, n \in \mathbb{N} \setminus \{0\}.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we obtain $Q(x) = Q'(x)$ for all $x \in G$. This completes the proof. □

3 Asymptotic behavior of the equation

As a consequences, we can prove some corollaries concerning the asymptotic behaviors of the functional equation (1.2).

Corollary 3.1. *Let $(G,+,\delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y,\|.\|)$ be a normed space. If a mapping $f : G \to Y$ satisfies*

$$\limsup_{\|x+y+z\|_\delta \to +\infty} \Delta_f(x,y,z) = 0, \tag{3.1}$$

then f is a quadratic function on X.

Proof. Let $f : G \to Y$ be a mapping satisfies (3.1). Then, there exists a sequence $\{d_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\|\Delta_f(x,y,z)\| \leq \frac{1}{n}, \quad \|x+y+z\|_\delta \geq d_n, \quad n > 1.$$

Let $\widetilde{Y}$ be the completion of Y. By Theorem 2.1, there exists a unique quadratic function $Q_n : G \to \widetilde{Y}$ solution of (1.2) and such that

$$\|Q_n(x) - f(x)\| \leq \frac{5}{8n}, \quad x \in G, \quad n > 1. \tag{3.2}$$

Let l and m be integers satisfying $m > l > 0$. From (3.2) we obtain

$$\|Q_m(x) - f(x)\| \leq \frac{5}{8m} \leq \frac{5}{8l}, \quad x \in G.$$

Hence, the uniqueness of Q_n implies that $Q_l = Q_m$ holds for any $l, m \in \mathbb{N}$. Taking the limit as $n \longrightarrow \infty$ in (3.2), we infer that f is quadratic. Then the result follows. □

Using Theorem 2.1, we obtain the results.

Corollary 3.2. *Let $(G,+,\delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y,\|.\|)$ be a Banach space. Let $\psi\colon G \times G \times G \to [0,+\infty)$. If a mapping $f : G \to Y$ satisfies*

$$\begin{cases} \lim\limits_{\|x+y+z\|_\delta \to +\infty} \psi(x,y,z) = +\infty; \\ \limsup\limits_{\|x+y+z\|_\delta \to +\infty} \psi(x,y,z)\,\|\Delta_f(x,y,z)\| < \infty, \end{cases} \tag{3.3}$$

then f is a quadratic function on G.

Proof. It follows from (3.3) that there exist constants $s > 0$ and R such that

$$\psi(x,y,z)\,\|\Delta_f(x,y,z)\| < R, \quad \|x+y+z\|_\delta \geq s.$$

Since $\lim_{\|x+y+z\|_\delta \to +\infty} \psi(x,y,z) = +\infty$, then for an arbitrary $\varepsilon > 0$ there is $M > 0$ such that

$$\psi(x,y,z) \geq \frac{R}{\varepsilon}, \quad \|x+y+z\|_\delta \geq M.$$

Then,

$$\|\Delta_f(x,y,z)\| < \varepsilon, \quad \|x+y+z\|_\delta \geq \max\{s, M\}.$$

Let $\widetilde{Y}$ be the completion of Y. Using theorem 2.1, there exists a unique quadratic function $Q: G \to Y$ solution of (1.2) and such that

$$\|Q(x) - f(x)\| \leq \frac{5\varepsilon}{8}, \quad x \in G.$$

Since ε is arbitrary, we infer that $Q(x) = f(x)$ for all $x \in G$. □

Corollary 3.3. *Let $(G, +, \delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y, \|.\|)$ be a Banach space. Let $p < 0$ and $\lambda > 0$ be arbitrary real numbers. If a mapping $f: G \to Y$ satisfies*

$$\|\Delta_f(x,y,z)\| \leq \lambda \|x+y+z\|_\delta^p, \quad x,y,z \in G \setminus \{0\}.$$

Then f is a quadratic function on $G \setminus \{0\}$.

4 Application

Several functional equations were used to characterize inner product spaces from normed spaces, for instance, see [14, 2, 6, 3, 16]. Quadratic functional equation was used to characterize inner product spaces by making use the parallelogram equality [14]:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

This characterization gave rise to what we now call Jordan-von Neumann characterization. Other characterization is given by Fréchet in [11], he proved that a normed space $(X, \|.\|)$ is an inner product space if and only if

$$\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2, \quad x,y,z \in X.$$

Now, we can apply the functional equation (1.2) in a characterizations of inner product spaces.

Let $\mathbb{K}$ be the field of real or complex numbers. Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$ and $X_0 := X \setminus \{0\}$. Write

$$D(x,y,z) = \|x+y+z\|^2 + \|x+y-z\|^2 + \|x-y+z\|^2 + \|-x+y+z\|^2 - 4\|x\|^2 - 4\|y\|^2 - 4\|z\|^2.$$

Theorem 4.1. *Let* $(X, \|.\|)$ *be a normed space over* $\mathbb{K}$. *Suppose that*

$$D(x,y,z) = 0, \quad x,y,z \in X.$$

Then X *is an inner product space.*

Proof. Let $X \neq \{0\}$ be a normed space over $\mathbb{K}$ such that

$$\|x+y+z\|^2 + \|x+y-z\|^2 + \|x-y+z\|^2 + \|-x+y+z\|^2 - 4\|x\|^2 - 4\|y\|^2 - 4\|z\|^2 = 0, \quad (4.1)$$

for $x, y, z \in X$. Putting $z = 0$ in (4.1), we get

$$\|x+y\|^2 + \|x+y\|^2 + \|x-y\|^2 + \|-x+y\|^2 - 4\|x\|^2 - 4\|y\|^2 = 0,$$

for $x, y \in X$, then the Jordan-von Neumann characterization holds

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Consequently, X is an inner product space. □

Theorem 4.2. *Let* $(X, \|.\|)$ *be a normed space over* $\mathbb{K}$. *Suppose that*

$$\sup_{x,y,z \in X} \frac{|D(x,y,z)|}{\lambda \|x+y+z\|^p} < \infty, \quad x+y+z \in X_0, \quad p < 0, \quad \lambda > 0.$$

Then X *is an inner product space.*

Proof. Write $f(x) = \|x\|$ for $x \in X$. From Corollary 3.3 we easily derive that f is a quadratic function, which yields the statement. □

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