A homotopy based computational scheme for local fractional Helmholtz and Laplace equations

Jagdev Singh[*], Ritin Babu[†], Ved Prakash Dubey[‡], Devendra Kumar[§¶]

December 31, 2022

In this work, we investigate solutions for the local fractional Helmholtz and Laplace equations on Cantor set having importance in electrostatics, gravitation and fluid dynamics. To find exact solutions, the q-local fractional homotopy analysis transform method (q-LFHATM) has been used. The numerical results computed with the aid of the applied scheme shows that it is an efficient and accurate tool to solve differential equations with local fractional derivatives.

Keywords: Local fractional derivative operator; Partial differential equations; Laplace equation; Helmholtz equation; q-local fractional homotopy analysis transform method.

1 Introduction

The concept of local fractional calculus (LFC) has been used to model and analyze numerous fractal equations some of which are Fokker-Planck equation [1, 2], fractal wave equations [3], fractal-time dynamical systems [4, 5], the local fractional stress strain relations [6], the local fractional heat conduction model [7], local fractional Tricomi equation [8], local fractional Laplace equations [9], the Helmholtz equation associated with local fractional operator [10], fractal signals [11, 12], fractal Fourier analysis [13], Yang Fourier transform [14, 15, 16], Yang-Laplace transform [15, 17], fractal vehicular traffic flow [18], local fractional modelling in growths of population [19], and Boussinesq equation containing local fractional operator [20], etc. Some recent outcomes of different authors on local fractional methods involving local fractional integral transforms can be seen in a series of articles [21, 22, 23].

[*]Department of Mathematics, JECRC University, Jaipur-303905, Rajasthan, India
[†]Department of Mathematics, JECRC University, Jaipur-303905, Rajasthan, India
[‡]Faculty of Mathematical and Statistical Sciences, Shri Ramswaroop Memorial University, Barabanki-225003, Uttar Pradesh, India
[§]Department of Mathematics, University of Rajasthan, Jaipur-302004, Rajasthan, India
[¶]Corresponding author: devendra.maths@gmail.com

This work presents a very useful scheme known as the q-local fractional homotopy analysis transform method (q-LFHATM), which is a combination of q-HAM and the local fractional Laplace transform (LFLT). The proposed q-LFHATM is implemented to analyze the local fractional Helmholtz and Laplace equations. The merger of q-HAM and LFLT resulted in lesser C.P.U time (RAM-1 GB or more and Processor 2.65 GHz or more) while solving fractional-order nonlinear problems. The ability of the proposed method to achieve the series solution of local fractional Helmholtz and Laplace equations over a vast domain by picking approximate values for parameters is one of its advantages. El-Tavil & Huseen proposed the q-HAM [24, 25] which is a smooth generalization of the HAM. The HAM was first proposed and used by Liao [26, 27] to solve several problems found in engineering, science and finance.
The rest of the article is organized as follows: Section 2 reports basic definitions and formulae of LFC and LFLT. Section 3, the working of q-LFHATM is explained. The q-LFHATM is utilized to derive the solutions of local fractional Helmholtz and Laplace equations under different fractal conditions in Section 4. Section 5 presents the glimpse of numerical simulation for fractal order $\rho = \ln 2/\ln 3$. At the end, conclusion is reported in Section 6.

2 Preliminaries

Here, we provide certain important concepts of LFC and LFLT.
Definition 2.1. If we have a relation [12, 28]

$$|\theta(t) - \theta(t_0)| < \varepsilon^\alpha, 0 < \alpha \leq 1, \tag{1}$$

with $|\theta(t) - \theta(t_0)| < \delta$, for $\varepsilon, \alpha \in R$, then the function $\theta(t)$ is said to be local fractional continuous (LFC) at $t = t_0$ and is indicated by $\lim_{t \to t_0} \theta(t) = \theta(t_0)$. Here, $\theta(t)$ is called LFC on (a, b) and is expressed as

$$\theta(t) \in C_\alpha(a, b). \tag{2}$$

Definition 2.2. A function $\theta(t)$ is a nondifferentiable function of exponentα $(0 < \alpha \leq 1)$ if it satisfies the Hölder function of the exponent α. Then for $t, s \in T$, we have [12, 28]

$$|\theta(t) - \theta(s)| < C |t - s|^\alpha. \tag{3}$$

Definition 2.3. $\theta(t)$ is said to be continuous of α, $0<\alpha \leq 1$, or α continuous if there exists the following condition [12] $|\theta(t) - \theta(t_0)| < \varepsilon^\alpha$,

$$\theta(t) - \theta(t_0) = o((t - t_0)^\alpha). \tag{4}$$

In view of (4), Eq. (1) presents the standard form of local fractional continuity.
Definition 2.4. If $\theta(t) \in C_\alpha(a, b)$, then local fractional derivative (LFD) of $\theta(t)$ of order α at $x = x_0$ is written as [12, 28]:

$$\theta^{(\alpha)}(t_0) = \frac{d^\alpha}{dt^\alpha}\theta(t)|_{t=t_0} = \lim_{t \to t_0} \frac{\Delta^\alpha(\theta(t) - \theta(t_0))}{(t - t_0)^\alpha}, \tag{5}$$

where $\Delta^{\alpha}(\theta(t)-\theta(t_0)) \cong \Gamma(\alpha+1)(\theta(t)-\theta(t_0))$.
For any $t \in (a,b)$, we have $\theta^{(\alpha)}(t) = D_t^{\alpha}\theta(t)$. The LFD of $m\alpha$ order is expressed as:

$$\theta^{(m\alpha)} \equiv \overbrace{D_t^{\alpha}....D_t^{\alpha}\theta(t)}^{\text{m times}},$$

whereas the local fractional partial derivative (LFPD) of $m\alpha$ order is expressed as:

$$\frac{\partial^{m\alpha}\theta(t)}{\partial t^{m\alpha}} \equiv \overbrace{\frac{\partial^{\alpha}}{\partial t^{\alpha}}....\frac{\partial^{\alpha}}{\partial t^{\alpha}}\theta(t)}^{\text{m times}}.$$

Definition 2.5. Let $\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}|\theta(t)|(dt)^{\alpha} < m < \infty$. Then the Yang-Laplace (YL) transform [29, 30] of $\theta(t)$ is defined as:

$$L_{\alpha}\{\theta(t)\} = \theta_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty} E_{\alpha}(-s^{\alpha}t^{\alpha})\,\theta(t)\,(dt)^{\alpha}, 0 < \alpha \leq 1. \quad (6)$$

Here, the latter integral converges and $s^{\alpha} \in R^{\alpha}$.
Definition 2.6. The inverse of YL transform of $\theta(t)$ is stated as

$$L_{\alpha}^{-1}\left(\theta_s^{L,\alpha}(s)\right) = \theta(t) = \frac{1}{(2\pi)^{\alpha}}\int_{\upsilon-i\omega}^{\upsilon+i\omega} E_{\alpha}(s^{\alpha}t^{\alpha})\,\theta_s^{L,\alpha}(s)\,(ds)^{\alpha}, 0 < \alpha \leq 1, \quad (7)$$

where $s^{\alpha} = \upsilon^{\alpha} + i^{\alpha}\omega^{\alpha}$; fractal imaginary unit i^{α} and $Re(s) = \alpha > 0$.
Some useful formulae of LFLT [11, 12] are mentioned here:

$$L_{\alpha}\{a\theta(t) + b\phi(t)\} = a\theta_s^{L,\alpha}(s) + b\phi_s^{L,\alpha}(s), \quad (8)$$

$$L_{\alpha}\{E_{\alpha}(c^{\alpha}t^{\alpha})\,\theta(t)\} = \theta_s^{L,\alpha}(s-c), \quad (9)$$

$$L_{\alpha}\left\{\theta^{(m\alpha)}(t)\right\} = s^{m\alpha}\theta_s^{L,\alpha}(s)$$

$$-s^{(m-1)\alpha}\theta(0) - s^{(m-2)\alpha}\theta^{(\alpha)}(0) - \cdots - \theta^{((m-1)\alpha)}(0), \quad (10)$$

$$L_{\alpha}\{E_{\alpha}(c^{\alpha}t^{\alpha})\} = \frac{1}{s^{\alpha}-c^{\alpha}}, \quad (11)$$

$$L_{\alpha}\{\sin_{\alpha}(c^{\alpha}t^{\alpha})\} = \frac{c^{\alpha}}{s^{2\alpha}+c^{2\alpha}}, \quad (12)$$

$$L_{\alpha}\{t^{m\alpha}\} = \frac{\Gamma(1+m\alpha)}{s^{(m+1)\alpha}}. \quad (13)$$

Definition 2.7. The Mittag-Leffler function (MLF) is formulated as [12, 28]

$$E_{\alpha}(t^{\alpha}) = \sum_{m=0}^{\infty}\frac{t^{m\alpha}}{\Gamma(1+m\alpha)}, 0 < \alpha \leq 1. \quad (14)$$

The following results hold true;

$$\sin_\alpha(t^\alpha) = \sum_{m=0}^{\infty} (-1)^m \frac{t^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)},$$

$$\cos_\alpha(t^\alpha) = \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m\alpha}}{\Gamma(1+2m\alpha)}, 0<\alpha\le 1. \tag{15}$$

Certain fundamental formulas and results used in the work are presented below:

$$\frac{d^\alpha t^{m\alpha}}{dt^\alpha} = \frac{\Gamma(1+m\alpha)\, t^{(m-1)\alpha}}{\Gamma(1+(m-1)\alpha)}, \tag{16}$$

$$\frac{d^\alpha E_\alpha(t^\alpha)}{dt^\alpha} = E_\alpha(t^\alpha), \tag{17}$$

$$\frac{d^\alpha E_\alpha(mt^\alpha)}{dt^\alpha} = mE_\alpha(mt^\alpha). \tag{18}$$

3 Working plan of q-LFHATM

To elucidate the procedure of q-LFHATM, the following nonlinear local fractional partial differential equation (LFPDE) is investigated

$$Ł_\alpha\, \varepsilon(\eta,\kappa) + R_\alpha\, \varepsilon(\eta,\kappa) + N_\alpha \varepsilon(\eta,\kappa) = h(\eta,\kappa), \quad n-1<\alpha\le n, \tag{19}$$

where $Ł_\alpha$ indicates the linear local fractional operator, R_α indicates the remaining local linear differential operator, N_α stands for the nonlinear differential operator and $h(\eta,\kappa)$ signifies the source term.
Operating the LFLT on Eq. (19), we obtain

$$\begin{gathered} L_\alpha[\varepsilon(\eta,\kappa)] - s^{-\alpha}\varepsilon(\eta,0) - s^{-2\alpha}\varepsilon^{(\alpha)}(\eta,0) - \cdots - s^{-t\alpha}\varepsilon^{((t-1)\alpha)}(\eta,0) \\ +s^{-t\alpha} L_\alpha[R_\alpha\varepsilon(\eta,\kappa) + N_\alpha\varepsilon(\eta,\kappa) - h(\eta,\kappa)] = 0. \end{gathered} \tag{20}$$

We describe the nonlinear operator as:

$$\begin{aligned} N[\psi(\eta,\kappa;l)] = L_\alpha[\psi(\eta,\kappa;l)] - s^{-\alpha}\psi(\eta,\kappa;l)(0^+) - s^{-2\alpha}\psi^{(\sigma)}(\eta,\kappa;l)(0^+) - \cdots \\ -s^{-t\alpha}\psi^{((t-1)\alpha)}(\eta,\kappa;l)(0^+) + s^{-t\alpha}L_\alpha[R_\alpha\varepsilon(\eta,\kappa) + N_\alpha\varepsilon(\eta,\kappa) - h(\eta,\kappa)]. \end{aligned} \tag{21}$$

In Eq. (21), $l\in[0,\,1/n]$ and $\psi(\eta,\kappa;l)$ is a real valued function of η,κ & l. Now, the homotopy is framed as:

$$(1-nl)\, L_\alpha\left[[\psi(\eta,\kappa;l) - \varepsilon_0(\eta,\kappa)\right] = \hbar\, l\, N\left[\varepsilon(\eta,\kappa)\right]. \tag{22}$$

In Eq. (22), L_α stands for the LFLT operator, $n\ge 1$,$l\in[0,\,1/n]$ is an embedding variable, $\hbar\ne 0$ stands for an auxiliary parameter, $\varepsilon_0(\eta,\kappa)$ denotes initial guess (IG) of $\varepsilon(\eta,\kappa)$ and $\psi(\eta,\kappa;l)$ is an unidentified function. Clearly, for $l=0$ and $l=\frac{1}{n}$, the results obtained are

$$\psi(\eta,\kappa;0) = \varepsilon_0(\eta,\kappa), \qquad \psi(\eta,\kappa\,;\frac{1}{n}) = \varepsilon(\eta,\kappa), \tag{23}$$

4

respectively. Therefore, when l approaches from 0 to $\frac{1}{n}$, $\psi(\eta,\kappa;l)$ changes from the IG $\varepsilon_0(\eta,\kappa)$ to solution $\varepsilon(\eta,\kappa)$. Taylor series expansion of $\psi(\eta,\kappa;l)$ provides

$$\psi(\eta,\kappa;l)=\sum_{m=0}^{\infty}\varepsilon_m(\eta,\kappa)l^{\,m}. \tag{24}$$

where

$$\varepsilon_m(\eta,\kappa)=\frac{1}{m\,!}\frac{\partial^m\psi(\eta,\kappa;l)}{\partial l^m}\,|_{l=0}. \tag{25}$$

For proper values of $u_0(x,t)$,n and $\hbar$, the series (10) converges for $l=\frac{1}{n}$, then we obtain

$$\varepsilon(\eta,\kappa)=\sum_{m=0}^{\infty}\varepsilon_m(\eta,\kappa)(\frac{1}{n})^m. \tag{26}$$

Now, the set of vectors is characterized as

$$\overrightarrow{\varepsilon}_m=\{\varepsilon_0(\eta,\kappa),\ \varepsilon_1(\eta,\kappa),\cdots,\ \varepsilon_m(\eta,\kappa)\}. \tag{27}$$

Next, the mth-order deformation equation is composed as

$$L_\alpha\left[\varepsilon_m(\eta,\kappa)-\chi_m\varepsilon_{m-1}(\eta,\kappa)\right]=\hbar\,\Re_m(\overrightarrow{\varepsilon}_{m-1}). \tag{28}$$

Operating the inverse LFLT on Eq. (28), we obtain

$$\varepsilon_m(\eta,\kappa)=\chi_m\varepsilon_{m-1}(\eta,\kappa)+\hbar\,{L_\alpha}^{-1}[\Re_m(\overrightarrow{\varepsilon}_{m-1})]. \tag{29}$$

In Eq. (29), the value of $\Re_k(\vec{\phi}_{k-1})$ and χ_k are presented below

$$\Re_m(\overrightarrow{\varepsilon}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N\left[\varepsilon(\eta,\kappa;l)\right]}{\partial l^{m-1}}\,|_{l=0}, \tag{30}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ n, & m>1.\end{cases} \tag{31}$$

4 Illustrative examples

Example 4.1 Let us take the local fractional Helmholtz equation [10] as follows

$$\frac{\partial^{2\rho}\phi(u,v)}{\partial u^{2\rho}}+\frac{\partial^{2\rho}\phi(u,v)}{\partial v^{2\rho}}=\phi(u,v),0<\rho\le 1 \tag{32}$$

with initial-boundary conditions given as:

$$\phi\left(0,v\right)=0,\frac{\partial^\rho\phi(0,v)}{\partial u^\rho}=E_\rho\left(v^\rho\right). \tag{33}$$

Applying LFLT on Eq. (32), we obtain

$$L_\rho\left\{\phi(u,v)\right\}-s^\rho\phi(0,v)-s^{-2\rho}\frac{\partial^\rho\phi(0,v)}{\partial u^\rho}+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}-\phi(u,v)\right\}=0,$$

5

or

$$L_\rho\left\{\phi(u,v)\right\}-s^{-2\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}-\phi(u,v)\right\}=0. \tag{34}$$

The nonlinear operator is defined as

$$N[\psi(\eta,\kappa;l)]=L_\rho\left[\psi(\eta,\kappa;l)\right]-s^{-2\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\beta\left\{\frac{\partial^{2\rho}\psi(\eta,\kappa;l)}{\partial^{2\rho}v}-\psi(\eta,\kappa;l)\right\}, \tag{35}$$

and so

$$\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right)=L_\rho\left\{\phi_{m-1}\left(u,v\right)\right\}$$
$$-\left(1-\frac{\chi_m}{n}\right)s^{-2\rho}E_\rho\left(v^\rho\right)+s^{-2\rho}L_\rho\left[\frac{\partial^{2\rho}\phi_{m-1}\left(u,v\right)}{\partial v^{2\rho}}-\phi_{m-1}\left(u,v\right)\right]. \tag{36}$$

The mth-order deformation equation is built as

$$L_\rho\left\{\phi_m\left(u,v\right)-\chi_m\phi_{m-1}\left(u,v\right)\right\}=\hbar\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right). \tag{37}$$

Applying inverse LFLT on Eq. (37), we obtain

$$\phi_m\left(u,v\right)=\chi_m\phi_{m-1}\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right)\right\}. \tag{38}$$

For $m=1$, we have

$$\phi_1\left(u,v\right)=\chi_1\phi_0\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_1\left(\vec{\phi}_0\left(u,v\right)\right)\right\}$$

or

$$\phi_1\left(u,v\right)=-\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}. \tag{39}$$

For $m=2$, we have

$$\phi_2\left(u,v\right)=\chi_2\phi_1\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_2\left(\vec{\phi}_1\left(u,v\right)\right)\right\},$$

or

$$\phi_2\left(u,v\right)=-\left(n+\hbar\right)\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}. \tag{40}$$

For $m=3$, we have

$$\phi_3\left(u,v\right)=\chi_3\phi_2\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_3\left(\vec{\phi}_2\left(u,v\right)\right)\right\},$$

or

$$\phi_3\left(u,v\right)=-\left(n+\hbar\right)\hbar^2E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}, \tag{41}$$

& so on.

Hence, the nondifferentiable solution is expressed as

$$\phi(u,v)=\sum_{m=0}^{\infty}\phi_m(u,v)(\frac{1}{n})^m.$$

$$\phi(u,v)=\phi_0(u,v)+\frac{\phi_1(u,v)}{n}+\frac{\phi_2(u,v)}{n^2}+\frac{\phi_3(u,v)}{n^3}+\cdots$$

or

$$\phi(u,v)=\frac{1}{n}\left[\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}\right]-\frac{1}{n^2}\left[\left(n+\hbar\right)\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}\right]$$
$$-\frac{1}{n^3}\left[\left(n+\hbar\right)\hbar^2 E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}\right]+\cdots. \quad (42)$$

On letting $\hbar=-1$ and $n=1$, one can achieve the result

$$\phi\left(u,v\right)=\frac{u^\rho}{\Gamma\left(1+\rho\right)}E_\rho\left(v^\rho\right). \quad (43)$$

which is the solution of fractal problem (32).

Example 4.2 Take the following local fractional Laplace equation [9]

$$\frac{\partial^{2\rho}\phi(u,v)}{\partial u^{2\rho}}+\frac{\partial^{2\rho}\phi(u,v)}{\partial v^{2\rho}}=0, 0<\rho\le 1, \quad (44)$$

with initial-boundary conditions given as:

$$\phi\left(0,v\right)=-E_\rho\left(v^\rho\right), \frac{\partial^\rho\phi(0,v)}{\partial u^\rho}=0. \quad (45)$$

Applying LFLT on Eq. (44), we obtain

$$L_\rho\left\{\phi(u,v)\right\}-s^{-\rho}\phi(0,v)-s^{-2\rho}\frac{\partial^\rho\phi(0,v)}{\partial u^\rho}+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0,$$

or

$$L_\rho\left\{\phi(u,v)\right\}+s^{-\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0. \quad (46)$$

The nonlinear operator is

$$N[\psi(\eta,\kappa;l)]=L_\rho\left[\psi(\eta,\kappa;l)\right]+s^{-\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\beta\left\{\frac{\partial^{2\rho}\psi(\eta,\kappa;l)}{\partial^{2\rho}v}\right\}, \quad (47)$$

and so

$$\Re_m\left(\vec{\phi}_{m-1}(u,v)\right)=L_\rho\left\{\phi_{m-1}(u,v)\right\}$$
$$+\left(1-\frac{\chi_m}{n}\right)s^{-\rho}E_\rho\left(v^\rho\right)+s^{-2\rho}L_\rho\left[\frac{\partial^{2\rho}\phi_{m-1}(u,v)}{\partial v^{2\rho}}\right]. \quad (48)$$

The mth-order deformation equation is constituted as:

$$L_\rho \left\{\phi_m(u,v) - \chi_m \phi_{m-1}(u,v)\right\} = \hbar \Re_m \left(\vec{\phi}_{m-1}(u,v)\right). \tag{49}$$

Operating the inverse LFLT, we obtain

$$\phi_m(u,v) = \chi_m \phi_{m-1}(u,v) + \hbar L_\rho^{-1} \left\{\Re_m \left(\vec{\phi}_{m-1}(u,v)\right)\right\}. \tag{50}$$

Taking $m = 1, 2, 3, \ldots$, we get

For $m = 1$

$$\phi_1(u,v) = -\hbar E_\rho(v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)}. \tag{51}$$

For $m = 2$

$$\phi_2(u,v) = -(n+\hbar)\hbar E_\rho(v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} - \hbar^2 E_\rho(v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)}. \tag{52}$$

For $m = 3$

$$\phi_3(u,v) = -(n+\hbar)^2 \hbar E_\rho(v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} - 2(n+\hbar)\hbar^2 E_\rho(v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)}$$
$$-\hbar^3 E_\rho(v^\rho) \frac{u^{6\rho}}{\Gamma(1+6\rho)}, \tag{53}$$

& so on.

Hence, the nondifferentiable solution is presented as

$$\phi(u,v) = -\frac{1}{n}\left[\hbar E_\rho(v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)}\right] - \frac{1}{n^2}\left[(n+\hbar)\hbar E_\rho(v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} - \hbar^2 E_\rho(v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)}\right]$$
$$-\frac{1}{n^3}\left[(n+\hbar)^2 \hbar E_\rho(v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} + 2(n+\hbar)\hbar^2 E_\rho(v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)} + \hbar^3 E_\rho(v^\rho) \frac{u^{6\rho}}{\Gamma(1+6\rho)}\right] + \cdots. \tag{54}$$

Setting $\hbar = -1$ and $n = 1$, one can have

$$\phi(u,v) = E_\rho(v^\rho)\left(-1 + \frac{u^{2\rho}}{\Gamma(1+2\rho)} - \frac{u^{4\rho}}{\Gamma(1+4\rho)} + \frac{u^{6\rho}}{\Gamma(1+6\rho)} - \cdots\right).$$

It can be written as

$$\phi(u,v) = E_\rho(v^\rho)\left(\sum_{m=0}^{\infty} (-1)^m \frac{u^{2m\rho}}{\Gamma(1+2m\rho)}\right).$$

The solution of Eq. (44) is constituted as

$$\phi(u,v) = E_\rho(v^\rho) \cos_\rho(u^\rho). \tag{55}$$

Example 4.3 Finally, the following Laplace equation with LFD [9] is investigated

$$\frac{\partial^{2\rho}\phi(u,v)}{\partial u^{2\rho}}+\frac{\partial^{2\rho}\phi(u,v)}{\partial v^{2\rho}}=0, 0<\rho\leq 1 \tag{56}$$

with initial-boundary conditions given as:

$$\phi\left(0,v\right)=0, \frac{\partial^{\rho}\phi(0,v)}{\partial u^{\rho}}=-E_{\rho}\left(y^{\rho}\right). \tag{57}$$

Applying LFLT on Eq. (56), we obtain

$$L_{\rho}\left\{\phi(u,v)\right\}-s^{-\rho}\phi(0,v)-s^{-2\rho}\frac{\partial^{\rho}\phi(0,v)}{\partial u^{\rho}}+s^{-2\rho}L_{\rho}\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0$$

or

$$L_{\rho}\left\{\phi(u,v)\right\}+s^{-2\rho}E_{\rho}\left(y^{\rho}\right)+s^{-2\rho}L_{\rho}\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0. \tag{58}$$

The nonlinear operator is constituted as

$$N[\psi(\eta,\kappa;l)]=L_{\rho}\left[\psi(\eta,\kappa;l)\right]+s^{-2\rho}E_{\rho}\left(y^{\rho}\right)+s^{-2\rho}L_{\beta}\left\{\frac{\partial^{2\rho}\psi(\eta,\kappa;l)}{\partial^{2\rho}v}\right\}, \tag{59}$$

and so

$$\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right)=L_{\rho}\left\{\phi_{m-1}\left(u,v\right)\right\}$$

$$+\left(1-\frac{\chi_m}{n}\right)s^{-2\rho}E_{\rho}\left(v^{\rho}\right)+s^{-2\rho}L_{\rho}\left[\frac{\partial^{2\rho}\phi_{m-1}\left(u,v\right)}{\partial v^{2\rho}}\right]. \tag{60}$$

Next, we present the mth-order deformation equation as

$$L_{\rho}\left\{\phi_m\left(u,v\right)-\chi_m\phi_{m-1}\left(u,v\right)\right\}=\hbar\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right). \tag{61}$$

Applying the inverse LFLT, we obtain

$$\phi_m\left(u,v\right)=\chi_m\phi_{m-1}\left(u,v\right)+\hbar L_{\rho}^{-1}\left\{\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right)\right\}. \tag{62}$$

Taking $m=1,2,3,\ldots$, we get
For $m=1$, we have

$$\phi_1\left(u,v\right)=\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}. \tag{63}$$

For $m=2$, we obtain

$$\phi_2\left(u,v\right)=\left(n+\hbar\right)\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}+\hbar^2E_{\rho}\left(v^{\rho}\right)\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}. \tag{64}$$

For $m=3$, we find

$$\phi_3\left(u,v\right)=\left(n+\hbar\right)^2\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}$$

$$+2\left(n+\hbar\right)\hbar^{2}E_{\rho}\left(v^{\rho}\right)\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}+\hbar^{3}E_{\rho}\left(v^{\rho}\right)\frac{u^{5\rho}}{\Gamma\left(1+5\rho\right)}, \tag{65}$$

Hence, the nondifferentiable solution is

$$\phi(u,v)=\sum_{m=0}^{\infty}\phi_{m}(u,v)(\frac{1}{n})^{m}.$$

or

$$\phi(u,v)=\frac{1}{n}\left[\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}\right]+\frac{1}{n^{2}}\left[\left(n+\hbar\right)\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}-\hbar^{2}E_{\rho}\left(v^{\rho}\right)\left(\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}\right)\right]$$

$$+\frac{1}{n^{3}}\left[\left(n+\hbar\right)^{2}\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}+2\left(n+\hbar\right)\hbar^{2}E_{\rho}\left(v^{\rho}\right)\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}+\hbar^{3}E_{\rho}\left(v^{\rho}\right)\frac{u^{5\rho}}{\Gamma\left(1+5\rho\right)}\right]+\cdots. \tag{66}$$

On using the values $\hbar=-1$ and $n=1$, we have

$$\phi\left(u,v\right)=E_{\rho}\left(v^{\rho}\right)\left(-\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}+\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}-\frac{u^{5\rho}}{\Gamma\left(1+5\rho\right)}+\cdots\right).$$

The solution of Eq. (56) in closed form is expressed as

$$\phi\left(u,v\right)=E_{\rho}\left(v^{\rho}\right)\left(\sum_{m=0}^{\infty}\left(-1\right)^{m}\frac{u^{(2m+1)\rho}}{\Gamma\left(1+\left(2m+1\right)\rho\right)}\right).$$

or

$$\phi\left(u,v\right)=E_{\rho}\left(v^{\rho}\right)\sin_{\rho}\left(u^{\rho}\right). \tag{67}$$

5 Numerical simulation

This section presents numerical outcomes for fractal problem given in Examples 4.1-4.3 under fractal initial-boundary conditions. The 3D graphs for the local fractional Helmholtz and Laplace equations are demonstrated on the Cantor set for the fractal order $\rho=\ln 2/\ln 3$ via MATLAB. The graphics authenticates that the achieved solutions for Examples 4.1-4.3 depend on the fractal order ρ of the LFD. The 3D graphical visuals show the fractal pattern of the nondifferentiable function $\phi\left(u,\,v\right)$ in Examples 4.1-4.3.

6 Conclusions

In this work, the q-LFHATM is utilized to obtain the nondifferentiable solutions for the Helmholtz and Laplace equation in fractal media. The computed results establish the reliability and efficiency of the proposed technique and the applied method can be used to solve many other LFPDEs arising in fractal media. Finally, the computer simulations are also presented for fractal analysis of local fractional Helmholtz and Laplace models.

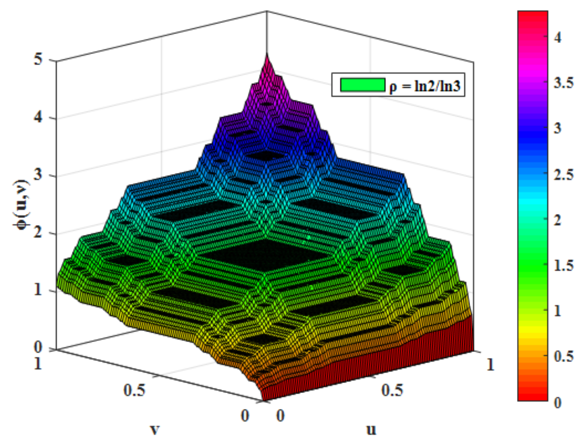

Figure 1: 3D nature of $\phi(u,v)$ w.r.t. u and v for Example 4.1

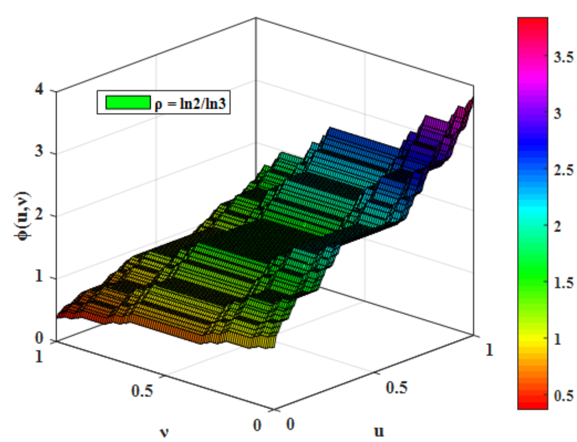

Figure 2: 3D behavior of $\phi(u,v)$ w.r.t. u and v for Example 4.2

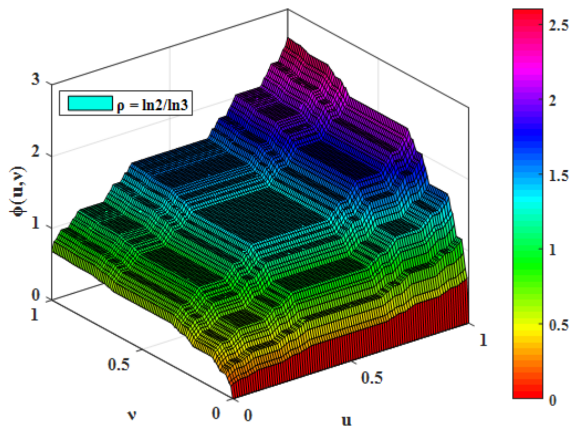

Figure 3: 3D pattern of $\phi\,(u,v)$ w.r.t. u and v for Example 4.3

References

[1] Kolwankar K.M., Gangal A.D., Hölder exponents of irregular signals and local fractional derivatives, *Pramana J. Phys.*, **48** (1997) 49-68.

[2] Kolwankar K.M., Gangal A.D., Local fractional Fokker-Planck equation, *Phys. Rev. Lett.*, **80** (1998) 214-217.

[3] Ahmad J., Mohyud-Din S.T., Solving wave and diffusion equations on Cantor sets, Proc. *Pakistan Acad. Sci.*, **52** (1) (2015) 81-87.

[4] Parvate A., Gangal A.D., Fractal differential equations and fractal time dynamical systems, *Pramana J. Phys*, **64** (3)(2005) 389- 409.

[5] Parvate A., Gangal A.D., Calculus on fractal subsets of real line I: formulation, *Fractals*, **17** (1) (2009) 53- 81.

[6] Carpinteri A., Chiaia B., Cornetti P., Static–kinematic duality and the principle of virtual work in the mechanics of fractal media, *Comput. Method. Appl. Mech. Eng.*, **191** (1-2) (2001) 3-19.

[7] Zhao D., Singh J., Kumar D., Rathore S., Yang X.-J., An efficient computational technique for local fractional heat conduction equations in fractal media, *J. Nonlinear Sci. Appl.*, **10** (4) (2017) 1478-1486.

[8] Singh J., Kumar D., Nieto J.J., A reliable algorithm for local fractional Tricomi equation arising in fractal transonic flow, *Entropy*, **18** (6) 206 (2016) 1-8.

[9] Dubey, V.P., Singh, J., Alshehri, A.M., Dubey, S., Kumar, D., A comparative analysis of two computational schemes for solving local fractional Laplace equations, *Math. Meth. Appl. Sci.*, **44** (17) (2021) 13540–13559.

[10] Yang A.M., Chen, Z., Srivastava, H.M., Yang X.-J., Application of the local fractional series expansion method and the variational iteration method to the Helmholtz equation involving local fractional derivative operators, *Abstr. Appl. Anal.*, **2013** (2013) 259125.

[11] Yang X.-J., Local Fractional Integral Transforms, *Progr. In nonlinear Sci.*, **4** (2011) 12-25.

[12] Yang X.-J., *Local Fractional Functional Analysis and Its Applications*, Asian Academic publisher Limited, Hong Kong (2011).

[13] Yang X.-J., Generalized sampling theorem for fractal signals, *Advances in Digital Multimedia*, **1** (2) (2012) 88-92.

[14] Yang X.-J., The discrete Yang-Fourier transforms in fractal space, *Advances in Electrical Engineering Systems*, **1** (2) (2012) 78-81.

[15] Yang X.-J., Local fractional partial differential equations with boundary problems, *Adv. Comput. Math. Appl.*, **1** (1) (2012) 60-63.

[16] Zhong W.P., Gao F., Shen X.M., Applications of Yang-Fourier transform to local fractional equations with local fractional derivative and local fractional integral, *Advanced Materials Research*, **416** (2012) 306-310.

[17] Guo Y., Local fractional Z transform in fractal space, *Advances in Digital Multimedia*, **1** (2) (2012) 96-102.

[18] Kumar D., Tchier F., Singh J., Baleanu D., An efficient computational technique for fractal vehicular traffic flow, *Entropy* **20**(4) (2018) 259.

[19] Yang X.-J., Machado JAT. A new insight into complexity from the local fractional calculus view point: modelling growths of populations, *Math Methods Appl Sci.*, **40** (17) (2017) 6070-6075.

[20] Yang X.-J., Machado ZAT, Baleanu D. Exact travelling-wave solution for local fractional Boussinesq equation in fractal domain, *Fractals*, **25** (04) (2017) 1740006.

[21] Dubey, V.P., Kumar, D., Singh, J., Alshehri, A.M., Dubey, S., Analysis of local fractional Klein-Gordon equations arising in relativistic fractal quantum mechanics, *Wave Random Complex Media*, (2022) 1-21, doi: 10.1080/17455030.2022.2112993.

[22] Dubey, V.P., Kumar, D., Alshehri, H.M., Dubey, S., Singh, J., Computational analysis of local fractional LWR model occurring in a fractal vehicular traffic flow, *Fractal Fract.*, **6** (2022), 1-19.

[23] Dubey, S., Dubey, V.P., Singh, J., Alshehri, A.M., Kumar, D., Computational study of a local fractional Tricomi equation occurring in fractal transonic flow, *J. Comput. Nonlinear Dynam.*, **17** (2022) 081006, doi: 10.1115/1.4054482.

[24] El-Tawil M.A., Huseen S.N., The q-Homotopy Analysis Method (q-HAM), *Int. J. Appl. Maths. Mech.*, **8** (2012) 51-75.

[25] El-Tawil M.A., Huseen S.N., On convergence of the q-Homotopy Analysis Method(q-HAM), *Int. J. Contemp. Math. Sci.*, **8** (2013) 481-497.

[26] Liao S.J., *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. Thesis, Shanghai Jiao Tong Univ. (1992).

[27] Liao S.J., *Beyond Perturbation: Introduction to the Homotopy analysis method*, Chapman and Hall/CRC Press, Boca Raton (2003).

[28] Yang, X.-J., *Advanced Local Fractional Calculus and Its Applications*, World Science, New York (2012).

[29] Zhang Y.Z., Yang A.M., Long Y., Initial boundary value problem for fractal heat equation in the semi-infinite region by Yang-Laplace transform, *Thermal Science*, **18** (2) (2014) 677–681.

[30] Zhao C.G., Yang A.M., Jafari H., Haghbin A., The Yang-Laplace transform for solving the IVPs with local fractional derivative, *Abstr. Appl. Anal.*, **2014** (2014) 1-5.