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Two-State Retrial Queueing Model with Catastrophe

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Abstract

The present paper discusses a two-state retrial queueing system with catastrophe. If a customer on arrival finds the server free then it is served immediately. Such customer is known as primary customer. Moreover, if the server is busy then the customer joins virtual queue and retries for service after a random amount of time. This customer is called a secondary customer. Primary and secondary customers both follow Poisson processes. Inter arrival times and service times both follow exponential distribution. Catastrophe occurs on a busy server and its occurrence follows a Poisson process. Catastrophe causes the failure of server and so the server is sent for repair after occurrence of catastrophe. The repair times are also exponentially distributed. Time dependent probabilities for exact number of arrivals in the system and departures from the system when the server is idle or busy are obtained by using recursive approach. The probability of server being under repair is also obtained. Verification of results is done. Numerical results are generated and represented graphically to study the effect of various parameters.

Keywords: Retrial, Arrivals, Departures, Catastrophe, Repair.

1 Introduction

In many real life situations it has been observed that a customer does not get the service instantly on arrival. So he tries again for the service after a random period of time which is popularly known as retrial. The queueing systems with these repeated attempts have been used in many fields such as telecommunication, computer networks, data transmission, etc. Analysis of such systems developed a new class of queueing systems which is known as retrial queueing systems. Retrial queue is a model of the system with finite capacity where if the arrival finds a free server, it is served immediately. However, if the server is not free, the customer leaves the service area and joins the virtual queue known as orbit. Thereafter it retries from the orbit after a random amount of time to get service.

Call centers serve as a basic example to retrial systems where call agent is the server and a person who is calling is the customer. If a customer is able to connect the call agent immediately after making a call, he is answered else he has to repeat the call.

[6] is the early work done on retrial queues. [16] discussed some important single server retrial queuing models and represented analytic results. [10] analyzed the single server retrial queue with finite number of sources and established customer's arrival distribution, busy period and waiting time process. An explanation of the retrial queueing system is shown in the following diagram:

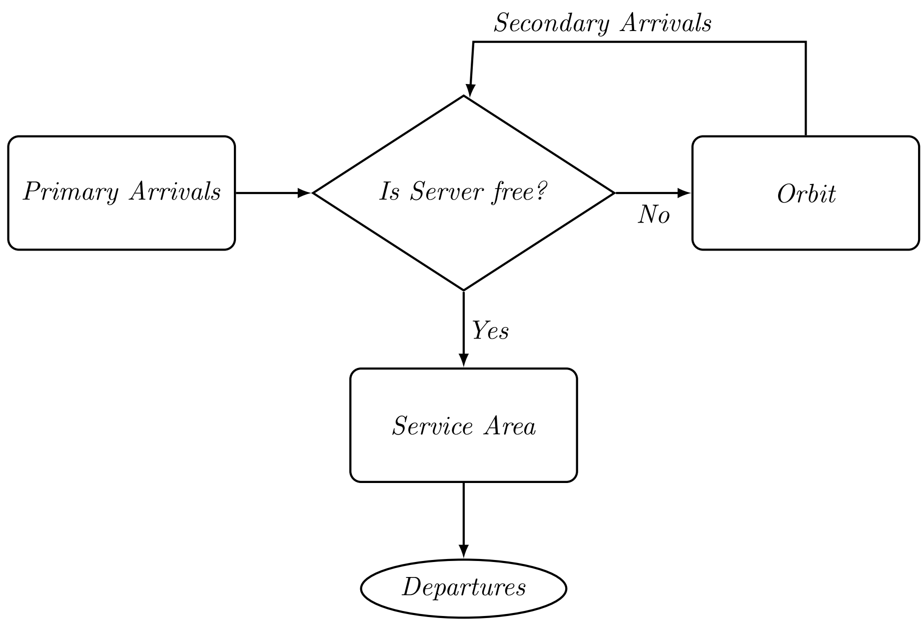

Figure 1. Basic Structure of a Retrial Queueing System

[13] was the first who introduced the concept of two-state in 'Some New Results for the M/M/1 Queue'. In this paper they obtained a closed form solution for the probability that exactly i arrivals and j services occur over a time interval of length t. [14] studied the two-state single server retrial queuing model in which the time dependent probabilities of exact number of arrivals and departures in the system are obtained when the server is free or busy. [15] developed 'A two-state retrial queueing model with feedback having two identical parallel servers' in which transient solution is obtained for the retrial queueing model.

2

In recent years many researchers have shown interest towards the concept of catastrophe. Catastrophe is a sudden, unexpected failure in a machine, computer network, electronic system, communication system, etc. Catastrophe occurs at random, deletes all the customers present in the system and inactivates the service facilities for a short period of time. Catastrophe resets the system from current state to zero state at random time intervals. Catastrophe may come from outside the system or from another service station. Retrial queueing system with catastrophe can be seen in call centers, computer networks and in telecommunication networks. In population dynamics, catastrophe can be considered as the natural disasters such as floods, storms, etc. On the other hand in the queueing models, catastrophe makes the system empty and causes server's breakdown.

For example: In call centers with the occurrence of catastrophic events like power failure, virus attacks will result in loss of all the calls present at that time and breakdown of the network.

[8], [5] are the works done on Catastrophe occurring in a simple Markovian queue. [11] discussed the asymptotic behavior of the probability of server being free. [12] worked on M/M/1 queuing system with catastrophes. Transient solution is obtained for system with server failure and non-zero repair time.

Furthermore, the server is sent for repair immediately when the failure occurs. After getting repaired, the server comes back to its working position and the system becomes ready to serve new customers. [4] proposed $M/M/\infty$ queueing system with catastrophe and repairable servers.

Following diagram shows the basic structure of retrial queueing system with catastrophe.

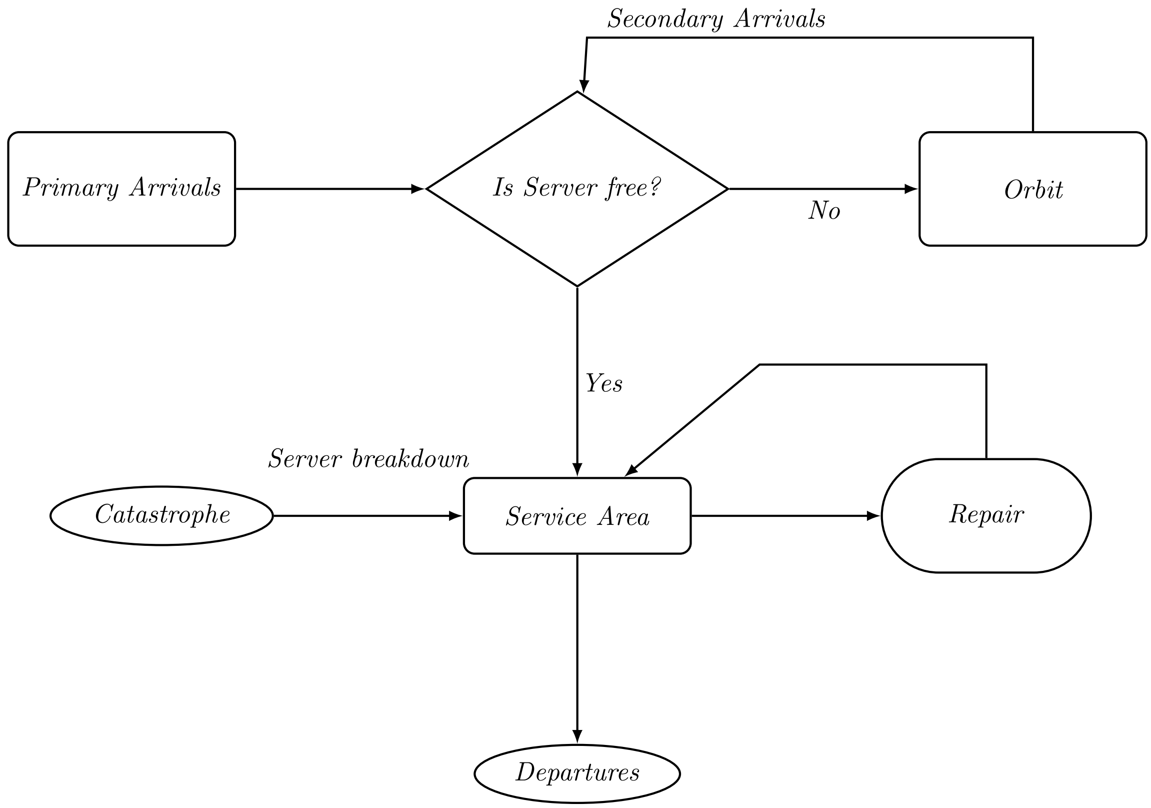

Figure 2. Basic Structure of a Retrial Queue With Catastrophe.

[1] studied the transient behaviour of two-processor heterogeneous system with catastrophes, server failures and repairs. [7] studied a fractional $M/M/1$ queue with catastrophe. [3] obtained transient solution of markovin queues with catastrophe having infinite servers.

The transient solutions are used to study the dynamic behaviour of a system. They are useful to study the characteristics of a system on different time points. Therefore, transient analysis of queueing systems is extremely important from theoretical and practical perspective.

In this paper, we derive two-state time-dependent probabilities for exact number of arrivals to the system and departures from the system by time t when the server is idle or busy for a single server retrial queueing system. The factor two-state makes the results well-quantified as in the case of [13]. Also we obtain the probability of server being under repair when the server fails due to catastrophic events at random time intervals. Besides these theoretical solutions, we present some numerical results graphically to study the effect of various parameters and the behaviour of probabilities with respect to average service times.

The paper has been organized in the following sections:

In section 2 the complete mathematical description of the model is defined. Also, the difference-differential equations are derived in this section. Solution of the model is given in section 3 in which we obtained the transient state probabilities and the probability of server being under repair. In section 4 verification of results is done. The numerical results are obtained and represented graphically in section 5. In section 6 the busy period probabilities of system and the server are obtained numerically and presented graphically. Section 7 discusses the conclusion and in section 8 acknowledgment is given. Finally the references are listed.

2 Model Description

In this paper, we are considering a two-state single server retrial queueing system with catastrophe. In this system, customers arrive according to Poisson process. If on arrival customer finds the server busy, he joins the orbit and retries from the orbit. These retrials are considered to be secondary arrivals. Catastrophe occurs according to Poisson process. We are assuming in our model that catastrophe occurs only when the system is non-empty and when the server is busy. It has no effect on the system when the system is empty. Catastrophe makes system empty and also causes server's breakdown by deleting all the customers present in the system. Once the system becomes empty or when the server breaks down, it is sent for repair immediately. Further, it is assumed that during the repair time no arrival can take place. The detailed description of the model is given as follows:

- Arrival Process: The primary customers arrive at the system according to Poisson process with mean arrival rate λ.
- Retrial Process: The secondary customers arrive at the system according to Poisson process with mean retrial rate θ.
- Service Process: The service times are exponentially distributed with parameter μ.
- Catastrophe: Catastrophe follow Poisson process with mean rate ξ.
- Repair: The repair time is exponentially distributed with parameter τ.

Also, the primary and secondary arrivals, inter-arrival times, service times, departures and catastrophes are mutually independent.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt; \quad Re(s) > 0$$

The Laplace inverse of

$$\frac{Q(p)}{P(p)} = \sum_{k=1}^{n}\sum_{l=1}^{m_k} \frac{t^{m_k-l}e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}}\left(\frac{Q(p)}{P(p)}\right)(p-a_k)^{m_k} \quad \forall p = a_k, \quad a_i \neq a_k \; for \; i \neq k$$

where

$$P(p) = (p-a_1)^{m_1}(p-a_2)^{m_2}.......(p-a_n)^{m_n}$$

Q(p) is a polynomial of degree $< m_1 + m_2 + m_3 + ... + m_n - 1$

$$If \;\; L^{-1}\{f(s)\} = F(t) \;\; and \;\; L^{-1}\{g(s)\} = G(t) \;\; then$$

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = F * G$$

$F * G$ is called the convolution of F and G.

2.1 The Two-Dimensional State Model

2.1.1 Notations

$P_{i,j,0}(t)$ = Probability that there are exactly i number of arrivals in the system and j number of departures from the system by time t and the server is free.

$P_{i,j,1}(t)$ = Probability that there are exactly i arrivals, j departures from the system by time t and the server is busy.

$Q(t)$=Probability that the server is under repair by time t.

$P_{i,j}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t) \qquad \forall\, i,j \;\; i \geq j;$$

$$and \qquad P_{i,j,1}(t) = 0, \; i \leq j; \qquad P_{i,j,0}(t) = 0, \quad i < j;$$

Initially

$$P_{0,0,0}(0) = 1; \qquad P_{i,j,0}(0) = 0, \quad P_{i,j,1}(0) = 0, \; i,j \neq 0; \qquad Q(0) = 0;$$

2.1.2 The Difference-Differential equations governing the system are:

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu P_{i,j-1,1}(t) \qquad i \geq j > 0 \tag{1}$$

$$\frac{d}{dt}P_{0,0,0}(t) = -\lambda P_{0,0,0}(t) + \tau Q(t) \tag{2}$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda + \mu + \xi)P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda P_{i-1,j,1}(t)(1 - \delta_{i-1,j}) + (i-j)\theta P_{i,j,0}(t) \qquad i > j \geq 0 \tag{3}$$

6

$$\frac{d}{dt}Q(t) = -\tau Q(t) + \xi \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} P_{i,j,1}(t) \tag{4}$$

where

$$\delta_{i-1,j} = \begin{cases} 1 & \text{when } i-1=j \\ 0 & \text{otherwise} \end{cases}$$

Using the Laplace transformation $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad Re(s) > 0$$

in the equations (1)-(4) along with the initial conditions, we have

$$(s+\lambda+(i-j)\theta)\bar{P}_{i,j,0}(s) = \mu\bar{P}_{i,j-1,1}(s) \qquad i \geq j > 0 \tag{5}$$

$$\begin{aligned}(s+\lambda+\mu+\xi)\bar{P}_{i,j,1}(s) = \lambda\bar{P}_{i-1,j,0}(s) + \lambda\bar{P}_{i-1,j,1}(s)(1-\delta_{i-1,j})+ \\ (i-j)\theta\bar{P}_{i,j,0}(s) \qquad i > j \geq 0\end{aligned} \tag{6}$$

$$(s+\lambda)\bar{P}_{0,0,0}(s) = 1 + \tau\bar{Q}(s) \tag{7}$$

$$(s+\tau)\bar{Q}(s) = \xi\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \bar{P}_{i,j,1}(s) \tag{8}$$

where

$$\delta_{i-1,j} = \begin{cases} 1 & \text{when } i-1=j \\ 0 & \text{otherwise} \end{cases}$$

3 Solution of the Problem

Solving equations (5)-(8) recursively, we have

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} + \frac{\tau}{s+\lambda}\bar{Q}(s) \tag{9}$$

$$\bar{P}_{i,0,1}(s) = \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^i \left(\frac{1}{s+\lambda} + \frac{\tau}{s+\lambda}\bar{Q}(s)\right) \qquad i \geq 1 \tag{10}$$

$$\bar{P}_{i,1,0}(s) = \left[\frac{\mu}{s+\lambda+(i-1)\theta}\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^i \left(\frac{1}{s+\lambda} + \frac{\tau}{s+\lambda}\bar{Q}(s)\right)\right] \quad i \geq 1 \tag{11}$$

$$\bar{P}_{i,i,0}(s) = \frac{\mu}{s+\lambda}\left[\frac{\lambda}{s+\lambda+\mu+\xi}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda+\mu+\xi}\bar{P}_{i,i-1,0}(s)\right] \quad i \geq 2 \tag{12}$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{s+\lambda+\mu+\xi}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda+\mu+\xi}\bar{P}_{i,i-1,0}(s) \qquad i \geq 2 \tag{13}$$

7

$$\bar{P}_{i,j,0}(s) = \frac{\mu}{s+\lambda+(i-j)\theta}\Bigg[\sum_{k=1}^{i-j+1}\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j-k+1}\eta_k^{'}(s)\bar{P}_{j+k-1,j-1,0}(s) + \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j}\bar{P}_{j,j-1,1}(s)\Bigg] \qquad i > j > 1 \tag{14}$$

where

$$\eta_k^{'} = \begin{cases} 1 & \text{if } k = 1 \\ 1 + \dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = 2 \; to \; i-j \\ \dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = i-j+1 \end{cases}$$

$$\bar{P}_{i,j,1}(s) = \sum_{k=1}^{i-j}\Bigg[\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j-k}\eta_k^{'}(s)\bar{p}_{j+k,j,0}(s)\Bigg] + \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j-1}\bar{P}_{j+1,1,1}(s) \qquad i \geq j+2, j \geq 1 \tag{15}$$

where

$$\eta_k^{'} = \begin{cases} 1 & \text{if } k = 1 \\ 1 + \dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = 2 \; to \; i-j-1 \\ \dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = i-j \end{cases}$$

$$\bar{Q}(s) = \frac{\xi}{s+\tau}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\bar{P}_{i,j,1}(s) \tag{16}$$

Taking the Inverse Laplace transform of equations $(9)-(16)$, we have

$$P_{0,0,0}(t) = e^{-\lambda t} + \tau e^{-\lambda t} * Q(t) \tag{17}$$

$$P_{i,0,1}(t) = \lambda^i e^{-\lambda t}\Bigg[\frac{1}{(\mu+\xi)^i} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-r}}\Bigg] + \tau\lambda^i \Bigg[\frac{1}{(\mu+\xi)^i} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-r}}\Bigg] * Q(t) \qquad i \geq 1 \tag{18}$$

8

$$P_{i,1,0}(t) = \mu e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t) \qquad i \geq 1 \tag{19}$$

$$P_{i,i,0}(t) = \mu\lambda e^{-\lambda t}\left[\frac{1}{\mu+\xi} - \frac{e^{-(\mu+\xi)t}}{\mu+\xi}\right] * P_{i-1,i-1,0}(t) + \mu\theta e^{-\lambda t} \left[\frac{1}{\mu+\xi} - \frac{e^{-(\mu+\xi)t}}{\mu+\xi}\right] * P_{i,i-1,0}(t) \qquad i > 1 \tag{20}$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu+\xi)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu+\xi))t} * P_{i,i-1,0}(t) \qquad i > 1 \tag{21}$$

$$\begin{aligned} P_{i,j,0}(t) &= \mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t}\left[\frac{1}{(\mu+\xi)^{i-j}} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-j-r}}\right] * P_{j,j-1,0}(t) \\ &+ e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}\mu\lambda^{i-j-k+1}\left[\frac{1}{(\mu+\xi)^{i-j-k+1}} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-j-k-r+1}}\right] \\ &* P_{j+k-1,j-1,0}(t) + e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(\mu k\theta)\lambda^{i-j-k+1}\left[\frac{1}{(\mu+\xi)^{i-j-k+2}}\right. \\ &\left. - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-k+1}\frac{t^r}{r!}\frac{1}{\mu^{i-j-k-r+2}}\right] * P_{j+k-1,j-1,0}(t) + e^{-(\lambda+(i-j)\theta)t} \\ &((i-j+1)\mu\theta)\left[\frac{1}{\mu+\xi} - \frac{e^{-(\mu+\xi)t}}{\mu+\xi}\right] * P_{i,j-1,0}(t) + \mu\lambda^{i-j}e^{-(\lambda+(i-j)\theta)t} \\ &\left[\frac{1}{(\mu+\xi)^{i-j}} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-j-r}}\right] * P_{j,j-1,1}(t) \qquad i > j > 1 \end{aligned} \tag{22}$$

$$\begin{aligned} P_{i,j,1}(t) &= \lambda^{i-j-1}e^{-(\lambda+\mu+\xi)t}\frac{t^{i-j-2}}{(i-j-2)!} * P_{j+1,j,0}(t) + e^{-(\lambda+\mu+\xi)t}\sum_{k=2}^{i-j-1}\lambda^{i-j-k} \\ &\frac{t^{i-j-k-1}}{(i-j-k-1)!} * P_{j+k,j,0}(t) + e^{-(\lambda+\mu+\xi)t}\sum_{k=2}^{i-j-1}k\theta\lambda^{i-j-k}\frac{t^{i-j-k}}{(i-j-k)!} * P_{j+k,j,0}(t) + \\ &(i-j)\theta e^{-(\lambda+\mu+\xi)t} * P_{i,j,0}(t) + \lambda^{i-j-1}e^{-(\lambda+\mu+\xi)t}\frac{t^{i-j-2}}{(i-j-2)!} * P_{j+1,j,1}(t) \\ &\qquad i \geq j+2, j \geq 1 \end{aligned} \tag{23}$$

$$Q(t) = \xi e^{-\tau t}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} P_{i,j,1}(t) \tag{24}$$

4 Verification of Results

- Summing equations (9)-(16) over i and j we get,

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left[\bar{P}_{i,j,0}(s)+\bar{P}_{i,j,1}(s)\right]+\bar{Q}(s)=\frac{1}{s}$$

  and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left[P_{i,j,0}(t)+P_{i,j,1}(t)\right]+Q(t)=1$$

  which is the verification of our results.

- Define $U_{n,l}(t)$= Probability that there are n customers in the orbit at time t. The server is idle when $l=0$ and server is busy when $l=1$.
  When the server is idle, it is represented as $U_{n,0}(t)$:

$$U_{n,0}(t)=\sum_{j=0}^{\infty}P_{j+n,j,0}(t)$$

  where n is the number of customers in the orbit, which can be calculated by using the following formula:
  n=(number of arrivals - number of departures).
  When the server is busy, it is represented as $U_{n,1}(t)$:

$$U_{n,1}(t)=\sum_{j=0}^{\infty}P_{j+n+1,j,1}(t)$$

  In this case:
  n= (number of arrivals - number of departures - 1).
  Using the above definitions in (1)-(4) and let $\xi=0$, $\tau=0$ the equations in statistical equilibrium are:

$$(\lambda+n\theta)\,U_{n,0}=\mu\,U_{n,1} \qquad n\geq 0$$

$$(\lambda+\mu)\,U_{n,1}=\lambda(U_{n,0}+U_{n-1,1})+(n+1)\theta\,U_{(n+1),0} \quad n\geq 2$$

  which coincides with the results (1.5) and (1.6) of [9]

5 Numerical Solution and Graphical Representation

The Numerical results are generated using MATLAB programming for the case $\rho=\left(\frac{\lambda}{\mu}\right)$=0.5, $\eta=\left(\frac{\theta}{\mu}\right)$=0.6, $\tau^{'}=\left(\frac{\tau}{\mu}\right)$=0.4, $\xi^{'}=\left(\frac{\xi}{\mu}\right)$=0.3. In following tables, we observe some significant probabilities at various time instants whose sum approaches to 1.

Table I. At time t =1

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,1,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,1,0}$	$P_{4,2,0}$	$P_{1,0,1}$	$P_{2,0,1}$
1	0.6118	0.1029	0.0117	0.0043	0.0012	0.0007	0.0001	0.0001	0.0001	0.1702	0.0335

$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	Sum
0.0153	0.0049	0.0032	0.96

Table II. At time t =5

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{4,1,0}$	$P_{4,2,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,1,0}$	$P_{5,2,0}$
5	0.1769	0.1584	0.0205	0.099	0.0009	0.0058	0.0128	0.0096	0.0004	0.0024

$P_{5,3,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,0,1}$
0.0063	0.0073	0.0032	0.0546	0.0174	0.0553	0.0057	0.0258	0.0332	0.0019

$P_{4,1,1}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{5,0,1}$	$P_{5,1,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$Q(t)$	Sum
0.0098	0.0182	0.0118	0.0008	0.0047	0.0104	0.0105	0.0041	0.1663	0.934

Table III. At time t =15

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,1,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{5,1,0}$	$P_{5,2,0}$	$P_{5,3,0}$
15	0.1544	0.0866	0.0109	0.0575	0.002	0.011	0.0433	0.0001	0.0006	0.0025

$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,3,0}$	$P_{7,4,0}$	$P_{7,5,0}$	$P_{7,6,0}$	$P_{7,7,0}$	$P_{1,0,1}$
0.0086	0.0271	0.0079	0.0186	0.0002	0.0008	0.0024	0.0061	0.0108	0.043

$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,0,1}$	$P_{4,1,1}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{6,0,1}$
0.0119	0.0277	0.0033	0.012	0.0199	0.0009	0.0043	0.0104	0.0157	0.0001

$P_{6,1,1}$	$P_{6,2,1}$	$P_{6,3,1}$	$P_{6,4,1}$	$P_{6,5,1}$	$P_{8,2,1}$	$P_{8,4,1}$	$P_{8,5,1}$	$P_{8,6,1}$	$P_{8,7,1}$	$Q(t)$
0.0005	0.0016	0.004	0.0078	0.0102	0.0003	0.0026	0.0056	0.0095	0.0102	0.1914

$P_{5,0,1}$	$P_{5,1,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{7,3,1}$	$P_{7,4,1}$	$P_{7,5,1}$	$P_{7,6,1}$	Sum
0.0003	0.0014	0.0042	0.0089	0.0129	0.0016	0.0037	0.0065	0.0073	0.8911

Table IV. At time t =25

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,8,0}$	$P_{1,0,1}$	$P_{5,0,1}$
25	0.1391	0.0798	0.0525	0.0369	0.027	0.0204	0.0158	0.0125	0.1515	0.0389	0.0002

$P_{7,5,1}$	$P_{7,6,1}$	$P_{6,4,1}$	$P_{5,3,1}$	$P_{5,4,0}$	$P_{7,6,0}$	$P_{7,5,0}$	$P_{5,4,1}$	$P_{4,1,1}$	$P_{6,5,1}$
0.0048	0.0059	0.0061	0.0077	0.007	0.0044	0.0017	0.0099	0.004	0.0076

$P_{7,4,1}$	$P_{2,1,1}$	$P_{4,2,1}$	$P_{5,2,1}$	$P_{6,5,0}$	$P_{2,0,1}$	$P_{3,2,1}$	$P_{6,3,1}$	$Q(t)$	Sum
0.0029	0.0257	0.0095	0.0039	0.0056	0.0109	0.0181	0.0034	0.1694	0.8831

Table V. At time t =40

t	$P_{6,6,0}$	$P_{7,7,0}$	$P_{5,5,0}$	$P_{5,4,0}$	$P_{3,2,0}$	$P_{7,6,0}$	$P_{4,4,0}$	$P_{6,5,1}$	$P_{7,3,1}$	$P_{7,4,1}$	$P_{7,6,1}$
40	0.0131	0.0101	0.0171	0.0059	0.0085	0.0037	0.0227	0.0064	0.0011	0.0024	0.0049

$P_{0,0,0}$	$P_{1,1,0}$	$P_{1,0,1}$	$P_{2,1,1}$	$P_{3,1,1}$	$P_{4,3,1}$	$P_{4,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{7,5,1}$	$P_{2,1,0}$
0.1153	0.0661	0.0323	0.0213	0.0093	0.0111	0.0079	0.0064	0.0083	0.0024	0.0083

$P_{2,2,0}$	$P_{6,4,0}$	$P_{3,3,0}$	$Q(t)$	$P_{3,1,0}$	$P_{4,3,0}$	$P_{5,2,0}$	$P_{5,3,0}$	$P_{8,8,0}$	$P_{2,0,1}$	$P_{3,0,1}$
0.0436	0.0017	0.0309	0.1406	0.0015	0.0073	0.0005	0.0019	0.295	0.0.009	0.0025

$P_{3,2,1}$	$P_{4,0,1}$	$P_{4,1,1}$	Sum
0.0151	0.0.0007	0.0033	0.9382

The probabilities against time are represented graphically in the following figures.

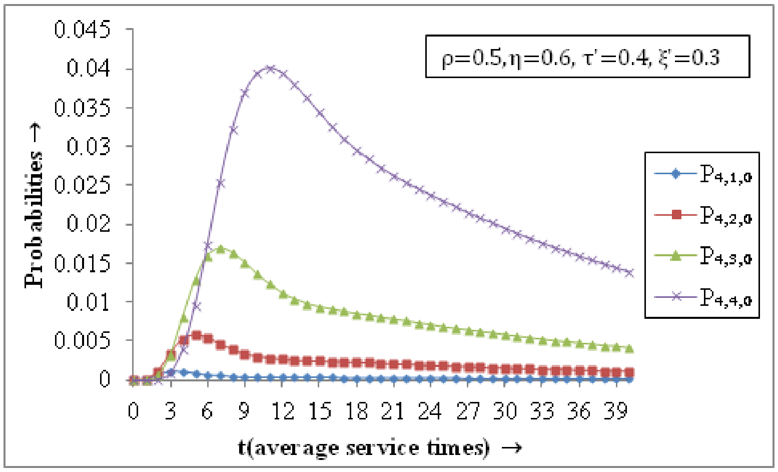

Figure 3. Probabilities $P_{4,1,0}$, $P_{4,2,0}$, $P_{4,3,0}$ and $P_{4,4,0}$ against average service times

In figure 3, the probabilities $P_{4,1,0}$, $P_{4,2,0}$, $P_{4,3,0}$ and $P_{4,4,0}$ are plotted against time t for the given case. It is observed that all the probabilities increase initially and then decrease. Also it can be seen that the probabilities attain higher values for greater number of departures.

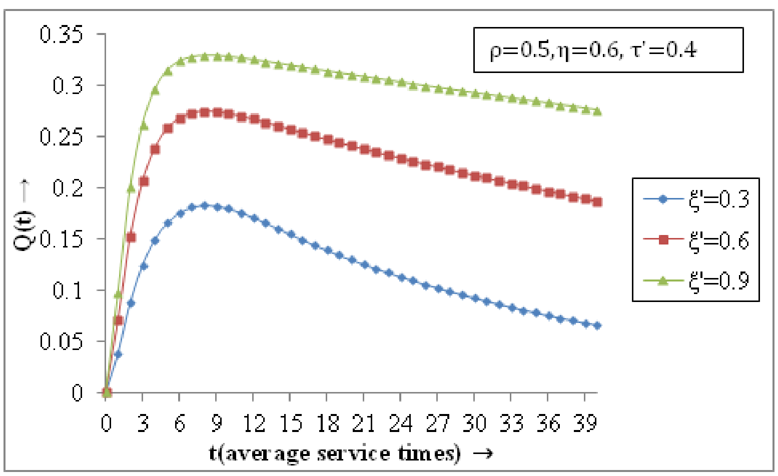

Figure 4. Effect of change in $\xi^{'}$ on the probability $Q(t)$

In figure 4, we study the effect of change in $\xi^{'}$(catastrophe rate per unit service time) on the probability $Q(t)$(probability of server being under repair). From the graph it can be seen that whenever the catastrophe rate per unit service time increases, the probability $Q(t)$ also increases which is as desired.

13

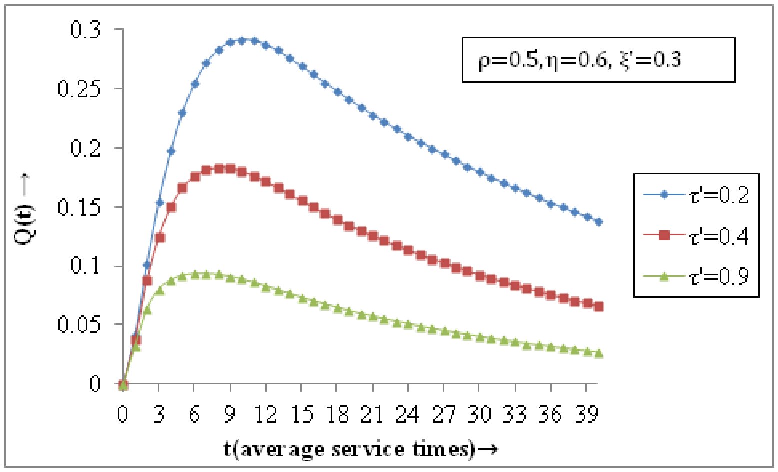

Figure 5. Effect of change in $\tau^{'}$ on the probability $Q(t)$

In figure 5, the effect of change in $\tau^{'}$(repair rate per unit service time) on the probability $Q(t)$ is studied. From the graph it is clearly visible that whenever the repair rate per unit service time increases, the probability $Q(t)$ decreases.

6 Busy Period Probabilities

In this section we discuss the busy period probabilities of the server and the system.
The Probability of busy server is given by:

$$P(Server\ is\ busy) = \sum_{i>j\geq 0} P_{i,j,1}(t) \tag{25}$$

The Probability of busy system is given by:

$$P(System\ is\ busy\ ) = \sum_{i>j\geq 0} \Big(P_{i,j,0}(t) + P_{i,j,1}(t)\Big) + Q(t) \tag{26}$$

6.1 Numerical and Graphical Representation of Busy Period Probabilities

The numerical results are obtained using MATLAB programming and following [2]. The Probabilities of system busy and server busy are obtained for different values of ρ keeping other parameters constant and are presented in the table given below.

Table VI. Probabilities of System Busy and Server Busy

t	Probability(System Busy)			Probability(Server Busy)		
	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
0	0	0	0	0	0	0
1	0.2788	0.369	0.4485	0.2286	0.2967	0.3546
2	0.3858	0.5009	0.5956	0.2626	0.3298	0.3827
3	0.4432	0.5683	0.6651	0.2678	0.3315	0.3785
4	0.4779	0.6067	0.7017	0.2689	0.3286	0.3685
5	0.5005	0.6311	0.724	0.2684	0.3226	0.3544
6	0.5171	0.6494	0.7405	0.2662	0.3137	0.338
7	0.5314	0.6654	0.7545	0.2622	0.3026	0.3208
8	0.5453	0.6804	0.7671	0.2565	0.2902	0.3042
9	0.5594	0.6949	0.7786	0.2495	0.2773	0.2885
10	0.5741	0.7086	0.7891	0.2416	0.2645	0.2741

The probabilities of system busy and server busy are also represented graphically.

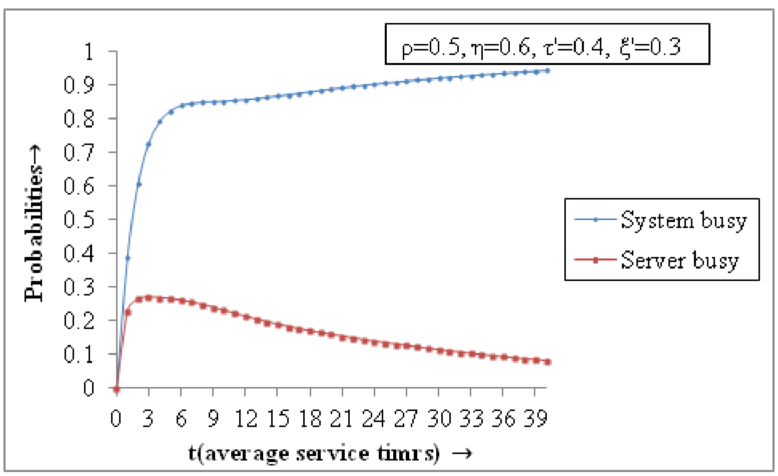

Figure 6. Probabilities of system busy and server busy against average service times

In Figure 6, the probabilities of system busy and server busy are plotted against time t for the case $\rho = 0.7$, $\eta = 0.6$, $\tau^{'}$=0.4, $\xi^{'}$=0.3. It is clear from the graph that probability of system busy is higher than probability of server busy. The probability of system busy increases rapidly with the increase in time. However, the probability of server busy increases first and then decreases gradually with time.

7 Conclusion

In this paper, we studied a two-state single server retrial queueing system with catastrophe. The catastrophes have significant impact on businesses, computer networks, etc. It is very important to manage the risk of catastrophe for the smooth functioning of the system. Moreover, the two-dimensional state queueing model has been proven to be a viable tool for understanding and quantifying factors. The proposed method is highly applicable in modeling many practical situations like in submitting any application online, ticket booking services using telephone facility, withdrawing cash at an ATM, manufacturing sectors, call centers, etc. In this paper, the transient state probabilities and the probability of server being under repair are obtained. Numerical results and graphical representations are also given.

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