# Generalization of Interval Jacobi and Gauss-Seidel Methods for Interval Linear System

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**Abstract**

The paper presents iterative methods for solving interval linear system of equations. We present a generalization of interval Jacobi method and interval Gauss-Seidel method by generalizing interval diagonal matrices to band interval matrices, and discuss the convergence analysis of the proposed methods. More specifically, we prove that both the proposed methods converge for any initial approximation if the coefficient interval matrix of the system is either an *interval strictly diagonally dominant matrix*, or *interval $M$-matrix* or *interval $H$-matrix*. Numerical experiment are carried out to assess the effectiveness of the proposed methods.

**Keywords:** Convergence, Generalized interval Jacobi method, Generalized interval Gauss-Seidel method, Linear interval systems
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## 1 Introduction

Many practical problems involving uncertainties, such as uncertainty in engineering or design problems, global optimization, mathematical programming problems etc., get reduced to solving system of interval equations. We refer to [12, 15, 16, 17, 20, 26] to find the literature on interval analysis dealing with uncertainty. It is worthwhile to mention that the solution set enclosure of interval linear systems, plays a significant role as data are impacted by uncertainty in many real-world problems that involves interval linear systems. However, it is well-known that the interval computations are NP-hard problems. In other words, one cannot expect an algorithm for computing all computations for the interval in less than exponential running time. So the research has been driven

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for finding a solution set enclosure for the interval linear equations with less computations.

Interval Jacobi, interval Gauss-Seidel, Bauer-Skeel, Hansen-Bliek-Rohn, Krawczyk iteration methods are among the oldest well-known iterative methods for solving interval linear systems. In [2, 7, 19, 21], authors proved that the above mentioned methods may not produce an optimal enclosure. However Hladík [19] in 2014, proposed a new interval operator that generalizes the interval Gauss-Seidel method, by introducing a new parameter. He further proved both theoretically and numerically that incorporation of such parameter is more effective than the Gauss-Seidel method, and provides tightening solution set enclosure of interval linear systems. Parametric interval linear system of equations were investigated in [18, 6]. In [21], author studied the Hansen-Bliek-Rohn method and the Bauer-Skeel method and their modification based on the preconditioning of the system and on the residual form. The paper aims to develop the iterative methods and their convergency to solve interval linear systems with uncertain coefficients. More specifically, we generalize the interval Jacobi method and interval Gauss-Seidel method for solving interval linear systems and analyze the convergence of these methods.

Throughout the paper, the sets of all real intervals, the set of $n$-dimensional real interval vectors, and the set of $m \times n$ real interval matrices are denoted by $\mathbb{IR}$, $\mathbb{IR}^n$ and $\mathbb{IR}^{m,n}$, respectively. We write bold letters to represent interval matrices/vectors, whereas normal letters are used to represent real matrices/vectors. A real interval matrix of order $m \times n$ for two real matrices $\underline{A}$ and $\overline{A}$, is defined as $\mathbf{A} = \{A \in \mathbb{R}^{m,n} : \underline{A} \leq A \leq \overline{A}\}$, with componentwise inequality $\underline{A} \leq \overline{A}$.

Consider the following system of interval linear systems of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

with $\mathbf{A} \in \mathbb{IR}^{n,n}$ and $\mathbf{b} \in \mathbb{IR}^n$ are given interval matrix and interval vector respectively, $\mathbf{x} \in \mathbb{IR}^n$ is unknown. The solution set of (1) is enclosed by

$$\Sigma(\mathbf{A}, \mathbf{b}) := \{\tilde{x} \in \mathbb{R}^n : \tilde{A}\tilde{x} = \tilde{b} \text{ for some } \tilde{A} \in \mathbf{A},\ \tilde{b} \in \mathbf{b}\}$$

The smallest interval enclosure of $\Sigma(\mathbf{A}, \mathbf{b})$ with regard to inclusion is represented by the interval $\mathbf{\Sigma} := \square\Sigma(\mathbf{A}, \mathbf{b}) = [\inf(\Sigma(\mathbf{A}, \mathbf{b})), \sup(\Sigma(\mathbf{A}, \mathbf{b}))]$, is known as the interval hull of the solution set $\Sigma(\mathbf{A}, \mathbf{b})$.

Let $\mathbf{D}$, $-\mathbf{E}$ and $-\mathbf{F}$ be respectively, represent the diagonal part, strictly lower triangular and strictly upper triangular parts of the interval matrix $\mathbf{A}$, so that $\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{F}$. If $0 \notin \mathbf{A}_{ii}$, then the interval Jacobi method and interval Gauss-Seidel method (see [3, 30]) for solving (1) are respectively, given by

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \mathbf{D}^{-1}(\mathbf{E} + \mathbf{F})\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b} \\ \mathbf{x}^{(k+1)} &= (\mathbf{D} - \mathbf{E})^{-1}\mathbf{F}\mathbf{x}^{(k)} + (\mathbf{D} - \mathbf{E})^{-1}\mathbf{b}, \end{aligned} \tag{2}$$

We write $\mathbf{H}_J = \mathbf{D}^{-1}(\mathbf{E} + \mathbf{F})$ and $\mathbf{H}_{GS} = (\mathbf{D} - \mathbf{E})^{-1}\mathbf{F}$ to represent the iteration matrices for the interval Jacobi and interval Gauss-Seidel method, respectively. Details of interval Jacobi and Gauss-Seidel method for solving interval linear equations can be found in [3].

It is known from the literature that $M$-matrices, $L$-matrices, strictly diagonally dominant (SDD) and symmetric positive definite (SPD) matrices are among the classes of matrices for which both Jacobi and Gauss-Seidel methods converge for any initial guess for a given linear system of equations $Ax = b$. In [4], Salkuyeh generalized the Jacobi and Gauss-Seidel method by generalizing the diagonal matrix to a band matrix, which are given by the following iteration relations

$$
\begin{aligned}
x^{(k+1)} &= T_m^{-1}(E_m + F_m)x^{(k)} + T_m^{-1}b \qquad (3)\\
x^{(k+1)} &= (T_m - E_m)^{-1}F_m x^{(k)} + (T_m - E_m)^{-1}b \qquad (4)
\end{aligned}
$$

where $m \geq 0$ and $F_m = A - T_m - E_m$ and

$$
T_m = \begin{bmatrix} a_{1,1} & \dots & a_{1,m+1} & 0 \\ \vdots & \ddots & \ddots & \\ a_{m+1,1} & & & a_{n-m,n} \\ & \ddots & \ddots & \\ 0 & & a_{n,n-m} & a_{n,n} \end{bmatrix}, \quad E_m = \begin{bmatrix} 0 & \dots & 0 \\ -a_{m+2,1} & & \\ \vdots & \ddots & \vdots \\ -a_{n,1} & \dots & -a_{n-m-1,n} \end{bmatrix}
$$

Salkuyeh proved that if the coefficient matrix of a system of linear equations is either SDD or an $M$-matrix, then the generalized Jacobi (GJ) and generalized Gauss-Seidel (GGS) methods converge. In [22], authors proved the convergence of GJ and GGS methods for $H$-matrices, however both methods may fail to converge for SPD and for $L$-matrices.

In this paper we generalize the interval Gauss-Seidel and interval Jacobi methods similar to equation (3) and (4), respectively, to obtain a tighter enclosure of the solution set. As mentioned earlier, the iteration schemes defined in (3) and (4) converge for SDD, $M$-matrices and for $H$-matrices, so motivated by these results we verify the convergence criteria of both generalized interval Jacobi and interval Gauss-Seidel methods for these classes of interval coefficient matrices.

This paper is organized as follows: In Section 2, we provide the notations and basic definitions related to interval analysis and define various classes of interval matrices under consideration. We also listed a few well-known results that are used in our study. Section 3 introduces the generalization of interval Jacobi method and discuss the convergence analysis of the method for solving (1) for various classes of interval coefficient matrices. In section 4, we describe the generalized interval Gauss-Seidel method and its convergence analysis for various classes of interval coefficient matrices. Numerical experiments are carried out for the proposed methods in Section 5. Finally, Section 6 ends with some concluding remarks.

## 2 Notation and preliminaries

In accordance with the standard notations, intervals are marked by boldface throughout this article. To represent the lower and upper bounds of intervals respectively, the underscores and overscores notations are used. So, any interval $\mathbf{x}$ is written as $\mathbf{x} = [\underline{x}, \overline{x}]$. For the interval $\mathbf{x}$, *magnitude* and *mignitude* are defined respectively as, $|\mathbf{x}| := \max\{|x| : x \in \mathbf{x}\} = \max\{|\underline{x}|, |\overline{x}|\}$ and $\langle \mathbf{x} \rangle := \min\{|x| : x \in \mathbf{x}\} = \min\{|\underline{x}|, |\overline{x}|\}$. *Magnitude* and *mignitude* of interval matrix $\mathbf{A}$ are defined componentwise, and denoted by corresponding notations as defined for intervals. For a given interval matrix $\mathbf{A} = (\mathbf{A}_{ij}) \in \mathbb{IR}^{n,n}$, we denote $|\mathbf{A}| := (|\mathbf{A}|_{ij}) \in \mathbb{R}^{n,n}$, and the comparison matrix of $\mathbf{A}$ is represented by $\langle \mathbf{A} \rangle$, which has entries $\langle \mathbf{A} \rangle_{ii} = \langle \mathbf{A}_{ii} \rangle$ and $\langle \mathbf{A} \rangle_{ij} = -|\mathbf{A}_{ij}|$, for $i \neq j$. Note that both $|\mathbf{A}|$ and $\langle \mathbf{A} \rangle$ are real matrices. Next we provide some properties of the interval matrices, and define various classes of interval matrices under consideration.

**Definition 2.1.** [3, 10, 11] An interval matrix $\mathbf{A} \in \mathbb{IR}^{n,n}$ is said to be *regular* if every $A \in \mathbf{A}$ is nonsingular. For a *regular* $\mathbf{A}$, the inverse $\mathbf{A}^{-1}$ is defined as

$$\mathbf{A}^{-1} := \square\{A^{-1} : A \in \mathbf{A}\}$$

where $\square\Sigma := [\inf \Sigma,\ \sup \Sigma]$ denotes the hull of $\Sigma$, which is the tightest enclosure for $\Sigma$. It is to be noted that the smallest interval matrix $\mathbf{A}^{-1}$ includes the set $\{A^{-1} : A \in \mathbf{A}\}$.

**Definition 2.2.** [3] For any real intervals $\mathbf{x} = [\underline{x}, \overline{x}], \mathbf{y} = [\underline{y}, \overline{y}]$, interval addition, subtraction and multiplication are defined as

(i) $\mathbf{x} + \mathbf{y} = [\underline{x} + \underline{y},\ \overline{x} + \overline{y}]$

(ii) $\mathbf{x} - \mathbf{y} = [\underline{x} - \overline{y},\ \overline{x} - \underline{y}]$

(iii) The interval multiplication $\mathbf{xy}$ is displayed in the following table:

| $*$ | $\mathbf{y} \geq 0$ | $\mathbf{y} \ni 0$ | $\mathbf{y} \leq 0$ |
|---|---|---|---|
| $\mathbf{x} \geq 0$ | $[\underline{x}\underline{y}, \overline{x}\overline{y}]$ | $[\overline{x}\underline{y}, \overline{x}\overline{y}]$ | $[\overline{x}\underline{y}, \underline{x}\overline{y}]$ |
| $\mathbf{x} \ni 0$ | $[\underline{x}\overline{y}, \overline{x}\overline{y}]$ | $[\min\{\underline{x}\overline{y}, \overline{x}\underline{y}\}, \max\{\underline{x}\underline{y}, \overline{x}\overline{y}\}]$ | $[\overline{x}\underline{y}, \underline{x}\underline{y}]$ |
| $\mathbf{x} \leq 0$ | $[\underline{x}\overline{y}, \overline{x}\underline{y}]$ | $[\underline{x}\overline{y}, \underline{x}\underline{y}]$ | $[\overline{x}\overline{y}, \underline{x}\underline{y}]$ |

**Definition 2.3.** [3] If $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{m,n}$, addition and subtraction for interval matrices are defined as

(i) $\mathbf{A} + \mathbf{B} = \square\{A + B :\ A \in \mathbf{A},\ B \in \mathbf{B}\}$

(ii) $\mathbf{A} - \mathbf{B} = \square\{A - B :\ A \in \mathbf{A},\ B \in \mathbf{B}\}$

If $\mathbf{A} = [\underline{A}, \overline{A}]$ and $\mathbf{B} = [\underline{B}, \overline{B}]$, then

$$\mathbf{A} + \mathbf{B} = [\underline{A} + \underline{B},\ \overline{A} + \overline{B}] \quad \text{and} \quad \mathbf{A} - \mathbf{B} = [\underline{A} - \overline{B},\ \overline{A} - \underline{B}\ ]$$

**Definition 2.4.** [3] If $\mathbf{A} \in \mathbb{IR}^{m,n}$ and $\mathbf{B} \in \mathbb{IR}^{n,p}$, then $\mathbf{AB} \in \mathbb{IR}^{m,p}$ is defined as

$$\mathbf{AB} = \square\{\tilde{A}\tilde{B} : \tilde{A} \in \mathbf{A},\ \tilde{B} \in \mathbf{B}\}$$

If $\mathbf{A} = (\mathbf{A}_{ij})$ and $\mathbf{B} = (\mathbf{B}_{ij})$ , then $(\mathbf{AB})_{ik} = \sum_{j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{jk}$

**Definition 2.5.** [3, 14] Let $\mathbf{A} \in \mathbb{IR}^{n,n}$ and $0 \notin \mathbf{A}_{ii}$ for all $i$. If the comparison matrix $\langle \mathbf{A} \rangle$ of $\mathbf{A}$ is strictly diagonally dominant,that is, if for all $i$, $\langle \mathbf{A}_{ii} \rangle > \sum_{j \neq i} |\mathbf{A}_{ij}|$ then $\mathbf{A}$ is known to be an interval *strictly diagonally dominant* (SDD) matrix.

**Definition 2.6.** [1, 3] A real matrix $A \in \mathbb{R}^{n,n}$ is called an *L-matrix* if it has positive diagonal entries and nonpositive off-diagonal entries. An interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is an *interval L-matrix* if each $A \in \mathbf{A}$ is an $L$-matrix, equivalently, if $\underline{A}_{ii} > 0$ for all $i$ and $\overline{A}_{ij} \leq 0$, for $i \neq j$.

**Definition 2.7.** [1] A matrix $A \in \mathbb{R}^{n,n}$ is said to be a *Z-matrix* if $A$ has nonpositive off-diagonal entries. If a $Z$-matrix $A$ can be written as $A = \alpha I - B$, where $\alpha > \rho(B)$, the spectral radius of $B$, then $A$ is called an *M-matrix.* Instead of nonsingular $M$-matrix we write $M$-matrix for convenience. A *Z-matrix A* becomes an *M-matrix* if and only if there exists a $u > 0$ such that $Au > 0$.

We now state the characterization of $M$-matrices.

**Theorem 2.8.** [1] Let $A \in \mathbb{R}^{n,n}$ be a $Z$-matrix. Then following equivalent conditions hold:

(i) $A$ is an $M$-matrix.

(ii) $A^{-1} \geq 0$.

(iii) There exists $u > 0$ such that $Au > 0$.

**Definition 2.9.** [3] An *interval M-matrix* is a square interval matrix $\mathbf{A} \in \mathbb{IR}^{n,n}$ such that $\mathbf{A}_{ik} \leq 0$, that is, every element in $\mathbf{A}_{ik}$ is nonpositive, for all $i \neq k$ and $\mathbf{A}u > 0$ for some real $u > 0$.

**Definition 2.10.** [3] An *interval H-matrix* $\mathbf{A} \in \mathbb{IR}^{n,n}$ is an interval matrix whose comparison matrix $\langle \mathbf{A} \rangle$ is an $M$-matrix. Equivalently, we say that $\mathbf{A}$ is an interval $H$-matrix if and only if $\langle \mathbf{A} \rangle u > 0$ for some $u > 0$.

**Definition 2.11.** [1] A splitting of a real $n \times n$ matrix $A$ is defined as $A = M - N$, with nonsingular $M$. A splitting $A = M - N$ of the matrix $A$ is said to be

(i) *regular* if $M^{-1} \geq 0$ and $N \geq 0$.

(ii) *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

(iii) *M-splitting* if $M$ is a $M$-matrix and $N \geq 0$.

**Definition 2.12.** A *splitting* of a square interval matrix $\mathbf{A} \in \mathbb{IR}^{n,n}$ is defined as $\mathbf{A} = \mathbf{M} - \mathbf{N}$, with regular $\mathbf{M}$.

Next we state few basic results on matrices, that are required to establish our results in the subsequent sections.

**Proposition 1.** [3, 23] If $A, B \in \mathbb{R}^{n,n}$, and $|A| \leq B$, then $\rho(A) \leq \rho(B)$.

**Theorem 2.13.** [1] Let $A$ be an $M$-matrix and let $A = M - N$ be a regular or weak regular splitting of $A$, then $\rho(M^{-1}N) < 1$.

**Theorem 2.14.** [27] Let $M$-splitting of $A$ be $A = M - N$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is a nonsingular $M$-matrix.

**Theorem 2.15.** [1, 13, 23] Let $A$ be a nonnegative matrix. Then

(i) If $\alpha \in \mathbb{R}$ and $Ax \geq \alpha x$, for some positive $x \in \mathbb{R}^n$, then $\rho(A) \geq \alpha$ .

(ii) If $Ax \leq \alpha x$ for some $x \geq (\neq)0$ , then $\rho(A) \leq \alpha$.

**Theorem 2.16.** [23] If $A$ is nonnegative matrix, then $\rho(A)$ is an eigenvalue of $A$ and there is a nonnegative nonzero vector $x$ such that $Ax = \rho(A)x$.

For convenience we have provided some well-known results on interval matrices that will be used to check the convergence of the mentioned methods in next sections of this paper.

**Theorem 2.17.** [3] Let $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{n,n}$. Then following conditions hold:

(i) If $\mathbf{A}$ is an $M$-matrix and $\mathbf{B} \subseteq \mathbf{A}$, then $\mathbf{B}$ is an $M$-matrix. Each $\tilde{A} \in \mathbf{A}$ in particular, is an $M$-matrix.

(ii) $\mathbf{A} = [\underline{A},\ \overline{A}]$ is an $M$-matrix if and only if $\underline{A}$ and $\overline{A}$ are $M$-matrices.

(iii) Every $M$-matrix $\mathbf{A} = [\underline{A},\ \overline{A}]$ is regular with $\mathbf{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}] \geq 0$ and $|\mathbf{A}^{-1}| = \langle \mathbf{A} \rangle^{-1}$.

**Theorem 2.18.** [14] For an interval matrix $\mathbf{A}$ we have

(i) if $\mathbf{A}$ is interval triangular (lower/upper) matrix, then $\mathbf{A}$ is an interval $H$-matrix.

(ii) if $\mathbf{A}$ is an interval $H$-matrix, then $|\mathbf{A}^{-1}| \leq \langle \mathbf{A} \rangle^{-1}$. Equality holds if $\mathbf{A}$ is an interval $M$-matrix.

**Proposition 2.** [3] For $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{n,n}$ and $\mathbf{C} \in \mathbb{IR}^{n,q}$, the properties listed below hold:

(i) $\langle \mathbf{A} \rangle = \langle \tilde{A} \rangle$, for some $\tilde{A} \in \mathbf{A}$.

(ii) $|\mathbf{A}\mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|$

(iii) $\langle \mathbf{A} \pm \mathbf{B} \rangle \geq \langle \mathbf{A} \rangle - |\mathbf{B}|$.

(iv) $|\mathbf{A}| - |\mathbf{B}| \leq |\mathbf{A} \pm \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$.

(v) $|\mathbf{AC}| \geq \langle \mathbf{A} \rangle |\mathbf{C}|$.

**Theorem 2.19.** [14] Let $\mathbf{C}, \mathbf{D} \in \mathbb{IR}^{n,n}$ satisfy $\rho(|\mathbf{C}||\mathbf{D}|) < 1$. Then for any $\mathbf{g} \in \mathbb{IR}^n$, the following statements hold:

(i) The equation $\mathbf{x} = \mathbf{C}(\mathbf{Dx} + \mathbf{g})$ has a unique solution $\mathbf{x} \in \mathbb{IR}^n$

(ii) For any initial vector $\mathbf{x}^0 \in \mathbb{IR}^n$, the iteration

$$\mathbf{x}^{(k+1)} = \mathbf{C}(\mathbf{Dx}^{(k)} + \mathbf{g}),\ k = 0, 1, ...$$

converges to the solution $\mathbf{x}$ of the equation $\mathbf{x} = \mathbf{C}(\mathbf{Dx} + \mathbf{g})$.

# 3 Generalized Interval Jacobi method

In this section, we propose a generalization of the interval Jacobi method for solving interval linear system similar to that of generalized Jacobi method introduced by Salkuyeh [5] and Saha *et al.* [22] for solving linear systems of matrix equations. Furthermore, we study the convergence properties of the proposed method for solving interval linear system with the coefficient matrix as either an interval SDD matrix, an interval $M$-matrix or an interval $H$-matrix.

In Section 2 the splitting of $\mathbf{A}$ for interval Jacobi method for solving (1) is given in equation (2) as $\mathbf{A} = \mathbf{M} - \mathbf{N}$ with $\mathbf{M} = \mathbf{D},\ \mathbf{N} = \mathbf{E} + \mathbf{F}$.

We now propose generalized interval Jacobi (GIJ) method for solving interval linear system similar to (3) and (4), which was introduced by Salkuyeh [4] for general matrices.

Let $\mathbf{A} = [\underline{a_{ij}}, \overline{a_{ij}}]$ be a square interval matrix of order $n$. Consider an interval band matrix $\mathbf{T}_m = [\underline{t_{ij}}, \overline{t_{ij}}]$ of $2m + 1$ bandwidth, which is characterized as

$$\mathbf{t}_{ij} = \begin{cases} [\underline{a_{ij}}, \overline{a_{ij}}], & \text{if } |i - j| \leq m \\ 0, & \text{elsewhere} \end{cases}$$

For $1 \leq m < n$, interval matrix $\mathbf{A}$ is decomposed as $\mathbf{A} = \mathbf{T}_m - \mathbf{E}_m - \mathbf{F}_m$, with strict lower part -$\mathbf{E}_m$ and strict upper part -$\mathbf{F}_m$ of $\mathbf{A}$. The interval matrices $\mathbf{T}_m,\ \mathbf{E}_m$ and $\mathbf{F}_m$ are expressed as follows

$$\mathbf{T}_m = \begin{bmatrix} & [\underline{a_{11}}, \overline{a_{11}}] & \cdots & [\underline{a_{1,m+1}}, \overline{a_{1,m+1}}] & & 0 \\ & \vdots & \ddots & & \ddots & \\ [\underline{a_{m+1,1}}, \overline{a_{m+1,1}}] & & & & & [\underline{a_{n-m,n}}, \overline{a_{n-m,n}}] \\ & & \ddots & & \ddots & \\ & 0 & & [\underline{a_{n,n-m}}, \overline{a_{n,n-m}}] & & [\underline{a_{n,n}}, \overline{a_{n,n}}] \end{bmatrix},$$

$$\mathbf{E}_m = \begin{bmatrix} & 0 & \cdots & & 0 \\ -[\underline{a_{m+2,1}}, \overline{a_{m+2,1}}] & & & & \\ & \vdots & \ddots & & \vdots \\ & -[\underline{a_{n,1}}, \overline{a_{n,1}}] & \cdots & -[\underline{a_{n-m-1,n}}, \overline{a_{n-m-1,n}}] & \end{bmatrix},$$

$$\mathbf{F}_m = \begin{bmatrix} 0 & -[\underline{a_{1,m+2}}, \overline{a_{1,m+2}}] & & \cdots - [\underline{a_{1,n}}, \overline{a_{1,n}}] \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & -[\underline{a_{n-m-1,n}}, \overline{a_{n-m-1,n}}] \\ & & & \end{bmatrix} \tag{5}$$

**Definition 3.1.** Let $\mathbf{T}_m$, $\mathbf{E}_m$ and $\mathbf{F}_m$ be the interval matrices specified in (5). For any $1 \leq m < n$ decompose $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{T}_m - \mathbf{E}_m - \mathbf{F}_m \tag{6}$$

which is corresponding to splitting

$$\mathbf{A} = \mathbf{M}_m - \mathbf{N}_m \tag{7}$$

with $\mathbf{M}_m = \mathbf{T}_m$ and $\mathbf{N}_m = \mathbf{E}_m + \mathbf{F}_m$. Then *generalized interval Jacobi* (GIJ) method to solve (1) is defined as,

$$\mathbf{x}^{(k+1)} = \mathbf{M}_m^{-1} \left( \mathbf{N}_m \mathbf{x}^{(k)} + \mathbf{b} \right) \tag{8}$$

For GIJ method $\mathbf{L} = \mathbf{T}_m^{-1}(\mathbf{E}_m + \mathbf{F}_m)$ is the iteration interval matrix. By decomposing $\mathbf{T}_m = \mathbf{D} + \mathbf{R}_m$, $\mathbf{A}$ can also be written as

$$\mathbf{A} = \mathbf{D} + \mathbf{R}_m - \mathbf{E}_m - \mathbf{F}_m \tag{9}$$

*Remark* 3.2. From (6), we can decompose $\langle \mathbf{A} \rangle$ as

$$\langle \mathbf{A} \rangle = \langle \mathbf{T}_m \rangle - |\mathbf{E}_m| - |\mathbf{F}_m| = \langle \mathbf{D} \rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - |\mathbf{F}_m| \tag{10}$$

and is associated with the splitting

$$\langle \mathbf{A} \rangle = \langle \mathbf{M}_m \rangle - |\mathbf{N}_m| \tag{11}$$

with $\widetilde{M}_1 = \langle \mathbf{M}_m \rangle = \langle \mathbf{T}_m \rangle = \langle \mathbf{D} \rangle - |\mathbf{R}_m|$ and $\widetilde{N}_1 = |\mathbf{N}_m| = |\mathbf{E}_m| + |\mathbf{F}_m|$.

8

**Notation.** Throughout the paper following notations are used:

$$\mathbf{R}_m = (\mathbf{R}_{ij}), \quad \mathbf{E}_m = (\mathbf{E}_{ij}), \quad \mathbf{F}_m = (\mathbf{F}_{ij})$$
$$\widetilde{R}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |\mathbf{R}_{ij}|, \quad \widetilde{E}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |\mathbf{E}_{ij}|, \quad \widetilde{F}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |\mathbf{F}_{ij}|$$

## 3.1 Convergence analysis of GIJ method

In this section we discuss the convergence criterion of GIJ method for solving interval linear system (1) with various classes of coefficient interval matrices. In particular we show that the GIJ method is convergent for interval SDD matrices, interval $M$-matrices and interval $H$-matrices using the idea of interval splitting as well as the various characterizations of interval $M$- and interval $H$-matrices.

Throughout the section we consider the splitting of $\mathbf{A}$ defined in (7) and (10). More specifically, we write $\mathbf{M}_m := \mathbf{T}_m$ and $\mathbf{N}_m := \mathbf{E}_m + \mathbf{F}_m$. It is known that GIJ method converges if $\rho(|\mathbf{M}_m^{-1}||\mathbf{N}_m|) < 1$ due to Theorem 2.19. Since computing the inverse of interval matrix is NP-hard, so we use the matrix $\widetilde{L}_m = \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \tilde{M}_1^{-1}\tilde{N}_1$ to check the convergence of GIJ method. The following theorem provides a relation between the spectral radius of the iteration matrix $\widehat{L}_m = |\mathbf{M}_m^{-1}||\mathbf{N}_m|$ with $\widetilde{L}_m = \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \tilde{M}_1^{-1}\tilde{N}_1$.

**Theorem 3.3.** Let $\widehat{L}_m = |\mathbf{M}_m^{-1}||\mathbf{N}_m|$ and $\widetilde{L}_m = \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \widetilde{M}_1^{-1}\widetilde{N}_1$. If $\mathbf{M}_m$ is an interval $H$-matrix (or interval $M$-matrix) then the following results hold

(i) $\widehat{L}_m \leq \widetilde{L}_m$ (equality holds if $\mathbf{M}_m$ is an interval $M$-matrix)

(ii) $\rho(\widehat{L}_m) \leq \rho(\widetilde{L}_m)$.

*Proof.* (i) As $\mathbf{M}_m$ is given an interval $H$-matrix, from Theorem 2.18 we have that

$$\widehat{L}_m = |\mathbf{M}_m^{-1}||\mathbf{N}_m| \leq \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \widetilde{L}_m$$

(ii) It follows from (i) and from Proposition 1. □

*Remark* 3.4. The above theorem shows that if $\rho(\widetilde{L}_m) < 1$, then GIJ converges. This idea has been used to check the convergence of GIJ method in case $\mathbf{M}_m$ is an interval $H$-(or $M$-) matrix.

*Remark* 3.5. [3, 29] Interval SDD matrices are a special case of interval $H$-matrices, that is, interval SDD matrices $\mathbf{A} = (\mathbf{a}_{ij})$ that satisfy for all $i$, $\langle \mathbf{A}_{ii} \rangle > \sum_{j\neq i} |\mathbf{A}_{ij}|$, are $H$-matrices.

Following theorem gives an upper bound for the spectral radius of the matrix $\widetilde{L}_m$ of GIJ method to solve linear interval equations with interval SDD matrix as coefficient interval matrix.

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**Theorem 3.6.** For any $1 \leq i \leq n$, let $\langle \mathbf{T} \rangle_i = \langle \mathbf{D}_{ii} \rangle - \widetilde{R}_i > 0$. Then

$$\rho(\widetilde{L}_m) \leq \max_{i \in \mathbb{N}} \frac{\widetilde{E}_i + \tilde{F}_i}{\langle \mathbf{T} \rangle_i}$$

*Proof.* Suppose that $\lambda$ is an eigenvalue of the matrix $\widetilde{L}_m$ satisfying

$$|\lambda| > \max_{i \in \mathbb{N}} \frac{\widetilde{E}_i + \widetilde{F}_i}{\langle \mathbf{T} \rangle_i} = \frac{\widetilde{E}_i + \widetilde{F}_i}{\langle \mathbf{D}_{ii} \rangle - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} |\mathbf{R}_{ij}|}, \text{ for all } 1 \leq i \leq n \tag{12}$$

which implies that

$$|\lambda \langle \mathbf{D}_{ii} \rangle| = |\lambda| \langle \mathbf{D}_{ii} \rangle > \widetilde{E}_i + \widetilde{F}_i + |\lambda| \sum_{\substack{j=1 \\ j \neq i}}^{n} |\mathbf{R}_{ij}| \tag{13}$$

Now $\det(\lambda I - \widetilde{L}_m) = 0$ implies $\det(\langle \mathbf{T}_m \rangle^{-1}) \det(C_1) = 0$ where $C_1 = \lambda \langle \mathbf{T}_m \rangle - |\mathbf{E}_m| - |\mathbf{F}_m|$, which again implies that $\det(C_1) = 0$. This is a contradiction to the fact that $C_1$ is SDD and hence nonsingular. Thus,

$$\rho(\widetilde{L}_m) \leq \max_{1 \leq i \leq n} \frac{\tilde{E}_i + \tilde{F}_i}{\langle \mathbf{T} \rangle_i}$$

□

Following results show the convergence of GIJ for interval SDD matrices.

**Theorem 3.7.** For interval SDD matrix $\mathbf{A}$, GIJ method (8) converges for any initial guess.

*Proof.* Let $\mathbf{A}$ be an interval SDD matrix. Decompose $\mathbf{A}$ as in (6). As $\mathbf{A}$ is a SDD matrix, so is the matrix $\mathbf{M}_m$, that is

$$\langle \mathbf{D}_{ii} \rangle > \sum_{j \neq i} |\mathbf{R}_{ij}|$$

Then $\mathbf{M}_m$ is an interval SDD matrix and hence an $H$-matrix due to the Remark 3.5.

Suppose $\lambda$ is an eigenvalue of $\widetilde{L}_m$ and $|\lambda| \geq 1$. Then we have that

$$\begin{aligned}
\det(\lambda I - \widetilde{L}_m) = 0 &\Rightarrow \det(\lambda \langle \mathbf{M}_m \rangle - |\mathbf{N}_m|) = 0 \\
&\Rightarrow \det(\lambda \langle \mathbf{D} \rangle - \lambda |\mathbf{R}_m| - |\mathbf{N}_m|) = 0 \\
&\Rightarrow \det\left( \langle \mathbf{D} \rangle - |\mathbf{R}_m| - \frac{1}{\lambda} |\mathbf{N}_m| \right) = 0 \\
&\Rightarrow \det\left( \langle \mathbf{D} \rangle - |\mathbf{R}_m| - \frac{1}{\lambda} |\mathbf{E}_m| - \frac{1}{\lambda} |\mathbf{F}_m| \right) = 0
\end{aligned}$$

This shows that the matrix $Q = (\langle \mathbf{D} \rangle - |\mathbf{R}_m| - \frac{1}{\lambda}|\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|)$ is singular. As $|\lambda| \geq 1$, that is, $\frac{1}{|\lambda|} \leq 1$ and hence $\mathbf{A}$ is SDD implies that $Q$ is SDD, a contradiction to the fact that $Q$ is singular. Thus $\rho(\widetilde{L}_m) < 1$ and the result holds for interval SDD matrix. □

Next two theorems provide the convergence criteria of GIJ method for two important classes under consideration, namely, the classes of interval $M$-matrices and interval $H$-matrices.

**Theorem 3.8.** GIJ method converges for interval $M$-matrix $\mathbf{A}$, for any $m \leq n$.

*Proof.* Let $\mathbf{A}$ be an interval $M$-matrix of order $n$. Then by Theorem 2.17 (iii), $\langle \mathbf{A} \rangle^{-1} \geq 0$. Since $\langle \mathbf{A} \rangle$ is a $Z$-matrix, Theorem 2.8 implies that $\langle \mathbf{A} \rangle$ is an $M$-matrix. Let $\langle \mathbf{A} \rangle = \widetilde{M}_1 - \widetilde{N}_1$ be the splitting of $\langle \mathbf{A} \rangle$ defined in (11). As $\langle \mathbf{A} \rangle$ is $M$-matrix, there exists $u > 0$ such that $\langle \mathbf{A} \rangle u > 0$ which implies that $\langle \mathbf{M}_m \rangle u > 0$, that is, $\widetilde{M}_1 u > 0$. Thus $\widetilde{M}_1$ is an $M$-matrix, by Theorem 2.8. Also, $\widetilde{N}_1 \geq 0$. Therefore, $\langle \mathbf{A} \rangle = \widetilde{M}_1 - \widetilde{N}_1$ is regular splitting, and hence by Theorem 2.13, $\rho(\widetilde{L}_m) < 1$. It is obvious that $\mathbf{M}_m$ is an interval $M$-matrix and hence Remark 3.4 implies that GIJ converges, for any initial guess. □

**Theorem 3.9.** GIJ method for solving interval linear system converges for interval $H$-matrix $\mathbf{A}$.

*Proof.* Let $\mathbf{A}$ be an interval $H$-matrix, so that the matrix $\langle \mathbf{A} \rangle$ is an $M$-matrix. Then as shown in Theorem 3.8, it can be proved that $\rho(\widetilde{L}_m) < 1$, which implies that GIJ method converges for interval $H$-matrix. □

# 4 Generalized interval Gauss-Seidel method

The interval Gauss-Seidel method for solving system of interval linear equations was introduced by Neumaier [3] and Moore [25]. In [4, 22], authors considered generalized Gauss-Seidel method (a particular case of AOR method) and discussed convergence properties thoroughly for various classes of matrices, like, SDD, SPD, $M$-matrices, $L$-matrices and for $H$-matrices as the coefficient matrices of the linear system. Using the similar approach we now propose generalized version of interval Gauss-Seidel (GIGS) method for solving interval linear systems. It is shown in [22] that generalized Gauss-Seidel may not converge for SPD and for $L$-matrices. Since general matrices are particular case of interval matrices, so this section is emphasised on checking the convergence of generalized interval Gauss-Seidel (GIGS) method, only for interval SDD matrices, interval $M$-matrices and for interval $H$-matrices. We now begin with defining iteration steps for generalized interval Gauss-Seidel method.

**Definition 4.1.** Consider the decomposition of $\mathbf{A}$, defined in equation (7) and the splitting

$$\mathbf{A} = \mathbf{M}_m - \mathbf{N}_m \tag{14}$$

where $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ and $\mathbf{N}_m = \mathbf{F}_m$. Then the iteration step for GIGS method to solve the interval linear system (1), is defined as,

$$\mathbf{x}^{(k+1)} = \mathbf{M}_m^{-1}\left(\mathbf{N}_m\mathbf{x}^{(k)} + \mathbf{b}\right) \tag{15}$$

Further decompose $\langle\mathbf{A}\rangle$ same as in (10) and consider the associated splitting (11) where

$$M_1 = \langle\mathbf{M}_m\rangle = \langle\mathbf{T}_m\rangle - |\mathbf{E}_m|, \quad N_1 = |\mathbf{N}_m| = |\mathbf{F}_m| \tag{16}$$

Now we emphasize on the convergence of GIGS method for interval SDD-matrices, interval matrices $H$-matrices and for interval $M$-matrices.

### 4.1 Convergence analysis of GIGS method

Convergence analysis of GIGS method is similar to that of GIJ method discussed in Section 3. For interval Gauss-Seidel method the splitting of $\mathbf{A}$ is considered as $\mathbf{A} = \mathbf{M} - \mathbf{N}$ with $\mathbf{M} = \mathbf{D} - \mathbf{E}$ and $\mathbf{N} = \mathbf{F}$ and the iteration matrix is given by $\mathbf{M}^{-1}\mathbf{N}$. If $\acute{G} := |\mathbf{M}^{-1}||\mathbf{N}|$ and $C := \langle\mathbf{M}\rangle^{-1}|\mathbf{N}|$, then from Theorem 2.19, it is known that interval Gauss-Seidel method converges if $\rho(\acute{G}) < 1$.

Following two results are immediate consequences of Theorem 2.18 and hence the proofs are skipped.

**Theorem 4.2.** If $\mathbf{M}$ is interval $H$-matrix then the following results hold

(i) $\acute{G} \leq C$.

(ii) $\rho(\acute{G}) \leq \rho(C)$ (equality holds if $\mathbf{M}$ is an interval $M$-matrix).

**Theorem 4.3.** Consider the splitting as given in equation (14) and (16). Let $\widetilde{G} = |\mathbf{M}_m{}^{-1}| \cdot |\mathbf{N}_m|$ and $C_m = \langle\mathbf{M}_m\rangle^{-1}|\mathbf{N}_m| = M_1^{-1}N_1$. If $\mathbf{M}_m$ is an interval $H$-matrix (or interval $M$-matrix) then the following results hold:

(i) $\widetilde{G} \leq C_m$ (equality holds if $\mathbf{M}_m$ is an interval $M$-matrix).

(ii) $\rho(\widetilde{G}) \leq \rho(C_m)$.

*Remark* 4.4. From the above results, it is obvious that if $\rho(C_m) < 1$ (with $C_0 = C$), then GIGS method converges, which will be used to prove the convergence of GIGS method in the succeeding results.

All through this section, we stick to the following notations:

(i) $\acute{G} := |\mathbf{M}^{-1}||\mathbf{N}|$, with $\mathbf{M} = \mathbf{D} - \mathbf{E}$ and $\mathbf{N} = \mathbf{F}$

(ii) $C := \langle\mathbf{M}\rangle^{-1}|\mathbf{N}|$.

(iii) $\widetilde{G} := |\mathbf{M}_m{}^{-1}| \cdot |\mathbf{N}_m|$, with $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ and $\mathbf{N}_m = \mathbf{F}_m$

(iv) $C_m := \langle\mathbf{M}_m\rangle^{-1}|\mathbf{N}_m| = M_1^{-1}N_1$

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We begin our results with the following theorem that presents a spectral bound of $C_m$ and hence for $\widetilde{G}$.

**Theorem 4.5.** Let $\mathbf{A}$ be an interval SDD matrix. Suppose $C_m = \langle \mathbf{T}_m - \mathbf{E}_m\rangle^{-1}|\mathbf{F}_m|$. Let $\widetilde{R}_i, \widetilde{E}_i, \widetilde{F}_i$ be defined same as in Notation 3, and, let $\langle \mathbf{T}\rangle_i = \langle \mathbf{D}_{ii}\rangle - \widetilde{R}_i$. If $\langle \mathbf{T}\rangle_i > \widetilde{E}_i,\ \forall i \in N$ then $\rho(C_m) \leq \max\limits_{i\in\langle n\rangle}\left(\dfrac{\widetilde{F}_i}{\langle \mathbf{T}\rangle_i - \widetilde{E}_i}\right)$, where $\langle n\rangle = \{1, 2, \dots, n\}$.

*Proof.* Let $\lambda$ be an eigenvalue $C_m$. Choose $x \neq 0 \in \mathbb{R}^n$ such that

$$\begin{aligned} C_m x = \lambda x \quad &\Rightarrow \quad \langle \mathbf{T}_m - \mathbf{E}_m\rangle^{-1}|\mathbf{F}_m|x = \lambda x \\ &\Rightarrow \quad (\lambda\langle \mathbf{T}_m\rangle - \lambda|\mathbf{E}_m| - |\mathbf{F}_m|)x = 0 \\ &\Rightarrow \quad [(\lambda\langle \mathbf{D}\rangle - \lambda|\mathbf{R}_m| - \lambda|\mathbf{E}_m| - |\mathbf{F}_m|]\, x = 0 \\ &\Rightarrow \quad \left[\langle \mathbf{D}\rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|\right] x = 0 \end{aligned}$$

Therefore the matrix $Q = \langle \mathbf{D}\rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|$ is singular, which implies $Q$ is not SDD. Hence there exists an $i \in N$ such that

$$\langle \mathbf{D}_{ii}\rangle \leq \widetilde{R}_i + \widetilde{E}_i + \left|\frac{1}{\lambda}\right| \widetilde{F}_i$$

After simplification we get

$$|\lambda| \leq \max_{i\in\langle n\rangle} \frac{\widetilde{F}_i}{(|\mathbf{D}_{ii}| - \widetilde{R}_i) - \widetilde{E}_i}$$

which implies that

$$\rho(C_m) \leq \max_{i\in\langle n\rangle} \frac{\widetilde{F}_i}{\langle \mathbf{T}\rangle_i - \widetilde{E}_i}$$

□

Note that above theorem provides a spectral upper bound of $C_m$ and hence of the iteration matrix $\widetilde{G}$ of the GIGS method.

**Lemma 4.6.** If $\mathbf{A} = \mathbf{T}_m - \mathbf{E}_m - \mathbf{F}_m$ be an interval SDD-matrix, then $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ is an interval $H$-matrix.

*Proof.* As $\mathbf{A}$ is an interval SDD matrix, the comparison matrix $\langle \mathbf{A}\rangle$ of $\mathbf{A}$ satisfies $\langle \mathbf{A}_{ii}\rangle > \sum\limits_{j\neq i} |\mathbf{A}_{ij}|$, for all $i$. Then

$$\langle \mathbf{A}_{ii}\rangle > \sum_{j\neq i} |\mathbf{A}_{ij}| \geq \sum_{\substack{j\neq i\\ |i-j|\leq m}} |\mathbf{A}_{ij}| + \sum_{i>j+m} |\mathbf{A}_{ij}|$$

which shows that $\mathbf{M}_m$ is an interval SDD matrix, hence an interval $H$-matrix by Remark 3.5. □

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**Theorem 4.7.** GIGS method given in (2) converges for interval SDD matrix $\mathbf{A}$, for any initial guess.

*Proof.* Let $\mathbf{A}$ be an interval SDD matrix. Consider the splitting of $\mathbf{A}$ as mentioned in equation (14). Then $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ is an interval $H$-matrix by Lemma 4.6. Thus it suffices to show $\rho(C_m) < 1$.

Suppose that $\lambda$ is an eigenvalue of $C_m$ and $|\lambda| \geq 1$. Take $Q = \langle \mathbf{D} \rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|$. Now $\frac{1}{|\lambda|} \leq 1$ and $\mathbf{A}$ is SDD imply that $Q$ is SDD. Again as shown in Theorem 4.5, we can prove that $\det(Q) = 0$, which contradicts the fact that $Q$ is SDD. Hence $\rho(C_m) < 1$ and thus the result holds for interval SDD matrix. $\square$

The successive two theorems analyze the convergence of GIGS method for interval $M$-matrices.

**Theorem 4.8.** GIGS method for solving (1) converges for interval $M$-matrix $\mathbf{A}$.

*Proof.* Let $\mathbf{A}$ be an interval $M$-matrix, then $\langle \mathbf{A} \rangle$ is an $M$-matrix by Theorem 2.17(iii). Consider the decomposition of $\mathbf{A}$ and $\langle \mathbf{A} \rangle$ respectively, defined in (6) and (10). As $\mathbf{A} = \mathbf{M}_m - \mathbf{N}_m$ is interval $M$-matrix, there exists $v > 0$ such that $\mathbf{A}v > 0$ which signifies that $\mathbf{M}_m v > 0$, because $\mathbf{N}_m \geq 0$. Hence $\mathbf{M}_m$ is an interval $M$-matrix. We need to show $\rho(\widetilde{G}) < 1$.

Since $\langle \mathbf{A} \rangle = M_1 - N_1$, where $M_1$ and $N_1$ mentioned in equation (16), is an $M$-matrix, we can choose $u > 0$ such that $\langle \mathbf{A} \rangle u > 0$ which leads to $M_1 u > 0$ that shows $M_1$ is an $M$-matrix. By Definition 2.11, $\langle \mathbf{A} \rangle = M_1 - N_1$ is an $M$-splitting with nonsingular $M_1$, so Theorem 2.14 gives $\rho(C_m) < 1$. Also using Theorem 4.3 for interval $M$-matrix $\mathbf{M}_m$ we have $\rho(\widetilde{G}) = \rho(C_m)$. Thus we get $\rho(\widetilde{G}) = \rho(C_m) < 1$. $\square$

**Theorem 4.9.** Let $\mathbf{A}$ be an interval $M$-matrix. Then $\rho(C_m) \leq \rho(C)$, for any $m \geq 1$.

*Proof.* By Lemma 4.6, $C = \langle \mathbf{M} \rangle^{-1}|\mathbf{N}|$ is a nonnegative matrix and hence by Perron-Frobenius Theorem $\rho(C)$ is an eigenvalue of $C$ and there is an $x \geq (\neq) 0$ such that $Cx = \lambda x$, that is, $\lambda \langle \mathbf{M} \rangle x = |\mathbf{N}|x$.

Let us write $|\mathbf{E}| = |\mathbf{E}_m| + |\mathbf{R}_m^E|$ and $|\mathbf{F}| = |\mathbf{F}_m| + |\mathbf{R}_m^F|$. We now have that

$$
\begin{aligned}
\lambda \langle \mathbf{M}_m \rangle x &= \lambda \left( \langle \mathbf{D} \rangle - |\mathbf{E}_m| \right) x \\
&= \lambda \left( \langle \mathbf{D} \rangle - |\mathbf{E}| + |\mathbf{R}_m^E| \right) x \\
&= \lambda \langle \mathbf{M} \rangle x + \lambda |\mathbf{R}_m^E| x \\
&\geq |\mathbf{N}|x = |\mathbf{F}|x \\
&= \left( |\mathbf{F}_m| + |\mathbf{R}_m^F| \right) x \geq |\mathbf{F}_m|x = |\mathbf{N}_m|x
\end{aligned}
$$

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As $\langle \mathbf{M}_m \rangle^{-1} \geq 0$, so we have that $\lambda x \geq \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| x$ and hence by Theorem 2.15 we get

$$\rho(\mathbf{C}_m) = \rho\left(\langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m|\right) \leq \lambda = \rho(C).$$

□

Note that Theorem 4.9 leads to the fact that if interval GS method converges, then GIGS method converges for any choice of bandwidth $m$.

Next theorem is for special case of interval $M$-matrices, which provides a comparison of spectral radii of iterative matrices of GIGS for different bandwidth. A similar result for AOR method for linear system was presented by Salkuyeh [5].

**Theorem 4.10.** Let $\mathbf{A}$ be an interval $M$-matrix. If $C_k = \langle \mathbf{M}_k \rangle^{-1} |\mathbf{N}_k|$, then for any $m \geq p$, $\rho(C_m) \leq \rho(C_p)$.

*Proof.* As $\mathbf{A}$ is interval $M$-matrix and so is $\langle \mathbf{M}_p \rangle$, so $C_p$ is nonnegative matrix. Thus by Perron-Frobenius theorem we can choose an eigenvector $x \geq (\neq)0$ associated with the eigenvalue $\lambda = \rho(C_p)$, so that $C_p x = \lambda x$, that is, $(|\mathbf{N}_p| - \lambda \langle \mathbf{M}_p \rangle) x = 0$.

If we write $\mathbf{T}_p = \mathbf{D} + \mathbf{R}_p$ and $\mathbf{T}_m = \mathbf{D} + \mathbf{R}_m$, then

$$|\mathbf{A}| = |\mathbf{D}| + |\mathbf{R}_m| - |\mathbf{E}_m| - |\mathbf{F}_m| = |\mathbf{D}| + |\mathbf{R}_p| - |\mathbf{E}_p| - |\mathbf{F}_p|$$

which implies that $R + L + U = 0$, where $R = |\mathbf{R}_m| - |\mathbf{R}_p|$, $L = |\mathbf{E}_p| - |\mathbf{E}_m|$ and $U = |\mathbf{F}_p| - |\mathbf{F}_m|$. Since $m \geq p$, so we must have $R \leq 0,\ L \geq 0$ and $U \geq 0$. We now have that

$$\begin{aligned}
C_m x - \lambda x &= \langle \mathbf{M}_m \rangle^{-1} \left( |\mathbf{N}_m| - \lambda \langle \mathbf{M}_m \rangle \right) x \\
&= \langle \mathbf{M}_m \rangle^{-1} \left( |\mathbf{F}_m| - \lambda \langle \mathbf{D} \rangle + \lambda |\mathbf{R}_m| + \lambda |\mathbf{E}_m| \right) x \\
&= \langle \mathbf{M}_m \rangle^{-1} \left( |\mathbf{F}_p| - U - \lambda \langle \mathbf{D} \rangle + \lambda R + \lambda |\mathbf{R}_p| + \lambda |\mathbf{E}_p| - \lambda L \right) x \\
&= \langle \mathbf{M}_m \rangle^{-1} \left( |\mathbf{N}_p| - \lambda \langle \mathbf{M}_p \rangle \right) x - \langle \mathbf{M}_m \rangle^{-1} \left( U + \lambda L - \lambda R \right) x \\
&= -\langle \mathbf{M}_m \rangle^{-1} \left( U + \lambda L - \lambda R \right) x \\
&\leq 0
\end{aligned}$$

Hence $C_m x \leq \lambda x$ and hence $\rho(C_m) \leq \lambda = \rho(C_p)$ by Theorem 2.15. □

Following example validates the above two theorems.

**Example 4.11.** Consider the interval $M$-matrix

$$\mathbf{A} = \begin{pmatrix} 4 & [-1,0] & [-1,0] & [-1,0] \\ [-1,0] & 5 & [-1,0] & [-1,0] \\ [-1,0] & [-1,0] & 4 & [-1,0] \\ [-1,0] & [-1,0] & [-1,0] & 5 \end{pmatrix}$$

Then we have that $\rho(C) = 0.4640$, $\rho(C_1) = 0.2749 < 1$ and $\rho(C_2) = 0.1111 < 1$ that is $\rho(C_2) < \rho(C_1) < \rho(C) < 1$ which shows that the above results hold.

We conclude the section by checking the convergence property of GIGS method for interval linear system with coefficient matrix as interval $H$-matrices.

**Theorem 4.12.** GIGS method converges for interval $H$-matrices.

*Proof.* Let $\mathbf{A}$ be an interval $H$-matrix, then $\langle \mathbf{A} \rangle$ is an $M$-matrix. It suffices to prove $\rho(\widetilde{G}) < 1$, where $\widetilde{G} = |{\mathbf{M}_m}^{-1}| \cdot |\mathbf{N}_m|$.

As $\langle \mathbf{A} \rangle = M_1 - N_1$, with $M_1$ and $N_1$ are defined in (16), is an $M$-matrix, so as shown in Theorem 4.8, we can find $u > 0$ such that $M_1 u > 0$. Thus $\langle \mathbf{A} \rangle = M_1 - N_1$ is regular splitting with $M_1^{-1} \geq 0$ and $N_1 > 0$. Hence by Theorem 2.13, $\rho(M_1^{-1} N_1) < 1$. Since $\mathbf{M}_m$ is an interval $H$-matrix, we have $\rho(\widetilde{G}) \leq \rho(C_m)$ due to Theorem 4.3. Thus we get $\rho(\widetilde{G}) \leq \rho(C_m) < 1$ by Remark 3.4. Hence GIGS method converges for any initial guess. □

However, for interval $L$-matrices GIGS and GIJ methods may not converge. For simplicity we consider examples of interval $L$-matrix with constant entries to see the convergence behavior of both the methods for interval $L$-matrices.

**Example 4.13.** Consider the interval $L$-matrix (with constant entries)

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & -6 \\ -3 & 1 & -4 \\ -4 & -5 & 3 \end{pmatrix}$$

If $m = 1$ then for GIGS method we get $\rho(\widetilde{G}) = 1.0459 > 1$ and for GIJ method we get $\rho(\widehat{L}_m) = 2.0725 > 1$. Thus it shows GIGS method and GIJ method do not converge for $\mathbf{A}$.

**Example 4.14.** Consider the interval $L$-matrix (with constant entries)

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ -3 & 2 & -3 \\ -2 & -1 & 2 \end{pmatrix}$$

If $m = 1$ then GIGS method gives $\rho(\widetilde{G}) = 0.6364 < 1$ but GIJ method gives $\rho(\widehat{L}_m) = 1$. In this case GIGS converges but GIJ method doesn't converge for $\mathbf{A}$.

# 5 Numerical Illustration

In this section numerical examples are considered to compare the convergence of generalized interval Jacobi method and generalized interval Gauss-Seidel method. In particular, examples are considered with coefficient matrix $\mathbf{A}$ as an interval SDD matrix, interval $M$-matrix, and an interval $H$-matrix. The computations are carried out in MATLAB(2021b) with the interval toolbox INTLAB v12 [28] and on a PC-Intel(R) Core(TM) i7-5700U CPU @1.80 GHz, 8 GB RAM. The computations are rounding to four digits and the stopping criteria is chosen as $\|\mathsf{qdist}(\mathbf{x}^{(k+1)}, \mathbf{x}^{(k)})\| \le 10^{-6}$, where $\mathsf{qdist}(\mathbf{x}, \mathbf{y}) := \max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}$ represents a measure of distance between the intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$, and in case $\mathbf{x},\ \mathbf{y} \in \mathbb{IR}^n$, then $\mathsf{qdist}(\mathbf{x}, \mathbf{y}) = [\mathsf{qdist}(\mathbf{x}_1, \mathbf{y}_1), \dots, \mathsf{qdist}(\mathbf{x}_n, \mathbf{y}_n)] \in \mathbb{R}^n$, where $\mathbf{x}_i$ represents the $i$-th entry of the interval vector $\mathbf{x}$.

We now furnish examples for various classes of interval coefficient matrices, and provide comparisons of our proposed methods with IJ and IGS methods , in terms of no. of iterations, time (in seconds) and $r_k = \|(\mathsf{qdist}(\mathbf{x}^{(\mathsf{k})}, \mathbf{x})\|$, where $\mathbf{x}$ is the enclosure obtained by using verifylss.

**Example 5.1.** Consider the interval linear system (1) with the coefficient interval matrix as interval strictly diagonally dominant matrix mentioned in Neumaier book [3] .

$$\mathbf{A} := \begin{pmatrix} 3 & [-2,2] & 0 \\ 0 & 3 & [-2,2] \\ [-2,2] & 0 & 3 \end{pmatrix}$$

and

$$\mathbf{b} := \begin{pmatrix} [-1,1] \\ [-1,1] \\ 2 \end{pmatrix}$$

Then the function verifylss from the package INTLAB produces the enclosure

$$\mathbf{x} = ([-1.2282, 1.2282], [-1.3423, 1.3423], [-0.1599, 1.4932])^T$$

Taking the initial guess as $\mathbf{x}_0 = ([0,3],[0,3],[0,3])^T$, generalized interval Jacobi converges after 13 iterations and yields the enclosure

$$\mathbf{x}_{\mathrm{GIJ}} = ([-1.2106, 1.2106], [-1.3158, 1.3158], [-0.1404, 1.4737])^T$$

whereas the generalized interval Gauss-Seidel converges after 1 iteration and produces the enclosure

$$\mathbf{x}_{\mathrm{GIGS}} = ([-1.2398, 1.2398], [-1.3481, 1.3481], [-0.1714, 1.5048])^T$$

Table 1 compares GIJ and GIGS methods (taking $m = 1$) with IJ and IGS methods (with $m = 0$) with the initial guess taken as $\mathbf{x}_0 = ([0,3],[0,3],[0,3])^T$.

| Iterative method | No. of iterations | $r$ | time in second |
|---|---|---|---|
| GIJ | 13 | 0.0372 | 0.0225 |
| GIGS | 1 | 0.0173 | 0.0147 |
| IJ | 35 | 0.0372 | 0.0437 |
| IGS | 24 | 0.0372 | 0.0237 |

Table 1: Numerical result for the interval SDD-matrix with $m = 1$

From the above table we can see that the generalized interval Jacobi method gives the tightest solution set enclosure.

**Example 5.2.** Consider the interval linear system (1) with the coefficient interval $M$-matrix

$$\mathbf{A} := \begin{pmatrix} 4 & [-2,0] & [-1,0] \\ [-1,0] & [3,4] & -1 \\ [-2,-1] & [-1,0] & 5 \end{pmatrix}$$

and

$$b := \begin{pmatrix} 1.2 \\ 1.5 \\ 5 \end{pmatrix}$$

then the verifylss function yields the enclosure

$$\mathbf{x} = ([-0.2935, 1.6658], [0.0677, 1.7258], [0.5795, 2.0115])^T$$

Taking initial guess as $\mathbf{x}_0 = ([-1,2], 1, [2,3])^T$, generalized interval Jacobi converges after 16 iterations and provides the enclosure

$$\mathbf{x}_{\mathrm{GIJ}} = ([-0.2935, 1.6658], [0.0677, 1.7258], [0.5795, 2.0115])^T$$

whereas generalized interval Gauss-Seidel converges after 9 iterations and gives the enclosure

$$\mathbf{x}_{\mathrm{GIGS}} = ([-0.2936, 1.6658], [0.0676, 1.7259], [0.5795, 2.0116])^T$$

Table 2 provides a comparison of GIJ and GIGS methods (taking $m = 1$) with IJ and IGS methods (with $m = 0$) with the initial guess taken as $\mathbf{x}_0 = ([-1,2], 1, [2,3])^T$.

| Iterative method | No. of iterations | $r$ | time in seconds |
|---|---|---|---|
| GIJ | 16 | $3.23 \times 10^{-5}$ | 0.0272 |
| GIGS | 9 | $1.25 \times 10^{-4}$ | 0.0211 |
| IJ | 30 | $2.38 \times 10^{-5}$ | 0.0443 |
| IGS | 18 | $2.46 \times 10^{-5}$ | 0.0236 |

Table 2: Numerical result for the interval $M$-matrix with $m = 1$

This shows that in case of interval $M$-matrices, generalized interval Jacobi method gives the tightest enclosure of the solution set.

**Example 5.3.** Consider the interval linear system (1) with the following coefficient interval $H$-matrix

$$\mathbf{A} := \begin{pmatrix} [4,5] & [-2,2] & [-1,0] \\ [0,1] & [3,5] & [-1,1] \\ [-1,1] & [1,3] & 5 \end{pmatrix}$$

and

$$\mathbf{b} := \begin{pmatrix} [0.1, 0.5] \\ [-0.3, -0.1] \\ [0.3, 0.4] \end{pmatrix}$$

Then the function verifylss from the package INTLAB generates the enclosure

$$\mathbf{x} = ([-0.2425, 0.3967], [-0.3567, 0.2375], [-0.1857, 0.3734])^T$$

Taking the initial guess as $\mathbf{x}_0 = (1, 5, 4)^T$, generalized interval Jacobi converges after 15 iteartions and yields the enclosure

$$\mathbf{x}_{\text{GIJ}} = ([-0.2426, 0.3968], [-0.3567, 0.2375], [-0.1857, 0.3734])^T$$

whereas the generalized interval Gauss-Seidel converges after 9 iterations and produces the enclosure

$$\mathbf{x}_{\text{GIGS}} = ([-0.2425, 0.3967], [-0.3567, 0.2374], [-0.1857, 0.3734])^T$$

We now produce a comparison in Table 3 of GIJ and GIGS methods (taking $m = 1$) with IJ and IGS methods (with $m = 0$) with the initial guess taken as $\mathbf{x}_0 = (1, 5, 4)^T$.

| Iterative method | No. of iterations | $r$ | time in seconds |
|---|---|---|---|
| GIJ | 15 | $3.96 \times 10^{-5}$ | 0.0247 |
| GIGS | 9 | $9.73 \times 10^{-6}$ | 0.0209 |
| IJ | 45 | $1.11 \times 10^{-5}$ | 0.0475 |
| IGS | 24 | $1.27 \times 10^{-5}$ | 0.0294 |

Table 3: Numerical result for the interval $H$-matrix with $m = 1$

For this example the most tightest enclosure of solution set provided by the generalized interval Gauss-Seidel method.

## 6 Conclusion

In this paper, we proposed a generalized interval Jacobi (GIJ) method and generalized interval Gauss-Seidel method (GIGS). These methods are generalization of interval Jacobi and interval Gauss-Seidel methods, discussed by Neumaier [3, 25] to solve interval linear system. The GIJ and GIGS methods are proposed similar to that introduced by Salkuyeh in [4], by generalizing the diagonal interval matrix to a band interval matrix. We proved that both the proposed methods converge for interval SDD matrix, interval $M$-matrix, and for interval $H$-matrix. Further we found that for interval $M$-matrices, GIGS method converges for any choice of bandwidth $m$ if interval GS method converges. At last we consider numerical examples to observe that GIJ gives a tighter enclosure for interval $M$- coefficient matrices, whereas GIGS provides a tighter enclosure of the solution set for interval $H$-matrices. This leads to the open problem that the same can be concluded in general.

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