A Homeomorphic Image of the Space of Quasi- Continuous Functions by

Bhagath kumar Soma[1] , V. Srinivasa kumar[2], KusumaTummala[3]

[1]Soma Assistant Professor, Department of Basic Sciences, Gokaraju Rangaraju Institute of Engineering and Technology, Bachupally,Kukatpally,Hyderabad-500090,Telangana State, India.
Email: somabagh@gmail.com
[2]Assistant Professor, Department of Mathematics, JNTUH College of Engineering, JNTU, Kukatpally, Hyderabad-500085, Telangana State, India.
Email: srinu_vajha@yahoo.co.in
[3]Assistant Professor, Department of Mathematics, VNR Vignana Jyothi Inistitute of Engineering and Technology, Hyderabad,500090, Telangana State, India.
Email: kusumatummala9@gmail.com

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ABSTRACT
In this present work, the concept of Quasi- continuity is defined on a closed and bounded interval $[0,1]$. It is established that the set of all such Quasi- continuous functions forms a commutative Banach algebra under the supremum norm. Properties of some maps on the space of Quasi-continuous functions are established. An isometrical isomorphic image of the space of Quasi-continuous functions is investigated. The notion of sequential quasi- continuity is introduced.

Keywords: Quasi-continuity, Continuity, Sequential quasi –continuity, Banach space, Linear map, Isometrical isomorphism.

1. INTRODUCTION

The notion of Quasi-continuous real functions of several real variable was introduced firstly by Kempistyin his classical research article[1]of 1932. Later, many authors defined Quasi-continuity according to their convenience and proposed many results. So one can find various types of Quasi-continuous functions in the literature $[4]$. Basically any type of Quasi-continuity is a weaker form of continuity, i.e., every continuous function is Quasi-continuous but the converse is not necessarily true.

The present work also introduces a notion of Quasi-continuity on the closed unit interval $[0,1]$ .Throughout this work, I stands for the closed unit interval $[0,1]$. Let X be a commutative Banach algebra over the field $\mathbb{R}$ of all real numbers. Let X^3 denote the product space $X \times X \times X$ with coordinate wise linear operations and norm $\|(x,y,z)\| = \max\{\|x\|,\|y\|,\|z\|\}$. Clearly the space X^3 forms a commutative Banach algebra over$\mathbb{R}$.

The main aim of this work is to investigate a homeomorphic image of the space of all Quasi-continuous bounded functions from I into X and to find an isometrical isomorphic image of this space.

2. Preliminaries

This section is devoted to introduce Quasi-continuity from I into X and to present a few definitions which are needed for further study of this paper.

Definition-1.1: Let $f : I \to X$. We say that $f(p+)$ exists at a point $P \in [0,1)$ and write $f(p+) = L$ if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$\|f(x) - L\| < \varepsilon \quad \forall \quad x \in (p, p+\delta) \subset I$,where $L \in X$.

We say that $f(p-) = l$ at a point $p \in (0,1]$ if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$\|f(x) - l\| < \varepsilon \quad \forall \quad x \in (p-\delta, p) \subset I$, where $l \in X$.

Definition-1.2: By a Quasi-continuous function, we mean that a function $f : I \to X$ satisfying the following conditions.

(a) $f(0+)$ and $f(1-)$ exist.

(b) $f(p+)$ and $f(p-)$ exist at every $p \in (0,1)$.

Notation-1.3:

1.We denotethe set of all quasi-continuous bounded functions from I into X by the symbol $\mathcal{Q}(I, X)$ and the set of all continuous bounded functions from I into X by the symbol $C(I, X)$

2. For our convenience, we take $f(0-) = f(0)$ and $f(1+) = f(1)$ where $f \in \mathcal{Q}(I, X)$

Definition-1.4 [2]: Let V and W be any two linear spaces over the same field K of scalars. A mapping $T : V \to W$ is said to be linearif $T(cx + dy) = cT(x) + dT(y)$ for all x and y in V and scalars c and d in K.

Definition -1.5[2]:Let V and W be any two normed linear spaces over the same field K of scalars. A linear map $T : V \to W$ is said to be bounded if there exists a real number $M \geq 0$

Suchthat $\|T(x)\| \leq M\|x\| \quad \forall \quad x \in V$.

Definition – 1.6 [5]: Let N and N' be normed linear spaces. An isometric isomorphism of N into N' is a one-to-one linear transformation T of N into N' such that $\|T(x)\| = \|x\|$ for every x in N and N is said to be isometrically isomorphic to N' if there exists an isometric isomorphism of N into N'.

3.The Space of Quasi- continuous functions

In this section, we investigate an isometrical isomorphic image of the space of all quasi-continuous bounded real functions from I into X.

Proposition-2.1: The set $\mathcal{Q}(I, X)$ of all Quasi-continuous bounded functions from I into X forms a commutative Banach algebra with identity under the supremum norm over the field $\mathbb{R}$ of real numbers with respect to pointwise linear operations.

Definition-2.2: Fix $f \in \mathcal{Q}(I, X)$. Define $\Psi_f : I \to X^3$ by $\Psi_f(x) = (f(x), f(x+), f(x-))$ for all $x \in I$.

Proposition-2.3: Let $f \in \mathcal{Q}(I, X)$. If $\Psi_f \in C(I, X^3)$ then $f \in C(I, X)$.

Proof: Suppose that $\Psi_f \in C(I, X^3)$. Let $p \in I$ and $\varepsilon > 0$ be given.

Since Ψ_f is continuous at p, there exists a $\delta > 0$ such that

$\|\Psi_f(x) - \Psi_f(p)\| < \varepsilon \quad \forall \quad x \in (p - \delta, p + \delta) \subset I$

$\Rightarrow \|(f(x), f(x+), f(x-)) - (f(p), f(p+), f(p-))\| < \varepsilon \quad \forall \quad x \in (p - \delta, p + \delta)$

$\Rightarrow \|(f(x) - f(p), f(x+) - f(p+), f(x-) - f(p-))\| < \varepsilon \quad \forall \quad x \in (p - \delta, p + \delta)$

$\Rightarrow \max\{\|f(x) - f(p)\|, \|f(x+) - f(p+)\|, \|f(x-) - f(p-)\|\} < \varepsilon \quad \forall \quad x \in (p - \delta, p + \delta)$

$\Rightarrow \|f(x) - f(p)\| < \varepsilon \quad \forall \quad x \in (p - \delta, p + \delta)$

Hence for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$\|f(x) - f(p)\| < \varepsilon \quad \forall \quad x \in (p - \delta, p + \delta)$

$\Rightarrow$ f is continuous at p.

Thus the continuity of Ψ_f at p implies the continuity of f at p.

Remark-2.4: If f is continuous at every point of I then it can be easily seen that Ψ_f is continuous on I.

Proposition-2.5: Let $\mathcal{B} = \{\Psi_f : f \in \mathcal{Q}(I,X)\}$. Then $\mathcal{B}$ forms a normed linear space withthe norm $\|\Psi_f\| = \sup\{\|\Psi_f(x)\| : x \in I\}$.

Proposition-2.6: Define $F : \mathcal{Q}(I,X) \to \mathcal{B}$ by $F(f) = \Psi_f$. Then F is a one-to-one continuous linear mapping from $\mathcal{Q}(I,X)$ onto $\mathcal{B}$ .Also $\|F(f)\| = \|f\|$.**Proof:**Clearly $F : \mathcal{Q}(I,X) \to \mathcal{B}$ is surjective.

For $f,g \in \mathcal{Q}(I,X)$, $\Psi_{f+g}(x) = ((f+g)(x),(f+g)(x+),(f+g)(x-))$

$= (f(x), f(x+), f(x-)) + (g(x), g(x+), g(x-)) = \Psi_f(x) + \Psi_g(x) \quad \forall \ x \in I$

Hence $\Psi_{f+g} = \Psi_f + \Psi_g \quad \forall \quad f,g \in \mathcal{Q}(I,X)$

$\Rightarrow F(f+g) = F(f) + F(g) \quad \forall \ f,g \in \mathcal{Q}(I,X)$.

Let c ϵ R. Then it is easy to see that $F(cf) = \Psi_{cf} = c\Psi_f = cF(f) \quad \forall \quad f \in \mathcal{Q}(I,X)$.

Hence F is linear.

Also we have $\Psi_{fg}(x) = ((fg)(x),(fg)(x+),(fg)(x-))$

$= (f(x), f(x+), f(x-)) \ (g(x), g(x+), g(x-))$

$= \Psi_f(x)\Psi_g(x) \quad \forall \quad x \in I$.

Hence $F(fg) = \Psi_{fg} = \Psi_f \Psi_g = F(f)F(g)$.

Now we prove that F is $1-1$. For this, suppose that

$F(f) = F(g) \Rightarrow \Psi_f = \Psi_g$

$\Rightarrow \Psi_f(x) = \Psi_g(x) \ \forall \ x \in I$

$\Rightarrow (f(x), f(x+), f(x-)) = (g(x), g(x+), g(x-)) \ \forall \ x \in I$

$\Rightarrow f(x) = g(x) \quad \forall \quad x \in I$

$\Rightarrow f = g$.

Hence F is $1-1$.

Now we prove that F is continuous. Supposethat $f_n \in \mathcal{Q}(I,X), n = 1,2,3,\ldots\ldots$, and $f \in \mathcal{Q}(I,X)$.

Let $f_n \to f$ uniformly on I. Then for a given $\varepsilon > 0$ there exists an integer $N > 0$ such that $\|f_n(x) - f(x)\| < \frac{\varepsilon}{3}$ for all $n \geq N$ and all $x \in I$.

Fix $x \in I$ and also fix a positive integer n such that $n \geq N$. Since f_n is Quasi-continuous at x, both $f_n(x+)$ and $f_n(x-)$ exist. Then there exists a $\delta_1 > 0$ such that $\|f_n(t) - f_n(x+)\| < \frac{\varepsilon}{3} \ \forall \ t \in (x, x+\delta_1) \subset I$.

Since f is also Quasi-continuous at x, both $f(x+)$ and $f(x-)$ exist. So there exists a $\delta_2 > 0$ such that $\|f(t) - f(x+)\| < \frac{\varepsilon}{3} \quad \forall \quad t \in (x, x+\delta_2) \subset I$.

Put $\delta = \min\{\delta_1, \delta_2\}$. Then for $t \in (x, x+\delta)$, we have

$\|f_n(x+) - f(x+)\| = \|f_n(x+) - f_n(t) + f_n(t) - f(t) + f(t) - f(x+)\|$

$\leq \|f_n(x+) - f_n(t)\| + \|f_n(t) - f(t)\| + \|f(t) - f(x+)\|$

$< \dfrac{\varepsilon}{3}+\dfrac{\varepsilon}{3}+\dfrac{\varepsilon}{3}=\varepsilon$.

Hence $\| f_n(x+)-f(x+)\| < \varepsilon$.

This holds for every $x \in I$ and for every positive integer $n \geq N$.

Similarly $\| f_n(x-)-f(x-)\| < \varepsilon$ for all $n \geq N$ and all $x \in I$.

$n \geq N \Rightarrow \|F(f_n)-F(f)\| = \|\Psi_{f_n}-\Psi_f\|$

$=\sup\left\{\|\Psi_{f_n}(x)-\Psi_f(x)\| : x \in I\right\}$

$=\sup\left\{\|(f_n(x), f_n(x+), f_n(x-))-(f(x), f(x+), f(x-))\| : x \in I\right\}$

$\leq \varepsilon$

$\Rightarrow \quad F(f_n) \to F(f)$ as $n \to \infty$

Hence F is continuous on $\mathscr{Q}(I, X)$.

Hence F is a one-to-one continuous linear mapping from $\mathscr{Q}(I, X)$ onto $\mathscr{B}$. This shows that $\mathscr{Q}(I, X)$ is homeomorphic to $\mathscr{B}$.

Now it remains to prove that $\|F(f)\|=\|f\|$ for every $f \in \mathscr{Q}(I, X)$. We have

$\|F(f)\|=\|\Psi_f\| \quad = \quad \sup\left\{\|\Psi_f(x)\| : x \in I\right\}$

$\geq \|\Psi_f(x)\|$ for all $x \in I$

$=\|(f(x), f(x+), f(x-))\|$

$=\max\left\{\|f(x)\|,\|f(x+)\|,\|f(x-)\|\right\}$

$\geq \|f(x)\|$ for all $x \in I$

$\Rightarrow \|F(f)\| \geq \sup\left\{\|f(x)\| : x \in I\right\}=\|f\| \qquad \to (1)$

Now, $\|f(x)\| \;\leq\; \max\left\{\|f(x)\|,\|f(x+)\|,\|f(x-)\|\right\}$

$=\|\Psi_f(x)\|$

$\leq \sup\left\{\|\Psi_f(x)\| : x \in I\right\}$

$=\|\Psi_f\|$

$=\|F(f)\|$

$\Rightarrow \quad \|f(x)\| \leq \|F(f)\|$ for all $x \in I$

$\Rightarrow \quad \|f\| \;=\; \sup\left\{\|f(x)\| : x \in I\right\} \leq \|F(f)\| \qquad \to (2)$

From (1) and (2), we have $\|F(f)\|=\|f\|$.

This shows that F is an Isometrical isomorphism from $\mathscr{Q}(I, X)$ onto $\mathscr{B}$.

Proposition-2.7: The set $\mathscr{B}=\{\Psi_f : f \in \mathscr{Q}(I, X)\}$ is a commutative Banach algebra with identity Ψ_e under the norm defined by $\|\Psi_f\|=\sup\left\{\|\Psi_f(x)\| : x \in I\right\}$, where $\Psi_e(x)=(1,1,1) \quad \forall \; x \in I$.

Proposition-2.8: Fix $x \in I$. Define $\varphi_x(f)=\Psi_f(x)=(f(x), f(x+), f(x-))$ $\forall f \in \mathscr{Q}(I, X)$.

Then

(a) $\varphi_x : \mathcal{Q}(I,X) \to X^3$ is linear

(b) $\|\varphi_x\| \geq \|x\| \quad \forall \; x \in I$.

(c) φ_x is bounded

Proof: (a) The linearity of φ_x is easy to verify.

(b) We have $\|\varphi_x(f)\| = \|(f(x), f(x+), f(x-))\|$

$= \max\{\|f(x)\|, \|f(x+)\|, \|f(x-)\|\}$

$\geq \|f(x)\| \quad \forall \; f \in \mathcal{Q}(I,X)$

$\Rightarrow \|\varphi_x(f)\| \geq \|f(x)\| \quad \forall \; f \in \mathcal{Q}(I,X)$

Now it follows that $\|\varphi_x\| = \sup\{\|\varphi_x(f)\| : \|f\| \leq 1\}$

$\geq \|\varphi_x(f)\| \; \forall \; f \in \mathcal{Q}(I,X)$ with $\|f\| \leq 1$

$\geq \|f(x)\| \; \forall \; f \in \mathcal{Q}(I,X)$ with $\|f\| \leq 1$

$\Rightarrow \|\varphi_x\| \geq \|f(x)\| \; \forall \; f \in \mathcal{Q}(I,X)$ with $\|f\| \leq 1 \rightarrow \qquad (1)$

Let $u : I \to X$ be defined by $u(x) = x \; \forall \; x \in I$

Clearly $u \in \mathcal{Q}(I,X)$. It is easy to see that $\|u\|$ =1.

From (1), $\|\varphi_x\| \geq \|u(x)\| = \|x\|$

$\Rightarrow \|\varphi_x\| \geq \|x\|$

Also $\|\varphi_x(f)\| = \|(f(x), f(x+), f(x-))\|$

$= \|\varphi_f(x)\|$

$\leq \sup\{\|\Psi_f(x)\| : x \in I\}$

$= \|\Psi_f\|$

$= \|F(f)\|$

$\Rightarrow \|\Psi_x(f)\| \leq \|F(f)\| = \|f\|$

$\Rightarrow \|\Psi_x(f)\| \leq \|f\| \quad \forall \; f \in \mathcal{Q}(I,X)$

$\Rightarrow \varphi_x$ is bounded.

4.Sequential Quasi-continuity

In this section, the notion of sequential quasi-continuity is introduced and it is shown to be equivalent to quasi-continuity.

Definition-3.1: We say that a function $f : I \to X$ is sequentially quasi-continuous at a point $x \in I$ if for any sequence $x_n \to x$ in I , the subsequential limits of the sequence $\{f(x_n)\}$ are $f(x), f(x+)$ *or* $f(x-)$ only. If f is sequentially quasi-continuous at every point of I then we say that f is sequentially quasi-continuouson I .

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