On minimal non-(Hypercentral-By-Cernikov) Groups

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ABSTRACT
If X is a class of groups, then a group is said to be minimal non-X if it is not an X-group, while all its proper subgroups belong to X. The main result of this note is if G is a minimal non-ZAC group , then G is a finitely generated perfect group which has no proper subgroup of finite index and such that G/Frat(G) is an infinite simple group, where ZA (respectively, C) denotes the class of hypercentral groups, (respectively, the class of Cernikov groups), and Frat(G) stands for the Frattini subgroup of G.

Keywords: Hypercentral, Cernikov, Hypercentral-by-Cernikov, Minimal non- Ω

1. INTRODUCTION

If X is a class of groups, then a group G is said to be minimal non -X if all its proper subgroups are in the class X, but G itself is not an X-group. We will denote minimal non-X-groups by MNX-groups. Many results have been obtained on minimal non-X-groups, for various choices of X, especially in the case of finite groups. For instance, finite minimal non-abelian, minimal non-nilpotent and minimal non-supersoluble groups have been completely described by G.A.Miller and H.C.Moreno [2], O..Schmidt [3] and K.Doerk[4] In particular, in [6] (respectively, in [5]) it is proved that if G is a finitely generated minimal non-nilpotent (respectively, non-(finite-by-nilpotent) group, then G is a perfect group which has no proper subgroup of finite index and such that G/Frat(G) is an infinite simple group, where Frat(G) denotes the Frattini subgroup of G.We generalize this last result to minimal non-hypercentral groups. We will prove. If G is a finitely generatednon- (Cernikov-by-hypercentral) group, then G is a perfect group which has no proper subgroup of finite index and such that G/Frat(G) is an infinite simple group.

2. Finitely generated hypercentral-by-finite groups

Lemma 2.1 Let G be a group whose proper subgroups are hypercentral-by-Cernikov. If N be a propre normal subgroup of G such that N is hypercentral group, Then G/N is an MNZAF.

Proof: Let G be a group whose proper subgroups are hypercentral-by-Cernikov, let N be a propre normal hypercentral subgroup of G. Suppose that G/N is hypercentral-by-Cernikovgroup.
Therefore there exists $(K/N)\lhd(G/N)$ such that (K/N) is hypercentral group and (G/K) is Cernikov , we obtain that G is hypercentral-by-Cernikov group, as required

Lemma 2.2 Let G be a group periodic whose proper subgroups are hypercentral-by-Cernikov. Then G it is locally finite.

Proof: Let H be an finitely generated subgroup of G,so proper subgroup, then H is hypercentral-by-Cernikov subgroup,there exists K is hypercentral subgroup of H and (H/K) is Cernikovgroup. K an finitely generated hypercentral group so finite, and H/K is finitely generated Cernikov group then H is finite.

Lemma 2.3 If G is a finitely generated minimal non ZAC-group, then G has no non-trivial finite factor hypercentral-by-Cernikov groups.

Proof: Let G be an finitely generated MNZAC-group. Suppose that G has a normal proper subgroup N such that G/N is hypercentral-by-Cernikov group,So that G/N is finitely generated hypercentral-by-Cernikov group, so it is locally nilpotent-by-finite group, since it is exist $M\lhd G$ such that M is hypercentral, $M\neq G$ and G/M is finite. Thus we deduce that G is hypercentral-by-Cernikov group. Contradiction.

Theorem 2.1 Let G be a finitely generated minimal non-ZAC-group. Then:
i) G has no non-trivial finite factor.

ii) G is perfect group
iii) G/Frat(G) is an infinite simple group.
Proof: Let G be a finitely generated minimal non-ZAC-group.

i) Suppose that G is finitely generated and admits N a proper normal subgroup of finite index in G. So N is hypercentral-by-Cernikovproper subgroup and it is also finitely generated. Hence N is hypercentral-by-Cernikov subgroup, so N is locally nilpotent-by-finite a finitely generated subgroup then there exists characteristic hypercentral subgroups M and N/M is finitely generated finite, so it is Cernikov group is it self G is hypercentral-by-Cernikov group .

ii) Suppose this statement is false. Then $G \neq G'$, so G' is hypercentral-by-Cernikov. Hence G/G' is abelian, since it is locally graded. Now if G/G' is finitely generated, then there exists $H \lhd G$ such that $H \neq G$ and G/H is finite. So G has a proper subgroup of finite index. Thus we deduce from (i) that G is hypercentral-by-finite. Contradiction.

iii) Let G be a minimal non-ZAC-group. It follows que G is a finitely generated perfect group which has no non trivial finite factor. Now we prove that G/Frat(G) is an infinite simple group. Since finitely generated groups have maximal subgroups, G/Frat(G) is non trivial and therefore infinite. Let N be a proper normal subgroup of G properly containing Frat(G). Then N is hypercentral-by-Cernikov. Hence there is a maximal subgroup M of G such that N is not contained in M. Then G=NM and we have $(G/N)=(MN)/N) \simeq (M/(M \cap N)$

so,G/N is hypercentral-by-Cernikov group. This is contradiction.Then G/Frat(G) is a simple group.

3. Infinitely generated hypercentral-by-finite groups

Lemma 3.1 Let G be a infinitely generated MNZAC-group. Then G is F-perfect

Proof: Let G be a infinitely generated whose proper subgroups are in the class ZAC. Suppose that G admits a proper normal subgroup N of finite index in G , then N belongs to ZAC . so there exists K be a proper normal subgroup of N such that K is hypercentral subgroup and N/K is Cernikov. Let L be a Core of K in G such that $L=K^{g}$; We have N/K is Cernikov so N^x/k^x is Cernikov, and $N \lhd G$ so $N=N^X$ then, N/K^x is Cernikov, so $N/\cap K^x$ is Cernikov, hence N/L is Cernikov, so there exists normal subgroup L of G such that L is hypercentral, and $G/N \approx (G/N)/(N/L)$ is finite, so G/L is Cernikov, then G is hypercentral-by-Cernikov, contradiction

lemma 3.2 The class ZAF of hypercentral-by-finite groups is $N_{\cdot}\{0\}$-closed.

proof: Let H and K be normal hypercentral-by-Cernikov subgroups of a group G, there exist hypercentral subgroups H_1 and K_1 of H and K, respectively, such that $(H/(H_1)$ and $(K/(K_1)$ are Cernikov. We put $N=H_{1H}=\cap hH_1h^{-1}$(respectively $M=K_{1K}=\cap kK_1k^{-1}$) so $N \lhd H$ (respectively $M \lhd k$) and H/M (respectively K/M is finite, so NM is a normal Cernikov subgroup of HK. we have

$(HN/NM) \simeq (H/(H \cap NM) \simeq (H/N)/(H \cap NM)/N)$and $((KM/NM)) \simeq (K/(K \cap NM)) \simeq (K/M)/(K \cap NM)/M)$

are Cernikov, so $HK/NM=(HM)/(NM)(NK)/(NM)$ is Cernikov as the class of Cernikov groups is $\{H, N_0\}$-closed. Therefore, HK is hypercentral-by-Cernikov, as required.

Remark 3.1 Let G be a infinitely generated MNZAF-group, then G/G' is quasicyclic group.

Proof: By lemma 3.2, G cannot be the product of two proper normal subgroups and by lemma 3.1. We deduce that G/G' is quasicyclic group.

Lemma 3.3 Let G be a group, if G/(Z(G) is hypercentral-by-Cernikov then, G is hypercentral-by-Cernikov.

Proof: If G/Z(G) is hypercentral-by-Cernikov, therefore there exists normal hypercentral subgroup N/Z(G) and G/N is Cernikov, so $(Z(N)/Z(G)) \lhd N/Z(G)$ and $(N/Z(G)/Z(N)/Z(G)) \approx N/Z(N)$ is hypercentral, hence

$Z_{\alpha}(N/Z(N)=N/Z(N) \Rightarrow Z_{\alpha+1}(N)/Z(N)=N/Z(N)$, so $Z_{\alpha+1}(N)=Z(N)$, we have N is hypercentral and G/N is Cernikov, then G is hypercentral-by-Cernikov group.

Theorem 3.1 Let G be infinitely generated, if G is MNZAF. Then G is MNZAC.

Proof: Let G be a locally nilpotent MNZAF -group, let N be a proper subgroup of G it is hypercentral-by-Cernikov. We have N is locally nilpotent and there exist normal hypercentral subgroup K of N such that N/K is Cernikov so it is finitely generated so N is hypercentral-by-Cernikov.

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