

Structure of Prime Near Rings with Generalized Derivations satisfying some Identities

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ABSTRACT

The purpose of the present paper is to obtain the commutativity of a prime near ring Ω with a generalized derivation τ associated with a nonzero derivation σ on Ω satisfying one of the following conditions: (i) $\tau([\lambda, \mu]) = \pm \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$, (ii) $\tau(\lambda \circ \mu) = \pm \lambda^\alpha [\lambda, \mu] \lambda^\beta$, (iii) $\tau([\lambda, \mu]) = \pm \mu^\alpha (\lambda \circ \mu) \mu^\beta$, (iv) $\tau(\lambda \circ \mu) = \pm \mu^\alpha [\lambda, \mu] \mu^\beta$ and (v) $\tau([\lambda, \mu]) = \pm \lambda^\alpha (\lambda \circ \mu) \mu^\beta$ for all $\lambda, \mu \in \Omega$ and $\alpha, \beta \in \mathbb{N}$, the set of nonnegative integers. Moreover, we give an example which shows the necessity of primeness hypothesis in the theorems.

Keywords: prime near ring, derivation, generalized derivation.

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1. INTRODUCTION

A right near ring Ω is a triplet $(\Omega, +, \cdot)$ where $+$ and \cdot are two binary operations such that (i) $(\Omega, +)$ is a group (not necessarily abelian) (ii) (Ω, \cdot) is a semigroup, and (iii) $(\lambda + \mu) \cdot \omega = \lambda \cdot \omega + \mu \cdot \omega$ for all $\lambda, \mu, \omega \in \Omega$. Analogously, if instead of (iii), Ω satisfies the left distributive law then, Ω is said to be a left near ring. A near ring Ω is said to be zero-symmetric if $\lambda 0 = 0$ for all $\lambda \in \Omega$ (right distributive yields that $0\lambda = 0$). Throughout the paper, Ω represents a zero-symmetric right near ring with multiplicative center $Z(\Omega)$. For any pair of elements $\lambda, \mu \in \Omega$, The symbols $[\lambda, \mu]$ and $(\lambda \circ \mu)$ denote the Lie product $\lambda\mu - \mu\lambda$ and the Jordan product $\lambda\mu + \mu\lambda$ respectively. A near ring Ω is known as prime if $\lambda\Omega\mu = 0$ for all $\lambda, \mu \in \Omega$ implies that $\lambda = 0$ or $\mu = 0$. A near ring Ω is known as 2-torsion free if $(\Omega, +)$ has no element of order 2. Throughout the paper \mathbb{N} represents the set of nonnegative integers. The notion of derivation in near rings was introduced by Bell and Mason [9]. An additive mapping $\sigma: \Omega \rightarrow \Omega$ is said to be a derivation on Ω if $\sigma(\lambda\mu) = \lambda\sigma(\mu) + \sigma(\lambda)\mu$ for all $\lambda, \mu \in \Omega$ or equivalently in [28], $\sigma(\lambda\mu) = \sigma(\lambda)\mu + \lambda\sigma(\mu)$ for all $\lambda, \mu \in \Omega$. Motivated by the definition of derivation in near rings, Gölbası [22] defined generalized derivation in near rings as follows: An additive mapping $\tau: \Omega \rightarrow \Omega$ is said to be a right (resp. left) generalized derivation associated with a derivation σ on Ω if $\tau(\lambda\mu) = \tau(\lambda)\mu + \lambda\sigma(\mu)$ (resp. $\tau(\lambda\mu) = \sigma(\lambda)\mu + \lambda\tau(\mu)$) for all $\lambda, \mu \in \Omega$. Moreover, τ is said to be a generalized derivation with associated with a derivation σ on Ω if it is both a right generalized derivation as well as a left generalized derivation on Ω . All derivations are generalized derivations. There has been a great deal of work by different authors with some suitable constraints on derivations and generalized derivations in prime and semiprime rings (see [1, 7, 8, 12, 13, 16, 17, 24]). A number of authors have been established some comparable results on near rings, (c.f. [2, 3, 9, 11, 14, 19, 21, 27, 28, 29]).

Daif and Bell [18] proved that if R is a prime ring and I is a nonzero ideal of a R . If σ is a derivation on R such that $\sigma([\lambda, \mu]) = \pm[\lambda, \mu]$ for all $\lambda, \mu \in I$, then R is commutative.

Further, Dhara [7] proved that if R is a semiprime ring with a generalized derivation τ associated with a derivation σ satisfying $\tau([\lambda, \mu]) = \pm[\lambda, \mu]$ or $\tau(\lambda \circ \mu) = \pm(\lambda \circ \mu)$ for all $\lambda, \mu \in I$, a nonzero ideal of R , then R must contain a nonzero central ideal, provided $\sigma(I) \neq \{0\}$. Moreover, he proved that in case R is a prime ring, R must be commutative, provided $\sigma \neq 0$. Motivated by above results, Boua and

Oukhtite[6] prove that a prime near ring Ω with a nonzero derivation σ is a commutative ring if one of the conditions holds: $\sigma([\lambda, \mu]) = \pm[\lambda, \mu]$ or $\sigma(\lambda \circ \mu) = \pm(\lambda \circ \mu)$ for all $\lambda, \mu \in \Omega$.

Recently, Shang [29] considered the more general situations (i) $\tau([\lambda, \mu]) = \pm \lambda^\alpha [\lambda, \mu] \lambda^\beta$, (ii) $\tau(\lambda \circ \mu) = \pm \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$, for all $\lambda, \mu \in \Omega$, $\alpha \geq 0$, $\beta \geq 0$ nonnegative integers and proved that the prime near ring Ω is a commutative ring if it satisfies one of the above conditions.

More recently, Miyan et. al [26] established the following results: A prime near ring Ω equipped with a generalized derivation τ associated with a nonzero derivation σ is a commutative ring if it satisfies any one of the following conditions: (i) $[\tau(\lambda), \mu] = \pm \mu^\alpha (\lambda \circ \mu) \mu^\beta$, (ii) $[\lambda, \tau(\mu)] = \pm \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$, (iii) $\tau(\lambda) \circ \mu = \pm \mu^\alpha [\lambda, \mu] \mu^\beta$, (iv) $\lambda \circ \tau(\mu) = \pm \lambda^\alpha [\lambda, \mu] \lambda^\beta$, (v) $\tau(\lambda) \circ \mu = \pm \mu^\alpha (\lambda \circ \mu) \mu^\beta$, (vi) $[\lambda, \tau(\mu)] = \pm \lambda^\alpha [\lambda, \mu] \lambda^\beta$, (vii) $[\tau(\lambda), \mu] = \pm \mu^\alpha [\lambda, \mu] \mu^\beta$ and (viii) $\lambda \circ \tau(\mu) = \pm \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$, for all $\lambda, \mu \in \Omega$ and for some nonnegative integers α and β .

In this line of investigation, we prove that, let Ω be a prime near ring. If Ω admits a generalized derivation τ associated with a nonzero derivation σ satisfying anyone of the following conditions: (i) $\tau([\lambda, \mu]) = \pm \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$, (ii) $\tau(\lambda \circ \mu) = \pm \lambda^\alpha [\lambda, \mu] \lambda^\beta$, (iii) $\tau([\lambda, \mu]) = \pm \mu^\alpha (\lambda \circ \mu) \mu^\beta$, (iv) $\tau(\lambda \circ \mu) = \pm \mu^\alpha [\lambda, \mu] \mu^\beta$ and (v) $\tau([\lambda, \mu]) = \pm \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$ for all $\lambda, \mu \in \Omega$ and $\alpha, \beta \in \mathbb{N}$, the set of nonnegative integers, then Ω is a commutative ring.

2. Preliminaries

For developing the proof of our theorems, we shall need the following lemmas. These results appear in the case of left near rings and so it is easy to observe that they also hold for right near rings as well.

Lemma 2.1. [10, Lemma 1.2] Let Ω be a prime near ring.

- (i) If $\lambda \in Z(\Omega)/\{0\}$, then λ is not a zero divisor.
- (ii) If $Z(\Omega)/\{0\}$ contains an element λ such that $\lambda + \lambda \in Z(\Omega)$, then $(\Omega, +)$ is abelian.
- (iii) If $\lambda \in Z(\Omega)/\{0\}$, and μ is an element of Ω such that $\mu \lambda \in Z(\Omega)$, then $\mu \in Z(\Omega)$.

Lemma 2.2. [10, Theorem 2.1] If a prime near ring Ω admits a nonzero derivation σ for which $\sigma(\Omega) \subseteq Z(\Omega)$, then Ω is a commutative ring.

Lemma 2.3. [26, Lemma 2.2] Let Ω be a near ring admitting a generalized derivation τ associated with a derivation σ . Then

- (i) $\lambda(\tau(\mu)\xi + \mu\sigma(\xi)) = \lambda\tau(\mu)\xi + \lambda\mu\sigma(\xi)$ for all $\lambda, \mu, \xi \in \Omega$,
- (ii) $\lambda(\mu\sigma(\xi) + \tau(\mu)\xi) = \lambda\mu\sigma(\xi) + \lambda\tau(\mu)\xi$ for all $\lambda, \mu, \xi \in \Omega$.

3. Main Results

Let \mathbb{N} be the set of nonnegative integers throughout in this section.

Theorem 3.1. Let Ω be a prime near ring. If there exist $\alpha, \beta \in \mathbb{N}$ such that Ω admits a right generalized derivation τ associated with a nonzero derivation σ satisfying either

- (i) $\tau([\lambda, \mu]) = \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$ or
 - (ii) $\tau([\lambda, \mu]) = -\lambda^\alpha (\lambda \circ \mu) \lambda^\beta$ for all $\lambda, \mu \in \Omega$,
- then Ω is a commutative ring.

Proof. (i) Suppose

$$(3.1) \quad \tau([\lambda, \mu]) = \lambda^\alpha (\lambda \circ \mu) \lambda^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Replacing μ by $\mu\lambda$ in (3.1), we find that

$$\tau([\lambda, \mu\lambda]) = \lambda^\alpha (\lambda \circ \mu\lambda) \lambda^\beta$$

$$\tau([\lambda, \mu]\lambda) = \lambda^\alpha (\lambda \circ \mu) \lambda^{\beta+1} \text{ which implies that}$$

$$\tau([\lambda, \mu])\lambda + [\lambda, \mu]\sigma(\lambda) = \lambda^\alpha (\lambda \circ \mu) \lambda^{(\beta+1)} \text{ for all } \lambda, \mu \in \Omega$$

Applying (3.1) in the above expression, we arrive at

$$(3.2) \quad [\lambda, \mu]\sigma(\lambda) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

Now replacing μ by $\mu\omega$, for any $\omega \in \Omega$ in (3.2) and using (3.2) again, we obtain $[\lambda, \mu]\omega\sigma(\lambda) = \{0\}$ for all $\lambda, \mu, \omega \in \Omega$. This means that

$$[\lambda, \mu]\Omega\sigma(\lambda) = \{0\} \text{ for all } \lambda, \mu \in \Omega.$$

In view of the primness of Ω , the last expression yields that for each $\lambda \in \Omega$,

$$(3.3) \quad \sigma(\lambda) = 0 \text{ or } \lambda \in Z(\Omega).$$

By a one-line calculation, we know that if $\lambda \in Z(\Omega)$, then $\sigma(\lambda) \in Z(\Omega)$. Hence (3.3) forces that for all $\lambda \in \Omega$, then $\sigma(\lambda) \in Z(\Omega)$, i.e., $\sigma(\Omega) \subseteq Z(\Omega)$. It then follows from Lemma 2.2 that Ω is a commutative ring.

(ii) By hypothesis,

$$(3.4) \quad \tau([\lambda, \mu]) = -\lambda^\alpha (\lambda\sigma\mu)\lambda^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Replacing μ by $\mu\lambda$ in (3.4), we find that

$$\tau([\lambda, \mu\lambda]) = -\lambda^\alpha (\lambda\sigma\mu\lambda)\lambda^\beta$$

$$\tau([\lambda, \mu]\lambda) = -\lambda^\alpha (\lambda\sigma\mu)\lambda^{\beta+1} \text{ which implies that}$$

$$\tau([\lambda, \mu])\lambda + [\lambda, \mu]\sigma(\lambda) = -\lambda^\alpha (\lambda\sigma\mu)\lambda^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Applying (3.4) in the above expression, we arrive at

$$(3.5) \quad [\lambda, \mu]\sigma(\lambda) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

Equation (3.5) is the same as equation (3.2). Now arguing in the similar manner, as we have done in case of (i), we obtain the result.

Theorem 3.2. Let Ω be a prime near ring. If there exist $\alpha, \beta \in \mathbb{N}$ such that Ω admits a right generalized derivation τ associated with a non zero derivation σ satisfying either

$$(i) \quad \tau(\lambda\sigma\mu) = \lambda^\alpha [\lambda, \mu]\lambda^\beta \text{ or}$$

$$(ii) \quad \tau(\lambda\sigma\mu) = -\lambda^\alpha [\lambda, \mu]\lambda^\beta \text{ for all } \lambda, \mu \in \Omega,$$

then Ω is a commutative ring.

Proof. (i) Suppose

$$(3.6) \quad \tau(\lambda\sigma\mu) = \lambda^\alpha [\lambda, \mu]\lambda^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Replacing μ by $\mu\lambda$ in (3.6), we find that

$$\tau((\lambda\sigma\mu)\lambda) = \lambda^\alpha [\lambda, \mu]\lambda^{\beta+1} \text{ which gives}$$

$$(3.7) \quad \tau((\lambda\sigma\mu)\lambda) + (\lambda\sigma\mu)\sigma(\lambda) = \lambda^\alpha [\lambda, \mu]\lambda^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.6) in (3.7), we obtain

$$(3.8) \quad (\lambda\sigma\mu)\sigma(\lambda) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

This implies that

$$\lambda\mu\sigma(\lambda) = -\mu\lambda\sigma(\lambda) \text{ for all } \lambda, \mu \in \Omega.$$

Substituting $\omega\mu$, where $\omega \in \Omega$ in place of μ in

the last expression and using it again, we get

$$\lambda\omega\mu\sigma(\lambda) = (-\omega)(-\lambda)\mu\sigma(\lambda)$$

which implies that

$$(3.9) \quad (\lambda\omega - (-\lambda)\omega)\mu\sigma(\lambda) = 0 \text{ for all } \lambda, \mu, \omega \in \Omega.$$

Replcing λ by $-\lambda$ in (3.9), we get

$$((-\lambda)\omega - (-\omega)(\lambda))\mu\sigma(-\lambda) = 0$$

$$((-\lambda)\omega + \omega\lambda)\mu\sigma(-\lambda) = 0 \text{ for all } \lambda, \mu, \omega \in \Omega.$$

Which implies that $[\omega, \lambda]\mu\sigma(-\lambda) = 0$ for all $\lambda, \mu, \omega \in \Omega$,

i.e., $[\lambda, \omega] \Omega \sigma(\lambda) = \{0\}$, for all $\lambda, \omega \in \Omega$.

By virtue of the primness of Ω , we have that for each $\lambda \in \Omega$,

$$(3.10) \quad \sigma(\lambda) = 0 \text{ or } \lambda \in Z(\Omega).$$

Since equation (3.10) is the same as equation (3.3), arguing as in the proof of Theorem 3.1 (i), we obtain that Ω is a commutative ring.

(ii) By hypothesis,

$$(3.11) \quad \tau(\lambda\sigma\mu) = -\lambda^\alpha [\lambda, \mu]\lambda^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Replacing μ by $\mu\lambda$ in (3.11), we find that

$$\tau((\lambda\sigma\mu)\lambda) = -\lambda^\alpha [\lambda, \mu]\lambda^{\beta+1} \text{ which gives}$$

$$(3.12) \quad \tau(\lambda\sigma\mu)\lambda + (\lambda\sigma\mu)\sigma(\lambda) = -\lambda^\alpha [\lambda, \mu]\lambda^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.11) in (3.12), we obtain

$$(3.13) \quad (\lambda\sigma\mu)\sigma(\lambda) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

Equation (3.13) is the same as equation (3.8). Now arguing in the similar manner, as we have done in case of (i), we obtain the result.

Theorem 3.3. Let Ω be a prime near ring. If there exist $\alpha, \beta \in \mathbb{N}$ such that Ω admits a right generalized derivation τ associated with a nonzero derivation σ satisfying either

$$(i) \quad \tau([\lambda, \mu]) = \mu^\alpha (\lambda\sigma\mu)\mu^\beta \text{ or}$$

$$(ii) \quad \tau([\lambda, \mu]) = -\mu^\alpha (\lambda\sigma\mu)\mu^\beta \text{ for all } \lambda, \mu \in \Omega,$$

then Ω is a commutative ring.

Proof. (i) Suppose

$$(3.14) \quad \tau([\lambda, \mu]) = \mu^\alpha (\lambda\sigma\mu)\mu^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Substituting $\lambda\mu$ for λ in (3.14), we get

$$\tau([\lambda, \mu]\mu) = \mu^\alpha(\lambda\sigma\mu)\mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Applying the definition of τ , we find that

$$(3.15) \quad \tau[\lambda, \mu]\mu + [\lambda, \mu]\sigma(\mu) = \mu^\alpha(\lambda\sigma\mu)\mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.14) in (3.15), we obtain

$$(3.16) \quad [\lambda, \mu]\sigma(\mu) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

This can be written as

$$\lambda\mu\sigma(\mu) = \mu\lambda\sigma(\mu) \text{ for all } \lambda, \mu \in \Omega.$$

Now replacing λ by $\omega\lambda$ in the above expression and using it again, we arrive at

$$[\mu, \omega]\Omega\sigma(\mu) = 0 \text{ for all } \mu, \omega \in \Omega.$$

By virtue of the primness of Ω , we have that for each $\mu \in \Omega$,

$$(3.17) \quad \sigma(\mu) = 0 \text{ or } \mu \in Z(\Omega).$$

Since equation (3.17) is the same as equation (3.3), arguing as in the proof of Theorem 3.1 (i), we obtain that Ω is a commutative ring.

(ii) By hypothesis,

$$(3.18) \quad \tau([\lambda, \mu]) = -\mu^\alpha(\lambda\sigma\mu)\mu^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Substituting $\lambda\mu$ for λ in (3.18), we get

$$\tau([\lambda, \mu]\mu) = -\mu^\alpha(\lambda\sigma\mu)\mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Applying the definition of τ , we find that

$$(3.19) \quad \tau([\lambda, \mu])\mu + [\lambda, \mu]\sigma(\mu) = -\mu^\alpha(\lambda\sigma\mu)\mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.18) in (3.19), we obtain

$$(3.20) \quad [\lambda, \mu]\sigma(\mu) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

Equation (3.20) is the same as equation (3.16). Now arguing in the similar manner, as we have done in case of (i), we obtain the result.

Theorem 3.4. Let Ω be a prime near ring. If there exist $\alpha, \beta \in \mathbb{N}$ such that Ω admits a right generalized derivation τ associated with a non zero derivation σ satisfying either

$$(i) \quad \tau(\lambda\sigma\mu) = \mu^\alpha[\lambda, \mu]\mu^\beta \text{ or}$$

$$(ii) \quad \tau(\lambda\sigma\mu) = -\mu^\alpha[\lambda, \mu]\mu^\beta \text{ for all } \lambda, \mu \in \Omega,$$

then Ω is a commutative ring.

Proof. (i) Suppose

$$(3.21) \quad \tau(\lambda\sigma\mu) = \mu^\alpha[\lambda, \mu]\mu^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Replacing λ by $\lambda\mu$ in (3.21), we find that

$$\tau((\lambda\sigma\mu)\mu) = \mu^\alpha[\lambda, \mu]\mu^{\beta+1} \text{ which gives}$$

$$(3.22) \quad \tau(\lambda\sigma\mu)\mu + (\lambda\sigma\mu)\sigma(\mu) = \mu^\alpha[\lambda, \mu]\mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.21) in (3.22), we obtain

$$(3.23) \quad (\lambda\sigma\mu)\sigma(\mu) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

Which implies that

$$(3.24) \quad \lambda\mu\sigma(\mu) = -\mu\lambda\sigma(\mu) \text{ for all } \lambda, \mu \in \Omega.$$

Replacing λ by $\omega\lambda$ in (3.24) and using (3.24), we get

$$(\omega(-\mu) + \mu\omega)\lambda\sigma(\mu) = 0, \text{ for all } \lambda, \mu, \omega \in \Omega.$$

Substituting $-\mu$ for μ in the last expression, we obtain

$$[\omega, \mu]\lambda\sigma(-\mu) = 0, \text{ for all } \lambda, \mu, \omega \in \Omega,$$

$$\text{i.e., } [\mu, \omega]\Omega\sigma(\mu) = \{0\}, \text{ for all } \mu, \omega \in \Omega.$$

By virtue of the primness of Ω , we have that for each $\mu \in \Omega$,

$$(3.25) \quad \sigma(\mu) = 0 \text{ or } \mu \in Z(\Omega).$$

Since equation (3.25) is the same as equation (3.3), arguing as in the proof of Theorem 3.1 (i), we obtain that Ω is a commutative ring.

(ii) By hypothesis,

$$(3.26) \quad \tau(\lambda\sigma\mu) = -\mu^\alpha[\lambda, \mu]\mu^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Replacing λ by $\lambda\mu$ in (3.26), we find that

$$\tau((\lambda\sigma\mu)\mu) = -\mu^\alpha[\lambda, \mu]\mu^{\beta+1} \text{ which gives}$$

$$(3.27) \quad \tau(\lambda\sigma\mu)\mu + (\lambda\sigma\mu)\sigma(\mu) = -\mu^\alpha[\lambda, \mu]\mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.26) in (3.27), we obtain

$$(3.28) \quad (\lambda\sigma\mu)\sigma(\mu) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

Equation (3.28) is the same as equation (3.23). Now arguing in the similar manner, as we have done in case of (i), we obtain the result.

Theorem 3.5. Let Ω be a prime near ring. If there exist $\alpha, \beta \in \mathbb{N}$ such that Ω admits a right generalized derivation τ associated with a nonzero derivation σ satisfying either

- (i) $\tau([\lambda, \mu]) = \lambda^\alpha (\lambda \circ \mu) \mu^\beta$ or
(ii) $\tau([\lambda, \mu]) = -\lambda^\alpha (\lambda \circ \mu) \mu^\beta$ for all $\lambda, \mu \in \Omega$,
then Ω is a commutative ring.

Proof. (i) Suppose

$$(3.29) \quad \tau([\lambda, \mu]) = \lambda^\alpha (\lambda \circ \mu) \mu^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Substituting $\lambda\mu$ for λ in (3.29), then we get

$$\tau([\lambda, \mu]\mu) = \lambda^\alpha (\lambda \circ \mu) \mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega,$$

which implies that

$$(3.30) \quad \tau[\lambda, \mu]\mu + [\lambda, \mu]\sigma(\mu) = \lambda^\alpha (\lambda \circ \mu) \mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.29) in (3.30), we arrive at

$$(3.31) \quad [\lambda, \mu]\sigma(\mu) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

This can be written as

$$\lambda\mu\sigma(\mu) = \mu\lambda\sigma(\mu) \text{ for all } \lambda, \mu \in \Omega.$$

Now replacing λ by $\omega\lambda$ in the above expression and using it again, we arrive at

$$[\mu, \omega] \Omega \sigma(\mu) = 0 \text{ for all } \mu, \omega \in \Omega.$$

In view of the primness of Ω , the last expression yields that for each $\mu \in \Omega$,

$$(3.32) \quad \sigma(\mu) = 0 \text{ or } \mu \in Z(\Omega).$$

Since equation (3.32) is the same as equation (3.3), arguing as in the proof of Theorem 3.1 (i), we obtain that Ω is a commutative ring.

(ii) By hypothesis,

$$(3.33) \quad \tau([\lambda, \mu]) = -\lambda^\alpha (\lambda \circ \mu) \mu^\beta \text{ for all } \lambda, \mu \in \Omega.$$

Substituting $\lambda\mu$ for λ in (3.33), then we find that

$$\tau([\lambda, \mu]\mu) = -\lambda^\alpha (\lambda \circ \mu) \mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

which implies that

$$(3.34) \quad \tau[\lambda, \mu]\mu + [\lambda, \mu]\sigma(\mu) = -\lambda^\alpha (\lambda \circ \mu) \mu^{\beta+1} \text{ for all } \lambda, \mu \in \Omega.$$

Using (3.33) in (3.34), we arrive at

$$(3.35) \quad [\lambda, \mu]\sigma(\mu) = 0 \text{ for all } \lambda, \mu \in \Omega.$$

Equation (3.35) is the same as equation (3.31). Now arguing in the similar manner, as we have done in case of (i), we obtain the result.

The following example shows that the primness hypothesis in Theorems 3.1, 3.2, 3.3, 3.4 and 3.5 cannot be omitted.

Example 3.1. Suppose that Y is a zero-symmetric right near ring. Let us consider

$$\Omega = \left\{ \begin{pmatrix} 0 & \lambda & \mu \\ 0 & 0 & 0 \\ 0 & \xi & 0 \end{pmatrix} \mid 0, \lambda, \mu, \xi \in Y \right\}.$$

It is easy to verify that Ω is a non-prime zero-symmetric right near ring with respect to matrix addition and matrix multiplication.

Define mappings $\tau, \sigma: \Omega \rightarrow \Omega$ by

$$\tau \begin{pmatrix} 0 & \lambda & \mu \\ 0 & 0 & 0 \\ 0 & \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \sigma \begin{pmatrix} 0 & \lambda & \mu \\ 0 & 0 & 0 \\ 0 & \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then τ is a nonzero generalized derivation associated with a nonzero derivation σ on Ω satisfying

- (i) $\tau([\lambda, \mu]) = \pm \lambda^\alpha (\lambda \circ \mu) \lambda^\beta$, (ii) $\tau(\lambda \circ \mu) = \pm \lambda^\alpha [\lambda, \mu] \lambda^\beta$, (iii) $\tau([\lambda, \mu]) = \pm \mu^\alpha (\lambda \circ \mu) \mu^\beta$,
(iv) $\tau(\lambda \circ \mu) = \pm \mu^\alpha [\lambda, \mu] \mu^\beta$ and (v) $\tau([\lambda, \mu]) = \pm \lambda^\alpha (\lambda \circ \mu) \mu^\beta$ for all $\lambda, \mu \in \Omega$ and $\alpha, \beta \in \mathbb{N}$, the set of nonnegative integers. However, Ω is not commutative.

4. Concluding Remarks

In this paper, the structure of near rings involving generalized derivations satisfying some identities has been studied. We proved commutativity of prime near rings with identities on generalized derivations. This work can be further studied by considering generalized semiderivations and multiplicative

generalized derivations on prime near rings and semiprime near rings along with examples that illustrates the necessity of the assumptions used which is left for future work.

CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest.

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