Twain Secure Perfect Dominating Sets and Twain Secure Perfect Domination Polynomials of Stars

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ABSTRACT
Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of G, if for every vertex in $V\backslash S$ is adjacent to atleast one vertex in S. A subset S of V is called a twain secure perfect dominating set of G (TSPD-set) if for every vertex $v \in V\backslash S$ is adjacent to exactly on evertex $u \in S$ and $(S\backslash\{u\}) \cup \{v\}$is a dominating set of G. The minimum cardinality of a twain secure perfect dominating set of G is called the twain secure perfect domination number of G and is denoted by $\gamma_{tsp}(G)$.Let $D_{tsp}(S_n, i)$ denote the family of all twain secure perfect dominating sets of S_n with cardinality i, for $\gamma_{tsp}(S_n) \leq i \leq n$. Let $d_{tsp}(S_n, i) = |D_{tsp}(S_n, i)|$. In this article, we derive a recursive formula for $d_{tsp}(S_n, i)$and construct $D_{tsp}(S_n, i)$. We consider the polynomial $D_{tsp}(S_n, x) = \sum_{i=\gamma_{tsp}(S_n)}^{n} d_{tsp}(S_n, i)\, x^i$, which we refer to as the twain secure perfect domination polynomial of stars using this recursive formula. In this research, we use a recursive technique to generate all twain secure perfect dominating sets of stars and twain secure perfect domination polynomials of stars.

Keywords: Star, twain secure perfect dominating set, twain secure perfect domination number, twain secure perfect domination polynomial.

1. INTRODUCTION

A finite undirected connected graph without loops or multiple edges is referred to as a graph $G = (V, E)$. G's order and size are shown by the numbers n and m, respectively. For fundamental terms and definitions, see [2]. There are two vertices. If uv is one of G's edges, then u and v are considered adjacent. A vertexv in a graph G has an open neighborhood defined as the set $N_G(V) = \{u \in V(G) : uv \in E(G)\}$, and a closed neighborhood defined as $N_G[V] = N_G(V) \cup \{v\}$. A subset $S \subseteq V(G)$ is called a dominating set if every vertex $v \in V(G)\backslash S$ is adjacent to a vertex $u \in S$.The domination number, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G.A minimum dominating set of a graph G is hence often called as a γ−set of G [1]. A dominating set S is called a secure dominating set if foreach $v \in V\backslash S$ thereexists $u \in N(v) \cap S$ suchthat $(S\backslash\{u\}) \cup \{v\}$ is a dominating set. The secure domination number $\gamma_s(G)$ is the minimum cardinality of a secure dominating set of G. Cockayne et al introduce the concept of secure domination of graphs[3]. A dominating set S is called a perfect dominating set if every vertex in $V\backslash S$ isadjacenttoexactlyonevertexinS.Theperfect domination number$\gamma_p(G)$is the minimum cardinality of a perfect dominating set of G.The concept of perfect domination of graphs is introduced by Weichsel [4]. In this sequel we introduce the concept of twain secure perfect domination of stars in this work. A dominating set S is called a twain secure perfect dominating set of G (TSPD-set) if for every vertex $v \in V\backslash S$ isadjacenttoexactlyonevertex $u \in S$ and $(S\backslash\{u\}) \cup \{v\}$isadominating setof G.Thetwain secure perfect domination numberof G, representedas $\gamma_{tsp}(G)$, isthelowestcardinalityofatwainsecureperfectdominatingsetof G.Considerthe star graph S_n, which has n vertices. The twain secure perfect domination number of S_n is denoted by$\gamma_{tsp}(S_n)$. Denote by $D_{tsp}(S_n, i)$ the family of all twain secure perfect dominating sets of S_n with cardinality i, where $\gamma_{tsp}(S_n) \leq i \leq n$. Let $d_{tsp}(S_n, i) = |D_{tsp}(S_n, i)|$. Following that $D_{tsp}(S_n, x) = \sum_{i=\gamma_{tsp}(S_n)}^{n} d_{tsp}(S_n, i)\, x^i$is the twain secure perfect domination polynomial of S_n. The families of the twain secure perfect dominating sets of stars are built using a recursive technique in the following section. We refer to the set $\{1,2,\ldots,n\}$ in this article as $[n]$.

2. Twain Secure Perfect Dominating Sets of Stars

The family of twain secure perfect dominating sets of S_n with cardinality i is denoted by $D_{tsp}(S_n, i)$. Also, twain secure perfect dominating sets of stars will be examined. The following lemmas are necessary to support the primary findings of this article.

Lemma 2.1.

For every, $n \in N, \gamma_{tsp}(S_n) = \begin{cases} 1 & \text{for } n = 1 \\ n-1 & \text{for } n > 1 \end{cases}$

Lemma 2.2.

Let S_n be a star with n vertices and $D_{tsp}(S_n, i)$ bethefamilyoftwain secure perfect dominating sets with cardinality i.Then $D_{tsp}(S_n, i) \neq \emptyset$ if and only if $n-1 \leq i \leq n$.Also $D_{tsp}(S_n, i) = \emptyset$ if and only if $i < n-1$ or $i > n$.

Proof:

By the definition of twain secure perfect domination number, the cardinality of the minimum twain secure perfect dominating set of star is $n-1$.Therefore $D_{tsp}(S_n, i) \neq \emptyset$ if and only if $n-1 \leq i \leq n$.Suppose, $i < n-1$, then there is no twain secure perfect dominating sets in $D_{tsp}(S_n, i)$. Therefore $D_{tsp}(S_n, i) = \emptyset$ if and only if $i < n-1$.Clearly if $i > n$, then $D_{tsp}(S_n, i) = \emptyset$.

Lemma 2.3.

In the event that Y is inside $D_{tsp}(S_{n-1}, i-1)$,then there is a $\{x\} \in [n]$ such that $Y \cup \{x\} \in D_{tsp}(S_n, i)$.

Proof:

The twain secure perfect dominating set of G are indicated by Y. Y has at least one vertex labeled $n-1$ or $n-2$, Since $Y \in D_{tsp}(S_{n-1}, i-1)$. In the event where $n-1 \in Y$, $Y \cup \{x\} \in Z_1$(say),a twain secure perfect dominating set of S_n.Thus $Z_1 \in D_{tsp}(S_n, i)$. In the event where $n-2 \in Y$, $Y \cup \{x\} \in Z_2$(say),a twain secure perfect dominating set of S_n. Thus $Z_2 \in D_{tsp}(S_n, i)$. $Y \cup \{x\}$ is a twain secure perfect dominating set of S_n, in each scenario. Consequently, $Y \cup \{x\} \in D_{tsp}(S_n, i)$.

Lemma 2.4.

For every $n \geq 4$,

(i) If $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$,$D_{tsp}(S_{n-2}, i-1) \neq \emptyset$,then $D_{tsp}(S_n, i) \neq \emptyset$.

(ii) If $D_{tsp}(S_{n-1}, i-1) = \emptyset$,$D_{tsp}(S_{n-2}, i-1) = \emptyset$,then $D_{tsp}(S_n, i) = \emptyset$.

Proof:

(i) Suppose $D_{tsp}(S_n, i) = \emptyset$, then by lemma 2.2, $i < n-1$ or $i > n$. If $i > n$, then $i-1 > n-1$, which gives $D_{tsp}(S_{n-1}, i-1) = \emptyset$. Also, $i-1 > n-1 > n-2$, which implies $i-1 > n-2$, which gives $D_{tsp}(S_{n-2}, i-1) = \emptyset$. This is a contradiction to $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(S_{n-2}, i-1) \neq \emptyset$.

Hence $D_{tsp}(S_n, i) \neq \emptyset$.

(ii) Suppose $D_{tsp}(S_n, i) \neq \emptyset$, then by lemma 2.2, $n-1 \leq i \leq n$. Which implies $n-2 \leq i-1 \leq n-1$. This gives $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$. Also $n-3 \leq i-2 \leq n-2$.Which implies $n-3 \leq i-2 < i-1 \leq n-2$. Which gives $n-3 \leq i-1 \leq n-2$.Therefore, $D_{tsp}(S_{n-2}, i-1) \neq \emptyset$. Which is a contradiction to $D_{tsp}(S_{n-1}, i-1) = \emptyset$ and $D_{tsp}(S_{n-2}, i-1) = \emptyset$.

Hence $D_{tsp}(S_n, i) = \emptyset$.

Lemma 2.5.

In case $D_{tsp}(S_n, i) \neq \emptyset$,

(i) $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$and $D_{tsp}(S_{n-2}, i-1) = \emptyset$ if and only if $i = n$.

(ii) $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$and $D_{tsp}(S_{n-2}, i-1) \neq \emptyset$ if and only if $i = n-1$.

Proof:

(i) Assume that $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$and $D_{tsp}(S_{n-2}, i-1) = \emptyset$. Since $D_{tsp}(S_{n-2}, i-1) = \emptyset$, by lemma 2.2, $i-1 > n-2$ or $i-1 < n-3$. Since $i-1 > n-2$,

$i \geq n$ (1)

Since $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$, by lemma 2.2, we have $n-2 \leq i-1 \leq n-1$. Which gives

$i \leq n$ (2)

From (1) and (2), $i = n$.

Conversely, assume that, $i = n$. Which gives $i-1 = n-1 > n-2$.Which implies $i-1 > n-2$, bylemma2.2, $D_{tsp}(S_{n-2}, i-1) = \emptyset$.Since $i-1 = n-1$, by lemma 2.2,$D_{tsp}(S_{n-1}, i-1) \neq \emptyset$.

(ii) Assume that $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$and $D_{tsp}(S_{n-2}, i-1) \neq \emptyset$. By lemma 2.2, we have $n-2 \leq i-1 \leq n-1$ and $n-3 \leq i-1 \leq n-2$. Which gives $n-2 \leq i-1 \leq n-2$. This impliesi $= n-1$.
Conversely, assume that $i = n-1$, which implies $i-1 = n-2$. Which gives $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$and $D_{tsp}(S_{n-2}, i-1) \neq \emptyset$.

Theorem 2.6

For every $n \geq 3$ and $i \geq n-1$

(i) If $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$and $D_{tsp}(S_{n-2}, i-1) = \emptyset$, then $D_{tsp}(S_n, i) = \{[n]\}$

(ii) If $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(S_{n-2}, i-1) \neq \emptyset$ then $D_{tsp}(S_n, i) = \{\{X \cup \{n\}\} \cup \{Y \cup \{n-1\}\}\}$, where $X \in D_{tsp}(S_{n-1}, i-1)$ and $Y \in D_{tsp}(S_{n-2}, i-1)$.

Proof:

(i) We have $D_{tsp}(S_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(S_{n-2}, i-1) = \emptyset$, by lemma 2.5(i), $i = n$. Therefore $D_{tsp}(S_n, i) = \{[n]\}$.

(ii) Constructionof $D_{tsp}(S_n, i)$ followsfrom $D_{tsp}(S_{n-1}, i-1)$and $D_{tsp}(S_{n-2}, i-1)$. Let X be the twain secure perfect dominating set of $S_{n-1}$with cardinality $i-1$.The elements of $D_{tsp}(S_{n-1}, i-1)$ belongs to $D_{tsp}(S_n, i)$ by adjoining n. Let Y be the twain secure perfect dominating set of S_{n-2} with cardinality $i-1$. The elements of $D_{tsp}(S_{n-2}, i-1)$ belongs to $D_{tsp}(S_n, i)$ by adjoining $n-1$. Thus

$$\{\{X \cup \{n\}\} \cup \{Y \cup \{n-1\}\}\} \subseteq D_{tsp}(S_n, i) \qquad (3)$$

Where $X \in D_{tsp}(S_{n-1}, i-1)$ and $Y \in D_{tsp}(S_{n-2}, i-1)$.

Conversely, Suppose $Z \in D_{tsp}(S_n, i)$. Here all the elements of $D_{tsp}(S_n, i)$ ends with $n-1$ or n.Suppose $n-1 \in Z$, then $Z = Y \cup \{n-1\}$, for some $Y \in D_{tsp}(S_{n-2}, i-1)$. Suppose $n \in Z$, then $Z = X \cup \{n\}$, for some $X \in D_{tsp}(S_{n-1}, i-1)$. Thus

$$D_{tsp}(S_n, i) \subseteq \{\{X \cup \{n\}\} \cup \{Y \cup \{n-1\}\}\} \qquad (4)$$

Where $X \in D_{tsp}(S_{n-1}, i-1)$ and $Y \in D_{tsp}(S_{n-2}, i-1)$.

From (3) and (4),

$D_{tsp}(S_n, i) = \{\{X \cup \{n\}\} \cup \{Y \cup \{n-1\}\}\}$, where $X \in D_{tsp}(S_{n-1}, i-1)$and $Y \in D_{tsp}(S_{n-2}, i-1)$.

Theorem2.7.

If$D_{tsp}(S_n, i)$isafamilyoftwainsecureperfectdominatingsetswith cardinality i,thenforevery $n \geq 4$, $|D_{tsp}(S_n, i)| = |D_{tsp}(S_{n-1}, i-1)| + |D_{tsp}(S_{n-2}, i-1)|$.

Proof:

It follows from Theorem 2.6.

3. Twain Secure Perfect Domination Polynomials of Stars

Definition 3.1.

The family of all twain secure perfect dominating sets of S_n with cardinality i is denoted as $D_{tsp}(S_n, i)$. Let $d_{tsp}(S_n, i)$ should equal $|D_{tsp}(S_n, i)|$. Then the twain secure perfect domination polynomial of S_n is defined as $D_{tsp}(S_n, x) = \sum_{i=\gamma_{tsp}(S_n)}^{n} d_{tsp}(S_n, i)\, x^i$, where $\gamma_{tsp}(S_n)$is the twain secure perfect domination number of S_n.

Theorem 3.2.

For every $n \geq 4$, $D_{tsp}(S_n, x) = x[D_{tsp}(S_{n-1}, x)] + x^{n-1}$ with initial value$D_{tsp}(S_3, x) = 2x^2 + x^3$.

Table 1. $d_{tsp}(S_n, i)$, the number of twain secure perfect dominating sets of S_n with cardinality i.

n\i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	1														
2	2	1													
3	0	2	1												
4	0	0	3	1											
5	0	0	0	4	1										
6	0	0	0	0	5	1									
7	0	0	0	0	0	6	1								
8	0	0	0	0	0	0	7	1							

9	0	0	0	0	0	0	0	8	1						
10	0	0	0	0	0	0	0	0	9	1					
11	0	0	0	0	0	0	0	0	0	10	1				
12	0	0	0	0	0	0	0	0	0	0	11	1			
13	0	0	0	0	0	0	0	0	0	0	0	12	1		
14	0	0	0	0	0	0	0	0	0	0	0	0	13	1	
15	0	0	0	0	0	0	0	0	0	0	0	0	0	14	1

Theorem 3.3.
For the coefficients of $D_{tsp}(S_n, x)$, the following characteristics are true.

(i) $d_{tsp}(S_n, n) = 1$, for every $n \in N$.
(ii) $d_{tsp}(S_{n+1}, n) = n$, for every $n \geq 2$.

Proof:

(i) Since $D_{tsp}(S_n, n) = \{[n]\}$, we have the result.
(ii) To prove $d_{tsp}(S_{n+1}, n) = n$, for every $n \geq 2$. We apply induction on n.
When n = 2.
L.H.S: $d_{tsp}(S_3, 2) = 2$ (From table)
R.H.S: $n = 2$.
L.H.S = R.H.S.
The result is true for $n = 2$.Now assume that the result is true for all natural numbers less than n and we prove that it for n.By theorem 2.7 and the induction hypothesis, $d_{tsp}(S_{n+1}, n) = d_{tsp}(S_n, n-1) + d_{tsp}(S_{n-1}, n-1) = n - 1 + 1 = n$.
Thus $d_{tsp}(S_{n+1}, n) = n$, for every $n \geq 2$.

CONCLUSION

Twain secure perfect dominating sets and polynomials of star graphs are examined and certain properties obtained in this study. Thus, the study can be applied to any S_n.

REFERENCES

[1] Alikhani S andPeng Y H,Dominating Sets and Domination Polynomials of Paths, International Journal of Mathematics and Mathematical Sciences, Vol.2009, article ID 542040.
[2] Buckley F and Harary F, Distance in Graphs, Addition-Wesley, Reduced City, CA.1990.
[3] CockayneEJ, GroblerPJP, GrundlinghWR,Mungannga J, and Van Vuuren J H, Protection of a Graph, Util. Math. 67(2005), 19 – 32.
[4] Lal Gipson K and Vijayan A, Dominating Sets and Domination Polynomials in Square of Paths, Open Journal in Discrete Mathematics, 3 (2013), 60 – 69.
[5] Lal Gipson K and Vijayan A, Dominating Sets and Domination Polynomials in Square of Cycles, IOSR Journal of Mathematics (IOSR-JM), 3(2012), 04 – 14.
[6] Lal Gipson K and Subha T,Secure Domination Polynomials of Paths, International Journal of Scientific & Technology Research, 9, (2020), 6517 – 6521.
[7] Lal Gipson, Subha K and Subha T, Secure Domination Polynomials of Wheels", IOSR Journal of Mathematics (IOSR-JM), 18(2022),21 – 26.
[8] Weichsel P W, Dominating Sets in n-cubes, J. Graph Theory 18(5) (1994), 479 – 488.