Norm retrieval by vectors and projections

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Abstract

Norm retrieval was introduced for Hilbert space frames for the first time by Bahmanpour et. al. in the year 2015. In order for a subspace as well as its orthogonal complement to do norm retrieval, it was proved by Bahmanpour et. al. that norm retrieval is a necessary requirement. Basically, norm retrieval refers to the process of reconstructing the signal's norm from the intensity measurements. We give a few characterizations for norm retrieval by vectors and subspaces under the action of bounded linear operators.

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1 Introduction

For any orthonormal basis $\{u_1, u_2, u_3, \dots\}$, a vector $v \in \mathcal{H}$ can be explicitly represented as $v = \sum_i \langle v, u_i \rangle u_i$. Thus orthonormal bases help to reconstruct a vector. In a similar manner, frames, having more flexible structure, also help to reconstruct a vector in a stable way. Duffin and Schaeffer [9] for the first time introduced frames for Hilbert spaces in the year 1952. Frames provides us with a reconstruction formula for lost signals. Daubechies et. al. popularized frames through their work in [7]. Over the last few decades, frame theory has become a prestigious area of research. Researchers worked various generalizations of frames, for instance, K-frame [13], fusion frame [5], wavelet frame [6] and many more. Basically, frames help us to recover and reconstruct the signal, that was lost or distorted, in a stable manner.

Reconstruction of signal is one of the important and significant problems in engineering especially in signal processing. Here a signal can be thought as a vector. This process of regaining the original signal becomes challenging when

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there is a partial loss of information. Sometimes it happens that we only have the intensity measurements or the phaseless measurements of the lost signal. In such case, phase retrieval sequences help to reconstruct or regain the signal from its intensity measurements or phaseless measurements. Phase retrieval was introduced by Balan et al. [2] for Hilbert space frames in the year 2006. Since then mathematicians have started to work in this area. Phase retrieval is one of the challenging engineering problems. It includes a broad range of applications in many fields, such as speech recognition technology, X-ray crystallography, etc.

Norm retrieval means regaining or reconstructing the lost signal's norm from its intensity measurements or phaseless measurements. Norm retrieval for Hilbert spaces was discussed for the first time by Bahmanpour et. al. [1] in the year 2015. It was proved in [1] that norm retrieval is the necessary requirement for a subspace so that the subspace along with its orthogonal complement do phase retrieval. We note that if a sequence does phase retrieval then it will always do norm retrieval. In the last few years, it is observed that researchers have worked on norm retrieval frames [10], norm retrieval subspaces in finite dimensional Hilbert spaces [4]; and in infinite dimensional Hilbert spaces [15]. Apart from these, pertubation of norm retrieval frames is discussed in [11]. Being highly influenced as well as encouraged by the above mentioned work we explore norm retrieval sequences for vectors under the action of bounded linear operators, T. We also provide a method for construction of norm retrieval subspaces.

We stick to the following notations throughout paper. $\mathcal{H}, \mathcal{K}$ represents separable Hilbert spaces, $\mathcal{B}(\mathcal{H})$ represents the space of linear and bounded operators from $\mathcal{H}$ to $\mathcal{H}$. I, Λ, Λ_i represents a countable index set.

The paper is organised as follows. In Section 2, we give some preliminary background on norm retrieval sequences for finite and infinite dimensional spaces and we highlight some of the important results in these fields. We provide characterizations of norm retrieval sequences and and norm retrieval subspaces in Section 3.

2 Preliminaries

We recall the fundamental definitions and basic results that will be helpful for the paper. Frames are mathematical tools that are used to reconstruct signals.

Definition 2.1. [6] Consider a sequence, say $\varphi = \{\varphi_i\}_{i\in I}$, in $\mathcal{H}$. If for all $x \in \mathcal{H}$, there exist constants $0 < A_1 \leq A_2 < \infty$ such that φ satisfies

$$A_1\|x\|^2 \leq \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \leq A_2\|x\|^2.$$

Then φ is called a frame for $\mathcal{H}$. Here the constants A_1 is known as the lower frame bound, A_2 is known as the upper frame bound. The frame φ is called Parseval frame if $A_1 = A_2 = 1$.

For example, consider an orthonormal basis, say, $\{e_n\}$ for $\mathcal{H}$, then the sequence $\{e_1, e_1, e_2, e_3, e_4, \dots\}$ is a frame for $\mathcal{H}$. The associated frame bounds are $A_1 = 1$, $A_2 = 2$.

The frame operator, S, is a mapping $S : \mathcal{H} \to \mathcal{H}$ defined as

$$Sx = \sum_{i \in I} \langle x, \varphi_i \rangle \varphi_i, \quad \forall x \in \mathcal{H}.$$

The reconstruction formula given by frame operator and frame elements is as follows:

$$x = \sum_{i \in I} \langle x, S^{-1}\varphi_i \rangle \varphi_i = \sum_{i \in I} \langle x, \varphi_i \rangle S^{-1}\varphi_i, \quad \forall x \in \mathcal{H}.$$

We note that this representation is not unique, owing to the fact that frame elements are not necessarily linearly independent. Frames are one of the essential tools for restoring a signal. There are many different types of frames. One special type of frame is the scalable frame [14]. A scalable frame is a frame, φ, for $\mathcal{H}$ such that there exists scalars, say $c_1, c_2, c_3, \dots$ with $c_i \geq 0$ for which $\{c_i \varphi_i\}_{i \in I}$ is a Parseval frame for $\mathcal{H}$. We refer the readers [6] for more information in frame theory.

Definition 2.2. [2] Consider a sequence $\varphi = \{\varphi_i\}_{i \in I} \in \mathcal{H}$. We say φ performs phase retrieval for $\mathcal{H}$, if for $x, y \in \mathcal{H}$, φ satisfies

$$|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle|, \quad \forall\ i \in I,$$

then $x = cy$ and c satisfies $|c| = 1$.

The sequence of vectors $\{e_i + e_j\}_{i<j}$, where e_i's are standard orthonormal basis, performs phase retrieval for ℓ_2. If a sequence does phase retrieval in a finite dimension space then it is also a frame, but it may not necessarily be a frame in an infinite dimension space.

In [3], Cahill et. al. thoroughly discussed phase retrieval by subspaces or projections.

Definition 2.3. [3] Suppose $W = \{W_i\}_{i \in I} \subset \mathcal{H}$ is a collection of closed subspaces with corresponding projections $P = \{P_i\}_{i \in I}$. Then W or P does phase retrieval whenever $x, y \in \mathcal{H}$, P satisfies

$$\|P_i x\| = \|P_i y\| \qquad \forall\ i \in I,$$

we have $x = cy$ and c satisfies $|c| = 1$.

Bahmanpour et. al. [1] introduced norm retrieval for frames in Hilbert spaces in the year 2015. In his attempt to pass the phase retrieval condition by subspaces to its orthogonal complements, Bahmanpour proved in [1] that the property of norm retrieval is a necessary requirement. A norm retrieval sequence helps to reconstruct partially lost signal's norm.

Definition 2.4. [1] A sequence of vectors $\varphi = \{\varphi_i\}_{i\in I}$ in $\mathcal{H}$ does norm retrieval if for $x, y \in \mathcal{H}$, φ satisfies

$$|\langle x, \varphi_i\rangle| = |\langle y, \varphi_i\rangle| \quad \forall\ i \in I,$$

then $\|x\| = \|y\|$.

It is obvious for scalable frames, parseval frames, tight frames to do norm retrieval. An orthonormal basis will always do norm retrieval for the corresponding space. It is to be noted that if a sequence performs phase retrieval for $\mathcal{H}$ then the sequence also performs norm retrieval for $\mathcal{H}$, however the converse is not true. For example, orthonormal bases do norm retrieval but not phase retrieval.

Norm retrieval by projections is defined as follows.

Definition 2.5. [1] Consider a family of subspaces, say $\{W_i\}_{i\in I}$, in an infinite dimensional Hilbert space $\mathcal{H}$ and define the orthogonal projections, say $\{P_i\}_{i\in I}$, onto $\{W_i\}_{i\in I}$. Then $\{W_i\}_{i\in I}$ (or $\{P_i\}_{i\in I}$) performs norm retrieval for $\mathcal{H}$ if for $x, y \in \mathcal{H}$, $\{P_i\}_{i\in I}$ satisfies $\|P_i x\| = \|P_i y\|, \ \forall\ i \in I$, we have $\|x\| = \|y\|$.

Norm retrieval can be thought as having an advantage of one free measurement when one tries to do phase retrieval.

The next proposition gives us a method to construct norm retrieval subspaces with the help of dimension of the subspaces.

Proposition 2.6. *[4] If $\{W_i\}_{i=1}^m$ are subspaces in $\mathbb{R}^n$ such that they do norm retrieval then $\sum\limits_{i=1}^m \dim W_i \geq n$. Moreover, if $\exists\ k_1, k_2, \ldots, k_m \in \mathbb{N}$ with $k_i \leq n$ such that for some $L \in \mathbb{N}$ $\sum\limits_{i=1}^m k_i = Ln$ then there exist subspaces $\{W_i\}_{i=1}^m$ that perform norm retrieval in $\mathbb{R}^n$ where $\dim W_i = k_i$ for $1 \leq i \leq m$.*

The above result can easily be generalized as follows.

Theorem 2.7. *Suppose $\{k_i\}_{i=1}^m$ are natural numbers such that $k_i \leq n$ and $\sum\limits_{i=1}^m k_i \geq n$. If for some $l \in \mathbb{N}$ with $1 \leq l \leq m$, $\sum\limits_{i=1}^l k_i$ is a multiple of n, then there exist subspaces $\{W_i\}_{i=1}^m$ in $\mathbb{R}^n$ satisfying $\dim W_i = k_i$ such that $\{W_i\}_{i=1}^m$ performs norm retrieval.*

We recall the following properties of projection operators.

Lemma 2.8. *[12] Consider any two Hilbert spaces, say, $\mathcal{H}_1$, $\mathcal{H}_2$ and $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Consider a closed subspace, say, W_1, of $\mathcal{H}_1$ and another closed subspace, say, W_2, of $\mathcal{H}_2$. Then the following statements are true.*

(i) $P_{W_1} T^* P_{W_2} = P_{W_1} T^*$ *if and only if* $TW_1 \subset W_2$.

(ii) $P_{W_1} T^* P_{\overline{TW_1}} = P_{W_1} T^*$

3 Main Results

We begin this section by studying norm retrieval sequences under the action of bounded linear operators.

In [1], it was shown that orthogonal projections preserve the norm retrieval property. However in [4], it is shown that the norm retrieval property is not preserved by invertible operators. For instance, $\varphi = \{(1,0),(0,1)\}$ does norm retrieval for $\mathbb{R}^2$; consider an invertible on $\mathbb{R}^2$ defined by $T(x_1,x_2) = (x_1+x_2, x_2)$; but $T\varphi = \{(1,0),(1,1)\}$ does not do norm retrieval for $\mathbb{R}^2$.

Remark 3.1. $\varphi = \{\varphi_i\}_{i\in I}$ perform norm retrieval for $\mathcal{H} \iff$ for $c_i \neq 0$, $c\varphi = \{c_i\varphi_i\}_{i\in I}$ perform norm retrieval for $\mathcal{H}$. Indeed, this can be easily verified from the fact that $|\langle x, c_i\varphi_i\rangle| = |\langle y, c_i\varphi_i\rangle| \iff |\langle x,\varphi_i\rangle| = |\langle y,\varphi_i\rangle|, \quad \forall i \in I$.

Theorem 3.2. *Suppose $\{\varphi_i\}_{i\in I}$ performs norm retrieval for $\mathcal{H}$. Consider $T \in \mathcal{B}(\mathcal{H})$, such that T is an isometry. Then $\{T^*\varphi_i\}_{i\in I}$ performs norm retrieval for $\mathcal{H}$.*

Proof. Suppose $x, y \in \mathcal{H}$ such that $|\langle x, T^*\varphi_i\rangle| = |\langle y, T^*\varphi_i\rangle| \implies |\langle Tx,\varphi_i\rangle| = |\langle Ty,\varphi_i\rangle|, \quad \forall\ i \in I$. Using the fact that $\{\varphi_i\}_{i\in I}$ performs norm retrieval for $\mathcal{H}$ and T is an isometry, we get $\|x\| = \|y\|$. □

Corollary 3.3. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is an unitary operator and let $\varphi = \{\varphi_i\}_{i\in I}$ be a sequence of vectors in $\mathcal{H}$. Then, φ doing norm retrieval for $\mathcal{H}$ is equivalent to $T\varphi$ doing norm retrieval for $\mathcal{H}$.*

In [8] it was shown that phase retrieval is preserved by non-zero idempotent operators for the range space. Theorem 3.4 shows that idempotent operators also preserves norm retrieval for the range space.

Theorem 3.4. *Consider $T \in \mathcal{B}(\mathcal{H})$, a non-zero idempotent operator and let $\varphi = \{\varphi_i\}_{i\in I}$ be a sequence of vectors in $\mathcal{H}$. Then φ doing norm retrieval for $R(T^*)$ is equivalent to $\{T\varphi_i\}_{i\in I}$ doing norm retrieval for $R(T^*)$.*

Proof. We note that for every $x_1, x_2 \in R(T^*)$, there exists $y_1, y_2 \in \mathcal{H}$ such that $T^*y_1 = x_1$, $T^*y_2 = x_2$. Then we have,

$$
\begin{aligned}
|\langle x_1, T\varphi_i\rangle| = |\langle x_2, T\varphi_i\rangle| &\iff |\langle T^*y_1, T\varphi_i\rangle| = |\langle T^*y_2, T\varphi_i\rangle| \\
&\iff |\langle T^*y_1, \varphi_i\rangle| = |\langle T^*y_2, \varphi_i\rangle| \\
&\iff |\langle x_1, \varphi_i\rangle| = |\langle x_2, \varphi_i\rangle|,
\end{aligned}
$$

for all $i \in I$. Hence the theorem holds. □

Theorem 3.5. *Given a closed subspace W of a Hilbert space $\mathcal{H}$, every norm sequence for $\mathcal{H}$ can be uniquely decomposed into norm retrieval sequences for W and $W^{\perp}$.*

Proof. Suppose $\varphi = \{\varphi_i\}_{i\in I}$ does norm retrieval for $\mathcal{H}$ and P_w is the orthogonal projection onto W. Then φ can be uniquely decomposed as $P_w\varphi$ and $(I-P_w)\varphi$, where $P_w\varphi = \{P_w\varphi_i\}_{i\in I}$. The conclusion follows from the facts that for $x, y \in W$,

$$|\langle x, \varphi_i\rangle| = |\langle x, P_w\varphi_i\rangle| = |\langle y, P_w\varphi_i\rangle| = |\langle y, \varphi_i\rangle|, \ \forall\ i \in I;$$

and for $x, y \in W^{\perp}$,

$$|\langle x, \varphi_i\rangle| = |\langle x, (I-P_w)\varphi_i\rangle| = |\langle y, (I-P_w)\varphi_i\rangle| = |\langle y, \varphi_i\rangle|, \ \forall\ i \in I.$$

□

Corollary 3.3 shows that the norm retrieval property for vectors is preserved by unitary operators. We now show that the norm retrieval property for subspaces is also preserved by unitary operators.

Theorem 3.6. *Consider $W = \{W_i\}_{i\in I}$ is a collection of closed subspaces in $\mathcal{H}$. Further, let $T : \mathcal{H} \to \mathcal{K}$ be unitary. If W does norm retrieval for $\mathcal{H}$, then TW does norm retrieval for $\mathcal{K}$.*

Proof. For $y_1, y_2 \in \mathcal{K}$, let $\|P_{TW_i}y_1\| = \|P_{TW_i}y_2\|$ for all $i \in I$. Since T is surjective, $\exists\ x_1, x_2 \in \mathcal{H}$ such that $Tx_1 = y_1$ and $Tx_2 = y_2$. We note that for $k = 1, 2$, we have $P_{TW_i}y_k = P_{TW_i}Tx_k = P_{TW_i}TP_{W_i}x_k + P_{TW_i}TP_{W_i^{\perp}}x_k = P_{TW_i}TP_{W_i}x_k = TP_{W_i}x_k$. Thus, we get $\|TP_{W_i}x_1\| = \|TP_{W_i}x_2\|$. Using the fact that T is isometry and $\{W_i\}_{i\in I}$ do norm retrieval, we obtain $\|y_1\| = \|y_2\|$. □

The following two examples show that if we drop the condition that T is isometry or the condition that T is surjective then we may lose the property of norm retrievality of $\{TW_i\}_{i\in I}$.

Example 3.7. Consider the subspaces $W_1 = x$-axis and $W_2 = y$-axis in $\mathbb{R}^2$. Clearly, $\{W_1, W_2\}$ does norm retrieval for $\mathbb{R}^2$. Define $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ as $T_1(x_1, x_2) = (x_1 + x_2, x_2)$. Thus T_1 is not an isometry. Now $T_1W_1 = x$-axis and $T_1W_2 = \text{span}\{(x, x) : x \in \mathbb{R}\}$. However $\{T_1W_1, T_1W_2\}$ does not do norm retrieval in $\mathbb{R}^2$. This can be easily verified at $(1, 1)$ and $(1, -3)$.

Example 3.8. Consider the subspaces $W_1 = x$-axis and $W_2 = y$-axis in $\mathbb{R}^2$. We note that $\{W_1, W_2\}$ does norm retrieval for $\mathbb{R}^2$. Define $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ as $T_2(x_1, x_2) = (x_1, x_2, 0)$. Clearly T_2 is not surjective. Now $T_2W_1 = x$-axis and $T_2W_2 = y$-axis in $\mathbb{R}^3$. But $\{T_2W_1, T_2W_2\}$ does not do norm retrieval in $\mathbb{R}^3$. This can be easily verified for $(0, 0, 1)$ and $(0, 0, 2)$.

Let $\{P_i\}_{i=1}^m$ be projections onto subspaces $\{W_i\}_{i=1}^m$ of $\mathbb{C}^n$. Consider any orthonormal bases $\{\varphi_{ij}\}_{j=1}^{I_i}$ of $\{W_i\}_{i=1}^m$ and a sub collection $S \subseteq \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq I_i\}$. It was shown in [4] that if $\{P_i\}_{i=1}^m$ does norm retrieval and $x \perp \text{span}\{\varphi_{ij}\}_{(i,j)\in S}$, $y \perp \text{span}\{\varphi_{ij}\}_{(i,j)\in S^c}$ then $\text{Re}\langle x, y\rangle = 0$. In fact $\langle x, y\rangle = 0$ for an arbitrary Hilbert space, this is eveident from the following result. A similar result for weaving norm retrival subspaces was proved in [8].

Theorem 3.9. *Let $\{P_i\}_{i\in\Lambda}$ be projections onto subspaces $\{W_i\}_{i\in\Lambda}$ of $\mathcal{H}$. Given any orthonormal bases $\{\varphi_{ij}\}_{j\in\Lambda_i}$ of $\{W_i\}_{i\in\Lambda}$ and a sub collection $S \subset \{(i,j) : i \in \Lambda, j \in \Lambda_i\}$. If $\{P_i\}_{i\in\Lambda}$ does norm retrieval then $\{\varphi_{ij}\}^{\perp}_{(i,j)\in S} \perp \{\varphi_{ij}\}^{\perp}_{(i,j)\in S^c}$.*

Proof. Given $S \subset \{(i,j) : i \in \Lambda, j \in \Lambda_i\}$. Let $x \in \{\varphi_{ij}\}^{\perp}_{(i,j)\in S}$ and $y \in \{\varphi_{ij}\}^{\perp}_{(i,j)\in S^c}$. We note that for each $i \in \Lambda$,

$$\begin{aligned}\|P_i(x+y)\|^2 = \sum_{j\in\Lambda_i} |\langle x+y, \varphi_{ij}\rangle|^2 &= \sum_{\substack{j\in\Lambda_i\\(i,j)\in S^c}} |\langle x, \varphi_{ij}\rangle|^2 + \sum_{\substack{j\in\Lambda_i\\(i,j)\in S}} |\langle y, \varphi_{ij}\rangle|^2.\\ &= \sum_{j\in\Lambda_i} |\langle x-y, \varphi_{ij}\rangle|^2 \\ &= \|P_i(x-y)\|^2.\end{aligned}$$

Therefore, we get $\|x+y\|^2 = \|x-y\|^2$ for all $i \in \Lambda$. Thus $\mathrm{Re}\langle x, y\rangle = 0$.

Similarly, we obtain $\|P_i(x+iy)\|^2 = \|P_i(x-iy)\|^2 \implies \|x+iy\|^2 = \|x-iy\|^2 \implies \mathrm{Im}\langle x, y\rangle = 0$ for all $i \in \Lambda$. Hence, $x \perp y$. □

Corollary 3.10. *Consider a sequence of vectors $\varphi = \{\varphi_i\}_{i\in I}$ in $\mathcal{H}$. For non-trivial $J \subset I$, let $W_1 = span\{\varphi_i\}_{i\in J}$ and $W_2 = span\{\varphi_i\}_{i\in J^c}$. If φ does norm retrieval then $W_1^{\perp} \subset W_2$.*

Proof. Since φ does norm retrieval, so by Theorem 3.9 we have $W_1^{\perp} \perp W_2^{\perp}$. Hence the conclusion follows. □

In [4], it has been proved that corollary 3.11 is true for $\mathbb{R}^n$. We extend it to $\mathcal{H}^n$ where $\mathcal{H}^n$ is an n-dimensional Hilbert space.

Corollary 3.11. *Every norm retrieval set with n elements is orthogonal in $\mathcal{H}^n$, where $\mathcal{H}^n$ is an n-dimensional Hilbert space.*

Proof. Consider a norm retrieval collection $\{\varphi_i\}_{i=1}^n$ in $\mathcal{H}^n$. If possible, suppose for some k with $1 \le k \le n$, φ_k is not orthogonal to another element of this collection. Let $W_1 = \mathrm{span}\{\varphi_i\}_{i\neq k}$ and $W_2 = \mathrm{span}\{\varphi_k\}$. Then $W_1^{\perp}$ can not be a subset W_2, a contradiction to Corollary 3.10. □

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