Optimizing Quasi-Interior Ideals and Fuzzy Soft Quasi-Interior Ideals of Ternary Semirings

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ABSTRACT
This study explores the algebraic structure of Ternary semirings and introduces the novel concepts of "quasi-interior ideals" (QIIs) and "fuzzy soft quasi-interior ideals" (FS QIIs). It starts with an overview of semirings, elaborating on their development and relevance in algebra and computer science—the job ventures into QIIs and their properties, including FS QIIs in T-semirings. The definitions, examples, and theorems clarify the conditions when such ideals work and their roles in characterizing regular T-semirings. Fuzzy set theory is further explored through its application to algebraic structures and problems in logic, set theory, and optimization. Such extensive analysis considerably expands the knowledge about semirings and their practical use in different mathematics and theoretical fields.

Keywords : Ternary semirings, QIIs, Fuzzy set theory, FS QIIs, Algebraic structures

1. INTRODUCTION

The semiring is the most essential algebraic structure explaining the broadspread principle of the semiring (duality of trees), with Vandiver introducing it in 1934. Some researchers argue that Dedekind's 19th-century work on commutative ring principles founded and developed the notion of semigroups. One operation within a semiring functions as the distributor of the other; it is an abstract algebraic structure composed of two such operations. The operations are addition and multiplication. An archetype for a semiring is the collection of natural numbers and standard additive and multiplicative operations. Most importantly, the transformation to the line segment subset with one as its base is a semiring for which the maximum and minimum operations work. The employees of this company calculate the difference and addition functions, represented by the additive unit 0 and multiplicative unit 1, respectively.

The semiring theory is rich since it is somewhere between semigroups and rings and the characteristic operation of their behavior - the centralizing property of semigroups and distributive properties of the rings. The semiring maintains the most by sourcing the portion of the overall organization's architecture that focuses on this element. As a theoretical idea in mathematics and computer science, semirings have ushered improvements in graph theory, constrained optimization, automata, and encoding on size and the study of formal languages.

Semirings first showed up indirectly by Dedekind and later more explicitly through researchers such as Macaulay, Neither, Lorenzen, and Vandiver, particularly in the arithmetic axiomatization of natural numbers. Many these days, the exploration of semirings is developed widely, although, since 1950, it has been vastly more popular to employ those methods stemming from semigroup and ring theories. The additive structure is not a "free" one, nor is it independent, as it relies upon the additive framework in semirings, while in rings, it is just multiplicative.

The reciprocal exchange between ring and semigroup theorems is what —one amongst others— deeply marked the expansion of semiring since it has been a vital discovering-oriented domain, continuously used in theoretical computer science, optimization problems and graph theory. The semiring multiplication operation and its properties relate semirings to areas of mathematics that are outsidethe ring theory framework.

Initially, the construction of the meaningless algebraic structures was carried out through approximate analogies with the earlier ideal notions. The development was complete with these elements in the stage, and they incorporated the ideas, adding on some as jumping to the conceptual level. Right-side ideals or bi-morphisms, the generalization of the mentioned terms within the semigroups theory, were firstly

introduced by Good and Hughes in 1952. This idea was later expanded to rings and semirings by Lájos and Sárdi Sárosi.
The issue of constructing quasi-ideals for semirings was further studied by Steinfeld in 1956, firstly for the semigroups and later on for the rings. By extending their results on these concepts to the framework of semirings, Iseki (1960) further refined them by introducing an original notion of quasi-ideals. Moreover, Henriksen approached the subject extemporaneously and examined ideals in semirings. On this basis, Jagtap and Pawar then explored quasi-ideals of semirings of Γ, which has dramatically impacted the structure of these objects.
To broaden the scope of his work, Rao introduced the topic fuzzy bi-quasi-ideals in Γ-semirings, which underlined the lattice-theoretic connection among rings and other more general algebraic structures. The present manuscript seeks to further the array of generalizations of the already introduced concepts of bi-ideals and interior ideals by giving rise to the quasi-internal ideals of Γ-semi-rings, one of which is the property of the binary intersection with ideals being ideal. On the one hand, it also proceeds to broaden fuzzy logic recognized as a subject by introducing fuzzy quasi-interior ideals to characterize Γ-regular semirings with the help of those new ideals.
The fuzzy set theory has received more highlighting lately in various fields of mathematics, so outlines the applicability of this theory in terms of logic, set theory, and other mathematical disciplines. Then, the fuzzification of algebraic structures resulted in the development of fuzzy subgroups based on Rosenfeld's definition of fuzzy subgroups.
The exploration of fuzzy structures continued with studies on fuzzy prime ideals and fuzzy subrings by Swamy and Liu, respectively. More specific to semirings, Mandal and Rao have explored fuzzy ideals within ordered semirings, further expanding the application of fuzzy logic in algebra.
This current study extends the theoretical framework for Γ- semirings by introducing and discussing the quasi-interior and Fuzzy SoftQuasi-Interior Ideals. Moreover, it uses these concepts to provide a characterization of regular Γ- semirings, showcasing the depth and breadth of applications of these advanced mathematical concepts in theoretical studies.

2. Preliminaries

This section includes the elements set and necessary terms for the discussion.

Definition 2.1. The associative binary functions of the structure are represented by the symbols + and ∗, respectively.

(i) Commutative laws allow simultaneous addition of two elements from a group, corresponding to s + t = t + s for each combination of sas well as t in T.
(ii) Since the distributive law over the addition operation of T is valid from both sides, it follows that the multiplication of þ∗(ƀ + π) = (þ ∗ ƀ) + (þ ∗ π) and (ƀ + π) ∗ þ = (ƀ ∗ þ) for any partþ, ƀ, and π from T.
(iii) There is a rule in T for an element p that says £ ∗ ¥ = £ and £ ∗ ¥ = ¥ ∗ £ = ¥. This value is called an identity element.

Definition 2.2. Consider V and Θ as two S≠∅ sets. V qualifies as a Θ-semigroup if it fulfills the conditions below:

(i) uξv belongs to V,
(ii) uξ(vψw) equals (uξv)ψw for any u, v, w in V and ξ, ψ in Θ.

Definition 2.3. Suppose (V, ⊕) and (Θ, ⊕) are commutative semigroups. V is considered a Θ-sr if it satisfies these axioms for any u, v, w ,ξ and ψ are in Θ:

(i) (uvw)ξψ = u(vwξ)ψ=uv(wξψ),
(ii) uξ(v ⊕ w) = uξv⊕uξw,
(iii) (u ⊕v)ξw = uξw⊕vξw,
(iv) u(ξ ⊕ ψ)v = uξv⊕uψv.

A standard sr V turns into a Θ-sr when Θ = V, using the typical sr multiplication as the ternary operation.

Definition 2.4. The zero element denoted by z of the Θ-sr V has the following properties: z ⊕ u = u = u ⊕ z and zξu = uξz = z for all u in V.

Definition 2.5. Using the standard matrix multiplication operation, let V be a set of matrices with p x q non-negative rational elements. Let Θ be a subset of q x p matrices with non-negative integer values, utilising the standard matrix multiplication. Under this conventional matrix multiplication operation, V is a sr known as Θ.

Definition 2.6. If y in S and σ, τ in T exist to the extent that c = cσyτc, then a factor c in T-sr S is said to be regular.

Definition 2.7. In the context of S being a T-sr, C is considered a right (orleft) principle whether it assures closure under addition and the condition STC⊆C (or CTS⊆C) holds. If C possesses the characteristics of both the right and Left Ideals, it will be regarded simply as an principle of S.

Definition 2.8. The regular elements of the T-srR, satisfying the conditions outlined in the definition, will correspond to the regular elements of the T.

Definition 2.9. A function g: D → [1, 0] is referred to as a fuzzy subset of D when considered within the context of a Λ-sr representation of D.

Definition 2.10.A threshold v that falls within the range of [1, 0], then the subset h_v = {t∈F | h(u) ≥ v} is considered to be a point subset of F from the perspective of h.

Definition 2.11. ω is a fuzzy in a T-srD for any elements u, v that are contained within N and ξ that is within T:
i.e., ω(u ⊕ v) is greater than or equal to the minimum of ω(u) and ω(v); i.e., ω(uξv) is greater than or equal to ω(v) (or ω(u)).

Definition 2.12. When a pair of fuzzy subsets ρ and ω of N is stated to be ⊆ ω, it implies that ρ(u) is less than or equal to ω(u) for any u that is a member of N.

Definition of 2.13. For any element i in I, the operations Б∘p, Б⊕€, and Б⊗€ are defined as explained in the following manner

$$Б\circ€(i)=\begin{cases}\{\sup\{\min[Б(¥),€(¢)]\}\text{if}¥,¢\in I \\ \qquad k,¥\in H \\ 0 \qquad\qquad \text{otherwise}\end{cases}$$

$$Б\oplus€(i)=\begin{cases}\{\sup\{\min[Б(¥),€(¢)]\}\text{if}\ \exists\ ¥=¢\in I \\ \qquad ¥,¢\in I \\ 0 \qquad\qquad \text{otherwise}\end{cases}$$

Definition 2.14. Assume that two T-srs, namely T and U. If the mapping i: T → U meets a requirement, then it is said to be a Λ-sr homomorphism.
(i) i(€ξ¥) = i(€)ξi(¥), €, ¥ and ξ in T respectively.

Definition of 2.15. The S≠∅ of set M is signified by D. With the following notation and definition, the distinctive position of D, which is a fuzzy subset on D, is established.

$$\chi B(n)=\begin{cases}1 & \text{if } n\in D,\\ 0 & \text{if } n\in D.\end{cases}$$

3. "Quasi-Interior Ideals" (QIIs)

Definition 3.1: If Γ is a Σ-ssr of Θ and ΘΣΓΘΣΘΓΘ⊆Γ, then Γ is a S≠∅ of Θ and is hence considered a left QII of Θ.

Definition 3.2: If Γ is a Σ-ssr of Θ and ΓΘΣΘΓΘΣΘ⊆Γ, then Γ is a S≠∅ of Θ and Γ is a RQII of Θ.

Definition 3.3: If Γ is a Σ-ssr of Θ and L/RQIIs of Θ, then Γ is a S≠∅ of Θ and a QII of Θ.

Remark 3.4: It is not necessary for a QII of a Σ-sr Θ to also be an interior ideal, BII, bi-QI, or QIof Σ-sr Θ.

Example 3.5. If Π represents the set of all numbers that are rational, we can write

Θ = (αγβδ)|α,β,δ,δ∈Π. As a result, Θ is a Σ-sr with standard matrix multiplication as the ternary operation and standard matrix addition. If Γ is equal to (00 β0)|β∈Π}, then Γ is not a bi-ideal of Θ but rather a RQII.Defining a left QII as the set Γ of Θ that is not S≠∅, we know that Γ is a Σ-ssr of Θ and ΘΣΓΘΣΘΓΘ⊆Γ.

Theorem 3.6. Within any Δ-sr Θ, the subsequent propositions are established:
1. All LIs inherently qualify as LQIIs of Θ.
2. All RIs inherently qualify as RQIIs of Θ.
3. All quasi ideals naturally qualify as QIIs of Θ.
4. All ideals by their nature qualify as QIIs of Θ.
5. The union of a RI with a LI within Θ constitutes a QII of Θ.
6. If Λ represents a LI and Ψ represents a RI of Θ, then the composite set Γ=ΨΔΛ is recognized as a QII of Θ.
7. Supposing Γ epitomizes a QII and ϒ represents a Δ-ssr of Θ, then the conjunction Γ∩ϒ is acknowledged as a QII of Θ.
8. Presuming Γ constitutes a Δ-ssr of Θ, and given ΘΔΘΔΘΔΓ is included in Γ, then Γ is affirmed as a left QII of Θ.
9. Assuming Γ is a Δ-ssr of Θ, and if ΘΔΘΔΘΔΓ is encompassed by Γ as well as ΓΔΘΔΘΔΘ is contained within Γ, then Γ is validated as a QII of Θ.
10. The convergence of a RQII with a left QII inside ΘΘ is recognized as a QII of Θ.
11. If Λ is a LI and Ψ is a RI within Θ, then the set Γ=Ψ∩Λ is considered a QII ofΘ.

Theorem 3.7.Λ is a bi-LQI of Ψ if it is a LQII of a T-sr Ψ.
Proof. Assume Λ is a LQII of Ψ. Then we have ΨΓΛΓΨΓΛ⊆Λ. Thus, we can deduce that ΛΓΨΓΛ⊆ΨΓΛΓΨΓΛ which, based on our assumption, is contained in Λ. This implies
ΨΓΛ∩ΛΓΨΓΛ
⊆ΛΓΨΓΛ
⊆ΨΓΛΓΨΓΛ
⊆Λ

Theorem 3.8. If Λ is LQII of a T-srΨ, then Λ is a BII of Ψ.
Proof. Let Λ be a LQII of Ψ. ΨΓΛΓΨΓΛ⊆Λ.
Thus we get ΨΓΛΓΨ∩ΛΓΨΓΛ⊆ΛΓΨΓΛ⊆ΨΓΛΓΨΓΛ⊆Λ. So Λ is a BII of Ψ.

Corollary 3.9. If Ω is a RQII of a T-srΨ, then Ω is a BII of Ψ.

Corollary 3.10. If Ω is a QII of a T-srΨ, then Ω is a BII of Ψ.

Theorem 3.11. Every LQII of a T-srΨ is also a BII of Ψ.
Proof. Given Λ is a LQII of Ψ, we have that ΛΓΨΓΛ⊆Λ, which infers that ΨΓΛΓΨΓΛ⊆Λ. Since the latter is within Λ, it follows that ΛΓΨΓΛ⊆Λ is also within Λ, and thus Λ qualifies as a BII of Ψ.

Corollary 3.12. For any T-srΨ, every RQII is also a BII of Ψ.

Corollary 3.13. If a subset Λ within a T-srΨ is a QII, then it naturally holds the property of being a BII of Ψ.

4. “Fuzzy Soft” (FS)QIIs of “T-Semiring”(T-sr)

Definition 4.1: If we say that (G,ν) is an FS set in V then (F,ρ) is an FS of (G,ν), which we denote as $(F,\rho)\subseteq(G,\nu)$, for all $\rho\in\Omega$, if $A\subseteq B$ and $F(\rho)\leq G(\nu)$.

Definition 4.2: Given two FS sets (F,λ) and (G,μ), their intersection is defined by (H,ζ) where $H:\Omega\to[0,1]^{P(V)}$ such that for each $\chi\in\Omega$:

$$H(\zeta)(\chi)=\begin{cases}F(\lambda)(\chi), & \text{if } \chi\in A\setminus B;\\ G(\mu)(\chi), & \text{if } \chi\in B\setminus A;\\ \min\{F(\lambda)(\chi),G(\mu)(\chi)\}, & \text{if } \chi\in A\cap B.\end{cases}$$

Definition 4.3: If (F,ρ) and (G,ν) are FS sets over V, the 'greater than or equal to' relation between them, indicated by $(F,\rho)\geq(G,\nu)$, is delineated by (J,τ) where $J:\Omega\to[0,1]^{P(V)}$ such that for each $\omega\in\Omega$, $J(\tau)(\omega)=max\{F(\rho)(\omega),G(\nu)(\omega)\}$, where $C=A\times B$, $\tau=(\rho\times\nu)$, and A,B are in Ω.

Definition 4.4: Assume a T-sr H, a parameter set Θ, a subset C⊆H, and a map h:C→[0,1]H.
$h\lambda(r\oplus s)\geq min\{h\lambda(r),h\lambda(s)\}$,
$h\lambda(r\odot s)\geq max\{h\lambda(r),h\lambda(s)\}$.

Definition4.5: Let $h\xi:H\rightarrow[0,1]$ is a FII of H, in which any r,s in H, ξ in Θ:
$h\xi(r\oplus s)\geq min\{h\xi(r),h\xi(s)\}$,
$h\xi(r\odot s)\geq max\{h\xi(r)\}$.

Definition 4.6: Assume T-sr K as a parameter set φ and Z⊆K as a subset. Let K:: [0,1]J be a function. Then, (j,) is known as a FS QI over K, if for each ρ in Φ, the associated fuzzy subset Kρ:K→[0,1] is a fuzzy QI of K, where for each m,n in K, ρ in Φ:
$K(m\oplus n)\geq min\{K\rho(m),K\rho(n)\}$

Definition 4.7: If a T-sr Λ, a parameter set Φ, and a subset Z⊆Λ. If α:Z→[0,1]Λ is a mapping, then (α,ϕ) is a FS L/RQII over Λ, if for every ϕ in Φ, the related fuzzy subsetαϕ:Λ→[0,1] is a fuzzy L/RQII of Λ, that is, for all μ,ν in Λ, ϕ in Φ:
$\alpha\phi(\mu\oplus\nu)\geq\min\{\alpha\phi(\mu),\alpha\phi(\nu)\}$
FS set (α,β) of T-sr Λ is called a FS QII, if it is both a FS L/RQII of Λ.

Example 4.8: Consider Ω as set of numbers:
$\Lambda = \left\{\begin{pmatrix} \xi & \rho \\ \sigma & \tau \end{pmatrix} \mid \xi,\rho,\sigma,\tau \in \Omega\right\}$
Then Λ is a T-sr.
$Z = \left\{\begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} \mid \rho \in \Omega\right\}$

Theorem 4.9. AssumeNisT-sr, Ψ is parameter set and X⊆N. Whether (γ,ϕ) is FS RI over N, then (γ,ψ) is a FS RQII over N.
Proof. Assume (γ,ϕ) is a FS RI over N. For every ψ in Ψ, γψ is a FS RI of the T-sr. For any ν in N:
$\gamma_\psi\circ\chi(n) = \sup_{b\in X} \min\{\gamma_\psi(b),\chi X(n)\}$,
implies
$\gamma_\psi\circ\chi_X(n) \leq \sup_{b\in X} \gamma_\psi(b)$.
Hence, γψ∘χ(ν)≤γψ(ν). Thus, we derive that:
$\gamma\psi\circ\chi X\circ\chi(\nu)\leq\min\{\gamma\psi\circ\chi X(\nu),\gamma\psi\circ\chi X(\nu)\}$,
establishing (γ,ψ) as a FS RQII of N.

Example 4.10 Suppose N is T-srand μ is S≠∅ of N, and η is level subset representing μ. If (ν)≥τ, then η includes ν. If ν is a part of η, then for any σ in N, (σ∘ν) should be greater than or equal to min{μ(σ),μ(ν)} implying that η includesσ∘ν. Thus, η is a LQII of N. Assuming ν,μ∈N and η as a level subset, it's deduced that for every σ∈N, μ(σ⊕ν)≥min{μ(σ),μ(ν)}, thus encompassing σ⊕ν within η. Hence, by definition, η acts as a left QII of N, fulfilling η∘η⊆η, which proves that μ itself is a FS left QII.

Corollary 4.11 For all σ∈N, the inequality (ν⊕σ)≥min{μ(ν),μ(σ)} is maintained, thereby ensuring $\nu\oplus\sigma\nu\oplus\sigma$ is in η. Likewise, (ν∘μ)≥μ(μ) assures that η is closed under the RQI operation in N. Therefore, ξ upholds the properties of a fuzzy RQII when each level subset η qualifies as a RQII of N.

Theorem 4.12 Let Pis T-sr, Ξ is parameter set, and A⊂P. (η,ϕ) is a soft RQII of P if and only if for each ξ in Ξ, ηξ is a FS RQII of P.
Proof. Assume (η,ϕ) is a soft RQII of P. For each ξ in Ξ, ηξ is a FS RI of P. Let λ be in P. Then we have:
ηξ∘χ(λ)=ν∈Asupmin{ηξ(ν),χA(λ)}
ηξ∘χA∘χA(λ)≤ηξ(λ)
Thus,ηξ is a fuzzy RQII of P. Thus, (η,ξ) is a FS RQII of P. On the contrary, infer that (η,ϕ) is a FS QII of P.
ηξ∘χA∘χA(λ)≤ηξ(λ)
Hence, ηξ is a RQII of P. Therefore, (η,ϕ) is a FS RQII of P.

Theorem 4.13 Assume$N$ is a T-srwith a set of parameters Φ, and subsets Δ⊆Φ, Σ⊆Φ. If(θ,Δ) and (ι,Σ) signify FS LQIIs of N, then the intersection(θ,Δ)∩(ι,Σ) forms a FS LQII of N.
Proof. Given (θ,Δ) and(ι,Σ) as FS left QIIs of N, definition 4.9 dictates that (θ,Δ)∩(ι,Σ) equals(κ,Λ), where Λ=Δ∪Σ.

Case (i): If γ is an element of Δ\Σ, then $\gamma\kappa\gamma$=θγ. Therefore, κγ maintains its status as a LQII of N, attributable to (θ,Δ)'s properties.
Case (ii): For ηη in Σ\Δ, κη=ιη, and hence κη upholds its function as a fuzzy LQII of N, given (ι,Σ)'s qualities.
Case (iii): If ζ is in Δ∩Σ and given any ξ in N, μ in Γ, then, κζ=θζ∩ιζ.

Corollary 4.14 If (θ,Δ) and (ι,Σ) are identified as FS RQIIs of N, the intersection (θ,Δ)∩(ι,Σ) is established as a FS RQII of N.
Corollary 4.15 If(θ,Δ) and (ι,Σ) be designated as FS R/LIs of N, respectively, then their intersection (θ,Δ)∩(ι,Σ) is ascertained to be a FS QII of N.

Corollary 4.16 Given(θ,Δ) as a fuzzy RI and (ι,Σ) as a fuzzy LI of N, the intersection (θ,Δ)∩(ι,Σ) constitutes a right fuzzy QII of N.

Corollary 4.17 If(θ,Δ) is a FS QI within a regular T-srN, then a FSI of N.

Theorem 4.18 AssumeT-srN is regular for every FS RI(θ,Δ) and FS LI (ι,Σ) of N, the operation θδ∘ισ is equivalent to the intersection (θ,Δ)∩(ι,Σ).

Proof4.19 If (π,Δ) is a FS LQII of Σ and let κ∈Σ. Consider any ν in Φ, we find that Σ∘πν∘Σ∘πν⊆π.
Suppose Σ∘π(κ)>πν(κ) and πν∘Σ∘πν(κ)>πν(κ). Since Σ is customary, there subsists some υ∈Σ, α, in Γ to the extend that κ=κ∘α∘υ∘β∘κ, we can then write:
πν∘Σ∘π(κ)=supξ∈Σmin{πν(κ),Σ∘πν(ξ)}
=supξ∈Σmin{πν(κ),1}
=supξ∈Σπν(κ)>πν(κ).

Corollary 4.20 For a regular T-srΣ, the pair (ν,Δ) is considered a FS RQII of Σ if and only if it is a FS QII of Σ.

Theorem 4.21 A T-srΣ is classified as regular precisely when ΔΓΣ=Δ∩Σ for any RI Δ and LI Σ within Σ.

Theorem 4.22 A T-srΣ exhibits regularity if and only if ΣΓΔΓΣΓΔ=Δ (or ΔΓΣΓΔΓΣ=Δ) is established as a L/RQII of Σ.
Proof. Given a regular T-srΣ and assuming Δ is a LQII in Σ, and any element η∈Δ, it follows that ΣΓΔΓΣΓΔ⊆Δ. This guaranteed to find elements ω∈Σ and parameters δ,δ,ϵ such that η=$\eta\delta\omega\epsilon\eta$ in ΣΓΔΓΣΓΔ. Thus, this leads to η being within ΣΓΔΓΣΓΔ, thereby confirming that ΣΓΔΓΣΓΔ=Δ. An analogous process can verify the same for a RQII in Σ, where ΔΓΣΓ.
Conversely, if we presume ΣΓΔΓΣΓΔ=Δ for all LQIIs Δ of Σ, let Δ=Σ∩Λ and Φ=ΣΓΛ, with Σ as a RI and Λ as a LI in Σ. Both Δ and Φ then qualify as QIIs of Σ. This establishes that (Σ∩Λ)ΓΣΓ(Σ∩Λ)ΓΣ=Σ∩Λ.
Expanding further, we discover:
Σ∩Λ=(Σ∩Λ)ΓΣΓ(Σ∩Λ)
⊆ΣΓΛΓΣ
⊆ΣΓΛ
alike
Σ∩Λ=(Σ∩Λ)ΓΣΓ(Σ∩Λ)ΓΣ
⊆ΣΓΛΓΣΓΛΓΣ
⊆ΣΓΛ.
Since ΣΓΛ is a subset of both Λ and Σ, it leads us to conclude that Σ∩Λ=ΣΓΛ.

Theorem 4.23 If Π represent a T-sr, πψ∘Π∘ψ∘Π∘ψ holds for any FS LQII (π,Δ) of Π.
Proof. Assume Π is regular. Let (π,Δ)(π,Δ) be a FS LQII of Π and consider arbitrary ξ,κξ,κ in Π, and α,βα,β in Γ. Then we have:$\pi\psi$∘Π∘ψ∘Π∘(ξ)
=π(ξ).πψ∘Π∘ψ∘Π∘(ξ)
=supω∈Πmin{πψ∘ψ(ξ),πψ∘ψ(ω)}
≥supω∈Πmin{πψ(ξ),πψ(ω)}
=πψ(ξ).
Therefore, πψ∘Π∘ψ∘Π∘ψ=π.

Conversely, suppose πψ=Π∘ψ∘Π∘ψ for any FS QII (π,Δ) of Π. Let Ω be a QII of the T-srΠ. By Theorem 4.19, a regular T-srupholds specific multiplication properties for its FS QIIs, and conversely, these properties can confirm the regularity of the T-sr.

Theorem 4.24 For a T-srΣ, it is regular precisely when *νι∩ψκ⊆νι∘ψκ∘νι∘ψ*, for every *ι*∈Φ, *κ*∈Λ, given a FS left QII (*ν*,Φ)and a FS ideal (ψ,Λ) of Σ.
Proof. Let's assume Σ is a regular T-srand select an element ρ from Σ. There will then exist an element ω in Σ, and parameters *δ*,δ,ϵ such that *ρ*=*ρδωϵρ*ρ=ρδωϵρ. This leads to:
νι∘ψκ∘νι∘ψκ(ρ)=min{supξ∈Σmin{νι∘ψκ(ξ),νι∘ψκ(ρ)},
≥min{min{supξ∈Σνι(ξ),supξ∈Σψκ(ωϵξ)},
min{supω∈Σνι(ω),supω∈Σψκ(ωδρ)}
=min{νι(ρ),ψκ(ω)}=νι∩ψκ(ρ).
Hence, νι∩ψκ is contained within νι∘ψκ∘νι∘ψκ.
Conversely, consider the conditions to be met, letting (*ν*,Φ) be a FS LQII of Σ. By Theorem 4.31, this implies Σ must be regular, completing the proof.

Corollary 4.25 Within a T-srΣ, the condition of regularity is equivalent to the inclusion νι∩ψκ⊆νι∘ψκ∘νι ∘ψκ for every *ι* in the set Φ, and *κ* in the set Λ, provided that (ν,Φ) represents a FS RQII and (*ψ*,Λ) stands as a FS ideal in Σ.

5. CONCLUSION

In this exploration, the concept of FSR/LQIIs, as well as the overarching category of FS QIIs within the structure of a T-srhas been ventured. The characteristics of these QIIs, bringing to light their intrinsic properties have been explored. Furthermore, a clear connection among these ideals, delineating a T-sr's regularity through the prism of FS R/LQIIs has been established, and various algebraic attributes pertinent to these structures have been illuminated.

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