A study of finite fuzzy group, finite fuzzy field and finite fuzzy vector space

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ABSTRACT
The present paper discusses some results regarding the membership values of elements of the finite fuzzy groups. Some results have been obtained for a finite fuzzy field including the membership values of the generator which is less than or equal to $\frac{1}{p-1}$. In addition, few observations have been made about the particular fields. Moreover, we studied finite fuzzy vector space that differs from the concept of finite fuzzy vector spaces which has been studied recently by the researchers.

Keywords: finite fuzzy group, finite fuzzy field and finite fuzzy vector space.

1. INTRODUCTION

The inclusion of fuzziness in the mathematical expressions real scientific or technological applications has raised a great need of correcting the concerned mathematical methodologies. Fuzzy algebras[1], [2]have many applications in the field of mathematics as well as various fields like informatics, coding theory, biology etc. Fuzzy algebras like fuzzy groups, fuzzy field and fuzzy vector spaces have been studied in detail[3]. We discussed mainly important problems of fuzzy group theory which is to categorize the fuzzy subgroups of a finite group[4], [5].
Since the inception of the concept of fuzzy sets, the theory of fuzzy sets has developed in many directions with applications in diverse fields[6], [7]. Rosenfeld [8]used this perception to describe the theory of fuzzy groups. Many basic properties in group theory are found to be carried over to fuzzy groups[9]. Moreover, he discusses a characterization of all fuzzy groups of a prime cyclic group in terms of the membership function[10]. We obtained a similar characterization of fuzzy fields and fuzzy vector spaces in this paper.
Recently, this topic has enjoyed a speedy development in the field having more applications[5]. Several other papers have treated the particular case of finite abelian groups. The present study concentrates on fuzzy finite groups, fuzzy finite fields and fuzzy finite vector spaces that is motivated by the foregoing applications[2], [3], [8].Some theoretical results have been obtained about fuzzy groups, fuzzy field and fuzzy vector spaces with mathematical applications[2], [3].

2. Preliminaries

In this part, we will set up some definitions.

2.1 Fuzzy subgroup[2]

Let fuzzy set A of group G is said to be fuzzy subgroup if
(i)$\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\} \forall x, y \in G(ii)\mu_A(x^{-1}) \geq \mu_A(x)$for all$x \in G$

2.2 Level subgroup[3]

The subgroup A_t of $G \ni t \leq \mu_A(e)$ is called a level subgroup of G. where $t \in [0,1]$.

2.3Fuzzy normal[1]

Fuzzy subgroup is normal if $\mu_A(xy) = \mu_A(yx)$

2.4 Fuzzy field[5]

Let μ_F be membership function on field F and

(i) $\mu_F(x+y) \geq \min\{\mu_F(x), \mu_F(y)\}$, $\forall$ x, y $\in$ X

(ii) $\mu_F(x^{-1}) \geq \mu_F(x)$, $\forall$ x, $\in$ X

(iii)$\mu_F(xy) \geq \min\{\mu_F(x), \mu_F(y)\}$, $\forall$ x, y $\in$ X

(iv)$\mu_F(x^{-1}) \geq \mu_F(x)$, $\forall$ x $\in$ X and x $\neq$ 0

(v) $\mu_F(0) = 1$, $\mu_F(1) = 1$

Then (F, μ_F) is fuzzy field.

3. Finite Additive Fuzzy Group

Let **G** be a finite abelian group of order p where p is prime, also g_0 is a generator of G .

$(G, +) \equiv (Z_p, +_p)$,

$$Z_p = \{[0], [1], [2], \ldots\ldots [p-1]\}$$
$$[m] +_p [n] = [l]$$

Where$(m+n) - l \equiv 0 \bmod p$.,

(i. e. , $0 \leq l \leq p-1$)

Let μ be a membership function on G then μ is said to be additively groupable if

$\mu[0] = 0, \mu[g_0] = q$ such that$\mu[mg_0] \leq mq \leq 1$, $q \leq 1$.

$G = \{0, g_0, 2g_0, 3g_0, \ldots, (p-1)g_0\}$with operation of addition, and

$pg_0 = 0$ in $+_p$Sense

Next we define modified membership function

$\mu^*: G \to [0,1]$ Such that

$$\mu^*(x) = \begin{cases} \mu(m' g_0) & \text{if } x = mg_0 \\ & \text{where } m \equiv m' (\text{mod } p) \\ & \text{and } 1 \leq m' \leq p-1 \\ 0 & \text{if } x = npg_0, n \in N \end{cases}$$

Then $(G, \mu^*)$is said to be finite additive fuzzy group.

And $\forall x \in G, \mu^*(x) \leq mq$, where $0 \leq m \leq p-1$.

4. Finite Multiplicative Fuzzy Group

Let $G = \{0, g_0, 2g_0, 3g_0, \ldots, (p-1)g_0\}$,p is prime with operation of multiplicative modulo p, denoted as p then ${g_0}^p = e$ in$._p$ sense and

$${g_0}^m = \begin{cases} {g_0}^n & \text{, where } m = tp + n, 1 \leq n \leq p-1 \\ e & , \quad \text{if } m = tp \end{cases}$$

Let $\mu: G \to [0,1]$ be a membership function such that $\mu(e) = 1$ and

$\mu({g_0}^n) \geq q^n$, where$\mu(g_0) = q$

and $1 \leq n \leq p-1$, here $q < 1$

thenμ is said to be multiplicatively groupable. Then we define modified membership function

$\mu^*: G \to [0,1]$ Such that

$$\mu^*(x) = \begin{cases} \mu({g_0}^n) & \text{if } x = {g_0}^m \\ & \exists \ m = tp + n \text{ and} \\ & 1 \leq n \leq p-1 \\ 1 & \text{if} x = {g_0}^{np}, n \in N \cup \{0\} \end{cases}$$

Then$(G, \mu^*)$is said to be finite multiplicative fuzzy group.

And $\forall x \in G, \mu^*(x) \geq q^{\,n}$

where $0 \leq n \leq p-1$.

Here$x = g^m = g^{tp+n}$,

$\mu^*(x) = \mu^*(g^{tp+n}) = \mu(g^n) \geq q^n$

$(\mathbf{t \in N \cup \{0\} \text{ and } 0 \leq n \leq p-1})$ **.**

5.Finite Fuzzy Field

Let F be a finite field of the type

$Z_p = \{[0], [1], [2], \ldots\ldots [p-1]\}$under$+_p$ and $._p$ in the usual sense, let g be a group generator of Z_p with respect to p-mod multiplication $._p$. See that gcan also be treated as a group generator with respect to p-mod addition $+_p$ then for any x $\in$ F

We can have two representations of x ; one with respect to $+_p$,
i.e., $x = n_1 g$ and another with respect to $\cdot_p$
i.e., $= g^{n_2}$,
Thus $x = n_1 g = g^{n_2}$
i.e., $xg^{-1} = n_1 e = g^{n_2 - 1}$ if $g \neq 0$.
If $\mu: G \rightarrow [0,1]$ be a membership function, then it is algebraic if
$q^{n_2} \leq \mu(x) = n_1 q$, $\forall\, x \in F$,
For some n_1 and n_2 such that
$1 \leq n_1, n_2 \leq p - 1$, then (F, μ) is said to be a finite fuzzy field.

Theorem: Let F be a finite field of the type $Z_p = \{[0], [1], [2], \ldots\ldots [p-1]\}$ under $+_p$ and $\cdot_p$ and g be a group generator of Z_p with respect to p-mod multiplication $\cdot_p$, $\mu(g) = q$ then $q \leq \frac{1}{p-1}$.
Proof: Let F be a finite field of the type
$Z_p = \{[0], [1], [2], \ldots\ldots [p-1]\}$,
That is
$F = \{0, g, 2g = g^2, 3g = g^3, \ldots\ldots ng = g^n, \ldots\ldots (p-1)g = g^{p-1}\}$
we must have $(p-1)g = g^{p-1} = e$ because we say that g is a generator of F with respect to $\cdot_p$
Then it means that g is a generator of $F' = \{e, g, g^2, \ldots\ldots, g^{p-2}\}$ and $g^{p-1} = e$
Therefore $\mu\{(p-1)g\} \leq (p-1)q \leq 1$
$q \leq \frac{1}{p-1}$.
(since $\mu\{(p-1)g\} = \mu\{g^{p-1}\} = \mu(e) = 1$)

6. Some Results And Observation
$Z_p = \{[0], [1], [2], \ldots\ldots [p-1]\}$
(1) if $\mu([0]) \neq \mu([r])$ and
Since $\mu([0]) \leq \mu([r])$, $0 \leq r \leq p-1$
We get $\mu([0]) < \mu([1])$ if $\mu([0]) \neq \mu([1])$
(2) if $0 \leq r \leq p-1$, $[p-r] = -[r]$
Additive inverse of [r]
This implies $\mu([p-r]) = \mu(-[r])$

$$Z_p = \{[0], [1], [2], \ldots\ldots, [p-2], [p-1]\}$$

(3) in Z_n , let [x], [y], [z] be such that $[x] + [y] = [z]$ with $x, y, z \leq \frac{n+1}{2}$ if n is odd and $x, y, z \leq \frac{n}{2}$ if n is even.
Then $\mu([z]) \leq \max\{\mu([x]), \mu([y])\}$ and
$\mu([x]), \mu([y])$ and $\mu([z])$ havethree possibilities;
(i) if $\mu([x]) \neq \mu([y])$ and $\mu([x]) < \mu([y])$ then $\mu([x]) < \mu([y]) = \mu([z])$
If $\mu([x]) = \mu([y])$, then(ii)..either $\mu([z]) < \mu([x]) = \mu([y])$(iii) … or $\mu([z]) = \mu([x]) = \mu([y])$
(4) if $\mu([1]) < \mu([x])$ then $\mu([p+1]) = \mu([p] + [1]) = \mu([p])$
Now since $[p+1] = [1]$ implies $\mu([p+1]) = \mu([1])$
So these two together implies that
$\mu([1]) = \mu([p])$
Analogously
$\mu([1]) = \mu([2]) = \cdots.. = \mu([p]) = \mu([0])$
That is μ is constant on Z_n
Thus μ is not constant on Z_n then
$\mu([1]) > \mu([x])$
(5) we have
$\mu([0]) < \mu([r]) < \mu([1]) = \mu([n-1])$then
$\mu([r+1]) \leq \max\{\mu([r], \mu([1]))\} = \mu([1])$
$\mu([0]) < \mu([r]) < \mu([1]) = \mu([r+1]) = \mu([n-1])$
and $\mu([r-1]) \leq \max\{\mu([r], \mu(-[1]))\}$
$$= \max\{\mu([r], \mu([n-1]))\}$$
$$= \mu([n-r])$$

$$= \mu([1])$$

Then $\mu([r-1]) = \mu([n-r]) = \mu([1])$
$\mu([0]) < \mu([r]) < \mu([1]) = \mu([r-1])$
(6) if$\mu([0]) = \mu([r]) < \mu([1])$ then
$\mu([r+1]) \leq \max\{\mu([r], \mu([1]))\} = \mu([1])$
$\Rightarrow \mu([r+1]) = \mu([1])$and
$\mu([r-1]) \leq \max\{\mu([r], \mu(-[1]))\} = \mu([1])$
$\Rightarrow \mu([r-1]) = \mu([1])$
Thus $\mu([0]) = \mu([r]) < \mu([1])$
$\Rightarrow \mu([r+1]) = \mu([r-1]) = \mu([1])$
(7)There are at most three membership vales,
$\mu([0]) = \mu([1])$or$\mu([0]) < \mu([1])$ or$\mu([0]) < \mu([r]) < \mu([1])$

Group	Posible valúesof μ
z_1	$\mu([0])$
z_2	$\mu([0]) = \mu([1])$, $\mu([0]) < \mu([1])$
z_3	$\mu([0]) = \mu([1]) = \mu([2])$,$\mu([0]) < \mu([1]) = \mu([2])$,$\mu([0]) < \mu([2]) < \mu([1])$
z_4	$\mu([0]) < \mu([2]) < \mu([1]) = \mu([3])$,$\mu([0]) < \mu([1]) = \mu([2]) = \mu([3])$
z_5	$\mu([0]) < \mu([2]) = \mu([3]) < \mu([1]) = \mu([4])$,$\mu([0]) < \mu([1]) = \mu([2]) = \mu([3]) = \mu([4])$
z_6	$\mu([0]) < \mu([1]) = \mu([2]) = \mu([3]) = \mu([4]) = \mu([5])$, $\mu([0]) < \mu([3]) < \mu([1]) = \mu([2]) = \mu([4]) = \mu([5])$, $\mu([0]) < \mu([2]) = \mu([4]) < \mu([1])\mu([3])\mu([5])$
z_7	$\mu([0]) < \mu([1]) = \mu([2]) = \mu([3]) = \mu([4]) = \mu([5]) = \mu([6])$, $\mu([0]) < \mu([2]) = \mu([5]) < \mu([1]) = \mu([3]) = \mu([4]) = \mu([6])$,

7. Finite Fuzzy Vector Space

Consider a basis $\{(1,0) = e_1,\ (0,1) = e_2\}$ to generate a finite vector space with the finite scalar field $F = \{[0], [1], [2], [.3], [4]\} = Z_5$ with the usual mod 5 operation + and
We define $[m]e_1 + [n]e_2 = ([m], 0) + (0, [n]) = ([m], [n])$ where $[m], [n] \in F$
Note that

$$[m_1] + [m_2] = [m_1 + m_2]$$
$$= [5k + m_0]$$
$$= [m_0]$$

Where $0 \leq m_0 \leq 4$
i.e.,
we take $[5k] = [0]$, similarly for $[x_1] + [x_2]$
Also we define scalar multiplication as
$[l]\bar{x} = [l]\{[m]e_1 + [n]e_2\}$

$$= [l]([m], [n])$$
$$= ([l][m], [l][n])$$
$$= ([lm], [ln])$$

$= ([5k + m_0], [5k + n_0])$ Where $0 \leq m_0, n_0 \leq 4$
$= ([5k_1], [5k_2]) + ([m_0], [n_0])$ ---- (1)
Next we define
$([5k_1], [5k_2]) = ([0], [0])$
Also define the vector addition as
$([M_1], [n_1]) + ([m_2], [n_2])$
$= ([m_1 + m_2], [n_1 + n_2])$
$= ([5k_1 + m_0], [5k_2 + n_0])$
$= ([5k_1], [5k_2]) + ([m_0], [n_0])$
$= ([0], [0]) + ([m_0], [n_0])$
$= ([m_0], [n_0])$
Where $0 \leq m_0, n_0 \leq 4$
Thus (1) can be written as
$[l]\bar{x} = ([m_0], [n_0])$,

Where $0 \leq m_0, n_0 \leq 4$ and
$\bar{x} = ([m],[n])$,
$m = 5k_1 + m_0$, $n = 5k_2 + n_0$
Under these definition , here generated finite vector space can be written as
$V = \{([m],[n])/$ m and n are integrs $\exists\ 0 \leq m, n \leq 4\}$
In all, Vhas 25 elements. Thus V can be compared with Z_{25}. And V can be seen as isomorphic to Z_{25} explicitly if we write the elements of Vand Z_{25}

$$V = \begin{Bmatrix} ([0],[0]),([0],[1]),([0],[2]),([0],[3]),([0],[4]),([0],[5]), \\ ([1],[0]),([1],[1]),([1],[2]),([1],[3]),([1],[4]),([1],[5]), \\ ([2],[0]),([2],[1]),([2],[2]),([2],[3]),([2],[4]),([2],[5]), \\ ([3],[0]),([3],[1]),([3],[2]),([3],[3]),([3],[4]),([3],[5]), \\ ([4],[0]),([4],[1]),([4],[2]),([4],[3]),([4],[4]),([4],[5]), \end{Bmatrix},$$

$Z_{25} = \{[0],[1],[2], \dots \dots . [24]\}$, Z_{25} is not an integral domain.
Now V is a abelian group under vector addition and its generator is $\{([1],[0]),([0],[1])\}$ and so V is a finitely generated abelian group. Then v is isomorphic to a direct product of cyclic groups in the form of
$Z_{(p_1)^{x_1}} \times Z_{(p_2)^{x_2}} \times \dots . . \times Z_{(p_n)^{x_n}} \times Z \times Z \times \dots . . \times Z$
The number of Z factors = Betti number of V
Then V is decomposable
Every finite indecomposable abelian group is cyclic of order equal to a power of a prime.
For a simple membership value
$\mu: V \rightarrow [0,1]$, since V is a group we may use the groupableμ .
i.e., for any vector $x \in V$we have $x = m([1],[0]) + n([0],[1])$ as V is generated by $\{([1],[0]),([0],[1])\}$
Therefore
$\mu(x) \leq m\ \mu([1],[0]) + n\ \mu([0],[1]) \leq 1$, for$0 \leq m, n \leq 5$
This gives as a condition on m
$\mu([1],[0])$and$\mu([0],[1])$
if$\mu([1],[0]) = p_1$ and $\mu([0],[1]) = p_2$ then $0 \leq mp_1 + np_2 \leq 1$,
$0 \leq m, n \leq 5$is the condition and $0 \leq m + n \leq 5$.
For a decomposable finite abelian group $G \equiv G_1 \otimes G_2$, g_1 and g_2 are generators of $G_1$and G_2 respectively. Also μ_1 and μ_2 membership functions of $G_1$and G_2 respectively, ${\mu_1}^*$ and ${\mu_2}^*$ modified membership functions of $G_1$and G_2 respectively. Then μ^* on G defined by
$\mu^*(g) \equiv \mu^*(mg_1 + ng_2)$
$\leq \max\{{\mu^*}_1(mg_1), {\mu^*}_2(ng_2)\}$
where $1 \leq m \leq p_1 - 1$, $1 \leq n \leq p_2 - 1$
And $\mu^*(g) \equiv \mu^*(p_1g_1 + p_2g_2) = 0$
Where$p_1q_1 + p_2q_2 \leq 1$
$\mu^*$is a membership function on G.
Thus, (G, μ^*) is a fuzzy group.

8. CONCLUSIONS

In this paper, we revealed the new results regarding the membership values of elements of the finite fuzzy additive and multiplicative groups. The newly established finite fuzzy field and some results and observations about the field have been detailed in the paper. From the new result about this field, the fact that there are at most three membership vales for a finite fuzzy field z_p, have been discussed. We defined the concept of the finite fuzzy vector space employing addition and scalar multiplication that differs from finite fuzzy vector spaces which the researchers have studied recently.

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