L^1-Convergence of Newly Defined Trigonometric Sums Under Some New Class of Fourier Coefficients

Priyanka, Karannvir Singh
Maharaja Ranjit Singh Punjab Technical University, Bathinda, India
priyanka.baghla@gmail.com

Tough difficulties in the trigonometric series convergence in L^1 norm is appearance of trigonometric series as Fourier series, and its L^1- convergence. Many academics investigated trigonometric series separately by examining the cosine & sine series , so as a result, modified cosine sums and sine sums were developed to assess the sharp consequences on trigonometric series's integrability & L^1-convergence, as improved sums approach respective limits closer than traditional trigonometric sums. This work presents 'KP', a new class of Fourier Coefficients, as well as advanced cosine and sine sums of trigonometric series with real coefficients. As a result, necessary & sufficient criterion for Integrability and L^1-normed convergence for trigonometric functions is achieved. Here, authors also discuss about L^1-convergence of r^{th} differential of newly defined improved trigonometric sums with Fourier coefficients are from an enlarged class KP_r.
Keywords: L^1- convergence; Integrability; Modified Sums; Dirichlet Kernel
Mathematics Subject Classifications: 42A20; 42A32

1 Introduction

Take a look at sine & cosine series

$$\sum_{\kappa=1}^{\infty} c_\kappa^* \sin \kappa y \tag{1.1}$$

$$\frac{c_0^*}{2} + \sum_{\kappa=1}^{\infty} c_\kappa^* \cos \kappa y \tag{1.2}$$

and these equations collectively written as

$$\sum_{\kappa=1}^{\infty} c_\kappa^* \psi y \tag{1.3}$$

where ψy is $\sin \kappa y$ or $\cos \kappa y$ respectively.
η^{th} sum of $\sum_{\kappa=1}^{\infty} c_\kappa^* \psi y$ is represented as $S_\eta(y)$. So $\lim_{\eta\to\infty} S_\eta(y) = Z(y)$.
Kano's[1] outcome is popularly known as sequence $\{c_\kappa^*\}$ fulfilling $\{c_\kappa^*\} \to 0$ as $\kappa \to \infty$ & $\sum_{\kappa=1}^{\infty} \kappa^2 |\Delta^2 \left(\frac{c_\kappa^*}{\kappa}\right)| < \infty$ then $\sum_{\kappa=1}^{\infty} c_\kappa^* \sin \kappa y$ and $\frac{c_0^*}{2} + \sum_{\kappa=1}^{\infty} c_\kappa^* \cos \kappa y$ are known to us as Fourier Series.

Definitions:

Convex Sequence: $\{c_\tau^*\}$ is called a convex sequence(seq.) satisfying

$$\Delta^2 c_\tau^* \geq 0, \quad \text{where} \quad \Delta c_\tau^* = c_\tau^* - c_{\tau+1}^* \quad \text{and} \quad \Delta^2 c_\tau^* = \Delta c_\tau^* - \Delta c_{\tau+1}^*.$$

Quasi-Convex Sequence([2],Vol.2, page 204): A seq. $\{c_\tau^*\}$ is called quasi-convex satisfying

$$\sum_{\tau=1}^{\infty} (\tau+1)|\Delta^2 c_\tau^*| < \infty.$$

Sequence $\{c_\tau^*\}$ is known as generalised quasi-convex satisfying

$$\sum_{\tau=1}^{\infty} \tau^{\varkappa} |\Delta^2 c_\tau^*| < \infty : \varkappa = 0, 1, 2, ...$$

'S' Class([4]: sequence $\{c_\tau^*\}$ follow class S by satisfying $c_\tau^* = 0(1)$, τ monotonically decreasing seq. converging to $0 \to \infty$ and $\exists$ a sequence $\{A_\tau^*\}$ s.t.
(a) A_τ^* is monotonically decreasing seq. converging to 0, as$\tau \to \infty$, (b) $\sum\limits_{\tau=0}^{\infty} A_\tau^* < \infty$,
(c) $|\Delta c_\tau^*| \leq A_\tau^* \quad \forall \quad \tau$.
Convergence in L^1-norm: The series L^1-converges in (0,π) if $||f^* - S_\tau^*|| = o(1), \tau \to \infty$.
Young[5] began to work on this issue in 1913 by examining a class of convex seq., which was followed by Kolmogorov[6] in 1923 by addressing a general class of quasi-convex seq.Then Telyakovskii[4] analysed Sidon's significantly weaker class S rather than the previously defined classes for L^1- normed convergence(cgs.) of trigonometric series. Following theorems are famous about the L^1- normed cgs. of Fourier series:

Theorem 1.1:[2], Vol.2, page 204

If $\{c_\kappa^*\}$ is monotonically decreasing and $\{c_\kappa^*\}$ is convex/quasi-convex seq. , then necessary & sufficient condition for L^1-normed convergence of $\frac{c_0^*}{2} + \sum_{\kappa=1}^{\infty} c_\kappa^* \cos \kappa y$ is $c_\kappa^* \log \kappa = o(1) \quad \kappa \to \infty$.
Telyakovsk˘ii generalised Theorem 1.1 for expression (1.2) where the coefficients of series (1.2) satisfy the requirements of class S[7] as follows:

Theorem 1.2:[4]

When coefficients of $\frac{c_0^*}{2} + \sum_{\kappa=1}^{\infty} c_\kappa^* \cos \kappa y$ satisfying criterion of class S[7] then criterion of its L^1 convergence is that $c_\kappa^* \log \kappa = o(1) \quad \text{as} \quad \kappa \to \infty$

Many writers examined and generalised these findings by examining various generalisations of seq. classes.Recently,the coefficient seq. SJ[8] was introduced to study the integrability and L^1-cgs. of modified cosine and sine sums, which was further generalied by Krasniqi[9]. A contemporary class of Fourier coefficients is formulated in this study as:

Definition 1.3: A monotonically decreasing seq. $\{c_\eta^*\}$ with $c_\eta^* \to 0 \quad \text{as} \quad \eta \to \infty$ is follow a new class KP if $\exists$ a seq. $\{A_\eta^*\}$ satisfying

$$(i) A_\eta^* \downarrow 0 \tag{1.4}$$

$$(ii) \sum \eta A_\eta^* < \infty \tag{1.5}$$

$$(iii) \left| \Delta \left(\frac{c_\eta^*}{\eta^2} \right) \right| \leq \frac{A_\eta^*}{\eta^2} \tag{1.6}$$

Here, coefficient sequence KP_r will be formulated that is enlargement of coefficient sequence KP.
Definition 1.4:: A monotonically decreasing seq. $\{c_\eta^*\}$ with $c_\eta^* \to 0 \quad \text{as} \quad \eta \to \infty$ is from a new class KP_r if $\exists$ seq. $\{A_\eta^*\}$ satisfying

$$(i) A_\eta^* \downarrow 0 \tag{1.7}$$

$$(ii) \sum \eta^{r+1} A_\eta^* < \infty \tag{1.8}$$

$$(iii) \left| \Delta \left(\frac{c_\eta^*}{\eta^2} \right) \right| \leq \frac{A_\eta^*}{\eta^2} \tag{1.9}$$

Obviously, KP = KP_r when r = 0. It is obvious that $KP_{r+1} \subseteq KP_r$, but its reverse does not hold.
Example. Define $b_\eta = \frac{1}{\eta^{r+3}},$ r = 0,1,2,... Firstly we are going to demonstrate that$\{b_\eta\} \notin KP_{r+1}$
As, $b_\eta = \frac{1}{\eta^{r+3}} \to 0 \quad \text{as} \quad \eta \to \infty$.
Let $\exists\ A_\eta = \frac{1}{\eta^{r+3}}, r = 0, 1, 2, 3, ... s.t. \sum\limits_{\eta=1}^{\infty} \eta^{r+2} A_\eta = \sum\limits_{\eta=1}^{\infty} \frac{1}{\eta}$ is divergent, means $\{b_\eta\}$ does not belong to KP_{r+1}.
But, A_η is monotonically decreasing and converging to 0 $\quad \eta \to \infty, \quad \&$
$\sum\limits_{\eta=1}^{\infty} \eta^{r+1} A_\eta^* = \sum\limits_{\eta=1}^{\infty} \frac{1}{\eta^2} < \infty,$
Also $|\Delta(\frac{b_\eta}{\eta^2})| \leq \frac{A_\eta^*}{\eta^2}, \forall \eta.$
Therefore, $\{b_\eta\} \in KP_r$.

3

2 Main Results:

Now we will give proof of the succeeding statement:

Theorem 2.1: If the coefficients of series (1.3) meet the class KP criteria, then it will be a Fourier series.

Explanation

$$\sum_{\kappa=1}^{\infty}\kappa^2\left|\Delta^2\left(\frac{c_\kappa^*}{\kappa}\right)\right| = \sum_{\kappa=1}^{\infty}\kappa^2\left|\Delta\left(\frac{c_\kappa^*}{\kappa}\right)-\Delta\left(\frac{c_{\kappa+1}^*}{\kappa+1}\right)\right|$$

$$= \sum_{\kappa=1}^{\infty}\kappa^2\left|\frac{c_\kappa^*}{\kappa}-\frac{c_{\kappa+1}^*}{\kappa+1}-\frac{c_{\kappa+1}^*}{\kappa+1}+\frac{c_{\kappa+2}^*}{\kappa+2}\right|$$

$$\left\{\begin{array}{ll} c_{\kappa+2}^*<c_{\kappa+1}^* \quad and \quad \kappa+2>\kappa+1 \quad \text{therefore} & \dfrac{1}{\kappa+2}<\dfrac{1}{\kappa+1} \\ & \Rightarrow \dfrac{c_{\kappa+2}^*}{\kappa+2}<\dfrac{c_{\kappa+1}^*}{\kappa+1} \end{array}\right\}$$

$$\leq \sum_{\kappa=1}^{\infty}\kappa^2\left|\frac{c_\kappa^*}{\kappa}-\frac{c_{\kappa+1}^*}{\kappa+1}\right|$$

$$= \sum_{\kappa=1}^{\infty}\kappa^2\left|\kappa\frac{c_\kappa^*}{\kappa^2}-(\kappa+1)\frac{c_{\kappa+1}^*}{(\kappa+1)^2}\right|$$

$$< \sum_{\kappa=1}^{\infty}\kappa^3\left|\frac{c_\kappa^*}{\kappa^2}-\frac{c_{\kappa+1}^*}{\kappa+1^2}\right|$$

$$= \sum_{\kappa=1}^{\infty}\kappa^3\left|\Delta\left(\frac{c_\kappa^*}{\kappa^2}\right)\right|$$

$$\leq \sum_{\kappa=1}^{\infty}\kappa^3\frac{A_\kappa^*}{\kappa^2} \quad \text{by defined class KP of Fourier Coefficients.}$$

$$= \sum_{\kappa=1}^{\infty}\kappa A_\kappa^*<\infty$$

As c_κ^* is null sequence, So by the result given by Kano[1], Theorem 1 holds. In this study, we provide latest improved trigonometric sums.

$$Z_\eta(y) = \frac{c_0^*}{2}+\sum_{\kappa=1}^{\eta}\left[\sum_{j=\kappa}^{\eta}\Delta\left(\frac{c_j^*\cos jy}{j^2}\right)\right]\kappa^2,$$

$$r_\eta(y) = \sum_{\kappa=1}^{\eta}\left[\sum_{j=\kappa}^{\eta}\Delta\left(\frac{c_j^*\sin jy}{j^2}\right)\right]\kappa^2.$$

4

Also investigated their L^1-convergence following the newly established class KP of coefficient sequences

Theorem 2.2: Suppose that coefficients of series (1.3) follow class KP, then

$$\lim_{\eta\to\infty} Z_\eta(y) = Z(y), exists \quad for \quad y \in (o,\pi] \tag{2.2.1}$$

$$Z(y) \in L^1(0,\pi] \tag{2.2.2}$$

$$||Z(y) - S_\eta(y)|| = o(1), \eta \to \infty \tag{2.2.3}$$

Theorem 2.3: If coefficients of a sequence (1.3) are from a class KPr, then

$$\lim_{\eta\to\infty} Z^r{}_\eta(y) = Z^r(y), exists \quad for \quad y \in (o,\pi] \tag{2.3.1}$$

$$Z^r(y) \in L^1(0,\pi], \qquad (r = 0,1,2,...) \tag{2.3.2}$$

$$||Z^r(y) - S^r{}_\eta(y)|| = o(1), \eta \to \infty. \tag{2.3.3}$$

3 Lemmas:

The subsequent lemmas are required to prove our main results.

Lemma 3.1[3]

Let $\eta \geq 1$ & $r \in \mathbb{Z}^+ \cup 0$, y $\in$ [s,π] So $|\tilde{D}^r_\eta(y)| \leq C_s \frac{\eta^r}{y}$ Where C_s is +ve constant rely upon s, $0<s<\pi$ & $\tilde{D}^r_\eta(y)$ is conjugate Dirichlet kernel.

Lemma 3.2[4]

Suppose $\{c^*_\eta\}$ is a sequence of $\Re$ s.t. $|c^*_\eta| \leq 1$ forall η. So the relation

$$\int_{\frac{\pi}{\eta+1}}^{\pi} |\sum_{\kappa=0}^{\eta} c^*_\kappa \tilde{D}_\kappa(y)|dy \leq N(\eta+1)$$

exists, where N is perfectly constant.
By Bernstein's inequality,

$$\int_{\frac{\pi}{\eta+1}}^{\pi} |\sum_{\kappa=0}^{\eta} c^*_\kappa \tilde{D}^r_\kappa(y)|dx \leq N(\eta+1)^{s+1} \quad \text{for s = 0,1,2,...}$$

lemma 3.3[3]

$||D^s_\eta(y)||_{L^1} = o(\eta^s \log\eta) + o(\eta^s)$, $s = 0,1,2,....$, and $D^r_\eta(y)$ shows the r^{th} differentials of Dirichlet Kernel.

4 Proof of Main results:

4.1 Solution of theorem 2.1:

We will just show the evidence for cosine sums here, while the argument for sine sums will be shown on parallel paths.
To prove (2.2.1), we notice that

$$\begin{aligned} Z_\eta(y) &= \frac{c_0^*}{2} + \sum_{\kappa=1}^{\eta}\left[\sum_{j=\kappa}^{\eta}\Delta\left(\frac{c_j^*\cos jy}{j^2}\right)\right]\kappa^2 \\ &= \frac{c_0^*}{2} + \sum_{\kappa=1}^{\eta}\left[\sum_{j=\kappa}^{\eta}\left(\frac{c_j^*\cos jy}{j^2} - \frac{c_{j+1}^*\cos(j+1)y}{(j+1)^2}\right)\right]\kappa^2 \\ &= \frac{c_0^*}{2} + \sum_{\kappa=1}^{\eta} c_\kappa^*\cos\kappa y - \sum_{\kappa=1}^{\eta}\kappa^2\frac{c_{\eta+1}^*\cos(\eta+1)y}{(\eta+1)^2} \\ &= S_\eta(y) - \frac{c_{\eta+1}^*\cos(\{\eta+1\}y)\eta(\eta+1)(2\eta+1)}{6(\eta+1)^2} \end{aligned}$$

$$\lim_{\eta\to\infty} Z_\eta(y) = \lim_{\eta\to\infty} S_\eta(y) - \lim_{\eta\to\infty}\frac{c_{\eta+1}^*\eta(2\eta+1)\cos((\eta+1)y)}{6(\eta+1)}$$

Since $\cos(\eta+1)y$ is bounded in $(0,\pi]$ and $\lim_{\eta\to\infty}\frac{2\eta+1}{\eta+1} = 2$ and

$$\begin{aligned} \eta\left|c_\eta^*\right| = \frac{\eta^3 c_\eta^*}{\eta^2} &= \eta^3\sum_{\kappa=\eta}^{\infty}\left|\Delta\left(\frac{c_\kappa^*}{\kappa^2}\right)\right| \\ &\le \sum_{\kappa=\eta}^{\infty}\kappa^3\left|\Delta\left(\frac{c_\kappa^*}{\kappa^2}\right)\right| \\ &\le \sum_{\kappa=\eta}^{\infty}\kappa^3\frac{A_\kappa^*}{\kappa^2} = \sum_{\kappa=\eta}^{\infty}\kappa A_\kappa^* = 0(1) \\ &as\quad \eta\to\infty \end{aligned}$$

$\{if \sum c_\eta^*$ is convergent then $\lim_{\eta\to\infty} c_\eta^* = 0\}$

$$\text{So,}\quad \lim_{\eta\to\infty} Z_\eta(y) = \lim_{\eta\to\infty} S_\eta(y) = Z(y)\quad where$$

$$\begin{aligned} Z(y) &= \frac{c_0^*}{2} + \lim_{\eta\to\infty}\sum_{\kappa=1}^{\eta} c_\kappa^*\cos\kappa y \\ &= \lim_{\eta\to\infty} Z_\eta(y) = \lim_{\eta\to\infty} S_\eta(y) \\ &= \lim_{\eta\to\infty}(\frac{c_0^*}{2} + \sum_{\kappa=1}^{\eta} c_\kappa^*\cos\kappa y) \end{aligned}$$

Now $\lim_{\eta\to\infty}(\sum_{\kappa=1}^{\eta} c_\kappa^* \cos\kappa y)$

$$
\begin{aligned}
&= \lim_{\eta\to\infty}(\sum_{\kappa=1}^{\eta} \frac{c_\kappa^*}{\kappa^2}\kappa^2 \cos\kappa y) \\
&= \lim_{\eta\to\infty}(\sum_{\kappa=1}^{\eta-1} \Delta(\frac{c_\kappa^*}{\kappa^2}(-D_\kappa^{''}(y)) + \frac{c_\eta^*}{\eta^2}(-D_\eta^{''}(y))) \\
&= \sum_{\kappa=1}^{\infty} \Delta(\frac{c_\kappa^*}{\kappa^2})(-D_\kappa^{''}(y)) \\
&\le \sum_{\kappa=1}^{\infty} \Delta(\frac{A_\kappa^*}{\kappa^2})(-D_\kappa^{''}(y))
\end{aligned}
$$

According to the provided hypothesis & lemma 1, $\sum_{\kappa=1}^{\infty} \Delta(\frac{A_\kappa^*}{\kappa^2})(-D_\kappa^{''}(y))$ converges. Therefore Z(y)exists for y $\in$ (0,π]
This brings the proof of (2.2.1).

$$
\begin{aligned}
\text{Now}||Z(y) - Z_\eta(y)|| &= \int_0^\pi |Z(y) - Z_\eta(y)|dy \\
&= \int_0^\pi |\sum_{\kappa=\eta+1}^{\infty} c_\kappa^* \cos\kappa y + \frac{\eta(2\eta+1)c_{\eta+1}^* \cos(\eta+1)y}{6(\eta+1)}|dy \\
&= \lim_{m\to\infty}\int_0^\pi |\sum_{\kappa=\eta+1}^{m} \frac{c_\kappa^*\kappa^2\cos\kappa y}{\kappa^2} + \frac{\eta(2\eta+1)c_{\eta+1}^* \cos(\eta+1)y}{6(\eta+1)}|dy
\end{aligned}
$$

We obtain by employing Abel's Transformation

$$
\begin{aligned}
&= \int_0^\pi |\sum_{\kappa=\eta+1}^{\infty} \Delta\left(\frac{c_\kappa^*}{\kappa^2}\right)\left(-D_\kappa^{''}(y)\right) + \frac{c_{\eta+1}^* D_\eta^{''}(y)}{(\eta+1)^2} \\
&+ \frac{\eta(2\eta+1)c_{\eta+1}^* \cos(\eta+1)y}{6(\eta+1)}|dy \\
&\le \int_0^\pi |\sum_{\kappa=\eta+1}^{\infty} \Delta\left(\frac{c_\kappa^*}{\kappa^2}\right)\left(-D_\kappa^{''}(y)\right)|dy + \int_0^\pi |\frac{c_{\eta+1}^* D_\eta^{''}(y)}{(\eta+1)^2}|dy \\
&+ \int_0^\pi |\frac{\eta(2\eta+1)c_{\eta+1}^* \cos(\eta+1)y}{6(\eta+1)}|dy \\
&= (i) + (ii) + (iii)
\end{aligned}
$$

Evidence of part (i)

$$
\int_0^\pi |\sum_{\kappa=\eta+1}^{\infty} \Delta\left(\frac{c_\kappa^*}{\kappa^2}\right)\left(-D_\kappa^{''}(y)\right)|dy = \int_0^\pi |\sum_{\kappa=\eta+1}^{\infty} \frac{\frac{A_\kappa^*}{\kappa^2}\Delta\left(\frac{c_\kappa^*}{\kappa^2}\right)\left(-D_\kappa^{''}(y)\right)}{\frac{A_\kappa^*}{\kappa^2}}|dy
$$

7

Implementing Abel's Transformation Once More

$$= \int_0^{\pi} |\sum_{\kappa=\eta+1}^{\infty} \Delta \frac{A^*_\kappa}{\kappa^2}) \sum_{j=1}^{\kappa} \frac{\Delta \frac{c^*_j}{j^2}}{(\frac{A_j}{j^2})} (-D''_j(x))|dy$$

$$\leq \sum_{\kappa=\eta+1}^{\infty} \Delta \left(\frac{A^*_\kappa}{\kappa^2}\right) \int_0^{\pi} |\sum_{j=1}^{\kappa} \left(\frac{\Delta\left(\frac{c^*_j}{j^2}\right)}{\frac{A^*_j}{j^2}}\right) \left(D''_j(y)\right) |dy$$

Now by given assumption

$$\leq \sum_{\kappa=\eta+1}^{\infty} \Delta \left(\frac{A^*_\kappa}{\kappa^2}\right) M(\kappa+1)^3$$

$$= o\left(\sum_{\kappa=\eta+1}^{\infty} (\kappa+1)^3 \Delta \left(\frac{A^*_\kappa}{\kappa^2}\right)\right)$$

$$= o(1) \quad \text{as} \quad \{c^*_\kappa\} \in \quad \text{new defined class.}$$

Validation of (ii) component

$$\frac{c^*_{\eta+1}}{(\eta+1)^2} \int_0^{\pi} |D''_\eta(y)|dy = \frac{c^*_{\eta+1}}{(\eta+1)^2} \left(\frac{4}{\pi}(\eta^2 \log \eta) + O(\eta^2)\right)$$

$$\leq c^*_{\eta+1} \left(\frac{4}{\pi} \frac{\eta^2 \log \eta}{(\eta+1)^2} + \frac{1}{(\eta+1)^2} o(\eta^2)\right)$$

$$\leq c^*_{\eta+1} \left(\frac{4}{\pi} \frac{\eta^2 \log \eta}{(\eta+1)^2} + o(1)\right)$$

$$= o\left(c^*_{\eta+1} \log \eta\right)$$

Now $\log \eta \leq \eta \quad \forall \quad \eta \geq 1$
And $\eta c^*_\eta = o(1) \quad as \quad \eta \to \infty$ as already proved above.
Proof of (iii)part
(iii) part is equal to $o(\eta c^*_{\eta+1})$ which is equal to $o(1)$ as $\eta \to \infty$.
Therefore $||Z(y) - Z_\eta(y)|| = o(1)$ as $\eta \to \infty$
Therefore $Z(y) \in L^1(0, \pi]$
This concludes (2.2.2).
Now we shall provide evidence of (2.2.3)

$$||Z - S_\eta|| = ||Z - Z_\eta + Z_\eta - S_\eta||$$

$$\leq ||Z - Z_\eta|| + ||Z_\eta - S_\eta||$$

$$= ||Z - Z_\eta|| + ||\frac{\eta(2\eta+1)}{6(\eta+1)} c^*_{\eta+1} \cos(\eta+1)y||$$

$$\leq ||Z - Z_\eta|| + \frac{\eta(2\eta+1)}{6(\eta+1)} c^*_{\eta+1} \int_0^{\pi} |\cos(\eta+1)y|dy$$

$$\to \quad o(1) \quad as \quad \eta \to \infty$$

8

by employing the assertion (2.2.1) and (2.2.2). This brings the proof of (2.2.3) to a close. Apparently theorem 2 is developed for feeble class than class S, yet conclusions are produced for L^1 -convergence by not employing condition like $c^*_\eta \log\eta = o(1), \quad as \quad \eta\to\infty$.

4.2 Explanation of theorem 2.3:

We will just show the evidence for cosine sums here, while the argument for sine sums will be shown on parallel paths.

$$Z_\eta(y) = S_\eta(y) - \frac{c^*_{\eta+1}\cos((\eta+1)y)(\eta)(2\eta+1)}{6(\eta+1)}$$

$$Z^r_\eta(y) = S^r{}_\eta(y) - \frac{c^*_{\eta+1}\cos(((\eta+1)y)+r\frac{\pi}{2})(\eta)(2\eta+1)(\eta+1)^r}{6(\eta+1)}$$

Since A_κ is monotonically decreasing and converging to 0 as $\kappa\to\infty$ & $\sum\limits_{\kappa=1}^{\infty}\kappa^{r+1}A_\kappa<\infty$,
So, we got $\kappa^{r+2}A_\kappa\to 0$, as $\kappa\to\infty$ and

$$\eta^{r+1}c^*_\eta = \eta^{r+3}\sum_{\kappa=\eta}^{\infty}|\Delta(\frac{a_\kappa}{\kappa^2})| \le \sum_{\kappa=\eta}^{\infty}\kappa^{r+3}|\Delta(\frac{c^*_\kappa}{\kappa^2})| \le \sum_{\kappa=\eta}^{\infty}\kappa^{r+3}(\frac{A^*_\kappa}{\kappa^2}) = o(1), \eta\to\infty. \tag{4.2.1}$$

As $\cos((\eta+1)y+r\frac{\pi}{2})$ is finite in (0,π]. So,

$$\begin{aligned} z^r(y) &= \lim_{\eta\to\infty} z_\eta{}^r(y) \\ &= \lim_{\eta\to\infty} S_\eta{}^r(y) \\ &= \lim_{\eta\to\infty}(\sum_{\kappa=1}^{\eta}\kappa^r c^*_\kappa\cos(\kappa y + r\frac{\pi}{2})) \end{aligned}$$

After using Abel's Transformation, obtained as

$$\begin{aligned} \lim_{\eta\to\infty}(\sum_{\kappa=1}^{\eta}\kappa^r c^*_\kappa\cos(\kappa y + r\frac{\pi}{2})) &= \lim_{\eta\to\infty}[\sum_{\kappa=1}^{\eta-1}\Delta(\frac{c^*_\kappa}{\kappa^2})(-D^{r+2}{}_\kappa(y)) + \frac{c^*_\eta}{\eta^2}D^{r+2}{}_\eta(y)] \\ &= \sum_{\kappa=1}^{\infty}\Delta(\frac{c^*_\kappa}{\kappa^2})(-D^{r+2}{}_\kappa(y)) + \lim_{\eta\to\infty}\frac{c^*_\eta}{\eta^2}D^{r+2}{}_\eta(y) \\ &\le \sum_{\kappa=1}^{\infty}\frac{A^*_\kappa}{\kappa^2}(-D^{r+2}{}_\kappa(y)) + \lim_{\eta\to\infty}\frac{c^*_\eta}{\eta^2}D^{r+2}{}_\eta(y) \end{aligned}$$

Using the provided assumptions, lemma 1 & (4.2.1), the series $\sum\limits_{\kappa=1}^{\infty}\frac{A^*_\kappa}{\kappa^2}(-D^{r+2}{}_\kappa(y))$ converges.

So, the limit $z^r(y)$ exists for $y \in (0,\pi]$ and (2.3.1) follows.
Take the following consideration to establish (2.3.2).

$$\begin{aligned}
z^r(y) - {z_\eta}^r(y) &= \sum_{\kappa=\eta+1}^{\infty} \kappa^r c^*_\kappa \cos(\kappa y + r\frac{\pi}{2}) + \frac{c^*_{\eta+1}\cos{(\eta+1)y + r\frac{\pi}{2}}\eta(2\eta+1)(\eta+1)^r}{6(\eta+1)} \\
&= \sum_{\kappa=\eta+1}^{\infty} \Delta(\frac{c^*_\kappa}{\kappa^2})(-{D_\kappa}^{r+2}(y)) + \frac{c^*_{\eta+1}}{(\eta+1)^2} D^{r+2}_\eta(y) \\
&+ \frac{\eta(\eta+1)^r(2\eta+1)}{6(\eta+1)} c^*_{\eta+1}\cos((\eta+1)y + r\frac{\pi}{2}) \\
&= \sum_{\kappa=\eta+1}^{\infty} \frac{A^*_\kappa}{\kappa^2} \frac{\Delta(\frac{c^*_\kappa}{\kappa^2})}{\frac{A^*_\kappa}{\kappa^2}}(-{D_\kappa}^{r+2}(y)) + \frac{c^*_{\eta+1}}{(\eta+1)^2} D^{r+2}_\eta(y) \\
&+ \frac{\eta(\eta+1)^r(2\eta+1)}{6(\eta+1)} c^*_{\eta+1}\cos((\eta+1)y + r\frac{\pi}{2}) \\
&= \sum_{\kappa=\eta+1}^{\infty} \Delta(\frac{A^*_\kappa}{\kappa^2}) \sum_{j=1}^{\kappa} \frac{\Delta(\frac{c^*_j}{j^2})}{\frac{A^*_j}{j^2}}(-{D_j}^{r+2}(y)) + (\frac{A^*_{\eta+1}}{\eta+1}) \sum_{j=1}^{\eta} \frac{\Delta(\frac{c^*_j}{j^2})}{\frac{A^*_j}{j^2}}(-{D_j}^{r+2}(y)) \\
&+ \frac{c^*_{\eta+1}}{(\eta+1)^2} D^{r+2}_\eta(y) + \frac{\eta(\eta+1)^r(2\eta+1)}{6(\eta+1)} c^*_{\eta+1}\cos((\eta+1)y + r\frac{\pi}{2})
\end{aligned}$$

After applying the lemma 2 & lemma 3

$$\begin{aligned}
||z^r(y) - {z_\eta}^r(y)|| &\le \sum_{\kappa=\eta+1}^{\infty} \Delta(\frac{A^*_\kappa}{\kappa^2}) \int_0^\pi |\sum_{j=1}^{\kappa} \frac{\Delta(\frac{c^*_j}{j^2})}{\frac{A^*_j}{j^2}}(-{D_j}^{r+2}(y))|dy \\
&+ (\frac{A^*_{\eta+1}}{\eta+1}) \int_0^\pi |\sum_{j=1}^{\eta} \frac{\Delta(\frac{c^*_j}{j^2})}{\frac{A^*_j}{j^2}}(-{D_j}^{r+2}(y))|dy + \int_0^\pi |\frac{c^*_{\eta+1}}{(\eta+1)^2} D^{r+2}_\eta(y)|dy \\
&+ \frac{\eta(\eta+1)^r(2\eta+1)}{6(\eta+1)} |c^*_{\eta+1}| \int_0^\pi |\cos((\eta+1)y + r\frac{\pi}{2})|dy \\
&= O(\sum_{\kappa=\eta+1}^{\infty} \kappa^{r+3}\Delta(\frac{A^*_\kappa}{\kappa^2})) + O(\eta^{r+3}(\frac{A^*_{\eta+1}}{\eta+1^2})) + O(\eta^r c^*_{\eta+1}\log\eta) \\
&+ \frac{\eta(\eta+1)^r(2\eta+1)}{6(\eta+1)} \left| c^*_{\eta+1}| \int_0^\pi |\cos\left((\eta+1)y + r\frac{\pi}{2}\right)\right| dy
\end{aligned}$$

Using the reasoning provided in the explanation of theorem 2, researchers may conclude that $\sum\limits_{\kappa=\eta+1}^{\infty} \kappa^{r+3}\Delta(\frac{A_\kappa}{\kappa^2})$ converges.
$\int_0^\pi |\cos((\eta+1)y + r\frac{\pi}{2})|dy \le \frac{2}{\eta+1}$ and for $\eta \ge 1, \eta^{r+1}c^*_\eta \log\eta \le \eta^{r+2}c^*_\eta = o(1)$ as $\eta \to \infty$. This implies that

$$||z^r(y) - {z_\eta}^r(y)|| = 0(1) \qquad as \qquad \eta \to \infty. \tag{4.2.2}$$

Because,${z_\eta}^r(y)$ is a monomial, so $z^r(y) \in L^1(0,\pi]$ which completes (2.3.2). We are now proceeding on to the evidence of (2.3.3)

$$\begin{aligned}||z^r - {S_\eta}^r|| &= ||z^r - {z_\eta}^r + {z_\eta}^r - {S_\eta}^r|| \\ &\leq ||z^r - {z_\eta}^r|| + ||{z_\eta}^r - {S_\eta}^r|| \\ &= ||z^r - {z_\eta}^r|| + ||\frac{\eta(\eta+1)^r(2\eta+1)}{6(\eta+1)}|c^*_{\eta+1}\cos((\eta+1)y + r\frac{\pi}{2})|| \\ &\leq ||z^r - {z_\eta}^r|| + \frac{\eta(\eta+1)^r(2\eta+1)}{6(\eta+1)}|c^*_{\eta+1}|\int_0^\pi |\cos((\eta+1)y + r\frac{\pi}{2})|dy\end{aligned}$$

Further $||z^r(y) - {z_\eta}^r(y)|| = 0(1) \quad as \quad \eta \to \infty$ by using (1.11),$\int_0^\pi |\cos((\eta+1)y + r\frac{\pi}{2})|dy \leq \frac{2}{\eta+1}$ and c^*_η is a seq. converging to 0,so the (2.3.3)part of theorem 2.3 holds.

Note The scenario r = 0 in main result 2.3 gives output of main result 2.2.

Acknowledgment

Authors are thankful to MRSPTU Bathinda for enabling opportunities in paper preparation process.

References

[1] T.Kano,Coefficients of some trigonometric series,*Jour. Frac. Sci. Shihshu University*,**3**,153-621(968).

[2] N.K.Bary,*A Treatise on Trigonometric Veries* (vol 1 and vol 2),London,1964.

[3] S.Sheng,The extension of the theorems of C.V.Stanojevic and V.B.Stanojevic,*Proc. Amer. Math. Soc.*,**14**,158-60,(1939).

[4] S.A.Telyakovskii,A sufficient condition of Sidon for the integrability of trigonometric series,*Mat.Zametki*,**14**,317-28,(1973).

[5] W.H.Young, On the Fourier series of bounded functions,*Proc. London Math. Soc.*,**12**,41-70,(1993).

[6] A.N. Kolmogorov,Sur l'ordere de grandeur des coefficients de la sėrie de Fourier-Lebsque,*Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*,83-86,(1923).

[7] S.Sidon, Hinreichende Bedingungen f ur den Fourier-Charakter einer trigonometrischen Reihe,*J. London Math. Soc.*,**14**,579-86,(1939).

[8] J.Kaur and S.S. Bhatia,Convergence of new modified trigonometric sums in metric space L, *The journal of Non Linear Sciences and applications*,**1(3)**,179-188,(2008).

[9] Xh.Z. Krasniqi, Further results on L^1-convergence of some modified complex trigonometric sums, *J. Numer. Anal. Approx. Theory*,**44(2)**,180-189,(2015).