Analysis of Tripled System of Fractional Differential Equation using Certain Fixed Points Theorems with Fractional Boundary Condition

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Abstract

This paper presents the tripled system of differential equations of fractional type with fractional integral boundary conditions as well as integer and fractional derivative. Here the Banach fixed points theorem and Scheafer's fixed points theorem are used as a main tool. To justify the results we illustrate some examples.

Key Words and Phrases: Fixed points theorem, Banach fixed point, Fractional differential equations, Fractional integral boundary conditions.
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[1]

1 Introduction

Fractional differential equation are applicable in many streams of science and engineering like as fitting of experimental data, e electromagnetics, physics, viscoelasticity, lectro chemistry, biophysics, blood flow phenomena,porous media,biology, electrical circuits, etc. Therefore compare to models of integer order, fractional order model become more practical and realistic. Thus there has been

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a significant developments in problems of boundary value for the existence and uniqueness of fractional differential equations; see [1, 4, 5, 6, 8, 9, 10, 12]. and the references therein. Many authors have worked on existence and uniqueness of solution of tripled system of fractional differential equations [2, 3, 7, 11, 13, 14]. The tripled systems of fractional differential equation often exits in numerous models such as Chemostats and Microorganism Culturing, Brine Tanks, Irregular Heartbeats, Chemical Kinetics, Lidocaine and Pesticides, Predator Prey etc. [8] study fractional differential equations for Boundary value problems of nonlinear type and include nonlocal and integral boundary condition of fractional type. Inspired by the problem [9],

$$\begin{cases} {}^{C}D^{a_1}x_1(\alpha) = e_1(\alpha, x_2(\alpha), x_3(\alpha)), \alpha \in [0,1] \\ {}^{C}D^{a_2}x_2(\alpha) = e_2(\alpha, x_1(\alpha), x_3(\alpha)), \alpha \in [0,1] \\ {}^{C}D^{a_1}x_3(\alpha) = e_3(\alpha, x_1(\alpha), x_2(\alpha)), \alpha \in [0,1] \\ x_1(0) = x_1'(0) = x_1"(0) = 0, \\ {}^{C}D^{p_1}x_1(1) = \gamma_1(J^{q_1}x_1)(1), \\ x_2(0) = x_2'(0) = x_2"(0) = 0, \\ {}^{C}D^{p_2}x_2(1) = \gamma_2(J^{q_2}x_2)(1) \\ x_3(0) = x_3'(0) = x_3"(0) = 0, \\ {}^{C}D^{p_3}x_3(1) = \gamma_3(J^{q_3}x_3)(1) \end{cases}$$

Where ${}^{C}D^{a_i}$ Caputo fractional derivative with order a_i, J^q represent the Riemann-Liouville fractional integral whose order $a_1, a_2 \in (4,5], p_1, p_2, p_3 \in (0,4]$ $q_1, q_2, q_3 > 0, e_1, e_2, : [0,1] \times R \to R$ are smooth functions and $\gamma_i \neq \frac{\Gamma(q_i+5)}{\Gamma(5-p_i)}, i = 1,2,3.$
Existence and uniqueness of solution for the mentioned above tripled system of nonlinear fractional order differential equations is main focus of the paper.

2 Preliminaries

Firstly we introduce some notation, lemmas and definitions.
Definition 2.1 [6] Caputo derivative whose fractional order is a for smooth function $e : [0, \infty) \to R$ is define as

$${}^{C}D^{a}e(\alpha) = \frac{1}{\Gamma(n-a)} \int_0^\alpha (\alpha - t)^{n-a-1} e^{(n)}(t) dt$$

gives $e(n)(\alpha)$ exist, where [a] represents the integer part of the real number a and Γ is the Euler's Gamma function.
Definition 2.2 [12] Riemann-Liouville fractional integral of the order $a > 0$ for a smooth function

$$J^{a}e(\alpha) = \frac{1}{\Gamma(a)} \int_0^\alpha (\alpha - t)^{a-1} e(t) dt.$$

Lemma 2.1 [2] Let $f, g > 0$ and $e \in L_1[a,b]$ then $J^f J^g e = J^{f+g} e$

Lemma 2.2 [**2**] If e is continuous and $n \geq 0$, then

$$^{C}D^{n}J^{n}e = e$$

It follows from Lemmas 2.1 and 2.2 that if e is continuous and $\gamma > a$, then $^{C}D^{a}e = J^{\gamma-a}e$.
Lemma 2.3 [**2**] Let $\gamma > -1$ and $n > 0$. Then

$$J^{n}z^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(n+\gamma+1)}z^{n+\gamma}$$

Lemma 2.4 [**2**] Let $\gamma \geq 0$ and $m = [n]+1$, then

$$^{C}D^{n}x^{\gamma} = \begin{cases} 0, & \text{if}\gamma \in 0,1,2,\dots m-1 \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n)}(z-a)^{\gamma-n}, & \text{if}\gamma \epsilon N \text{and} \gamma \geq m \\ & \text{or}\gamma \notin N, \gamma > m-1 \end{cases}$$

Lemma 2.5 [**7**] Let $a > 0$ then,

$$J^{a\,C}D^{a}V(\alpha) = V(\alpha) + h_0 + h_1\alpha + h_2\alpha^2 + \cdots + h_{n-1}\alpha^{n-1}$$

for some $h_i \in \mathbb{R}, i = 0,1,2,\dots n-1$, n is smallest integer grater than or equal to a.

3 Supporting Result

In this part, we establish the result required in our main proofs.
Lemma 3.1 Let $y \in H([0,1],\mathbb{R})$ and $\gamma \neq \frac{\Gamma(q+5)}{\Gamma(5-p)}$. Then the problem

$$\begin{cases} ^{C}D^{a}x(\alpha) = y(\alpha)\alpha \in [0,1] \\ x(0) = x'(0) = x''(0) = x'''(0) = 0,^{C}D^{p}x(1) = \gamma(J^{q}x)(1) \end{cases} \tag{3.1}$$

has unique solution

$$\begin{aligned} x(\alpha) &= \frac{1}{\Gamma a}\int_0^{\alpha}(\alpha-t)^{\alpha-1}y(t)dt \\ -&\frac{\gamma\Gamma(5-p)\Gamma(5+q)\alpha^3}{24\Gamma(a-p)[\gamma\Gamma(5-p)-\Gamma(q+4)}\int_0^1(1-t)^{q+a-1}y(t)dt \\ +&\frac{\Gamma(5-p)\Gamma(q+5)\alpha^3}{24\Gamma(a-p)[\gamma\Gamma(5-p)-\Gamma(q+5)]}\int_0^1(1-t)^{a-p-1}y(t)dt \end{aligned} \tag{3.2}$$

Proof: From Lemma 2.2, (3.2) is similar to

$$x(\alpha) = J^{a}y(\alpha) - h_0 - h_1\alpha - h_2\alpha^2 - h_3\alpha^3 - h_4\alpha^4 \tag{3.3}$$

for some $h_i \in \mathbb{R}, i$ from $0to4$.
from $x(0) = 0$ it follows $h_0 = 0$ also $x'(0) = 0 \implies h_1 = 0, x''(0) = 0 \implies h_2 = 0$ and $x'''(0) = 0 \implies h_3 = 0$. Thus (3.3) becomes

$$x(\alpha) = J^a y(\alpha) - h_4 \alpha^4 \tag{3.4}$$

Now

$$({}^C D^p x) = J^{a-p} y(\alpha) - c_4 \frac{\Gamma 5}{\Gamma(5-p)} \alpha^{4-p}$$
$$J^q x(\alpha) = J^{p+q} y(\alpha) - c_4 \frac{\Gamma 5}{\Gamma(5+q)} \alpha^{4+q}$$

From the boundary condition,

$$({}^C D^p x)(1) = (J^q x)(1)$$
$$\implies J^{a-p} y(1) - c_4 \frac{\Gamma 5}{\Gamma(5-p)} = \gamma J^{p+q} y(1) - c_4 \frac{\Gamma 5}{\Gamma(5+q)}$$
$$\implies c_4 \left[\frac{\Gamma 5(\gamma \Gamma(5-p) - \Gamma(5+q))}{\Gamma(5+q)\Gamma(5-q)} \right] = \gamma J^{p+q} y(1) - J^{a-p} y(1)$$
$$\implies c_4 = \frac{\Gamma(5-q)\Gamma(5+q)}{24(\gamma\Gamma(5-p) - \Gamma(5+q)} \left[\gamma J^{p+q} y(1) - J^{a-p} y(1) \right].$$

On substituting the value of c_4 in (3.4) we find solution (3.2). It clear from lemma (3) that solution of the tripled system (1.1) is given by the integral equation,

$$x_1(\alpha) = \frac{1}{\Gamma a_1} \int_0^\alpha (\alpha - t)^{a_1 - 1} e_1(t, x_2(t), x_3(t)) dt$$
$$- \frac{\gamma_1 R_1 \alpha^3}{24\Gamma(q_1 + a_1)} \int_0^1 (1-t)^{q_1 + a_1 - 1} e_1(t, x_2(t), x_3(t)) dt$$
$$+ \frac{R_1 \alpha^3}{\Gamma(a_1 - p_1)} \int_0^1 (1-t)^{a_1 - p_1 - 1} e_1(t, x_2(t), x_3(t)) dt$$

$$x_2(\alpha) = \frac{1}{\Gamma a_2} \int_0^\alpha (\alpha - t)^{a_2 - 1} e_2(t, x_2(t), x_3(t)) dt$$
$$- \frac{\gamma_2 R_2 \alpha^3}{24\Gamma(q_2 + a_2)} \int_0^1 (1-t)^{q_2 + a_2 - 1} e_2(t, x_2(t), x_3(t)) dt$$
$$+ \frac{R_2 \alpha^3}{\Gamma(a_2 - p_2)} \int_0^1 (1-t)^{a_2 - p_2 - 1} e_2(t, x_2(t), x_3(t)) dt$$

$$x_3(\alpha) = \frac{1}{\Gamma a_3} \int_0^\alpha (\alpha - t)^{a_3 - 1} e_3(t, x_2(t), x_3(t)) dt$$
$$- \frac{\gamma_3 R_3 \alpha^3}{24\Gamma(q_3 + a_3)} \int_0^1 (1-t)^{q_3 + a_3 - 1} e_3(t, x_2(t), x_3(t)) dt$$
$$+ \frac{R_3 \alpha^3}{\Gamma(a_3 - p_3)} \int_0^1 (1-t)^{a_3 - p_3 - 1} e_3(t, x_2(t), x_3(t)) dt$$

Where

$$R_i = \frac{\Gamma(5-p_i)\Gamma(q_i+5)}{\gamma_i\Gamma(5-p_i) - \Gamma(q_i+4)},$$

for i =1, 2, 3.

Let $X = H[0,1]$ then $(X, \|.\|_X)$ is Banach space fit out with the norm.

$$\|X\|_X = (sup|x(\alpha)|\colon \alpha \in [0,1])$$

Let $B = X \times X \times X$ then $(B, \|.\|_B)$ is also a Banach space equipped with the norm.

$$\|(x_1, x_2, x_3)\|_B = \|x_1\|_X + \|x_2\|_X + \|x_3\|_X$$

Let us define an operation $F : B \to B$

$$f(x_1, x_2, x_3)(\alpha) = (f_1 x_2(\alpha)x_3(\alpha), f_2 x_1(\alpha)x_3(\alpha), f_3 x_1(\alpha)x_2(\alpha)$$

Where

$$\begin{aligned}
f_1 x_2(\alpha)x_3(\alpha) &= \frac{1}{\Gamma a_1}\int_0^\alpha (\alpha-t)^{a_1-1} e_1(t, x_2(t), x_3(t))dt \\
&- \frac{\gamma_1 R_1 \alpha^3}{24\Gamma(q_1+a_1)}\int_0^1 (1-t)^{q_1+a_1-1} e_1(t, x_2(t), x_3(t))dt \\
&+ \frac{R_1\alpha^3}{\Gamma(a_1-p_1)}\int_0^1 (1-t)^{a_1-p_1-1} e_1(t, x_2(t), x_3(t))dt \\
f_2 x_1(\alpha)x_3(\alpha) &= \frac{1}{\Gamma a_2}\int_0^\alpha (\alpha-t)^{a_2-1} e_2(t, x_2(t), x_3(t))dt \\
&- \frac{\gamma_2 R_2 \alpha^3}{24\Gamma(q_2+a_2)}\int_0^1 (1-t)^{q_2+a_2-1} e_2(t, x_2(t), x_3(t))dt \\
&+ \frac{R_2\alpha^3}{\Gamma(a_2-p_2)}\int_0^1 (1-t)^{a_2-p_2-1} e_2(t, x_2(t), x_3(t))dt \\
f_3 x_1(\alpha)x_2(\alpha) &= \frac{1}{\Gamma a_3}\int_0^\alpha (\alpha-t)^{a_3-1} e_3(t, x_2(t), x_3(t))dt \\
&- \frac{\gamma_3 R_3 \alpha^3}{24\Gamma(q_3+a_3)}\int_0^1 (1-t)^{q_3+a_3-1} e_3(t, x_2(t), x_3(t))dt \\
&+ \frac{R_3\alpha^3}{\Gamma(a_3-p_3)}\int_0^1 (1-t)^{a_3-p_3-1} e_3(t, x_2(t), x_3(t))dt
\end{aligned}$$

We see fixed point of F are solution of tripled system(1.1). To simplify and our convenience we put.

$$\Lambda_i = \frac{1}{\Gamma(a_i+1)} + \frac{\gamma|R_i|}{24\Gamma(q_i+a_i+1)} + \frac{|R_i|}{24\Gamma(a_i-p_i+1)}$$

for i = 1, 2, 3

4 Main Theorem

We will use well know Banach fixed points theorem to prove our first result.
Theorem 4.1 Suppose that $\gamma_i \neq \frac{\Gamma(q_i+5)}{\Gamma(5-p_i)}, i = 1, 2, 3$ and the following hypothesis holds. (H 1) Assume that a non-negative continuous functions $k_i \in C[0,1], i = 1, 2$ exist such that

$$|e_i(\alpha, y_1) - e_i(\alpha, y_2)| \leq k_i(\alpha)|y_1 - y_2|$$
$$|e_i(\alpha, y_2) - e_i(\alpha, y_3)| \leq k_i(\alpha)|y_2 - y_3|$$
$$|e_i(\alpha, y_3) - e_i(\alpha, y_1)| \leq k_i(\alpha)|y_3 - y_1|$$
$$\forall y_1, y_2, y_3 \in \mathbb{R} \text{and} \forall \alpha \in [0,1]$$

with $I_i = \sup k_i(\alpha) i = 1, 2, 3$ $\alpha \in [0,1]$ and $I = \underset{i}{max} I_i$ and if $I(\eta_1 + \eta_2 + \eta_3) < 1$ where $\eta_i, i = 1, 2, 3$ and defined by (7) then on $[0,1]$ the tripled system (1) has a unique. We shall show F is contraction.
Proof. Let $(x_1, x_2, x_3), (x'_1, x'_2, x'_3) \in B$ then $\forall \alpha \in [0,1]$

$$|f_1(x_2)(x_3)(\alpha) - f_1(x'_2)(x'_3)(\alpha)| \leq \frac{1}{\Gamma a_1} \int_0^\alpha (\alpha - t)^{a_1 - 1}$$
$$|e_1(t, x_2(t), x_3(t) - e_1(t, x'_2(t), x'_3(t)|dt + \frac{|R_1|\gamma_1}{24\Gamma(q_1 + a_1)}$$
$$\int_0^1 (1-t)^{q_1 + a_1 - 1} |e_1(t, x_2(t), x_3(t) - e_1(t, x'_2(t), x'_3(t)|dt$$
$$+ \frac{|R_1|}{\Gamma(a_1 - p_1)} \int_0^1 (1-t)^{a_1 - p_1 - 1} |e_1(t, x_2(t), x_3(t)$$
$$-e_1(t, x'_2(t), x'_3(t)|dt$$
$$\leq I\|x_2 x_3 - x'_2 x'_3\| \left[\frac{1}{\Gamma a_1} \int_0^\alpha (\alpha - t)^{a_1 - 1} dt + \frac{|R_1|\gamma_1}{24\Gamma(q_1 + a_1)} \right.$$
$$\left. \int_0^1 (1-t)^{q_1 + a_1 - 1} dt + \frac{|R_1|}{\Gamma(a_1 - p_1)} \int_0^1 (1-t)^{a_1 - p_1 - 1} dt \right]$$
$$\leq \|x_2 x_3 - x'_2 x'_3\| \times \left[\frac{1}{\Gamma a_1} + \frac{|R_1|}{24\Gamma(q_1 + a_1)} + \frac{|R_1|\gamma_1}{\Gamma(a_1 - p_1)} \right]$$

Thus

$$\|f_1(x_2)(x_3) - f_1(x'_2)(x'_3)\| \leq I\eta_1 \|x_2 x_3 - x'_2 x'_3\|_x$$

Similarly

$$\|f_2(x_1)(x_3) - f_2(x'_1)(x'_3) \leq I\eta_2 \|x_1 x_3 - x'_1 x'_3\|$$

and

$$\|f_2(x_1)(x_2) - f_2(x'_1)(x'_2) \leq I\eta_2 \|x_1 x_2 - x'_1 x'_2\|$$

$$\|f(x_1,x_2,x_3)-f(x'_1,x'_2,x'_3)\|_B\leq$$
$$I(\eta_1+\eta_2+\eta_3)\|(x_1,x_2,x_3)-(x'_1,x'_2,x'_3)\|_B$$

As $I(\eta_1+\eta_2+\eta_3)<1$ therefore f is a contradiction and by Banach fixed point result, f must have unique fixed point i.e. the tripled system (1.1) has unique solution.

Theorem 4.2 Assume $\gamma_i \neq \frac{\Gamma(q_i+5)}{\Gamma(5-p_i)}, i=1,2,3$ and the following hypothesis holds.

(H 2) there exist non negative continuous function $l_1,l_2,l_3 \in C[0,1]$ such that $|e_i(\alpha,y)|\leq l_i(\alpha)$ $\forall y\in\mathbb{R}$ and $\forall\alpha\in[0,1]$ with $L_i = \sup\limits_{\alpha\in[0,1]} l_i(\alpha), i=1,2,3.$

Then the tripled system (1.1) defined on [0, 1] has at least one solution

Proof: To prove this result we take help of Schaefer fixed point theorems.

Step-1 F is smooth.

Since e_1,e_2 and e_3 are smooth therefore f is also smooth.

Step-2 Under the mapping f bounded set of B are mapped into bounded sets of B.

Let $\omega_\xi=(x_1,x_2,x_3)\in B; \|(x_1,x_2,x_3)\|_B\leq\xi$

where $\xi>0$ Now for $(x_1,x_2,x_3)\in\omega_\xi$ and $\forall\alpha\in[0,1]$

$$\begin{aligned}|f_1(x_1)(x_2)(x_3)|\leq & \frac{1}{\Gamma a_1}\int_0^\alpha(\alpha-t)^{a_1-1}|e_1(t,x_2(t),x_3(t)|dt\\ &+\frac{|R_1|\gamma_1}{24\Gamma(q_1+a_1)}\int_0^1(1-t)^{q_1+a_1-1}|e_1(t,x_2(t),x_3(t)|dt\\ &+\frac{|R_1|}{\Gamma(a_1-p_1)}\int_0^1(1-t)^{a_1-p_1-1}|e_1(t,x_2(t),x_3(t)|dt\\ &\leq\omega_1\left[\frac{1}{\Gamma a_1}\int_0^\alpha(\alpha-t)^{a_1-1}dt+\frac{|R_1|\gamma_1}{24\Gamma(q_1+a_1)}\right.\\ &\left.\int_0^1(1-t)^{q_1+a_1-1}dt+\frac{|R_1|}{\Gamma(a_1-p_1)}\int_0^1(1-t)^{a_1-p_1-1}dt\right]\\ &\leq\omega_1\left[\frac{1}{\Gamma a_1}+\frac{|R_1|\gamma_1}{24\Gamma(q_1+a_1)}+\frac{|R_1|}{\Gamma(a_1-p_1)}\right]\end{aligned}$$

Thus

$$\|f_1(x_2)(x_3)\|_X\leq\omega_1\eta_1$$

similar

$$\|f_1(x_1)(x_3)\|_X\leq\omega_2\eta_2$$

and

$$\|f_1(x_1)(x_2)\|_X\leq\omega_3\eta_3$$

$$\implies \|f_1(x_1,x_2,x_3)\|_X\leq\omega_1\eta_1+\omega_2\eta_2+\omega_3\eta_3$$

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i.e. $\|f_1(x_1,x_2,x_3)\|_X \leq \infty$ Step-3. $F : B \to B$ is completely continuous operator. Let $(x_1,x_2,x_3) \in \omega_\xi$ and $\alpha_1,\alpha_2,\alpha_3 \in [0,1]$ with $\alpha_1 < \alpha_2 < \alpha_3$, then

$$|f_1(x_2)(\alpha_2) - f_1(x_2)(\alpha_1)| \leq \frac{\omega_1}{\Gamma a_1}\int_0^{\alpha_1}\left[(\alpha_2-t)^{a_1-1} - (\alpha_1-t)^{a_1-1}\right]dt$$

$$+\frac{\omega_1}{\Gamma a_1}\int_0^{\alpha_1}(\alpha_2-t)^{a_1-1} + \frac{\omega_1\gamma_1|R_1|\|\alpha_2^3-\alpha_1^3\|}{24\Gamma(q_1+a_1)}\int_0^1(1-t)^{q_1+a_1-1}dt$$

$$+\frac{\omega_1\gamma_1|R_1|\|\alpha_2^3-\alpha_1^3\|}{24\Gamma(q_2-p_1)}\int_0^1(1-t)^{a_1-p_1-1}dt \leq \frac{\omega_1}{\Gamma(a_1+1)}\left[(\alpha_2-\alpha_1)^{a_1}\right. \quad (4.1)$$

$$\left.+(\alpha_2^{a_1}-\alpha_1^{a_1})\right] + \frac{(\alpha_2-\alpha_1)^{a_1}}{\Gamma(a_1+1)} + \frac{\omega_1\gamma|R_1|\|\alpha_2^3-\alpha_1^3\|}{24\Gamma(q_1+a_1+1)} + \frac{\omega|R_1|\|\alpha_2^3-\alpha_1^3\|}{24\Gamma(a_1-p_1+1)} \quad (4.2)$$

right- hand side tends to zero when $\alpha_1 \to \alpha_2$.
Thus $\|f_1x_2(\alpha_2) - f_1x_2(\alpha_1)\|_X \to 0$ as $\alpha_1 \to \alpha_2$.
Similarly $\|f_2x_1(\alpha_2) - f_2x_1(\alpha_1)\|_X \to 0$ as $\alpha_1 \to \alpha_2$
$\|f_3x_1(\alpha_2) - f_3x_1(\alpha_1)\|_X \to 0$ as $\alpha_1 \to \alpha_2$.
Thus $\|f(x_1,x_2,x_3)(\alpha_2) - f(x_1,x_2,x_3)(\alpha_1)\|_B \to 0$ as $\alpha_1 \to \alpha_2$
Similarly $\|f(x_1,x_2,x_3)(\alpha_3) - f(x_1,x_2,x_3)(\alpha_1)\|_B \to 0$ as $\alpha_1 \to \alpha_3$
Combining step 1 to 3 and by reaction of Arzela - Ascoli theorem, $F : B \to B$ is completely continuous operation.
Step-4
Let

$$\psi = \{(x_1,x_2,x_3) \in B : (x_1,x_2,x_3) = \phi F(x_1,x_2,x_3)$$

for some $\phi \in (0,1)$ we shall show that set ψ is bounded. Let $(x_1,x_2,x_3) \in \psi \implies (x_1,x_2,x_3)(\alpha) = \phi f(x_1,x_2,x_3)(\alpha)$ for some $\phi \in (0,1)$. Then we have

$$\begin{aligned} x_1(\alpha) &= \phi f_1 x_2 x_3(\alpha) \\ x_2(\alpha) &= \phi f_2 x_2 x_3(\alpha) \\ x_3(\alpha) &= \phi f_3 x_2 x_3(\alpha), \forall \alpha \in [0,1] \end{aligned}$$

$$\|x_1(\alpha)\| = |\phi f_1 x_2 x_3(\alpha)| \leq \phi\omega_1\left[\frac{1}{\Gamma a_1}\int_0^\alpha(\alpha-t)^{a_1-1}dt\right.$$

$$+\frac{\gamma_1|R_1|}{24\Gamma(q_1+a_1)}\int_0^1(1-t)^{q_1+a_1-1}dt + \frac{|R_1|}{24\Gamma(a_1-p_1)}$$

$$\left.\int_0^1(1-t)^{a_1-p_1-1}dt\right]$$

$$\leq \omega_1\left[\frac{1}{\Gamma(a_1+1)} + \frac{\gamma_1|R_1|}{24\Gamma(q_1+a_1+1)} + \frac{|R_1|}{24\Gamma(a_1-p_1+1)}\right] \quad (4.3)$$

Thus

$$\|x_1\|_X \leq \omega_1\eta_1$$

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Similarly

$$\|x_2\|_X \leq \omega_2\eta_2$$

and

$$\|x_3\|_X \leq \omega_3\eta_3$$

Hence, we get

$$\begin{aligned}\|(x_1,x_2,x_3)\|_X &\leq \omega_1\eta_1+\omega_2+\omega_3\eta_3\eta_2\\ \|(x_1,x_2,x_3)\|_B &\leq \infty\end{aligned}$$

Thus Scheafer's fixed point result present ϕ is bounded set. f must have minimum one fixed point which is solution of tripled system (1.1).

Example 4.1. Take the following tripled system

$$\begin{cases} {}^C D^{\frac{17}{4}} x_1(\alpha) = \frac{1}{\alpha^2+16}\frac{|x_2(\alpha)x_3(\alpha|)}{1+|x_2(\alpha)x_3(\alpha)|} \\ {}^C D^{\frac{9}{2}} x_2(\alpha) = \frac{1}{\alpha^2+25}\tan^{-1}(x_1(\alpha)x_3(\alpha)), \alpha\in[0,1] \\ {}^C D^{\frac{13}{2}} x_3(\alpha) = \frac{1}{\alpha^2+49}\cot^{-1}(x_1(\alpha)x_3(\alpha)), \alpha\in[0,1] \\ x_1(0)=x_1'(0)=x_1''(0)=0, {}^C D^{\frac{1}{2}}x_1(1)=\frac{15}{16}(J^{\frac{5}{2}}x_1)(1) \\ x_2(0)=x_2'(0)=x_2''(0)=0, {}^C D^{\frac{3}{2}}x_2(1)=\frac{16}{17}(J^{\frac{7}{2}}x_2)(1) \\ x_3(0)=x_3'(0)=x_3''(0), {}^C D^{\frac{4}{3}}x_3(1)=\frac{17}{18}(J^{\frac{9}{2}}x_3)(1) \\ a_1=\frac{17}{4}, p_1=\frac{1}{2}, q_1=\frac{5}{2}, \gamma_1=\frac{15}{16}\neq\frac{\Gamma(q_1+5)}{\Gamma(5-p_1)}=160.875 \\ a_2=\frac{9}{2}, p_2=\frac{3}{2}, q_2=\frac{7}{2}, \gamma_2=\frac{16}{7}\neq\frac{\Gamma(q_2+5)}{\Gamma(5-p_2)}=422.96 \\ a_3=\frac{13}{2}, p_3=\frac{4}{3}, q_3=\frac{9}{2}, \gamma_3=\frac{17}{8}\neq\frac{\Gamma(q_3+5)}{\Gamma(5-p_3)}=4558.125 \end{cases} \tag{4.4}$$

for $\alpha \in [0,1]$ and $y_1, y_2, y_3 \in \mathbb{R}$.

$$|e_i(\alpha,y_1)-e_i(\alpha,y_2)| \leq \frac{1}{\alpha^2+16}|y_1-y_2|$$
$$|e_i(\alpha,y_2)-e_i(\alpha,y_3)| \leq \frac{1}{\alpha^2+25}|y_2-y_3|$$
$$|e_i(\alpha,y_3)-e_i(\alpha,y_1)| \leq \frac{1}{\alpha^2+49}|y_3-y_1|$$

So, we can take $K_1=\frac{1}{\alpha^2+16}, K_2=\frac{1}{\alpha^2+25}, K_3=\frac{1}{\alpha^2+49}$

$$I_1 = \sup_{\alpha\in[0,1]} K_1(\alpha) = \frac{1}{16}$$
$$I_2 = \sup_{\alpha\in[0,1]} K_2(\alpha) = \frac{1}{25}$$
$$I_3 = \sup_{\alpha\in[0,1]} K_3(\alpha) = \frac{1}{49}$$

and then, we have

$$I = \max\{I_1, I_2, I_3\} = \frac{1}{16}$$

Further,

$$\begin{aligned}
|R_1| &= \frac{\Gamma(5-p_1)\Gamma(q_1+5)}{|\Gamma(5-p_1)-\Gamma(q_1+5)|} = \frac{2786582\sqrt{\pi}}{1467322} = 3.37 \\
|R_2| &= \frac{\Gamma(5-p_2)\Gamma(q_2+5)}{|\Gamma(5-p_2)-\Gamma(q_2+5)|} = \frac{8968428\sqrt{\pi}}{9624241} = 1.65 \\
|R_3| &= \frac{\Gamma(5-p_3)\Gamma(q_3+5)}{|\Gamma(5-p_3)-\Gamma(q_3+5)|} = \frac{7525863\sqrt{\pi}}{9569341} = 1.39 \\
I\eta_1 &= I\left[\frac{1}{\Gamma(a_1+1)} + \frac{\alpha_1|R_1|}{24\Gamma(q_1+a_1+1)} + \frac{R_1}{24\Gamma(a_1-p_1+1)}\right] \\
&= \frac{1}{16}[0.078 + 0.0034 + 0.0007] \\
&= \frac{1}{16}[0.08211] \\
&= 0.00513 \\
I\eta_2 &= I\left[\frac{1}{\Gamma(a_2+1)} + \frac{\alpha_2|R_2|}{24\Gamma(q_2+a_2+1)} + \frac{R_2}{24\Gamma(a_2-p_2+1)}\right] \\
&= \frac{1}{16}[0.4357 + 0.0046 + 0.0036] \\
&= \frac{1}{16}[0.44066] \\
&= 0.027 \\
I\eta_3 &= I\left[\frac{1}{\Gamma(a_3+1)} + \frac{\alpha_3|R_3|}{24\Gamma(q_3+a_3+1)} + \frac{R_3}{24\Gamma(a_3-p_3+1)}\right] \\
&= \frac{1}{16}[0.00742 + 0.0000127 + 0.00332] \\
&= \frac{1}{16}[0.010752] \\
&= 0.005376
\end{aligned}$$

and then

$$I(\eta_1 + \eta_2 + \eta_3) = 0.005131 + 0.027 + 0.005376 = 0.0375087 < 1$$

Hence all assumptions of Theorem 4.1 are justify and consequently the tripled system (4.4) must have unique solution defined on $[0, 1]$.

Example 4.2. Now consider the following tripled system

$$\begin{cases} {}^{C}D^{\frac{5}{2}}x_1(\alpha) = \frac{\cos x_2 x_3(\alpha)}{7+\alpha} \\ {}^{C}D^{\frac{11}{4}}x_2(\alpha) = \frac{\sin x_1 x_3(\alpha)}{4+\alpha^2} \\ {}^{C}D^{\frac{17}{4}}x_3(\alpha) = \frac{\cos 2\pi x_2 x_3(\alpha)}{7+\alpha^3} \\ x_1(0) = x_1'(0) = x_1'''(0) = 0, {}^{C}D^{\frac{1}{2}}x_1(1) = \frac{13}{4}(J^{\frac{13}{2}}x_1)(1) \\ x_2(0) = x_2'(0) = x_2'''(0) = 0, {}^{C}D^{\frac{3}{2}}x_2(1) = \frac{9}{8}(J^{\frac{9}{2}}x_2)(1) \\ x_3(0) = x_3'(0) = x_3'''(0), {}^{C}D^{\frac{5}{2}}x_3(1) = \frac{6}{7}(J^{\frac{7}{2}}x_3)(1) \\ a_1 = \frac{5}{2}, p_1 = \frac{1}{2}, q_1 = \frac{13}{2}, \alpha_1 = \frac{13}{4} \neq \frac{\Gamma(q_1+5)}{\Gamma(5-p_1)} = 1023014.17 \\ a_2 = \frac{11}{4}, p_2 = \frac{3}{2}, q_2 = \frac{9}{2}, \alpha_2 = \frac{9}{8} \neq \frac{\Gamma(q_2+5)}{\Gamma(5-p_2)} = 35.895.23 \\ a_3 = \frac{17}{4}, p_3 = \frac{5}{2}, q_3 = \frac{7}{2}, \alpha_3 = \frac{6}{7} \neq \frac{\Gamma(q_3+5)}{\Gamma(5-p_3)} = 10557.42 \end{cases} \tag{4.5}$$

for $\alpha \in [0, 1]$ and $B \in R$, we get

$$|e_1(\alpha, B)| = |\frac{\cos B}{7+\alpha}| \leq \frac{1}{7+\alpha}$$
$$|e_2(\alpha, B)| = |\frac{\sin B}{4+\alpha^2}| \leq \frac{1}{4+\alpha^2}$$
$$|e_3(\alpha, B)| = |\frac{\cos 2\pi B}{7+\alpha}| \leq \frac{1}{9+\alpha^3}$$

so we can take $l_1(\alpha) = \frac{1}{7+\alpha}, l_2(\alpha) = \frac{1}{4+\alpha^2}, l_3(\alpha) = \frac{1}{9+\alpha^3}$ and then, we have

$$w_1 = \sup_{\alpha\in[0,1]} l_1(\alpha) = \frac{1}{7}$$
$$w_2 = \sup_{\alpha\in[0,1]} l_2(\alpha) = \frac{1}{4}$$
$$w_3 = \sup_{\alpha\in[0,1]} l_3(\alpha) = \frac{1}{9}$$

Hence all assumption of Theorem 4.2 are satisfied therefor the tripled solution (4.5).

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