Groups with few non-(Hypercentral-By-Finite) Subgroups

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ABSTRACT
In this note we study groups with few non-(hyper central-by-finite)subgroups and we prove if G is a minimal non-ZAF group, then G is a finitelygenerated perfect group which has no proper subgroup of finite index and suchthat G/Frat(G) is an infinite simple group, where ZA (respectively,F) denotesthe class of hypercentral groups, (respectively,the class of Finite groups), andFrat(G) stands for the Frattini subgroup of G.Moreover, we proved a infinitely generated F-perfect MNZAF-group. Gis MNZA-group if, and only if, G is MNFZA-group if, and only if, G is MNZAF

Keywords : Hypercentral, Finite-by-hypercentral, Hypercentral-by-finite, Minimal non-Ω

1. INTRODUCTION

The aim of this paper is to study groups that in some sense have few non-(hypercentral-by-finite) subgroups, namely minimal non-(hypercentral-by-finite) groups. If Ω is a class of groups, then a group G is said to be minimal non -Ω if all its proper subgroups are in the class Ω. Butitself is not an Ω-group. We will denote minimal non-Ω-groups by MNΩ -groups. Many results have been obtained on minimal non-Ω-groups, for various choices of Ω, especially in the case of finite groups. For instance, finite minimal non-abelian, minimal non-nilpotent and minimal non-supersoluble groups have been completely described by G.A.Miller and H.C.Moreno [5], O.I.Schmidt [8] and K.Doerk[3] In particular, in [10] (respectively, in [9]) it is proved that if G is a finitely generated minimal non-nilpotent (respectively, non-(finite-by-nilpotent) group, then G is a perfect group which has no proper subgroup of finite index and such that G/Frat(G) is an infinite simple group, where Frat(G) denotes the Frattini subgroup of G.We generalize this last result to minimal non-hypercentral groups. We will prove. If G is a finitely generated non- (hypercentral-by-finite) group, then G is a perfect group which has no proper subgroup of finite index and such that G/Frat(G) is an infinite simple group. Moreover, I proved a infinitely generated F-perfect MNZAF-group. The following conditions are equivalent

i) G is MNZA-group
ii) G is MNFZA-group
iii) G is MNZAF-group

2. Finitely generated hypercentral-by-finite groups

Lemma 2.1 Let G be a group whose proper subgroups are hypercentral-by-finite. If N be a propre normalsubgroup of G such that N is hypercentral group, Then G/N is an MNZAF.

Proof Let G be a group whose proper subgroups are hypercentral-by-finite, let N be a propre normal hypercentral subgroup of G. Suppose that G/N is hypercentral-by-finitegroup.
Therefore there exists (K/N)⊲ (G/N) such that (K/N) is Hypercentral group and (G/K) isfinite , we obtain that G is hypercentral-by-finite group, as required

Lemma 2.2 Let G be a group periodic whose proper subgroups are hypercentral-by-finite.
Then G it is locally finite.

Proof Let H be an finitely generated subgroup of G,so proper subgroup, then H is hypercentral-by-finite subgroup,there exists K is hypercentral subgroup of H and (H/K) is finite group. K an finitely generated hypercentral group so finite, and H/K is finite group then H is finite.

Lemma 2.3 If G is a finitely generated minimal non ZAF-group, then G has no non-trivial finite factor hypercentral-by-finite groups.

Proof Let G be an finitely generated MNZAF-group. Suppose that G has a normal proper subgroup N such that G/N is hypercentral-by-finite group,So that G/N is finitely generated hypercentral-by-finite group, so it is locally nilpotent-by-finite group, since it is exist M⊲G such that M is hypercentral, M≠G and G/M is finite. Thus we deduce that G is hypercentral-by-finite group. Contradiction.

Theorem 2.1 Let G be a finitely generated minimal non-ZAF-group. Then :

i) G has no non-trivial finite factor.

ii) G is perfect group

iii) G/Frat(G) is an infinite simple group.

Proof Let G be a finitely generated minimal non-ZAF-group.

i) Suppose that G is finitely generated and admits N a proper normal subgroup of finite index in G. So N is hypercentral-by-finite proper subgroup and it is also finitely generated. Hence N is hypercentral-by-finite subgroup, so N is locally nilpotent-by-finite a finitely generated subgroup then there exists characteristic hypercentral subgroups M and N/M is finitely generated finite, so it is finite group is it self G is hypercentral-by-finite group .

ii) Suppose this statement is false. Then G≠G′, so G′ is hypercentral-by-finite. Hence G/G′ is abelian, since it is locally graded. Now if G/G′ is finitely generated, then there exists H⊲G such that H≠G and G/H is finite. So G has a proper subgroup of finite index. Thus we deduce from (i) that G is hypercentral-by-finite. Contradiction.

iii) Let G be a minimal non-ZAF-group. It follows que G is a finitely generated perfect group which has no non trivial finite factor. Now we prove that G/Frat(G) is an infinite simple group. Since finitely generated groups have maximal subgroups, G/Frat(G) is non trivial and therefore infinite. Let N be a propernormal subgroup of G properly containing Frat(G). Then N is hypercentral-by-Cernikov. Hence there is a maximal subgroup M of G such that N is not contained in M. Then G=NM and we have

(G/N)=(MN)/N)≃(M/(M∩N)

so,G/N is hypercentral-by-finite group. This is contradiction.Then G/Frat(G) is a simple group.

3. Infinitely generated hypercentral-by-finite groups

Lemma 3.1 Let G be a infinitely generated MNZAF-group. Then G is F-perfect

Proof Let G be a infinitely generated whose proper subgroups are in the class ZAF. Suppose that G admits a proper normal subgroup N of finite index in G , then H belongs to ZAF . Let H be a proper normal subgroup of N such that H is hypercentral subgroup and N/H is finite. We have H⊲N⊲G so

|G:H|=|G:N||N:H|≤∞

then H is finite index, we put $H_G = gHg^{-1}$, so H_G⊲G such that $H_G$is hypercentral and G/$H_G$is finite so G is hypercentral-by-finite, as required

Lemma 3.2 The class ZAF of hypercentral-by-finite groups is N.{0 }-closed.

proof Let H and K be normal hypercentral-by-finite subgroups of a group G, there exist hypercentral subgroups H_1 and K_1 of H and K, respectively, such that (H/(H_1) and (K/(K_1) are finite. We put N=H_{1H}=∩hH_1h^{-1}(respectively M=K_{1K}=∩kK_1k^{-1}) so N⊲H (respectively M⊲k) and H/M (respectively K/M is finite, so NM is a normal finite subgroup of HK. we have

(HN/NM)≃(H/(H∩NM)≃(H/N)/(H∩NM)/N)and ((KM/NM))≃(K/(K∩NM))≃(K/M)/(K∩NM)/M)

are finite, so HK/NM=(HM)/(NM)(NK)/(NM) is finite as the class of finite groups is {H, N_0}-closed. Therefore, HK is hypercentral-by-finite, as required.

Remark 3.1 Let G be a infinitely generated MNZAF-group, then G/(G′ is quasicyclic group.

Proof By lemma 3.2, G cannot be the product of two proper normal subgroups and by lemma 3.1. We deduce that G/(G′ is quasicyclic group.

Lemma 3.3 Let G be a group, if G/(Z(G) is hypercentral-by-finite then, G is hypercentral-by-finite.

Proof If G/Z(G) is hypercentral-by-finite, therefore there exists normal hypercentral subgroup N/Z(G) and G/N is finite, by lemma 3.1 G is F-perfect, so N=G then G is hypercentral-by-finite.

Lemma3.4 Let G be a infinitely generated non-perfect MNZAF-group, then all these normal proper subgroups are hypercentral.

proof Let N be normal proper subgroup of a group G, so N is hypercentral-by-finite then there exist normal proper hypercentral subgroup K such that N/K is finite, we have that K=Z_α(K) such that

K/Z_α(K) is finite, so

$G/Z_\alpha(K)/C_{G/Z\alpha(K)}(N/Z_\alpha(K)) \simeq (N/Z_\alpha(K)(N/(Z\alpha(K))/(C_{G/Z\alpha(K)})(N/(Z_\alpha(K))$
$\simeq T \leq Aut\ (N/Z_\alpha(K)$
We have $N/Z_\alpha(K)$ is finite, so is $Aut\ (N/Z_\alpha(K)$ finite, hence $G/(Z_\alpha(K)/C_{G/(Z\alpha(K)}(N/(Z_\alpha(K)$ is finite, we have G is F-perfect, so $G/Z_\alpha(K)$ is F-perfect, so $G/Z_\alpha(K) = (C_{G/Z\alpha(K)})(N/(Z_\alpha(K))$
Then ,
$N/Z_\alpha(K) \leq Z(G/Z_\alpha(K))$
we have
$[(N/Z_\alpha(N),(G/Z_\alpha(K)]=Z_\alpha(K)/Z_\alpha(N)=1 \Rightarrow [G,N] \leq Z_\alpha(K)$
Especially
$[N,N] \leq Z_\alpha(K) \Rightarrow N' \leq Z_\alpha(K)$
hence $N/Z_\alpha(K)$ is abelian, so it is hypercentral and
$Z_\beta(N/Z_\alpha(K))=N/Z_\alpha(K) \Rightarrow Z_{\alpha+\beta}(N)/Z_\alpha(K)=N/Z_\alpha(K)$
$\Rightarrow N=Z_{\alpha+\beta}(N)$
then N is hypercentral.

Lemma3.5 Let G be infinitely generated MNZAF-group,and let N be a normal proper subgroup of G. Then $G' \lhd N$ if, and only if, G/N is MNZAF.
Proof Suppose that G/N is hypercentral-by-finite , so it is F-perfect hypercentral periodic so it is abelian then $G' \leq N$.
Lemma3.6 Let G be infinitely generated non-perfect MNZAF-group. Then G is a p-group for some prime p.
Proof Let G be infinitely generated non-perfect MNZAF-group.We have G′ is hypercentral "$G' \lhd G$", so G′ is locally nilpotent, and
$G'=Dr_p G_P \simeq G_{p_1}' \times G_{p_2}' \times \ldots \times G_{p_1}$
Suppose $N=Dr_{p\neq q} G_q$ for some prime $q \neq p$, so
$(G/N)'=Dr_p G_p'/\ Dr_{p\neq q} G_q' \simeq G_p'$ so G′/N is a p-group, we have G/N is MNZAF, hence G/N is locally nilpotent and $G'/N \lhd G/N$ such that $(G/N)/(G'/N) \simeq (G/G')$ is quasicyclic p-group for some prime p. so it is countable, by Kiguel that there exists p-subgroup (P/N) of (G/N) such that (G/N)=(G′/N)(P/N), we have $(G'/N) \lhd (G/N)$ such that (G′/N) is hypercentral, so (G/N)=(P/N) hence G=P then G is p-group

Theorem 3.1 Let G be infinitely generated, if G is MNZAF. Then G is MNZAF.
Proof Let G be a locally nilpotent MNZAF -group, let N be a proper subgroup of G it is hypercentral-by-finite. We have N is locally nilpotent and there exist normal hypercentral subgroup K of N such that N/K is finite so it is finitely generated so N is hypercentral

Theorem3.2 Let G be infinitely generated F-perfect . The following conditions are equivalent
i) G is MNZA-group
ii) G is MNFZA-group
iii) G is MNZAF-group
Proof Let G be infinitely generated F-perfect
i) ⇒ii) Let G be a MNZA-group. Suppose that G is finite-by-hypercentral, so $G/(Z_\alpha(G)$ is finite. As G is F-perfect, so $G = Z_\alpha(G)$ then G is hypercentral group
ii)⇒iii) Let G be a MNFZA-group. Suppose that G is hypercentral-by-finite, so it is hypercentral then is finite-by-hypercentral Contradiction.
iii)⇒ii) Let G be a locally nilpotent MNZAF, Let N be a proper subgroup of G, since N proper subgroup of G, so N is hypercentral-by-finite, we have N is locally nilpotent, there existsproper normal hypercentral K subgroup such that N/K is finite so it is finitely generated so N is hypercentral

Corollary 3.1 Let G be infinitely generated MNZAF-group if, and only if,G is MNZAF.
Lemma 3.7 Let G be infinitely generated MNZAF-group. Then G is MNZAC.
Proof Let G be infinitely generated whose proper subgroups are hypercentral -by-finite, so are hypercentral-by-Cernikov. Suppose that G is hypercentral-by-Cernikov,so there exist normal hypercentral subgroup N of G such that (G/N) is Cernikov, hence (G/N) is abelian, so $G' \leq N$, then G′ is minimal of G it is locally nilpotent then $G' \leq Z(G)$, hence $G \in N_2$.

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