An Approach to the Stability of Symmetric Bi-K-Derivations in Γ −Banach Algebras

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ABSTRACT
Given any Γ −Banach Algebra V over F, the binary operation D in V is called Symmetric bi-k-derivation if (a) D is bilinear over F, (b) $D(u,v) = D(v,u)$, (c) $D(u\gamma v, w) = D(u,w)k(\gamma)v + uk(\gamma)D(v,w)$, $D(u, v\gamma w) = D(u,v)k(\gamma)w + vk(\gamma)D(u,w)$ and (d) $k: \Gamma \to \Gamma$ is linear over F. We study on Stability of Symmetric bi-k-derivation in Γ −Banach Algebra V for the functional equation[1]$f(au + av, bw - bx) + f(au - av, bw + bx) = 2abf(u,w) - 2abf(v,x)$ for all $u, v, w, x \in V$ and $a, b \in T = \{x \in \mathbb{C} : |x| = 1\} \subseteq \mathbb{C}$.

Keywords: Stability, Γ −Banach Algebra, Symmetric bi-k-derivation, functional equation.

1. INTRODUCTION

The idea of Γ −ring was first given by N. Nobusawa[2]. W.E. Barnes [3] also defined Γ −ring by reducing criterions in the definition of Γ −ring. When J. Luh, S. Kyuno and W.E. Barnes explored the structure of Γ −ring, they discovered a number of generalizations that are comparable to related concepts in ring concepts. Maity and Bhattacharya [4] define Γ −Banach Algebra in 1989. They also generalized some results on Γ −Banach Algebra. In 2001, T.K. Dutta, R.C. Kalita and H.K. Nath [5]define projective tensor product of Γ −Banach Algebras. They also defined Γ −derivation on projective tensor product of Γ −Banach Algebras and proved some important results on Γ −derivations[5].

In 1940, S. M. Ulam mentioned the following question in his book entitled "Problems in Modern Mathematics", on related with the stability of function:

[6]Let $(G_1,*)$ be a group and $(G_2,\circ, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does their exist a $\delta(\varepsilon) > 0$ such that if a function $f: G_1 \to G_2$ satisfies the inequality

$d(f(x * y), f(x) \circ f(y)) < \delta$

for all $x, y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ such that

$d(f(x), H(x)) < \varepsilon$

for all $x \in G_1$?

If the above conditions are satisfied by the homomorphism $H: G_1 \to G_2$, then the group homomorphism H is called stable [6].

In the same year, D. H. Hyers [7] gave response to the problem raised by Ulam for additive mapping fromG_1 into $G_2$where $G_1$and $G_2$are Banach spaces. In 1978, Th. M. Rassias[8] made a broad statement of Hyers's stability for linear mapping by take into account of an unbounded Cauchy difference and which is called the Hyers-Ulam-Rassias stability. Many new concepts in connection with the Hyers-Ulam-Rassias stability are investigated by a large number of researchers (see[1], [9], [10]). In this paper, we study on Stability of Symmetric bi-k-derivation on Γ −Banach Algebra V for the functional equation $f(au + av, bw - bx) + f(au - av, bw + bx) = 2abf(u,w) - 2abf(v,x)$ for all $u, v, w, x \in V$ and $a, b \in T = \{x \in \mathbb{C} : |x| = 1\} \subseteq \mathbb{C}$.

2. Preliminaries

Definition 2.01 [11]For additive abelian groups V and Γ, if the mappings$V \times \Gamma \times V \to V, (x, \gamma, y) \mapsto x\gamma y$ and $\Gamma \times V \times \Gamma \to \Gamma, (\alpha, x, \beta) \mapsto \alpha x\beta$ satisfy the following conditions

N1. $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$

N2. $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$for all $x, y, z \in V$ and $\alpha, \beta \in \Gamma$.

N3. If $x\alpha y = 0$ for all $x, y \in V$, then $\alpha = 0$.

thenV is called a Γ-ring. This definition is given by N. Nobusawa in 1964[2].

W. E. Barnes [3] define Γ-ring as follows:

Let V and Γ be additive abelian groups. If the mapping $V \times \Gamma \times V \to V$ (the image of (x, γ, y) where $x, y \in V$ and $\gamma \in \Gamma$, being denoted by $x\gamma y$), satisfies the following conditions

B1. $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$

B2. $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$

for all $x, y, z \in V$ and $\alpha, \beta \in \Gamma$, then V is called a Γ-ring. This definition is due to Barnes, and is slightly weaker than the origin one due to Nobusawa.

Definition 2.02[11]Let V and Γ be additive abelian groups. Then Vis called a Γ-Banach algebra over a field F if

(a) Vis a Γ-ring due to Barnes.

(b) $a(x\gamma y) = (ax)\gamma y = x\gamma(ay)$ for all $x, y \in V$; $\gamma \in \Gamma$ and $a \in F$.

(c) V is a Banach space over F with the norm satisfies

$\|x\gamma y\| \le \|x\|\|\gamma\|\|y\|$ for all $x, y \in V$ and $\gamma \in \Gamma$.

Vis known as Γ_N-Banach algebra over F if V is a Γ-ring due to Nobusawa.

Definition 2.02[11]Given any Γ −Banach Algebra V over F, the binary operation D inV is called Symmetric bi-k-derivation if

(a) D is bilinear over F,

(b) $D(u, v) = D(v, u)$,

(c) $D(u\gamma v, w) = D(u, w)k(\gamma)v + uk(\gamma)D(v, w)$,

$$D(u, v\gamma w) = D(u, v)k(\gamma)w + vk(\gamma)D(u, w)$$

and (d) $k: \Gamma \to \Gamma$ is linear over F.

3. Main Results

Lemma 3.01[1] Let V and W be two $\mathbb{C}$ −linear spaces and let $f: V \times V \to W$ be a bi-additive mapping such that $f(au, bv) = abf(u, v)$ for all $a, b \in T = \{x \in \mathbb{C} : |x| = 1\}$ and $u, v \in V$, then f is $\mathbb{C}$ −linear.

Lemma 3.02[1] Let V and W be two $\mathbb{C}$ −linear spaces and let $f: V \times V \to W$ be a mapping such that $f(au + av, bw - bx) + f(au - av, bw + bx) = 2abf(u, w) - 2abf(v, x)$ for all $u, v, w, x \in V$ and $a, b \in T = \{x \in \mathbb{C} : |x| = 1\}$. Then f is $\mathbb{C}$ −linear.

Theorem 3.03 Let V be a Γ −Banach algebra over the complex field $\mathbb{C}$ with $\Gamma(\mathbb{C})$ is a Banach space.Suppose $f: V \times V \to V$ and $g: \Gamma \to \Gamma$ are functions with $p \in (-\infty, 2)$, $q \in (-\infty, 1)$ and $\theta \in (0, \infty)$ such that

$\|f(au + av, bw - bx) + f(au - av, bw + bx) - 2abf(u, w) + 2abf(v, x)\| \le \theta(\|u\|^p + \|v\|^p + \|w\|^p + \|x\|^p)$ …… (1)

$\|f(u, v) - f(v, u)\| \le \theta(\|u\|^p + \|v\|^p)$ …… (2)

$\|f(u\gamma v, w) - f(u, w)k(\gamma)v - uk(\gamma)f(v, w)\| + \|f(u, v\gamma w) - f(u, v)k(\gamma)w - vk(\gamma)f(u, w)\| \le \theta(\|u\|^p + \|v\|^p + \|w\|^p)$ …… (3)

$\|g(a\alpha + a\beta) - ag(\alpha) - ag(\beta)\| \le \theta(\|\alpha\|^q + \|\beta\|^q)$ ……. (4)

$\lim_{n\to\infty} \frac{1}{4^{3n}} f(2^{3n}u, 2^{3n}v) = \lim_{n\to\infty} \frac{1}{4^{3n}} f(2^{3n}u, 2^{n}v) = \lim_{n\to\infty} \frac{1}{4^{3n}} f(2^{n}u, 2^{3n}v)$ …… (5)

$\lim_{n\to\infty} \frac{1}{4^{2n}} f(2^{2n}u, 2^{2n}v) = \lim_{n\to\infty} \frac{1}{4^{2n}} f(2^{n}u, 2^{n}v)$ …….. (6)

for all $u, v, w, x \in V$; $a, b \in T = \{x \in \mathbb{C} : |x| = 1\}$ and $\alpha, \beta \in \Gamma$. Then there exist unique $\mathbb{C}$ −linear map $k: \Gamma \to \Gamma$ and symmetric bi-k-derivation inV such that

$$\|f(u, v) - D(u, v)\| \le \frac{5\theta}{4 - 2^p}(\|u\|^p + \|v\|^p) + \frac{3}{4}\|f(0,0)\|$$

$$\|g(\alpha) - k(\alpha)\| \le \frac{2\theta}{2 - 2^q}\|\alpha\|^q$$

Proof: Putting $a = b = 1$, $v = u$ and $x = -w$ in (1), we obtain

$\|f(2u, 2w) - 2f(u, w) + 2f(u, -w)\| \le 2\theta(\|u\|^p + \|w\|^p) + \|f(0,0)\|$ …… (7)

Again, Putting $a = b = 1$ and $u = w = 0$ in (1), we find

$\|f(v, -x) + f(-v, x) + 2f(v, x)\| \le \theta(\|v\|^p + \|x\|^p) + 2\|f(0,0)\|$…… (8)

Putting u instead of v and w instead of x in (8), we get

$\|f(u, -w) + f(-u, w) + 2f(u, w)\| \le \theta(\|u\|^p + \|w\|^p) + 2\|f(0,0)\|$ …… (9)

Setting $a = b = 1$ and Putting $-u$ instead of v and w instead of x in (1), we obtain

$\|f(2u, 2w) - 2f(u, w) + 2f(-u, w)\| \le 2\theta(\|u\|^p + \|w\|^p) + \|f(0,0)\|$ …… (10)

From (7) and (9), we obtain

$\|f(2u, 2w) - 4f(u, w) + f(u, -w) - f(-u, w)\| \le 3\theta(\|u\|^p + \|w\|^p) + 3\|f(0,0)\|$ …… (11)

From (7) and (10), we obtain

$\|f(u, -w) - f(-u, w)\| \le 2\theta(\|u\|^p + \|w\|^p) + \|f(0,0)\|$…… (12)

From (11) and (12), we obtain.

$\|f(2u, 2w) - 4f(u, w)\| \le 5\theta(\|u\|^p + \|w\|^p) + 4\|f(0,0)\|$ …… (13)

for all u, w ∈ V. Replacing u by $2^k u$ and w by $2^k w$ in (13), we get

$$\left\|f(2^{k+1}u, 2^{k+1}w) - 4f(2^k u, 2^k w)\right\| \le 5\theta\left(\left\|2^k u\right\|^p + \left\|2^k w\right\|^p\right) + 4\|f(0,0)\|$$

Dividing both sides of the above inequality by 4^{k+1}, we gain

$$\left\|\frac{1}{4^k} f(2^k u, 2^k w) - \frac{1}{4^{k+1}} f(2^{k+1}u, 2^{k+1}w)\right\| \le \frac{5\theta}{4}.2^{k(p-2)}(\|u\|^p + \|w\|^p) + \frac{1}{4^k}\|f(0,0)\|$$

for all u, w ∈ V and k(≥ 0) ∈ Z. For each pair of integers m, n (0 ≤ m < n), we can write

$$\left\|\frac{1}{4^m} f(2^m u, 2^m w) - \frac{1}{4^n} f(2^n u, 2^n w)\right\|$$

$\le \frac{5\theta}{4}.\sum_{k=m}^{n-1} 2^{k(p-2)}(\|u\|^p + \|w\|^p) + \sum_{k=m}^{n-1}\frac{1}{4^k}\|f(0,0)\|$ …… (14)

Since the sequence $\sum_{k=0}^{\infty} 2^{k(p-2)}$ converges for $p \in (-\infty, 2)$ and $\sum_{k=m}^{n-1}\frac{1}{4^k}$ is a convergence series, so from (14), wefind$\left\{\frac{1}{4^n} f(2^n u, 2^n w)\right\}$ is a Cauchy sequence in V for all $u, w \in V$. Completeness of V implies that there exists a binary operation$D$in Vsuch that

$$D(u,w) = \lim_{n\to\infty}\frac{1}{4^n} f(2^n u, 2^n w)$$

Putting $a = 1$ and $\beta = \alpha$ in (4), we gain

$\|g(2\alpha) - 2g(\alpha)\| \le 2\theta\|\alpha\|^q$ ……. (15)

Replacing α by $2^l\alpha$ and dividing by 2^{l+1}, we attain

$$\left\|\frac{1}{2^l} g(2^l\alpha) - \frac{1}{2^{l+1}} g(2^{l+1}\alpha)\right\| \le 2^{l(q-1)}\theta\|\alpha\|^q$$

for all $\alpha \in \Gamma$ and $l(\ge 0) \in \mathbb{Z}$. For each pair of integers r, s $(0 \le r < s)$, we can write

$\left\|\frac{1}{2^r} g(2^r\alpha) - \frac{1}{2^s} g(2^s\alpha)\right\| \le \sum_{l=r}^{s-1} 2^{l(q-1)}\theta\|\alpha\|^q$ …… (16)

Since for $q \in (-\infty, 1)$, the series $\sum_{l=0}^{\infty} 2^{l(q-1)}$ converges, so from (16), we can conclude that $\left\{\frac{1}{2^n} g(2^n\alpha)\right\}$ is a Cauchy sequence in Γ. Since Γ is a Banach space over $\mathbb{C}$, so we can define a map $k: \Gamma \to \Gamma$ such that

$$k(\alpha) = \lim_{n\to\infty}\frac{1}{2^n} g(2^n\alpha)$$

Now, we want to show that D is symmetric bi-k-derivation.

$$\|D(u,v) - D(v,u)\| = \left\|\lim_{n\to\infty}\frac{1}{4^n} f(2^n u, 2^n v) - \lim_{n\to\infty}\frac{1}{4^n} f(2^n v, 2^n u)\right\|$$

$$= \lim_{n\to\infty}\frac{1}{4^n}\|f(2^n u, 2^n v) - f(2^n v, 2^n u)\|$$

$\le lim_{n\to\infty} 2^{n(p-2)}\,\theta(\|u\|^p + \|v\|^p) = 0$as $p \in (-\infty, 2)$.

Using Lemma 3.02, we can show D is $\mathbb{C}$ −bilinear.

$$\|D(u\gamma v, w) - D(u,w)k(\gamma)v - uk(\gamma)D(v,w)\| + \|D(u, v\gamma w) - D(u,v)k(\gamma)w - vk(\gamma)D(u,w)\|$$

$$= \lim_{n\to\infty}[\| \frac{1}{4^{3n}} f(2^{3n}(u\gamma v), 2^{3n}w) - \frac{1}{4^{2n}} f(2^{2n}u, 2^{2n}w)\frac{1}{2^n} g(2^n\gamma)v - u\frac{1}{2^n} g(2^n\gamma)$$
$$\frac{1}{4^{2n}} f(2^{2n}v, 2^{2n}w) \| + \| \frac{1}{4^{3n}} f(2^{3n}u, 2^{3n}(v\gamma w)) - \frac{1}{4^{2n}} f(2^{2n}u, 2^{2n}v)\frac{1}{2^n} g(2^n\gamma)w -$$
$$v\frac{1}{2^n} g(2^n\gamma)\frac{1}{4^{2n}} f(2^{2n}u, 2^{2n}w) \|]$$

$$= \lim_{n\to\infty}\frac{1}{4^{3n}}[\| f(2^n u 2^n\gamma 2^n v, 2^3 w) - f(2^n u, 2^n w)g(2^n\gamma)2^n v - 2^n u g(2^n\gamma)$$
$$f(2^n v, 2^n w) \| + \| f(2^n u, 2^n v 2^n\gamma 2^n w) - f(2^n u, 2^n v)g(2^n\gamma)2^n w -$$

$2^n v g(2^n\gamma)f(2^n u, 2^n w) \|]$using (5) and (6).

$\le lim_{n\to\infty}\frac{1}{4^{3n}}\theta(\|2^n u\|^p + \|2^n v\|^p + \|2^n w\|^p)$using (3)

$$= \lim_{n\to\infty} 2^{n(p-6)}\theta(\|u\|^p + \|v\|^p + \|w\|^p)$$

$= 0$as$p \in (-\infty, 2)$.

Thus$D(u\gamma v, w) = D(u,w)k(\gamma)v + uk(\gamma)D(v,w)$ and $D(u, v\gamma w) = D(u,v)k(\gamma)w + vk(\gamma)D(u,w)$

Putting $\beta = 0$ in (4), we attain

$$\|g(a\alpha) - ag(\alpha) - ag(0)\| \le \theta\|\alpha\|^q$$
$$\Rightarrow \|g(a\alpha) - ag(\alpha)\| \le \theta\|\alpha\|^q + \|ag(0)\|$$

Putting $2^n\alpha$ instead of α, dividing by 2^n and passing limit $n \to \infty$, we gain

$$\lim_{n\to\infty}\left\|\frac{1}{2^n} g(2^n(a\alpha)) - a\frac{1}{2^n} g(2^n\alpha)\right\| \le \lim_{n\to\infty}[2^{n(q-1)}\theta\|\alpha\|^q + \frac{|a|}{2^n}\|g(0)\|$$

$= 0$as $q \in (-\infty, 1)$.

Thus $k(a\alpha) = ak(\alpha)$.

Similarly, by putting $a = 1$ in (4), we obtain

$$\|g(\alpha+\beta)-g(\alpha)-g(\beta)\| \leq \theta(\|\alpha\|^q+\|\beta\|^q)$$

Putting $2^n\alpha$ instead of α and $2^n\beta$ instead of β, we gain

$$\|g(2^n(\alpha+\beta))-g(2^n\alpha)-g(2^n\beta)\| \leq 2^{nq}\theta(\|\alpha\|^q+\|\beta\|^q)$$

Dividing both sides of the above inequality by 2^n and passing limit $n\to\infty$, we attain $k(\alpha+\beta)=k(\alpha)+k(\beta)$.

Hence D is a symmetric bi-k-derivation.

Now putting $m=0$ in (14) and passing limit $n\to\infty$, we get

$$\|f(u,v)-D(u,v)\| \leq \frac{5\theta}{4-2^p}(\|u\|^p+\|v\|^p)+\frac{3}{4}\|f(0,0)\|$$

Similarly, putting $r=0$ in (15) and passing limit $n\to\infty$, we obtain

$$\|g(\alpha)-k(\alpha)\| \leq \frac{2\theta}{2-2^q}\|\alpha\|^q$$

Theorem 3.04 Let V be a Γ −Banach algebra over the complex field $\mathbb{C}$ with $\Gamma(\mathbb{C})$ is a Banach space. Let$f: V\times V\to V$ be a mapping such that $f(0,0)=0$ satisfies (1), (2), (3) and $lim_{n\to\infty} 4^{3n} f(\frac{u}{2^{3n}},\frac{v}{2^{3n}}) = lim_{n\to\infty} 4^{3n} f(\frac{u}{2^{3n}},\frac{v}{2^{n}}) = lim_{n\to\infty} 4^{3n} f(\frac{u}{2^{n}},\frac{v}{2^{3n}})$ …… (17)

$lim_{n\to\infty} 4^{2n} f(\frac{u}{2^{2n}},\frac{v}{2^{2n}}) = lim_{n\to\infty} 4^{2n} f(\frac{u}{2^{n}},\frac{v}{2^{n}})$ …….. (18)

for $\theta\in(0,\infty)$and$p\in(6,\infty)$. Let$g:\Gamma\to\Gamma$ is a mapping such that $g(0)=0$ satisfies (4) for $\theta\in(0,\infty)$and $q\in(1,\infty)$. Then there exist unique $\mathbb{C}$ −linear map $k:\Gamma\to\Gamma$ and symmetric bi-k-derivation Din Vsuch that

$\|f(u,v)-D(u,v)\| \leq \frac{5\theta}{2^p-4}(\|u\|^p+\|v\|^p)$……. (19)

and$\|g(\alpha)-k(\alpha)\| \leq \frac{2\theta}{2^q-2}\|\alpha\|^q$ …… (20)

for all $u,v\in V$ and $\alpha\in\Gamma$.

Proof: Putting $\frac{u}{2}$ and $\frac{w}{2}$ instead of u and w respectively in (13), we obtain

$\left\|f(u,w)-4f\left(\frac{u}{2},\frac{w}{2}\right)\right\| \leq \frac{5\theta}{2^p}(\|u\|^p+\|w\|^p)asf(0,0)=0$.

for all $u,v\in V$. Replacing u by $\frac{u}{2^k}$ and w by $\frac{w}{2^k}$ and multiplying by 4^k, we attain

$$\left\|4^k f\left(\frac{u}{2^k},\frac{w}{2^k}\right)-4^{k+1}f\left(\frac{u}{2^{k+1}},\frac{w}{2^{k+1}}\right)\right\| \leq \frac{5\theta}{2^p}\cdot\frac{1}{2^{k(p-2)}}(\|u\|^p+\|w\|^p)$$

Which is true for all $k(\geq 0)\in\mathbb{Z}$. Thus for each pair of integers m,n $(0\leq m<n)$, we can write$\left\|4^m f\left(\frac{u}{2^m},\frac{w}{2^m}\right)-4^n f\left(\frac{u}{2^n},\frac{w}{2^n}\right)\right\| \leq \frac{5\theta}{2^p}.\sum_{k=m}^{n-1}\frac{1}{2^{k(p-2)}}(\|u\|^p+\|w\|^p)$ …… (21)

Since the sequence $\sum_{k=0}^{\infty}\frac{1}{2^{k(p-2)}}$ converges for $p\in(2,\infty)$ and $\sum_{k=0}^{\infty}\frac{1}{2^{k(p-2)}}=\frac{2^{p-2}}{2^{p-2}-1}$, so from (21), we find $\left\{4^n f\left(\frac{u}{2^n},\frac{w}{2^n}\right)\right\}$ is a Cauchy sequence in V for all $u,w\in V$. Completeness of V implies that there exists abinary operation D in Vsuch that

$$D(u,w)=\lim_{n\to\infty}4^n f\left(\frac{u}{2^n},\frac{w}{2^n}\right)$$

Raplacingα by $\frac{\alpha}{2}$ in (15), we gain

$$\left\|g(\alpha)-2g\left(\frac{\alpha}{2}\right)\right\| \leq \frac{\theta}{2^{q-1}}\|\alpha\|^q$$

Putting $\frac{\alpha}{2^l}$ instead of α and multiplying both sides of the above inequality by 2^l, we obtain

$$\left\|2^l g\left(\frac{\alpha}{2^l}\right)-2^{l+1}g\left(\frac{\alpha}{2^{l+1}}\right)\right\| \leq \frac{\theta}{2^{q-1}}\frac{1}{2^{l(q-1)}}\|\alpha\|^q$$

Which is true for all $\alpha\in\Gamma$and $l(\geq 0)\in\mathbb{Z}$. For each pair of integers r,s $(0\leq r<s)$, we can write

$\left\|2^r g\left(\frac{\alpha}{2^r}\right)-2^s g\left(\frac{\alpha}{2^s}\right)\right\| \leq \frac{\theta}{2^{q-1}}\sum_{l=r}^{s-1}\frac{1}{2^{l(q-1)}}\|\alpha\|^q$ ……. (22)

Since the series $\sum_{l=0}^{\infty}\frac{1}{2^{l(q-1)}}$ converges for $q\in(1,\infty)$, so from (22), we can conclude that $\left\{2^n g\left(\frac{\alpha}{2^n}\right)\right\}$ is a Cauchy sequence in Γ. Completeness of Γ implies that there is a mapping $k:\Gamma\to\Gamma$ such that

$$k(\alpha)=\lim_{n\to\infty}2^n g\left(\frac{\alpha}{2^n}\right)$$

We can easily show that k is a $\mathbb{C}$ −linear and D is symmetric as we done in theorem 3.03. Using Lemma 3.02, we can show D is $\mathbb{C}$ −bilinear.

$$\|D(u\gamma v,w)-D(u,w)k(\gamma)v-uk(\gamma)D(v,w)\|+\|D(u,v\gamma w)-D(u,v)k(\gamma)w-vk(\gamma)D(u,w)\|$$
$$=\lim_{n\to\infty}[\| 4^{3n}f\left(\frac{u\gamma v}{2^{3n}},\frac{w}{2^{3n}}\right)-4^{2n}f\left(\frac{u}{2^{2n}},\frac{w}{2^{2n}}\right)2^n g\left(\frac{\gamma}{2^n}\right)v-u2^n g\left(\frac{\gamma}{2^n}\right)4^{2n}f\left(\frac{v}{2^{2n}},\frac{w}{2^{2n}}\right)\|+$$
$$\| 4^{3n}f\left(\frac{u}{2^{3n}},\frac{v\gamma w}{2^{3n}}\right)-4^{2n}f\left(\frac{u}{2^{2n}},\frac{v}{2^{2n}}\right)2^n g\left(\frac{\gamma}{2^n}\right)w-v2^n g\left(\frac{\gamma}{2^n}\right)4^{2n}f\left(\frac{u}{2^{2n}},\frac{w}{2^{2n}}\right)\|]$$

$$= \lim_{n\to\infty} 4^{3n} [\| f\left(\frac{u}{2^n}.\frac{\gamma}{2^n}.\frac{v}{2^n},\frac{w}{2^n}\right) - f\left(\frac{u}{2^n},\frac{w}{2^n}\right) g\left(\frac{\gamma}{2^n}\right) 2^n v - 2^n ug\left(\frac{\gamma}{2^n}\right) f\left(\frac{v}{2^n},\frac{w}{2^n}\right) \| +$$

$\| f\left(\frac{u}{2^n},\frac{v}{2^n}.\frac{\gamma}{2^n}.\frac{w}{2^n}\right) - f\left(\frac{u}{2^n},\frac{v}{2^n}\right) g\left(\frac{\gamma}{2^n}\right) 2^n w - 2^n vg\left(\frac{\gamma}{2^n}\right) f\left(\frac{u}{2^n},\frac{w}{2^n}\right) \|]$by using (17) and (18).

$\leq lim_{n\to\infty} 4^{3n}\theta\left[\left\|\frac{u}{2^n}\right\|^p + \left\|\frac{v}{2^n}\right\|^p + \left\|\frac{w}{2^n}\right\|^p\right]$by using (3)

$= lim_{n\to\infty} \frac{\theta}{2^{n(p-6)}}[\|u\|^p + \|v\|^p + \|w\|^p] = 0$as $p \in (6,\infty)$.

Putting $m = 0$ in (21) and passing limit $n \to \infty$, we get the result (19). Similarly, Putting $r = 0$ in (22) and passing limit $s \to \infty$, we get the result (20).

Theorem 3.05 Let V be a Γ −Banach algebra over the complex field $\mathbb{C}$ with $\Gamma(\mathbb{C})$ is a Banach space. Suppose $f: V \times V \to V$ and $g: \Gamma \to \Gamma$ are functions with $p \in \left(-\infty,\frac{1}{2}\right)$, $q \in \left(-\infty,\frac{1}{2}\right)$ and $\theta \in (0,\infty)$ such that

$\|f(au + av, bw - bx) + f(au - av, bw + bx) - 2abf(u,w) + 2abf(v,x)\| \leq$
$\theta\|u\|^p.\|v\|^p.\|w\|^p.\|x\|^p$ …… (23)

$\|f(u,v) - f(v,u)\| \leq \theta\|u\|^p.\|v\|^p$ …… (24)

$\|f(u\gamma v,w) - f(u,w)g(\gamma)v - ug(\gamma)f(v,w)\| +$
$\|f(u,v\gamma w) - f(u,v)g(\gamma)w - vg(\gamma)f(u,w) \leq\|\theta\|u\|^p.\|v\|^p.\|w\|^p$ …… (25)

$\|g(a\alpha + a\beta) - ag(\alpha) - ag(\beta)\| \leq 2\theta\|\alpha\|^q.\|\beta\|^q$ ……. (26)

and f satisfies (5) and (6) for all $u,v,w,x \in V$; $a,b \in T = \{x \in \mathbb{C} : |x| = 1\}$ and $\alpha,\beta \in \Gamma$. Then there exist unique $\mathbb{C}$ −linear map $k: \Gamma \to \Gamma$ and symmetric bi-k-derivation D in Vsuch that

$\|f(u,v) - D(u,v)\| \leq \frac{\theta}{2-2^{4p-1}}.\|u\|^{2p}.\|v\|^{2p} + \frac{3}{4}\|f(0,0)\|$ …… (27)

$\|g(\alpha) - k(\alpha)\| \leq \frac{\theta}{1-2^{2q-1}}\|\alpha\|^{2q}$ ……. (28)

Proof: If we proceed as in Theorem 3.03, from (23), we find

$\|f(2u,2w) - 4f(u,w)\| \leq 2\theta.\|u\|^{2p}.\|w\|^{2p} + 4\|f(0,0)\|$……. (29)

for all $u,w \in V$.

Putting $2^k u$ and $2^k w$ instead of u and w respectively and dividing both sides of the above inequality by 4^{k+1}, we gain

$$\left\|\frac{1}{4^k}f(2^k u,2^k w) - \frac{1}{4^{k+1}}f(2^{k+1}u,2^{k+1}w)\right\| \leq \frac{\theta}{2}.2^{2k(2p-1)}.\|u\|^{2p}.\|w\|^{2p} + \frac{1}{4^k}\|f(0,0)\|$$

for all $u,w \in V$ and $k(\geq 0) \in Z$. For each pair of integers m,n $(0 \leq m < n)$, we can write

$\left\|\frac{1}{4^m}f(2^m u,2^m w) - \frac{1}{4^n}f(2^n u,2^n w)\right\| \leq \frac{\theta}{2}.\sum_{k=m}^{n-1}[2^{2k(2p-1)}.\|u\|^{2p}.\|w\|^{2p} + \frac{1}{4^k}\|f(0,0)\|]$…… (30)

Since the sequence $\sum_{k=0}^{\infty} 2^{2k(2p-1)}$ converges to $\frac{1}{1-2^{2(2p-1)}}$for $p \in \left(-\infty,\frac{1}{2}\right)$ and $\sum_{k=0}^{\infty}\frac{1}{4^k}$ converges to$\frac{4}{3}$, so from (30), we find that$\left\{\frac{1}{4^n}f(2^n u,2^n w)\right\}$ is a Cauchy sequence in V for all $u,w \in V$. Completeness of V implies that there exists a binary operation$D$in Vsuch that

$$D(u,w) = \lim_{n\to\infty}\frac{1}{4^n}f(2^n u,2^n w)$$

Applying (24), we can show

$\|D(u,v) - D(v,u)\| \leq lim_{n\to\infty} 2^{2n(p-1)}\theta.\|u\|^p.\|v\|^p = 0$as $p \in \left(-\infty,\frac{1}{2}\right)$.

That is $D(u,v) = D(v,u)$ for all $u,v \in V$.

If we proceed as in theorem 3.03, for each pair of integers r,s $(0 \leq r < s)$, we obtain from (26),$\left\|\frac{1}{2^r}g(2^r\alpha) - \frac{1}{2^s}g(2^s\alpha)\right\| \leq \sum_{l=r}^{s-1}2^{l(2q-1)}\theta\|\alpha\|^{2q}$ …… (31)

and there is a$\mathbb{C}$ −linear map $k: \Gamma \to \Gamma$ such that

$$k(\alpha) = \lim_{n\to\infty}\frac{1}{2^n}g(2^n\alpha)$$

Applying (5), (6) and (25), we can show

$\|D(u\gamma v,w) - D(u,w)k(\gamma)v - uk(\gamma)D(v,w)\| + \|D(u,v\gamma w) - D(u,v)k(\gamma)w - vk(\gamma)D(u,w)\| \leq$
$lim_{n\to\infty}\frac{\theta}{2^{3n(2-p)}}\|u\|^p.\|v\|^p.\|w\|^p = 0$as $p \in \left(-\infty,\frac{1}{2}\right)$.

That is $D(u\gamma v,w) = D(u,w)k(\gamma)v + uk(\gamma)D(v,w)$ and $D(u,v\gamma w) = D(u,v)k(\gamma)w + vk(\gamma)D(u,w)$.

Letting $m = 0$ and taking limit $n \to \infty$ on both sides of (30), we obtain the desired inequality (27). Similarly, letting $r = 0$ and taking limit $n \to \infty$ on both sides of (31), we obtain the desired inequality (28).

Theorem 3.06 Let V be a Γ −Banach algebra over the complex field $\mathbb{C}$ with $\Gamma(\mathbb{C})$ is a Banach space. Let $f: V \times V \to V$ be a mapping such that $f(0,0) = 0$ satisfies (17), (18), (23), (24) and (25) for $\theta \in$

$(0,\infty), p \in (3,\infty)$. Let $g: \Gamma \to \Gamma$ be a mapping such that $g(0)=0$ satisfies (26) for $q \in \left(\frac{1}{2},\infty\right)$. Then there exist unique $\mathbb{C}$ −linear map $k: \Gamma \to \Gamma$ and symmetric bi-k-derivation D in V such that

$\|f(u,v) - D(u,v)\| \le \frac{\theta}{2^{4p-1} - 2} . \|u\|^{2p} . \|v\|^{2p}$ ……. (32)

$\|g(\alpha) - k(\alpha)\| \le \frac{\theta}{2^{2q-1} - 1} \|\alpha\|^{2q}$ ……. (33)

Proof: Putting $\frac{u}{2}$ and $\frac{w}{2}$ instead of u and w respectively in (29), we obtain

$\left\|f(u,w) - 4f\left(\frac{u}{2},\frac{w}{2}\right)\right\| \le \frac{\theta}{2^{4p-1}} (\|u\|^{2p} . \|w\|^{2p})$ as $f(0,0) = 0$.

for all $u, v \in V$. Replacing u by $\frac{u}{2^k}$ and w by $\frac{w}{2^k}$ and multiplying by 4^k, we attain

$$\left\|4^k f\left(\frac{u}{2^k},\frac{w}{2^k}\right) - 4^{k+1} f\left(\frac{u}{2^{k+1}},\frac{w}{2^{k+1}}\right)\right\| \le \frac{\theta}{2^{4p} - 1} . \frac{1}{2^{2k(2p-1)}} (\|u\|^{2p} . \|w\|^{2p})$$

Which is true for all $k(\ge 0) \in \mathbb{Z}$. Thus for each pair of integers m, n $(0 \le m < n)$, we can write

$\left\|4^m f\left(\frac{u}{2^m},\frac{w}{2^m}\right) - 4^n f\left(\frac{u}{2^n},\frac{w}{2^n}\right)\right\| \le \frac{\theta}{2^{4p} - 1} . \sum_{k=m}^{n-1} \frac{1}{2^{2k(2p-1)}} (\|u\|^{2p} . \|w\|^{2p})$ …… (34)

Since the sequence $\sum_{k=0}^{\infty} \frac{1}{2^{2k(2p-1)}}$ converges for $p \in \left(\frac{1}{2},\infty\right)$ and $\sum_{k=0}^{\infty} \frac{1}{2^{2k(2p-1)}} = \frac{2^{4p-2}}{2^{4p-2} - 1}$, so from (34), we find $\left\{4^n f\left(\frac{u}{2^n},\frac{w}{2^n}\right)\right\}$ is a Cauchy sequence in V for all $u, w \in V$. Completeness of V implies that there exists a binary operation D in V such that

$$D(u,w) = \lim_{n\to\infty} 4^n f\left(\frac{u}{2^n},\frac{w}{2^n}\right)$$

Putting $a = 1$ and $\beta = \alpha$ in (26), we gain

$\|g(2\alpha) - 2g(\alpha)\| \le 2\theta \|\alpha\|^{2q}$ …… (35)

Raplacing α by $\frac{\alpha}{2}$ in (35), we gain

$$\left\|g(\alpha) - 2g\left(\frac{\alpha}{2}\right)\right\| \le \frac{\theta}{2^{2q-1}} \|\alpha\|^{2q}$$

Putting $\frac{\alpha}{2^l}$ instead of α and multiplying both sides of the above inequality by 2^l, we obtain

$$\left\|2^l g\left(\frac{\alpha}{2^l}\right) - 2^{l+1} g\left(\frac{\alpha}{2^{l+1}}\right)\right\| \le \frac{\theta}{2^{2q-1}} \frac{1}{2^{l(2q-1)}} \|\alpha\|^{2q}$$

Which is true for all $\alpha \in \Gamma$ and $l(\ge 0) \in \mathbb{Z}$. For each pair of integers r, s $(0 \le r < s)$, we can write

$\left\|2^r g\left(\frac{\alpha}{2^r}\right) - 2^s g\left(\frac{\alpha}{2^s}\right)\right\| \le \frac{\theta}{2^{2q-1}} \sum_{l=r}^{s-1} \frac{1}{2^{l(2q-1)}} \|\alpha\|^{2q}$ ……. (36)

Since the series $\sum_{l=0}^{\infty} \frac{1}{2^{l(2q-1)}}$ converges for $q \in \left(\frac{1}{2},\infty\right)$, so from (36), we can conclude that $\left\{2^n g\left(\frac{\alpha}{2^n}\right)\right\}$ is a Cauchy sequence in Γ. Completeness of Γ implies that there is a mapping $k: \Gamma \to \Gamma$ such that

$$k(\alpha) = \lim_{n\to\infty} 2^n g\left(\frac{\alpha}{2^n}\right)$$

We can easily show that k is a $\mathbb{C}$ −linear and D is symmetric as we done in theorem 3.03. Using Lemma 3.02, we can show D is $\mathbb{C}$ −bilinear.

$$\|D(u\gamma v, w) - D(u,w)k(\gamma)v - uk(\gamma)D(v,w)\| + \|D(u, v\gamma w) - D(u,v)k(\gamma)w - vk(\gamma)D(u,w)\|$$
$$= \lim_{n\to\infty} [\| 4^{3n} f\left(\frac{u\gamma v}{2^{3n}},\frac{w}{2^{3n}}\right) - 4^{2n} f\left(\frac{u}{2^{2n}},\frac{w}{2^{2n}}\right) 2^n g\left(\frac{\gamma}{2^n}\right) v - u2^n g\left(\frac{\gamma}{2^n}\right) 4^{2n} f\left(\frac{v}{2^{2n}},\frac{w}{2^{2n}}\right) \| +$$
$$\| 4^{3n} f\left(\frac{u}{2^{3n}},\frac{v\gamma w}{2^{3n}}\right) - 4^{2n} f\left(\frac{u}{2^{2n}},\frac{v}{2^{2n}}\right) 2^n g\left(\frac{\gamma}{2^n}\right) w - v2^n g\left(\frac{\gamma}{2^n}\right) 4^{2n} f\left(\frac{u}{2^{2n}},\frac{w}{2^{2n}}\right) \|]$$
$$= \lim_{n\to\infty} 4^{3n} [\| f\left(\frac{u}{2^n}.\frac{\gamma}{2^n}.\frac{v}{2^n},\frac{w}{2^n}\right) - f\left(\frac{u}{2^n},\frac{w}{2^n}\right) g\left(\frac{\gamma}{2^n}\right) 2^n v - 2^n u g\left(\frac{\gamma}{2^n}\right) f\left(\frac{v}{2^n},\frac{w}{2^n}\right) \| +$$

$\| f\left(\frac{u}{2^n},\frac{v}{2^n}.\frac{\gamma}{2^n}.\frac{w}{2^n}\right) - f\left(\frac{u}{2^n},\frac{v}{2^n}\right) g\left(\frac{\gamma}{2^n}\right) 2^n w - 2^n v g\left(\frac{\gamma}{2^n}\right) f\left(\frac{u}{2^n},\frac{w}{2^n}\right) \|]$ by using (17) and (18).

$\le \lim_{n\to\infty} 4^{3n} \theta \left[\left\|\frac{u}{2^n}\right\|^p . \left\|\frac{v}{2^n}\right\|^p . \left\|\frac{w}{2^n}\right\|^p\right]$ by using (25)

$= \lim_{n\to\infty} \frac{\theta}{2^{3n(p-3)}} [\|u\|^p . \|v\|^p . \|w\|^p] = 0$ as $p \in (3,\infty)$.

Putting $m = 0$ in (34) and passing limit $n \to \infty$, we get the result (32). Similarly, Putting $r = 0$ in (36) and passing limit $s \to \infty$, we get the result (33).

CONCLUSION

The above results have come from a preliminary knowledge of linear spaces, Banach spaces, Banach algebras, Γ −rings and Γ −Banach Algebras. In this article, we establish the Hyers-Ulam-Rassias Stability of Symmetric bi-k-derivation in Γ −Banach Algebra V for the functional equation $f(au + av, bw - bx) + f(au - av, bw + bx) = 2abf(u,w) - 2abf(v,x)$ for all $u, v, w, x \in V$ and $a, b \in T = \{x \in \mathbb{C} : |x| = 1\} \subseteq \mathbb{C}$. We have seen this study is very interesting and it has enhanced our knowledge in a fuller form. Possibly this study will inspire others to do further research in this direction.The authors also believes that by using different methods the results could be extended in the future.

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