Exploring Edge-Neighbor Distinguishing Proper Coloring in Neutrosophic Graphs: Theory and Applications

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ABSTRACT
The article introduces the novel concept of the chromatic number of neutrosophic graphs, expanding upon the theoretical framework of graph coloring from crisp graphs to incorporate elements of neutrosophic logic. It explores the theoretical underpinnings of neutrosophic graphs and their coloring theory, aiming to broaden the understanding of graph coloring in uncertain environments. Definitions of neighboring vertex distinguishing suitable edge coloring for neutrosophic graphs are provided, leveraging λ -cuts and distance functions. The article establishes lower bounds on the chromatic number based on edge coloring properties, showcasing distinctions from general chromatic numbers. Furthermore, it highlights differences between adjacent vertex distinguishing chromatic numbers and offers insights into creating (p,f)- edge-neighbor distinguishing proper coloring (ENDPC) for neutrosophic graphs. Recommendations for further research into differentiating coloring of neutrosophic graphs are provided, indicating the ongoing exploration and potential applications of this emerging area of study.

Keywords: Chromatic number, neutrosophic graphs, edge-neighbor distinguishing proper coloring (ENDPC), edge coloring

1. INTRODUCTION

Everyone is aware that graph coloring has always been a significant issue in the study of graph theory and has significant theoretical and practical implications [1, 2]. In real life, graph coloring is frequently the best way to solve a variety of practical issues. Points may represent individuals, for instance, and lines would connect pairs of friends; vertices could represent animals, and lines would connect creatures that have the same attributes; or points could represent communication centers, with lines representing communication linkages [3-5]. Keep in mind that the major focus of these kinds of diagrams is whether or not two specified locations are connected by a line; the specifics of that connection are irrelevant. In traditional graph theory, the notion of graphs originated from the mathematical abstraction of this circumstance. The degree of a relation cannot be determined by the edge of a graph in traditional graph theory; only the existence of a binary relation can [6] be shown.

As an expansion of standard graph theory, neutrosophic graph theory is developed to account for ambiguity and indeterminacy in the representation of real-world data [7, 8]. The ability to express ambiguous or partial information is made possible in neutrosophic networks by the fact that vertices and edges can have degrees of truth, indeterminacy, or falsehood [9, 10]. This approach finds use in a variety of contexts where uncertainties are present, including network analysis, image processing, and decision-making. Compared to conventional crisp or fuzzy graph models, neutrosophic graphs have the benefit of being able to capture and describe complicated uncertainty more correctly [11].

In previous studies employing fuzzy set theory, a significant limitation arises in the lack of distinction between adjacent vertices, hindering precise analysis and coloring of edges. This limitation stems from the inability of fuzzy sets to adequately represent the degree of adjacency between vertices, leading to ambiguities in edge coloring. Neutrosophic graph theory addresses this challenge by providing a more nuanced representation of adjacency relationships, thus enabling the development of adjacent vertex distinguishing edge coloring schemes. This advancement allows for more refined graph analysis and coloring techniques, particularly in scenarios where precise differentiation between adjacent vertices is crucial for decision-making processes [12]. By incorporating the concept of neutrosophic graphs, the problem of ambiguous edge coloring inherent in previous fuzzy set approaches is effectively overcome. Neutrosophic graph theory offers a more comprehensive framework for representing uncertainties, allowing for the precise differentiation of adjacent vertices and facilitating accurate edge coloring schemes. This enhancement not only improves the analysis and visualization of graph structures but also enhances decision-making processes in various applications, ultimately advancing the utility and effectiveness of graph-based methodologies in handling uncertain data [13,14].

With its ability to provide detailed representations of intricate uncertainties, neutrophilic graph theory has become a potent paradigm for describing uncertainty in graph structures.This body of literature highlights the advantages of neutrosophic graphs over traditional crisp or fuzzy graph models, particularly in scenarios where uncertainties are prevalent and require more sophisticated handling. Furthermore, recent studies within the field have focused on addressing limitations inherent in previous fuzzy set approaches by introducing novel techniques such as adjacent vertex distinguishing edge coloring schemes [15]. These advancements have significantly enhanced the accuracy and applicability of graph-based methodologies in handling uncertain data, underscoring the importance of neutrosophic graph theory in modern research.

Researchers Akramet al. [16] looked at a graph structure that is bipolar and neutrosophic. the concepts of bipolar neutrosophic subgraph structures B_k , edges, and strong and full bipolar fuzzy graph structures. Describe certain relations and isomorphism qualities of bipolar neutrosophic graph topologies φ that are self-complementary, fully self-complementary, and totally strong self-complementary, using the suffix - complement. The significance of bipolar neutrosophic graph formations, which have several practical implications in psychology, global politics, and security.

A research on generalized graph representations of complicated neutrosophic information has been given by Siddique et al. [17]. Give numerous instances to further illustrate these qualities, along with some intriguing findings. Additionally, we define the complicated neutrosophic sets' accuracy function and scoring function. that explain decision-making analysis using type 1 generalized complex neutrosophic graphs. Using a traffic light system, Sudha et al. [18] proposed a bipolar triangular neutrosophic chromatic numbers. A minimal spanning tree issue on connections with neutrosophic numbers has been described by Adhikaryet al. [19].

In their presentation of pentapartitionedneutrosophic graphs (PPNGs), Quek et al. [20] highlight the value these graphs are for interpreting highly heterogeneous data that is frequently encountered in daily life, especially data collected from multiple sources, which is becoming more and more common in today's world. The suggested PPNG's applicability is shown by using it in a hypothetical real-world situation to combat the COVID-19 pandemic. In this scenario, the PPNGs are employed to identify the safest routes to travel and locations to stay in order to reduce the risk of contracting the virus. Both of these pieces of information have shown to be essential in the fight against the COVID-19 pandemic's spread as well as in giving the local economies—the majority of which are now experiencing a recession as a result of the pandemic's negative effects—the help they need.

A coloring of the spherical neutrosophic graph has been published by Akalyadeviet al. [21], who also describe several of its features, including union, complement of the strong neutrosophic graph, full spherical neutrosophic graph, and strong spherical neutrosophic graph, with an illustrative example. In neutrosophic graphs, Majeed et al. [22] have introduced a closed neutrosophic dominant set.

The cited research works collectively tackle the challenges posed by uncertainty and imprecision in diverse real-world scenarios through the development of advanced mathematical frameworks and models. Akram et al. delve into bipolar neutrosophic graph structures, Siddique et al. explore generalized graph representations of complex neutrosophic information, Sudha et al. introduce bipolar triangular neutrosophic chromatic numbers with practical applications in traffic systems, and Adhikary et al. address optimization problems in networks with neutrosophic numbers. Quek et al. propose pentapartitionedneutrosophic graphs for interpreting heterogeneous data, particularly demonstrating their utility in COVID-19 response strategies. Akalyadevi et al. discuss properties of spherical neutrosophic graphs, while Majeed et al. present closed neutrosophic dominating sets in neutrosophic graphs. These works collectively underline the significance of accommodating uncertainty in mathematical models, aiding decision-making processes across domains such as international relations, psychology, pandemic management, and network optimization.

The remainder of this paper is organized as follows. In Section 1: Introduction of review some concept of the chromatic number of neutrosophic graphs and the motivation behind the research. Basic concepts and lemmas in graph theory, neutrosophic set theory, and neutrosophic graph coloring are reviewed in

Section 2. These serve as the foundation for subsequent sections. We also introduce various types of neutrosophic graphs to contextualize the proposed methodologies. Section 3: Proposed Neutrosophic Chromatic Number presents our main contribution—the introduction of the proposed (p,f)- edge-neighbor distinguishing proper coloring. Subsection 3.1 further elaborates on the specific methodology of the proposed (p,f)-edge-neighbor distinguishing proper coloring.Section 4: Open Problems and Applications identifies open problems and potential research directions in neutrosophic graph coloring. It further highlights the usefulness and relevance of the suggested approaches by discussing their implementation in actual situations. Finally, Section 5: Conclusion summarizes the key findings and contributions of the paper, reflecting on their implications and suggesting future avenues for research in neutrosophic graph theory and coloring techniques.

2. Basic Concepts

To construct the chromatic number of neutrosophic graphs, we first review certain concepts and lemmas in graph theory, neutrosophic set theory, and neutrosophic graph coloring in this section. Following that, a few neutrosophic graphs are defined.

Explanation 2.1.A **neutrosophic graph** is a sharp graph $G \to (V,E)$ where $\delta = (\delta_1, \delta_2, \delta_3) : V \to [0,1]$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3) : E \to [0,1]$ is sufficient $\gamma(ab) \le \delta(a) \wedge \delta(b)$ for every $ab \in E$.

Explanation 2.2. If a **neutrosophicgraph** contains no edges, it is referred to as neutrosophic empty. Another name for it is neutrosophic trivial. Neutronosophic nontrivial refers to a neutrosophic graph that is not neutrosophic empty [23].

Explanation 2.3. When a neutrosophic graph $G \to (V,E)$ is complete $\gamma(ab) \le \delta(a) \wedge \delta(b)$ and for all $ab \in E$, it is referred to as a neutrosophic complete.

Explanation 2.4.A **neutrosophic strong** where $\gamma(ab) \le \delta(a) \vee \delta(b)$ for every $ab \in E$ is a neutrosophic graph $G \to (\delta, \gamma)$.

Explanation 2.5. The term "neutronosophic path" refers to a path $v_0, v_1, \ldots, v_x$, while "x-path," also known as $\gamma(v_x v_{x+1}) > 0, x = 0,1,\ldots, x-1$ "length of path," is a path with x edges.

Explanation 2.6. The term "neutosophic cycle" $v_0, v_1, \ldots, v_x, v_0$ refers to a crisp cycle with two edges ab and ij that is as $\gamma(ab) = \gamma(ij) = \bigcap_{a=0,1,\ldots x-1} \gamma(v_a v_{a+1})$.

Explanation 2.7. Assume $G \to (\delta, \gamma)$ we have a neutrosophic graph. The symbols g and h stand for the number of vertices and edges, respectively.

Explanation 2.8. Consider $\square = (\delta, \gamma)$ a graph that is neutrosophic. If there is at least one neutrosophic path connecting any two vertices, the connection is said to as neutrosophic linked.

Explanation 2.9. Consider $\square = (\delta, \gamma)$ a graph that is neutrosophic [24]. Assume that the path $W : v_0, v_1, \ldots, v_{x-1}, v_x$ after v_0 to v_x is $Min_{x=0,1,2,\ldots,i-1} \gamma(v_x v_{x+1})$ known as the neutrosophic strength of W, and that it is represented by $\square_S(W)$.

Explanation 2.10. Assume that $\square = (\delta, \gamma)$ the graph is neutrosophic. Among all vertices, the maximum number of edges for a vertex is represented by $\Omega(\square)$.

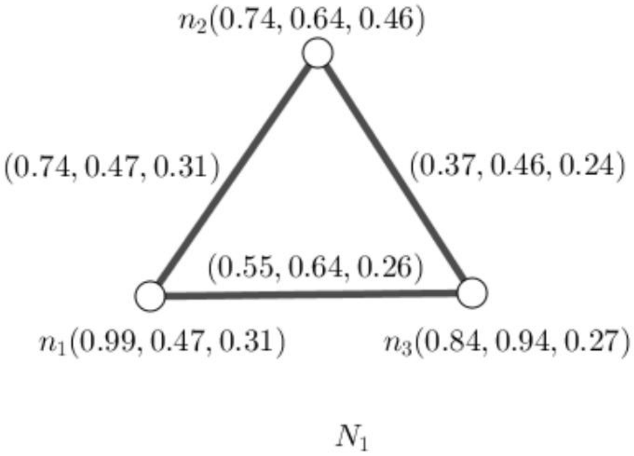


Figure 1. Neutrosophic Graph □$_1$

3. Proposed Neutrosophic Chromatic Number Based (P,F)- Edge-Neighbor Distinguishing Proper Coloring

The λ-cuts family of a neutrosophic set R definite on M may be used to describe it $R\lambda = \left\{ m \in M \left| \lambda R(M) \geq \lambda \right. \right\}, \lambda \in X$. This set family is monotone, meaning it $R\lambda \subseteq R\mu$ checks aimed at slightly $\lambda, \mu \in [0,1]$ fulfilling $\lambda \geq \mu$.

Consider $\left\{ NG_\lambda = (V, E_\lambda) \middle| \lambda \in X \right\}$ the set of λ-cuts family of □NG, where the fuzzy graph $NG_\lambda = (V, E_\lambda)$ with $E_\lambda = \left\{ v_{p,q} \left| v_p, v_q \in V, \alpha\left(v_{p,q}\right) \geq \lambda \right. \right\}$ is the λ-cut of a neutrosophic graph. The backing set E_0 of the neutrosophic edge set □E, or $E_0 = \sup p$□E is known as the support graph of the neutrosophic graph □NG, is specifically denoted as $NG_0 = (V, E_0)$ with $E_0 = \left\{ v_{p,q} \left| v_p, v_q \in V, \alpha\left(v_{p,q}\right) > 0 \right. \right\}$.

Definition 3.1. Assume $R = (\lambda, \mu)$ we have a neutrosophic graph. The chromatic number is the smallest number of unique colors that may be used to color vertices with strong neutrosophic edges. When the number of unique hues in the set is at its lowest among all of these sets, it is referred to as the neutrosophic chromatic number in relation to the initial degree.

Example 3.2. Examine Figure 1. Regarding the first order, the chromatic number is three and the neutrosophic chromatic number is 1.84.

Suggestion3.3. Consider $R = (\lambda, \mu)$ a comprehensive neutrosophic. In such case, r is the chromatic number and neutrosophic order is the neutrosophic chromatic number [25].

Proof. Every edge is very neutrosophic. With r-1 vertices, every vertex has an edge. Hence, the chromatic number is r. Because each vertex differs in color from every other vertex, V's neutrosophic cardinality corresponds to its neutrosophic chromatic number. Consequently, neutrosophic order equals neutrosophic chromatic number.

Suggestion 3.4. Assume $R = (\lambda, \mu)$ a strong neutrosophic route. subsequently the neutrosophic chromatic number is four and the number of chromatic molecules is $\underset{i,j,k \text{ and } l \text{ have different colors}}{Min} \left\{ \lambda(i) + \lambda(j) + \lambda(k) + \lambda(l) \right\}$.

Verification. Every vertex on the neutrosophic strong route that shares two edges has a unique color thanks to alternative colors. As a result, if i, j, k, and l are four vertices sharing a single edge, then each of them has a distinct hue. Thus, four is the chromatic number. The vertex with the lowest value among all the other vertices that share the same color is the representation of all colors. Consequently,

$$\underset{i,j,k \text{ and } l \text{ have different colors}}{Min} \left\{ \lambda(i) + \lambda(j) + \lambda(k) + \lambda(l) \right\}.$$

Suggestion 3.5. Assume $R=(\lambda,\mu)$ a strong neutrosophic cycle that is even. Then, the neutrosophic chromatic number is four and the number of chromatic molecules is

$$\underset{i,j,k\,and\;l\;have\;different\;colors}{Min}\left\{\lambda(i)+\lambda(j)+\lambda(k)+\lambda(l)\right\}.$$

Verification. Every edge is very neutrosophic. The vertices with a shared edge have a distinct hue since the cycle's vertices are even and have alternate coloring. So, four is the chromatic number. Every color is represented by the vertex with the lowest value among vertices that share that color. Consequently,
$\underset{i,j,k\,and\;l\;have\;different\;colors}{Min}\left\{\lambda(i)+\lambda(j)+\lambda(k)+\lambda(l)\right\}$.

Suggestion 3.6. Assume $R=(\lambda,\mu)$ that the N center is a neutrosophic strong star. Then, the neutrosophic chromatic number is four and the number of chromatic molecules is

$$\underset{i,j\;and\;k\;non-center\,vertex}{Min}\left\{\lambda(\mathrm{N})+\lambda(i)+\lambda(j)+\lambda(k)\right\}.$$

Verification. Every edge is very neutrosophic. Every vertex, including the center vertex, has a shared edge. Consequently, its hue differs from that of the other vertices. Thus, two colors have two vertices colors [26]. There is not a single common edge among any of the non-center vertices. These then have the same hue. A non-center vertex with the lowest value among all non-center vertices serves as the representation for this color. Therefore,
$\underset{i,j\;and\;k\;non-center\,vertex}{Min}\left\{\lambda(\mathrm{N})+\lambda(i)+\lambda(j)+\lambda(k)\right\}$.

Suggestion 3.7. Envision $R=(\lambda,\mu)$ a powerful neutrosophic field. If $R=(\lambda,\mu)$ only it is neutrosophic complete bipartite, the number of chromatic molecules becomes 4.

Verification. (⃞). Specify four as the chromatic number. As a result, each vertex has two or three vertices that share an edge. Since there are four colors, there are four sets, and each set's vertex are all the same color. Two vertices lack a shared edge if their colors are the same. As a result, every set is a part of it, and no vertex shares an edge. There are four of these sets. As a result, each of them has four components, and no vertex shares an edge with another. Is neutrosophic full bipartite $R=(\lambda,\mu)$ because $R=(\lambda,\mu)$ is a neutrosophic powerful.

(⃞). $R=(\lambda,\mu)$ is a full bipartite neutrosophicassumption. Then every edge is strongly neutrosophic. Each component has vertices that are not connected by edges. They are therefore given the same hue. There are four sections. As a result, there are four colors that may be applied to the vertices so that the vertices that share an edge have distinct colors. It results in a chromatic quantity of four.

Suggestion 3.8. Assume $R=(\lambda,\mu)$ we have a neutrosophic graph. At most, the number of vertices is the chromatic number, and the maximum value of the neutrosophic chromatic number is the neutrosophic order.

Proof. Chromatic number, or r, is the number of vertices in a chromatic number of neutrosophic complete when each vertex is a representation of each color. Neutrophic chromatic number is neutrosophic order and acute for neutrosophic complete when all vertices have different colors.

The relationship between the primary parameters of neutrosophic graphs and the neutrosophic chromatic number is calculated.

Suggestion 3.9. Allow $R=(\lambda,\mu)$ there be a powerful neutrosophic. When it happens, the number of chromatic molecules $\Omega+3$ is a minimum of 4.

Verification. Strong neutrosophicism is nontrivial neutrosophicism. It is therefore not neutrosophic empty, which implies that it lacks an edge. The chromatic number of two is implied. As the chromatic number is one if $R=(\lambda,\mu)$ and only if $R=(\lambda,\mu)$ it is neutrosophically trivial, and empty otherwise. A vertex with grade Ω is made up of Ω vertices that share edges with one another. In particular, a neutrosophic star has a chromatic number of four if these vertices have a single edge separating them. If not, then all vertices have edges that are close to one another, and the chromatic number is $\Omega+3$ particularly neutrosophic complete.

Suggestion 3.10. Consider $R=(\lambda,\mu)$ a standard neutrosophic Ψ. In that case, the chromatic number is maximum $\Psi+3$.

Verification. The Ψ neutrosophic regular is $R=(\lambda,\mu)$. Thus, every vertex has vertices Ψ that share an edge with it. In the case of a neutrosophic star, when these vertices share a single edge, the chromatic number is four. On the other hand, the number of chromatic particles is $\Psi+3$, for example, neutrosophic complete since the vertices share an edge.

Definition 3.11. In the case of a neutrosophic graph $NG=(V,E)$. For every $\lambda\in[0,1]$, assume $\{NG_\lambda=(V,E_\lambda)|\lambda\in[0,1]\}$ the family of λ -cuts set of. This results in a NG_λ simple graph devoid of isolated edges. The ψ_λ ENDPC function of the fuzzy graph $NG_\lambda=(V,E_\lambda)$ is an edge-neighbor distinguishing suitable coloring of NG a family of functions fulfilled $\{\psi_\lambda : E_\lambda \to \{1,2,...,n\}|\lambda\in[0,1]\}$.

Definition 3.12. An ENDPC chromatic number of NG is $\delta'_\lambda(NG)=best\{m_\lambda|\lambda\in[0,1]\}$ given a neutrosophic graph $NG=(V,E)$, $\{NG_\lambda=(V,E_\lambda)|\lambda\in[0,1]\}$ be, be it the family of λ -cuts set of NG. The chromatic quantity NG of in the ENDPC is $\delta'_\lambda(NG)$. This indicates that for any x minimum $\delta'_\lambda(NG)$, there exists a minimum of one α such that NG_λ x-ENDPC NG is absent. The collection of colors incident to the vertex v' of an x-ENDPC is v' indicated by $\Psi_\psi^\lambda(v')$, much like the symbols in a fuzzy graph. Thus, it is possible to define the x-ENDPC function ψ_λ^x on. Via this sequence NG_λ, the x-ENDPC function of NG is defined.

Remark. It is observed that in cases when the membership function α of V is a zero matrix, the neutrosophic graph $NG=(V,E)$ degenerates to a fuzzy graph ϕ known as a zero graph. The graph's chromatic number is zero since no two of its vertices are neighboring. Furthermore, the neutrosophic graph NG lacks the ENDPC specified in Definition 3.11 if it has isolated edges NG or NG_λ if a vertex in the λ -cuts set of includes isolated vertices. The explanation of ENDPC for neutrosophic graphs with rings and parallel edges is compatible with the case of different graphs. Consequently, only basic neutrosophic graphs with no isolated edges are covered by the ENDPC.

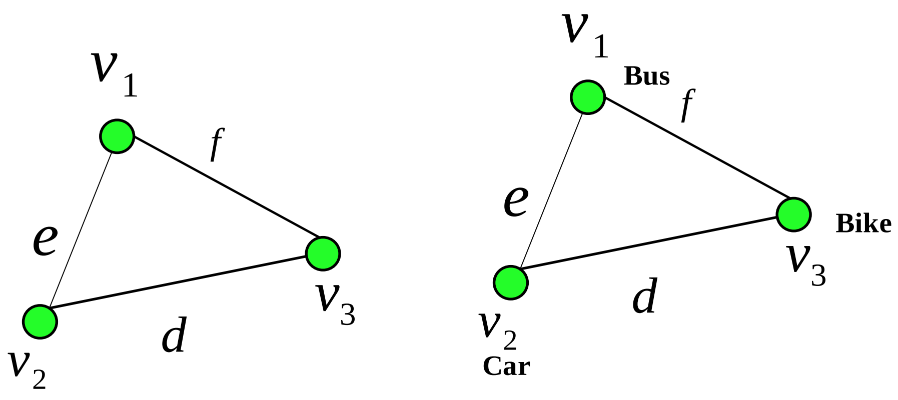
Example 1. The following $V=(v_1,v_2,v_3)$ is the membership function V of, which α is represented as a neutrosophic graph $NG=(V,E)$ in Figure 2.

$$\alpha=\begin{bmatrix} - & e & f \\ d & - & e \\ e & f & d \end{bmatrix}$$

The above vertex coloring NG of can be used as a model for the traffic light problem in certain ways; see [16]. The categories low, medium, and high are indicated by the letters d, e, and f, respectively; d < l < e < f, d, e, f ∈ [0, 1]. A vertex NG in an ENDPC of a neutrosophic network might be seen as an item, such a bus, automobile, or bike, that the artificial intelligence apparatus needs to be able to identify.

Finding a neutrosophic graph's chromatic number and related coloring function is the goal of the neutrosophiccoloring issue. Our method will calculate λ the bare minimum of colors required to color the fuzzy graph NG_λ at any level. The λ -cuts of the ENDPC will therefore specify its chromatic number.

Three fuzzy graphs $NG_\lambda=(V,E_\lambda)$ in the neutrosophic graph NG are produced by taking the values $\lambda\in X$ into account $\lambda\in X$. Table 1 displays the color set $\Psi_\psi^\lambda(v)$ of each vertex V under the coloring function ψ_λ as well as the ENDPC chromatic number δ'_λ of each NG_λ.



(a) (b)

Figure 2. (a) Example 1's neutrosophic graph NG and (b) the neutrosophic graph X

Table 1. The graph's NG_λ chromatic number δ'_λ and the color set $\Psi^\lambda_\psi(v')$ of its vertices

λ	NG_λ	δ'_λ	$\Psi^\lambda_\psi(v_1)$	$\Psi^\lambda_\psi(v_2)$	$\Psi^\lambda_\psi(v_3)$
d	NG_d	2	$\{1,2,3\}$	$\{2,3\}$	$\{1,2\}$
e	NG_e	1	$\{1\}$	$\{1,2\}$	$\{2,3\}$
f	NG_f	0	ϕ	$\{1,2\}$	ϕ

Figure 3 displays the particular coloring function (or approach ψ_λ) for the neutrophilic graph NG_λ found in column 2 of Table 1.

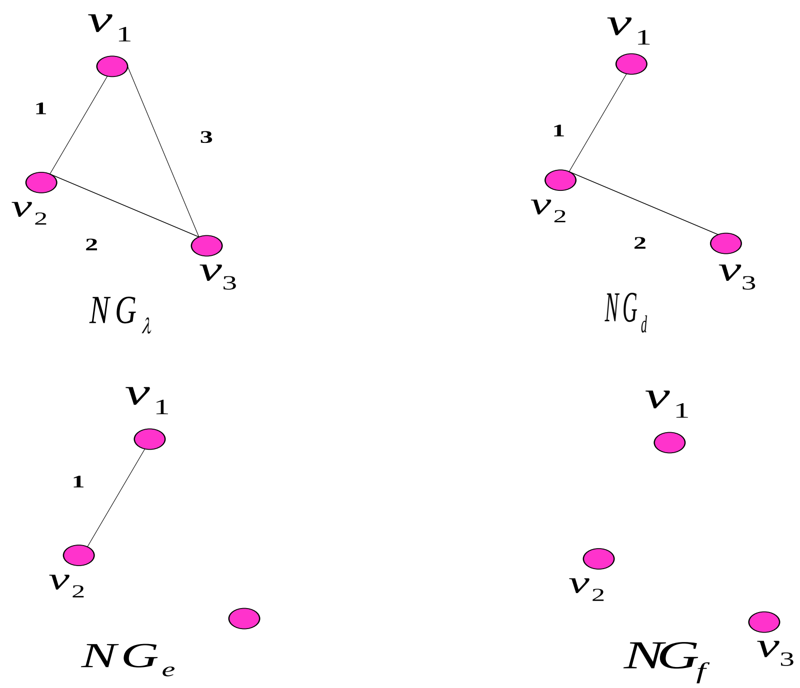


Figure 3. Neutrosophic graph NG_λ and the ENDPC function of NG_λ in Example 1

The fuzzy graph NG in Figure 2's vertices v_1, v_2, v_3 represent animals in an artificial intelligence recognition task. The degree of the neutrosophic edge reflects the degree of conflict. Let the edges represent the conflicting nature of the same attribute between pairs of vertices. Assigning a High (f), Low (d), or Medium degree of conflict of edges is based on the objective qualities of various animals (e). Let the scenario be represented by the following neutrosophic graph X in Figure 2. The neutrosophic graph's NG color number $\delta'_\lambda(NG) = Max\{\delta'_\lambda(NG_\lambda) | \lambda \in [0,1]\}$ may be understood as follows: a higher chromatic number indicates that artificial intelligence equipment need more information (i.e., color) to discriminate between two vehicles, while lower values λ are related with smaller attribute differences. Conversely, greater values λ of indicate that less information is required to identify the two vehicles when the chromatic number is smaller.

Any algorithm that calculates the chromatic number of each neutrosophic graph can be used to solve the neutrosophic ENDPC issue.

Theorem 3.1. Let $NG_1(V_1, E_1)$ and $NG_2(V_2, E_2)$ be two neutrosophic graphs. If $NG_1 \cong NG_2$, then $\delta'_\lambda(NG_1) = \delta'_\lambda(NG_2)$.

Proof. As $NG_1 \cong NG_2$ there is a bijection $\rho : v_1 \to v_2$ that allows $\alpha_1(u,v) = \alpha_2(\rho(u), \rho(v))$ for any $u, v \in v_1$. A bijection $v(NG_\lambda^1) \to v(NG_\lambda^2)$ exists such that $\alpha_1(u,v) = \alpha_2(\rho^*(u), \rho^*(v))$ for any $u, v \in (NG_\lambda^1)$, i.e. $NG_1 \cong NG_2, \lambda \in [0,1]$, because of $v_1 = v(NG_\lambda^1)$ and $v_2 = v(NG_\lambda^2)$.

Definition 3.13

Assume $NG(V, E)$ that is a neutrosophic graph and NG_0 that is its sustaining graph NG. The number $\Omega(NG) = \Omega(NG_0)$ represents the highest degree of NG.

Certain findings drawn from the neutrosophic graph may be naturally extended to the neutrosophic graphs that have neutrosophic edges and vertices, based on Definition 3.11. Two isomorphic neutrosophic graphs have an ENDPC chromatic number of equal, and neither neutrosophic graph's ENDPC chromatic number is smaller than its highest degree, as demonstrated by Theorems 3.1 and 3.2.

Theorem 3.2. In the case of a neutrosophic graph $NG(V, E)$. Consider $\{NG_\lambda = (V, E_\lambda) | \lambda \in [0,1]\}$ the λ-cuts set of NG-family members. If neutrosophic graph NG_α without isolated edges for all $\lambda \in [0,1]$, then $\delta'_\lambda(NG) \geq \Omega(NG)$.

Proof. According to Definition 3.13, $\Omega(NG) = \Omega(NG_0) = Max\{\Omega(NG_\lambda) | \lambda \in [0,1]\}$. It is clear that $\delta'_\lambda(NG) \geq \Omega(NG)$ by Lemma 1 and the definition of the ENDPC chromatic number of the neutrosophic graph.

3.1. Proposed Edge Coloring with (p,f)- ENDPC Algorithm

Definition 3.14. Be the R collection of colors that are accessible. p is a measure of distance that $p : R \times R \to [0, \infty]$ has the characteristics listed below.

(a) $p(a,b) \geq 0, \forall a, b \in R$,

(b) $p(a,b) = 0 \Leftrightarrow a = b, \forall a, b \in R$,

(c) $p(a,b) = 0, \forall a, b \in R$.

The essence of the graph coloring problem lies in minimizing the grouping of edges or vertices within a graph. Objects, whether edges or vertices, are deemed incompatible if they are subject to different constraints and thus cannot be grouped together. The function p, serving as a distance measure, quantifies the level of incompatibility between adjacent neutrosophicedges. Consequently, the greater the incompatibility between two edges, the further apart their associated colors will be in the extended coloring function.

The membership function image of a neutrosophic graph $NG = (V, E)$ is given $\Psi \subseteq [0,1]$. To define the scale function, we suppose that there is an order < specified on the elements of Ψ.

Definition 3.15. Give it a go $\Psi \subseteq [0,1]$. If the function $f(\alpha) \le f(\alpha')$, $\forall \alpha, \alpha' \in \Psi$ is such that $\alpha < \alpha'$, then $f : \Psi \to [0, \infty]$ it is referred to as a scale function.

The definition that follows follows from the introduction of the scale function and distance measure above.

Definition 3.16. Suppose we have a neutrosophic network $NG = (V, E, D, f)$ with α a membership function of E. A $M_{p,f} : E \to X$ mapping meeting the specified criteria is called a (p, f)-extended ENDPC function of NG, indicated by $M_{p,f}$.

(a) $p\left(M_{p,f}(v_{x,y}), M_{p,f}(v_{x,n})\right) \ge \wedge\left\{f(\alpha_{xy}), f(\alpha_{xn})\right\}$ for everyedges $v_{x,y}$ and $v_{x,n}$,

(b) $X_{M_{p,f}}(v_x) \ne X_{M_{p,f}}(v_y)$ if $\alpha_{xy} > 0$.

$\delta'_{\lambda_{p,f}}(NG)$ it is known as the (p, f) -ENDPC chromatic number of NG, and it is the lowest value x for which a (p, f)-extended x- ENDPC of NG exists.

Remark 2. Additionally, it should be emphasized that there is no distinction between the two ways that the same neutrosophic edge may be written in the neutrosophic graph $NG = (V, E)$; that is, for any $p, q \in V(NG)$, pq and qp, they denote the same neutrosophic edge. The values of pq, pr and qp, rp show that the two neutrosophic edges, pq and pr, are next to one another. As a result, another way to express condition

(a) in Definition 3.15 is as follows.

(b) $p\left(M_{p,f}(v_{x,y}), M_{p,f}(v_{x,n})\right) \ge \wedge\left\{f(\alpha_{xy}), f(\alpha_{xn})\right\}$ for everyedges $v_{x,y}$ and $v_{n,y}$

An (p, f) ENDPC function $M^k_{p,f}$ is (p, f) one that can accept a maximum of k distinct colors. Put otherwise, $M^k_{p,f} : E \to Z$ a situation when $Z = \{1, 2, \ldots, k\}$ the following requirements are met.

(a) $p\left(M^t_{p,f}(v_{x,y}), M^t_{p,f}(v_{x,n})\right) \ge \wedge\left\{f(\alpha_{xy}), f(\alpha_{xn})\right\}$ for eachedges $v_{x,y}$ and $v_{n,y}$

(b) $X_{M^t_{p,f}}(v_x) \ne X_{M^t_{p,f}}(v_y)$ if $\alpha_{xy} > 0$.

One might envision of the (p, f) ENDPC of a neutrosophic graph $NG = (V, E)$ as an extension of the ENDPC of a fuzzy graph $FG = (V, E)$. Let's say that $\Psi = \{0,1\}$, $f(0) = 0$, $f(1) = 1$ and $p = p_0$ is p_0 defined as

$$p_0(a,b) = \begin{cases} 1, & if \;\; a \ne b \\ 0, & if \;\; a = b \end{cases}$$

In the following sense, the coloring provided in Definition 3.16 is not the same as the ENDPC of a fuzzy graph. Using a color that doesn't beyond its matching chromatic number is the standard procedure for graph coloring theory.

Algorithm: The algorithm for $x-(p, f)$ - ENDPC of neutrosophic graphs
Source:ENDPC's adjacency matrix; **Result:**The adjacent matrix of $x-(p, f)$ ENDPC.
Step 1: The greatest value of each element in the matrix $M[\;][\;]$ is then stored in x after we input the adjacency matrix $M[\;][\;]$ of the ENDPC of the provision graph NG_0 of the neutrosophic graph NG; **Step 2:**Examine the matrix $M[\;][\;]$,

In order x=1 with $\left|V\left(\tilde{N}G\right)\right|$ initiate

In order y=1 with $\left|V\left(\tilde{N}G\right)\right|-1$ initiate

In order n=y+1 with $\left|V\left(\tilde{N}G\right)\right|$ initiate

After $M[x][y]>0, M[x][n]>0$, computed $p\left(M[x][y], M[x][n]\right)$,

If $p\left(M[x][y], M[x][n]\right)<\wedge\left\{f\left(\alpha_{xy}\right), f\left(\alpha_{xn}\right)\right\}$

Then $M[x][y]\leftarrow x+1, M[y][x]\leftarrow x+1$ *While* $M[x][y]>M[x][n]$

Else $M[x][n]\leftarrow x+1, M[n][x]\leftarrow x+1$

$x\leftarrow x+1$

End
End
End

Step 3: Step 2 is re-implemented until the elements of the new matrix $M[\ ][\ ]$ all satisfy condition (i);

Step 4: $x = Max\left(Max\left(M[\ ][\ ]\right)\right)$;

Step 5: Output $M[\ ][\ ]$, x.

4. Open Problems

The definitions of the chromatic number and neutrosophic chromatic number in neutrosophic graphs rely on distinguishing adjacent vertices through appropriate edges. The presence of neutrosophic strong vertex differentiating suitable edges plays a crucial role in establishing these concepts.

Query 1: In neutrosophic graphs, can different kinds of neighboring edges be used to define the chromatic number as well as the neutrosophic chromatic number?

Query 2: Can the number of colors and neutrosophic chromatic number be determined using any other numerical methods?

Query 3: What specific classes of neutrosophic graphs are suitable for conducting independent studies in this context?

Query 4: In what practical applications could an independent study on defining the chromatic number and neutrosophic chromatic number find relevance?

Problem 5: What methodologies can be employed to develop applications for conducting independent studies on defining the chromatic number and neutrosophic chromatic number?

Problem 6: How can definitions be formulated that utilize all three arrays and their interrelations, rather than relying solely on one array, to facilitate independent studies?

4.1. Application

Social Network Analysis

Problem Statement

In social network analysis, researchers aim to understand the structure, dynamics, and behavior of social networks. One key aspect is community detection, which involves identifying groups of individuals with dense connections within the network. Traditional graph coloring methods can be applied to partition the network into distinct communities. However, uncertainty and ambiguity in social relationships necessitate the use of more advanced techniques like neutrosophic graphs and ENDPC.

Neutrosophic Graph Representation

In a social network represented as a neutrosophic graph, each vertex corresponds to an individual, and edges represent relationships between them. It is unclear if social links exist or are strong, and this is captured by the degrees of truth, indeterminacy, and falsehood corresponding to each edge. This representation allows for a more nuanced analysis of social networks, considering the varying levels of trust, influence, and ambiguity in relationships.

Edge-Neighbor Distinguishing Proper Coloring
ENDPC involves coloring the edges of the neutrosophic graph in such a way that each vertex is uniquely distinguished from its neighbors by the colors of its incident edges. This coloring scheme enables the identification of individuals who play distinct roles within their social circles, such as opinion leaders, mediators, or connectors between communities.

Example 2
Consider a small social network of individuals A, B, C, D, and E, with the following relationships:

Table 2. Social Network Relationships

Individual	Friends
A	B, C
B	A, D
C	A, E
D	B, E
E	C, D

This social network can be represented as a neutrosophic graph, where each vertex represents an individual, and edges represent relationships between them. The ambiguity around the presence or strength of social links is captured by the degrees of truth, indeterminacy, and falsehood corresponding to each edge.
Now, let's apply ENDPC to identify key individuals and communities within this social network.

Working Process

- **Data Collection and Preprocessing:**
  Gather data on social interactions and preprocess it to construct the neutrosophic graph representation. Depending on which you think the correlations are, give each edge a degree of truth, indeterminacy, or falsehood.
- **Edge-Neighbor Distinguishing Proper Coloring:**
  Utilize ENDPC-specific methods to color the neutrosophic graph's edges. Ensure that each vertex is uniquely identifiable based on the colors of its incident edges.
- **Community Detection and Analysis:**
  Analyze the coloredneutrosophic graph to detect communities or clusters of individuals with dense intra-group connections. Identify vertices with similar edge colors, indicating cohesive social groups.
- **Interpretation and Insights:**
  Interpret the results of community detection and vertex roles based on the ENDPC coloring. Gain insights into the underlying social dynamics, such as the formation of cliques, the spread of information or influence, and the emergence of social hierarchies or subgroups.

  Benefits and Implications:
- **Improved Community Detection:**Neutrosophic graphs and ENDPC enable more accurate and nuanced community detection by considering uncertainty and ambiguity in relationships.
- **Identification of Key Individuals:** ENDPC facilitates the identification of key individuals who play unique roles within their social circles, providing valuable insights for targeted interventions or social interventions.
- **Enhanced Understanding of Social Dynamics:** By incorporating uncertainty-aware graph coloring techniques, researchers can gain a deeper understanding of the complex dynamics shaping social networks, including the influence of trust, ambiguity, and diverse perspectives on social interactions.

This example demonstrates how neutrosophic graphs and ENDPC can be applied in social network analysis to uncover hidden patterns, dynamics, and influential actors within social networks, leading to more informed decision-making and interventions.

5. CONCLUSION

In summary, the development and implementation of the (p,f)-edge-neighbor distinguishing proper coloring (ENDPC) algorithm for neutrosophic graphs not only represent a significant advancement in graph theory but also offer practical solutions for modeling uncertainty in real-world networks. By leveraging concepts from neutrosophic logic and graph theory, this algorithm provides a systematic approach to differentiate colors in uncertain graph structures, addressing the challenges posed by indeterminate elements. Through our research, we have not only expanded the theoretical framework of graph coloring but also established practical techniques for addressing uncertainty in complex systems. The insights gained from this study, including lower bounds on the chromatic number and disparities between adjacent vertex distinguishing chromatic numbers, provide valuable contributions to both theoretical understanding and practical applications. Looking ahead, further exploration and refinement of the ENDPC algorithm hold great potential for advancing our understanding of uncertain environments and fostering innovation in graph theory and related disciplines.

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