Multi Sum Identities Involving q-Basic Series

Priya Wadhwa[1], A. K. Shrivastav[2*], Harwinder Kaur[3]

[1,3]Department of Mathematics, RIMT University, FGS, Punjab, India147301
[2]Department of Mathematics, MMEC, Maharishi Markandeshwar (Deemed to be University), Mullana-Ambala, Haryana, India 133207
Email: priyapuri093@gmail.com[1], aakkaasshhkumar8888@gmail.com[2], harwinderkaur808@gmail.com[3]

Received: 18.04.2024 Revised : 24.05.2024 Accepted: 26.05.2024

ABSTRACT
The q-basic infinite products can be used to express certain multi-summation identities of Rogers-Ramanujan. This work attempts to prove certain results concerning the ratio of q-basic infinite products and multi summation expressions utilizing well-known m- dissections of the power series.

Keywords: Bailey pair's, bailey lemma, m-dissection, RR-type identities.

1. INTRODUCTION

Both applied mathematicians and pure mathematicians from earlier ages have been drawn to continued fractions. Numerous conclusions from earlier centuries have been produced in terms of continuous fraction. It's additionally a tool that serves as a conduit between applied and pure mathematicians. Thus, the continuous fraction has also become more appealing to mathematicians nowadays.

Ramanujan's second notebook [5] contains numerous continuing fraction identities, as he was a trailblazer in the field of continued fraction theory. The techniques employed by the renowned Indian mathematician SrinivasaRamanujan to arrive at numerous intriguing findings are still a mystery. Several authors, including Andrews, have addressed and developed this portion of Ramanujan's work. Simultaneously Number theory, orthogonal polynomials, affine root systems, Lie algebras and groups, physics, and other sciences all heavily rely on q-Series, often known as basic hypergeometric series.

Among the helpful resources for studying special functions isthe inequality method; see [17] for a list of numerous works discussing q-integral and inequalities.

The m-dissection of the power series;

$$P=\sum_{n=0}^{\infty} a_n q^n$$

is the representation of P as

$$P = P_0+P_1 \dots\dots\dots + P_{m-1}, \text{where } P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$$

From last 3 decades, a number of researchers [7], [16] & [20] have given the 2 dissection, 4 dissection and 5 dissections of the continued fraction. Denis et al. [19] gave equivalent continued fraction representations for ratios of infinite products.

$$S(q)=\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}$$
$$=1+\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^5}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+}\frac{q^{12}}{1+\cdots}$$
$$=1+\frac{q+q^2}{1-q+}\frac{q+q^4}{1-q+}\frac{q+q^6}{1-q+\cdots}$$
$$=1+q+\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^5}{1+q^7+\cdots}$$

One well-known and often applied method in the theory of partitions is the Bailey chain. It enlightens from W. N. Bailey's formula [21] that the Rogers-Ramanujan identities might be obtained from the simple observation that if $\{\alpha_0,\alpha_1,\dots\}$ and $\{\delta_0,\delta_1,\dots\}$are sequences that satisfy;

$$\beta_k=\sum_{r=0}^{k}\alpha_r u_{k-r} v_{k+r}$$

and

$$\gamma_k = \sum_{r=k}^{\infty} \delta_r u_{r-k} v_{r+k},$$

Then

$$\sum_{k=0}^{\infty} \alpha_k \gamma_k = \sum_{k=0}^{\infty} \beta_k \delta_k,$$

which stipulated consistent convergence of all infinite sums.
His concept served as the basis for L.J. Slater's list of 130 identities that fit the Rogers-Ramanujan type [13, 14]. Over the past twenty years or more, several authors have explored intriguing and practical applications of Bailey's identity (see for e.g., [1–4]; see also [9-12], the very recent work).
Here an attempt has been made to obtain certain results involving the multi summation expressions and ratio of infinite products by using well known m dissections of the power series.

2. Notation

Suppose that $|q| < 1$, where q is non-zero complex number, this condition ensures that all the infinite products that we will converge. We will use the notation as follows;

$$(z;q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n),$$

(2.1)

$$[z;q]_\infty = (z;q)_\infty (z^{-1}q;q)_\infty; \text{ (for } z \neq 0\text{)and often we write,}$$

(2.2)

$$[z_1, z_2, \dots \dots \dots z_n; q]_\infty = [z_1;q]_\infty [z_2;q]_\infty \dots \dots \dots [z_n;q]_\infty.$$

(2.3)

And also the following fact can be easily verified;

$$[z^{-1};q]_\infty = -z^{-1}[z;q]_\infty = [zq;q]_\infty,$$

(2.4)

$$[z, zq; q^2]_\infty = [z;q]_\infty,$$

(2.5)

$$[z, -z; q]_\infty = [z^2; q^2]_\infty,$$

(2.6)

$$[z^{-1}q; q]_\infty = [z;q]_\infty,$$

(2.7)

$$[-1;q]_\infty [q;q^2]_\infty = 2.$$

(2.8)

We have the following general relations;
Suppose $a_1, a_2, \dots \dots \dots, a_n; b_1, b_2, \dots \dots \dots, b_n \in C \setminus \{0\}$ satisfy
i) $a_i \neq q^n a_j$ for $i \neq j$ and $n \in z$,
ii) $a_1, a_2, \dots \dots \dots, a_n = b_1, b_2, \dots \dots \dots, b_n$,
Then

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n} [a_i^{-1} b_j; q]_\infty}{\prod_{j=1, j\neq i}^{n} [a_i^{-1} a_j; q]_\infty} = 0$$

(2.9)

This theorem appears without proof as given by Slater [4] and with a proof as given by Lewis [8].
Also, we have the following well known Rogers- Ramanujan identity as;

$$\sum_{s_1,s_2,s_3 \geq 0} \frac{q^{9s_1^2+3s_2^2+s_3^2} (q^3;q^3)_{s_1-s_2} (q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1} (q^9;q^9)_{s_1-s_2} (q^3;q^3)_{2s_2} (q^3;q^3)_{s_2-s_3} (q;q)_{s_3}} = \frac{(q^{29}, q^{14}, q^{15}; q^{29})_\infty}{(q^9;q^9)_\infty}$$

(2.10)

$$\sum_{s_1,s_2,s_3 \geq 0} \frac{q^{9s_1^2+3s_2^2+s_3^2} (q^3;q^3)_{s_1-s_2} (q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1} (q^9;q^9)_{s_1-s_2} (q^3;q^3)_{2s_2} (q^3;q^3)_{s_2-s_3} (q;q)_{s_3}} = \frac{(q^{29}, q^{14}, q^{15}; q^{29})_\infty}{(q^9;q^9)_\infty}$$

(2.11)

[D.Bressoud,1;p452]

3. Main Result

$$\sum_{s_1,s_2,s_3\geq 0} \frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}}$$
$$= \frac{(q^5,q^{24},q^{29};q^{29})_\infty}{(q^9;q^9)_\infty}\Bigg[\frac{(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{28},q^{88};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{10},q^{18},q^{58},q^{58},q^{98},q^{106};q^{116})_\infty}$$
$$+\frac{q^9(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{30},q^{86};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{48},q^{58},q^{58},q^{68},q^{98};q^{116})_\infty}$$
$$+\frac{q^9(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{30},q^{86};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{48},q^{58},q^{58},q^{68},q^{98};q^{116})_\infty}$$
$$-\frac{q^2(q^3,q^4,q^{22},q^{36},q^{54},q^{55};q^{58})_\infty(q^{32},q^{34},q^{84},q^{86};q^{116})_\infty}{(q^{16},q^{17},q^{41},q^{42};q^{58})_\infty(q^8,q^{10},q^{48},q^{58},q^{58},q^{68},q^{106},q^{108};q^{116})_\infty}\Bigg]$$

(3.1)

$$\frac{(q^5,q^{24},q^{29};q^{29})_\infty}{(q^9;q^9)_\infty} = \Bigg[\frac{(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{48},q^{68};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{30},q^{58},q^{58},q^{86},q^{98};q^{116})_\infty}$$
$$+\frac{q^9(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{10},q^{106};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{30},q^{58},q^{58},q^{86},q^{98};q^{116})_\infty}$$
$$+\frac{q^2(q^3,q^4,q^{22},q^{36},q^{54},q^{55};q^{58})_\infty(q^{32},q^{34},q^{82},q^{84};q^{116})_\infty}{(q^{16},q^{17},q^{41},q^{42};q^{58})_\infty(q^8,q^{28},q^{30},q^{58},q^{58},q^{86},q^{88},q^{108};q^{116})_\infty}$$
$$-\frac{q^2(q^4,q^{12},q^{13},q^{45},q^{46},q^{54};q^{58})_\infty(q^{14},q^{52},q^{64},q^{102};q^{116})_\infty}{(q^7,q^{26},q^{32},q^{51};q^{58})_\infty(q^8,q^{28},q^{30},q^{58},q^{58},q^{86},q^{88},q^{108};q^{116})_\infty}\Bigg]$$
$$\times\sum_{s_1,s_2,s_3\geq 0} \frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}}$$

(3.2)

4. Proofs

On considering

$$\mathrm{C(q)} = \frac{(q^{14},q^{15};q^{29})_\infty}{(q^5,q^{24};q^{29})_\infty},$$

which can be written as

$$\mathrm{C(q)} = \frac{[q^{14},q^{43};q^{58}]_\infty}{[q^5,q^{34};q^{58}]_\infty}$$

by using (2.2).

Now, setting $(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4)$ = $(1,\text{-}1,q^5,q^{34};q^{14},q^{43},-q^{-9},q^{-9})$ and taking q^{58} for q in (2.9);

$$\frac{[q^{14},q^{43},-q^{-9},q^{-9};q^{58}]_\infty}{[-1,q^5,q^{34};q^{58}]_\infty}+\frac{[-q^{14},-q^{43},-q^{-9},q^{-9};q^{58}]_\infty}{[-1,-q^5,-q^{34};q^{58}]_\infty}$$
$$+\frac{[q^9,q^{38},-q^{-14},q^{-14};q^{58}]_\infty}{[q^{-5},-q^{-5},q^{29};q^{58}]_\infty}+\frac{[q^9,q^{-20},-q^{-43},q^{-43};q^{58}]_\infty}{[q^{-34},-q^{-34},q^{-29};q^{58}]_\infty}=0$$

(4.1)

By using (2.6) and (2.8) in (4.1)

$$\frac{[q^{14},q^{43};q^{58}]_\infty}{[q^5,q^{34};q^{58}]_\infty}+\frac{[-q^{14},-q^{43};q^{58}]_\infty}{[-q^5,-q^{34};q^{58}]_\infty}$$
$$=\frac{2}{[q^{18},q^{58};q^{116}]_\infty(-q^{-18})}\Bigg[-\frac{[q^9,q^{38};q^{58}]_\infty[q^{-28};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{-10};q^{116}]_\infty}-\frac{[q^9,q^{-20};q^{58}]_\infty[q^{-86};q^{116}]_\infty}{[q^{-29};q^{58}]_\infty[q^{-68};q^{116}]_\infty}\Bigg]$$

By applying (2.4),

$$\frac{[q^{14},q^{43};q^{58}]_\infty}{[q^5,q^{34};q^{58}]_\infty}+\frac{[-q^{14},-q^{43};q^{58}]_\infty}{[-q^5,-q^{34};q^{58}]_\infty}=\frac{2}{[q^{18},q^{58};q^{116}]_\infty(-q^{-18})}$$
$$\times\Bigg[-\frac{[q^9,q^{38};q^{58}]_\infty[q^{28};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{10};q^{116}]_\infty}(q^{-18})-\frac{[q^9,q^{20};q^{58}]_\infty[q^{86};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{68};q^{116}]_\infty}(q^{-9})\Bigg]$$

Or,

$$\frac{[q^{14},q^{43};q^{58}]_\infty}{[q^5,q^{34};q^{58}]_\infty}+\frac{[-q^{14},-q^{43};q^{58}]_\infty}{[-q^5,-q^{34};q^{58}]_\infty}$$

$$=\frac{2}{[q^{18},q^{58};q^{116}]_\infty}\left[\frac{[q^9,q^{38};q^{58}]_\infty[q^{28};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{10};q^{116}]_\infty}+\frac{q^9[q^9,q^{20};q^{58}]_\infty[q^{86};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{68};q^{116}]_\infty}\right]$$

Thus,

$$\mathrm{C(q)}+\mathrm{C}'(\mathrm{q})=\frac{2[q^9,q^{38};q^{58}]_\infty[q^{28};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{10},q^{18},q^{58};q^{116}]_\infty}+\frac{2q^9[q^9,q^{20};q^{58}]_\infty[q^{86};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{68};q^{116}]_\infty} \tag{4.2}$$

$$\alpha_1(q)\frac{1}{2}\left[\mathrm{C(q)}+\mathrm{C}'(\mathrm{q})\right]$$
$$=\frac{[q^9,q^{38};q^{58}]_\infty[q^{28};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{10},q^{18},q^{58};q^{116}]_\infty}+\frac{q^9[q^9,q^{20};q^{58}]_\infty[q^{86};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{68};q^{116}]_\infty}$$

(4.3)

Again, setting $(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4)$ = (1, -1,$q^{56},-q^{60};q^{44},q^{43},-q^5,-q^{24}$) and taking$q^{58}$ for q in (2.9),

$$\frac{[q^{44},q^{43},-q^5,-q^{24};q^{58}]_\infty}{[-1,q^{56},-q^{60};q^{58}]_\infty}+\frac{[-q^{44},-q^{43},q^5,q^{24};q^{58}]_\infty}{[-1,-q^{56},q^{60};q^{58}]_\infty}$$
$$+\frac{[q^{-12},q^{-13},-q^{-51},-q^{-32};q^{58}]_\infty}{[q^{-56},-q^{-56},-q^4;q^{58}]_\infty}+\frac{[-q^{-16},-q^{-17},q^{-55},q^{-36};q^{58}]_\infty}{[q^{-60},-q^{-60},-q^{-4};q^{58}]_\infty}=0$$

By applying (2.4) and (2.7),

$$\frac{q^2[[q^{14},q^{43},-q^5,-q^{34};q^{58}]_\infty-[-q^{14},-q^{43},q^5,q^{34};q^{58}]_\infty]}{[-1,q^2,-q^2;q^{58}]_\infty}$$
$$-\frac{q^4[q^{12},q^{13},-q^{32},-q^{51};q^{58}]_\infty}{[q^{56},-q^{56},-q^4;q^{58}]_\infty}-\frac{[-q^{16},-q^{17},q^{36},q^{55};q^{58}]_\infty}{[q^{60},-q^{60},-q^4;q^{58}]_\infty}=0$$

By using (2.6)

$$[[q^{14},q^{43},-q^5,-q^{34};q^{58}]_\infty-[-q^{14},-q^{43},q^5,q^{34};q^{58}]_\infty]$$
$$=\frac{2[q^4;q^{116}]_\infty}{q^2[q^{58};q^{116}]_\infty}\left[\frac{q^4[q^{12},q^{13},-q^{32},-q^{51};q^{58}]_\infty}{[q^{56},-q^{56},-q^4;q^{58}]_\infty}+\frac{[-q^{16},-q^{17},q^{36},q^{55};q^{58}]_\infty}{[q^{60},-q^{60},-q^4;q^{58}]_\infty}\right]$$

Or,

$$[[q^{14},q^{43},-q^5,-q^{34};q^{58}]_\infty-[-q^{14},-q^{43},q^5,q^{34};q^{58}]_\infty]$$
$$=\left[\frac{2q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{58};q^{116}]_\infty}-\frac{2q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{58};q^{116}]_\infty}\right]$$

Dividing by $[-q^5,q^5,q^{34},-q^{34};q^{58}]_\infty$ and using (2.6),

$$\frac{[q^{14},q^{43};q^{58}]_\infty}{[q^5,q^{34};q^{58}]_\infty}-\frac{[-q^{14},-q^{43};q^{58}]_\infty}{[-q^5,-q^{34};q^{58}]_\infty}$$
$$=\frac{1}{[q^{10},q^{68};q^{116}]_\infty}\left[\frac{2q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{58};q^{116}]_\infty}-\frac{2q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{58};q^{116}]_\infty}\right]$$

Or,

$$\frac{[q^{14},q^{43};q^{58}]_\infty}{[q^5,q^{34};q^{58}]_\infty}-\frac{[-q^{14},-q^{43};q^{58}]_\infty}{[-q^5,-q^{34};q^{58}]_\infty}$$
$$=\left[\frac{2q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}-\frac{2q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}\right]$$

Thus,

$$\mathrm{C(q)}-\mathrm{C}'(\mathrm{q})=\left[\frac{2q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}-\frac{2q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}\right],$$

(4.4)

$$i.e.$$

$$\beta_1(\mathrm{q})=\mathrm{C(q)}-\mathrm{C}'(\mathrm{q})=\left[\frac{q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}-\frac{q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}\right]$$

(4.5)

By adding (4.3) and (4.5) as

$$C(q)=\alpha_1(q)+\beta_1(q),$$
$$i.e.$$

$$\mathrm{C(q)}=\frac{[q^9,q^{38};q^{58}]_\infty[q^{28};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{10},q^{18},q^{58};q^{116}]_\infty}+\frac{q^9[q^9,q^{20};q^{58}]_\infty[q^{86};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{68};q^{116}]_\infty}$$
$$+\frac{q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}$$

$$-\frac{q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{10},q^{58},q^{68};q^{116}]_\infty}$$

(4.6)

By applying (2.2) in (4.6),

$$\mathrm{C(q)} = \frac{(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{28},q^{88};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{10},q^{18},q^{58},q^{58},q^{98},q^{106};q^{116})_\infty} + \frac{q^9(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{30},q^{86};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{48},q^{58},q^{58},q^{68},q^{98};q^{116})_\infty} + \frac{q^2(q^4,q^{12},q^{13},q^{45},q^{46},q^{54};q^{58})_\infty(q^{14},q^{52},q^{64},q^{102};q^{116})_\infty}{(q^7,q^{26},q^{32},q^{51};q^{58})_\infty(q^8,q^{10},q^{48},q^{58},q^{58},q^{68},q^{106},q^{108};q^{116})_\infty} - \frac{q^2(q^3,q^4,q^{22},q^{36},q^{54},q^{55};q^{58})_\infty(q^{32},q^{34},q^{84},q^{86};q^{116})_\infty}{(q^{16},q^{17},q^{41},q^{42};q^{58})_\infty(q^8,q^{10},q^{48},q^{58},q^{58},q^{68},q^{106},q^{108};q^{116})_\infty}$$

(4.7)

By taking known identity (2.10)

$$\sum_{s_1,s_2,s_3\geq 0}\frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}} = \frac{(q^{29},q^{14},q^{15};q^{29})_\infty}{(q^9;q^9)_\infty}$$

or,

$$\sum_{s_1,s_2,s_3\geq 0}\frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}} = \frac{(q^5,q^{24},q^{29};q^{29})_\infty}{(q^9;q^9)_\infty}\frac{(q^{14},q^{15};q^{29})_\infty}{(q^5,q^{24};q^{29})_\infty}$$

or,

$$\sum_{s_1,s_2,s_3\geq 0}\frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}} = \frac{(q^5,q^{24},q^{29};q^{29})_\infty}{(q^9;q^9)_\infty}\mathrm{C(q)},$$

(4.8)

where

$$\mathrm{C(q)} = \frac{(q^{14},q^{15};q^{29})_\infty}{(q^5,q^{24};q^{29})_\infty}$$

Now, by putting the value of C(q) from (4.7) in (4.8),

$$\sum_{s_1,s_2,s_3\geq 0}\frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}} = \frac{(q^5,q^{24},q^{29};q^{29})_\infty}{(q^9;q^9)_\infty}\Bigg[\frac{(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{28},q^{88};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{10},q^{18},q^{58},q^{58},q^{98},q^{106};q^{116})_\infty} + \frac{q^9(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{30},q^{86};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{48},q^{58},q^{58},q^{68},q^{98};q^{116})_\infty} + \frac{q^2(q^4,q^{12},q^{13},q^{45},q^{46},q^{54};q^{58})_\infty(q^{14},q^{52},q^{64},q^{102};q^{116})_\infty}{(q^7,q^{26},q^{32},q^{51};q^{58})_\infty(q^8,q^{10},q^{48},q^{58},q^{58},q^{68},q^{106},q^{108};q^{116})_\infty} - \frac{q^2(q^3,q^4,q^{22},q^{36},q^{54},q^{55};q^{58})_\infty(q^{32},q^{34},q^{84},q^{86};q^{116})_\infty}{(q^{16},q^{17},q^{41},q^{42};q^{58})_\infty(q^8,q^{10},q^{48},q^{58},q^{58},q^{68},q^{106},q^{108};q^{116})_\infty}\Bigg]$$

(4.9)

This is the main result (3.1)

Now, considering (4.2)

$$\mathrm{C(q)} + \mathrm{C}'(\mathrm{q}) = \frac{2[q^9,q^{38};q^{58}]_\infty[q^{28};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{10},q^{18},q^{58};q^{116}]_\infty} + \frac{2q^9[q^9,q^{20};q^{58}]_\infty[q^{86};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{68};q^{116}]_\infty}$$

Multiplying on both sides of (4.2) by

$$\frac{[q^{10},q^{68};q^{116}]_\infty}{[q^{28},q^{86};q^{116}]_\infty}$$

The following obtained as;

$$\frac{[-q^5,-q^{34};q^{58}]_\infty}{[-q^{14},-q^{43};q^{58}]_\infty} + \frac{[q^5,q^{34};q^{58}]_\infty}{[q^{14},q^{43};q^{58}]_\infty} = \frac{2[q^9,q^{38};q^{58}]_\infty[q^{68};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{86};q^{116}]_\infty} + \frac{2q^9[q^9,q^{20};q^{58}]_\infty[q^{10};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{28},q^{58};q^{116}]_\infty}$$

or,

$$\mathrm{C(q)}^{-1} + \mathrm{C}'(q)^{-1} = \frac{2[q^9,q^{38};q^{58}]_\infty[q^{68};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{86};q^{116}]_\infty} + \frac{2q^9[q^9,q^{20};q^{58}]_\infty[q^{10};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{28},q^{58};q^{116}]_\infty}.$$

(4.10)

Next, on arranging (4.10) as;

$$\alpha_2(q)=\frac{1}{2}[\mathrm{C}'(\mathrm{q})^{-1}+\mathrm{C}(\mathrm{q})^{-1}]=\frac{[q^9,q^{38};q^{58}]_\infty[q^{68};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{86};q^{116}]_\infty}+\frac{q^9[q^9,q^{20};q^{58}]_\infty[q^{10};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{28},q^{58};q^{116}]_\infty}$$

(4.11)

Again, multiplying (4.4) by,

$$\frac{[q^{10},q^{68};q^{116}]_\infty}{[q^{28},q^{86};q^{116}]_\infty},$$

get the following as;

$$\mathrm{C}(\mathrm{q})^{-1}-\mathrm{C}'(\mathrm{q})^{-1}=\left[\frac{2q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{28},q^{58},q^{86};q^{116}]_\infty}-\frac{2q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{28},q^{58},q^{86};q^{116}]_\infty}\right]$$

(4.12)

And, on arranging (4.12) as;

$$\beta_2(q)=\frac{1}{2}[\mathrm{C}(\mathrm{q})^{-1}-\mathrm{C}'(\mathrm{q})^{-1}]$$
$$=\left[\frac{q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{28},q^{58},q^{86};q^{116}]_\infty}-\frac{q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{28},q^{58},q^{86};q^{116}]_\infty}\right]$$

(4.13)

By adding (4.11) and (4.13),

$$\mathrm{C}(\mathrm{q})^{-1}=\alpha_2(q)+\beta_2(q)$$

Then

$$\mathrm{C}(\mathrm{q})^{-1}=\frac{[q^9,q^{38};q^{58}]_\infty[q^{68};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{58},q^{86};q^{116}]_\infty}+\frac{q^9[q^9,q^{20};q^{58}]_\infty[q^{10};q^{116}]_\infty}{[q^{29};q^{58}]_\infty[q^{18},q^{28},q^{58};q^{116}]_\infty}$$
$$+\frac{q^2[q^3,q^4,q^{22};q^{58}]_\infty[q^{32},q^{34};q^{116}]_\infty}{[q^{16},q^{17};q^{58}]_\infty[q^8,q^{28},q^{58},q^{86};q^{116}]_\infty}-\frac{q^2[q^4,q^{12},q^{13};q^{58}]_\infty[q^{14},q^{52};q^{116}]_\infty}{[q^7,q^{26};q^{58}]_\infty[q^8,q^{28},q^{58},q^{86};q^{116}]_\infty}$$

(4.14)

By applying (2.2) in (4.14),

$$C(q)^{-1}=\frac{(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{48},q^{68};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{30},q^{58},q^{58},q^{86},q^{98};q^{116})_\infty}$$
$$+\frac{q^9(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{10},q^{106};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{30},q^{58},q^{58},q^{86},q^{98};q^{116})_\infty}$$
$$+\frac{q^2(q^3,q^4,q^{22},q^{36},q^{54},q^{55};q^{58})_\infty(q^{32},q^{34},q^{82},q^{84};q^{116})_\infty}{(q^{16},q^{17},q^{41},q^{42};q^{58})_\infty(q^8,q^{28},q^{30},q^{58},q^{58},q^{86},q^{88},q^{108};q^{116})_\infty}$$
$$-\frac{q^2(q^4,q^{12},q^{13},q^{45},q^{46},q^{54};q^{58})_\infty(q^{14},q^{52},q^{64},q^{102};q^{116})_\infty}{(q^7,q^{26},q^{32},q^{51};q^{58})_\infty(q^8,q^{28},q^{30},q^{58},q^{58},q^{86},q^{88},q^{108};q^{116})_\infty}$$

(4.15)

Now, from (4.8)

$$C(q)^{-1}\sum_{s_1,s_2,s_3\geq 0}\frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}}=\frac{(q^5,q^{24},q^{29};q^{29})_\infty}{(q^9;q^9)_\infty}$$

(4.16)

By using (4.15) in (4.16)

$$\frac{(q^5,q^{24},q^{29};q^{29})_\infty}{(q^9;q^9)_\infty}$$
$$=\left[\frac{(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{48},q^{68};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{30},q^{58},q^{58},q^{86},q^{98};q^{116})_\infty}\right.$$
$$+\frac{q^9(q^9,q^{20},q^{38},q^{49};q^{58})_\infty(q^{10},q^{106};q^{116})_\infty}{(q^{29},q^{29};q^{58})_\infty(q^{18},q^{30},q^{58},q^{58},q^{86},q^{98};q^{116})_\infty}$$
$$+\frac{q^2(q^3,q^4,q^{22},q^{36},q^{54},q^{55};q^{58})_\infty(q^{32},q^{34},q^{82},q^{84};q^{116})_\infty}{(q^{16},q^{17},q^{41},q^{42};q^{58})_\infty(q^8,q^{28},q^{30},q^{58},q^{58},q^{86},q^{88},q^{108};q^{116})_\infty}$$
$$\left.-\frac{q^2(q^4,q^{12},q^{13},q^{45},q^{46},q^{54};q^{58})_\infty(q^{14},q^{52},q^{64},q^{102};q^{116})_\infty}{(q^7,q^{26},q^{32},q^{51};q^{58})_\infty(q^8,q^{28},q^{30},q^{58},q^{58},q^{86},q^{88},q^{108};q^{116})_\infty}\right]$$
$$\times\sum_{s_1,s_2,s_3\geq 0}\frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}}$$

(4.17)
This is main result (3.2)

5. CONCLUSION

This paper outlines the concept of m-dissections to establish our results involving q-basic infinite products, which expressed in terms of RR identities. These expansions will shape the subject matter of our ensuing communications.

REFERENCES

[1] A. K. Shrivastav, N. A. Herbert, H. Kaur, Certain Rogers-Ramanujan type Multi Sum Identities and Ratio of Infinite Products, South East Asian Journal of Mathematics and Mathematical Sciences, 15(2), (2019), 85–96.

[2] A. K. Shrivastav, N. A. Herbert, On Certain Results Involving Bilateral Basic Hypergeometric Functions and Ratio of Infinite Products, International J. of Scientificand Engineering Research, 8(9), (2017),1598-1612.

[3] A. K. Shrivastav, N. A. Herbert, On Transformation Formulae for Basic Hypergeometric Functions by using WP Baley's Pairs, International Journal of Mathematics Trends and Technology, 57(2), (2018), 122-127.

[4] A. K. Shrivastav, N. A. Herbert, On Bailey's Transform and Transformation Formulae for Basic Hypergeometric Functions, Journal of Computer and Mathematical Sciences, 9(7), (2018), 880-885.

[5] B. C. Berndt, P. T. Joshi, B. M. Wilson, Chapter 2 of Ramanujan's second notebook. Glasgow Mathematical Journal, 22(2), (1981), 199-216.

[6] D. Bressoud, M. E. H. Ismail, and D. Stanton, Change of Base in Bailey Pairs, TheRamanujan Journal, 4, (2000), 435-453.

[7] G. E. Andrews, Multiple series Rogers-Ramanujan type identities, Pac. J. Math.114, (1984), 267-283.

[8] G. E. Andrews, Ramanujan's lost notebook III-The Rogers-Ramanujans Continued Fraction, Adv. Math,41 (1981), 186-208.

[9] HarwnderKaur, and A. K. Shrivastav, Summation formulae involving basic hypergeometric and truncated basic hypergeometric functions, Journal of Mathematical Sciences and Computational Mathematics, 1(4), (2020), 456-461.

[10] HarwnderKaur, and A. K. Shrivastav, Certain q-series by using 2Ψ2 Basic Bilateral Hypergeometric identity, Journal of Information and Computational Science, 10(9), (2020),638- 643.

[11] HarwnderKaur, and A. K. Shrivastav, On transformations formulae and certaintransformations, Journal of Physics: Conference Series (Iraqi Academics SyndicateInternational Conference for Pure and Applied Sciences), (2021), 1- 7.

[12] KuljotKaur, PriyaWadhwa, HarwnderKaur, and A. K. Shrivastav, On identitiesfor basic hypereometric series through WP-bailey's pairs, Journal of Critical Review,7(19), (2020), 3669- 3674.

[13] L. J. Slater, A new proof of Roger's transformations of infinite series, Proc. London Math. Soc. (2) 53, (1951), 460-475.

[14] L. J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc. (2) 54, (1952), 147-167.

[15] L. J. Slater, Generalized Hyper geometric Function, Cambridge Univ. Press, Cambridge, (1996).

[16] M. D. Hirschhorn, on the the expansion of Ramanujan's Continued Fraction, TheRamanujan's Journal, 2 (1998), 521-527.

[17] PriyaWadhwa, A. K. Shrivastav, An Aspect of Basic (q-) Integral for q-Basic Series, Advances in Nonlinear Variational Inequalities, 26(2), (2023), 41-45.

[18] R. P. Lewis, On the Ranks of Partitions Modulo9, Bull. London Math. Soc., 23 (1991), 417-421.

[19] R. Y. Denis, S. N. Singh and N. A. Herbert, On certain Bilateral q-series and Ramanujan's Continued Fraction, South East Asia J. Math. and Math Sc., 3(1), (2004),49-61.

[20] Richard Lewis and Zhi-Guo Liu, A Conjecture of Hirschhorn on the 4-dissection of Ramanujan's Continued Fraction, The Ramanujan's Journal, 4 (2001), 347-352.

[21] W. N. Bailey, Identities of the Rogers-Ramanujan type, Proc. London Math. Soc. (2) 50, (1949), 1-10.