Nearly Projective Semimodules

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ABSTRACT
Other researchers have previously proposed and studied the concept of a nearly projective module. In this paper, the above concept will be analyzed for semimodules, along with related ideas. A semimodule Pis considered to beroughly projective if ∀ surjective-homα:A→B, where A,B are any two semimodules, and ∀ homβ: P→B, ∃γ: P→A homs.t παγ=πβ where π:B→B/J(B) is the natural map.

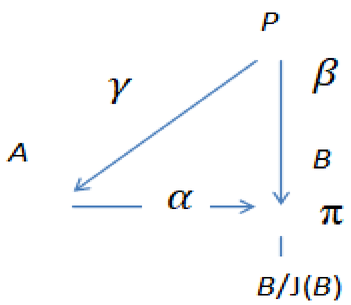
Keywords: studied, semimodule, concept

1. INTRODUCTION

Naoum and Al-Mothafar [9] proposed the notion of a nearly projective module as a generalization of a projective module relative to Jacobson radical. A module P is nearly projective if ∀ epimf:M→N and homg:P→N, ∃a homh:P→M such that (fh-g)(a)∈J(N), ∀ a∈P.M and N are arbitrary modules, and J(N) is the Jacobsonradical of the module N. It is clear that an equivalent condition to the condition "(h-g)(a)∈J(N), for each a∈P" is πfh=πg where π is the natural map of N onto N/J(N). This fact helps to define nearly projective semimodule. Characterizations of this concept are given with some properties. Some results relating nearly projective with projective semimodules(presented in a previous paper[8]) are concluded. Conditions for nearly projective semimodule to beprojective are proved. Finally, a nearly-quasi-projectivesemimodule is defined, and some properties are proved. It is proved that for the class of semimodules over a semiring in which the direct sum of any two nearly-quasi-projectivesemimodules is nearly-quasi-projective, in such class any nearly-quasi-projectiveis nearly projective.
R is a semiring with identity in what follows, and the semimodules are unitary left R-semimodules.

2. Nearly Projective semimodules

Definition2.1. A semimodule P is said to be nearly projective (N-projective) if ∀ surjective homα:A→B, where A,B are any two semimodules and ∀ homβ: P→B, ∃ a homeγ: P→A such that παγ=πβ where π:B→B/J(B) is the natural map.



Definition2.2. [5]. An R-semimodule Ais Artinian if any non-empty set of
S-subsemimodules of A have minimal membersconcerning set inclusion.
Definition2.3. [6] An epimα:P→M is called a projective cover of M↔P is projective, and α is a small epim.
Definition2.4. [5]. A left R-semimodule N is retracted of a left R-semimodule M↔ ∀ a surjective R-homθ:M→N and an R-hom
δ:N→M satisfying the condition that $\theta\delta=1_N$.
Remark2.5.
1- Every projective semimodule is nearly projective.

2- If a semimodule P has no maximalsubsemimodule,e.g.,$J(P)=P$, $\beta(P)=\beta(J(P))\subseteq J(B)$, hence $\pi\beta=0$, that is,Pis nearly projective. Thus, an almost projective semimodule may not be projective.

Lemma2.6. If $\alpha:F\to P$ is a surjective hom of semimodule and $\theta\in End(F)$ $\exists$ $\ker\alpha\subseteq\ker\theta$, then $\exists\ \alpha':P\to F$ such that $\alpha'\alpha=\theta$.

Proof: Since $\alpha:F\to P$ is surjective, then P is isomorphic to $F/\ker\alpha$, so $\exists\delta:P\to F/\ker\alpha$,which is an isomorphism. Define β: $F/\ker\alpha\to F/\ker\theta$ as$\beta(x/\ker\alpha)=x/\ker\theta$ where is well defined, and define $\sigma:F/\ker\theta\to\theta(F)\leq F$ as$\sigma(x/\ker\theta)=\theta(x)\leq F$.

Now $\alpha'=\sigma\beta\delta:P\to F$, then $\alpha'\alpha=\sigma\beta\delta\alpha=\theta$. Thus $\alpha'\alpha=\theta$.[$\theta\in End(F)$].

Lemma2.7. If $g:P\to B$ is a hom, then $\exists g':P/J(P)\to B/J(B)$ such that $g'\pi_1=\pi_2 g$ where $\pi_1:P\to P/J(P)$ and $\pi_2:B\to B/J(B)$ are the natural maps.

Proof: Assum π_1,π_2be two natural maps as$\pi_1:x\mapsto x/J(P)$,$\pi_2:y\mapsto y/J(B)$and g is a hom of semimodules by assumption.

Define $g':P/J(P)\to B/J(B)$ as $g'(x/J(P))=g(x)/J(B)$, since $g(J(P))\subseteq J(B)$, then g'Is well-defined. $g'\pi_1(x)=g'(x/J(P))=g(x)/J(B)=\pi_2 g(x)$, so $g'\pi_1=\pi_2 g$.

Theorem 2.8. AssumePbe R-semimodule:

1. If P is nearly projective,then for each exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\ \alpha\ } P \longrightarrow 0$$

Such that F is free and $K=\ker\alpha$, $\exists$a home$\theta\in End(F)$ satisfies:

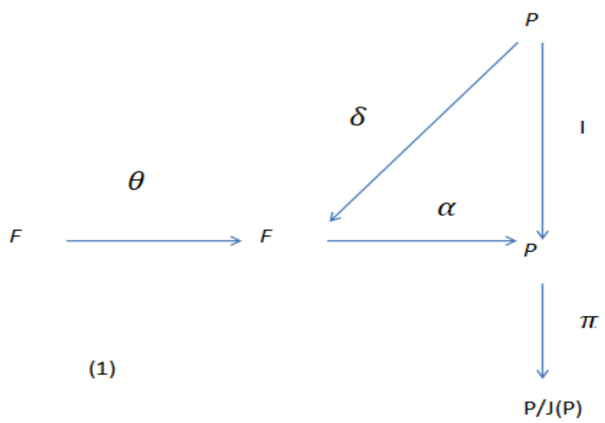
a. $\pi\alpha\theta=\pi\alpha$ where $\pi:P\to P/J(P)$ is the natural map.

b. $\ker\alpha\subseteq\ker\theta$.

2- If there is anexact sequence as above with k-regularα and $\theta\in End(F)$ satisfying (a) and (b) of (1), then P is N-projective.

Proof(1): Assume that P is nearly projective,in the diagram (1),$\exists\delta$: $P\to F$hom, such that $\pi\alpha\delta=\pi I=\pi$.....(1),take $\theta=\delta\alpha$.....(2).

Then$\pi\alpha\theta=\pi\alpha\delta\alpha=\pi\alpha$. For every $a\in\ker\alpha$ implies $a\in\ker\delta\alpha=\ker\theta$, thus $\ker\alpha\subseteq\ker\theta$.



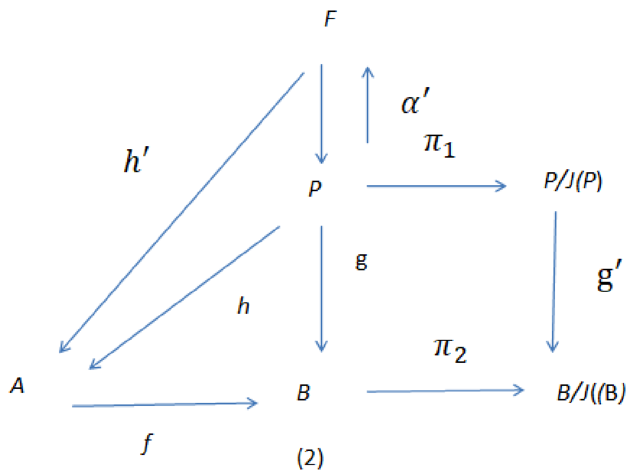
(2) By assumption $\exists\theta\in End(F)$ satisfying (a) and (b).

Define $\alpha':P\to F$by $\alpha'(a)=\theta(x_a)$ where $\{x_a\}$ is a basis for F and $\{\alpha(x_a)=a\}$ is a generating set of P; if$\alpha(x_a)=\alpha(y_a)=a$, then $x_a+k=y_a+k'$ for some $k,k'\in\ker\alpha$ (Since $\alpha:F\to P$ is k-regular). So,$\theta(x_a)=\theta(y_a)$, that is,α'Is well-defined.

Since F is a projective R-semimodule,$\exists h'$: $F\to A$ such that $fh'=g\alpha$. Define $h=h'\alpha'$.

In diagram (2):

$\pi_1:P\to P/J(P)$ and $\pi_2:B\to B/J(B)$ are the natural maps where $g'\pi_1=\pi_2 g$ by [Lemma1.3.]$\pi_2 fh(a)=\pi_2 fh'\alpha'(a)=\pi_2 fh'\theta(x_a)=\pi_2\,\theta(x_a)=g'\pi_1\,\alpha\,\theta(x_a)=g'\pi_1\,\alpha(x_a)=g'\pi_1(a)=\pi_2 g(a)$, that is, $\pi_2 fh=\pi_2 g$. So P is N-projective.



(2)

Definition.2.9.[5].A left R-semimodule N is a retract of R-semimodule M if and only if ∃a surjective R-homθ:M→N and an R-homδ:N→Msatisfying the condition that θδ is the identity map on N.

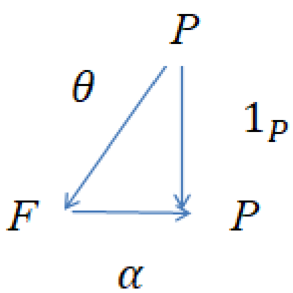
$$N \xrightarrow{\delta} M \xrightarrow{\theta} N$$

Lemma 2.10. If θδis an isomorphism where then there exist β:N→N such that $\theta\delta\beta=1_N$.
Lemma 2.11. If P is a retract of a free semimodule, then P is projective. [Golan]
Proposition2.12.AssumePisan N-projective R-semimodule. If J(P) is small in P, P is projective[8].
Proof: Since P is N-projective, then by [Theorem 1.4.] for each exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\alpha} P \longrightarrow 0$$

such t

F is free R-semimodule and K=ker α∃a homθ ∈ End(F) satisfying παθ = πα where π:P→P/J(P) is the natural map and since α surjective P=α(F), π(α(F))= π(α(θ(F))) implies P=α(F)= α(θ(F))+J(P) but J(P) is small, so α(F)= α(θ(F)), that is,Pis projective.
Corollary2.13.AssumePis a Hopfian N-projective R-semimodule; if J(P) is small,P is projective.Where Ris a semiring and MR-semimodule. If every surjectiveR-endomorphism of M is an isomorphism, one calls M Hopfian.
Proof: In the diagram,P is a Hopfian N-projective,andF is a free semimodule with α is surjective then by (Theorem 1.4.) $\alpha(\theta(P)=1_P(P)=P$, that is,αθ is surjective.Since P Hopfianand αθ∈ End(P), then αθis an isomorphismthat is P is a retract of F. By Lemma.1.5.P is projective.



Lemma2.14. If Pis a finitely generated projective semimodule,J(P)is small in P.
Proof: Similar to the case's evidence in modules, see [7.p.159.].
Corollary2.15.AssumePis a finitely generated R-semimodule;P is projective only if P is N-projective.
Proof: Since Pis projective, then P is N-projective.
Conversely, Since P is finitely generated, it is Hopfian and J(P) is small in P. By [Lemma1.5],P is projective.
Corollary2.16.AssumePis an N-projective R-semimodule with J(P) is small in P; if P is a multiplication semimodule, P is projective.

Proof: Since P is an N-projective R-semimodule and J(P) is small in P, if we prove P Hopfian, then it will be projective by[Lemma 1.7.], now assumefP→P be a surjective map, then f(P)=P, assum K=ker f, then K=JP for some J≤R, then f(K)=J f(P)=JP=K=0, so f is one-one, that is P is Hopfian. Hence,P is projective.

Theorem2.17. For any R-semimodule P, the following statements are equivalent.

1- P is an N-projective R-semimodule.

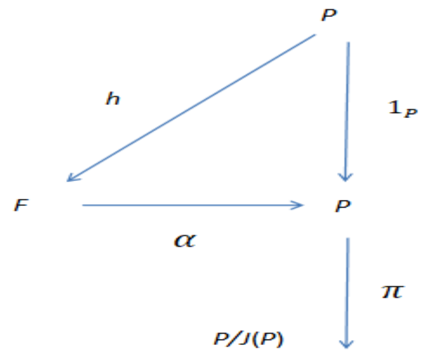
2- For every family $\{a_i: i\in I\}$ of generators of P over R,∃a family $\{f_i:i\in I\}$, $f_i \in P^* = Hom(P,R)$ with
   - a- For all a∈P, $f_i(a)\neq 0$ only for finitely many i∈ I.
   - b- For all a∈P, $\pi(\sum f_i(a)a_i)= \pi(a)$ where$\pi:P\to P/J(P)$ is the surjective map.

Proof: Assume that P is N-projective, and assumeFto be afree R-semimodule over P, assum $\{x_i:i\in I\}$ be a basis for F, and assum $\alpha:F\to P$ be surjective map $\alpha(x_i)=a_i$ For all i.

Define homes$\alpha_i:F\to R$ as follows $\alpha_i(\sum r_j x_j)=r_i$ For all I∈I.

It is clear that α_iIt is well-defined, and if we put $r_j=0$ in the case that the index does not appear in $\sum r_j x_j$ then for all x in F, $x=\sum r_j x_j$ and $\alpha_i(x)\neq 0$ for only finitely many i∈I. Moreover, $x=\sum \alpha_j(x)x_j$.......(*).

Now, since P is N-projective, then by definition [N-projective], ∃a homh:P→F such that $\pi(\alpha(h(a)))= \pi(a)$, for all an in P (in the diagram) put $f_i= \alpha_i h$, i∈I, then $f_i \in P^*$ and for all a∈P, $f_i(a)= \alpha_i h(a)\neq 0$ for only finitely many i∈I, furthermore for all a∈P, $a=\sum r_i a_i=\sum r_i\alpha(x_i)$. Thus $\pi(\sum f_i(a)a_i)= \pi(a)$ implies $\pi(\sum(\alpha_i h)(a)\alpha(x_i)=\pi(a)$, so $\pi(\alpha\sum \alpha_i h(a)\, x_i)= \pi(a)$.



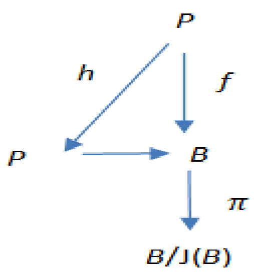
For the converse,assume $\{a_i:i\in I\}$ be a set of generators for P. Assum $\{x_i:i\in I\}$ be a basis for F such that $\alpha(x_i)=a_i$. Define a map θ:F→F:assumenext show that kerα=kerθ. Assum x∈F such that α(x)=0, then $\alpha(x)=0=\sum f_i(0)a_i+t$. But $f_i(0)=0$ for all I; thus, it is clear that θ(x)=0. It follows from theorem (2.8.) that P is an N-projective R-semimodule.

3. Nearly-Quasi-Projective Semimodules

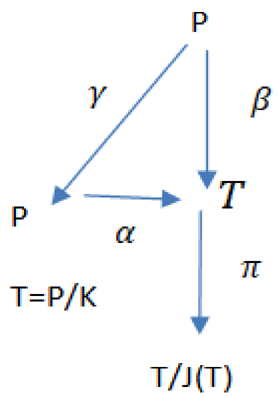
Definition3.1. An R-semimodule P is called nearly-quasi-projective
(NQ-projective), if for every R-semimodule B and surjective
homg:P→B
and any homf:P→B∃R-hom
h:P→P such that πgh=πf, where π is the natural surjective map
of B onto B/J(B).

Remark 3.1. Every nearly projective semimodule is nearly quasi-projective.

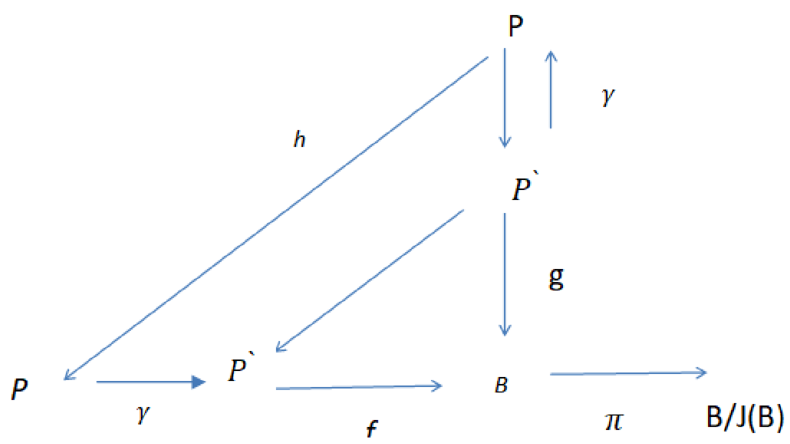
Remark 3.2. The retraction of the N-quasi projective is the N-quasi projective.



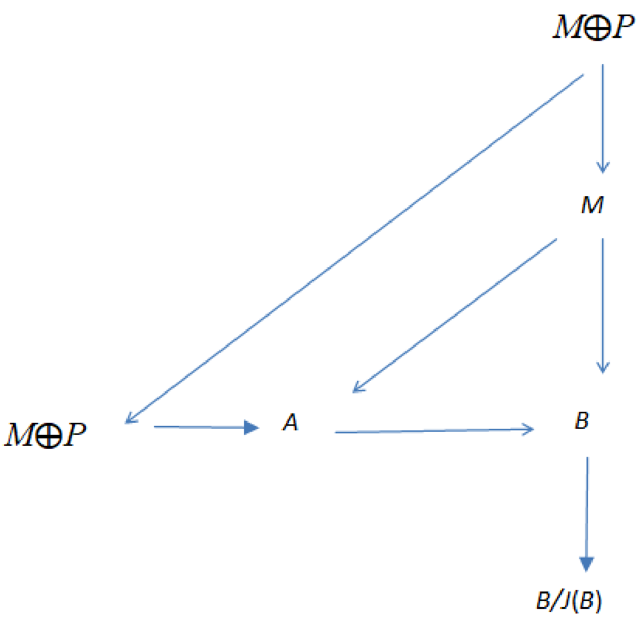
Proposition3.4.An R-semimodule P is nearly quasi-projective if and only if for any K≤P; β:P→ P/K , ∃γ:P→P such that παγ=πβ.
Proof: In the diagram,assumePto be quasi-projective and K≤P ;
αsurjectivehom and β any home, then there
exist γ S-hom such that παγ=πβ.



Proposition3.5.Any R-semimodule retract of any N-quasi projective
is N-quasi projective.
Proof: Consider the diagram P is N-quasi projective P`Is a semimodule such that ∃γ:P→ P` and γ`:P` → P with $γγ`=1_{P`}$, then P` is N-quasi projective such that πfγh=πgγ, assum h`= γhγ` so πfγ`=πfγhγ`=πgγγ`=πg, then P is N-projective.



Proposition 3.6. Assum R be a semiring such thatany R-semimodule has a projective cover, and the direct sum of any two N-quasi-projective R-semimodule is N-quasi-projective. Then any N-quasi-projectiveR-semimodule is N-projective R-semimodule.
Proof. AssumeMisthe N-quasi-projective R-semimodule and P is the projective cover of M.
We must prove that M is N-projective.In the diagram:
M⊕P is N-quasi-projective, then
πgαh`=πfα.
Assum h= αh`i, then πgh=πgαh`i=πfαi=πf, then M is N-projective.



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