An Application of Incomplete I-Functions with Two Variables to Solve the Nonlinear Differential Equations Using S-Function

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Abstract

In this article, we evaluate the approximate solutions of Nonlinear Differential Equations (NoLDEs) with the association of S-function, incomplete H-functions (IHFs) and incomplete I-functions (IIFs) with two variables by using the Hermite, Legendre and Jacobi polynomials. Here, we introduce incomplete I-functions with two variables. The NoLDEs are significantly applicable in fluid dynamics, vibration problems, population dynamics, electromagnetism, chemical kinetics, combustion theory, economics and finance. Recently, it was implemented to solve the resistance less circuit with a nonlinear capacitor under influence of external periodic force.

This method is established as an application of improper integrals, polynomials and special functions. The obtained results are helpful to get the solution of various problems of mathematical physics and engineering in approximate aspects.

Keywords: Incomplete H-functions, Incomplete I-functions, q-Gamma functions, S-function, Laplace transform, Improper Integral.
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1 Introduction and Preliminaries:

Earlier, Nonlinear differential equations play remarkable roles in the field of Engineering and Physics. Numerous authors have given their outputs to solving

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these equations by several methods. In the nineteen century, many Mathematicians like Gadre [9], Saxena et al. [16], Srivastava [24] and Srivastava et al. [25] worked on NoLDEs system. In last two decades many more authors such as Chaurasia et al. [5], Sharma [17], Singh [20], Singh et al. [21, 22, 23], Bansal et al. [1, 2, 3, 4] and Kumar et al. [11, 12] have also paid their attention in this branch of Applied Mathematics.

In the section 1, we defined the incomplete H-functions, incomplete I-functions with two variables and S-function. Section 2, shows some important theorems that will be used to solve the NoLDEs given in the section 3.

Special functions are well known tool which are uses in various fields of Engineering and Physics. The incomplete Gamma functions (IGFs) $\gamma(s,y)$ and $\Gamma(s,y)$ are investigated by Prym [14]. The incomplete Gamma functions are base of the recently developed incomplete forms of special functions like incomplete H-functions, incomplete $\overline{H}$-functions, incomplete I-functions and incomplete $\aleph$-functions.

The incomplete Gamma functions $\gamma(s,y)$ and $\Gamma(s,y)$ are defined, by

$$\gamma(s,y)=\int_0^y t^{s-1}e^{-t}dt, \quad (\mathbb{R}(s)>0;\ y\geq 0). \tag{1}$$

$$\Gamma(s,y)=\int_y^\infty t^{s-1}e^{-t}dt, \quad (y\geq 0;\ \mathbb{R}(s)>0\ when\ y=0). \tag{2}$$

The IGFs holds the decomposition formula $\gamma(s,y)+\Gamma(s,y)=\Gamma(s)$, here $\Gamma(.)$ is well known gamma function.

Pochhammer symbol $(\mu)_\lambda$ defined as:

$$(\mu)_\lambda=\frac{\Gamma(\mu+\lambda)}{\Gamma(\mu)}=\begin{cases}1, & \text{if} \quad \lambda=0;\mu\in\mathbb{C}\backslash\{0\}\\ (\mu)(\mu+1)\dots(\mu+n-1), & \text{if} \quad \lambda=n\in\mathbb{N};\mu\in\mathbb{C},\end{cases} \tag{3}$$

provided $\Gamma(\mu)$ exists. Here $\mathbb{C}$ and $\mathbb{N}$ are as usual denote the set of complex and natural numbers respectively.

Definition 1: In terms of the incomplete gamma functions (IGFs) $\Gamma(s,x)$ and $\gamma(s,x)$, the IHFs [26] is defined $\gamma_{P,Q}^{M,N}(z)$ and $\Gamma_{P,Q}^{M,N}(z)$ as follows:

$$\begin{aligned}\gamma_{P,Q}^{M,N}(z)&=\gamma_{P.Q}^{M,N}\left[z\left|\begin{array}{l}(f_1,F_1,t),(f_j,F_j)_{2,P}\\(w_j,W_j)_{1,Q}\end{array}\right.\right]\\&:=\frac{1}{2\pi i}\int_L\varphi(s,t)z^s ds,\end{aligned} \tag{4}$$

where

$$\varphi(s,t)=\frac{\gamma(1-f_1+F_1s,t)\prod_{j=1}^M\Gamma(w_j-W_js)\prod_{j=2}^N\Gamma(1-f_j+F_js)}{\prod_{j=M+1}^Q\Gamma(1-w_j+W_js)\prod_{j=N+1}^P\Gamma(f_j-F_js)}, \tag{5}$$

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and

$$\Gamma_{P,Q}^{M,N}(z) = \Gamma_{P,Q}^{M,N}\left[z \left| \begin{matrix} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{matrix} \right. \right]$$
$$:= \frac{1}{2\pi i}\int_L \phi(s,t) z^s ds, \tag{6}$$

where

$$\phi(s,t) = \frac{\Gamma(1 - f_1 + F_1 s, t)\prod_{j=1}^{M}\Gamma(w_j - W_j s)\prod_{j=2}^{N}\Gamma(1 - f_j + F_j s)}{\prod_{j=M+1}^{Q}\Gamma(1 - w_j + W_j s)\prod_{j=N+1}^{P}\Gamma(f_j - F_j s)}, \tag{7}$$

The IHFs $\gamma_{P,Q}^{M,N}(z)$ and $\Gamma_{P,Q}^{M,N}(z)$ are exist for all $t \geq 0$ and for more existing conditions (see, [26]).

Definition 2: We introduce the incomplete I-functions with two variables ${}^{(\Gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l^{(1)}},q_{l^{(1)}},r_{(1)};p_{l^{(2)}},q_{l^{(2)}},r_{(2)}}$ as follows:

$${}^{(\Gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l^{(1)}},q_{l^{(1)}},r_{(1)};p_{l^{(2)}},q_{l^{(2)}},r_{(2)}}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} \left| \begin{matrix} \left(e_1, E_1^{(1)}, E_1^{(2)}, x\right), \left(e_j, E_j^{(1)}, E_j^{(2)}\right)_{2,n} \\ \ldots, \left(f_{jl}, F_{jl}^{(1)}, F_{jl}^{(2)}\right)_{m+1,q_l}, \end{matrix} \right. \right.$$

$$, \left(e_{jl}, E_{jl}^{(1)}, E_{jl}^{(2)}\right)_{n+1,p_l}, (e_j^{(1)}, E_j^{(1)})_{1,n_1}, \left(e_{jl^{(1)}}^{(1)}, E_{jl^{(1)}}^{(1)}\right)_{n_1+1,p_l^{(1)}},$$
$$\left(f_j^{(1)}, F_j^{(1)}\right)_{1,m_1}, \left(f_{jl^{(1)}}^{(1)}, F_{jl^{(1)}}^{(1)}\right)_{m_1+1,q_l^{(1)}},$$

$$\left. \begin{matrix} (e_j^{(2)}, E_j^{(2)})_{1,n_2}, \left(e_{jl^{(2)}}^{(2)}, E_{jl^{(2)}}^{(2)}\right)_{n_2+1,p_l^{(2)}} \\ \left(f_j^{(2)}, F_j^{(2)}\right)_{1,m_2}, \left(f_{jl^{(2)}}^{(2)}, F_{jl^{(2)}}^{(2)}\right)_{m_2+1,q_l^{(2)}} \end{matrix} \right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(\xi_1, \xi_2, x) z_1^{\xi_1} z_2^{\xi_2}\theta_1(\xi_1)\theta_2(\xi_2) d\xi_1 d\xi_2 \quad (\omega = \sqrt{-1}), \tag{8}$$

where
$$\theta(\xi_1, \xi_2, x) = \frac{\Gamma(1 - e_1 + \sum_{i=1}^{2} E_j^{(i)}\xi_i, x)\prod_{j=2}^{n}\Gamma(1 - e_j + \sum_{i=1}^{2} E_j^{(i)}\xi_i)}{\sum_{i=1}^{r}\left[\prod_{j=n+1}^{p_l}\Gamma\left(e_{jl} - \sum_{i=1}^{2} E_{jl}^{(i)}\xi_i\right)\prod_{j=1}^{q_l}\Gamma\left(1 - f_{jl} + \sum_{i=1}^{2} F_{jl}^{(i)}\xi_i\right)\right]},$$

and,
$$\theta_i(\xi_i) = \frac{\prod_{j=1}^{m_i}\Gamma(f_j^{(i)} - F_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1 - e_j^{(i)} + E_j^{(i)}\xi_i)}{\sum_{i^{(i)}=1}^{r^{(i)}}\left[\prod_{j=m_i+1}^{q_{l^{(i)}}}\Gamma\left(1 - f_{jl^{(i)}}^{(i)} + F_{jl^{(i)}}^{(i)}\xi_i\right)\prod_{j=n_i+1}^{p_{i^{(i)}}}\Gamma\left(e_{jl^{(i)}}^{(i)} - E_{jl^{(i)}}^{(i)}\xi_i\right)\right]}, \ (i = 1, 2).$$

Now, We can define lower form of the incomplete I-function with two variables

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${}^{(\gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l(1)},q_{l(1)},r_{(1)};p_{l(2)},q_{l(2)},r_{(2)}}$ as follows:

$$
{}^{(\gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l(1)},q_{l(1)},r_{(1)};p_{l(2)},q_{l(2)},r_{(2)}}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle|\begin{array}{l} \left(e_1,E_1^{(1)},E_1^{(2)},x\right),\left(e_j,E_j^{(1)},E_j^{(2)}\right)_{2,n} \\ \ldots,\left(f_{jl},F_{jl}^{(1)},F_{jl}^{(2)}\right)_{m+1,q_l}, \end{array}\right.
$$

$$
\begin{array}{l} ,\left(e_{jl},E_{jl}^{(1)},E_{jl}^{(2)}\right)_{n+1,p_l},(e_j^{(1)},E_j^{(1)})_{1,n_1},\left(e_{jl^{(1)}}^{(1)},E_{jl^{(1)}}^{(1)}\right)_{n_1+1,p_l^{(1)}}, \\ \left(f_j^{(1)},F_j^{(1)}\right)_{1,m_1},\left(f_{jl^{(1)}}^{(1)},F_{jl^{(1)}}^{(1)}\right)_{m_1+1,q_l^{(1)}}, \end{array}
$$

$$
\left.\begin{array}{l} (e_j^{(2)},E_j^{(2)})_{1,n_2},\left(e_{jl^{(2)}}^{(2)},E_{jl^{(2)}}^{(2)}\right)_{n_2+1,p_l^{(2)}} \\ \left(f_j^{(2)},F_j^{(2)}\right)_{1,m_2},\left(f_{jl^{(2)}}^{(2)},F_{jl^{(2)}}^{(2)}\right)_{m_2+1,q_l^{(2)}} \end{array}\right]
$$

$$
=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta\left(\xi_1,\xi_2,x\right)z_1^{\xi_1}z_2^{\xi_2}\theta_1(\xi_1)\theta_2(\xi_2)d\xi_1 d\xi_2 \quad (\omega=\sqrt{-1}),
$$

where
$\theta\left(\xi_1,\xi_2,x\right)=\frac{\gamma(1-e_1+\sum_{i=1}^2 E_j^{(i)}\xi_i,x)\prod_{j=2}^n\Gamma(1-e_j+\sum_{i=1}^2 E_j^{(i)}\xi_i)}{\sum_{i=1}^r\left[\prod_{j=n+1}^{p_l}\Gamma\left(e_{jl}-\sum_{i=1}^2 E_{jl}^{(i)}\xi_i\right)\prod_{j=1}^{q_l}\Gamma\left(1-f_{jl}+\sum_{i=1}^2 F_{jl}^{(i)}\xi_i\right)\right]}$,

and,
$\theta_i(\xi_i)=\frac{\prod_{j=1}^{m_i}\Gamma(f_j^{(i)}-F_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1-e_j^{(i)}+E_j^{(i)}\xi_i)}{\sum_{i^{(i)}=1}^{r^{(i)}}\left[\prod_{j=m_i+1}^{q_{l^{(i)}}}\Gamma\left(1-f_{jl^{(i)}}^{(i)}+F_{jl^{(i)}}^{(i)}\xi_i\right)\prod_{j=n_i+1}^{p_{i^{(i)}}}\Gamma\left(e_{jl^{(i)}}^{(i)}-E_{jl^{(i)}}^{(i)}\xi_i\right)\right]}$, $(i=1,2)$.

Decomposition formula satisfying in case of incomplete I-functions with two variables defined in (1) and (2) as ${}^{(\gamma)}I_Q^P[z_i]+{}^{(\Gamma)}I_Q^P[z_i]=I_Q^P[z_i]$.

Here, $z_i\neq 0$; f_j $(j=1,\ldots,p)$; e_j $(j=1,\ldots,q)$; $e_j^{(i)}$ $(j=1,\ldots,n_i)$; $e_{jl^{(i)}}^{(i)}$ $(j=n_i+1,\ldots,p_{l^{(i)}})$; $f_j^{(i)}$ $(j=1,\ldots,m_i)$; $f_{jl^{(i)}}^{(i)}$ $(j=m_i+1,\ldots,q_{l^{(i)}})$; $i=1,\ 2$ are complex numbers and $E_j,\ F_j,\ E_{jl^{(i)}}^{(i)},\ F_{jl^{(i)}}^{(i)}$ are positive real numbers for standardization purpose such that

$$
A_l^{(i)}=\sum_{j=1}^{n}E_j^{(i)}+\sum_{j=n+1}^{p_i}E_{jl}^{(i)}+\sum_{j=1}^{n_i}E_j^{(2)}+\sum_{j=n_i+1}^{p_{i^{(i)}}}E_{jl^{(i)}}^{(i)}
$$

$$
-\sum_{j=1}^{q_i}F_{jl^{(i)}}^{(i)}-\sum_{j=1}^{m_i}F_j^{(i)}-\sum_{j=m_i+1}^{q_{i^{(i)}}}F_{jl^{(i)}}^{(i)}\leq 0 \quad (i=1,2),
$$

The integral path is a contour starting from $L-l\infty$ to $L+l\infty$ and the poles of $\Gamma(f_j^{(i)}-F_j^{(i)}\xi_i),\ j=1,\ldots,m_i,\ i=1,2$ are separated from those of $\Gamma(1-a_j+\sum_{i=1}^2 E_j^{(i)}\xi_i),\ j=1,\ldots,n$ and $\Gamma(1-e_j^{(i)}+E_j^{(i)}\xi_i),\ j=1,\ldots,n_i,\ i=1,2$

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to the left of the contour L_k. The existence conditions for multiple Mellin-Barnes contours can be obtained with the help of two variables I-function [18] as $|arg z_i| < \frac{\pi}{2}\bar{A}_i^{(i)}$, where

$$\bar{A}_l^{(k)} = \sum_{j=1}^{n} E_j^{(i)} - \sum_{j=n+1}^{p_l} E_{jl}^{(i)} - \sum_{j=1}^{q_l} F_{jl}^{(i)} + \sum_{j=1}^{n_i} E_j^{(i)} - \sum_{j=n_i+1}^{p_{l^{(i)}}} E_{jl^{(i)}}^{(i)} + \sum_{j=1}^{m_i} F_j^{(i)} - \sum_{j=m_i+1}^{q_{l^{(i)}}} F_{jl^{(i)}}^{(i)} > 0 \quad (i = 1, 2).$$

for more detail conditions (see [13, 18, 10, 19]).

Definition 3: The S-function introduced and investigated by Saxena et al. [7] and defined as:

$$\underset{(p,q)}{\overset{(a,b,c,d,e)}{\mathrm{S}}}(y) = \underset{(p,q)}{\overset{(a,b,c,d,e)}{\mathrm{S}}}(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; y) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{(c)_{nd,e}}{\Gamma_e(na+b)} \frac{y^n}{n!}, \quad (9)$$

where $e \in \mathbb{R}$; $a, b, c, d \in \mathbb{C}$; $Re(a) > 0, \alpha_i$ $(i = 1, 2, \ldots, p)$, β_j $(j = 1, 2, \ldots, q)$, $Re(a) > eRe(d)$ and $p < q + 1$. Pochhammer symbol $(\mu)_\lambda$ defined in (3).
Here, the k-Pochhammer symbol $(y)_{n,k}$ and k-Gamma function $\Gamma_q(y)$ defined by Diaz et al. [6].
If we put $c = d = e = 1$ in S-function, it reduces to the generalized M-series. Similarly, we can convert S-function to other functions named Generalized k-Mittag-Leffler function, k-function, generalized Mittag-Leffler function and Mittag-Leffler function.

2 Theorems

In this section, we use the linear approximation of the Hermite, Legendre and Jacobi polynomials to obtain the approximate solution of general NoLDEs which given below:

$$\ddot{\mathrm{x}} + \omega \underset{(p,q)}{\overset{(a,b,c,d,e)}{\mathrm{S}}} \left[y \left(\frac{x}{L} \right)^{2\Lambda'} \right] \Gamma_{P,Q}^{M,N} \left[z \left(\frac{x}{L} \right)^{2\Lambda} \middle| \begin{matrix} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{matrix} \right] = NF(t), \quad (10)$$

$$\ddot{\mathrm{x}} + \omega \underset{(p,q)}{\overset{(a,b,c,d,e)}{\mathrm{S}}} \left[y \left(1 + \frac{x}{L} \right)^{\nu'} \right] \Gamma_{P,Q}^{M,N} \left[z \left(1 + \frac{x}{L} \right)^{\nu} \middle| \begin{matrix} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{matrix} \right] = NF(t), \quad (11)$$

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$$\ddot{\mathrm{x}}+\omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}}\left[y\left(\frac{x}{L}\right)^{\Lambda}\right]{}^{(\Gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l^{(1)}},q_{l^{(1)}},r^{(1)};p_{l^{(2)}},q_{l^{(2)}},r^{(2)}}\left[\begin{array}{c|c} z_1\left(\frac{x}{L}\right)^{\nu} & A^* \\ z_2\left(\frac{x}{L}\right)^{\lambda} & B^* \end{array}\right] = NF(t), \quad (12)$$

where

$$A^*=\left(e_1,E_1^{(1)},E_1^{(2)},s\right),\left(e_j,E_j^{(1)},E_j^{(2)}\right)_{2,n},\left(e_{jl},E_{jl}^{(1)},E_{jl}^{(2)}\right)_{n+1,p_l},$$
$$(e_j^{(1)},E_j^{(1)})_{1,n_1},\left(e^{(1)}_{jl^{(1)}},E^{(1)}_{jl^{(1)}}\right)_{n_1+1,p_l^{(1)}},(e_j^{(2)},E_j^{(2)})_{1,n_2},\left(e^{(2)}_{jl^{(2)}},E^{(2)}_{jl^{(2)}}\right)_{n_2+1,p_l^{(2)}}$$

and

$$B^*=\ldots,\left(f_{jl},F_{jl}^{(1)},F_{jl}^{(2)}\right)_{m+1,q_l},\left(f_j^{(1)},F_j^{(1)}\right)_{1,m_1},\left(f^{(1)}_{jl^{(1)}},F^{(1)}_{jl^{(1)}}\right)_{m_1+1,q_l^{(1)}},$$
$$\left(f_j^{(2)},F_j^{(2)}\right)_{1,m_2},\left(f^{(2)}_{jl^{(2)}},F^{(2)}_{jl^{(2)}}\right)_{m_2+1,q_l^{(2)}}.$$

Under the effect of external periodic force, these NoLDEs defined in (10), (11) and (12) used in the theory of resistance less circuits. To solve these Nonlinear differential equations, we use Hermite, Legendre and Jacobi polynomials.
Now, we derive some new integrals as theorems that will use to solve the above given NoLDEs.

Theorem 1:

$$\int_{-\infty}^{\infty} x^{2\sigma}e^{-x^2}H_n(x)\overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}}\left(yx^{2\rho'}\right)$$
$$\Gamma^{M,N}_{P,Q}\left[zx^{2\rho}\left|\begin{array}{l}(f_1,F_1,t),(f_j,F_j)_{2,P}\\(w_j,W_j)_{1,Q}\end{array}\right.\right]dx=\sqrt{\pi}2^{n-2\sigma}L_1(k)$$
$$\Gamma^{M,N+1}_{P+1,Q+1}\left[z2^{-2\rho}\left|\begin{array}{l}(-2\sigma-2\rho'k,2\rho),(f_1,F_1,t),(f_j,F_j)_{2,P}\\(w_j,W_j)_{1,Q},(n/2-\sigma-\rho'k,\rho)\end{array}\right.\right], \quad (13)$$

where,

$$L_1(k)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{(c)_{kd,e}}{\Gamma_e(ka+b)}\frac{y^k}{k!}2^{-2\rho'k}.$$

Proof: By using the results of (6) and (9) in (13), we arrive at

$$\int_{-\infty}^{\infty} x^{2\sigma}e^{-x^2}H_n(x)$$
$$\left[\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{(c)_{kd,e}}{\Gamma_e(ka+b)}\frac{\left(yx^{2\rho'}\right)^k}{k!}\frac{1}{2\pi i}\int_L\phi(s,t)\left(zx^{2\rho}\right)^s ds\right]dx,$$

here $\phi(s,t)$ defined in (7).
Provided under the given condition, interchange the order of contour integral

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and integral. We get,

$$\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\frac{(c)_{kd,e}}{\Gamma_e(ka+b)}\frac{y^k}{k!}$$
$$\frac{1}{2\pi i}\int_L \phi(s,t)z^s\left[\int_{-\infty}^{\infty}x^{2(\sigma+\rho' k+\rho s)}e^{-x^2}H_n(x)dx\right]ds.$$

By using improper integral given below

$$\int_{-\infty}^{\infty}x^{2\sigma}e^{-x^2}H_n(x)dx=2^{n-2\sigma}\sqrt{\pi}\frac{\Gamma(2\sigma+1)}{(\sigma-n/2+1)}. \quad (14)$$

After little simplification we get the desire result.

Theorem 2:

$$\int_{-1}^{1}(1+x)^{\Lambda-1}P_n(x)\overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}}\left[y\,(1+x)^{\nu'}\right]$$
$$\Gamma^{M,N}_{P,Q}\left[z\,(1+x)^{\nu}\left|\begin{array}{l}(f_1,F_1,t),(f_j,F_j)_{2,P}\\(w_j,W_j)_{1,Q}\end{array}\right.\right]dx=2^{\Lambda}L_2(k)\Gamma^{M,N+2}_{P+2,Q+2}$$
$$\left[z2^{\nu}\left|\begin{array}{l}(1-\Lambda-\nu' k,\nu),(1-\Lambda-\nu' k,\nu),(f_1,F_1,t),(f_j,F_j)_{2,P}\\(w_j,W_j)_{1,Q},(\Lambda-\nu' k-n,\nu),(1-\Lambda-\nu' k+n,\nu)\end{array}\right.\right], \quad (15)$$

where,

$$L_2(k)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\frac{(c)_{kd,e}}{\Gamma_e(ka+b)}\frac{y^k}{k!}2^{\nu' k}.$$

Proof: By using the known result given in (p.316, eq.15, [8]), we can get the proof of the Theorem 2 in similar manner as we did in the Theorem 1 .

Theorem 3:

$$\int_{-1}^{1}(1-x)^{\lambda}(1+x)^{\delta}x^nP_n^{(\lambda,\delta)}(x)\overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}}(yx^{\rho})$$
$${}^{(\Gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l^{(1)}},q_{l^{(1)}},r^{(1)};p_{l^{(2)}},q_{l^{(2)}},r^{(2)}}\left[\begin{array}{c}z_1x^{\nu}\\z_2x^{\mu}\end{array}\left|\begin{array}{c}A^*\\B^*\end{array}\right.\right]=2^{\lambda+\delta+n+1}L_3(k)$$
$${}^{(\Gamma)}I^{0,n+2:m_1,n_1;m_2,n_2}_{p_l+2,q_l+1,r;p_{l^{(1)}},q_{l^{(1)}},r^{(1)};p_{l^{(2)}},q_{l^{(2)}},r^{(2)}}\left[\begin{array}{c}z_12^{\nu}\\z_22^{\mu}\end{array}\left|\begin{array}{c}C^*\\D^*\end{array}\right.\right], \quad (16)$$

where,

$A^*=\left(e_1,E_1^{(1)},E_1^{(2)},s\right),\left(e_j,E_j^{(1)},E_j^{(2)}\right)_{2,n},\left(e_{jl},E_{jl}^{(1)},E_{jl}^{(2)}\right)_{n+1,p_l},$
$(e_j^{(1)},E_j^{(1)})_{1,n_1},\left(e^{(1)}_{jl^{(1)}},E^{(1)}_{jl^{(1)}}\right)_{n_1+1,p_l^{(1)}},(e_j^{(2)},E_j^{(2)})_{1,n_2},\left(e^{(2)}_{jl^{(2)}},E^{(2)}_{jl^{(2)}}\right)_{n_2+1,p_l^{(2)}}$
$B^*=\dots,\left(f_{jl},F_{jl}^{(1)},F_{jl}^{(2)}\right)_{m+1,q_l},\left(f_j^{(1)},F_j^{(1)}\right)_{1,m_1},\left(f^{(1)}_{jl^{(1)}},F^{(1)}_{jl^{(1)}}\right)_{m_1+1,q_l^{(1)}},$

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$\left(f_j^{(2)}, F_j^{(2)}\right)_{1,m_2}, \left(f_{jl^{(2)}}^{(2)}, F_{jl^{(2)}}^{(2)}\right)_{m_2+1,q_l^{(2)}}$

$C^* = (-\lambda - n - \rho k, \nu, \mu), (-\delta - n - \rho k, \nu, \mu), \left(e_1, E_1^{(1)}, E_1^{(2)}, s\right),$
$\left(e_j, E_j^{(1)}, E_j^{(2)}\right)_{2,n}, \left(e_{jl}, E_{jl}^{(1)}, E_{jl}^{(2)}\right)_{n+1,p_l}, (e_j^{(1)}, E_j^{(1)})_{1,n_1},$
$\left(e_{jl^{(1)}}^{(1)}, E_{jl^{(1)}}^{(1)}\right)_{n_1+1,p_l^{(1)}}, (e_j^{(2)}, E_j^{(2)})_{1,n_2}, \left(e_{jl^{(2)}}^{(2)}, E_{jl^{(2)}}^{(2)}\right)_{n_2+1,p_l^{(2)}}$

$D^* = \ldots, \left(f_{jl}, F_{jl}^{(1)}, F_{jl}^{(2)}\right)_{m+1,q_l}, (-1 - \lambda - \delta - 2(n - \rho k), 2\nu, 2\mu),$
$\left(f_j^{(1)}, F_j^{(1)}\right)_{1,m_1}, \left(f_{jl^{(1)}}^{(1)}, F_{jl^{(1)}}^{(1)}\right)_{m_1+1,q_l^{(1)}}, \left(f_j^{(2)}, F_j^{(2)}\right)_{1,m_2},$
$\left(f_{jl^{(2)}}^{(2)}, F_{jl^{(2)}}^{(2)}\right)_{m_2+1,q_l^{(2)}}$

and

$L_3(k) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \ldots (\beta_q)_k} \frac{(c)_{kd,e}}{\Gamma_e(ka+b)} \frac{y^k}{k!} 2^{\rho k}.$

Proof: By using the result given in (p.261, eq.15, [15]), we can get the proof of the Theorem 3 in similar manner as we did in the Theorem 1 .

3 Polynomials and Linear Approximation

Here, we solve the NoLDEs given in (10), (11) and (12) by using Hermite, Legendre and Jacobi polynomials respectively.

3.1 Hermite Polynomials and Linear Approximation

Here, we solve the NoLDE given in equation (10) as follows:

$$\ddot{\mathrm{x}} + f(x) = NF(t), \tag{17}$$

where

$$f(x) = \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left[y \left(\frac{x}{L}\right)^{2\Lambda'} \right] \Gamma_{P,Q}^{M,N} \left[z \left(\frac{x}{L}\right)^{2\Lambda} \middle| \begin{array}{l} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{array} \right], \tag{18}$$

which can be written in terms of Hermite polynomials in the interval (–L, L), We get

$$f(x) = \sum_{n=0}^{\infty} \eta_n H_n\left(\frac{x}{L}\right), \tag{19}$$

where coefficient η_n is defined by

$$\eta_n = \frac{\int_{-\infty}^{\infty} f(Lx) H_n(x) x^{2\rho} e^{-x^2} dx}{\int_{-\infty}^{\infty} \left[H_n(x)\right]^2 e^{-x^2} dx}. \tag{20}$$

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The series given in equation (20) truncated after two terms then, we get a linear approximation $f^*(x)$ as follows:

$$f^*(x) = \eta_0 H_0\left(\frac{x}{L}\right) + \eta_1 H_1\left(\frac{x}{L}\right). \tag{21}$$

Now, we can write linear approximation $f^*(x)$ by using equation (18) as

$$f^*(x) = \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left[y\left(\frac{x}{L}\right)^{2\Lambda'} \right] \Gamma_{P,Q}^{M,N}\left[z\left(\frac{x}{L}\right)^{2\Lambda} \middle| \begin{matrix} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{matrix} \right], \tag{22}$$

putting $H_0(x) = 1$ and $H_1(x) = 2x$ in (21), we have

$$f^*(x) = \eta_0 + 2\eta_1\left(\frac{x}{L}\right). \tag{23}$$

To obtain the values of η_0 and η_1, we consider $n = 0, n = 1$ and using equation (18) in (22), then we have

$$\eta_0 = \frac{\int_{-\infty}^{\infty} \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left(yx^{2\Lambda'}\right) A_H H_0(x) x^{2\rho} e^{-x^2} dx}{\int_{-\infty}^{\infty} \left[H_0(x)\right]^2 e^{-x^2} dx}, \tag{24}$$

and

$$\eta_1 = \frac{\int_{-\infty}^{\infty} \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left(yx^{2\Lambda'}\right) A_H H_1(x) x^{2\rho} e^{-x^2} dx}{\int_{-\infty}^{\infty} \left[H_1(x)\right]^2 e^{-x^2} dx}, \tag{25}$$

where
$A_H = \Gamma_{P,Q}^{M,N}\left[zx^{2\Lambda} \middle| \begin{matrix} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{matrix} \right]$
Further, using Theorem 1 and result (p.193, eq.(6), [15]) with $n = 0$ in equation (24), we get

$$\eta_0 = 2^{-2\sigma} \omega L_1(k) \Gamma_{P+1,Q+1}^{M,N+1}\left[z2^{-2\rho} \middle| \begin{matrix} (-2\sigma - 2\rho' k, 2\rho), (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q}, (-\sigma - \rho' k, \rho) \end{matrix} \right], \tag{26}$$

similarly, we can obtained

$$\eta_1 = 2^{-2\sigma} \omega L_1(k) \Gamma_{P+1,Q+1}^{M,N+1}\left[z2^{-2\rho} \middle| \begin{matrix} (-2\sigma - 2\rho' k, 2\rho), (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q}, (1/2 - \sigma - \rho' k, \rho) \end{matrix} \right]. \tag{27}$$

Now, on replacing $f(x)$ by $f^*(x)$ and using (23), we can express (17) as

$$\ddot{\mathrm{x}} + \eta_0 + 2\eta_1\left(\frac{x}{L}\right) = NF(t), \tag{28}$$

If $\delta^2 = 2\eta_1/L$ and $\delta_1^2 = \eta_0$, then (28) can be written as

$$\ddot{\mathrm{x}} + \delta^2 x + \delta_1^2 = NF(t). \tag{29}$$

Apply Laplace transform in equation (29) to find the approximate solution under the given constraints

$$y = L(L-1) \text{ and } \dot{\mathrm{x}} = 0 \text{ if } t = 0,$$

$$x^* = \left[L(L-1) + \frac{\delta_1^2}{\delta}\right] \cos \delta t - \frac{\delta_1^2}{\delta} + \frac{N}{\delta} \int_0^t F(u) \sin \delta(t-u) du. \tag{30}$$

The obtained approximate solution is general in nature.

3.2 Legendre Polynomials and Linear Approximation

The main objective of this section is to solve the NoLDE defined in (11) as follows:

$$\ddot{\mathrm{x}} + f(x) = NF(t), \tag{31}$$

where

$$f(x) = \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left[y \left(1 + \frac{x}{L}\right)^{\nu'} \right] \Gamma_{P,Q}^{M,N} \left[z \left(1 + \frac{x}{L}\right)^{\nu} \middle| \begin{array}{l} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{array} \right], \tag{32}$$

which can be written in terms of Legendre polynomials in the interval (–L, L). We get

$$f(x) = \sum_{n=0}^{\infty} \tau_n P_n \left(\frac{x}{L}\right), \tag{33}$$

where coefficient τ_n is defined by

$$\tau_n = \frac{\int_{-1}^{1} f(Lx) P_n(x) (1+x)^{\Lambda-1} dx}{\int_{-1}^{1} \left[P_n(x)\right]^2 dx}. \tag{34}$$

Truncated the series (34) after two terms. We get a linear approximation $f^*(x)$ as follows:

$$f^*(x) = \tau_0 P_0 \left(\frac{x}{L}\right) + \tau_1 P_1 \left(\frac{x}{L}\right). \tag{35}$$

Now, we can write linear approximation $f^*(x)$ by using equation (32) by

$$f^*(x) = \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left[y \left(1 + \frac{x}{L}\right)^{\nu'} \right] \Gamma_{P,Q}^{M,N} \left[z \left(1 + \frac{x}{L}\right)^{\nu} \middle| \begin{array}{l} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{array} \right], \tag{36}$$

Putting $P_0(x) = 1$ and $P_1(x) = x$ in (35), we have

$$f^*(x) = \tau_0 + \tau_1 \left(\frac{x}{L}\right). \tag{37}$$

Use equation (36) in (34) with $n = 0$ and $n = 1$, to obtain the values of τ_0 and τ_1 respectively as

$$\tau_0 = \frac{\int_{-1}^{1} \omega \underset{(p,q)}{\overset{(a,b,c,d,e)}{\mathrm{S}}} \left[y\,(1+x)^{\nu'}\right] A_L P_0(x)(1+x)^{\Lambda-1} dx}{\int_{-1}^{1} \left[P_0(x)\right]^2 dx}, \tag{38}$$

and

$$\tau_1 = \frac{\int_{-1}^{1} \omega \underset{(p,q)}{\overset{(a,b,c,d,e)}{\mathrm{S}}} \left[y\,(1+x)^{\nu'}\right] A_L P_1(x)(1+x)^{\Lambda-1} dx}{\int_{-1}^{1} \left[P_1(x)\right]^2 dx}, \tag{39}$$

where
$A_L = \Gamma^{M,N}_{P,Q}\left[z\,(1+x)^{\nu} \,\middle|\, \begin{matrix} (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q} \end{matrix}\right]$
Further, using Theorem 2 and result (p.175, eq.(12), [15]) with $n = 0$ in (38), we get

$$\begin{aligned}\tau_0 = {} & \omega 2^{\Lambda-1} L_2(k) \Gamma^{M,N+2}_{P+2,Q+2} \\ & \left[z2^{\nu} \,\middle|\, \begin{matrix} (1-\Lambda-\nu' k, \nu), (1-\Lambda-\nu' k, \nu), (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q}, (\Lambda-\nu' k, \nu), (1-\Lambda-\nu' k, \nu) \end{matrix}\right],\end{aligned} \tag{40}$$

Similarly, we can evaluate for $n = 1$

$$\begin{aligned}\tau_1 = {} & 3\omega 2^{\Lambda-1} L_2(k) \Gamma^{M,N+2}_{P+2,Q+2} \\ & \left[z2^{\nu} \,\middle|\, \begin{matrix} (1-\Lambda-\nu' k, \nu), (1-\Lambda-\nu' k, \nu), (f_1, F_1, t), (f_j, F_j)_{2,P} \\ (w_j, W_j)_{1,Q}, (\Lambda-\nu' k-1, \nu), (2-\Lambda-\nu' k, \nu) \end{matrix}\right].\end{aligned} \tag{41}$$

Now, on replacing $f(x)$ by $f^*(x)$ and use (37), we can express in (31) as

$$\ddot{\mathrm{x}} + \delta_1^2 + \delta^2 x = NF(t), \tag{42}$$

where $\delta^2 = \tau_1/L$ and $\delta_1^2 = \tau_0$. Apply the Laplace transform in (42) to find the approximate solution under the constraints

$$x = L(L-1) \text{ and } \dot{\mathrm{x}} = 0 \text{ if } t = 0,$$

$$x^* = \left[L(L-1) + \frac{\delta_1^2}{\delta}\right] \cos \delta t - \frac{\delta_1^2}{\delta} + \frac{N}{\delta} \int_0^t F(u) \sin \delta (t-u) du. \tag{43}$$

The obtained approximate solution is general in nature.

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3.3 Jacobi Polynomials and Linear Approximation

Here, our aim is to solve the NoLDE given in (12) is as follows:

$$\ddot{\mathrm{x}} + f(x) = NF(t), \tag{44}$$

where

$$f(x) = \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left[y \left(\frac{x}{L}\right)^{\Lambda} \right] {}^{(\Gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l(1)},q_{l(1)},r_{(1)};p_{l(2)},q_{l(2)},r_{(2)}} \left[\begin{array}{c} z_1 \left(\frac{x}{L}\right)^{\nu} \\ z_2 \left(\frac{x}{L}\right)^{\mu} \end{array} \middle| \begin{array}{c} A^* \\ B^* \end{array} \right], \tag{45}$$

where A^* and B^* are defined in (12), which can be written in terms of Jacobi polynomials in the interval (–L, L). We have

$$f(x) = \sum_{n=0}^{\infty} C_n^{(\iota,\kappa)} P_n^{(\iota,\kappa)} \left(\frac{x}{L}\right), \tag{46}$$

where coefficient $C_n^{(\iota,\kappa)}$ is defined by

$$C_n^{(\iota,\kappa)} = \frac{\int_{-1}^{1} f(Lx) P_n^{(\iota,\kappa)}(x)(1-x)^{\iota}(1+x)^{\kappa} dx}{\int_{-1}^{1} \left[P_n^{(\iota,\kappa)}(x)\right]^2 (1-x)^{\iota}(1+x)^{\kappa} dx}. \tag{47}$$

Truncated the above given series after two terms. We get a linear approximation $f^*(x)$ as follows:

$$f^*(x) = C_0^{(\iota,\kappa)} P_0^{(\iota,\kappa)} \left(\frac{x}{L}\right) + C_1^{(\iota,\kappa)} P_1^{(\iota,\kappa)} \left(\frac{x}{L}\right). \tag{48}$$

Now, we can write linear approximation $f^*(x)$ by using equation (45) by

$$f^* = \omega \overset{(a,b,c,d,e)}{\underset{(p,q)}{\mathrm{S}}} \left[y \left(\frac{x}{L}\right)^{\Lambda} \right] {}^{(\Gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l(1)},q_{l(1)},r_{(1)};p_{l(2)},q_{l(2)},r_{(2)}} \left[\begin{array}{c} z_1 \left(\frac{x}{L}\right)^{\nu} \\ z_2 \left(\frac{x}{L}\right)^{\mu} \end{array} \middle| \begin{array}{c} A^* \\ B^* \end{array} \right], \tag{49}$$

Putting $P_0^{(\iota,\kappa)} = 1$ and $P_1^{(\iota,\kappa)} = \frac{\iota-\kappa}{2} + \frac{(2+\iota+\kappa)x}{2}$ in (48), we have

$$f^*(x) = C_0^{(\iota,\kappa)} + C_1^{(\iota,\kappa)} \left[\frac{\iota-\kappa}{2} + \frac{(2+\iota+\kappa)x}{2} \right]. \tag{50}$$

Use equation (47) in (45) with $n = 0$ and $n = 1$, to obtain the values of $C_0^{(\iota,\kappa)}$ and $C_1^{(\iota,\kappa)}$ with the help of Theorem 3 and aid a result (p.260, eq.(11), [15]) with $n=0$ in equation (47), we get $C_0^{(\iota,\kappa)}$ as follows:

$$C_0^{(\iota,\kappa)} = \frac{\omega L_3(k)\Gamma(2+\iota+\kappa)}{\Gamma(1+\iota)\Gamma(1+\kappa)} \, {}^{(\Gamma)}I^{0,n+2:m_1,n_1;m_2,n_2}_{p_l+2,q_l+1,r;p_{l^{(1)}},q_{l^{(1)}},r^{(1)};p_{l^{(2)}},q_{l^{(2)}},r^{(2)}} \left[\begin{array}{c|c} z_1 2^{\nu} & C^{**} \\ z_2 2^{\mu} & D^{**} \end{array} \right], \quad (51)$$

where
$C^{**} = (-\iota-\rho k,\nu,\mu), (-\kappa-\rho k,\nu,\mu), \left(e_1, E_1^{(1)}, E_1^{(2)}, s\right), \left(e_j, E_j^{(1)}, E_j^{(2)}\right)_{2,n},$
$\left(e_{jl}, E_{jl}^{(1)}, E_{jl}^{(2)}\right)_{n+1,p_l}, (e_j^{(1)}, E_j^{(1)})_{1,n_1}, \left(e_{jl^{(1)}}^{(1)}, E_{jl^{(1)}}^{(1)}\right)_{n_1+1,p_l^{(1)}}, (e_j^{(2)}, E_j^{(2)})_{1,n_2},$
$\left(e_{jl^{(2)}}^{(2)}, E_{jl^{(2)}}^{(2)}\right)_{n_2+1,p_l^{(2)}}$
$D^{**} = \ldots, \left(f_{jl}, F_{jl}^{(1)}, F_{jl}^{(2)}\right)_{m+1,q_l}, (-1-\iota-\kappa-2\rho k, 2\nu, 2\mu), \left(f_j^{(1)}, F_j^{(1)}\right)_{1,m_1}$
$, \left(f_{jl^{(1)}}^{(1)}, F_{jl^{(1)}}^{(1)}\right)_{m_1+1,q_l^{(1)}}, \left(f_j^{(2)}, F_j^{(2)}\right)_{1,m_2}, \left(f_{jl^{(2)}}^{(2)}, F_{jl^{(2)}}^{(2)}\right)_{m_2+1,q_l^{(2)}}.$

Similarly, we can obtain $C_1^{(\iota,\kappa)}$ as

$$C_1^{(\iota,\kappa)} = \frac{2\omega L_3(k)\Gamma(3+\iota+\kappa)}{\Gamma(2+\iota)\Gamma(2+\kappa)} \, {}^{(\Gamma)}I^{0,n+2:m_1,n_1;m_2,n_2}_{p_l+2,q_l+1,r;p_{l^{(1)}},q_{l^{(1)}},r^{(1)};p_{l^{(2)}},q_{l^{(2)}},r^{(2)}} \left[\begin{array}{c|c} z_1 2^{\nu} & C^{***} \\ z_2 2^{\lambda} & D^{***} \end{array} \right], \quad (52)$$

where
$C^{***} = (-\iota-1-\rho k,\nu,\mu), (-\kappa-1-\rho k,\nu,\mu), \left(e_1, E_1^{(1)}, E_1^{(2)}, s\right),$
$\left(e_j, E_j^{(1)}, E_j^{(2)}\right)_{2,n}, \left(e_{jl}, E_{jl}^{(1)}, E_{jl}^{(2)}\right)_{n+1,p_l}, (e_j^{(1)}, E_j^{(1)})_{1,n_1},$
$\left(e_{jl^{(1)}}^{(1)}, E_{jl^{(1)}}^{(1)}\right)_{n_1+1,p_l^{(1)}}, (e_j^{(2)}, E_j^{(2)})_{1,n_2}, \left(e_{jl^{(2)}}^{(2)}, E_{jl^{(2)}}^{(2)}\right)_{n_2+1,p_l^{(2)}}$
$D^{***} = \ldots, \left(f_{jl}, F_{jl}^{(1)}, F_{jl}^{(2)}\right)_{m+1,q_l}, (-3-\iota-\kappa-2\rho k, 2\nu, 2\mu), \left(f_j^{(1)}, F_j^{(1)}\right)_{1,m_1}$
$, \left(f_{jl^{(1)}}^{(1)}, F_{jl^{(1)}}^{(1)}\right)_{m_1+1,q_l^{(1)}}, \left(f_j^{(2)}, F_j^{(2)}\right)_{1,m_2}, \left(f_{jl^{(2)}}^{(2)}, F_{jl^{(2)}}^{(2)}\right)_{m_2+1,q_l^{(2)}}.$

Now, on replacing $f(x)$ by $f^*(x)$ and use equation (50), we can write equation (44) as

$$\ddot{x} + \delta^2 x + \frac{\iota-\kappa}{2+\iota+\kappa}\left(\delta^2-\delta_1^2\right) = NF(t), \quad (53)$$

where $\delta^2 = \frac{2+\iota+\kappa}{2L}$ and $\delta_1^2 = \frac{2+\iota+\kappa}{(\kappa-\iota)L} C_0^{(\iota,\kappa)}$.
Apply the Laplace transform in equation (53) to find the approximate solution under the constraints

$$x = L(L-1) \text{ and } \dot{x} = 0 \text{ if } t = 0,$$

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$$x^* = \left[L(L-1) + \frac{(\iota-\kappa)\delta}{2+\iota+\kappa}\left(1-\frac{\delta_1^2}{\delta}\right)\right]$$
$$\cos\delta t - \frac{(\iota-\kappa)\delta}{2+\iota+\kappa}\left(1-\frac{\delta_1^2}{\delta}\right) + \frac{N}{\delta}\int_0^t F(u)\sin\delta(t-u)du. \quad (54)$$

The obtained approximate solution is general in nature.

Similarly, we can prove all of the above results and theorems for lower forms of the incomplete H-function $\gamma_{P,Q}^{M,N}[z]$ and incomplete I-function with two variables ${}^{(\gamma)}I^{0,n:m_1,n_1;m_2,n_2}_{p_l,q_l,r;p_{l(1)},q_{l(1)},r_{(1)};p_{l(2)},q_{l(2)},r_{(2)}}[z_i]$.

4 Conclusion

In this article, we introduced the approximate solution of NoLDE associated with incomplete H-functions, incomplete I-functions with two variables and S-function with the help of Hermite, Legendre and Jacobi polynomials. These obtained results are general and effectively used in the field of Science, Mathematics, Statistics, Economics and finance. These findings can be used to solve the problem of a resistance less circuit involving a nonlinear capacitor under the influence of external periodic force.

References

[1] Bansal, M. K., Jolly, N., Jain, R., and Kumar, D. (2019). An integral operator involving generalized Mittag-Leffler function and associated fractional calculus results. *The Journal of Analysis*, 27(3), 727-740.

[2] Bansal, M. K., Kumar, D., and Jain, R. (2019). Interrelationships Between Marichev–Saigo–Maeda Fractional Integral Operators, the Laplace Transform and the $\overline{H}$-Function. *International Journal of Applied and Computational Mathematics*, 5(4), 1-13.

[3] Bansal, M. K., Kumar, D., Khan, I., Singh, J., and Nisar, K. S. (2019). Certain unified integrals associated with product of M-series and incomplete H-functions. *Mathematics*, 7(12), 1191.

[4] Bansal, M. K., Kumar, D., Singh, J., Tassaddiq, A., and Nisar, K. S. (2020). Some new results for the Srivastava–Luo–Raina M-transform pertaining to the incomplete H-functions. *AIMS Math*, 5(1), 717-722.

[5] Chaurasia V. B. L., and Saxena, V. (2007). Applications of certain special functions to non-linear differential equation involving $\bar{H}$-function. *Acta Ciencia Indica.*, XXXIVM(1), 307-318.

[6] Diaz, R., and Pariguan, E. (2004). On hypergeometric functions and Pochhammer k-symbol. *arXiv preprint math/0405596.*

[7] Daiya, J., and Saxena, R. K. (2015). Integral transforms of the S-functions. *Le Matematiche*, 70(2), 147-159.

[8] Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1953). *Higher Transcendental Functions Vol. II.* New York, Toronto and London: McGraw-Hill Book Company.

[9] Garde R. M. (1967). Applications of Jacobi polynomials to non-linear oscillations–1, Free oscillations. *Proc Nat Acad Sci India*. 37, 119-120.

[10] Jayarama, P., Madam, V. N. T., and Kurumujji, S. K. (2014). A study of I-function of several complex variables. *International Journal of Engineering Mathematics*, 2014.

[11] Kumar, D., Singh, J., Purohit, S. D., and Swroop, R. (2019). A hybrid analytical algorithm for nonlinear fractional wave-like equations. *Mathematical Modelling of Natural Phenomena*, 14(3), 304.

[12] Kumar, D., Singh, J., Prakash, A., and Swroop, R. (2019). Numerical simulation for system of time-fractional linear and nonlinear differential equations. *Progr. Fract. Differ. Appl*, 5(1), 65-77.

[13] Prasad, Y. N. (1986). Multivariable I-function. *Vijnana Parishad Anusandhan Patrika*. 29(4), 231-235.

[14] Prym, F. E. (1877). Zur theorie der gamma function. *The Journal fur die Reine und Angewandte Mathematik*. 82, 165–172.

[15] Rainville, E. D. (1960). *Special functions (Vol. 5)*. New York.

[16] Saxena, R. K., and Kushwaha, R. S. (1970). Applications of Jacobi polynomials to non-linear oscillations. *In Proceedings of the National Academy of Sciences, India*. 40, 65-72.

[17] Sharma, M. (2008). Fractional integration and fractional differentiation of the M-series. *Fractional calculus and applied analysis*, 11(2), 187-191.

[18] Sharma, C. K., and Mishra, P. L. (1991). On the I-function of two variables and its properties. *Acta Ciencia Indica Math*. 17, 667-672.

[19] Sharma, R., Singh, J., Kumar, D., and Singh, Y. (2022). Certain unified integrals associated with product of the general class of polynomials and incomplete I-functions. *International Journal of Applied and Computational Mathematics*, 8(1), 1-11.

[20] Singh, P. (2007). Applications of Hermite, Legendre and Jacobi polynomials and linear approximation to some non-linear differential equations associated with H-function. *Vijnana Parishad Anusandhan Patrika*. 50(1), 41-61.

[21] Singh, J., Kumar, D., and Bansal, M. K. (2019). Solution of nonlinear differential equation and special functions. *Mathematical Methods in the Applied Sciences*, 43(5), 2106-2116.

[22] Singh, J., Kumar, D., Baleanu, D., and Rathore, S. (2018). An efficient numerical algorithm for the fractional Drinfeld–Sokolov–Wilson equation. *Applied Mathematics and Computation*. 335, 12-24.

[23] Singh, J., Secer, A., Swroop, R., and Kumar, D. (2019). A reliable analytical approach for a fractional model of advection-dispersion equation. *Nonlinear Engineering*, 8(1), 107-116.

[24] Srivastava, B. M. (1975). Solvable potentials for the Chebyshev and associated Laguerre equations. *Jnanabha Sect A*. 5, 181-187.

[25] Srivastava, B. M., and Singh, F. (1976). Application of Jacobi polynomials to non-linear differential equation associated with H-function. *Vijnana Parishad Anusandhan Patrika*. 19(1), 25-34.

[26] Srivastava, H. M., Saxena, R. K., and Parmar, R. K. (2018). Some Families of the Incomplete H-Functions and the Incomplete $\overline{H}$-Functions and Associated Integral Transforms and Operators of Fractional Calculus with Applications. *Russian Journal of Mathematical Physics*, 25(1), 116-138.