Direct approach to the stability of various functional equations in Felbin's type non-archimedean fuzzy normed spaces

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ABSTRACT. Using the direct approach, the authors find the Ulam stability of the septic functional equation and octic functional equation in Felbin's type non-Archimedean fuzzy normed space.

1. INTRODUCTION AND PRELIMINARIES

The emergence of functional equations coincided with the modern formulation of the function concept. The first publications regarding functional equations were authored by D'Alembert [1] during the period between 1747 and 1750. Due to their apparent simplicity and harmonic characteristics, functional equations have captured the interest of numerous renowned mathematicians. Notable figures such as Rassias [5], Aoki [4], Găvruta [6] and Jakhar [11, 12] have all engaged with this area of study.

The foundational concept of Hyers-Ulam stability for functional equations traces back to a renowned problem centered on group homomorphisms("Let G be a group and G' be a metric group with metric $d(.,.)$. Given $\epsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists homomorphism $H : G \to G'$ with $d(f(x), H(x)) < \epsilon$ for all $x \in G$?"), successfully addressed by Ulam [2] and Hyers [3]. Over the past decades, a substantial volume of literature has been devoted to addressing the stability problem in the context of functional equations, with significant focus on crucial issues within this domain (see [7, 8, 9, 10, 11, 12]). Consequently, numerous effective techniques have been detailed in various papers (such as [10, 18–24, 27]), encompassing approaches like the direct method, fixed point method. Notably, the

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direct method consistently emerges as the primary investigative tool for exploring functional equations of diverse kinds.

The idea of fuzzy criteria on a set of data is used in the field of fuzzy functional analysis. In 1984, Katsaras [13] was the first to suggest this concept while researching fuzzy topological vector spaces, his groundbreaking work [14, 15, 16] being a motivating factor for many mathematicians. The authors Cheng & Mordsen [19] introduced an alternative form of fuzzy norm for linear spaces utilizing a distinct technique . In a related context, Michalek and Kramosil [20] further investigated the associated fuzzy metric in 1994 . The concept of a fuzzy real number's criterion, as articulated by Gähler and Gähler [21], quantifies the discrepancy between its negative and positive components.

Interestingly, Samantha and Bag [22] identified an enigmatic criterion that diverged somewhat from Mordsen & Cheng's established criterion. They subsequently demonstrated an applicable decomposition theorem for this distinctive criterion. This concept has found utility in the advancement and execution of fuzzy functional analysis, leading to an array of publications from diverse researchers.

Of particular significance is the work conducted by Xiao and Zhu[26] in this domain . They explored into various aspects of fuzzy norm linear spaces, encompassing the consideration of Felbin-type fuzzy norms in their generalized manifestation. Bag and Samantha, in their contribution [27], presented a minor alteration to Felbin's concept of a fuzzy standardized linear space.

The functional equation

$$\begin{aligned}&g(u+4v)-7g(u+3v)+21g(u+2v)-35g(u+v)-21g(u-v)\\&+7g(u-2v)-g(u-3v)+35g(u)=5040g(v)\end{aligned}$$

is known as septic functional equation since cu^7 is the solution.

Similarly, the functional equation

$$\begin{aligned}&g(u+4v)-8g(u+3v)+28g(u+2v)-56g(u+v)\\&-56g(u-v)+28g(u-2v)-8g(u-3v)+g(u-4v)\\&+70g(u)=40320g(v)\end{aligned}$$

is known as octic functional equation since cu^8 is the solution. Each solution to a octic functional equation in particular is referred to as a octic mapping.

Now, the authors will address the definitions, notations, and fundamental characteristics of a non-Archimedean fuzzy normed linear space in the Felbin's type framework.

Definition 1.1. [26] A function $\sigma : R \to [0,1]$ is termed a fuzzy real number if its α-level set is represented as $[\sigma]_\alpha = \{s : \sigma(s) \geq \alpha\}$ and function satisfies two conditions:

(1) there exist $s_0 \in R$ such as $\sigma(s_0) = 1$.
(2) $[\sigma]_\alpha = [\sigma^1_\alpha, \sigma^2_\alpha]$ for each $\alpha \in (0,1]$
where $-\infty < [\sigma^1_\alpha] \leq [\sigma^2_\alpha] < +\infty$.

$\mathcal{F}$ denotes the set of all fuzzy real numbers.

Definition 1.2. [19] Let $\sigma, \varsigma \in \mathcal{F}$ and $[\sigma]_\alpha = [l^1_\alpha, m^1_\alpha]$, $[\varsigma]_\alpha = [l^2_\alpha, m^2_\alpha]$, $\alpha \in (0,1]$. Then $[\varsigma \oplus \sigma]_\alpha = [l^1_\alpha + l^2_\alpha, m^1_\alpha + m^2_\alpha]$.

Definition 1.3. [17] A partial order denoted by "$\preceq$" is established within the set $\mathcal{F}$ as follows: For any σ and ς in $\mathcal{F}$, $\sigma \preceq \varsigma$ holds if and only if, for all $\alpha \in (0,1]$, it satisfies $\sigma^1_\alpha \leq \varsigma^1_\alpha$ and $\sigma^2_\alpha \leq \varsigma^2_\alpha$, where $[\sigma]_\alpha = [\sigma^1_\alpha, \sigma^2_\alpha]$ and $[\varsigma]_\alpha = [\varsigma^1_\alpha, \varsigma^2_\alpha]$. Furthermore, a stricter inequality, denoted by " $<$ ", is defined within $\mathcal{F}$: $\sigma < \varsigma$ if and only if, for all $\alpha \in (0,1]$, the conditions $\sigma^1_\alpha < \varsigma^1_\alpha$ and $\sigma^2_\alpha < \varsigma^2_\alpha$ are satisfied.

Definition 1.4. [17] Consider a vector space U over R and $||.|| : U \to R^*(I)$(set of all upper semi continuous normal convex fuzzy real numbers) and let the mappings $\mathcal{L}, \mathcal{R} : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $\mathcal{L}(0,0) = 0$ and $\mathcal{R}(1,1) = 0$. Write

$$[||u||]_\alpha = [||u||^\alpha_1, ||u||^\alpha_2] \quad \forall \quad u \in U \quad \& \quad 0 < \alpha \leq 1$$

and suppose for all $u \in U, u \neq 0$, there exists $\alpha_0 \in (0,1]$ independent of u such that for all $\alpha \leq \alpha_0$
(I) $\quad ||u||^\alpha_2 < \infty$,
(II) $\quad ||u||^\alpha_2 > 0$. The quadruple $(U, ||.||, \mathcal{L}, \mathcal{U})$ is called a fuzzy normed linear space and $||.||$ is a fuzzy norm if

(1) $||u|| = \overline{0} \iff u = \underline{0}$.
(2) $||ru|| = |r|||u|| \quad \forall \quad u \in U, r \in R$.
(3) For all $u, v \in U$
  (a) whenever

$$p \leq ||u||^1_1, q \leq ||v||^1_1 \quad and \quad p + q \leq ||u+v||^1_1,$$
$$||u+v||(p+q) \geq \mathcal{L}(||u||(p), ||v||(q)),$$

  (b) whenever

$$p \geq ||u||^1_1, q \geq ||v||^1_1 \quad and \quad p + q \geq ||u+v||^1_1,$$
$$||u+v||(p+q) \leq \mathcal{U}(||u||(p), ||v||(q)).$$

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Fuzzy norm on a linear space is defined in [17] as stated by C. Felbin. Now, as stated in [17], we define the fuzzy norm of modified Felbin's type on a linear space.

Definition 1.5. [17] Consider a linear space U over R. Let $|| \quad || : U \to \mathcal{F}^+$ be a mapping satisfying

(1) $||u|| = \overline{0} \iff u = \underline{0}$.
(2) $||ru|| = |r|||u|| \quad \forall \quad u \in U, r \in R$.
(3) $||v + u|| \preceq ||v|| \oplus ||u|| \; \forall \; v, u \in U$, and
$u \neq \underline{0} \Rightarrow ||u||(s) = 0 \quad \forall \quad s \leq 0$.

Then $(U, || \quad ||)$ is known as fuzzy normed linear space and $|| \quad ||$ is known as fuzzy norm on U.

Definition 1.6. [12] Suppose $\mathcal{K}$ be a field. An absolute value on $\mathcal{K}$ is classified as non-Archimedean field if it satisfies the following conditions for any elements a and b in $\mathcal{K}$:

(1) $|a| \geq 0$ and $|a| = 0 \iff a = 0$.
(2) $|a + b| \leq \max\{|a|, |b|\}$.
(3) $|ab| = |a||b|$.
(4) There exists $a_0 \in \mathcal{K}$ such that $|a_0| \neq 0, 1$.

The main objective of this study is to establish the generalized Hyers-Ulam stability for septic and octic functional equations within a modified Felbin-type fuzzy normed linear space. The article is organized into three sections. In section 2, we examine the stability analysis of the septic functional equation within a non-Archimedean fuzzy normed linear space of the Felbin type. Moving on to section 3, our focus shifts to the generalized Hyers-Ulam stability of the octic functional equation. This investigation takes place within a non-Archimedean fuzzy normed linear space of the Felbin type.

2. Stability of septic functional equation

The stability problems of various septic functional equations in several spaces such as intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean spaces, Banach spaces, orthogonal spaces and many other spaces have been broadly investigated by a number of mathematicians. Motivated by the approach of research by various mathematicians, an effort has been made in this paper to obtain the stability of the following functional equations.

$$\begin{aligned} &g(u+4v) - 7g(u+3v) + 21g(u+2v) - 35g(u+v) - 21g(u-v) \\ &+7g(u-2v) - g(u-3v) + 35g(u) = 5040g(v). \end{aligned} \tag{2.1}$$

To simplify notation, let us introduce the "difference operator "denoted by $\Delta_{s'}$.

$$\begin{aligned}\Delta_{s'}g(u,v) &= g(u+4v)-7g(u+3v)+21g(u+2v)-35g(u+v)\\ &- 21g(u-v)+7g(u-2v)-g(u-3v)+35g(u)\\ &- 5040g(v).\end{aligned}$$

Theorem 2.1. *Suppose that U is a linear space and $(W, ||.||^{\sim})$ is a fuzzy normed space. Consider $\psi : U^2 \to W$ be a mapping such that*

$$\lim_{n\to\infty}\frac{||\psi(2^n u, 2^n v)||_{\alpha}^{\sim 1}}{|2^{7n}|} = \lim_{n\to\infty}\frac{||\psi(2^n u, 2^n v)||_{\alpha}^{\sim 2}}{|2^{7n}|} = 0, \tag{2.2}$$

for all $u, v \in U$ and $\alpha \in (0,1]$. Let $(V, ||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g : U \to V$ is such that

$$||\Delta_{s'}g(u,v)|| \preceq ||\psi(u,v)||^{\sim} \tag{2.3}$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$||S(u)-g(u)|| \preceq \frac{1}{2^7}\max\left\{\frac{H(2^k u)}{2^{7k}}; k\in N\cup 0\right\}, \tag{2.4}$$

where

$$\begin{aligned}H(2^k u) &= \left|\frac{1}{2520}\right|\left[\left|\frac{1}{10080}\right|\Big(||\psi(0,6.2^k u)||^{\sim}\oplus||\psi(6.2^k u,-6.2^k u)||^{\sim}\Big)\right.\\ &\oplus \left|\frac{1}{1440}\right|\Big(||\psi(0,4.2^k u)||^{\sim}\oplus||\psi(4.2^k u,-4.2^k u)||^{\sim}\Big)\\ &\oplus |\frac{1}{180}|\Big(||\psi(0,3.2^k u)||^{\sim}\oplus||\psi(3.2^k u,-3.2^k u)||^{\sim}\Big)\\ &\oplus \left|\frac{13}{288}\right|\Big(||\psi(0,2.2^k u)||^{\sim}\oplus||\psi(2.2^k u,-2.2^k u)||^{\sim}\Big)\\ &\oplus \left|\frac{373}{2520}\right|\Big(||\psi(0,2^k u)||^{\sim}\oplus||\psi(2^k u,-2^k u)||^{\sim}\Big)\\ &\oplus \left|\frac{1}{2}\right|||\psi(4.2^k u,2^k u)||^{\sim}\oplus\left|\frac{7}{2}\right|||\psi(3.2^k u,2^k u)||^{\sim}\\ &\oplus 11.||\psi(2.2^k u,2^k u)||^{\sim}\oplus 21.||\psi(2^k u,2^k u)||^{\sim}\\ &\oplus \left.\left|\frac{1}{2}\right|||\psi(0,2.2^k u)||^{\sim}\oplus 28.||\psi(0,2^k u)||^{\sim}\oplus\left|\frac{217}{720}\right|||\psi(0,0)||^{\sim}\right].\end{aligned}$$

Proof. Taking $u = 0 = v$ in (2.3), we obtain

$$||g(0)|| \preceq \frac{||\psi(0,0)||^{\sim}}{|5040|}. \tag{2.5}$$

Taking $(u,v)=(0,u)$ in (2.3), the authors get

$$||g(4u)-7g(3u)+21g(2u)-5075g(u)+35g(0)-21g(-u)$$
$$+7g(-2u)-g(-3u)|| \preceq ||\psi(0,u)||^{\sim}. \tag{2.6}$$

Putting $(u,v)=(u,-u)$ in (2.3), we get

$$||g(-3u)-7g(-2u)-5019g(-u)-35g(0)+35g(u)-21g(2u)$$
$$+7g(3u)-g(4u)|| \preceq ||\psi(u,-u)||^{\sim}. \tag{2.7}$$

By (2.6) and (2.7), we obtain

$$||g(u)-g(-u)|| \preceq \frac{1}{|5040|}(||\psi(0,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}). \tag{2.8}$$

Putting $(u,v)=(4u,u)$ in (2.3), we get

$$||g(8u)-7g(7u)+21g(6u)-35g(5u)+35g(4u)-21g(3u)$$
$$+7g(2u)-5041g(u)|| \preceq ||\psi(4u,u)||^{\sim}. \tag{2.9}$$

Taking $(u,v)=(0,2u)$ in (2.3), the authors get

$$||g(8u)-7g(6u)+21g(4u)-5075g(2u)+35g(0)-21g(-2u)$$
$$+7g(-4u)-g(-6u)|| \preceq ||\psi(0,2u)||^{\sim}. \tag{2.10}$$

By (2.9) and (2.10), we obtain

$$||7g(7u)-28g(6u)+35g(5u)-14g(4u)+21g(3u)$$
$$-5082g(2u)+5041g(u)+35g(0)-21g(-2u)$$
$$+7g(-4u)-g(-6u)|| \preceq (||\psi(4u,u)||^{\sim} \oplus ||\psi(0,2u)||^{\sim}). \tag{2.11}$$

Now, using (2.5), (2.8) and (2.11), we conclude

$$||7g(7u)-27g(6u)+35g(5u)-21g(4u)+21g(3u)-5061g(2u)$$
$$+5041g(u)|| \preceq \frac{1}{|5040|}(||\psi(0,6u)||^{\sim} \oplus ||\psi(6u,-6u)||^{\sim})$$
$$\oplus \frac{1}{|720|}(\psi(||0,4u)||^{\sim} \oplus ||\psi(4u,-4u)||^{\sim}) \oplus \frac{1}{|240|}(||\psi(0,2u)||^{\sim}$$
$$\oplus ||\psi(2u,-2u)||^{\sim}) \oplus ||\psi(4u,u)||^{\sim} \oplus ||\psi(0,2u)||^{\sim}$$
$$\oplus \frac{1}{144}||\psi(0,0)||^{\sim}. \tag{2.12}$$

Putting $(u,v)=(3u,u)$ in (2.3), we get

$$||g(7u)-7g(6u)+21g(5u)-35g(4u)+35g(3u)-21g(2u)$$
$$-5033g(u)-g(0)|| \preceq ||\psi(3u,u)||^{\sim}. \tag{2.13}$$

From (2.5), we obtain

$$||g(7u)-7g(6u)+21g(5u)-35g(4u)+35g(3u)-21g(2u)$$
$$-5033g(u)|| \preceq ||\psi(3u,u)||^{\sim} \oplus \frac{1}{|5040|}||\psi(0,0)||^{\sim}. \tag{2.14}$$

From (2.12) and (2.14), we obtain

$$||11g(6u)-56g(5u)+112g(4u)-112g(3u)-2457g(2u)$$
$$+20136g(u)|| \preceq \frac{1}{|10080|}(||\psi(0,6u)||^{\sim} \oplus ||\psi(6u,-6u)||^{\sim})$$
$$\oplus\frac{1}{|1440|}(||\psi(0,4u)||^{\sim} \oplus ||\psi(4u,-4u)||^{\sim}) \oplus \frac{1}{|480|}(||\psi(0,2u)||^{\sim}$$
$$\oplus||\psi(2u,-2u)||^{\sim}) \oplus \frac{1}{2}||\psi(4u,u)||^{\sim} \oplus \frac{7}{2}||\psi(3u,u)||^{\sim}$$
$$\oplus\frac{1}{2}||\psi(0,2u)||^{\sim} \oplus \frac{1}{240}||\psi(0,0)||^{\sim}. \tag{2.15}$$

Putting $(u,v)=(2u,u)$ in (2.3), we obtain

$$||g(6u)-7g(5u)+21g(4u)-35g(3u)+35g(2u)-5061g(u)$$
$$-g(-u)+7g(0)|| \preceq ||\psi(2u,u)||^{\sim}. \tag{2.16}$$

Now, by using (2.5), (2.8) and (2.16), we get

$$||g(6u)-7g(5u)+21g(4u)-35g(3u)+35g(2u)-5060g(u)||$$
$$\preceq ||\psi(2u,u)||^{\sim} \oplus \frac{1}{|5040|}(||\psi(0,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim})$$
$$\oplus\frac{1}{|720|}||\psi(0,0)||^{\sim}. \tag{2.17}$$

From (2.15)and (2.17), the authors obtain

$$||21g(5u)-119g(4u)+273g(3u)-2842g(2u)+75796g(u)||$$
$$\preceq \frac{1}{|10080|}(||\psi(0,6u)||^{\sim} \oplus ||\psi(6u,-6u)||^{\sim}) \oplus \frac{1}{|1440|}(||\psi(0,4u)||^{\sim}$$
$$\oplus||\psi(4u,-4u)||^{\sim}) \oplus \frac{1}{|480|}(||\psi(0,2u)||^{\sim} \oplus |\psi(2u,-2u)||^{\sim})$$
$$\oplus|\frac{11}{5040}|(||\psi(0,u)||^{\sim} \oplus |\psi(u,-u)||^{\sim}) \oplus \frac{1}{|2|}|\psi(4u,u)||^{\sim}$$
$$\oplus|\frac{7}{2}|\psi(3u,u)||^{\sim} \oplus 11||\psi(2u,u)||^{\sim} \oplus \frac{1}{|2|}|\psi(0,2u)||^{\sim}$$
$$\oplus|\frac{7}{360}|\psi(0,0)||^{\sim}. \tag{2.18}$$

Putting $(u,v)=(u,u)$ in (2.3), we get

$$||g(5u)-7g(4u)+21g(3u)-35g(2u)-g(-2u)-5005g(u)$$
$$+7g(-u)-21g(0)|| \preceq ||\psi(u,u)||^{\sim}. \tag{2.19}$$

With the help of (2.5), (2.8) and (2.19), we get

$$||g(5u)-7g(4u)+21g(3u)-34g(2u)-5012g(u)|| \preceq ||\psi(u,u)||^{\sim}$$
$$\oplus||\psi(u,u)||^{\sim} \oplus \frac{1}{|5040|}(||\psi(0,2u)||^{\sim} \oplus ||\psi(2u,-2u)||^{\sim})$$
$$\oplus\frac{1}{|720|}(||\psi(0,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \oplus \frac{1}{|240|}||\psi(0,0)||^{\sim}). \tag{2.20}$$

Now, from (2.18) and (2.20), we conclude

$$||28g(4u)-168g(3u)-21328g(2u)+181048g(u)||$$
$$\preceq \frac{1}{|10080|}(||\psi(0,6u)||^{\sim} \oplus |\psi(6u,-6u)||^{\sim}) \oplus \frac{1}{|1440|}(||\psi(0,4u)||^{\sim}$$
$$\oplus||\psi(4u,-4u)||^{\sim}) \oplus \frac{1}{|160|}(||\psi(0,2u)||^{\sim} \oplus ||\psi(2u,-2u)||^{\sim})$$
$$\oplus\frac{79}{2520}(||\psi(0,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \oplus \frac{1}{2}||\psi(4u,u)||^{\sim}$$
$$\oplus\frac{7}{2}||\psi(3u,u)||^{\sim} \oplus 11||\psi(2u,u)||^{\sim} \oplus 21||\psi(u,u)||^{\sim}$$
$$\oplus\frac{1}{2}||\psi(0,2u)||^{\sim} \oplus \frac{77}{720}||\psi(0,0)||^{\sim}. \tag{2.21}$$

Now, by using (2.5), (2.6) and (2.8), we obtain

$$||g(4u)-6g(3u)+14g(2u)-5054g(u)|| \preceq \frac{1}{|5040|}(||\psi(0,3u)||^{\sim}$$
$$\oplus||\psi(3u,-3u)||^{\sim}) \oplus \frac{1}{|720|}(||\psi(0,2u)||^{\sim} \oplus ||\psi(2u,-2u)||^{\sim})$$
$$\oplus\frac{1}{|240|}(||\psi(0,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \oplus ||\psi(0,u)||^{\sim}$$
$$\oplus\frac{1}{|144|}||\psi(0,0)||^{\sim}. \tag{2.22}$$

Now, by (2.21) and (2.22), the authors conclude that

$$||g(2u)-2^7g(u)|| \preceq H(u) \tag{2.23}$$

where

$$H(u) = \frac{1}{2520}\bigg[\frac{1}{10080}(||\psi(0,6u)||^{\sim} \oplus ||\psi(6u,-6u)||^{\sim})$$
$$\oplus\frac{1}{1440}(||\psi(0,4u)||^{\sim} \oplus ||\psi(4u,-4u)||^{\sim}) \oplus \frac{1}{180}(||\psi(0,3u)||^{\sim}$$
$$\oplus||\psi(3u,-3u)||^{\sim}) \oplus \frac{13}{288}(||\psi(0,2u)||^{\sim} \oplus ||\psi(2u,-2u)||^{\sim})$$
$$\oplus\frac{373}{2520}(||\psi(0,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \oplus \frac{1}{2}||\psi(4u,u)||^{\sim}$$
$$\oplus\frac{7}{2}||\psi(3u,u)||^{\sim} \oplus 11||\psi(2u,u)||^{\sim} \oplus ||\psi(u,u)||^{\sim} \oplus \frac{1}{2}||\psi(0,2u)||^{\sim}$$
$$\oplus 28||\psi(0,u)||^{\sim} \oplus \frac{217}{720}||\psi(0,0)||^{\sim}\bigg].$$

Hence from (2.23), we get

$$||g(2u) - 2^7 g(u)||_\alpha^1 \leq H_1(u), \tag{2.24}$$

and

$$||g(2u) - 2^7 g(u)||_\alpha^2 \leq H_2(u), \tag{2.25}$$

where for $\alpha \in (0,1]$, then

$$H_1(u) = \frac{1}{2520}\bigg[\frac{1}{10080}(||\psi(0,6u)||_\alpha^{1\sim} \oplus ||\psi(6u,-6u)||_\alpha^{1\sim})$$
$$\oplus\frac{1}{1440}(||\psi(0,4u)||_\alpha^{1\sim} \oplus ||\psi(4u,-4u)||_\alpha^{1\sim}) \oplus \frac{1}{180}(||\psi(0,3u)||_\alpha^{1\sim}$$
$$\oplus||\psi(3u,-3u)||_\alpha^{1\sim}) \oplus \frac{13}{288}(||\psi(0,2u)||_\alpha^{1\sim} \oplus ||\psi(2u,-2u)||_\alpha^{1\sim})$$
$$\oplus\frac{373}{2520}(||\psi(0,u)||_\alpha^{1\sim} \oplus ||\psi(u,-u)||_\alpha^{1\sim}) \oplus \frac{1}{2}||\psi(4u,u)||_\alpha^{1\sim}$$
$$\oplus\frac{7}{2}||\psi(3u,u)||_\alpha^{1\sim} \oplus 11||\psi(2u,u)||_\alpha^{1\sim} \oplus ||\psi(u,u)||_\alpha^{1\sim} \oplus \frac{1}{2}||\psi(0,2u)||_\alpha^{1\sim}$$
$$\oplus 28||\psi(0,u)||_\alpha^{1\sim} \oplus \frac{217}{720}||\psi(0,0)||_\alpha^{1\sim}\bigg] \tag{2.26}$$

and

$$
\begin{aligned}
H_2(u) = \frac{1}{2520}\Bigg[&\frac{1}{10080}(||\psi(0,6u)||_\alpha^{2\sim} \oplus ||\psi(6u,-6u)||_\alpha^{2\sim}) \\
&\oplus \frac{1}{1440}(||\psi(0,4u)||_\alpha^{2\sim} \oplus ||\psi(4u,-4u)||_\alpha^{2\sim}) \oplus \frac{1}{180}(||\psi(0,3u)||_\alpha^{2\sim} \\
&\oplus ||\psi(3u,-3u)||_\alpha^{2\sim}) \oplus \frac{13}{288}(||\psi(0,2u)||_\alpha^{2\sim} \oplus ||\psi(2u,-2u)||_\alpha^{2\sim}) \\
&\oplus \frac{373}{2520}(||\psi(0,u)||_\alpha^{2\sim} \oplus ||\psi(u,-u)||_\alpha^{2\sim}) \oplus \frac{1}{2}||\psi(4u,u)||_\alpha^{2\sim} \\
&\oplus \frac{7}{2}||\psi(3u,u)||_\alpha^{2\sim} \oplus 11||\psi(2u,u)||_\alpha^{2\sim} \oplus ||\psi(u,u)||_\alpha^{2\sim} \oplus \frac{1}{2}||\psi(0,2u)||_\alpha^{2\sim} \\
&\oplus 28||\psi(0,u)||_\alpha^{2\sim} \oplus \frac{217}{720}||\psi(0,0)||_\alpha^{2\sim}\Bigg]. \qquad (2.27)
\end{aligned}
$$

From (2.25), we conclude

$$ ||\frac{g(2u)}{2^7} - g(u)||_\alpha^{1\sim} \leq \frac{H_1(u)}{|2^7|}. \qquad (2.28) $$

Replacing u by $2^n u$ in (2.28) and dividing both side by 2^{7n}, we obtain

$$ \left|\left|\frac{g(2^{n+1}u)}{2^{7(n+1)}} - \frac{g(2^n u)}{2^{7n}}\right|\right|_\alpha^1 \leq \frac{H_1(2^n u)}{|2^{7(n+1)}|} \qquad (2.29) $$

for all non-negative integers n. Hence the sequence $\{\frac{g(2^n u)}{2^{7n}}\}$ is Cauchy. Every Cauchy sequence is convergent in Y, since Y is complete. So, the authors construct a mapping $S : U \to V$ such that

$$ S(u) = \lim_{n\to\infty} \frac{g(2^n u)}{2^{7n}}, \qquad (2.30) $$

i.e.,

$$ \lim_{n\to\infty} \left|\left|\frac{g(2^n u)}{2^{7n}} - S(u)\right|\right| = \overline{0}. \qquad (2.31) $$

Now for each non-negative integer n, the authors explore

$$
\begin{aligned}
\left|\left|\frac{g(2^n u)}{2^{7n}} - g(u)\right|\right|_\alpha^1 &= \left|\left|\sum_{k=0}^{n-1}\left(\frac{g(2^{k+1}u)}{2^{7(n+1)}} - \frac{g(2^k u)}{2^{7n}}\right)\right|\right|_\alpha^1 \\
&\leq \max\left\{\left|\left|\frac{g(2^{k+1}u)}{2^{7(n+1)}} - \frac{g(2^k u)}{2^{7n}}\right|\right|_\alpha^1 : 0 \leq k < n\right\} \\
&\leq \frac{1}{2^7}\max\left\{\frac{H_1(2^k u)}{|2^{7k}|} : 0 \leq k < n\right\}. \qquad (2.32)
\end{aligned}
$$

Similarly, the authors can show that

$$\left|\left|\frac{g(2^n u)}{2^{7n}} - g(u)\right|\right|_\alpha^2 \leq \frac{1}{2^7} \max\left\{\frac{H_2(2^k u)}{|2^{7k}|} : 0 \leq k < n\right\}. \quad (2.33)$$

Taking $n \to \infty$ in (2.32) and (2.33), the authors see that inequality (2.4) holds. Next we prove that $S : U \to V$ is a cubic mapping. Replacing (u, v) by $(2^n u, 2^n v)$ and divide by $|2^{7n}|$ in (2.3), we get

$$\begin{aligned}
&\frac{1}{2^{7n}}||g(2^n(u+4v)) - 7g(2^n(u+3v)) + 21g(2^n(u+2v)) \\
&-35(2^n(u+v)) - 21g(2^n(u-v)) + 7g(2^n(u-2v)) \\
&-g(2^n(u-3v)) + 35g(2^n u) - 5040g(2^n v)|| \\
&\preceq \left|\left|\frac{\psi(2^n u, 2^n v)}{2^{7n}}\right|\right|^\sim
\end{aligned} \quad (2.34)$$

Taking $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
&||S(u+4v) - 3S(u+3v) + 21S(u+2v) - 35S(u+v) - 21S(u-v) \\
&+7S(u-2v) - S(u-3v) + 35S(u) - 5040S(v)|| \preceq \overline{0}
\end{aligned}$$

this implies that

$$\begin{aligned}
&S(u+4v) - 3S(u+3v) + 21S(u+2v) - 35S(u+v) - 21S(u-v) \\
&+7S(u-2v) - S(u-3v) + 35S(u) - 5040S(v) = \underline{0}.
\end{aligned}$$

□

Therefore, the mapping $S : U \to V$ is septic. Next we shall prove uniqueness of mapping S. Now, consider another septic mapping $S' : U \to V$ which satisfies (2.1) and (2.4). For fix $u \in U$, certainly $S(2^n u) = 2^{7n} S(u)$ and $S'(2^n u) = 2^{7n} S'(u)$ for all $n \in N$. Therefore,

$$\begin{aligned}
&||S(u) - S'(u)|| = \lim_{n\to\infty} \frac{1}{2^{7n}}||S(2^n u) - S'(2^n u)|| \\
&= \lim_{n\to\infty} \frac{1}{2^{7n}}||S(2^n u) - g(2^n u) + g(2^n u) - S'(2^n u)|| \\
&\preceq \lim_{n\to\infty} \max\left\{\frac{1}{2^{7n}}||S(2^n u) - g(2^n u)||, \frac{1}{2^{7n}}||g(2^n u) - S'(2^n u)||\right\} \\
&\preceq \lim_{k\to\infty} \lim_{n\to\infty} \max\left\{\max\left\{\frac{1}{2^{7(n+1)}}\left\{\frac{H(2^{k+n}u)}{2^{7k}}, \frac{H(2^{k+n}u)}{2^{7k}}\right\}\right\}\right\} \\
&= \overline{0}.
\end{aligned}$$

Therefore, $S(u) - S'(u) = \underline{0}$. So, $S(u) = S'(u)$ Hence, we deduced that S is unique mapping.

Corollary 2.2. *Suppose that $(U, ||.||)$ is a normed space, $(W, ||\ \ ||^{\sim})$ is a fuzzy normed space, and $(V, ||\ \ ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $p < 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u, v)|| \preceq ||(||v||^p + ||u||^p)w_0||^{\sim} \tag{2.35}$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| \ &\preceq\ \frac{|| \, ||u||^p w_0||^{\sim}}{2^7}\left[\left|\frac{1}{2520}\right|\left(\left|\frac{50773}{840}\right| \oplus \left|\frac{61|2|^p}{96}\right| \oplus \left|\frac{|3|^p}{60}\right|\right.\right. \\ &\oplus\ \left|\frac{|4|^p}{480}\right| \oplus \left|\frac{|6|^p}{3360}\right| \oplus \left|\frac{|4|^p+1}{2}\right| \oplus \left|\frac{7(|3|^p+1)}{2}\right| \\ &\oplus\ \left.\left. |11(|2^p|+1)|\right)\right]. \end{aligned}$$

Corollary 2.3. *Suppose that $(U, ||.||)$ is a normed space, $(W, ||\ \ ||^{\sim})$ is a fuzzy normed space, and $(V, ||\ \ ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $p, q < 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u, v)|| \preceq ||(||v||^p ||u||^q)w_0||^{\sim}$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| \ &\preceq\ \frac{|| \, ||u||^{p+q} w_0||^{\sim}}{2^7}\left[\left|\frac{1}{2520}\right|\left(\left|\frac{52920}{2520}\right| \oplus \left|\frac{13|2|^{p+q}}{288}\right| \oplus \left|\frac{|3|^{p+q}}{180}\right|\right.\right. \\ &\oplus\ \left|\frac{|4|^{p+q}}{1440}\right| \oplus \left|\frac{|6|^{p+q}}{10080}\right| \oplus \left|\frac{|4|^p}{2}\right| \oplus \left|\frac{7(|3|^p)}{2}\right| \\ &\oplus\ \left.\left. |11(|2^p|)|\right)\right]. \end{aligned}$$

Corollary 2.4. *Suppose that $(U, ||.||)$ is a normed space, $(W, ||\ \ ||^{\sim})$ is a fuzzy normed space, and $(V, ||\ \ ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $\lambda = s + r < 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u, v)|| \preceq ||[||v||^s ||u||^r + (||u||^{s+r} + ||v||^{s+r})]w_0||^{\sim} \tag{2.36}$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| \ &\preceq \ \frac{|| \, ||u||^{\lambda} w_0||^{\sim}}{2^7} \Bigg[\left| \frac{1}{2520} \right| \Bigg(\left| \frac{6733}{630} \right| \oplus \left| \frac{49|2|^{\lambda}}{72} \right| \oplus \left| \frac{|3|^{\lambda}}{45} \right| \\ &\oplus \ \left| \frac{|4|^{\lambda}}{380} \right| \oplus \left| \frac{|6|^{\lambda}}{2520} \right| \oplus \left| \frac{7(|3|^{r} + |3|^{\lambda} + 1)}{2} \right| \\ &\oplus \ \left| \frac{|4|^{r} + |4|^{\lambda} + 1}{2} \right| \oplus |11(|2|^{r} + 2^{\lambda} + 1)| \Bigg) \Bigg]. \end{aligned}$$

Theorem 2.5. *Suppose that U is a linear space and $(W, ||.||^{\sim})$ is a fuzzy normed space. Consider $\psi : U^2 \to W$ be a mapping such that*

$$\lim_{n\to\infty} |2^{7n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 1} = \lim_{n\to\infty} |2^{7n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 2} = 0, \quad (2.37)$$

for all $u \in U$ and $\alpha \in (0, 1]$. Let $(V, ||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g : U \to V$ is such that

$$||\Delta_{s'} g(u, v)|| \preceq ||\psi(u, v)||^{\sim} \quad (2.38)$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$||S(u) - g(u)|| \preceq \frac{1}{2^7} \max \left\{ |2^{7(k+1)}| H\left(\frac{u}{2^{k+1}}\right), k \in N \cup \{0\} \right\} \quad (2.39)$$

where

$$\begin{aligned}
H(\frac{u}{2^{k+1}}) \quad = \quad & \left|\frac{1}{2520}\right|\Bigg[\left|\frac{1}{10080}\right|\left(||\psi\left(0,\frac{6u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{6u}{2^{k+1}},\frac{-6u}{2^{k+1}}\right)||^{\sim}\right)\\
\oplus \quad & \left|\frac{1}{1440}\right|\left(||\psi\left(0,\frac{4u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{4u}{2^{k+1}},\frac{-4u}{2^{k+1}}\right)||^{\sim}\right)\\
\oplus \quad & |\frac{1}{180}|\left(||\psi\left(0,\frac{3u}{2^{k+1}}\right)||^{\sim}\oplus||\psi(\frac{3u}{2^{k+1}},\frac{-3u}{2^{k+1}})||^{\sim}\right)\\
\oplus \quad & \left|\frac{13}{288}\right|\left(||\psi\left(0,\frac{2u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{2u}{2^{k+1}},\frac{-2u}{2^{k+1}}\right)||^{\sim}\right)\\
\oplus \quad & \left|\frac{373}{2520}\right|\left(||\psi\left(0,\frac{u}{2^{k+1}}\right)||^{\sim}\oplus||\psi\left(\frac{u}{2^{k+1}},\frac{-u}{2^{k+1}}\right)||^{\sim}\right)\\
\oplus \quad & \left|\frac{1}{2}\right|||\psi\left(\frac{4u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\oplus\left|\frac{7}{2}\right|||\psi\left(\frac{3u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\\
\oplus \quad & 11.||\psi\left(\frac{2u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\oplus 21.||\psi\left(\frac{u}{2^{k+1}},\frac{u}{2^{k+1}}\right)||^{\sim}\\
\oplus \quad & \left|\frac{1}{2}\right|||\psi\left(0,\frac{2u}{2^{k+1}}\right)||^{\sim}\oplus 28.||\psi\left(0,\frac{u}{2^{k+1}}\right)||^{\sim}\\
\oplus \quad & \left|\frac{217}{720}\right|||\psi(0,0)||^{\sim}\Bigg].
\end{aligned}$$

Proof. From (2.24), the authors get

$$\left|\left|g(u)-2^{7}g\left(\frac{u}{2}\right)\right|\right|_{\alpha}^{1}\leq H_{1}\left(\frac{u}{2}\right) \tag{2.40}$$

for $\alpha\in(0,1]$. Replacing u by $\frac{u}{2^n}$ and multiplying both side by $|2^{7n}|$ in (2.40), we get

$$\left|\left|2^{7n}g\left(\frac{u}{2^{n}}\right)-2^{7(n+1)}g\left(\frac{u}{2^{n+1}}\right)\right|\right|_{\alpha}^{1}\leq|2^{7n}|H_{1}\left(\frac{u}{2^{n+1}}\right) \tag{2.41}$$

for all negative integer n. Hence the sequence $g\{2^{7n}g(\frac{u}{2^n})\}$ is Cauchy by (2.37) and (2.41). Every Cauchy sequence is convergent in Y, since Y is complete. So, the authors construct a mapping $S:U\to V$ such that

$$S(u)=\lim_{n\to\infty}2^{7n}g\left(\frac{u}{2^{n}}\right)$$

for all $u\in U$. That is

$$\lim_{n\to\infty}||2^{7n}g\left(\frac{u}{2^{n}}\right)-S(u)||=\overline{0}$$

for all $u \in U$. Now, for each positive integer n, the authors have

$$\begin{aligned}\left|\left|2^{7n}g\left(\frac{u}{2^n}\right)-g(u)\right)\right|\right|_\alpha^1 &= \left|\left|\sum_{k=0}^{n-1}\left(2^{7(k+1)}g\left(\frac{u}{2^{k+1}}\right)-2^{7k}g\left(\frac{u}{2^k}\right)\right)\right|\right|_\alpha^1\\ &\leq \max\left\{\left|\left|\left(2^{7(k+1)}g\left(\frac{u}{2^{k+1}}\right)-2^{7k}g\left(\frac{u}{2^k}\right)\right)\right|\right|_\alpha^1; 0\leq k<n\right\}\\ &\leq \frac{1}{|2^7|}\max\left\{|2^{7(k+1)}|H_1\left(\frac{u}{2^{k+1}}\right):0\leq k<n\right\}. \qquad (2.42)\end{aligned}$$

Similarly , it can be shown from (2.25)

$$\begin{aligned}&\left|\left|2^{7n}g\left(\frac{u}{2^n}\right)-g(u)\right)\right|\right|_\alpha^2\\ &\leq \frac{1}{|2^7|}\max\left\{|2^{7(k+1)}|H_2\left(\frac{u}{2^{k+1}}\right):0\leq k<n\right\}. \qquad (2.43)\end{aligned}$$

Taking $n \to \infty$ in (2.42) and (2.43), the authors see that inequality (2.39) holds. The authors conclude that $S(u)$ is a unique cubic mapping holding (2.39) using the same procedure as in the demonstration of theorem (2.1).

□

Corollary 2.6. *Suppose that $(U, ||.||)$ is a normed space, $(W, ||\quad||^\sim)$ is a fuzzy normed space, and $(V, ||\quad||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $p > 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_s g(u,v)|| \preceq ||(||v||^p + ||u||^p)w_0||^\sim \qquad (2.44)$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned}||S(u)-g(u)|| \preceq\ & \frac{||\,||u||^p w_0||^\sim}{|2^p|}\left[\left|\frac{1}{2520}\right|\left(\left|\frac{50773}{840}\right|\oplus\left|\frac{61|2|^p}{96}\right|\oplus\left|\frac{|3|^p}{60}\right|\right.\right.\\ &\oplus \left|\frac{|4|^p}{480}\right|\oplus\left|\frac{|6|^p}{3360}\right|\oplus\left|\frac{|4|^p+1}{2}\right|\oplus\left|\frac{7(|3|^p+1)}{2}\right|\\ &\left.\left.\oplus\ |11(|2^p|+1)|\right)\right]\end{aligned}$$

for all $u \in U$.

Corollary 2.7. *Suppose that $(U, ||.||)$ is a normed space, $(W, ||\quad||^\sim)$ is a fuzzy normed space, and $(V, ||\quad||)$ is a non-Archimedean complete* *fuzzy normed space. Let $w_0 \in W$ and $p, q > 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||(||v||^p ||u||^q) w_0||^{\sim}$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| \;\; &\preceq \;\; \frac{|| \, ||u||^{p+q} w_0||^{\sim}}{2^7} \left[\left| \frac{1}{2520} \right| \left(\left| \frac{52920}{2520} \right| \oplus \left| \frac{13|2|^{p+q}}{288} \right| \oplus \left| \frac{|3|^{p+q}}{180} \right| \right.\right. \\ &\oplus \;\; \left| \frac{|4|^{p+q}}{1440} \right| \oplus \left| \frac{|6|^{p+q}}{10080} \right| \oplus \left| \frac{|4|^p}{2} \right| \oplus \left| \frac{7(|3|^p)}{2} \right| \\ &\oplus \;\; \left.\left. |11(|2^p|)| \right) \right]. \end{aligned}$$

Corollary 2.8. *Suppose that $(U, ||.||)$ is a normed space, $(W, || \;\; ||^{\sim})$ is a fuzzy normed space, and $(V, || \;\; ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $\lambda = s + r > 7$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that*

$$||\Delta_{s'} g(u,v)|| \preceq ||[||v||^s ||u||^r + (||v||^{s+r} + ||u||^{s+r})] w_0||^{\sim} \qquad (2.45)$$

for all $u, v \in U$, then there exists one and only one septic mapping $S : U \to V$ fulfilling the given condition

$$\begin{aligned} ||S(u) - g(u)|| \;\; &\preceq \;\; \frac{|| \, ||u||^{\lambda} w_0||^{\sim}}{|2|^{\lambda}} \left[\left| \frac{1}{2520} \right| \left(\left| \frac{6733}{630} \right| \oplus \left| \frac{49|2|^{\lambda}}{72} \right| \oplus \left| \frac{|3|^{\lambda}}{45} \right| \right.\right. \\ &\oplus \;\; \left| \frac{|4|^{\lambda}}{380} \right| \oplus \left| \frac{|6|^{\lambda}}{2520} \right| \oplus \left| \frac{7(|3|^r + |3|^{\lambda} + 1)}{2} \right| \\ &\oplus \;\; \left.\left. \left| \frac{|4|^r + |4|^{\lambda} + 1}{2} \right| \oplus |11(|2|^r + 2^{\lambda} + 1)| \right) \right] \end{aligned}$$

for all $u \in U$.

Counterexample 2.9. Consider a real Banach algebra $(U, ||.||)$ and a non-Archimedean complete fuzzy norm space $(U, || \;\; ||^{\sim})$ in which

$$||u||^{\sim}(t) = \begin{cases} \frac{||u||^7}{t}, & \textit{when } ||u||^7 < t, t \neq 0 \\ 1, & \textit{when } ||u||^7 = t = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

whose α-level set is defined as $[||u||^{\sim}]_{\alpha} = [||u||^7, \frac{||u||^7}{\alpha}]$. Construct a mapping $g : U \to U$ such that $g(u) = u^7 + ||u||^7 u_0$, where u_0 is a unit vector and

$$||\Delta_o g(u,v)||^{\sim} \preceq ||(128||u||^7 + 42560||v||^7) u_0||^{\sim},$$

then there does not exist a septic mapping $S : U \to U$ fulfilling the given condition

$$||S(u) - g(u)||^{\sim} \preceq 2^7||||u||^7 u_0||^{\sim}.$$

3. Stability of octic functional equation

The stability problems of various octic functional equations in several spaces such as intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean spaces, Banach spaces, orthogonal spaces and many other spaces have been broadly investigated by a number of mathematicians. Motivated by the approach of research by various mathematicians, an effort has been made in this paper to obtain the stability of the following functional equations.

$$\begin{aligned} &g(u+4v) - 8g(u+3v) + 28g(u+2v) - 56g(u+v) \\ &-56g(u-v) + 28g(u-2v) - 8g(u-3v) + g(u-4v) \\ &+70g(u) = 40320g(v). \end{aligned} \tag{3.1}$$

To simplify notation, let us introduce the "difference operator "denoted by Δ_o.

$$\begin{aligned} \Delta_o g(u,v) &= g(u+4v) - 8g(u+3v) + 28g(u+2v) - 56g(u+v) \\ &- 56g(u-v) + 28g(u-2v) - 8g(u-3v) + g(u-4v) \\ &+ 70g(u) - 40320g(v). \end{aligned}$$

Theorem 3.1. *Suppose that U is a linear space and $(W, ||.||^{\sim})$ is a fuzzy normed space. Consider $\psi : U^2 \to W$ be a mapping such that*

$$\lim_{n\to\infty} \frac{||\psi(2^n u, 2^n v)||_{\alpha}^{\sim 1}}{|2^{8n}|} = \lim_{n\to\infty} \frac{||\psi(2^n u, 2^n v)||_{\alpha}^{\sim 2}}{|2^{8n}|} = 0, \tag{3.2}$$

for all $u, v \in U$ and $\alpha \in (0, 1]$. Let $(V, ||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g : U \to V$ is such that

$$||\Delta_o g(u,v)|| \preceq ||\psi(u,v)||^{\sim} \tag{3.3}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$||O(u) - g(u)|| \preceq \frac{1}{2^8} \max\left\{\frac{H(2^k u)}{2^{8k}}; k \in N \cup 0\right\}, \tag{3.4}$$

where

$$\begin{aligned} H(2^k u) &= \left|\frac{1}{20160}\right|\Bigg[\left|\frac{1}{80640}\right|\Big(||\psi(8.2^k u, 8.2^k u)||^{\sim} \oplus ||\psi(8.2^k u, -8.2^k u)||^{\sim}\Big) \\ &\oplus \left|\frac{1}{10080}\right|\Big(||\psi(6.2^k u, 6.2^k u)||^{\sim} \oplus ||\psi(6.2^k u, -6.2^k u)||^{\sim}\Big) \\ &\oplus |\frac{7}{960}|\Big(||\psi(4.2^k u, 4.2^k u)||^{\sim} \oplus ||\psi(4.2^k u, -4.2^k u)||^{\sim}\Big) \\ &\oplus \left|\frac{1}{15}\right|\Big(||\psi(3.2^k, 3.2^k u)||^{\sim} \oplus ||\psi(3.2^k u, -3.2^k u)||^{\sim} \\ &\oplus \left|\frac{139}{480}\right|\Big(||\psi(2.2^k u, 2.2^k u)||^{\sim} \oplus ||\psi(2.2^k u, -2.2^k u)||^{\sim}\Big) \\ &\oplus \left|\frac{417}{560}\right|\Big((||\psi(2^k u, 2^k u)||^{\sim} \oplus ||\psi(2^k u, -2^k u)||^{\sim}\Big) \\ &\oplus ||\psi(4.2^k u, 2^k u)||^{\sim} \oplus |8|.||\psi(3.2^k u, 2^k u)||^{\sim} \oplus |28|.||\psi(2.2^k u, 2^k u)||^{\sim} \\ &\oplus |56|.||\psi(2^k u, 2^k u)||^{\sim} \oplus |35|.||\psi(0, 2^k u)||^{\sim} \oplus \left|\frac{851}{672}\right|||\psi(0,0||^{\sim}\Bigg] \end{aligned}$$

for all $u \in U$.

Proof. Taking $u = 0 = v$ in (3.3), we have

$$||g(0)|| \preceq \frac{1}{|40320|}||\psi(0,0)||^{\sim}. \tag{3.5}$$

Replacing (u, v) in (3.3) with (u,-v), we get

$$\begin{aligned} &||g(u+4v) - 8g(u+3v) + 28g(u+2v) - 56g(u+v) \\ &-56g(u-v) + 28g(u-2v) - 8g(u-3v) + g(u-4v) \\ &+70g(u) - 40320g(-v)|| \preceq ||\psi(u,-v)||^{\sim}. \end{aligned} \tag{3.6}$$

By using (3.3) and (3.6), we get

$$||g(u) - g(-u)|| \preceq \frac{1}{|40320|}(||\psi(u,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \tag{3.7}$$

Replacing (u, v) in (3.3) with (0,2u), the authors get

$$\begin{aligned} &||g(8u) - 8g(6u) + 28g(4u) - 40376g(2u) + 70g(0) - 56g(-2u) \\ &+28g(-4u) - 8g(-6u) + g(-8u)|| \preceq ||\psi(0,2u)||^{\sim}. \end{aligned} \tag{3.8}$$

Now, by using (3.5), (3.7) and (3.8), we obtain

$$\begin{aligned}
&||g(8u) - 8g(6u) + 28g(4u) - 20216g(2u)|| \preceq \frac{1}{|80640|}(||\psi(8u, 8u)||^{\sim} \\
&\oplus||\psi(8u, -8u)||^{\sim}) \oplus \frac{1}{|10080|}(||\psi(6u, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \\
&\oplus\frac{1}{|2880|}(||\psi(4u, 4u)||^{\sim} \oplus ||\psi(4u, -4u)||^{\sim})\frac{1}{|1440|}(||\psi(2u, 2u)||^{\sim} \\
&\oplus||\psi(2u, -2u)||^{\sim}) \oplus \frac{1}{|1152|}||\psi(0, 0)||^{\sim}. \qquad (3.9)
\end{aligned}$$

Replacing (u, v) in (3.3) with (4u,u), we get

$$\begin{aligned}
&||g(8u) - 8g(7u) + 28g(6u) - 56g(5u) + 70g(4u) - 56g(3u) \\
&+28g(2u) - 40328g(u) + g(0)|| \preceq ||\psi(4u, u)||^{\sim}. \qquad (3.10)
\end{aligned}$$

By using (3.5), (3.9) and (3.10), we obtain

$$\begin{aligned}
&||8g(7u) - 36g(6u) + 56g(5u) - 42g(4u) + 56g(3u) - 20244g(2u) \\
&+40328g(u)|| \preceq \frac{1}{|80640|}(||\psi(8u, 8u)||^{\sim} \oplus ||\psi(8u, -8u)||^{\sim}) \\
&\oplus\frac{1}{|10080|}(||\psi(6u, 6u)||^{\sim} \oplus ||\psi(6u, -6u)||^{\sim}) \oplus \frac{1}{|2880|}(||\psi(4u, 4u)||^{\sim} \\
&\oplus||\psi(4u, -4u)||^{\sim})\frac{1}{|1440|}(||\psi(2u, 2u)||^{\sim} \oplus ||\psi(2u, -2u)||^{\sim}) \\
&\oplus||\psi(4u, u)||^{\sim} \oplus \frac{1}{|1120|}||\psi(0, 0)||^{\sim}. \qquad (3.11)
\end{aligned}$$

Replacing (u, v) in (3.3) with (3u,u), we get

$$\begin{aligned}
&||g(7u) - 8g(6u) + 28g(5u) - 56g(2u) - 40292g(u) \\
&+g(-u) - 8g(0)|| \preceq ||\psi(3u, u)||^{\sim}. \qquad (3.12)
\end{aligned}$$

Now, by using (3.5), (3.7) and (3.12), we obtain

$$\begin{aligned}
&||g(7u) - 8g(6u) + 28g(5u) - 56g(4u) + 70g(3u) - 56g(2u) \\
&-40291g(u)|| \preceq \frac{1}{40320}(||\psi(u, u)||^{\sim} \oplus ||\psi(u, -u)||^{\sim}) \oplus ||\psi(3u, u)||^{\sim} \\
&\oplus\frac{1}{5040}||\psi(0, 0)||^{\sim}. \qquad (3.13)
\end{aligned}$$

By using (3.11) and (3.13), we get

$$\begin{aligned}&||2g(6u)-12g(5u)+29g(4u)-36g(3u)-1414g(2u)\\&+25904g(u)||\preceq\frac{1}{14}\Big[\frac{1}{80640}(||\psi(8u,8u)||^{\sim}\oplus||\psi(8u,-8u)||^{\sim})\\&\oplus\frac{1}{10080}(||\psi(6u,6u)||^{\sim}\oplus||\psi(6u,-6u)||^{\sim})\oplus\frac{1}{2880}(||\psi(4u,4u)||^{\sim}\\&\oplus||\psi(4u,-4u)||^{\sim})\oplus\frac{1}{1440}(||\psi(2u,2u)||^{\sim}\oplus||\psi(2u,-2u)||^{\sim})\\&\oplus\frac{1}{5040}(||\psi(u,u)||^{\sim}\oplus||\psi(u,-u)||^{\sim})\oplus||\psi(4u,u)||^{\sim}\oplus8||\psi(3u,u)||^{\sim}\\&\oplus\frac{5}{2016}||\psi(0,0)||^{\sim}\Big].\end{aligned}\tag{3.14}$$

Replacing (u,v) in (3.3) with (2u,u), we get

$$\begin{aligned}&||g(6u)-8g(5u)+28g(4u)-56g(3u)+70g(2u)+g(-2u)\\&-40376g(u)-8g(-u)+28g(0)||\preceq||\psi(2u,u)||^{\sim}.\end{aligned}\tag{3.15}$$

By using (3.5), (3.7) and (3.15), we obtain

$$\begin{aligned}&||g(6u)-8g(5u)+28g(4u)-56g(3u)+71g(2u)-40384g(u)||\\&\preceq\frac{1}{5040}(||\psi(2u,2u)||^{\sim}\oplus||\psi(2u,-2u)||^{\sim})\oplus\frac{1}{630}(||\psi(u,u)||^{\sim}\\&\oplus||\psi(u,-u)||^{\sim})\oplus||\psi(2u,u)||^{\sim}\oplus\frac{1}{180}||\psi(0,0)||^{\sim}.\end{aligned}\tag{3.16}$$

Now, from (3.14) and (3.16), we obtain

$$\begin{aligned}&||4g(5u)-27g(4u)+76g(3u)-1556g(2u)+106672g(u)||\\&\preceq\frac{1}{14}\Big[\frac{1}{80640}(||\psi(8u,8u)||^{\sim}\oplus||\psi(8u,-8u)||^{\sim})\\&\oplus\frac{1}{10080}(||\psi(6u,6u)||^{\sim}\oplus||\psi(6u,-6u)||^{\sim})\oplus\frac{1}{2880}(||\psi(4u,4u)||^{\sim}\\&\oplus||\psi(4u,-4u)||^{\sim})\oplus\frac{1}{160}(||\psi(2u,2u)||^{\sim}\oplus||\psi(2u,-2u)||^{\sim})\\&\oplus\frac{5}{112}(||\psi(u,u)||^{\sim}\oplus||\psi(u,-u)||^{\sim})\oplus||\psi(4u,u)||^{\sim}\oplus8||\psi(3u,u)||^{\sim}\\&\oplus28||\psi(2u,u)||^{\sim}\oplus\frac{177}{1120}||\psi(0,0)||^{\sim}\Big]\end{aligned}\tag{3.17}$$

Replacing (u,v) in (3.3) with (u,u), we get

$$\begin{aligned}&||g(5u)-8g(4u)+28g(3u)+g(-3u)-56g(2u)-8g(-2u)\\&-40250g(u)+28g(-u)-56g(0)||\preceq||\psi(u,u)||^{\sim}.\end{aligned}\tag{3.18}$$

With the help of (3.5), (3.7) and (3.18), we get

$$\begin{aligned}&||g(5u)-8g(4u)+29g(3u)-64g(2u)-40222g(u)||\\&\preceq \frac{1}{5040}(||\psi(3u,3u)||^{\sim}\oplus||\psi(3u,-3u)||^{\sim})\oplus\frac{1}{630}(||\psi(2u,2u)||^{\sim}\\&\oplus||\psi(2u,-2u)||^{\sim})\oplus\frac{1}{180}(||\psi(u,u)||^{\sim}\oplus||\psi(u,-u)||^{\sim})\\&\oplus||\psi(u,u)||^{\sim}\oplus\frac{1}{90}||\psi(0,0)||^{\sim}.\end{aligned}\tag{3.19}$$

By using (3.19) and (3.17), we get

$$\begin{aligned}&||g(4u)-8g(3u)-260g(2u)+53512g(u)||\\&\preceq\frac{1}{70}\Big[\frac{1}{80640}(||\psi(8u,8u)||^{\sim}\oplus||\psi(8u,-8u)||^{\sim})\\&\oplus\frac{1}{10080}(||\psi(6u,6u)||^{\sim}\oplus||\psi(6u,-6u)||^{\sim})\oplus\frac{1}{2880}(||\psi(4u,4u)||^{\sim}\\&\oplus||\psi(4u,-4u)||^{\sim})\oplus\frac{1}{90}(||\psi(3u,3u)||^{\sim}\oplus||\psi(3u,-3u)||^{\sim})\\&\oplus\frac{137}{1440}(||\psi(2u,2u)||^{\sim}\oplus||\psi(2u,-2u)||^{\sim})\\&\oplus\frac{1793}{5040}(||\psi(u,u)||^{\sim}\oplus||\psi(u,-u)||^{\sim})\oplus||\psi(4u,u)||^{\sim}\oplus 8||\psi(3u,u)||^{\sim}\\&\oplus 28||\psi(2u,u)||^{\sim}\oplus 56||\psi(u,u)||^{\sim}\oplus\frac{1573}{2016}||\psi(0,0)||^{\sim}\Big].\end{aligned}\tag{3.20}$$

Replacing (u, v) in (3.3) with (0,u), we get

$$\begin{aligned}&||g(4u)+g(-4u)-8g(3u)-8g(-3u)+28g(2u)+28g(-2u)\\&-40376g(u)-56g(-u)+70g(0)||\preceq||\psi(0,u)||^{\sim}.\end{aligned}\tag{3.21}$$

Using (3.5), (3.7) and (3.21), we obtain

$$\begin{aligned}&||g(4u)-8g(3u)+28g(2u)-20216g(u)||\preceq\frac{1}{2}\Big[\frac{1}{5040}(||\psi(4u,4u)||^{\sim}\\&\oplus||\psi(4u,-4u)||^{\sim})\oplus\frac{1}{630}(||\psi(3u,3u)||^{\sim}\oplus||\psi(3u,-3u)||^{\sim})\\&\oplus\frac{1}{180}(||\psi(2u,2u)||^{\sim}\oplus||\psi(2u,-2u)||^{\sim})\oplus\frac{1}{90}(||\psi(u,u)||^{\sim}\\&\oplus||\psi(u,-u)||^{\sim})\oplus||\psi(0,u)||^{\sim}\oplus\frac{1}{72}||\psi(0,0)||^{\sim}\Big].\end{aligned}\tag{3.22}$$

By (3.20) and (3.22), the authors conclude that

$$||g(2u)-2^8g(u)||\preceq H(u)\tag{3.23}$$

where,

$$\begin{aligned} H(u) &= \left|\frac{1}{20160}\right|\left[\left|\frac{1}{80640}\right|(||\psi(8u,8u)||^{\sim} \oplus ||\psi(8u,-8u)||^{\sim}) \right.\\ &\oplus \left|\frac{1}{10080}\right|(||\psi(6u,6u)||^{\sim} \oplus ||\psi(6u,-6u)||^{\sim}) \\ &\oplus \left|\frac{7}{960}\right|(||\psi(4u,4u)||^{\sim} \oplus ||\psi(4u,-4u)||^{\sim}) \\ &\oplus \left|\frac{1}{15}\right|(||\psi(3u,3u)||^{\sim} \oplus ||\psi(3u,-3u)||^{\sim}) \\ &\oplus \left|\frac{139}{480}\right|(||\psi(2u,2u)||^{\sim} \oplus ||\psi(2u,-2u)||^{\sim}) \\ &\oplus \left|\frac{417}{560}\right|(||\psi(u,u)||^{\sim} \oplus ||\psi(u,-u)||^{\sim}) \oplus ||\psi(4u,u)||^{\sim} \\ &\oplus |8|.||\psi(3u,u)||^{\sim} \oplus |28|.||\psi(2u,u)||^{\sim} \oplus |56|.||\psi(u,u)||^{\sim} \\ &\oplus \left. |35|.||\psi(0,u)||^{\sim} \oplus \frac{851}{672}||\psi(0,0)||^{\sim}\right]. \end{aligned}$$

Hence from (3.23), we have

$$||g(2u) - 2^8 g(u)||_{\alpha}^{1} \leq H_1(u), \tag{3.24}$$

and

$$||g(2u) - 2^8 g(u)||_{\alpha}^{2} \leq H_2(u), \tag{3.25}$$

where for $\alpha \in (0,1]$

$$\begin{aligned} H_1(u) &= \left|\frac{1}{20160}\right|\Bigg[\left|\frac{1}{80640}\right|(||\psi(8u,8u)||_\alpha^{1\sim} \oplus ||\psi(8u,-8u)||_\alpha^{1\sim}) \\ &\oplus \left|\frac{1}{10080}\right|(||\psi(6u,6u)||_\alpha^{1\sim} \oplus ||\psi(6u,-6u)||_\alpha^{1\sim}) \\ &\oplus \left|\frac{7}{960}\right|(||\psi(4u,4u)||_\alpha^{1\sim} \oplus ||\psi(4u,-4u)||_\alpha^{1\sim}) \\ &\oplus \left|\frac{1}{15}\right|(||\psi(3u,3u)||_\alpha^{1\sim} \oplus ||\psi(3u,-3u)||_\alpha^{1\sim}) \\ &\oplus \left|\frac{139}{480}\right|(||\psi(2u,2u)||_\alpha^{1\sim} \oplus ||\psi(2u,-2u)||_\alpha^{1\sim}) \\ &\oplus \left|\frac{417}{560}\right|(||\psi(u,u)||_\alpha^{1\sim} \oplus ||\psi(u,-u)||_\alpha^{1\sim}) \oplus ||\psi(4u,u)||_\alpha^{1\sim} \\ &\oplus |8|.||\psi(3u,u)||_\alpha^{1\sim} \oplus |28|.||\psi(2u,u)||_\alpha^{1\sim} \oplus |56|.||\psi(u,u)||_\alpha^{1\sim} \\ &\oplus |35|.||\psi(0,u)||_\alpha^{1\sim} \oplus \frac{851}{672}||\psi(0,0)||_\alpha^{1\sim}\Bigg] \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} H_2(u) &= \left|\frac{1}{20160}\right|\Bigg[\left|\frac{1}{80640}\right|(||\psi(8u,8u)||_\alpha^{2\sim} \oplus ||\psi(8u,-8u)||_\alpha^{2\sim}) \\ &\oplus \left|\frac{1}{10080}\right|(||\psi(6u,6u)||_\alpha^{2\sim} \oplus ||\psi(6u,-6u)||_\alpha^{2\sim}) \\ &\oplus \left|\frac{7}{960}\right|(||\psi(4u,4u)||_\alpha^{2\sim} \oplus ||\psi(4u,-4u)||_\alpha^{2\sim}) \\ &\oplus \left|\frac{1}{15}\right|(||\psi(3u,3u)||_\alpha^{2\sim} \oplus ||\psi(3u,-3u)||_\alpha^{2\sim}) \\ &\oplus \left|\frac{139}{480}\right|(||\psi(2u,2u)||_\alpha^{2\sim} \oplus ||\psi(2u,-2u)||_\alpha^{2\sim}) \\ &\oplus \left|\frac{417}{560}\right|(||\psi(u,u)||_\alpha^{2\sim} \oplus ||\psi(u,-u)||_\alpha^{2\sim}) \oplus ||\psi(4u,u)||_\alpha^{2\sim} \\ &\oplus |8|.||\psi(3u,u)||_\alpha^{2\sim} \oplus |28|.||\psi(2u,u)||_\alpha^{2\sim} \oplus |56|.||\psi(u,u)||_\alpha^{2\sim} \\ &\oplus |35|.||\psi(0,u)||_\alpha^{2\sim} \oplus \frac{851}{672}||\psi(0,0)||_\alpha^{2\sim}\Bigg]. \end{aligned} \tag{3.27}$$

From (3.24), we conclude

$$||\frac{g(2u)}{2^8} - g(u)||_\alpha^{1\sim} \le \frac{H_1(u)}{|2^8|}. \tag{3.28}$$

By substituting $2^n u$ for u in (3.28), then dividing both sides by 2^{8n}, we get

$$\left|\left|\frac{g(2^{n+1}u)}{2^{8(n+1)}} - \frac{g(2^n u)}{2^{8n}}\right|\right|_\alpha^1 \leq \frac{H_1(2^n u)}{|2^{8(n+1)}|} \tag{3.29}$$

for all non-negative integers n. Hence the sequence $\{\frac{g(2^n u)}{2^{8n}}\}$ is Cauchy by (3.2) and (3.29). Since Y is complete therefore, every Cauchy sequence is convergent in Y. So, the authors define a mapping $O : U \to V$ such that

$$O(u) = \lim_{n\to\infty} \frac{g(2^n u)}{2^{8n}}. \tag{3.30}$$

i.e.,

$$\lim_{n\to\infty} \left|\left|\frac{g(2^n u)}{2^{8n}} - O(u)\right|\right| = \bar{0}. \tag{3.31}$$

Now for each non-negative integer n, the authors have

$$\begin{aligned}
\left|\left|\frac{g(2^n u)}{2^{8n}} - g(u)\right|\right|_\alpha^1 &= \left|\left|\sum_{k=0}^{n-1}\left(\frac{g(2^{k+1}u)}{2^{8(n+1)}} - \frac{g(2^k u)}{2^{8n}}\right)\right|\right|_\alpha^1 \\
&\leq \max\left\{\left|\left|\frac{g(2^{k+1}u)}{2^{8(n+1)}} - \frac{g(2^k u)}{2^{8n}}\right|\right|_\alpha^1 : 0 \leq k < n\right\} \\
&\leq \frac{1}{2^8}\max\left\{\frac{H_1(2^k u)}{|2^{8k}|} : 0 \leq k < n\right\}. \qquad (3.32)
\end{aligned}$$

Similarly, the authors can show that

$$\left|\left|\frac{g(2^n u)}{2^{8n}} - g(u)\right|\right|_\alpha^2 \leq \frac{1}{2^8}\max\left\{\frac{H_2(2^k u)}{|2^{8k}|} : 0 \leq k < n\right\}. \tag{3.33}$$

Taking $n \to \infty$ in (3.32) and (3.33), the authors see that inequality (3.4) holds. Next we prove that $O : U \to V$ is a octic mapping. Replacing (u, v) by $(2^n u, 2^n v)$ and divide by $|2^{8n}|$ in (3.3), we get

$$\begin{aligned}
&\frac{1}{|2^{8n}|}||g(2^n(u + 4v)) - 8g(u + 3v) + 28g(u + 2v) - 56g(u + v) \\
&-56g(u - v) + 28g(u - 2v) - 8g(u - 3v) + g(u - 4v) + 70g(u) \\
&-40320g(v)|| \preceq ||\frac{\psi(2^n u, 2^n v)}{2^{8n}}||^\sim.
\end{aligned}$$

Taking $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
&||O(u + 4v) - 8O(u + 3v) + 28O(u + 2v) - 56O(u + v) - 56O(u - v) \\
&+28O(u - 2v) - 8O(u - 3v) + O(u - 4v) + 70O(u) - 40320O(v)|| \preceq \bar{0}
\end{aligned}$$

this implies that

$$O(u+4v)-8O(u+3v)+28O(u+2v)-56O(u+v)-56O(u-v)$$
$$+28O(u-2v)-8O(u-3v)+O(u-4v)+70O(u)-40320O(v)=\underline{0}.$$

Therefore, the mapping $O:U\to V$ is octic. Next we shall prove uniqueness of mapping O. Now, consider another octic mapping $O':U\to V$ which satisfies (3.1) and (3.4). For fix $u\in U$, certainly $O(2^nu)=2^{8n}O(u)$ and $O'(2^nu)=2^{8n}O'(u)$ for all $n\in N$. Therefore,

$$\begin{aligned}
&||O(u)-O'(u)||=\lim_{n\to\infty}\frac{1}{2^{8n}}||O(2^nu)-O'(2^nu)||\\
&=\lim_{n\to\infty}\frac{1}{2^{8n}}||O(2^nu)-g(2^nu)+g(2^nu)-O'(2^nu)||\\
&\preceq\lim_{n\to\infty}\max\left\{\frac{1}{2^{8n}}||O(2^nu)-g(2^nu)||,\frac{1}{2^{8n}}||g(2^nu)-O'(2^nu)||\right\}\\
&\preceq\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\max\left\{\frac{1}{2^{8(n+1)}}\left\{\frac{H(2^{k+n}u)}{2^{8k}},\frac{H(2^{k+n}u)}{2^{8k}}\right\}\right\}\right\}\\
&=\overline{0}.
\end{aligned}$$

Therefore, $O(u)-O'(u)=\underline{0}$. So, $O'(u)=O(u)$. Hence we deduced that O is unique mapping. □

Corollary 3.2. *Suppose that $(U,||.||)$ is a normed space, $(W,||\quad||^{\sim})$ is a fuzzy normed space, and $(V,||\quad||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0\in W$ and $p<8$ be non-negative real numbers. If the mapping $g:U\to V$ is such that*

$$||\Delta_o g(u,v)||\preceq||(||v||^p+||u||^p)w_0||^{\sim}\tag{3.34}$$

for all $u,v\in U$, then there exists one and only one octic mapping $O:U\to V$ fulfilling the given condition

$$\begin{aligned}
||O(u)-g(u)|| \;\preceq\; &\frac{||\,||u||^pw_0||^{\sim}}{2^8}\left[\left|\frac{1}{20160}\right|\left(\left|\frac{8089}{560}\right|\oplus\left|\frac{139|2|^p}{120}\right|\oplus\left|\frac{4|3|^p}{15}\right|\right.\right.\\
&\oplus\left|\frac{7|4|^p}{240}\right|\oplus\left|\frac{|6|^p}{2520}\right|\oplus\left|\frac{|8|^p}{20160}\right|\oplus\left||4|^p+1\right|\\
&\left.\left.\oplus\left|8(|3|^p+1)\right|\oplus|28(|2^p|+1)|\right)\right]
\end{aligned}$$

for all $u\in U$.

Corollary 3.3. *Suppose that $(U,||.||)$ is a normed space, $(W,||\quad||^{\sim})$ is a fuzzy normed space, and $(V,||\quad||)$ is a non-Archimedean complete*

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fuzzy normed space. Let $w_0 \in W$ and $p, q < 8$ be non-negative real numbers, respectively. If the mapping $g : U \to V$ is such that

$$||\Delta_{s'} g(u,v)|| \preceq ||(||v||^p ||u||^q) w_0||^{\sim}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned} ||O(u) - g(u)|| \preceq & \frac{|| \, ||u||^{p+q} w_0||^{\sim}}{2^8} \Bigg[\left| \frac{1}{20160} \right| \Bigg(\left| \frac{2|8|^{p+q}}{80640} \right| \oplus \left| \frac{2|6|^{p+q}}{10080} \right| \oplus \left| \frac{14|4|^{p+q}}{960} \right| \\ \oplus & \left| \frac{2|3|^{p+q}}{15} \right| \oplus \left| \frac{278|2|^{p+q}}{480} \right| \oplus \left| \frac{32192}{560} \right| \oplus |(|4^p|)| \oplus |8(|3^p|)| \\ \oplus & |28(|2^p|)| \Bigg) \Bigg]. \end{aligned}$$

Corollary 3.4. *Assume that $(U, ||.||)$ is a normed space, $(W, || \quad ||^{\sim})$ is a fuzzy normed space, and $(V, || \quad ||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $\lambda = s + r < 8$ be non-negative real numbers. If the mapping $g : U \to V$ is such that*

$$||\Delta_o g(u,v)|| \preceq ||[||v||^s ||u||^r + (||v||^{s+r} + ||u||^{s+r})] w_0||^{\sim} \qquad (3.35)$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned} ||O(u) - g(u)|| \preceq & \frac{|| \, ||u||^{\lambda} w_0||^{\sim}}{2^8} \Bigg[\left| \frac{1}{20160} \right| \Bigg(|13440|8|^{\lambda}| \oplus |1680|6|^{\lambda}| \\ \oplus & \left| \frac{7|4|^{\lambda}}{160} \right| \oplus \left| \frac{8|3|^{\lambda}}{5} \right| \oplus \left| \frac{139|2|^{\lambda}}{130} \right| \oplus \left| \frac{58091}{280} \right| \\ \oplus & |(|4|^r + |4|^{\lambda} + 1)| \oplus |8(|3|^r + |3|^{\lambda} + 1)| \\ \oplus & |(|2|^r + |2|^{\lambda} + 1) \Bigg) \Bigg] \end{aligned}$$

for all $u \in U$.

Theorem 3.5. *Assume that U is a linear space and $(W, ||.||^{\sim})$ is a fuzzy normed space. Consider $\psi : U^2 \to W$ be a function such that*

$$\lim_{n\to\infty} |2^{8n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 1} = \lim_{n\to\infty} |2^{8n}| \left|\left| \psi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) \right|\right|_{\alpha}^{\sim 2} = 0, \quad (3.36)$$

for all $u, v \in U$ and $\alpha \in (0, 1]$. Let $(V, ||.||)$ is a non-Archimedean fuzzy Banach space. If the mapping $g : U \to V$ is such that

$$||\Delta_o g(u,v)|| \preceq ||\psi(u,v)||^{\sim} \qquad (3.37)$$

for all $u \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$||O(u) - g(u)|| \preceq \frac{1}{2^8} \max\left\{|2^{8(k+1)}|H\left(\frac{u}{2^{k+1}}\right), k \in N \cup \{0\}\right\} \quad (3.38)$$

where

$$\begin{aligned} H\left(\frac{u}{2^{k+1}}\right) &= \left|\frac{1}{20160}\right|\left[\left|\frac{1}{80640}\right|\left(\left|\left|\psi\left(\frac{8u}{2^{k+1}}, \frac{8u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{8u}{2^{k+1}}, \frac{-8u}{2^{k+1}}\right)\right|\right|^{\sim}\right)\right. \\ &\oplus \left|\frac{1}{10080}\right|\left(\left|\left|\psi\left(\frac{6u}{2^{k+1}}, \frac{6u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{6u}{2^{k+1}}, \frac{-6u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{7}{960}\right|\left(\left|\left|\psi\left(\frac{4u}{2^{k+1}}, \frac{4u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{4u}{2^{k+1}}, \frac{-4u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{1}{15}\right|\left(\left|\left|\psi\left(\frac{3u}{2^{k+1}}, \frac{3u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{3u}{2^{k+1}}, \frac{-3u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{139}{480}\right|\left(\left|\left|\psi\left(\frac{2u}{2^{k+1}}, \frac{2u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{2u}{2^{k+1}}, \frac{-2u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\frac{417}{560}\right|\left(\left|\left|\psi\left(\frac{u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\left|\psi\left(\frac{u}{2^{k+1}}, \frac{-u}{2^{k+1}}\right)\right|\right|^{\sim}\right) \\ &\oplus \left|\left|\psi\left(\frac{4u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus |8|.\left|\left|\psi\left(\frac{3u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \\ &\oplus |28|.\left|\left|\psi\left(\frac{2u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus |56|.\left|\left|\psi\left(\frac{u}{2^{k+1}}, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \\ &\oplus \left.|35|.\left|\left|\psi\left(0, \frac{u}{2^{k+1}}\right)\right|\right|^{\sim} \oplus \left|\frac{851}{672}\right|||\psi(0,0||^{\sim}\right] \end{aligned}$$

for all $u \in U$.

Proof. From (3.24), the authors get

$$\left|\left|g(u) - 2^8 g\left(\frac{u}{2}\right)\right|\right|_{\alpha}^{1} \leq H_1\left(\frac{u}{2}\right) \quad (3.39)$$

for $\alpha \in (0, 1]$. Replacing u by $\frac{u}{2^n}$ and multiplying both side by $|2^{8n}|$ in (3.39), we get

$$\left|\left|2^{8n} g\left(\frac{u}{2^n}\right) - 2^{8(n+1)} g\left(\frac{u}{2^{n+1}}\right)\right|\right|_{\alpha}^{1} \leq |2^{8n}| H_1\left(\frac{u}{2^{n+1}}\right) \quad (3.40)$$

for all negative integer n. Hence the sequence $g\{2^{8n} g(\frac{u}{2^n})\}$ is Cauchy by (3.36) and (3.40). Every Cauchy sequence is convergent in Y since

Y is complete. So, the authors construct a mapping $O : U \to V$ such that

$$O(u) = \lim_{n\to\infty} 2^{8n} g\left(\frac{u}{2^n}\right)$$

for all $u \in U$. That is

$$\lim_{n\to\infty} ||2^{8n} g\left(\frac{u}{2^n}\right) - O(u)|| = \overline{0}$$

for all $u \in U$. Now, for each positive integer n, the authors have

$$\left|\left|2^{8n} g\left(\frac{u}{2^n}\right) - g(u)\right)\right|\right|_\alpha^1 = \left|\left|\sum_{k=0}^{n-1}\left(2^{8(k+1)} g\left(\frac{u}{2^{k+1}}\right) - 2^{8k} g\left(\frac{u}{2^k}\right)\right)\right|\right|_\alpha^1$$
$$\leq \max\left\{\left|\left|\left(2^{8(k+1)} g\left(\frac{u}{2^{k+1}}\right) - 2^{8k} g\left(\frac{u}{2^k}\right)\right)\right|\right|_\alpha^1 ; 0 \leq k < n\right\}$$
$$\leq \frac{1}{|2^8|} \max\left\{|2^{8(k+1)}| H_1\left(\frac{u}{2^{k+1}}\right) : 0 \leq k < n\right\}. \tag{3.41}$$

Similarly , it can be shown from (3.25)

$$\left|\left|2^{8n} g\left(\frac{u}{2^n}\right) - g(u)\right)\right|\right|_\alpha^2 \leq \frac{1}{|2^8|} \max\left\{|2^{8(k+1)}| H_2\left(\frac{u}{2^{k+1}}\right); 0 \leq k < n\right\} \tag{3.42}$$

Taking $n \to \infty$ in (3.41) and (3.42), the authors see that inequality (3.39) holds. The authors conclude that $O(u)$ is a unique cubic mapping holding (3.38) using same procedure as in the demonstration of theorem (3.1).

□

Corollary 3.6. *Assume that $(U, ||.||)$ is a normed space, $(W, ||\quad||^\sim)$ is a fuzzy normed space, and $(V, ||\quad||)$ is a non-Archimedean complete fuzzy normed space. Let $w_0 \in W$ and $p > 8$ be non-negative real numbers. If the mapping $g : U \to V$ is such that*

$$||\Delta_o g(u, v)|| \preceq ||(||v||^p + ||u||^p) w_0||^\sim \tag{3.43}$$

for all $u, v \in U$, then there exists one and only one octic mapping $O : U \to V$ fulfilling the given condition

$$\begin{aligned} ||O(u) - g(u)|| \ \preceq\ & \frac{||\,||u||^p w_0||^\sim}{|2^p|}\left[\left|\frac{1}{20160}\right|\left(\left|\frac{8089}{560}\right| \oplus \left|\frac{139|2|^p}{120}\right| \oplus \left|\frac{4|3|^p}{15}\right|\right.\right. \\ & \oplus \left|\frac{7|4|^p}{240}\right| \oplus \left|\frac{|6|^p}{2520}\right| \oplus \left|\frac{|8|^p}{20160}\right| \oplus \left||4|^p + 1\right| \\ & \left.\left.\oplus \left|8(|3|^p + 1)\right| \oplus |28(|2^p| + 1)|\right)\right] \end{aligned}$$

for all $u \in U$.

Corollary 3.7. *Suppose that* $(U, ||.||)$ *is a normed space,* $(W, ||\quad||^{\sim})$ *is a fuzzy normed space, and* $(V, ||\quad||)$ *is a non-Archimedean complete fuzzy normed space. Let* $w_0 \in W$ *and* $p, q > 8$ *be non-negative real numbers, respectively. If the mapping* $g : U \to V$ *is such that*

$$||\Delta_{s'} g(u, v)|| \preceq ||(||v||^p ||u||^q) w_0||^{\sim}$$

for all $u, v \in U$*, then there exists one and only one octic mapping* $O : U \to V$ *fulfilling the given condition*

$$\begin{aligned} ||O(u) - g(u)|| \;\; \preceq \;\; & \frac{|| \, ||u||^{p+q} w_0||^{\sim}}{2^8} \Bigg[\left| \frac{1}{20160} \right| \Bigg(\left| \frac{2|8|^{p+q}}{80640} \right| \oplus \left| \frac{2|6|^{p+q}}{10080} \right| \oplus \left| \frac{14|4|^{p+q}}{960} \right| \\ \oplus \;\; & \left| \frac{2|3|^{p+q}}{15} \right| \oplus \left| \frac{278|2|^{p+q}}{480} \right| \oplus \left| \frac{32192}{560} \right| \oplus |(|4^p|)| \oplus |8(|3^p|)| \\ \oplus \;\; & |28(|2^p|)| \Bigg) \Bigg]. \end{aligned}$$

Corollary 3.8. *Suppose that* $(U, ||.||)$ *is a normed space,* $(W, ||\quad||^{\sim})$ *is a fuzzy normed space, and* $(V, ||\quad||)$ *is a non-Archimedean complete fuzzy normed space. Let* $w_0 \in W$ *and* $\lambda = s + r > 8$ *be non-negative real numbers. If the mapping* $g : U \to V$ *is such that*

$$||\Delta_o g(u, v)|| \preceq ||[||v||^s ||u||^r + (||v||^{s+r} + ||u||^{s+r})] w_0||^{\sim} \qquad (3.44)$$

for all $u, v \in U$*, then there exists one and only one octic mapping* $O : U \to V$ *fulfilling the given condition*

$$\begin{aligned} ||O(u) - g(u)|| \;\; \preceq \;\; & \frac{|| \, ||u||^{\lambda} w_0||^{\sim}}{|2|^{\lambda}} \Bigg[\left| \frac{1}{20160} \right| \Bigg(|13440|8|^{\lambda}| \oplus |1680|6|^{\lambda}| \\ \oplus \;\; & \left| \frac{7|4|^{\lambda}}{160} \right| \oplus \left| \frac{8|3|^{\lambda}}{5} \right| \oplus \left| \frac{139|2|^{\lambda}}{130} \right| \oplus \left| \frac{58091}{280} \right| \\ \oplus \;\; & |(|4|^r + |4|^{\lambda} + 1)| \oplus |8(|3|^r + |3|^{\lambda} + 1)| \\ \oplus \;\; & |(|2|^r + |2|^{\lambda} + 1) \Bigg) \Bigg] \end{aligned}$$

for all $u \in U$.

Counterexample 3.9. Consider a real Banach algebra $(U, ||.||)$ and a non-Archimedean complete fuzzy norm space $(U, ||\quad||^{\sim})$ in which

$$||u||^{\sim}(t) = \begin{cases} \frac{||u||^8}{t}, & \textit{when } ||u||^8 < t, t \neq 0 \\ 1, & \textit{when } ||u||^8 = t = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

whose α-level set is defined as $[||u||^{\sim}]_\alpha = [||u||^8, \frac{||u||^8}{\alpha}]$. Construct a mapping $g : U \to U$ such that $g(u) = u^8 + ||u||^8 u_0$, where u_0 is a unit vector and

$$||\Delta_o g(u,v)||^{\sim} \preceq ||(256||u||^8 + 290816||v||^8)u_0||^{\sim},$$

then there does not exist an octic mapping $O : U \to U$ fulfilling the given condition

$$||O(u) - g(u)||^{\sim} \quad \preceq \quad 2^8||||u||^8 u_0||^{\sim}.$$

References

[1] J.D'Alembert, Addition au Mémoire sur la courbe que forme une corde tendue mise en vibration, Hist. Acad. Berlin, (1750) 355–360.

[2] S.M. Ulam, A collection of mathematical problems, Interscience Publ., New York, (1960).

[3] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci., USA, **27** (1941), 222-224.

[4] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, **2** (1950), 64-66.

[5] T.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., **72** (1978), 297-300.

[6] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., **184** (1994), 431-436.

[7] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta Appl. Math., **62** (2000), 23-130.

[8] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, Fuzzy stability of the Jensen functional equation, Fuzzy Sets Syst., **159** (6) (2008), 730-738.

[9] J. Jakhar, R. Chugh and J. Jakhar, Solution and intuitionistic fuzzy stability of 3- dimensional cubic functional equation: using two different methods, J. Math. Comp. Sci., **25** (2022), 103–114.

[10] K. Tamilvanan, A.H. Alkhaldi, J. Jakhar, R. Chugh, J. Jakhar and J.M. Rassias, Ulam stability results of functional equations in modular spaces and 2-Banach spaces. $\sum$- Mathematics, **11** (2) (2023) 371.

[11] J. Jakhar, J. Jakhar and R. Chugh, Fuzzy stability of mixed type functional equations in Modular spaces, Math. found. comput., (2023).

[12] J. Jakhar, R. Chugh, S. Jaiswal and R. Dubey, Stability of various functional equations in non-Archimedean (n, β) normed spaces, J. Anal., **30** (4) (2022), 1653-1669.

[13] A.K. Katsaras, Fuzzy topological vector space II, Fuzzy Sets syst., **12** (1984), 215-229.

[14] M. Amini and R. Saadati, Topics in fuzzy metric space, J. Fuzzy Math., **4** (2003), 765-768.

[15] A. George and P. Veeramani, On some results in fuzzy metric space, Fuzzy Sets Syst., **64** (1994), 395-399.

[16] B. Schweizer and A. Sklas, Statistical metric space, Pac. J. Math., **10** (1960), 314-344.

[17] C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets Syst., **48** (1992), 239-248.

[18] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets Syst., **12** (1984), 215-229.

[19] S.C. Cheng and J.N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Cal. Math. Soc., **86** (1994), 429-436.

[20] A.K. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica, **11** (1975), 326-334.

[21] S. Gähler and W. Gähler, Fuzzy real numbers, Fuzzy Sets Syst., **66** (1994), 137-158.

[22] T. Bag and S.K. Samanta, Finite dimensional fuzzy normed spaces, J. Fuzzy Math., **11** (3) (2003), 687-705.

[23] T. Bag and S.K. Samanta, Fixed point theorems on fuzzy normed linear spaces, Information Sci., **176** (2006), 2910-2931.

[24] A. Hasankhani, A. Nazari and M. Saheli, Some properties of fuzzy Hilbert spaces and norm of operatos, Iran. J. Fuzzy Syst., **7** (3) (2010), 129-157.

[25] J. Golet, On generalized fuzzy normed spaces and coincident point theorems, Fuzzy Sets Syst., **161** (2010), 1138-1144.

[26] J. Xiao and X. Zhu, On linearly topological structure and property of fuzzy norned linear space, Fuzzy Sets Syst., **125** (2002), 153-161.

[27] T. Bag and S.K. Samanta, Fuzzy biunded linear operators in Felbin's type fuzzy normed linear spaces, Fuzzy Sets Syst.,**159** (2008), 685-707.

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