

Sharp Constants in Fractional Ostrowski Inequalities for Convex Derivatives**Yunxuan Tan**

School of Health and Education, Mianyang City College

Mianyang 621000, Sichuan, China

Corresponding author: 18280971170@163.com

Tel.: +86 18280971170

Abstract

We determine the sharp constants of a fractional Ostrowski functional over bounded derivatives, nondecreasing derivatives, and continuous convex derivatives. The unconstrained and nondecreasing classes have the same constant. For the convex class, the constant is expressed in terms of the unique root of an integral equation. A proof uses the monotonicity of secant slopes and controls both signs of the functional. For evaluation points to the left of the midpoint, the unique extremal derivative decreases linearly from its upper bound to its lower bound and then remains constant. Reflection gives the extremal to the right of the midpoint; both reflected profiles attain equality at the midpoint. The convex constraint gives a strict improvement at every interior evaluation point. At fractional order one, the midpoint constant is approximately 0.106505512. A finite linear program with convex profiles provides numerical comparisons. The normalization relating the functional to the fractional mean error is stated.

Keywords: Ostrowski inequality; fractional integral; sharp constant; convex derivative; extremal function.

2020 Mathematics Subject Classification: 26D15; 26A33; 26A51.

1 Introduction

The classical Ostrowski inequality [4] bounds the difference between a function value and its integral mean. For an absolutely continuous function with $\|f'\|_\infty \leq M$, it gives

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{(x - (a+b)/2)^2}{(b-a)^2} \right]. \quad (1)$$

Shape restrictions on f and on f' lead to different questions. Convexity of f corresponds to a nondecreasing derivative, whereas convexity of the signed derivative f' imposes a separate restriction. Dragomir and Sofo [2] obtained Ostrowski bounds under the latter assumption. Anastassiou [1] studied fractional Ostrowski inequalities. Set [5] used a fractional integral identity to obtain inequalities under s -convexity assumptions. Liu [3] derived bounds when $|f'|$ or its powers are h -convex. Tariq, Ntouyas, and Ahmad [6] survey fractional Ostrowski inequalities for a range of operators and convexity conditions.

Here we fix the kernel arising from two truncated Riemann–Liouville integrals and optimize its integral against a bounded derivative. We compare the unconstrained, nondecreasing, and convex classes. The last two classes are each contained in the unconstrained class, but neither contains the other. Our main result gives the exact constant and equality profiles for the signed convex-derivative class. The proof does not require a classification of the extreme points of

bounded convex functions. Instead, it first computes the positive support value of the functional and then compares it with the support value at the reflected evaluation point.

The assumption on the signed derivative distinguishes this problem from Liu's assumptions on $|f'|$ and $|f'|^q$. Consequently, the result is not a direct sharpening of Liu's estimates obtained by setting $h(t) = t$. We use the earlier identities and bounds to specify the operator and the constraint being optimized.

2 Materials and methods

Let $a < b$, $\alpha > 0$, and $x \in [a, b]$. For an absolutely continuous function f , put

$$\mathcal{J}_a[f] = \int_a^x (t-a)^{\alpha-1} f(t) dt, \mathcal{J}_b[f] = \int_x^b (b-t)^{\alpha-1} f(t) dt,$$

$$S_\alpha(x) = (x-a)^\alpha + (b-x)^\alpha.$$

These integrals are $\Gamma(\alpha)J_{x-}^\alpha f(a)$ and $\Gamma(\alpha)J_{x+}^\alpha f(b)$, respectively, in the endpoint notation used in [5, 3]. Define

$$E_\alpha^x[f] = f(x) - \frac{\alpha}{S_\alpha(x)} (\mathcal{J}_a[f] + \mathcal{J}_b[f]).$$

Lemma 1 (Fractional Montgomery identity).

For $f \in AC[a, b]$, one has

$$E_\alpha^x[f] = \frac{1}{S_\alpha(x)} \int_a^b w_{\alpha,x}(t) f'(t) dt, \quad (2)$$

where $w_{\alpha,x}(t) = (t-a)^\alpha$ for $a \leq t < x$ and $w_{\alpha,x}(t) = -(b-t)^\alpha$ for $x < t \leq b$. The value at $t = x$ does not affect the integral.

Proof. Integration by parts on $[a, x]$ and $[x, b]$ gives

$$\int_a^x (t-a)^\alpha f'(t) dt = (x-a)^\alpha f(x) - \alpha \mathcal{J}_a[f],$$

$$- \int_x^b (b-t)^\alpha f'(t) dt = (b-x)^\alpha f(x) - \alpha \mathcal{J}_b[f].$$

The power factors are absolutely continuous for every $\alpha > 0$. Adding the identities and dividing by $S_\alpha(x)$ proves the result. $\square$

For the remainder of the proof we normalize to $[0,1]$ and $M = 1$, and write

$$w_x(t) = \begin{cases} t^\alpha, & 0 \leq t < x, \\ -(1-t)^\alpha, & x < t \leq 1, \end{cases} \quad L_x(g) = \int_0^1 w_x(t) g(t) dt.$$

The dependence on α is suppressed in w_x and L_x . The three feasible classes are

$$\begin{aligned}\mathcal{G}_u &= \{g \in L_\infty(0,1): \|g\|_\infty \leq 1\}, \\ \mathcal{G}_m &= \{g \in \mathcal{G}_u: g \text{ has a nondecreasing representative}\}, \\ \mathcal{G}_c &= \{g \in C[0,1]: g \text{ is convex and } |g(t)| \leq 1\}.\end{aligned}$$

For each class set $C_G(\alpha, x) = \sup_{g \in G} |L_x(g)|$. Every admissible g produces an absolutely continuous function $f(t) = f(0) + \int_0^t g(s) ds$. For the convex class, f is continuously differentiable. For the nondecreasing class, f need not be differentiable at every point.

The reflection $Rg(t) = g(1-t)$ preserves $\mathcal{G}_c$ and satisfies

$$L_x(Rg) = -L_{1-x}(g). \quad (3)$$

Thus $C_c(\alpha, x) = C_c(\alpha, 1-x)$. For the nondecreasing class the appropriate class-preserving transformation is $g(t) \mapsto -g(1-t)$.

3 Results: sharp constants and equality cases

Theorem 1.

For $\alpha > 0$ and $0 < x < 1$,

$$C_u(\alpha, x) = C_m(\alpha, x) = \frac{x^{\alpha+1} + (1-x)^{\alpha+1}}{\alpha+1}. \quad (4)$$

Proof. The inequality $|L_x(g)| \leq \int_0^1 |w_x(t)| dt$ gives the upper bound. The nondecreasing function equal to -1 on $[0, x)$ and 1 on $(x, 1]$ attains its negative. This profile belongs to both classes. At $\alpha = 1$ and $x = 1/2$, the constant is $1/4$. $\square$

Theorem 2.

Let $\alpha > 0$ and $0 < x \leq 1/2$. There is a unique $r = r(\alpha, x) \in (x, 1)$ satisfying

$$\int_x^r t(1-t)^\alpha dt = \frac{x^{\alpha+2}}{\alpha+2}. \quad (5)$$

The convex-class constant is

$$C_c(\alpha, x) = \frac{x^{\alpha+1} - (1-x)^{\alpha+1} + 2(1-r)^{\alpha+1}}{\alpha+1}. \quad (6)$$

For $1/2 < x < 1$, it is given by reflection. For $0 < x < 1/2$, the unique extremal derivative is

$$g_x(t) = \max\{-1, 1 - 2t/r(\alpha, x)\}. \quad (7)$$

For $1/2 < x < 1$, the unique extremal is $g_{1-x}(1-t)$. At $x = 1/2$, the extremals are exactly $g_{1/2}$ and $g_{1/2}(1-t)$. Moreover,

$$C_c(\alpha, x) < C_u(\alpha, x), \quad 0 < x < 1. \quad (8)$$

The proof is given in the next section. No overall change of sign is included in the equality cases: the negative of the nonaffine convex profile in (7) is concave.

Corollary 1.

Let $f \in AC[a, b]$ have a continuous convex derivative with $|f'| \leq M$, and put $\xi = (x - a)/(b - a)$. Then

$$|E_\alpha^x[f]| \leq \frac{M(b-a)}{\xi^\alpha + (1-\xi)^\alpha} C_c(\alpha, \xi), \quad a < x < b. \quad (9)$$

This bound is sharp. For $M > 0$, the equality derivatives are obtained from Theorem 2 by affine rescaling of the argument and multiplication by M .

Proof. Under $t = a + (b - a)s$, the integral in (2) scales by $M(b - a)^{\alpha+1}$, whereas $S_\alpha(x) = (b - a)^\alpha [\xi^\alpha + (1 - \xi)^\alpha]$. The equality profiles integrate to admissible functions f . ▀

4 A direct proof for the convex class

Define the positive support value

$$\Phi(x) = \sup_{g \in \mathcal{G}_c} L_x(g), \quad B(x) = \int_0^1 w_x(t) dt = \frac{x^{\alpha+1} - (1-x)^{\alpha+1}}{\alpha+1},$$

and the first moment $A(x) = \int_0^1 t w_x(t) dt$. Since

$$A'(x) = x[x^\alpha + (1-x)^\alpha] > 0,$$

A is strictly increasing. Its endpoint values have opposite signs, and

$$A(1/2) = \int_0^{1/2} (2t-1)t^\alpha dt < 0.$$

There is therefore a unique $q \in (1/2, 1)$ with $A(q) = 0$.

4.1 Computing the positive support value

Lemma 2.

For $0 < x < q$, let $r(x) \in (x, 1)$ be the unique solution of (5). Then

$$\Phi(x) = B(x) + \frac{2(1-r(x))^{\alpha+1}}{\alpha+1}. \quad (10)$$

10.48047/jocaaa.2026.35.10.04

ISSN:1521-1398,1572-9206

The unique function attaining this support value is g_x from (7). For $q \leq x < 1$, one has $\Phi(x) = B(x)$.

Proof. Suppose first that $A(x) < 0$. The partial moment $\int_0^u t w_x(t) dt$ is positive at $u = x$, strictly decreases for $u > x$, and is negative at $u = 1$. Hence it has a unique zero $r \in (x, 1)$, which is (5). Put

$$P = \int_0^r w_x(t) dt, \quad T = - \int_r^1 w_x(t) dt = \frac{(1-r)^{\alpha+1}}{\alpha+1}.$$

The moment condition gives

$$xP = - \int_0^r (t-x)w_x(t) dt > 0.$$

For a continuous convex function g , its secant slope

$$h(t) = \frac{g(t) - g(0)}{t}, \quad t > 0,$$

is nondecreasing. The integrand $tw_x(t)[h(t) - h(x)]$ is nonpositive on both $(0, x)$ and (x, r) . Consequently,

$$\begin{aligned} \int_0^r w_x(t)g(t) dt &= Pg(0) + \int_0^r tw_x(t)h(t) dt \\ &\leq Pg(0) + h(x) \int_0^r tw_x(t) dt \leq P. \end{aligned}$$

On $[r, 1]$, the bound $g \geq -1$ gives $\int_r^1 w_x g \leq T$. Thus $L_x(g) \leq P + T = B(x) + 2T$. The function g_x is affine on $[0, r]$ and equals -1 on $[r, 1]$, so it attains this bound.

Equality forces $g(0) = 1$ because $P > 0$, and $g = -1$ on $[r, 1]$. It also forces $h(t) = h(x)$ almost everywhere on $(0, r)$: the signs of the weights are strict away from the endpoints and x . Continuity then makes g affine on $[0, r]$. Its endpoint values determine the slope $-2/r$, proving uniqueness.

Now suppose $A(x) \geq 0$, so $x \geq q > 1/2$. The same secant-slope argument over $[0, 1]$ gives

$$L_x(g) \leq B(x)g(0) + A(x)h(x).$$

Convexity implies $h(x) \leq g(1) - g(0)$. Also $B(x) - A(x) > 0$ for $x \geq 1/2$: it is positive at $1/2$, where it equals $-A(1/2)$, and its derivative is $(1-x)[x^\alpha + (1-x)^\alpha] > 0$. Therefore

$$L_x(g) \leq [B(x) - A(x)]g(0) + A(x)g(1) \leq B(x).$$

The constant function $g = 1$ attains the bound. $\square$

4.2 Comparing the two signs

For $0 < s < q$, the root $r(s)$ is differentiable and

$$r'(s) = \frac{s[s^\alpha + (1-s)^\alpha]}{r(s)(1-r(s))^\alpha}.$$

If $s \leq 1/2$, then

$$\begin{aligned} \int_s^{2s} t(1-t)^\alpha dt &\geq \int_s^{2s} t(2s-t)^\alpha dt \\ &= \frac{(\alpha+3)s^{\alpha+2}}{(\alpha+1)(\alpha+2)} > \frac{s^{\alpha+2}}{\alpha+2}. \end{aligned}$$

Hence $r(s) < 2s$. For $s > 1/2$ this inequality follows from $r(s) < 1$. Differentiating (10) yields

$$\Phi'(s) = [s^\alpha + (1-s)^\alpha] \left(1 - \frac{2s}{r(s)}\right) < 0, \quad 0 < s < q. \quad (11)$$

Thus Φ is strictly decreasing on this interval.

By (3), the negative support value at x equals $\Phi(1-x)$, and

$$C_c(\alpha, x) = \max\{\Phi(x), \Phi(1-x)\}.$$

Let $0 < x < 1/2$ and $y = 1-x$. If $y < q$, (11) gives $\Phi(x) > \Phi(y)$. If $y \geq q$, Lemma 2 gives $\Phi(y) = B(y) = -B(x)$, whereas

$$\Phi(x) - \Phi(y) = 2[B(x) + T] = 2P > 0.$$

Thus the positive support value determines the absolute maximum for $x < 1/2$. Its unique maximizer is g_x . At the midpoint the two support values coincide, and their unique maximizers are the two reflected profiles. The case $x > 1/2$ follows by reflection. Finally,

$$C_u(\alpha, x) - C_c(\alpha, x) = \frac{2[(1-x)^{\alpha+1} - (1-r)^{\alpha+1}]}{\alpha+1} > 0$$

for $x \leq 1/2$, since $r > x$. This proves Theorem 2.

4.3 Evaluating the root

The equation can be evaluated without numerical integration by using

$$\int t(1-t)^\alpha dt = -\frac{(1-t)^{\alpha+1}}{\alpha+1} + \frac{(1-t)^{\alpha+2}}{\alpha+2}.$$

10.48047/jocaaa.2026.35.10.04

ISSN:1521-1398,1572-9206

For a general real order α , this is a scalar equation involving real powers; it need not be a polynomial equation. At $\alpha = 1$, $x = 1/2$, the root is specified by

$$2r^3 - 3r^2 + \frac{3}{4} = 0, \quad \frac{1}{2} < r < 1,$$

and

$$r \approx 0.673648177667, \quad C_c(1,1/2) = (1-r)^2 \approx 0.106505511940.$$

At the endpoints, $C_c(\alpha, 0) = C_c(\alpha, 1) = 1/(\alpha + 1)$, attained by constant derivatives. The same value is the interior limit as x tends to either endpoint.

5 Discussion and numerical comparisons

The supplementary scripts compute the root with a bracketed scalar solver and compare (6) with a finite linear program. For grid nodes $t_i = i/N$, the variables g_i define a continuous piecewise-linear function. Convexity is enforced by

$$g_{i-1} - 2g_i + g_{i+1} \geq 0, \quad 1 \leq i < N, \quad -1 \leq g_i \leq 1.$$

The objective coefficients are the exact integrals of w_x against the linear nodal basis functions. Both signs of the objective are solved with HiGHS. This discretization optimizes over admissible convex functions, so its exact-arithmetic optimum is a lower bound for the continuous supremum. Floating-point solver tolerances must also be taken into account.

Table 1 lists the root-based midpoint values. The parameter $c = 2/r$ specifies the slope of the decreasing equality profile. The ratio compares the functional constants, not differently normalized fractional mean errors.

Table 1. Midpoint constants and the equality-profile parameter.

α	C_u	C_c	C_c/C_u	r	$2/r$
0.5	0.4714045	0.2319226	0.49198	0.68840	2.9053
1.0	0.2500000	0.1065055	0.42602	0.67365	2.9689
1.5	0.1414214	0.0532852	0.37678	0.66162	3.0229
2.0	0.0833333	0.0282040	0.33845	0.65156	3.0696
3.0	0.0312500	0.0088223	0.28231	0.63554	3.1469

The comparison used the 63 parameter pairs drawn from

$$\alpha \in \{0.25, 0.5, 1, 1.5, 2, 3, 5\},$$

$$x \in \{0.05, 0.2, 0.35, 0.45, 0.5, 0.55, 0.65, 0.8, 0.95\},$$

on grids with $N = 1000$ and $N = 2000$. At $N = 2000$, the largest absolute difference between the root-based constant and the LP value was 7.60×10^{-7} . The runs used Python 3.13.12, NumPy 2.4.1, and SciPy 1.18.1, with primal and dual feasibility tolerances set to 10^{-10} . The

10.48047/jocaaa.2026.35.10.04

ISSN:1521-1398,1572-9206

CSV records both signed LP values and the comparison formula. The numerical checks concern those recorded parameter values; the proofs establish the results for all $\alpha > 0$ and interior x .

Figure 1 shows an equality profile and the functional constants at order one.

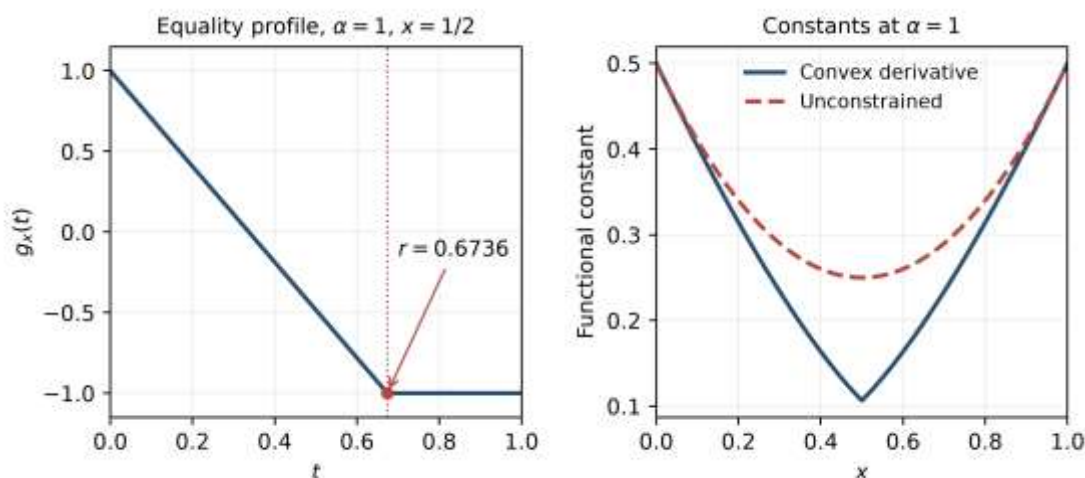


Figure 1. The decreasing equality profile at $\alpha = 1$, $x = 1/2$ (left), and the convex-class and unconstrained functional constants at $\alpha = 1$ (right).

6 Conclusion

The bounded nondecreasing-derivative class has the same sharp functional constant as the unconstrained class. Requiring the signed derivative to be convex gives a smaller constant at every interior evaluation point. The equality profile has one transition from a linear segment to a constant segment, and its transition point is determined by (5). The secant-slope argument supplies a global upper bound and identifies the equality cases. Numerical optimization over continuous convex grid profiles gives a separate comparison with the formula.

Declarations

Funding. This research received no external funding.

Conflict of interest. The author declares no conflict of interest.

Code availability. The root solver, convex linear-programming checker, figure-generation routine, and numerical comparison output accompany the manuscript. No external dataset was used.

AI-assisted preparation. WorkBuddy, using multiple models, and Codex were used for literature triage, drafting, mathematical analysis and proof revision, numerical workflow design, and submission-material checks. The author takes responsibility for reviewing and verifying the manuscript and for its final content.

References

- [1] G. A. Anastassiou, Fractional Ostrowski type inequalities, *Commun. Appl. Anal.* 7(2) (2003), 203–208.

10.48047/jocaaa.2026.35.10.04

ISSN:1521-1398,1572-9206

[2] S. S. Dragomir and A. Sofo, Ostrowski type inequalities for functions whose derivatives are convex, *RGMLA Res. Rep. Coll.* **5** (Supplement) (2002).
https://www.rgmia.org/papers/v5e/OTIDC2_col.pdf.

[3] W. Liu, Some Ostrowski type inequalities via Riemann–Liouville fractional integrals for h -convex functions, *J. Comput. Anal. Appl.* **16** (2014), 998–1004. Preprint:
<https://arxiv.org/abs/1210.3942>.

[4] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.* **10** (1937), 226–227.
<https://doi.org/10.1007/BF01214290>.

[5] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s -convex in the second sense via fractional integrals, *Comput. Math. Appl.* **63**(7) (2012), 1147–1154.
<https://doi.org/10.1016/j.camwa.2011.12.023>.

[6] M. Tariq, S. K. Ntouyas, and B. Ahmad, Ostrowski-Type Fractional Integral Inequalities: A Survey, *Foundations* **3**(4) (2023), 660–723. <https://doi.org/10.3390/foundations3040040>.