

Optimization of Adams-type difference formulas in Hilbert space $W_2^{(2,1)}(0, 1)$

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In this paper, we consider the problem of constructing new optimal explicit and implicit Adams-type difference formulas for finding an approximate solution to the Cauchy problem for an ordinary differential equation in a Hilbert space. In this work, I minimize the norm of the error functional of the difference formula with respect to the coefficients, we obtain a system of linear algebraic equations for the coefficients of the difference formulas. This system of equations is reduced to a system of equations in convolution and the system of equations is completely solved using a discrete analog of a differential operator $d^2/dx^2 - 1$. Here we present an algorithm for constructing optimal explicit and implicit difference formulas in a specific Hilbert space. In addition, comparing the Euler method with optimal explicit and implicit difference formulas, numerical experiments are given. Experiments show that the optimal formulas give a good approximation compared to the Euler method.

Keywords: Hilbert space; initial-value problem; multistep method; the error functional; optimal difference formula.

1 Introduction

It is known that the solutions of many practical problems lead to solutions of differential equations or their systems. Although differential equations have so

many applications and only a small number of them can be solved exactly using elementary functions and their combinations. Even in the analytical analysis of differential equations, their application can be inconvenient due to the complexity of the obtained solution. If it is very difficult to obtain or impossible to find an analytic solution to a differential equation, one can find an approximate solution.

In the present paper we consider the problem of approximate solution to the first order linear ordinary differential equation

$$y' = f(x, y), \quad x \in [0, 1] \quad (1)$$

with the initial condition

$$y(0) = y_0. \quad (2)$$

We assume that $f(x, y)$ is a suitable function and the differential equation (1) with the initial condition (2) has a unique solution on the interval $[0, 1]$.

For approximate solution of problem (1)-(2) we divide the interval $[0, 1]$ into N pieces of the length $h = \frac{1}{N}$ and find approximate values y_n of the function $y(x)$ for $n = 0, 1, \dots, N$ at nodes $x_n = nh$.

A classic method of approximate solution of the initial-value problem (1)-(2) is the Euler method. Using this method, the approximate solution of the differential equation is calculated as follows: to find an approximate value y_{n+1} of the function at the node x_{n+1} , it is used the approximate value y_n at the node x_n :

$$y_{n+1} = y_n + hy'_n, \quad (3)$$

where $y'_n = f(x_n, y_n)$, so that y_{n+1} is a linear combination of the values of the unknown function $y(x)$ and its first-order derivative at the node x_n .

Everyone are known that there are many methods for solving the initial-value problem for ordinary differential equation (1). For example, the initial-value problem can be solved using the Euler, Runge-Kutta, Adams-Bashforth and Adams-Moulton formulas of varying degrees [1]. In [2] by Ahmad Fadly Nurullah Rasedee, et al., research they discussed the order and stepsize strategies of the variable order stepsize algorithm. The stability and convergence estimations of the method are also established. In the work [3] by Adekoya Odunayo M. and Z.O.Ogunwobi, it was shown that the Adam-Bashforth-Moulton method is better than the Milne Simpson method in solving a second-order differential equation. Some studies have raised the question of whether Nordsieck's technique for changing the step size in the Adams-Bashforth method is equivalent to the explicit continuous Adams-Bashforth method. And in N.S.Hoang and R.B.Sidje's work [4] they provided a complete proof that the two approaches are indeed equivalent. In the works [5] and [6] there were shown the potential superiority of semi-explicit and semi-implicit methods over conventional linear multi-step algorithms.

However, it is very important to choose the right one among these formulas to solve the Initial-value problem and it is not always possible to do this. Also, in this work, in contrast to the above-mentioned works, exact estimates of the error of the formula is obtained.

Our aim, in this paper, is to construct new difference formulas that are exact for e^{-x} and optimal in the Hilbert space $W_2^{(2,1)}(0, 1)$. Also these formulas can be used to solve certain classes of problems with great accuracy.

The rest of the work is organized as follows. In the first paragraph, an algorithm for constructing an explicit difference formula in the space is given. The above algorithm is used to obtain an analytical formula for the optimal coefficients of an explicit difference formula. In the second section, the same algorithm is used to obtain an analytical formula for the optimal coefficients of the implicit difference formula. In the third and fourth sections, respectively, exact formulas are given for the square of the norm of the error functionals of explicit and implicit difference formulas. Numerical experiments are presented at the end of the work.

2 Optimal explicit difference formulas of Adams-Bashforth type in the Hilbert space $W_2^{(2,1)}(0, 1)$

We consider a difference formula of the following form for the approximate solution of the problem (1)-(2) [7, 8]

$$\sum_{\beta=0}^k C[\beta] \varphi[\beta] - h \sum_{\beta=0}^{k-1} C_1[\beta] \varphi'[\beta] \cong 0, \tag{1}$$

where $h = \frac{1}{N}$, N is a natural number, $C[\beta]$ and $C_1[\beta]$ are the coefficients, functions φ belong to the Hilbert space $W_2^{(2,1)}(0, 1)$. The space $W_2^{(2,1)}(0, 1)$ is defined as follows

$$W_2^{(2,1)}(0, 1) = \{\varphi : [0, 1] \rightarrow \mathbf{R} | \varphi' \text{ is abs.contunuous, } \varphi'' \in L_2(0, 1)\}$$

equipped with the norm [9, 10]

$$\|\varphi\|_{W_2^{(2,1)}} = \left\{ \int_0^1 (\varphi''(x) + \varphi'(x))^2 dx \right\}^{1/2}. \tag{2}$$

The following difference between the sums given in the formula (1) is called the error of the formula (1) [11]

$$(\ell, \varphi) = \sum_{\beta=0}^k C[\beta] \varphi(h\beta) - h \sum_{\beta=0}^{k-1} C_1[\beta] \varphi'(h\beta).$$

To this error corresponds the error functional [12]

$$\ell(x) = \sum_{\beta=0}^k C[\beta] \delta(x - h\beta) + h \sum_{\beta=0}^{k-1} C_1[\beta] \delta'(x - h\beta), \tag{3}$$

where $\delta(x)$ is Dirac's delta-function. We note that (ℓ, φ) is the value of the error functional ℓ at a function φ and it is defined as [13, 14]

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx.$$

It should be also noted that since the error functional ℓ is defined on the space $W_2^{(2,1)}(0, 1)$ it satisfies the following conditions

$$(\ell, 1) = 0, \quad (\ell, e^{-x}) = 0.$$

These give us the following equations with respect to coefficients $C[\beta]$ and $C_1[\beta]$:

$$\sum_{\beta=0}^k C[\beta] = 0, \tag{4}$$

$$\sum_{\beta=0}^k C[\beta]e^{-h\beta} + h \sum_{\beta=0}^{k-1} C_1[\beta]e^{-h\beta} = 0. \tag{5}$$

Based on the Cauchy-Schwartz inequality for the absolute value of the error of the formula (1) we have the estimation

$$|(\ell, \varphi)| \leq \|\varphi|W_2^{(2,1)}\| \cdot \|\ell|W_2^{(2,1)*}\|.$$

Hence, the absolute error of the difference formula (1) in the space $W_2^{(2,1)}$ is estimated by the norm of the error functional ℓ on the conjugate space $W_2^{(2,1)*}$. From this we get the following[15].

Problem 1. Calculate the norm $\|\ell|W_2^{(2,1)*}\|$ of the error functional ℓ .

From the formula (3) one can see that the norm $\|\ell|W_2^{(2,1)*}\|$ depends on the coefficients $C[\beta]$ and $C_1[\beta]$.

Problem 2. Find such coefficients $C_1[\beta] = \mathring{C}_1[\beta]$ that satisfy the equality

$$\|\mathring{\ell}|W_2^{(2,1)*}\| = \inf_{\mathring{C}_1[\beta]} \sup_{\|\varphi|W_2^{(2,1)}\| \neq 0} \frac{|(\ell, \varphi)|}{\|\varphi|W_2^{(2,1)}\|}.$$

In this case $\mathring{C}_1[\beta]$ are called the optimal coefficients and the corresponding difference formula (1) is called the *optimal difference formula*.

A function ψ_ℓ satisfying the following equation is called the extremal function of the difference formula (1) [13]

$$(\ell, \psi_\ell) = \|\ell|W_2^{(2,1)*}\| \cdot \|\psi_\ell|W_2^{(2,1)}\|. \tag{6}$$

Since the space $W_2^{(2,1)}(0, 1)$ is a Hilbert space, then from the Riesz theorem on the general form of a linear continuous functional on a Hilbert space there is a

function ψ_ℓ (which is the extremal function) that satisfies the following equation [16, 17]

$$(\ell, \varphi) = \langle \varphi, \psi_\ell \rangle_{W_2^{(2,1)}} \tag{7}$$

and the equality $\|\ell|W_2^{(2,1)*}\| = \|\psi_\ell|W_2^{(2,1)}\|$ holds, here $\langle \varphi, \psi_\ell \rangle_{W_2^{(2,1)}}$ is the inner product in the space $W_2^{(2,1)}(0, 1)$ and is defined as follows [18]

Theorem 2.1 *The solution of equation (7) has the form*

$$\psi_\ell(x) = \ell(x) * G_2(x) + de^{-x} + p_0 \tag{8}$$

and it is an extremal function for the difference formula (1), where $G_2(x) = \frac{\text{sgn}(x)}{2} \left(\frac{e^x - e^{-x}}{2} - x \right)$, d and p_0 are real numbers.

According to the above mentioned Riesz's theorem, the following equalities is fulfilled

$$\|\ell|W_2^{(2,1)*}\|^2 = (\ell, \psi_\ell) = \|\ell|W_2^{(2,1)*}\| \cdot \|\psi_\ell|W_2^{(2,1)}\|.$$

By direct calculation from the last equality for the norm of the error functional for the difference formula (1) we have the following result [18].

Theorem 2.2 *For the norm of the error functional of the difference formula (1) we have the following expression*

$$\begin{aligned} \|\ell|W_2^{(2,1)*}\|^2 = & \sum_{\gamma=0}^k \sum_{\beta=0}^k C[\gamma]C[\beta]G_2(h\gamma-h\beta) - 2h \sum_{\gamma=0}^{k-1} C_1[\gamma] \sum_{\beta=0}^k C[\beta]G_2'(h\gamma-h\beta) - \\ & - h^2 \sum_{\gamma=0}^{k-1} \sum_{\beta=0}^{k-1} C_1[\gamma]C_1[\beta]G_2''(h\gamma-h\beta), \end{aligned} \tag{9}$$

where $G_2'(x) = \frac{\text{sgn}(x)}{2} \left(\frac{e^x + e^{-x}}{2} - 1 \right)$ and $G_2''(x) = \frac{\text{sgn}(x)}{2} \left(\frac{e^x - e^{-x}}{2} \right)$.

It is known that stability in the Dahlquist sense, just like strong stability, is determined only by the coefficients $C[\beta]$, $\beta = \overline{0, k}$. For this reason, our search for the optimal formula is only related to finding $C_1[\beta]$. Therefore, in this subsection we consider difference formulas of the Adams-Bashforth type, i.e. $C[k] = -C[k-1] = 1$ and $C[k-i] = 0$, $i = \overline{2, k}$, [19, 20]. Then is easy to check, that the coefficients satisfy the condition (4).

In this work, we find the minimum of the norm (9) by the coefficients $C_1[\beta]$ under the condition (5) in the space $W_2^{(2,1)}(0, 1)$ [21]. Then using Lagrange method of undetermined multipliers we get the following system of linear equations with respect to the coefficients $C_1[\beta]$:

$$h \sum_{\gamma=0}^{k-1} C_1[\gamma] G_2''(h\beta-h\gamma) + de^{-h\beta} = - \sum_{\gamma=0}^k C[\gamma] G_2'(h\beta-h\gamma), \tag{10}$$

$$\beta = \overline{0, k-1},$$

$$h \sum_{\gamma=0}^{k-1} C_1[\gamma] e^{-h\gamma} = - \sum_{\gamma=0}^k C[\gamma] e^{-h\gamma}. \tag{11}$$

It is easy to prove that the solution of this system gives the minimum value to the expression (9) under the condition (5). Here d is an unknown constant, $\overset{\circ}{C}_1[\beta]$ are optimal coefficient. Given that $C[k] = 1, C[k-1] = -1, C[k-i] = 0, i = \overline{2, k}$ the system (10),(11) is reduced to the form,

$$h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] G_2''(h\beta - h\gamma) + de^{-h\beta} = f[\beta], \quad \beta = \overline{0, k-1} \tag{12}$$

$$h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] e^{-h\gamma} = g, \tag{13}$$

where

$$f[\beta] = \frac{1 - e^h}{4} (e^{h\beta - hk} - e^{-h\beta + hk - h}), \tag{14}$$

$$g = e^{-hk+h} - e^{-hk}. \tag{15}$$

Assuming that $C_1[\beta] = 0$, for $\beta < 0$ and $\beta > k-1$, we rewrite the system (12), (13) in the convolution form

$$\begin{cases} h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = f[\beta] \text{ for } \beta = \overline{0, k-1}, \\ h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] e^{-h\gamma} = g. \end{cases} \tag{16}$$

We denote first equation of the system (16) by U_{exp}

$$U_{exp}[\beta] = h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta}. \tag{17}$$

(12) implies that

$$U_{exp}[\beta] = f[\beta] \text{ for } \beta = \overline{0, k-1}. \tag{18}$$

Now calculating the convolution we have

$$U_{exp}[\beta] = \overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] G_2''(h\beta - h\gamma) + de^{-h\beta}.$$

For $\beta < 0$ we get

$$\begin{aligned} U_{exp}[\beta] &= h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] \frac{\text{sgn}(h\beta - h\gamma)}{2} \left(\frac{e^{h\beta - h\gamma} - e^{-h\beta + h\gamma}}{2} \right) + de^{-h\beta} \\ &= -\frac{e^{h\beta}}{4} h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] e^{-h\gamma} + \frac{e^{-h\beta}}{4} h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta} = -\frac{e^{h\beta}}{4} g + e^{-h\beta} (d + b). \end{aligned}$$

For $\beta > k - 1$

$$U_{exp}[\beta] = \frac{e^{h\beta}}{4}g + e^{-h\beta}(d - b).$$

Then $d^+ = d + b$ and $d^- = d - b$ the function $U_{exp}[\beta]$ becomes

$$U_{exp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4}g + e^{-h\beta}d^+ & \text{for } \beta > k - 1, \\ f[\beta] & \text{for } \beta = 0, k - 1, \\ \frac{e^{h\beta}}{4}g + e^{-h\beta}d^- & \text{for } \beta < 0. \end{cases} \quad (19)$$

We use to find the unknowns d^+ and d^- from the discrete analogue of the differential operator $\frac{d^2}{dx^2} - \frac{d}{dx}$ which is given below [22]

$$D_1[\beta] = \frac{1}{1 - e^{2h}} \begin{cases} -2e^h & \text{for } |\beta| = 1, \\ 2(1 + e^{2h}) & \text{for } \beta = 0, \\ 0 & \text{for } |\beta| \geq 2. \end{cases} \quad (20)$$

The unknowns d^+ and d^- are determined from the conditions

$$\overset{\circ}{C}_1[\beta] = h^{-1}D_1[\beta] * U_{exp}[\beta] = 0 \text{ for } \beta < 0 \text{ and } \beta > k - 1. \quad (21)$$

Calculate the convolution

$$\begin{aligned} & h^{-1}D_1[\beta] * U_{exp}[\beta] \\ &= h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta + \gamma] U_{exp}[-\gamma] + h^{-1} \sum_{\gamma=0}^{k-1} D_1[\beta - \gamma] U_{exp}[\gamma] \\ & \quad + h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta - k - \gamma + 1] U_{exp}[k + \gamma - 1]. \end{aligned}$$

From (19) with $\beta = k$ and $\beta = -1$, we have

$$\begin{cases} h^{-1}D_1[0] U_{exp}[-1] + h^{-1}D_1[1] U_{exp}[-2] + h^{-1}D_1[-1] U_{exp}[0] = 0, \\ h^{-1}D_1[0] U_{exp}[k] + h^{-1}D_1[1] U_{exp}[k - 1] + h^{-1}D_1[-1] U_{exp}[k + 1] = 0. \end{cases}$$

Hence, due to (21), we get

$$\begin{cases} 2(1 + e^{2h}) \left[-\frac{1}{4}e^{-h}g + e^h d^+ \right] - 2e^h \left[-\frac{1}{4}e^{-2h}g + e^{2h}d^+ \right] - 2e^h f[0] = 0 \\ 2(1 + e^{2h}) \left[\frac{1}{4}e^{hk}g + e^{-hk}d^- \right] - 2e^h \left[\frac{1}{4}e^{hk+h}g + e^{-hk-h}d^- \right] - 2e^h f[hk - h] = 0. \end{cases}$$

From the first equation d^+ is equal to the following

$$d^+ = \frac{e^{hk} - e^{hk-h}}{4}.$$

From the second equation d^- is equal to the following

$$d^- = \frac{e^{hk} - 3e^{hk-h} + 2e^{hk-2h}}{4}.$$

so

$$d = \frac{d_0^+ + d_0^-}{2} = \frac{e^{hk} - 2e^{hk-h} + e^{hk-2h}}{4} \quad \text{and} \quad b = \frac{d_0^+ - d_0^-}{2} = \frac{e^{hk-h} - e^{hk-2h}}{4}.$$

Now we calculate the optimal coefficients $\overset{\circ}{C}_1[\beta]$

$$\overset{\circ}{C}_1[\beta] = h^{-1} D_1[\beta] * U_{exp}[\beta] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[\beta - \gamma] U_{exp}[\gamma], \quad \beta = \overline{0, k-1}.$$

Let $\beta = k - 1$, then

$$\begin{aligned} \overset{\circ}{C}_1[k-1] &= h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[k-1-\gamma] U_{exp}[\gamma] \\ &= h^{-1} \{D_1[0] U_{exp}[k-1] + D_1[1] U_{exp}[k-2] + D_1[-1] U_{exp}[k]\} \\ &= \frac{h^{-1}}{(1-e^{2h})} \cdot \{1 - e^{-h} + e^h - e^{2h}\} = \frac{e^h - 1}{he^h}, \end{aligned}$$

thus, $\overset{\circ}{C}_1[k-1] = \frac{e^h - 1}{he^h}$ for $\beta = k - 1$.

Compute $\overset{\circ}{C}_1[0]$

$$\begin{aligned} \overset{\circ}{C}_1[0] &= h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[-\gamma] U_{exp}[\gamma] \\ &= h^{-1} \{D_1[0] U_{exp}[0] + D_1[1] U_{exp}[-1] + D_1[-1] U_{exp}[1]\} \\ &= \frac{h^{-1}}{2(1-e^{2h})} \cdot \{(1+e^{2h})(e^{-hk} - e^{hk-h} - e^{-hk+h} + e^{hk}) \\ &\quad - e^{-h}(-e^{-hk} + e^{-hk-h} - e^{hk+h} - e^{hk})\} \\ &\quad - \frac{h^{-1}}{2(1-e^{2h})} \cdot \{e^{-h}(e^{-hk+h} - e^{hk-2h} - e^{-hk+2h} + e^{hk-h})\} = \frac{h^{-1}}{2(1-e^{2h})} \cdot 0 = 0, \end{aligned}$$

hence, $\overset{\circ}{C}_1[0] = 0$ for $\beta = 0$.

Now calculate $\overset{\circ}{C}_1[\beta]$ for $\beta = \overline{1, k-2}$

$$\begin{aligned} \overset{\circ}{C}_1[\beta] &= h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[-\gamma] U_{exp}[\gamma] \\ &= h^{-1} \{D_1[0] U_{exp}[\beta] + D_1[1] U_{exp}[\beta-1] + D_1[-1] U_{exp}[\beta+1]\} \\ &= \frac{h^{-1}}{2(1-e^{2h})} \cdot \{(1+e^{2h})(1-e^h)(e^{-hk+h\beta} - e^{hk-h\beta-h})\} \\ &\quad - \frac{h^{-1}}{2(1-e^{2h})} \cdot \{e^{-h}(1-e^h)(e^{-hk+h\beta-h} - e^{hk-h\beta})\} \end{aligned}$$

$$-\frac{h^{-1}}{2(1 - e^{2h})} \cdot \{e^{-h}(1 - e^h)(e^{-hk+h\beta+h} - e^{hk-h\beta-2h})\} = \frac{h^{-1}}{2(1 - e^{2h})} \cdot 0 = 0,$$

thereby, $\overset{\circ}{C}_1[\beta] = 0$ for $\beta = \overline{1, k-2}$.

Finally, we have proved the following theorem.

Theorem 2.3 *In the Hilbert space $W_2^{(2,1)}(0, 1)$ there is a unique optimal explicit difference formula of the Adams-Bashforth type whose coefficients are determined by following expressions*

$$C[\beta] = \begin{cases} 1 & \text{for } \beta = k, \\ -1 & \text{for } \beta = k - 1, \\ 0 & \text{for } \beta = \overline{0, k - 2}, \end{cases} \quad (22)$$

$$\overset{\circ}{C}_1[\beta] = \begin{cases} \frac{e^h - 1}{he^h} & \text{for } \beta = k - 1, \\ 0 & \text{for } \beta = \overline{0, k - 2}. \end{cases} \quad (23)$$

Thus, the optimal explicit difference formula in $W_2^{(2,1)}(0, 1)$ has the form

$$\varphi_{n+k} = \varphi_{n+k-1} + \frac{e^h - 1}{e^h} \varphi'_{n+k-1}, \quad (24)$$

where $n = 0, 1, \dots, N - k, k \geq 1$.

3 Optimal implicit difference formulas of Adams-Moulton type in the Hilbert space $W_2^{(2,1)}(0, 1)$

Consider an implicit difference formula of the form

$$\sum_{\beta=0}^k C[\beta]\varphi[\beta] - h \sum_{\beta=0}^k C_1[\beta]\varphi'[\beta] \cong 0 \quad (1)$$

with the error function

$$\ell(x) = \sum_{\beta=0}^k C[\beta]\delta(x - h\beta) + h \sum_{\beta=0}^k C_1[\beta]\delta'(x - h\beta) \quad (2)$$

in the space $W_2^{(2,1)}(0, 1)$.

In this section, we also consider the case $C[k] = -C[k-1] = 1$, and $C[k-i] = 0, i = \overline{2, k}$, i.e. Adams-Moulton type formula. Minimizing the norm of the error functional (2) of an implicit difference formula of the form (1) with respect to the coefficients $C_1[\beta], \beta = \overline{0, k}$ in the space $W_2^{(2,1)}(0, 1)$ we obtain a system of linear algebraic equations

$$\begin{cases} h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] G_2''(h\beta - h\gamma) + de^{-h\beta} = f[\beta], & \beta = \overline{0, k} \\ h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{-h\gamma} = g. \end{cases}$$

Here $\overset{\circ}{C}_1[\beta]$ are unknowns coefficients of the implicit difference formulas (1), $\beta = \overline{0, k}$ and d is an unknown constant,

$$f[\beta] = G'_2(h\beta - hk + h) - G'_2(h\beta - hk)$$

$$= \begin{cases} \frac{1}{4}(1 - e^h)(e^{-hk+h\beta} - e^{hk-h\beta-h}), & \beta = \overline{0, k-1}, \\ \frac{1}{4}(e^h + e^{-h} - 2), & \beta = k, \end{cases} \quad (3)$$

$$g = e^{-hk+h} - e^{-hk}. \quad (4)$$

Assuming, in general, that

$$\overset{\circ}{C}_1[\beta] = 0, \text{ for } \beta < 0 \text{ and } \beta > k, \quad (5)$$

rewrite the system in the convolution form

$$\begin{cases} h\overset{\circ}{C}_1[\beta] * G''_2(h\beta) + de^{-h\beta} = f[\beta], & \beta = \overline{0, k}, \\ h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{-h\gamma} = g. \end{cases}$$

Denote by $U_{imp}[\beta] = h\overset{\circ}{C}_1[\beta] * G''_2(h\beta) + de^{-h\beta}$. Shows that

$$U_{imp}[\beta] = f[\beta] \text{ for } \beta = \overline{0, k} \quad (6)$$

Now we find $U_{imp}[\beta]$ for $\beta < 0$ and $\beta > k$. Let $\beta < 0$, then

$$U_{imp}[\beta] = h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] \frac{\text{sgn}(h\beta - h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2} \right) + de^{-h\beta}$$

$$= -\frac{e^{h\beta}}{4} h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{-h\gamma} + \frac{e^{-h\beta}}{4} h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}.$$

Here d^+ is defined by the equality

$$d^+ = \frac{e^{-h\beta}}{4} h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}. \quad (7)$$

Similarly, for $\beta > k$ we have

$$U_{imp}[\beta] = h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] \frac{\text{sgn}(h\beta - h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2} \right) + de^{-h\beta}$$

$$= \frac{e^{h\beta}}{4} h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{-h\gamma} - \frac{e^{-h\beta}}{4} h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}.$$

Here d^- is defined by the equality

$$d^- = -\frac{e^{-h\beta}}{4}h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] e^{h\gamma} + de^{-h\beta}. \tag{8}$$

(7) and (8) immediately imply that

$$d = \frac{d^+ + d^-}{2}. \tag{9}$$

So $U_{imp}[\beta]$ for any $\beta \in Z$ is defined by the formula

$$U_{imp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4}g + e^{-h\beta}d^+ & \text{for } \beta > k, \\ f[\beta] & \text{for } \beta = 0, k, \\ \frac{e^{h\beta}}{4}g + e^{-h\beta}d^- & \text{for } \beta < 0. \end{cases} \tag{10}$$

If we operate operator (20) on expression $U_{imp}[\beta]$, we get

$$\overset{\circ}{C}_1[\beta] = h^{-1}D_1[\beta] * U_{imp}[\beta], \beta \in Z. \tag{11}$$

Assuming that $\overset{\circ}{C}_1[\beta] = 0$ for $\beta < 0$ and $\beta > k$, we get a system of linear equations for finding the unknowns d^+ and d^- in the formula (10). Indeed, calculating the convolution, we have

$$\begin{aligned} h^{-1}D_1[\beta] * U_{imp}[\beta] &= h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[\beta - \gamma] U_{imp}[\gamma] \\ &= h^{-1} \sum_{\gamma=-\infty}^{-1} D_1[\beta - \gamma] U_{imp}[\gamma] + h^{-1} \sum_{\gamma=0}^k D_1[\beta - \gamma] U_{imp}[\gamma] \\ &\quad + h^{-1} \sum_{\gamma=k+1}^{\infty} D_1[\beta - \gamma] U_{imp}[\gamma] \\ &= h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta + \gamma] U_{imp}[-\gamma] + h^{-1} \sum_{\gamma=0}^k D_1[\beta - \gamma] U_{imp}[\gamma] \\ &\quad + h^{-1} \sum_{\gamma=1}^{\infty} D_1[\beta - k - \gamma] U_{imp}[k + \gamma]. \end{aligned} \tag{12}$$

Equating the expression (12) to zero with $\beta = -1, \beta = k + 1$ and using the formulas (10), (20) we get

$$\begin{cases} h^{-1}D_1[0] U_{imp}[-1] + h^{-1}D_1[1] U_{imp}[-2] + h^{-1}D_1[-1] U_{imp}[0] = 0, \\ h^{-1}D_1[0] U_{imp}[k + 1] + h^{-1}D_1[1] U_{imp}[k] + h^{-1}D_1[-1] U_{imp}[k + 2] = 0 \end{cases}$$

or

$$\begin{cases} 2(1 + e^{2h}) \left[-\frac{1}{4}e^{-h}g + e^h d^+ \right] - 2e^h \left[-\frac{1}{4}e^{-2h}g + e^{2h}d^+ \right] - 2e^h f[0] = 0, \\ 2(1 + e^{2h}) \left[\frac{1}{4}e^{hk+h}g + e^{-hk-h}d^- \right] - 2e^h \left[\frac{1}{4}e^{hk+2h}g + e^{-hk-2h}d^- \right] - 2e^h f[hk] = 0. \end{cases}$$

By virtue of the formulas (3) and (4), finally, we find

$$d^+ = \frac{1}{4} (e^{hk} - e^{hk-h}), \tag{13}$$

$$d^- = \frac{1}{4} (e^{hk-h} - e^{hk}). \tag{14}$$

Then from (9) we find that $d = 0$.

As a result, we rewrite $U_{imp}[\beta]$ through the (13) and (14) as follows

$$U_{imp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4}g + \frac{e^{-h\beta}}{4}(e^{hk} - e^{hk-h}) & \text{for } \beta > k, \\ f[\beta] & \text{for } \beta = \overline{0, k}, \\ \frac{e^{h\beta}}{4}g + \frac{e^{-h\beta}}{4}(e^{hk-h} - e^{hk}) & \text{for } \beta < 0. \end{cases} \tag{15}$$

Now we turn to calculating the optimal coefficients of implicit difference formulas $\overset{\circ}{C}_1[\beta]$, $\beta = \overline{0, k}$ according to the formula (11)

$$\begin{aligned} \overset{\circ}{C}_1[k] &= h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[k - \gamma]U_{imp}[\gamma] = \\ &= h^{-1} \{D_1[0]U_{imp}[k] + D_1[1]U_{imp}[k - 1] + D_1[-1]U_{imp}[k + 1]\} = \\ &= \frac{h^{-1}}{2(1 - e^{2h})} (-2e^{2h} + 4e^h - 2) = \frac{e^h - 1}{h(e^h + 1)}. \end{aligned}$$

So $\overset{\circ}{C}_1[k] = \frac{e^h - 1}{h(e^h + 1)}$.

Calculate the next optimal coefficient

$$\begin{aligned} \overset{\circ}{C}_1[k - 1] &= h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[k - \gamma - 1]U_{imp}[\gamma] = \\ &= h^{-1} \{D_1[0]U_{imp}[k - 1] + D_1[1]U_{imp}[k - 2] + D_1[-1]U_{imp}[k]\} = \\ &= \frac{h^{-1}}{2(1 - e^{2h})} (-2e^{2h} + 4e^h - 2) = \frac{e^h - 1}{h(e^h + 1)}. \end{aligned}$$

Thus $\overset{\circ}{C}_1[k - 1] = \frac{e^h - 1}{h(e^h + 1)}$.

Go to computed $\overset{\circ}{C}_1[\beta]$ when $\beta = \overline{1, k - 2}$

$$\overset{\circ}{C}_1[\beta] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[\beta - \gamma]U_{imp}[\gamma]$$

$$\begin{aligned}
 &= h^{-1} \{D_1[0]U_{imp}[\beta] + D_1[1]U_{imp}[\beta - 1] + D_1[-1]U_{imp}[\beta + 1]\} \\
 &= \frac{h^{-1}}{2(1 - e^{2h})} \{(1 + e^{2h}) (e^{-hk+h\beta} - e^{hk-h\beta-h} - e^{-hk+h\beta+h} + e^{hk-h\beta})\} \\
 &\quad - \frac{h^{-1}}{2(1 - e^{2h})} \{e^h (e^{-hk+h\beta-h} - e^{hk-h\beta} - e^{-hk+h\beta} + e^{hk-h\beta+h})\} \\
 &\quad - \frac{h^{-1}}{2(1 - e^{2h})} \{e^h (e^{-hk+h\beta+h} - e^{hk-h\beta-2h} - e^{-hk+h\beta+2h} + e^{hk-h\beta-h})\} \\
 &= \frac{h^{-1}}{2(1 - e^{2h})} \cdot 0 = 0.
 \end{aligned}$$

Thereby, $\mathring{C}_1[\beta] = 0$, for $\beta = \overline{1, k - 2}$.

Then calculate $\mathring{C}_1[0]$

$$\begin{aligned}
 \mathring{C}_1[0] &= h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[-\gamma]U_{imp}[\gamma] \\
 &= h^{-1} \{D_1[0]U_{imp}[0] + D_1[1]U_{imp}[-1] + D_1[-1]U_{imp}[1]\} \\
 &= \frac{h^{-1}}{2(1 - e^{2h})} \{(1 + e^{2h}) (e^{-hk} - e^{hk-h} - e^{-hk+h} + e^{hk})\} \\
 &\quad - \frac{h^{-1}}{2(1 - e^{2h})} \{e^h (-e^{-hk} + e^{-hk-h} + e^{hk+h} - e^{hk})\} \\
 &\quad - \frac{h^{-1}}{2(1 - e^{2h})} \{e^h (e^{-hk+h} - e^{hk-2h} - e^{-hk+2h} + e^{hk-h})\} = \frac{h^{-1}}{2(1 - e^{2h})} \cdot 0 = 0.
 \end{aligned}$$

hence $\mathring{C}_1[0] = 0$.

Finally, we have proved the following.

Theorem 3.1 *In the Hilbert space $W_2^{(2,1)}(0, 1)$, there exists a unique optimal implicit difference formula, of Adams-Moulton type, whose coefficients are determined by formulas*

$$C[\beta] = \begin{cases} 1 & \text{for } \beta = k, \\ -1 & \text{for } \beta = k - 1, \\ 0 & \text{for } \beta = \overline{0, k - 2}, \end{cases} \quad (16)$$

$$\mathring{C}_1[\beta] = \begin{cases} \frac{e^h - 1}{h(e^h + 1)} & \text{for } \beta = k, \\ \frac{e^h - 1}{h(e^h - 1)} & \text{for } \beta = k - 1, \\ 0 & \text{for } \beta = \overline{0, k - 2}. \end{cases} \quad (17)$$

Consequently, the optimal implicit difference formula in $W_2^{(2,1)}(0, 1)$ has the form

$$\varphi_{n+k} = \varphi_{n+k-1} + \frac{e^h - 1}{e^h + 1} (\varphi'_{n+k} + \varphi'_{n+k-1}), \quad (18)$$

where $n = 0, 1, \dots, N - k$, $k \geq 1$.

4 Norm of the error functional of the optimal explicit difference formula

The square of the norm of an explicit Adams-Bashforth type difference formula is expressed by the equality

$$\begin{aligned} \left\| \ell \left| W_2^{(2,1)*}(0,1) \right. \right\|^2 &= \sum_{\gamma=0}^k \sum_{\beta=0}^k C[\gamma]C[\beta]G_2[\gamma-\beta] - \\ &- 2h \sum_{\gamma=0}^{k-1} C_1[\gamma] \sum_{\beta=0}^k C[\beta]G_2'[\gamma-\beta] - h^2 \sum_{\gamma=0}^{k-1} \sum_{\beta=0}^{k-1} C_1[\gamma]C_1[\beta]G_2''[\gamma-\beta]. \end{aligned} \quad (1)$$

In this section, we deal with the calculation of the squared norm (1) in the space $W_2^{(2,1)}(0,1)$. For this we use the coefficients $C[\beta]$ and $\overset{\circ}{C}_1[\beta]$, which is detected in the formulas (22) and (23).

Then we calculate (1) in sequence as follows.

$$\begin{aligned} \left\| \overset{\circ}{\ell} \left| W_2^{(2,1)*}(0,1) \right. \right\|^2 &= \sum_{\gamma=0}^k C[\gamma] \{G_2[\gamma-k] - G_2[\gamma-k+1]\} \\ &- 2h \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] \{G_2'[\gamma-k] - G_2'[\gamma-k+1]\} \\ &- h^2 \frac{e^h - 1}{he^h} \sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma] \{G_2''[\gamma-k+1]\} \\ &= G_2[0] - G_2[1] - G_2[-1] + G_2[0] - \frac{2(e^h - 1)}{e^h} \{G_2'[-1] - G_2'[0]\} - \frac{(e^h - 1)^2}{e^{2h}} G_2''[0] \\ &= -2G_2[1] + \frac{2(e^h - 1)}{e^h} G_2'[1] = \frac{2(e^h - 1)}{e^h} \cdot \frac{\text{sgn}(h)}{2} \left(\frac{e^h + e^h}{2} - 1 \right) \\ &- 2 \cdot \frac{\text{sgn}(h)}{2} \left(\frac{e^h - e^h}{2} - h \right) = \frac{e^h - 1}{e^h} \cdot \frac{e^{2h} - 2e^h + 1}{2e^h} - \frac{e^{2h} - 1}{2e^h} + h \\ &= h - \frac{(e^h - 1)(3e^h - 1)}{2e^{2h}}. \end{aligned}$$

As a result, we get the following outcome.

Theorem 4.1 *The square of the norm of the optimal error functional of an explicit difference formula of the form (1) in the quotient space $W_2^{(2,1)}(0,1)$ is expressed as formula*

$$\left\| \overset{\circ}{\ell} \left| W_2^{(2,1)*}(0,1) \right. \right\|^2 = h - \frac{(e^h - 1)(3e^h - 1)}{2e^{2h}}.$$

5 Norm of the error functional of the implicit optimal difference formula

In this case, the square of the norm of the error functional of an implicit Adams-Moulton type difference formula of the form (1) is expressed by the equality

$$\begin{aligned} \left\| \ell \left| W_2^{(2,1)*}(0, 1) \right. \right\|^2 &= \sum_{\gamma=0}^k \sum_{\beta=0}^k C[\gamma]C[\beta]G_2[\gamma - \beta] - \\ &- 2h \sum_{\gamma=0}^k C_1[\gamma] \sum_{\beta=0}^k C[\beta]G_2'[\gamma - \beta] - h^2 \sum_{\gamma=0}^k \sum_{\beta=0}^k C_1[\gamma]C_1[\beta]G_2''[\gamma - \beta]. \end{aligned} \quad (1)$$

Here we use the optimal coefficients of an implicit difference formula of the form (1), which is detected in the formulas (16) and (17).

Then, we calculate (1) as follows

$$\begin{aligned} \left\| \ell \left| W_2^{(2,1)*}(0, 1) \right. \right\|^2 &= \sum_{\gamma=0}^k C[\gamma] \{G_2[\gamma - k] - G_2[\gamma - k + 1]\} \\ &- 2h \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] \{G_2'[\gamma - k] - G_2'[\gamma - k + 1]\} \\ &- h^2 \frac{e^h - 1}{h(e^h + 1)} \sum_{\gamma=0}^k \overset{\circ}{C}_1[\gamma] \{G_2''[\gamma - k] + G_2''[\gamma - k + 1]\} \\ &= G_2[0] - G_2[1] - G_2[-1] + G_2[0] - \frac{2h(e^h - 1)}{h(e^h + 1)} \{G_2'[0] - G_2'[1] + G_2'[-1] - G_2'[0]\} \\ &- \frac{h^2(e^h - 1)^2}{h^2(e^h + 1)} \{G_2''[0] + G_2''[1] + G_2''[-1] + G_2''[0]\} \\ &= -2G_2[1] + \frac{4(e^h - 1)}{e^h + 1} G_2'[1] - 2 \frac{(e^h - 1)^2}{(e^h + 1)} G_2''[1] \\ &= \frac{4(e^h - 1)}{e^h + 1} \cdot \frac{\operatorname{sgn}(h)}{2} \left(\frac{e^h + e^h}{2} - 1 \right) - 2 \cdot \frac{\operatorname{sgn}(h)}{2} \left(\frac{e^h - e^h}{2} - h \right) \\ &- 2 \frac{(e^h - 1)^2}{(e^h + 1)} \cdot \frac{\operatorname{sgn}(h)}{2} \left(\frac{e^h - e^h}{2} \right) \\ &= h - \frac{e^{2h} - 1}{2e^h} + \frac{2(e^h - 1)^2(e^h - 1)}{2e^h(e^h + 1)} - \frac{(e^h - 1)^2(e^h - 1)}{2e^h(e^h + 1)} \\ &= h + \frac{(e^h - 1)}{2e^h} \left(\frac{(e^h - 1)^2}{e^h + 1} - e^h - 1 \right) = h - \frac{2(e^h - 1)}{e^h + 1}. \end{aligned}$$

Consequently, we get the following result.

Theorem 5.1 Among all implicit difference formulas of the form (1) in the Hilbert space $W_2^{(2,1)}(0,1)$, there is a unique implicit optimal difference formula square the norms of the error functional of which is determined by the equality

$$\left\| \ell \right\|_{W_2^{(2,1)*}(0,1)}^2 = h - \frac{2(e^h - 1)}{e^h + 1}.$$

6 Numerical results

In this section, we give some numerical results in order to show tables and graphs of solutions and errors of our optimal explicit difference formulas (24) and optimal implicit difference formulas (18), with coefficients given correspondingly in Theorem 2.3 and Theorem 3.1. We show the results of the created formulas in some examples in the form of tables and graphs. Here, of course, the results presented in the table are then shown in the graph.

Example: $y' = x \sin(-3x) - 2$, $N = 10$, $h = 0.1$, $y_0 = 1$; exact solution: $1/3x \cos(-3x) - 1/3 \cos(-3x) - 2x + 4/3$
 $N=10$
 $y_0=1$

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal exp.diff formula	Error of the Optimal exp.diff formula
0.0	1.0	1.0	0.0	1.0	0.0
0.1	0.846732386595652	0.8	0.046732386595651	0.809674836071919	0.037057550523732
0.2	0.713243836024086	0.597044797933387	0.116199038098699	0.616537425554994	0.096706410469892
0.3	0.588291007483512	0.385751948465486	0.202539858938026	0.415465694495946	0.172825312907565
0.4	0.468061782437999	0.162252141176661	0.298609641261337	0.202777507195246	0.258084275242752
0.5	0.321543799722049	-0.075029422262028	0.396573221984077	-0.023025795097653	0.344569554819702
0.6	0.163626945959078	-0.32490417159223	0.488531117551308	-0.268813018236448	0.424439964195526
0.7	-0.016182056206681	-0.58335029444922	0.567152973238241	-0.506742495160889	0.498560438954208
0.8	-0.217507085630584	-0.843759685110343	0.62625259947976	-0.754569321563865	0.537062235932482
0.9	-0.436530928599431	-1.097796739554436	0.661265810955004	-0.996317541717459	0.559786613118027
1.0	-0.666666666666667	-1.33626092877548	0.669594262108813	-1.22324622123982	0.556579554573153

Figure 1:

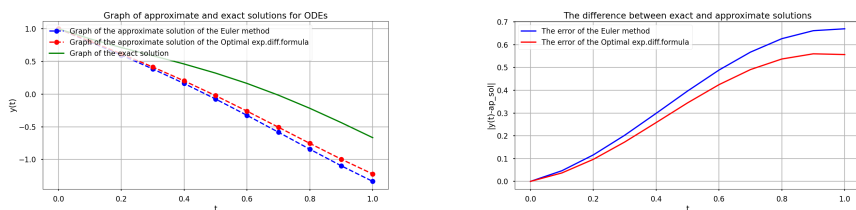


Figure 2:

Example: $y' = x \sin(-3x) - 2$, $N = 10$, $h = 0.1$, $y_0 = 0$; exact solution: $1/3x \cos(-3x) - 1/3 \cos(-3x) - 2x + 4/3$
 $N=10$
 $y_0=0$

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal exp.diff. formula	Error of the Optimal exp.diff. f. formula
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.177582570455131	0.2	0.022417429544869	0.190325163928081	0.01274259347295
0.2	0.318937085892713	0.35751574640716	0.038578740514447	0.340221215209065	0.021284209316352
0.3	0.433883226340232	0.48432171812668	0.050438491786449	0.460893851981951	0.02700982564172
0.4	0.529205824465697	0.588522961386072	0.059317136920375	0.560053645506219	0.030847821840522
0.5	0.609635519478213	0.675744617789102	0.066109098310888	0.643056825771146	0.033420506292933
0.6	0.67850887558362	0.749931212084026	0.071422336580406	0.713653904373384	0.035145028789763
0.7	0.738212180552286	0.81388417102237	0.075671990470004	0.774513191341514	0.036301018789229
0.8	0.790480484888187	0.869623707664033	0.079143222775846	0.827556373584513	0.037075888696326
0.9	0.836599409647578	0.918632824473592	0.082033414826014	0.874194714538262	0.037595304890684
1.0	0.877541649186374	0.962822162692381	0.084480513586007	0.915405129084372	0.037943479977998

Figure 3:

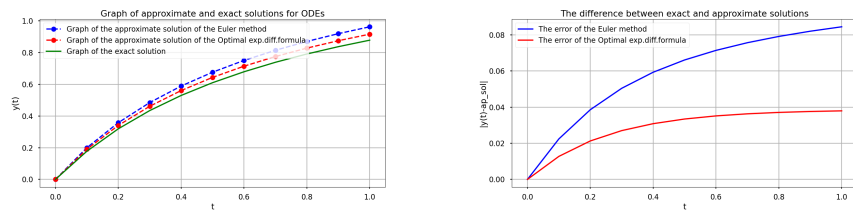


Figure 4:

The tables in Figures 1, 3, and 5 show the exact and approximate solutions and the differences between the exact and approximate solutions.

According to the tables in Figures 1 and 3 on the left side of these Figures 2 and 4 are graphs of approximate and exact solutions, and on the right side of these Figures 2 and 4 graphs of the difference between the actual and approximate solutions are shown. As can be seen from the results presented above, in a certain sense, the optimal explicit formula gives better results than the classical Euler formula.

In accordance with the table, shown in Figure 5, on the left side of these Figure 6, graphs of approximate and exact solutions are shown, and on the right side of these Figure 6, graphs of the difference between the authentic and approximate solutions. As can be seen from the results presented above, in a certain sense, optimal explicit and implicit difference formulas give better results than the classical Euler formula.

It should also be noted that with the help of newly constructed difference schemes, it is possible to obtain approximate solutions with good accuracy.

Example: $y' = y - x^2 + 1$, $N = 10$, $h = 0.1$, $y_0 = 0.5$; exact solution: $y(x) = (x+1)^2 - 0.5e^x(x)$
 $N=10$
 $y_0=0.5$

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal imp.diff. formula	Error of the Optimal imp.diff. formula
0.0	0.5	0.5	0.0	0.5	0.0
0.1	0.657414541	0.65	0.007414541	0.6572305225	0.0001840184
0.2	0.8292986209	0.814	0.0152986209	0.8288937051	0.0004049158
0.3	1.0150705962	0.9914	0.0236705962	1.0144040254	0.0006665708
0.4	1.2140876512	1.18154	0.0325476512	1.2131143814	0.0009732698
0.5	1.4256393646	1.383694	0.0419453646	1.4243096145	0.0013297501
0.6	1.6489405998	1.5970634	0.0518771998	1.6471993523	0.0017412475
0.7	1.8831236463	1.82076974	0.0623539063	1.8809100984	0.0022135479
0.8	2.1272295358	2.053846714	0.0733828218	2.1244764896	0.0027530461
0.9	2.3801984444	2.2952313854	0.084967059	2.3768316349	0.0033660895
1.0	2.6408598858	2.5437545239	0.0971045618	2.6367964373	0.0040626485

Figure 5:

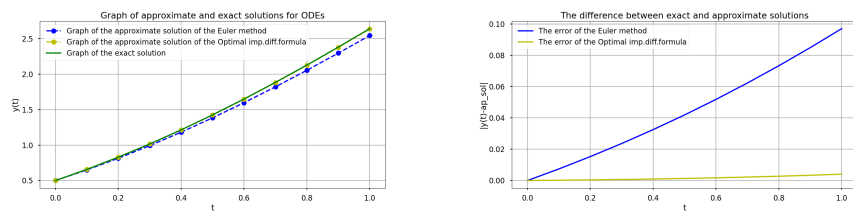


Figure 6:

Also, with the help of these methods, it is possible to solve problems in various fields of mechanics. The cited numerical results were obtained using the Python programming language.

7 Conclusion

In conclusion, In this paper, new Adams-type optimal difference formulas are constructed and exact expressions for the exact estimation of their error are obtained. Moreover, we have shown that the results obtained by the optimal explicit difference formulas constructed in the $W_2^{(2,1)}(0,1)$ Hilbert space are better than the results obtained by the Euler formula. In addition, the optimal implicit formula is more accurate than the optimal explicit formula and the effectiveness of the new optimal difference formulas was shown in the numerical results.

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