

## Integrating CAS into teaching and learning of advanced mathematics

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### Abstract

Mathematics, at its core deals with abstract concepts. However integrating technology in teaching mathematics can effectively enhance the learning experience. With graphing technologies, students can visualize and understand the concepts much more easily. In this paper we highlight the importance and use of CAS in teaching of higher mathematics. We explain with various examples from the fields of calculus and analysis that how certain topics can be better understood when classroom teaching is combined with the use of the software Mathematica.

**Keywords:** Mathematical software, CAS, Mathematica, technology, higher mathematics teaching, functions, calculus, sequences

## 1 Introduction

The use of Computer Algebra System (CAS) like Mathematica, Maple, MuPAD and Maxima etc. puts the dream of modern, interactive and effective mathematics education within our grasp. These softwares promote an interactive approach to learning, where students are not only the passive recipients but regard themselves as the active participants. Oftenly in an undergraduate mathematics classroom where the curriculum comprises of definitions, theorems, formulae and algorithms, students view doing mathematics as a routine and monotonous task. The use of CAS in classroom teaching helps to solve this problem.

A computer algebra system is any mathematical software with the ability to perform mathematical computations, including solving equations, plotting of 2D and 3D graphs, performing algebraic manipulations and many more. It is for sure that a classroom teacher plays a crucial role in introducing and

explaining a new concept or method to the students and in our opinion there is no substitute of a classroom teacher. However, few mathematical concepts and proofs involve abstract and complex approaches which can put a significant challenge for many students. In the teaching of mathematics, though chalk and board method remains a foundational approach but it has certain limitations while teaching concepts like plotting complex 2-dimensional graphs and 3-dimensional geometry. Quite often it is challenging to accurately represent 3-dimensional structures on a 2-dimensional surface, making it difficult for students to understand spatial relationships. In this paper our aim is to demonstrate how CAS, in particular the software Mathematica can be used to explain some of the abstract mathematical concepts and can serve as a teaching aid in conjunction with traditional classroom teaching. The topics or concepts we aim to demonstrate are mainly from the branch of calculus.

The paper in total comprises of five sections including the introductory section. In section 2, we intend to give a brief introduction about the mathematical software Mathematica. Sections 3 and 4 present a range of examples illustrating how this software can be integrated with conventional class room teaching to enhance the effectiveness of teaching and understanding. The concluding section of the paper presents selected examples illustrating that in certain cases, Mathematica may yield incorrect or misleading results. These instances highlight the necessity of exercising caution and applying verification techniques when interpreting the outputs generated by Computer Algebra Systems (CAS).

## 2 What is Mathematica

Mathematica is a mathematical computation program which is not only used in many scientific, mathematical, engineering, computing fields but also in various areas of humanities including social sciences, psychology etc. It provides a single integrated and ever expanding system with over 6000 built-in functions encompassing all areas of technical computing and is effortlessly accessible in the cloud through any web browser as well as natively on all modern desktop systems. It was conceived by Stephan Wolfram and is developed by Wolfram Research, an American multinational company which creates computational technology. Mathematica 1.0 was released on June 23,

1988 [7]. Recently on January 9, 2024, Wolfram announced the release of its version 14.0 [8].

### 3 Visualizing and understanding the concept of functions through the use of Mathematica

Functions play a central role in calculus and is an essential tool for understanding basic concepts of mathematics where there is a physical dependence of one quantity on the other. A function  $f$  is a rule that assigns to each element  $x$  of a set  $P$  a unique element  $y$  of a set  $Q$  [6]. While teaching the concept of functions to undergraduate students we often ask students to draw the graph of a function which further helps in deriving various other characteristics of a function. Curve tracing is an important part of calculus and in performing tracing of graph of a function, certain concepts such as increasing/decreasing behaviour of function, the first derivative test, the second derivative test, concavity and inflection points, etc. are used primarily. In this section we aim to demonstrate how using graphing utilities of Mathematica aids in studying above mentioned characteristics of functions with the help of following example.

*Example 3.1.* We analyze the graph of the polynomial  $f(x) = -38080 + 2044x$

$$+ 5573x^2 - 299x^3 - 19x^4 + x^5 \quad (3.1)$$

and find important characteristics of it using Mathematica. We see that  $f(x)$  is a polynomial of degree five and its  $y$ -intercept can be obtained by substituting  $x = 0$  in the equation (3.1). Thus we obtain that the  $y$ -intercept is -38080.

Fundamental theorem of algebra ensures that  $f(x)$  has five roots but computing the roots of  $f(x)$ , that is, its  $x$ -intercepts is an arduous task. However, the use of Mathematica makes this task easier. This can be achieved using the following commands:

input: `f[x_]:=-38080+2044 x+5573 x^2-299 x^3-19x^4 +x^5 ; Solve[f[x]==0,x]`

Executing above inputs generates the following output:

output:  $\{\{x > -17\}, \{x > 16\}, \{x > 20\}, \{x > -\sqrt{7}\}, \{x > \sqrt{7}\}\}$

This is immediate to see that the five roots of  $\sqrt{\quad} \sqrt{\quad} f(x)$  occur at the positions

$-17, 16, 20, -\sqrt{7}$  and  $\sqrt{7}$  on  $x$ -axis. We can further evaluate the limit of  $f(x)$  as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  as shown below:

input: `Limit[-38080 + 2044 x + 5573 x^2 - 299 x^3 - 19 x^4 + x^5, x -> Infinity]`

input: `Limit[-38080 + 2044 x + 5573 x^2 - 299 x^3 - 19 x^4 + x^5, x -> -Infinity]`

output:  $\infty$  output:

$-\infty$

This suggests that the graph of  $f(x)$  should increase without bound as  $x \rightarrow \infty$  and should decrease without bound as  $x \rightarrow -\infty$ . We will verify this characteristic of  $f(x)$  later in the graph of it.

For tracing of graph of  $f(x)$  we now work towards investigating the increasing/decreasing behaviour of  $f(x)$ . For this, we first calculate the critical numbers of  $f(x)$  which are obtained by equating the first derivative of  $f(x)$  to zero. This task has been accomplished by using the following code:

input: `D[f[x],x]`

output: `2044+11146x- 897x^2-76x^3+ 5x^4` input: `N[Solve[D[f[x],x]==0,x]]` output:

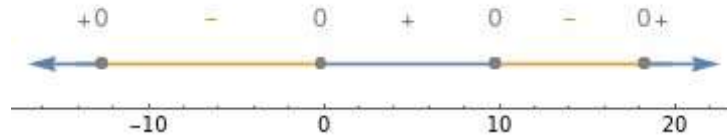
$\{\{x > -12.6641\}, \{x > -0.180794\}, \{x > 9.76997\}, \{x > 18.275\}\}$  In the above output, we find that first derivative of  $f(x)$  has four real roots and they occur approximately at the positions  $x = -12.6641, -0.180794, 9.76997$  and  $18.275$  (we note here that while these values are not exact but for the purposes of this analysis, we will proceed with these available approximations). In order to continue our investigation of increasing/decreasing behaviour of  $f(x)$ , we now check the sign pattern and sign graph of  $f'(x)$  followed by plotting the graph of  $f(x)$ , with the help of following codes and obtain Figures 1, 2 and 3 respectively.

input: `ResourceFunction["FunctionSigns"][D[f[x],x], x]` output:

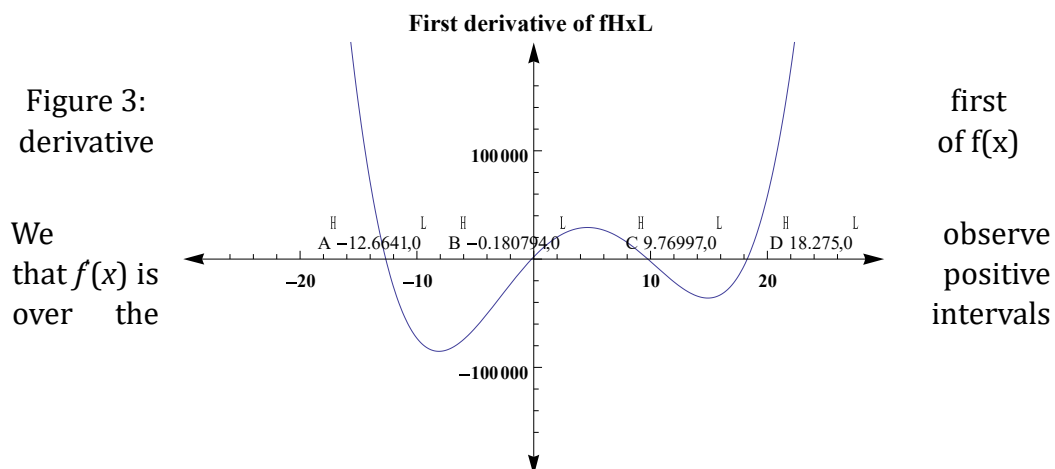
$\langle \text{Positive} \rightarrow \{x < -12.7 \dots \mid -0.181 \dots < x < 9.77 \dots \mid x > 18.3 \dots\},$   
 $\text{Negative} \rightarrow \{-12.7 \dots < x < -0.181 \dots \mid 9.77 \dots < x < 18.3 \dots\},$   
 $\text{Zeros} \rightarrow \{x = -12.7 \dots \mid x = -0.181 \dots \mid x = 9.77 \dots \mid x = 18.3 \dots\} \rangle$

Figure 1: sign pattern of  $f'(x)$ 

input: ResourceFunction["FunctionSigns"][D[f[x],x], x, "SignChart"] output:

Figure 2: sign chart of  $f'(x)$ 

input: Plot[2044 + 11146\*x - 897\*x^2 - 76\*x^3 + 5\*x^4, {x, -25, 25},  
 PlotRange -> {{-30, 30}, {-200000, 200000}},  
 AxesStyle -> Arrowheads[{-0.03, 0.03}],  
 PlotLabel -> "First derivative of f(x) ",  
 Epilog -> {Text["A(-12.6641,0)", {-13.84, 20000}],  
 Text["B(-0.180794,0)", {-2.3, 20000}],  
 Text["C(9.76997,0)", {12, 20000}],  
 Text["D(18.275,0)", {24, 20000}]],  
 LabelStyle -> Directive[Bold, Black]]  
 output:



$(-\infty, -12.6641)$ ,  $(-0.180794, 9.76997)$  and  $(18.275, \infty)$ . Hence, the graph of  $f(x)$  must be increasing over these intervals and  $f'(x)$  is negative over the intervals

$(-12.6641, -0.180794)$ ,  $(9.76997, 18.275)$ , indicating the decreasing nature of graph of  $f(x)$  over here. This too will be verified later in the graph of  $f(x)$ . We can see that there is a change of sign of  $f'(x)$  at  $x = -12.6641$  and  $x = 9.76997$  from positive to negative; at  $x = -0.180794$  and  $x = 18.275$  from negative to positive. Therefore, using the first derivative test we can conclude that all these points are relative extremum points of  $f(x)$ . More specifically, the points  $A(-12.6641, 0)$  and  $C(9.76997, 0)$  will be relative maximum points while the points  $B(-0.180794, 0)$  and  $D(18.275, 0)$  will be relative minimum points.

We now proceed further and perform second derivative test analysis of  $f(x)$ . We first calculate the inflection points of  $f(x)$  by equating  $f''(x)$  to zero.

input:  $N[\text{Solve}[D[f[x], \{x, 2\}] == 0, x]]$  output:  $\{\{x \rightarrow -8.11531\}, \{x \rightarrow 4.60604\}, \{x \rightarrow 14.9093\}\}$  Thus, we see that  $f(x)$  has inflection points approximately at  $x = -8.11531$ ,  $4.60604$  and at  $x = 14.9093$ . This help us to know that the behaviour of concavity of  $f(x)$  changes at these positions. Like in the first derivative case, here we again check the sign pattern of  $f''(x)$  and plot the graph of it to obtain Figures 4, 5 and 6.

Positive  $\rightarrow \{ -8.12... < x < 4.61... \mid x > 14.9... \}$ ,  
 Negative  $\rightarrow \{ x < -8.12... \mid 4.61... < x < 14.9... \}$ ,  
 Zeros  $\rightarrow \{ x = -8.12... \mid x = 4.61... \mid x = 14.9... \}$

Figure 4: sign pattern of  $f''(x)$

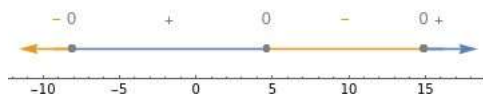
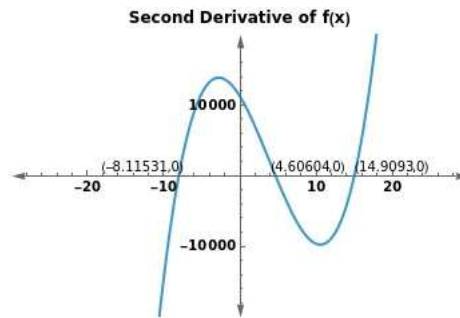


Figure 5: sign graph of  $f''(x)$

Figure 6: second derivative of  $f(x)$ 

We see that  $f''(x)$  is positive over the intervals  $(-8.11531, 4.60604)$  and  $(14.9093, \infty)$ . So, the graph of  $f(x)$  should be concave upwards over these intervals. Further,  $f''(x)$  is negative over the intervals  $(-\infty, -8.11531)$  and  $(4.60604, 14.9093)$ ; therefore the graph of  $f(x)$  should be concave downwards over these intervals. Thus we see that there is a change of concavity at  $x = -8.11531$ ,  $4.60604$  and at  $x = 14.9093$ .

Furthermore, we calculate the value of  $f''(x)$  at the critical numbers of  $f$  using the following code:

```
input: Map[D[f[x],{x,2}],{-12.6641,9.76,-0.180794,18.275}] output: {-43322.3,-9487.89,11462.8,24282.3}
```

The output above and the second derivative test reassure that the points  $x = -12.6641$  and  $9.76$  correspond to the points of relative maxima while the points  $x = -0.180794$  and  $18.275$  correspond to the points of relative minima for  $f$ .

We finally plot the graph of  $f(x)$  and confirm all our observations derived from the analysis of behaviour of  $f'(x)$  and  $f''(x)$ .

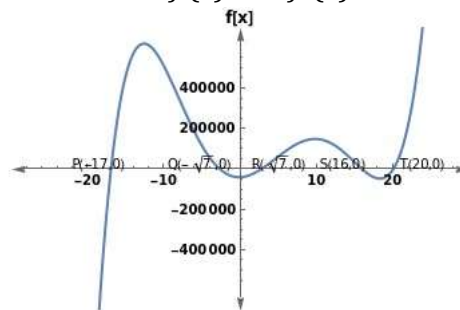


Figure 7: graph of  $f(x)$ 

We conclude our section with the plotting of graph of  $f$  and its first derivative on one set of axes and the graph of  $f$  and its second derivative together on one set of axes in the following figures.

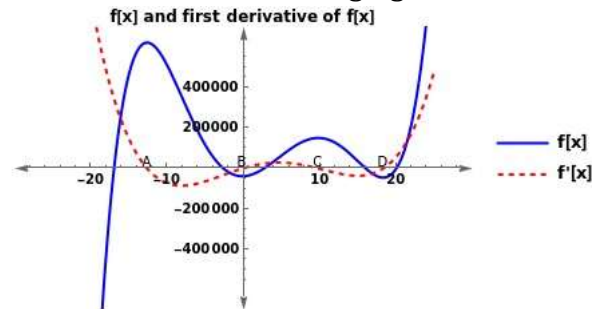
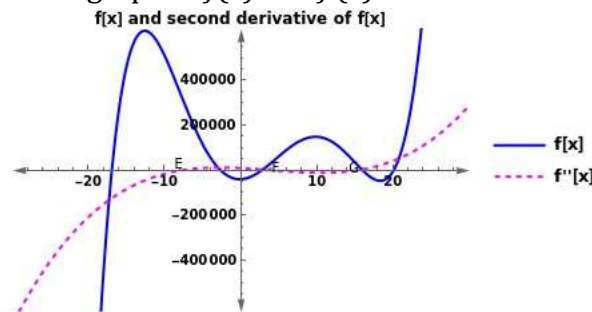
Figure 8: graph of  $f(x)$  and  $f'(x)$  on same set of axesFigure 9: graph of  $f(x)$  and  $f''(x)$  on same set of axes

Figure 8 and Figure 9 help us to analyze the behaviour of  $f(x)$  and its derivatives simultaneously.

## 4 Sequences

In this section, we aim to demonstrate how Mathematica can help in understanding the concept of convergence of sequences of real numbers. A real sequence/sequence is a function whose domain is  $\mathbb{N}$ , the set of natural numbers and range is contained in  $\mathbb{R}$ , the set of real numbers [1]. An

undergraduate student is often taught the concept of convergence of a sequence. Below is the formal definition of convergence of a sequence.

*Definition 4.1.* [1] A sequence  $\{a_n\}$  is said to converge to a real number  $l$ , if for each  $\epsilon > 0$ , there exists a positive integer  $m$  such that

$$|a_n - l| < \epsilon \quad \text{for all } n \geq m.$$

In a classroom setting, a student generally find this concept abstract, challenging and rigorous. However the graphical representation of a sequence with the use of Mathematica facilitates the student to visualize the sequence and its behaviour. We elaborate our point of view with the help of some examples and illustrate their convergence through plotting their corresponding points.

*Example 4.2.* We consider the sequence  $\{a_n\}$ , where

$$a_n = \frac{n}{e^n}.$$

We plot the above sequence using the following code:

input:

```
ListPlot[Table[n/Exp[n],{n, 1, 50}],Ticks->{Range[0, 50, 3],All},
PlotRange -> {{-6,54},{-0.1,0.5}},
AxesStyle-> Arrowheads[{-0.03,0.03}],
PlotStyle -> {Red}, PlotLabel -> "Sequence {(n)/(Exp[n])}",
LabelStyle -> Directive[Bold, Black]]
```

With the help of above code we obtain Figure 10.

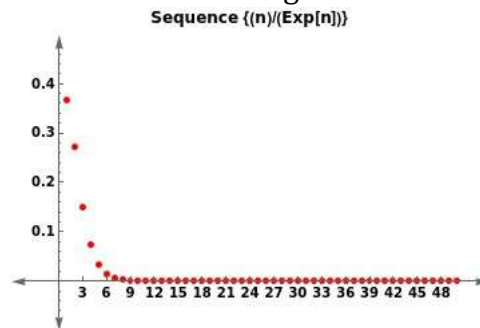


Figure 10: sequence  $\frac{n}{e^n}$

We observe in the above figure that the value of  $a_n$ , that is, the terms of this sequence are approaching towards zero as  $n$  is increasing. This gives an idea to the students that the limit of the above sequence could possibly be zero. We can further evaluate the values of some terms of the sequence which is manually difficult and cumbersome to compute. We achieve this task in the following code:

```
input:N[Table[(n)/(Exp[n]), {n, 1, 20}]]
output: 0.367879,0.270671,0.149361,0.0732626,0.0336897,0.0148725,
0.00638317,0.0026837,0.00111069,0.000453999,0.000183719,
0.0000737305,0.0000293843,0.0000116414,4.58853\times10^-6,
1.80056\times10^-6,7.03789x\times10^-7,2.7414\times10^-7,
1.06453\times10^-7,4.12231\times10^-8
```

We see that the numerical values of first 20 terms of the sequence are becoming closer to zero. This further strengthens the idea of taking the possible value of  $l$  in the Definition 4.1 for this given sequence as zero.

Next, we analyze the behaviour of following sequence.

*Example 4.3.* Consider the sequence  $\{b_n\}$  where,

$$b_n = \frac{10^n}{n!}.$$

Let us compute numerical values of some terms of this sequence using the following code.

```
input:N[Table[((10)^n)/Factorial[n],{n,1,30}]]
output:10.,50.,166.667,416.667,833.333,1388.89,1984.13,2480.16,
2755.73,2755.73,2505.21,2087.68,1605.9,1147.07,764.716,477.948,
281.146,156.192,82.2064,41.1032,19.5729,8.89679,3.86817,1.61174,
0.644695,0.24796,0.0918369,0.0327989,0.01131,0.00376999
```

Here, we have calculated first 30 terms of this sequence, but these terms do not exhibit any regular pattern or approach towards a definite number. Hence, the behaviour of a sequence is unpredictable just by calculating few initial terms. To get more accurate idea, we plot the terms of  $\{b_n\}$  for larger

values of  $n$ , in Figure 11 and gets an idea that the terms of the sequence  $\{b_n\}$  eventually converge to zero.

input:

```
ListPlot[Table[((10)^n)/Factorial[n], {n, 1, 50}], Ticks -> {Range[0, 50, 3],
Range[0, 10000, 200]}, AxesStyle -> Arrowheads[{-0.03, 0.03}],
PlotLabel -> "Sequence {((10)^n)/n!}",
PlotRange -> {{-8, 54}, {-1000, 3500}},
PlotStyle -> {Red}, LabelStyle -> Directive[Bold, Black]]
```

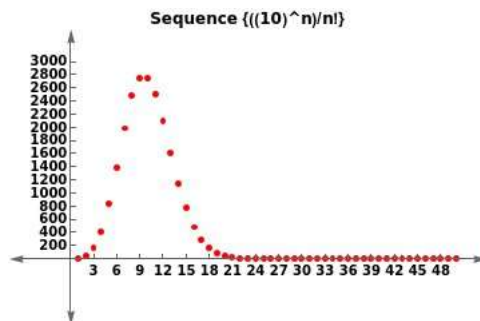


Figure 11: sequence  $\frac{10^n}{n!}$

In the following example we study a sequence which exhibits non-convergent behaviour.

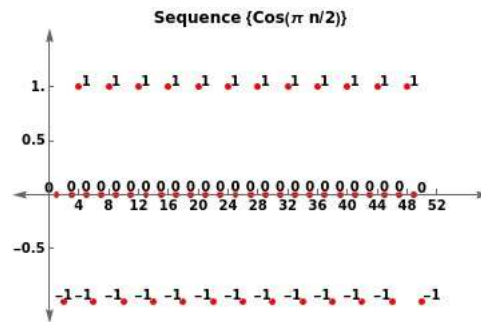
*Example 4.4.* Let us consider the sequence  $\{a_n\}$  where

$$a_n = \cos\left(\frac{n\pi}{2}\right).$$

Analytic examination of this sequence yields the following information:

1. For odd values of  $n$ , the value of  $a_n$  is 0 and,
2. for even values of  $n$ , that is, when  $n = 2m$  ( $m \in \mathbb{N}$ ), the value of  $a_n$  is  $-1$ ; if  $m$  is odd and it is 1 if  $m$  is even.

However plotting the graph of the sequence (see Figure 12) provides an immediate idea to the students regarding three subsequences of the given sequence and hence appreciating the oscillatory nature of the sequence.

Figure 12: sequence  $\cos\left(\frac{n\pi}{2}\right)$ 

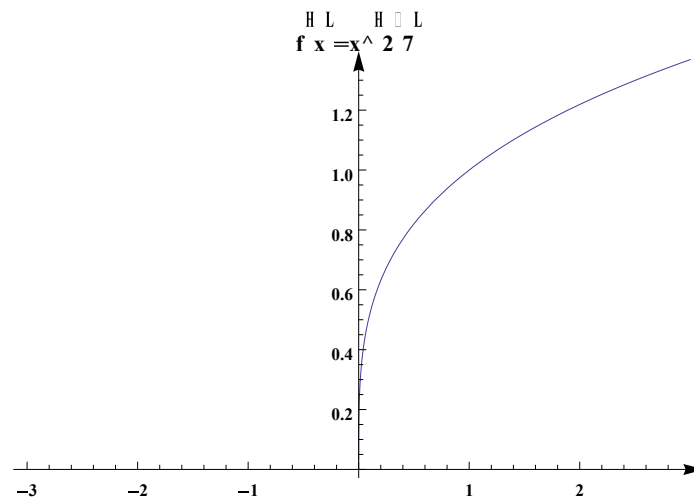
In all of the above examples, we do not undervalue the importance of abstract and analytical mathematical methods. Our aim is to highlight how the usage of computing technology can enhance the appeal of an abstract approach and thus generating greater interest and curiosity in the minds of the students.

## 5 A word of caution

While mathematica is very useful in understanding and visualizing mathematical concepts, it still has certain limitations. Most graphing softwares employ logarithms to compute functions with fractional exponents. For instance,  $f(x) = x^{\frac{2}{7}}$ . When we plot the graph of this function for  $x$  in  $[-3, 3]$  in mathematica, we get the following plot (see Figure 13):

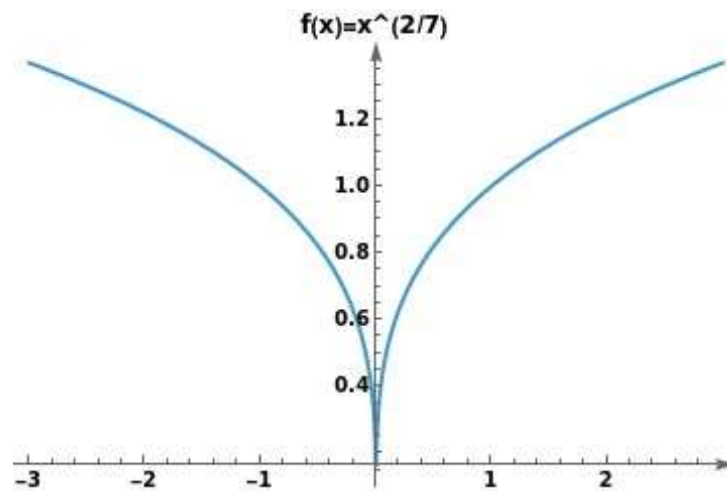
input: `Plot[x^(2/7),{x,-3,3},AxesStyle->Arrowheads[{0.0,0.03}], PlotLabel ->`

`"f(x)=x^(2/7)", LabelStyle -> Directive[Bold, Black]]` output:

Figure 13: incorrect graph of  $f(x) = x^{2/7}$ 

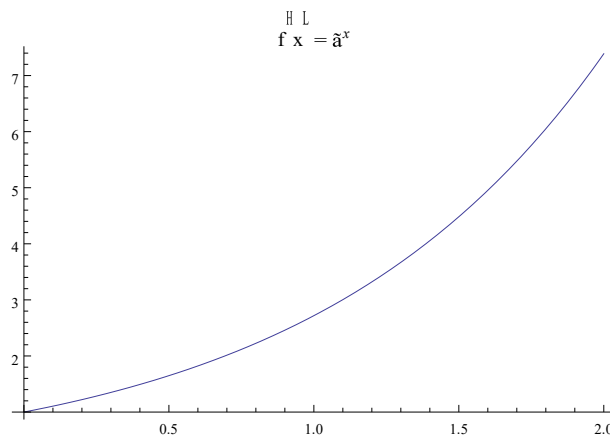
Since logarithms are defined only for positive values so mathematica omit portion of the graph for nonpositive values of  $x$ . So we see that the caution must be exercised to plot such graphs. The correct plot of this function has been drawn using the following command:

```
input: Plot[(Abs[x])^(2/7),{x,-3,3},AxesStyle->
Arrowheads[{0.0,0.03}],
PlotLabel -> "f(x)=x^(2/7)", LabelStyle->Directive[Bold, Black]] output:
```

Figure 14: correct graph of  $x^{2/7}$ 

Thus we see that the implementation of technology alone is not sufficient to explain all mathematical concepts; a thorough mathematical analysis alongwith usage of CAS is necessary in order to obtain accurate results.

To illustrate another instance where caution is necessary we consider the plot of the exponential function over the interval  $[0, 2]$ . Writing the following code generates the graph in Figure 15. input: `Plot[Exp[x], {x, 0, 2}, PlotLabel->"f(x) = e^x"]`

Figure 15:  $f(x) = e^x$

In the above graph, one gets an impression that the value of  $e^x$  is zero at  $x = 0$ . However this is not the case. In the above figure the intersection point of  $x$ -axis and the  $y$ -axis is not  $(0,0)$ , giving the false impression of value being 0 at  $x = 0$ . We redraw the graph of the exponential function using the following command and obtain Figure 16.

```
input: Plot[Exp[x], {x, 0, 2}, AxesOrigin -> {0, 0},
PlotLabel-> "f(x) = e^x"]
```

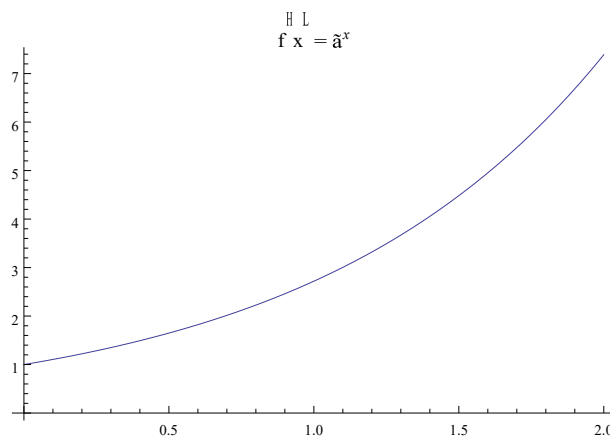


Figure 16:  $f(x) = e^x$

*AxesOrigin* command in mathematica is an option for graphing functions which specifies where axes should intersect. Using the command *AxesOrigin*  $\rightarrow \{0,0\}$ , ensures that the  $x$  and  $y$  axes cross at  $(0,0)$ .

Thus we can say that mathematica can definitely suggest shapes of the graphs but cannot determine them with full accuracy and there are instances where the graph may be misinterpreted. However, at the same time we see that it plays a crucial role in deepening students' understanding and sparking their interest. Therefore, we conclude by saying that the integration of technology and tools like Mathematica along with mathematical knowledge is needed to significantly enrich the teaching and learning experience, providing an effective and dynamic approach to mathematics education and to enhance students engagement and interactive visualizations.

## Declarations

### Data availability statement

We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

### Competing and financial interest

The authors have no relevant financial or non-financial interests to disclose.

### Author contribution information

Both authors contributed equally to this work.

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