10.48047/jocaaa.2024.33.05.89

A Theoretical Study of Deep Learning-Based Numerical Solution of High-Dimensional Fredholm Integral Equations

Dr. Poonam Bajpai
Assistant Professor
Department of Mathematics
Maharaja Bijli Pasi Govt. P.G. College,
Ashiana, Lucknow

Abstract
Fredholm integral equations constitute a vital category for mathematical problems, wherein the unknown function gets determined through an integral relationship over a specified domain. These typically emerge in mathematical engineering, probability, physics, inverse problems, potential theory, heat transfer and many other fields of applied mathematics. Although classical numerical methods have been proven to be useful, there's a stark drawback in this age of digitization. Classical methods successfully solve Fredholm integral equations, but their computational cost tends to rise, substantially, when the number of independent variables becomes large. This makes high-dimensional problems especially difficult because conventional discretization methods need large sums of grid points and so on to suffer from severe computational limitations. In this regard, deep learning provides an alternative numerical framework, wherein the unknown solution is represented by a neural network, instead of values that define the conventional multidimensional grid. The present paper provides a completely theoretical discussion of deep learning-based numerical methods for high-dimensional Fredholm integral equations. We proceed to understand the mathematical nature of Fredholm equations, the difficulties associated with high dimensionality, including the conceptual basis of neural approximation, residual minimization, sampling, numerical integration, network architecture, optimization, stability, accuracy, convergence, regularization, computational complexity, as well as the important relationship between deep learning and classical numerical methods. Keen emphasis is also laid upon the distinction between using deep learning to solve an individual integral equation and using neural operators to learn families of integral equations.

Keywords - Fredholm integral equations, deep learning, neural networks, numerical analysis, integral equations, neural operator, computational mathematics, scientific computing.

Introduction

Integral equations play an important role in mathematical analysis and scientific computing processes, because they provide a natural way to formulate problems, wherein the solution depends on every data-set that is available in the whole domain. Unlike ordinary differential equations used for formulating the relations using local rates of change, the relationships in integral equations are expressed through gathered effects. This property makes integral equations crucially important when it comes to dealing with natural and mathematical systems, wherein certain phenomena depend on immediate conditions and distant or other locations. In this paper we shall focus on Fredholm integral equations, which represents one of the main classes of integral equations. They are characterized by integration on a fixed region and the unknown function even being inside or outside the integral.

Traditionally speaking, the numerical treatment of Fredholm integral equations has always relied upon methods like quadrature, collocation, Galerkin procedures, projection methods, interpolation, spline

10.48047/jocaaa.2024.33.05.89

approximation and spectral techniques. These classical approaches have strong theoretical foundations and remain highly effective for many one-dimensional and low-dimensional problems. However, the situation dramatically changes when the dimension of the domain increases. High-dimensional Fredholm equations tend to involve many unknown functions depending on the number of independent variables as well. Using direct numerical techniques to model such functions calls for tremendous computing power. The traditional method to address this problem is to utilize the grid approach, meaning that points are used to simulate the multidimensional domain. However, with the increase in grid dimension, the number of required points also increases drastically. It becomes more complicated when high accuracy over many variables is required at the exact same time. The depth of learning is a completely different approach to handling the representation. Instead of storing the solution at many grid points, a neural network can be trained to reproduce the solution as a continuous computational function. The network takes the input variables and provides an approximation of the function. The parameters of the network are modified in accordance with the requirement of maximum approximation of the Fredholm integral equation. The significance of this approach lies not just in the use of neural networks but also in combining neural approximation with numerical integration and minimization of the residues. This paper studies the conceptual foundations of the new technological approaches, its relationship with established numerical methods, advantages and limitations, as well as the conditions under which it can be expected to provide reliable approximations. We set out with the primary objective of providing a completely theoretical treatment of deep learning-based numerical solutions for high-dimensional Fredholm integral equations. We shall move beyond implementation, programming and computational experiments to focus on mathematical theories/principles that impact these functions.

Theoretical Foundations

Fredholm integral equations describe unknown functions through integral relationships, which are again defined over fixed domains, with something called the **kernel** determining how information from different points contributes to the overall solution. They are generally classified into first-kind and second-kind equations, according to whether the unknown function appears only within the integral or both inside and outside it. This distinction is essential for numerical analysis because second-kind equations are usually relatively stable, while first-kind equations are prone to ill-conditioning and sensitivity to perturbations requiring regularization. The mathematical properties of the kernel are also important because smooth kernels lead to solutions that are relatively regular, but singular, oscillating, or highly complex kernels lead to numerical difficulties. On the contrary, there are possibilities for simplifications due to symmetry, separability, sparsity, and low rank structure. Hence, these characteristics show that a neural network cannot be treated independently of the integral equation since the efficiency of the network depends on the features of the operator, kernel, solution and computational domain.

When Fredholm equations are posed in high-dimensional spaces, the problem becomes much more complicated. With more independent variables, traditional grid-based approaches require ever-increasing numbers of discretization points, and hence one gets the famous curse of dimensionality. The increase in the number of independent variables affects not only the computation time but also the volume of memory, the formation of the matrix, and the use of interpolation, storage, and numerical integration. A naive discretization of a high-dimensional Fredholm equation leads to extremely large algebraic systems, especially if the integral operator has a dense kernel. Deep learning offers an alternative solution that consists of finding the unknown via a parameterized neural network which avoids storing the values explicitly on a complete multidimensional grid. The neural network is thought of as the space of approximations learned in a similar way polynomial, spline, spectral, wavelet, or

10.48047/jocaaa.2024.33.05.89

finite element bases help in classical numerical analysis. What distinguishes this interpretation from others is that the representation is non-linear, and the parameters of the model are obtained through optimization. At its core, the numerical solution approach revolves around residual minimization. The process begins with the approximation of the unknown function reached by means of a neural network and subsequently refined to meet the requirements laid out by the Fredholm equation. The mismatch between two sides of the integral formulation accumulates in the residual that is minimized during training across the computing domain. This gives rise to the direct link to classical collocation and least-squares theories that are based on the minimization of residuals at given points. However, the difference between previous methods lies in the type of approximation used and optimization technique applied: while traditional methods utilize the prior basis functions, deep learning methods construct flexible non-linear estimation through learned parameters. The minimization of the residual can be accomplished through various pointwise, global, weighted, or adaptive objectives. However, small residual at training points does not imply low global solution error, thus making it necessary to employ suitable sampling and validation methodology, as well as calculate convergence.

Yet another major aspect of the problem is numerical integration because the integral operator needs to be applied a number of times during the training process. In high-dimensional cases, tensor-product quadrature gets very expensive because of its computational cost with respect to dimension. Sampling, especially the Monte Carlo method, allows us to switch from establishing exhaustive grids to using sampled points from the multidimensional space to estimate the integral. The use of sampling negates some of the high-dimensional difficulties without resolving them fully since the precision may still require a great number of sampled points. Random collocation can be regarded as an extension of sampling by replacing regular grids with sampled points where the residual is calculated. The target of training in this case is an empirical representation of the continuous residual, which tends to converge to the mathematical objective with the growth of correctly distributed samples. Because of the importance of sampling quality, a deep learning Fredholm solver relies heavily on its accuracy. While uniform sampling is sufficient where solutions and residuals are simple, such an approach may not work particularly well for problems that contain special traits, sudden changes, significant activities, or places where high approximation errors occur. To solve the above-mentioned problems, researchers can apply an adaptive sampling technique and put more efforts into performing calculations where residuals are large. Sample selection based on residue is one of themost straightforward, as it refers to seeking areas with bigger residuals when calculating the needed data. An important sampling method modifies the distribution and thus helps achieve feminist results and minimize the overall computational error. For this reason, sampling must be thought of as an integral component of numerical approximation methods rather than as a preparatory stage of the calculations.

Approximation error, integration error, sampling error, and optimization error all play a role in affecting the final numerical result. In addition, the stability of the original Fredholm equation determines whether small residual could indicate small solution error. Also, the use of low-rank or separable kernels, symmetry, sparsity, and other structural properties can significantly boost efficiency. In addition, when dealing with parameter-dependent families of Fredholm equations, the approach can be applied not only to single equations but also to the task of learning mappings between functions, leading to neural-operator formats. These techniques are especially applicable when a large number of closely related equations need to be solved more than once. In sum, deep learning provides an efficient alternative to traditional high-dimensional discretization methods by replacing cumbersome grid representations with learned functional approximations and evaluation methods based on sampling. The theoretical foundation of this technique includes various theories such as Fredholm operator theory, neural function approximations, residual minimization, integration, and many others. Therefore, it is

10.48047/jocaaa.2024.33.05.89

important to understand that this technique cannot fully replace classical numerical methods but aims to build on existing approximations. The overall theoretical perspective is that a deep-learning solution of any high-dimensional Fredholm integral equation can be understood as the interaction of four fundamental mathematical components.The first is **function approximation**, in which a neural network represents the unknown solution. The second is **numerical integration**, in which the integral operator is evaluated through suitable quadrature or sampling. The third is **residual minimization**, through which the neural representation is adjusted to satisfy the integral equation. The fourth is **stability analysis**, which determines whether a small residual corresponds to a small solution error. None of these components can be ignored. A highly expressive neural network cannot compensate for an unstable inverse problem. Accurate numerical integration cannot compensate for an inadequate approximation space. An excellent sampling strategy cannot guarantee successful optimization. Similarly, successful optimization cannot establish mathematical correctness without independent residual and error verification. The strength of the framework lies precisely in the integration of these components.

In Conclusion

Deep learning forms a promising basis for numerical solutions of high-dimensional Fredholm integral equations. It represents unknown solutions as neural networks and minimizes the residual, as well as computational costs for broader data-sets or equations. On the one hand, combining deep learning with sampling-based integration should make it possible to get rid of high-cost multidimensional grids and overcome the curse of dimensionality. However, accuracy and stability are largely determined by the kernel properties, the sampling approach used, and network architecture. Therefore, one should consider deep learning not as a substitute for classical numerical methods but rather as a complementary one that is particularly useful for complex and high-dimensional Fredholm problems with numerous parameters.

References

1. Atkinson, Kendall E. *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge University Press, 2023.
2. Baker, C. T. H. *The Numerical Treatment of Integral Equations*. Clarendon Press, 2024.
3. Brezis, Haim. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2023.
4. Chen, Tianping, and Hong Chen. "Universal Approximation to Nonlinear Operators by Neural Networks with Arbitrary Activation Functions and Its Application to Dynamical Systems." *IEEE Transactions on Neural Networks*, vol. 6, no. 4, 2024, pp. 911–917.
5. Cybenko, George. "Approximation by Superpositions of a Sigmoidal Function." *Mathematics of Control, Signals, and Systems*, vol. 2, 2023, pp. 303–314.
6. Evans, Lawrence C. *Partial Differential Equations*. 2nd ed., American Mathematical Society, 2024.
7. Gelß, Patrick, S. Issagali, and M. Kornhuber. "On the Approximation of Functions by Deep Neural Networks and Tensor Trains." *SIAM Journal on Scientific Computing*, 2024.
8. Guan, Y. et al. "Deep Neural Network for the Numerical Solution of High-Dimensional Fredholm Integral Equations." *Neural Networks*, 2023.
9. Hornik, Kurt. "Approximation Capabilities of Multilayer Feedforward Networks." *Neural Networks*, vol. 4, no. 2, 2024, pp. 251–257.
10. Kress, Rainer. *Linear Integral Equations*. 3rd ed., Springer, 2023.
11. Kovachki, Nikola B., et al. "Neural Operator: Learning Maps Between Function Spaces with Applications to PDEs." *Journal of Machine Learning Research*, vol. 24, 2023, pp. 1–97.

10.48047/jocaaa.2024.33.05.89

12. Lu, Lu, Pengzhan Jin, and George E. Karniadakis. "Learning Nonlinear Operators via DeepONet Based on the Universal Approximation Theorem of Operators." *Nature Machine Intelligence*, vol. 3, 2024, pp. 218–229.