Power Dominator Chromatic Number of Cartesian and Indu-Bala Product of Some Graphs

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ABSTRACT
A graph G with a power dominator coloring is a proper coloring that ensures that every vertex has power dominates over every other vertex in a specific color class. This paper focuses on determining the power dominator chromatic number for the Cartesian and Indu-Bala product of various graph classes, such as complete graphs, paths and Cycle, paths and star graphs as well as paths and wheel graphs.

Keywords: Power dominator chromatic number, Cartesian product, Indu-Bala Product

1. INTRODUCTION

The problems of domination theory and graph coloring are highly interesting within the realms of graph theory, algorithms, and combinatorial optimizations. In the applied sciences, both areas are enhanced by a broad depth of research. We refer [[9],[13],[14],[15]] accordingly for comprehensive results on coloring and domination in graphs. The concept of domination is associated with a dominating set, which needs to consist of the smallest possible number of vertices. This set ensures that every vertex in a graph not included in the set has at least one neighboring vertex within it.While coloring the vertices in a network requires applying distinct colors to each one such that the two end vertices of edges are colored differently. Domination problems and graph colouring problems are frequently related to one another.
The graph G=(V(G), E(G)) is simple, connected, undirected, and finite, with a vertex set of V(G) and an edge set E(G).Graph coloring is the well-known process of assigning colors to the vertices of a graph in such a way that adjacent vertices do not have the same color. The function g : $(G) \rightarrow \{1,2, ... , m\}$ is considered a proper m-coloring of a graph G if g(x) ≠ g(y) for all x y $\in E(G)$, where x and y are adjacent vertices in G.The more compact set of colors for which such an assignment can be done is the chromatic number $\chi(G)$ of a graph G.The group of vertices sharing the same color is referred to as the color class. If x is an element of G, the open neighborhood is the set N(x) which includes all y in V(G) such that xy is an edge in G, and the closed neighborhood is the set N[x] which includes N(x) as well as x. All vertices in a graph G that are neighbors of at least one element of S or that are elements of S are referred to as dominating sets. The domination number $\gamma(G)$ represents the smallest number of vertices in a dominating set. If each vertex in the graph dominates every other vertex in the same color class, the coloring is known as dominator coloring.The dominator chromatic number, represented by $\chi_d(G)$, is the minimal cardinality of colors utilized in the graph for dominator coloring. For comprehensive results on dominator coloring we refer [[2],[9],[10],[13]]respectively.
Electricity providers must regularly monitor the condition of their system based on many state variables, including the phase angle of the generator machine and the voltage at loads. Placing phase measuring units (PMU's) at specific locations within the system is one method for monitoring these variables. Reducing the number of PMU's while maintaining the capability to watch (monitor) the whole system is desirable due to their high cost. Finding the minimum number of PMU's needed to continuously monitor the entire system is a problem in graph theory that has strong ties to the prominent vertex covering and domination problems. Haynes et al. [15] introduced a modified version of graph domination, known as power domination, to address the issue of monitoring electric power system states. Power dominator coloring of a graph G is a new kind of coloring that was transplanted in [7] and is more precisely discussed here. It is built upon the concepts of coloring and power domination. Consider a connected graph G and let S be a subset of its vertices. The set being monitored by S is represented by M(S), and the algorithmic outline for this process is as follows:
1. M(S) ← S ∪ N(S)(domination)

2. As long as there is some element u belonging to the set M(S)such that
N(u) ∩ (V (G) − M(S)) = {x},set M(S) ← M(S) ∪ {x}(propagation).
Thus, the set M(S) can be acquired as follows from S. First, add the vertices to M(S) from its closed neighbours .Subsequently, add vertices w to M(S) so many times that every neighbour of v in M(S) is already in M(S).The set observed by S has been built once there doesn't appear to a vertex w of that kind.A set S is defined as a power dominating set of G if M(S) is equal to V(G), and the smallest size of a power dominating set is denoted by $\gamma_p(G)$,the power domination number. The creation of a new concept, known as power dominator coloring, involves the integration of power domination and coloring conceptsrepresented by$\chi_{pd}(G)$, requiring for each vertex to power dominate every other vertex in a color class. Satheeshkumar et al. introduced the concept of a power dominator chromatic number, and they analyzed the power dominator chromatic number of various graph families in [7].A. Uma Maheswari [1] has examined the power dominator chromatic number for certain unique graphs as well. I.Chandramani et al. [4]has also explored power dominator chromatic number ofJahangir and associated graphs.
The graphs G_1 and G_2 used to find the Cartesian product, indicated by $G = G_1 \square G_2$, has V(G) = V(G_1) × V(G_2) = {(x_1, x_2) | x_i∈ V(G_i) for i = 1, 2}, andtwo vertices (u_1, u_2) and (v_1, v_2) in G are connected only if u_1 = v_1 and u_2v_2 is an edge in E(G_2), or u_2 = v_2 and u_1v_1∈ E(G_1).Power domination of the Cartesian product of graphs was derived by K.M. Koh [6].
The Indu-Bala product of two graphs was given by G. Indulal and R. Balakrishnan [3]. This graph product is predicted on classic loop switching problem analyzed by Graham and Pollack [12]. The classic loop switching problem was used for wireless communication network theory. The Indu-Bala product G_1▼G_2 of graphs G_1 and G_2 is formed by combining two disjoint copies of the join G_1∨ G_2 where the corresponding vertices of two copies of G_2 are connected by an edge. It is obvious that |V (G_1▼G_2)| = 2|V (G_1∨ G_2)| = 2 (n_1 + n_2) and that |E (G_1▼G_2)| = 2|E(G_1∨ G_2)| + n_2 = 2 (q_1) + 2 (q_2) + 2 (n_1n_2) + n_2. M. Priyadharshini et al. [11] investigated the independent strong domination number of the Indu-Bala product of various distinct graphs.After reviewing the existing literature, we have discovered the power dominator coloring of certain graphs formed by the Cartesian and Indu-Bala products.

2. Power Dominator Chromatic Number of Carteasian Product of Some Graph

Theorem 2.1

For any integer $m > n \geq 2$, $\chi_{pd}(K_m \square K_n) = \begin{cases} m, & if\ n = 2 \\ m+n-1, & if n \geq 3 \end{cases}$.

Proof

Assume that the graph G= $K_m \square K_n$ has a vertexset $V = \{(i,j): 1 \leq i \leq m\ and\ 1 \leq j \leq n\}$.
If we take n=2 we get K_2 which is a path of order 2. So G contains 2 cliques of order m. Thus we have $\chi_{pd}(G) \geq m$.For a Latin rectangle $B = (\beta_{ij})$of order $m \times 2$of the set {1,2,3,....,m} define a function g on V for each vertex $(i,j), g(i,j) = \beta_{ij}$. The mapping g, which is power dominator coloring in G with m color classes, indicates that$\chi_{pd}(G) = m$.Our next assumption is $n \geq 3$. Let $g = \{v_1, v_2, \ldots, v_p\}$ be a arbitrary power dominator coloring in G. Let $S = \{i: |v_i| = 1\}$ be a set of cardinality p. We assume that $p < m + n - 1$.
If p=0, then $|v_i| \geq 2$ for each i and $p \geq m + n - 1$. Let $1 \leq s \leq m - 1$. There exist a K_n- layer of $\{i\} \times K_n$ such that no vertices is in $T = \bigcup_{i \in S} V_i$. Consider$C_1, C_2, \ldots, C_n$are n new colorsnecessary to color $\{i\} \times K_n$.If the condition is $n \geq 3$, it means that the color classes$C_1, C_2, \ldots, C_n$has cardinality more than 2. To color the remaining vertices, there is a requirement of at-least m-1 new colors. Thus $p \geq s + m + n - 1 \geq m + n - 1$.
If $m \leq s \leq m + n - 2$, then, according to the principle of pigeonhole, there exist a K_n- layer of $\{j\} \times K_n$ such that and to color$(\{j\} \times K_n) \cap (V - T)$, it is necessary to have at least n-1 new color. So we get $p \geq m + n - 1$. Since we assumed that g is an arbitrary power dominator coloring in G, we get $\chi_{pd}(G) \geq m + n - 1$.Let $B = (\beta_{ij})$ be a Latin rectangular of order $(m-1) \times n$ on the set $\{1,2,3, \ldots, m-1\}$.A function g on V is defined so that for a vertex (i, j),

$$g(i,j) = \begin{cases} j + m - 1, & i = 1 \\ \beta_{i,j}, & otherwise \end{cases}.$$

Thus, the function g is a power dominator coloring in G with n+m-1 color classes. This gives that $\chi_{pd}(G) \leq m + n - 1$. Thus $\chi_{pd}(G) = m + n - 1$.

Observation 2.1.1

For any integer $n > 2$, $\chi_{pd}(K_n \square K_n) = 2n - 1$ can be shown by taking m = n in theorem 2.1.

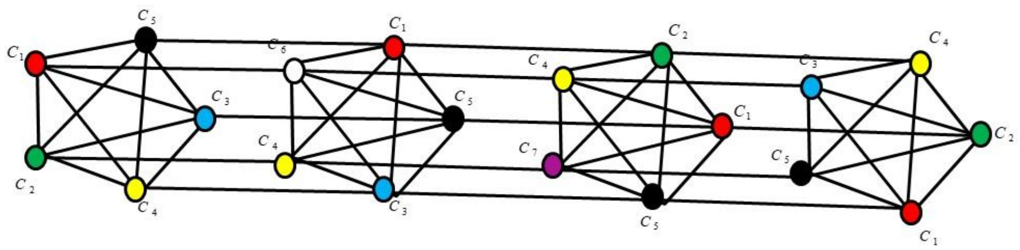
Theorem2.2

For any integer $m \geq 2$, $n \geq 2$, $\chi_{pd}(P_m \square K_n) = \begin{cases} n, & if\ m = 2 \\ m + n - 2, & otherwise \end{cases}$

Proof

We name the vertices of P_m by $x_1, x_2, \ldots, x_m$ and K_n by $y_1, y_2, \ldots. y_n$. According to the definition of the Cartesian product of two graphs, the vertices of $P_m \square K_n$ may be labeled as (x_i, y_j) where $1 \leq i \leq m\ and\ 1 \leq j \leq n$.

Every vertices of K_n power dominates and adjacent to each other. So we have to assign n colors for K_n. In $P_m \square K_n$ we have m cliques. The same n color classes can be used for the end clique in $P_m \square K_n$ because every vertex in those cliques power dominates all the remaining vertices of $P_m \square K_n$.So n colorclasses are enough when m takes the value two. While $m > 2$, the vertices in the clique which are appeared in middle power dominates only the vertices of the same. Along with the existing *n-1* color classes *m-2* new colors can be used for the cliques in the middle.



$\chi_{pd}(P_4 \square K_5) = 7$

Theorem 2.3

For any integer $n, m \geq 2$, $\chi_{pd}(P_n \square K_{1,m}) = \begin{cases} 2, & n = m = 2 \\ n + 2, & otherwise \end{cases}$.

Proof

We name the vertices of P_n by $x_1, x_2, \ldots, x_n$ and m+1 vertices of $K_{1,m}$ by $y_1, y_2, \ldots. y_{m+1}$. Where y_1 is the middle vertex. Cartesian product of these graphs denoted by $P_n \square K_{1,m}$ can be named as (x_i, y_j) where $1 \leq i \leq n\ and\ 1 \leq j \leq m + 1$.While n and m take the value 2 by theorem 2.1,two colors are enough to power dominate the graph. For the cases $n, m > 2$, n new colors can be assigned to all the middle vertex of $P_n \square K_{1,m}$ because every vertex in $K_{1,m}$ power dominates its middle vertex. Two more new color classes can be used for the remaining vertices by assigning different color class to adjacent vertices.

Theorem 2.4

For any integer $n \geq 2\ and\ k \geq 4$, $\chi_{pd}(P_n \square W_k) = \begin{cases} n + 2, & k\ is\ odd \\ n + 3, & k\ is\ even \end{cases}$.

Proof

We name the vertices of P_n by $x_1, x_2, \ldots, x_n$ and the vertices of W_k by $y_1, y_2, \ldots. y_k$ in which y_1 is connected to all the other (k-1) vertices of a cycle. Cartesian product of these graph denoted by $P_n \square W_k$ can be named as (x_i, y_j) where $1 \leq i \leq n\ and\ 1 \leq j \leq k$.Every wheel graph occur in $P_n \square W_k$ is power dominated by its middle vertex. So n new colors are needed for middle vertex of every wheel graph to power dominate remaining vertices. For coloring the (k-1) vertices in every cycles two more color classes can be used while k is odd and 3 color classes can be used while n is even.

Theorem 2.5

For any integer $m, n \geq 3$, $\chi_{pd}(C_m \square K_n) = n + m - 1$

Proof

Indicate the vertices of C_m with $x_1, x_2, \ldots, x_m$ and the vertices of K_n with $y_1, y_2, \ldots. y_n$ where $y_j (1 \leq j \leq n)$ are adjacent with each (n-1) vertices in it. Cartesian product of these graph denoted by $C_m \square K_n$ can be labeled as (x_i, y_j) where $1 \leq i \leq m\ and\ 1 \leq j \leq n$. The graph $C_m \square K_n$ contains m number of cliques of order n. Every vertices of the clique power dominates the vertices in the corresponding clique. To power dominate $C_m \square K_n$, m number of colors can be used to each clique and n-1 color classes can be used for the remaining vertices.

Theorem 2.6

For any integer $m, n \geq 3$, $\chi_{pd}\left(C_m \square\ K_{1,n}\right) = \begin{cases} m+2, m\ is\ even \\ m+3, m\ is\ odd \end{cases}$.

Proof

The vertices of C_m are labeled as $x_1, x_2, \dots, x_m$ and the vertices of $K_{1,n}$ with $y_1, y_2, \dots. y_{n+1}$ where y_1 is the middle vertex. Cartesian product of these graph denoted by $C_m \square K_{1,n}$ can be labeled as (x_i, y_j) where $1 \leq i \leq m\ and\ 1 \leq j \leq n+1$. Every y_1 in $K_{1,n}$ of $C_m \square K_{1,n}$ power dominate $y_j\ (2 \leq j \leq n+1)$. m number of color can be assigned to each y_1 in $K_{1,n}$ and two more color classes are assigned alternatively to $y_j\ (2 \leq j \leq n+1)$ when n is even or two more color classes can be assigned alternatively to $y_j\ (2 \leq j \leq n+1)$ when n is odd.

Theorem 2.7

For any integer $m, n \geq 3$, $m+2 \leq \chi_{pd}\ (C_m \square W_k) \leq m+4$

Proof

The vertices of C_m aredesignated as $x_1, x_2, \dots, x_m$ and the vertices of W_k by $y_1, y_2, \dots. y_{k+1}$ where y_1 is the middle vertex. Cartesian product of these graph denoted by $C_m \square W_k$ can be named as (x_i, y_j) where $1 \leq i \leq m\ and\ 1 \leq j \leq k+1$.

Case 1

Suppose $\chi_{pd}(G) < m+2$, all the vertices occurred in the cycle of each W_k power dominates its own middle vertex. So for cycle of order m we need m different colors.For the remaining vertices occur in the cycle,at least two color classes are required to power dominates the graph. So $\chi_{pd}(G) \geq m+2$.

Case 2

Suppose $\chi_{pd}(G) > m+4$, as a continuation of case 1, when m is odd, assigning two color classes is not satisfying the condition of power dominator coloring. so, one more new color is needed. Also if k even means one more additional color class will power dominate the same. More than m+4 color class will violate the condition.

3. Power Dominator Chromatic Number of Indu-Bala Product of Some Graphs:

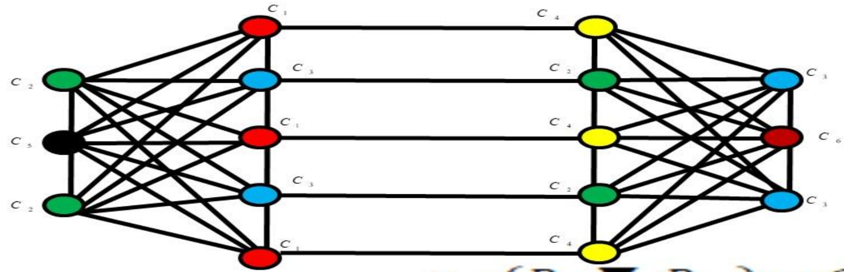
Theorem 3.1

For any integer $k \geq 2, n > 2 and k < n, \chi_{pd}$ (P_k ▼P_n) = 6.

Proof

Consider the Indhu-Bala product of two paths P_k ▼P_n. Let $(x_1, x_2, \dots, x_k), (x_1', x_2', \dots, x_k')$ be the vertices of two copies of P_k and $(z_1, z_2, \dots, z_n), (z_1', z_2', \dots, z_n')$ be the vertices of two copies of $P_n, \forall k < n$. By the definition of the Indu-Bala product , all the vertices of P_k are adjacent with n vertices of P_n.Then the vertices $(z_1, z_2, \dots, z_n)$ and $(z_1', z_2', \dots, z_n')$ connected by an edge. Between the two copies, it forms a ladder. The vertices of ladder dominates all vertices of both copies and itself.The degree of P_k is always more than or equal to the degree of P_n since k < n.

Assign colour class C_1 and C_2 alternatively to $x_i (1 \leq i \leq k)$ and the colour classes C_3 and C_4 alternatively to $x_i' (1 \leq i \leq k)$ and the color classes C_4 and C_5 alternatively to $z_j (1 \leq j \leq n)$ and the colour classes C_2 and C_6 alternatively to $z_j' (1 \leq j \leq n)$ will make the possible power dominator coloring of P_k ▼P_n.



$\chi_{pd}(P_5$ ▼ $P_3) = 6$

Observation 3.1.1

Let the Indu-Bala product of cycle and Path C_k and P_n be $C_k▼P_n$. Then for $k \geq 3$ and $n \geq 2$, we have χ_{pd} $(C_k▼P_n)= \begin{cases} 7, k \text{ is odd} \\ 6, otherwise \end{cases}$.

Observation 3.1.2

Let the Indu-Bala product of two cycles C_k and C_n be $C_k▼C_n$. Then we have

$$\chi_{pd} (C_k▼C_n)= \begin{cases} 6, both\ k\ and\ n\ are\ even \\ 7, either\ k\ or\ n\ is\ odd \\ 8, both\ k\ and\ n\ are\ odd \end{cases}.$$

CONCLUSION

In this article, the power dominator chromatic number of Cartesian and Indu-Bala Product of Some Graphs are obtained. It will be interesting to find power dominator chromatic numbers of Indu-Bala Product of Some of other classes of graphs.

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