10.48047/jocaaa.2026.35.07.17

AN INTRODUCTION TO NOVEL GRAPH-THEORETIC OPERATIONS

DEEPA S. NAIR[1]

[1] Department of Mathematics, Devaswom Board Pampa College, Parumala, Pathanamtitta, Kerala 689626, India, Email: deepamtcr@gmail.com

ABSTRACT. In this article, new graphs are derived from existing ones by performing novel graph-theoretic operations associated with subdivision and maximum, minimum degree of the vertices of the given graph G.Two new subdivision graphs viz., Arithmetic mean and geometric mean subdivisions graph are introduced in the former part whereas a few degree-derived graphs for various classes of graphs are discussed in the later part.

Key words: Arithmetic Mean Subdivision Graph, Geometric Mean Subdivision Graph, Degree-derived Graphs

1. INTRODUCTION

Graph theory is a fundamental branch of discrete mathematics that provides powerful tools for modelling and analysing relationships among objects. A significant amount of the subject graph theory focuses on creating new graphs from pre-existing ones via well defined operations, rather than just examining individual graphs. In addition to enhancing the theoretical framework of the field, these operations are essential to applications in chemistry, biology, social sciences, computer networks, and communication systems. Graph operations can be broadly divided into two categories: unary operations, which act on a single graph and binary operations, which join two graphs to create a new graph. By methodically altering a given graph while maintaining or emphasizing specific structural characteristics, a number of derived graphs are also produced in addition to these operations. By breaking down difficult problems into simpler or more understandable forms, the study of such operations and derived graphs aids in the comprehension of graph invariants, the characterization of graph classes, and problem solving.

A graph G consists of a finite non-empty set V of objects called vertices and a set E of 2-element subsets of V called edges. The sets V and E are the vertex set and edge set of G, respectively. So a graph G is a pair of two sets V and E. For this reason, some write $G = (V, E)$. At times, it is useful to write $V(G)$ and $E(G)$ rather than V and E to emphasize that these are the vertex and edge sets of a particular graph G. The number of vertices in G is often called the order of G, while the number of edges is its size. Since, the vertex set of every graph is non-empty, the order of every graph is at least 1. A graph with exactly one vertex is called a trivial graph, implying that the order of a non-trivial graph is at least 2. A $u-v$ walk W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as $W = (u = v_0, v_1, \ldots, v_k = v)$, where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, 2, \ldots, k-1$. Each vertex v_i $(0 \leq i \leq k)$ and each edge v_iv_{i+1} $(0 \leq i \leq k-1)$ is said to lie on or belong to W. If $u = v$, then the walk W is closed; while if $u \neq v$, then W is open. A $u-v$ trail in a graph G to be a $u-v$ walk in which no edge is traversed more than once. A $u-v$ walk in a graph in which no vertices are repeated is a $u-v$ path.

A circuit in a graph G is a closed trail of length 3 or more. Hence a circuit begins and ends at the same vertex but repeats no edges. A circuit can be described by choosing any of its vertices as the beginning (and ending) vertex provided the vertices are listed in the same cyclic order. In a circuit, vertices can be repeated, in addition to the first and last. A circuit that repeats no vertex, except for the first and last, is a cycle. A k-cycle is a cycle of length k.

The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by $deg_G v$ or simply by $d(v)$ if the graph G is clear from the context. Also, $d(v)$ is the number of vertices adjacent to v. If G contains a $u-v$ path, then u and v are said to be connected and u is connected to v (and v is connected to u). A graph G is connected if and only if G contains a $u-v$ walk for every pair u, v of vertices of G. Since every vertex is connected to itself, the trivial graph is connected. A graph G that is not connected is called disconnected [3]. The star graph S_n is isomorphic to the complete bipartite graph $K_{1,n-1}$. The greatest integer less than or equal to a number x is represented as $\lfloor x \rfloor$. For more definitions, notations and terminologies not covered here, follow [1, 2, 3, 4, 5].

2. Arithmetic Mean and Geometric Mean Subdivision Graphs

If $x = uv$ is a line of G and w is a point of G, then x is subdivided when it is replaced by the lines uw and uv. If every line of G is subdivided, then resulting graph is the subdivision graph $S(G)$. Let G be a graph with n vertices and m edges. Here we are defining two novel subdivisions called Arithmetic Mean Subdivision and Geometric Mean Subdivision of the given graph G.

Definition 2.1. *Let $v_1, v_2, \ldots v_n$ be the vertices of the graph G. The* **Arithmetic Mean Subdivision Graph** *of G denoted by $AMS(G)$ is the graph obtained from G by replacing each edge v_iv_j by $\left\lfloor \frac{d(v_i)+d(v_j)}{2} \right\rfloor$ number of vertices of degree 2 for each edge, where $i \neq j$ and $i, j = 1, 2, \ldots, n$.*

In a similar manner we can define the Geometric Mean Subdivision graph.

Definition 2.2. *Let $v_1, v_2, \ldots v_n$ be the vertices of the graph G. The* **Geometric Mean Subdivision Graph** *of G denoted by $GMS(G)$ is the graph obtained from G by replacing each edge v_iv_j by $\lfloor \sqrt{d(v_i)d(v_j)} \rfloor$ number of vertices of degree 2 for each edge, where $i \neq j$ and $i, j = 1, 2, \ldots, n$.*

An illustration for the Arithmetic Mean (AM) Subdivision and Geometric Mean (GM) Subdivision graph for a given graph G is given here.

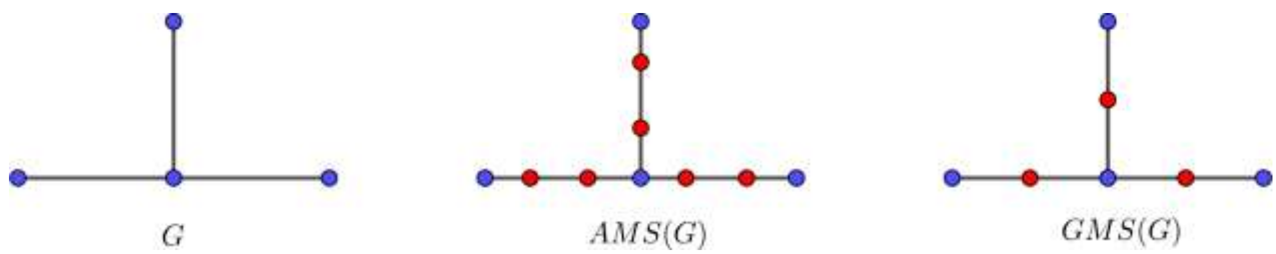

FIGURE 1. Arithmetic and Geometric Mean Subdivision graph of G

3. AM and GM Subdivision Graph of Some Classes of Graphs

In this section, the Arithmetic Mean and Geometric Mean subdivision graphs of some standard graphs namely Path, Cycle, d-regular graph are derived.

Theorem 3.1. *For the path P_n,*

$$AMS(P_n) \cong GMS(P_n) \cong \begin{cases} P_3 & , n = 2 \\ P_{3n-4} & , n \geq 3. \end{cases} \tag{1}$$

Proof. The path P_n has n vertices and $n-1$ edges. For $n = 2$, the result holds trivially. For $n \geq 3$, P_n has $n-3$ edges with degree of end vertices as 2 and two edges with degree of end vertices 1 and 2. Hence, when we perform the arithmetic mean subdivision operation in the graph G, we add 2 vertices between the $n-3$ edges and 1 vertex to the 2 pendant edges, since $\lfloor \frac{2+2}{2} \rfloor = 2$ and $\lfloor \frac{1+2}{2} \rfloor = 1$. Hence, $AMS(G)$ has $3n-4$ vertices and $3n-5$ edges, which is isomorphic to P_{3n-4}.

The fact $\lfloor \sqrt{2 \times 2} \rfloor = 2$ and $\lfloor \sqrt{2 \times 1} \rfloor = 1$ along with the above explanation is enough to prove the case of GMS of P_n. This completes the proof. □

An illustration of Theorem 3.1 is given here.

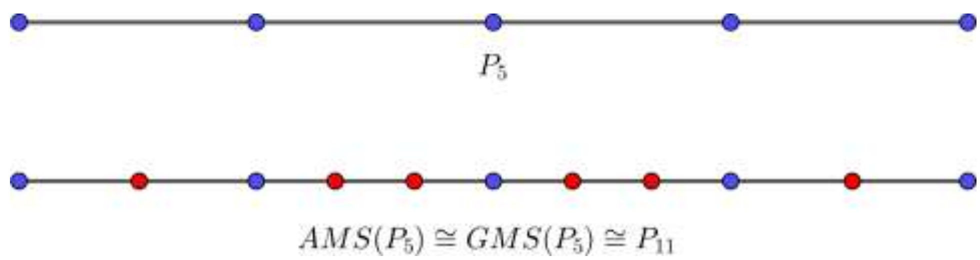

FIGURE 2. Arithmetic Mean and Geometric Mean subdivision graph of P_5

Theorem 3.2. *For the cycle C_n, $n \geq 3$,*

$$AMS(C_n) \cong GMS(C_n) \cong C_{3n}. \tag{2}$$

Proof. We know that C_n has n vertices and n edges and it is 2-regular. Also we have $\frac{2+2}{2} = \sqrt{2 \times 2} = 2$. This implies, 2 vertices in each edge of C_n has to be inserted. Hence, $AMS(C_n)$ and $GMS(C_n)$ has $3n$ vertices and $3n$ edges. Again one can notice that the so obtained graph is of 2-regular. Therefore, the resulting graph is a cycle on $3n$ vertices. □

An illustration of Theorem 3.2 is given here.

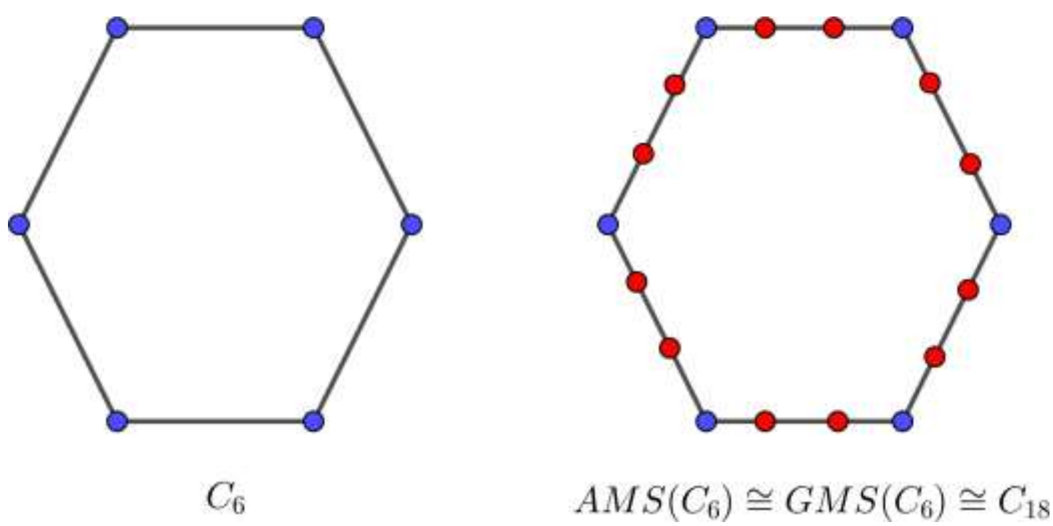

FIGURE 3. Arithmetic Mean and Geometric Mean subdivision graph of C_6

Theorem 3.3. *For a d-regular graph G on n vertices, AMS(G) and GMS(G) is a graph with order* $\frac{n(d^2+2)}{2}$ *and size* $\frac{(d+1)nd}{2}$.

Proof. Since G is a $d-$regular graph n vertices it has $\frac{nd}{2}$ edges. Also,

$$\left\lceil \frac{d+d}{2} \right\rceil = \lfloor \sqrt{d \times d} \rfloor = d. \tag{3}$$

Hence, the number of vertices in AMS(G)

$$\begin{aligned} &= n + \frac{nd}{2} \times d \\ &= \frac{2n+nd^2}{2} = \frac{n(d^2+2)}{2}. \end{aligned} \tag{4}$$

Number of edges in AMS(G) = $\frac{(d+1)nd}{2}$.

The same result holds for GMS(G). Hence the result is obtained. □

The following are immediate results from the above theorem.

Graph	Degree	Order	Size
Complete graph, K_n	$n-1$	$\frac{n\,[(n-1)^2+2]}{2}$	$\frac{n^2(n-1)}{2}$
Petersen graph	3	55	60

TABLE 1. Order and Size of graphs obtained through AMS and GMS

4. Some Novel Degree-Derived Graphs

A derived graph is obtained from a given graph through a systematic transformation that reflects specific structural features. Derived graphs play a central role in modern graph theory by providing systematic transformations that generate new graphs from existing ones while preserving or highlighting specific structural properties. Some important classes of derived graphs are: line graph, total graph, middle graph.

In this section, two new degree-derived graphs from the given graphs are introduced. Here Δ and δ denotes the maximum and minimum vertex degree of the graph respectively.

Definition 4.1. *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two connected graphs. Then the* $\Delta-\Delta$ **Degree-Derived Graph** *of* G_1 *and* G_2 *denoted by* $G_1 \,\Delta\Delta\, G_2$ *is a graph* $G = (V, E)$ *with vertex set* $V = V_1 \cup V_2$ *and edge set consists of set of those edges joining the maximum degree vertices of* G_1 *to every maximum degree vertices of* G_2.

In a similar manner, we can define the case of minimum degree.

Definition 4.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. Then the $\delta - \delta$* **Degree-Derived Graph** *of G_1 and G_2 denoted by $G_1\ \delta\delta\ G_2$ is a graph $G = (V, E)$ with vertex set $V = V_1 \cup V_2$ and edge set consists of set of those edges joining the minimum degree vertices of G_1 to every minimum degree vertices of G_2.*

Example 4.1. *Consider the graphs P_4 and P_3. The $\Delta - \Delta$ and $\delta - \delta$ degree-derived graphs of P_4 and P_3 are illustrated as follows:*

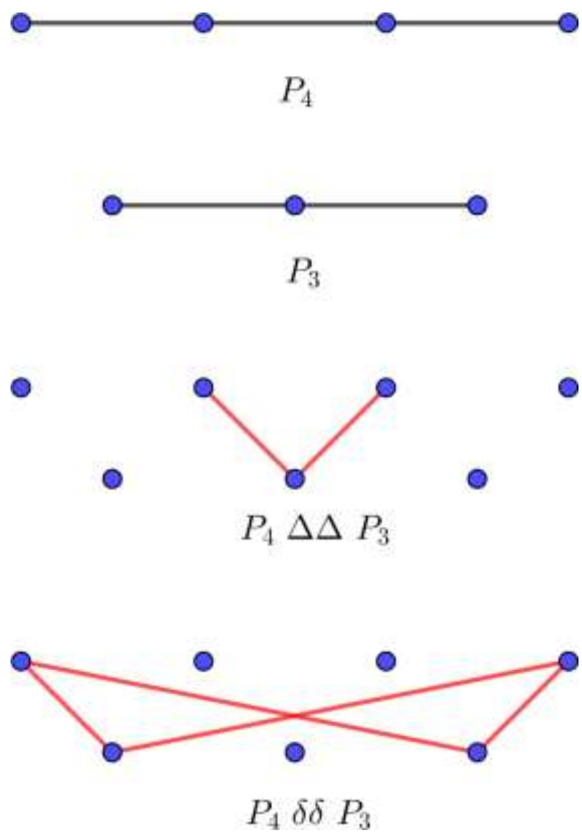

FIGURE 4. Degree-derived graphs of P_4 and P_3

Remark 4.1. *The following are some immediate observations:*

(1) *The operations $G_1\ \Delta\Delta\ G_2$ and $G_1\ \delta\delta\ G_2$ are commutative.*
(2) *$G_1\ \Delta\Delta\ G_2$ and $G_1\ \delta\delta\ G_2$ need not be connected even though G_1 and G_2 are connected (See Figure 4).*
(3) *There exists graphs with $G_1\ \Delta\Delta\ G_2 \cong G_1\ \delta\delta\ G_2$ (See Figure 5).*

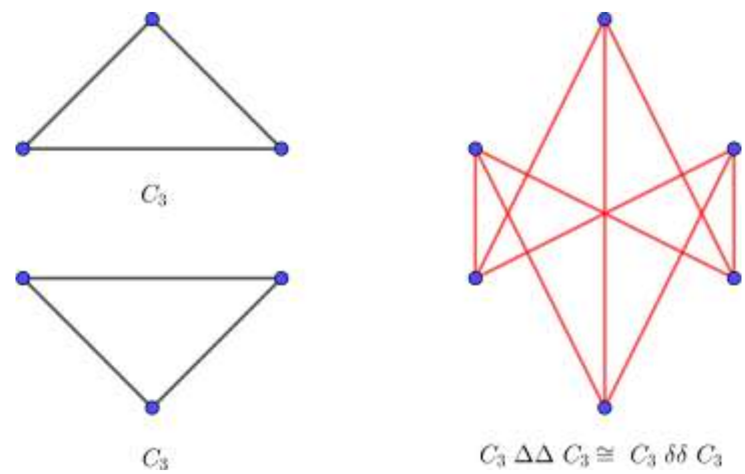

FIGURE 5. Example for a graph with isomorphic degree- derived graphs

Theorem 4.1. *Let G_1 be a d-regular graph on n vertices and G_2 be an r-regular graph on m vertices, where $d, r \geq 2$. Then*

$$G_1\ \Delta\Delta\ G_2 \cong G_1\ \delta\delta\ G_2 \cong K_{n,m}.$$

Proof. Since G_1 and G_2 are regular graphs, for the graph G_1, $\Delta = \delta = d$ and for the graph G_2, $\Delta = \delta = r$ respectively. Hence all the n vertices of G_1 will be adjacent to the m vertices of G_2. This forms a complete bipartite graph $K_{n,m}$. This completes the proof. □

Corollary 4.1. *For $m, n \geq 3$,*

$$C_m\ \Delta\Delta\ C_n \cong C_m\ \delta\delta\ C_n \cong K_{m,n}.$$

Corollary 4.2. *For $m, n \geq 1$,*

$$K_m\ \Delta\Delta\ K_n \cong K_m\ \delta\delta\ K_n \cong K_{m,n}.$$

Corollary 4.3. *For $m \geq 3, n \geq 1$,*

$$K_n\ \Delta\Delta\ C_m \cong K_n\ \delta\delta\ C_m \cong K_{n,m}.$$

Corollary 4.4. *For the Petersen graph G,*

$$G\ \Delta\Delta\ G \cong G\ \delta\delta\ G \cong K_{10,10}.$$

Theorem 4.2. *For $m, n \geq 2$*

(a) $P_m\ \Delta\Delta\ P_n = K_{m-2,n-2} \cup 4K_1$.
(b) $P_m\ \delta\delta\ P_n = (n + m - 4)K_1 \cup K_{2,2}$.

Proof. Since P_m and P_n has $m-2$ and $n-2$ internal vertices (vertices with degree 2) and 2 pendant vertices each, while calculating $P_m\ \Delta\Delta\ P_n$, the $m-2$ internal vertices of P_n will be adjacent to the $n-2$ internal vertices of P_n. The remaining 4 vertices are isolated. Hence, $P_m\ \Delta\Delta\ P_n = K_{m-2,n-2} \cup 4K_1$.

A similar explanation gives the result, $P_m\ \delta\delta\ P_n = (n+m-4)K_1 \cup K_{2,2}$. This completes the proof. □

Theorem 4.3. *For $m, n \geq 2$,*

(a) $S_n\ \Delta\Delta\ S_m = K_2 \cup (m+n-2)K_1$.

(b) $S_n\ \delta\delta\ S_m = K_{n-1,m-1} \cup 2K_1$.

Proof. The graph S_n has 1 vertex of degree $n-1$ and $n-1$ vertices of degree 1. Hence, in $S_n\ \Delta\Delta\ S_m$, there is only one edge joining the vertex of degree $n-1$ and the vertex of degree $m-1$. Therefore, $S_n\ \Delta\Delta\ S_m = K_2 \cup (m+n-2)K_1$. Similarly, in $S_n\ \delta\delta\ S_m$, the $n-1$ pendant vertices of S_n will be adjacent to the $m-1$ pendant vertices of S_m. Hence, $S_n\ \delta\delta\ S_m = K_{n-1,m-1} \cup 2K_1$. This completes the proof. □

Theorem 4.4. *For $n \geq 2, m \geq 1$,*

(a) $P_n\ \Delta\Delta\ K_m = K_{n-2,m} \cup 2K_1$.

(b) $P_n\ \delta\delta\ K_m = K_{2,m} \cup (n-2)K_1$.

Proof. Since P_n has $n-2$ internal vertices and K_m is $m-1$ regular, the $n-2$ vertices of P_n are adjacent to the m vertices of K_m. Hence, $P_n\ \Delta\Delta\ K_m = K_{n-2,m} \cup 2K_1$. The minimum degree of P_n is 1 and there are 2 pendant vertices. Since K_m is regular $\Delta = \delta = m$. Hence, the 2 pendant vertices of K_m will be adjacent to the m vertices of K_m. Remaining $n-2$ vertices are isolated. Hence, $P_n\ \delta\delta\ K_m = K_{2,m} \cup (n-2)K_1$. Hence the theorem. □

Theorem 4.5. *For $n \geq 2$ and G an r-regular graph with m vertices*

(a) $P_n\ \Delta\Delta\ G = K_{n-2,m} \cup 2K_1$.

(b) $P_n\ \delta\delta\ G = K_{2,m} \cup (n-2)K_1$.

Proof. Trivially holds in view of the above theorems. □

Conclusion

In this paper, new graphs are derived using graph theoretical operations. Also presented results on Arithmetic Subdivision and Geometric subdivision of graphs. Some novel degree-derived graphs associated with maximum degree, minimum degree are derived. There are wide areas related to subdivision of graphs and derived graphs. There is great scope for further research.

Acknowledgement

The author acknowledges with sincere gratitude the valuable guidance and support received in the preparation of this research article.

References

[1] Bondy, J. A., and Murty, U. S. R., Graph Theory. Springer, Graduate Texts in Mathematics, 2008.

[2] Chartrand, G., Lesniak, L., and Zhang, P., Graphs and Digraphs (5th ed.). CRC Press, 2010.

[3] Gary Chartrand and Ping Zhang, A First Course in Graph Theory, Dover Publications, INC, Mineola, New York, 2012.

[4] Harary, F., Graph Theory. Addison-Wesley Publishing Company, 1969.

[5] West, D. B., Introduction to Graph Theory. Prentice Hall, Upper Saddle River, NJ, 2001.