Numerical investigation of zeros of the fully modified (p, q)-poly-Euler polynomials

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The aim of this paper is to introduce a fully modified (p,q)-poly-Euler polynomials and numbers of the first type. We investigate some properties that is related with (p, q)-Gaussian binomials coefficients. We also construct (p, q)-analogue of the Stirling numbers of the second kind and fully modified (p,q)-poly-Euler polynomials and numbers of the first type with two variables.

1 Introduction

Many researchers are interested in the applications of q-numbers and (p, q)-numbers. In areas of quantum mechanics, physics and mathematics, the applying theory is studied and extended actively. Especially, Mathematicians in the fields of combinatorics, number theory and special functions, frequently explorer that(cf [2], [3], [4], [7], [8],[9], [10], [11], [12], [13]). We also obtain the generalization of poly Bernoulli polynomials and poly tangent polynomials involving (p, q)-numbers. In this paper, we use the following notations. $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of complex numbers, respectively.

For $0 < q < p \leq 1$, the (p, q)-numbers are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q},$$

where $p = 1, [n]_{p,q} = [n]_q$ and $\lim_{q\to 1}[n]_q = n$.

The (p, q)-factorial of n of order k is defined as

$$[n]_{p,q}^{(\underline{k})} = [n]_{p,q}[n-1]_{p,q} \cdots [n-k+1]_{p,q},$$

for $k = 1, 2, 3, \cdots$. If $k = n$, it is denoted $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}$ that is called (p, q)-factorial of n.

The (p,q)-Gaussian binomial formula is defined by

$$(x+a)_{p,q}^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} a^{n-k} x^k,$$

with the (p,q)-Gaussian binomial coefficient, $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$ $(n \geq k)$.

In [13] , two type of the (p,q)-exponential functions are given as below

$$\begin{aligned} e_{p,q}(x) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}, \\ E_{p,q}(x) &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}. \end{aligned} \tag{1.1}$$

In [8], [10], the (p,q)-analogue of polylogarithm function $Li_{k,p,q}$ is known by

$$Li_{k,p,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_{p,q}^k}, (k \in \mathbb{Z}).$$

In [5], we defined the fully modified q-poly-Bernoulli polynomials $\widetilde{B}_{n,q}^{(k)}(x)$ of the first type and the fully modified q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x)$ of the first type.

Definition 1.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, we define fully modified q-poly-Bernoulli polynomials $\widetilde{B}_{n,q}^{(k)}(x)$ of the first type and the fully modified q-poly-tangent polynomials $\widetilde{T}_{n,q}^{(k)}(x)$ of the first type by

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{B}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!} &= \frac{Li_{k,q}(1-e_q(-t))}{(e_q(t)-1)} e_q(xt), \\ \sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(t)+1)} e_q(xt). \end{aligned} \tag{1.2}$$

When $x=0$, $\widetilde{B}_{n,q}^{(k)} = \widetilde{B}_{n,q}^{(k)}(0)$, $\widetilde{E}_{n,q}^{(k)} = \widetilde{E}_{n,q}^{(k)}(0)$ are called fully modified q-poly-Bernoulli numbers of the first type and fully modified q-poly-Euler numbers of the first type. If $q \to 1$ in (1.2), we get the poly-Bernoulli polynomials $B_n^{(k)}(x)$ and poly-Euler polynomials $E_n^{(k)}(x)$, respectively.

Substitute $k=1, q \to 1$ in (1.2), we have Bernoulli polynomials $B_n(x)$and Euler polynomials $E_n(x)$, respectively.

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \left(\frac{t}{e^t - 1} \right) e^{xt}, \qquad \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \left(\frac{2}{e^t + 1} \right) e^{xt}.$$

2 Some properties of the fully modified (p,q)-poly-Euler polynomials of the first type

In this section, we introduce fully modified (p,q)-poly-Euler numbers and polynomials of the first type by the generating functions. We explore some identities of the polynomials and find a relation connected with (p,q)-analogue of the ordinary Euler polynomials.

Definition 2.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}, p,q \in \mathbb{R}$ such that $0 < q < p \leq 1$, we define a fully modified (p,q)-poly-tangent polynomials $\widetilde{E}_{n,p,q}^{(k)}(x)$ of the first type by

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,p,q}^{(k)}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q} Li_{k,p,q}(1 - e_{p,q}(-t))}{t(e_{p,q}(t)+1)} e_{p,q}(xt).$$

When $x = 0$, $\widetilde{E}_{n,p,q}^{(k)} = \widetilde{E}_{n,p,q}^{(k)}(0)$ are called fully modified (p,q)-poly-Euler numbers of the first type. Note that $p = 1, [n]_{p,q} = [n]_q$, and $\widetilde{E}_{n,p,q}^{(k)}(x) = \widetilde{E}_{n,q}^{(k)}(x)$. If we set $k = 1, p = 1, q \to 1$ in Definition 2.1, then the Euler polynomials $E_n(x)$.

Theorem 2.2. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $p,q \in \mathbb{R}$ such that $0 < q < p \leq 1$, the following result holds*

$$\widetilde{E}_{n,p,q}^{(k)}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} p^{\binom{n-l}{2}} \, \widetilde{E}_{l,p,q}^{(k)} \, x^{n-l}.$$

In [5], the generating series of (p,q)-Stirling numbers of the second kind is defined by

$$\frac{(e_{p,q}(t)-1)^m}{[m]_{p,q}!} = \sum_{n=m}^{\infty} S_{p,q}(n,m) \frac{t^n}{[n]_{p,q}!}.$$

We also ontain

$$\frac{Li_{k,p,q}(1 - e_{p,q}(-t))}{t} = \sum_{n=0}^{\infty} \sum_{l=1}^{n+1} \frac{[l]_{p,q}!}{[l]_{p,q}^k [n+1]_{p,q}} (-1)^{l+n+1} S_{p,q}(n+1,l) \frac{t^n}{[n]_{p,q}!}.$$

Using the above identity, we derive the following result which is connected with (p,q)-Stirling numbers of the second kind and (p,q)-Euler polynomials.

Theorem 2.3. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $p,q \in \mathbb{R}$ such that $0 < q < p \leq 1$, the following identity holds*

$$\widetilde{E}_{n,p,q}^{(k)}(x) = \sum_{a=0}^{n} \sum_{l=1}^{a+1} \begin{bmatrix} n \\ a \end{bmatrix}_{p,q} \frac{[l]_{p,q}!}{[l]_{p,q}^k [a+1]_{p,q}} (-1)^{l+a+1} S_{p,q}(a+1,l) E_{n-a,p,q}(x).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < p \leq 1$. By the recomposition of (p,q)-polylogarithm function in (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{E}_{n,p,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(t)+1)}e_{p,q}(xt)\\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+l}[l]_{p,q}!}{[l]_{p,q}^k[n+1]_{p,q}}S_{p,q}(n+1,l)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\\ &= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n\\ a\end{bmatrix}_{p,q}\frac{(-1)^{l+a+1}[l]_{p,q}!}{[l]_{p,q}^k[a+1]_{p,q}}S_{p,q}(a+1,l)E_{n-a,p,q}(x)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Comparing the coefficient both sides, we get

$$\widetilde{E}_{n,p,q}^{(k)}(x) = \sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n\\ a\end{bmatrix}_{p,q}\frac{[l]_{p,q}!}{[l]_{p,q}^k[a+1]_{p,q}}(-1)^{l+a+1}S_{p,q}(a+1,l)E_{n-a,p,q}(x).$$

Now, we introduce fully modified (p,q)-poly-Euler polynomials of the first type with two variables by using two generating functions.

Definition 2.4. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}, p,q \in \mathbb{R}$ and $0 < q < p \leq 1$, the fully modified (p,q)-poly-Euler polynomials $\widetilde{E}_{n,p,q}^{(k)}(x,y)$ of the first type with two variables by

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt).$$

Theorem 2.5. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. Then we have the addition theorem.*

$$\widetilde{E}_{n,p,q}^{(k)}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}\widetilde{E}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}.$$

Proof. Let n be a nonnegative integer and $k \in \mathbb{Z}$. Then we get

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{E}_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!} &= \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}\widetilde{E}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Thus, we have

$$\widetilde{E}_{n,p,q}^{(k)}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}\widetilde{E}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}.$$

Theorem 2.6. *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $p,q \in \mathbb{R}$ such that $0 < q < p \leq 1$. We have*

$$\widetilde{E}_{n,p,q}^{(k)}(x,y) - \widetilde{E}_{n,p,q}^{(k)}(x) = \sum_{l=0}^{n-1}\begin{bmatrix} n\\ l\end{bmatrix}_{p,q}q^{\binom{n-l}{2}}\ \widetilde{E}_{l,p,q}^{(k)}(x)\ y^{n-1}.$$

3 Distribution of zeros of the fully modified (p,q)-poly-Euler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the fully modified (p,q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,p,q}(x)$. The fully modified (p,q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{1,p,q}(x)$ can be determined explicitly. A few of them are

$$\widetilde{E}^{(k)}_{0,p,q}(x)=\frac{p+q}{2},$$

$$\widetilde{E}^{(k)}_{1,p,q}(x)=-\frac{3p}{4}-\frac{q}{4}+\frac{p+q}{2[2]^k_{p,q}}+\frac{px+qx}{2},$$

$$\begin{aligned}
\widetilde{E}^{(k)}_{2,p,q}(x)&=\frac{p^3}{8(p-q)}+\frac{p^2q}{8(p-q)}-\frac{pq^2}{8(p-q)}-\frac{q^3}{8(p-q)}\\
&-\frac{p^4}{4(p-q)^2[2]^k_{p,q}}+\frac{p^2q^2}{2(p-q)^2[2]^k_{p,q}}-\frac{q^4}{4(p-q)^2[2]^k_{p,q}}-\frac{p^5}{(p-q)^2(p+q)[2]^k_{p,q}}\\
&+\frac{2p^3q^2}{(p-q)^2(p+q)[2]^k_{p,q}}-\frac{pq^4}{(p-q)^2(p+q)}+\frac{p^5}{2(p-q)(p^2+pq+q^2)}\\
&-\frac{p^3q^2}{2(p-q)(p^2+pq+q^2)}+\frac{p^7}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}}\\
&-\frac{p^5q^2}{(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}}-\frac{p^4q^3}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}}\\
&+\frac{p^3q^4}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}}+\frac{p^2q^5}{(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}}\\
&-\frac{q^7}{2(p-q)^3(p^2+pq+q^2)[3]^k_{p,q}}-\frac{3p^3x}{4(p-q)}-\frac{p^2qx}{4(p-q)}\\
&+\frac{3pq^2x}{4(p-q)}+\frac{q^3x}{4(p-q)}+\frac{p^4x}{2(p-q)^2[2]^k_{p,q}}-\frac{p^2q^2x}{(p-q)^2[2]^k_{p,q}}\\
&+\frac{q^4x}{2(p-q)^2)[2]^k_{p,q}}+\frac{p^3x^2}{2(p-q)}-\frac{pq^2x^2}{2(p-q)}.
\end{aligned}$$

We investigate the zeros of the fully modified (p,q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,p,q}(x)$ by using a computer. We plot the zeros of the (p,q)-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,p,q}(x)$ for $n = 20$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we

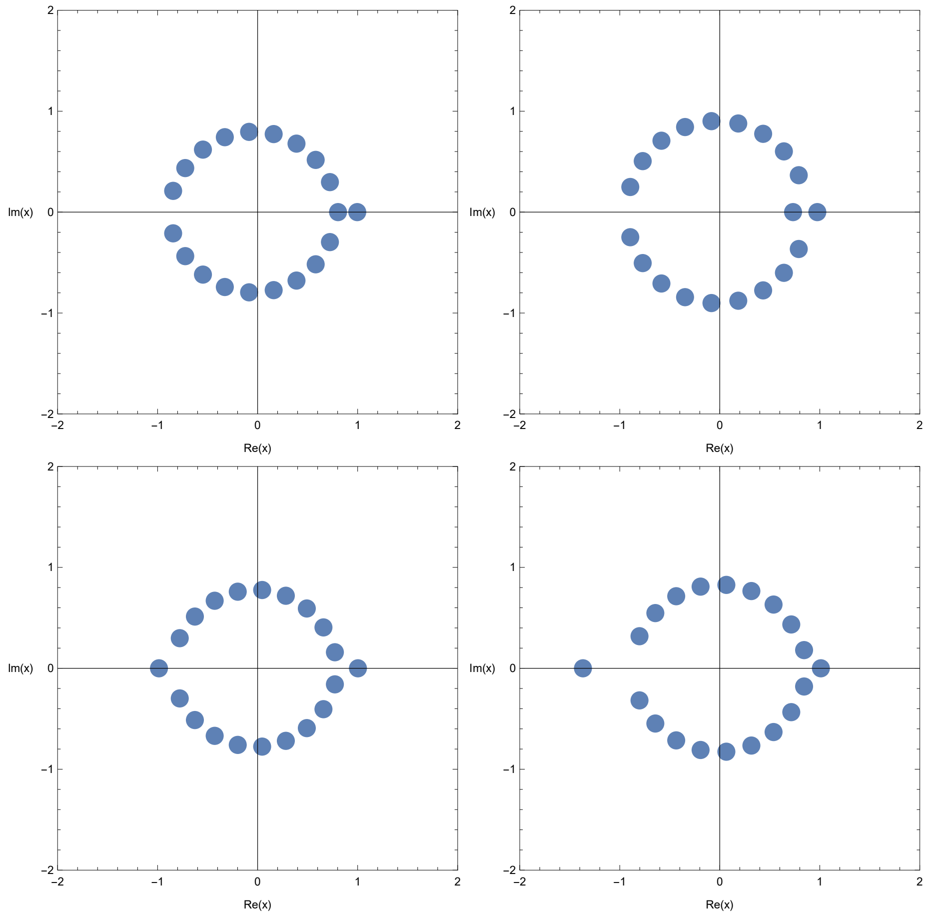

Figure 1: Zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$

choose $n = 20, p = 9/10, q = 1/10$, and $k = 1$. In Figure 1(top-right), we choose $n = 20, p = 9/10, q = 1/10$, and $k = 5$. In Figure 1(bottom-left), we choose $n = 20, p = 9/10, q = 1/10$, and $k = -1$. In Figure 1(bottom-right), we choose $n = 20, p = 9/10, q = 1/10$, and $k = -5$.

Stacks of zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$ for $1 \leq n \leq 20$ from a 3-D structure are presented(Figure 3). In Figure 3(top-left), we choose $p = 9/10, q = 1/10$, and

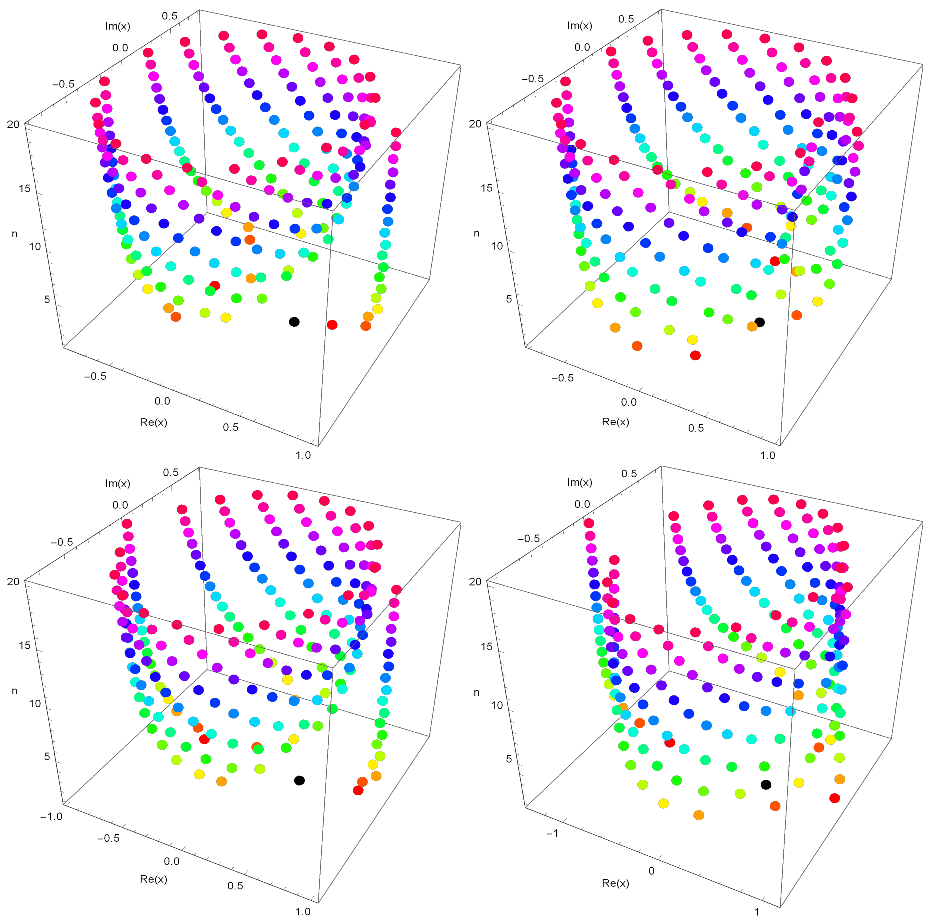

Figure 2: Stacks of zeros of $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$ for $1 \leq n \leq 20$

$k = 1$. In Figure 3(top-right), we choose $p = 9/10, q = 1/10$, and $k = 5$. In Figure 3(bottom-left), we choose $p = 9/10, q = 1/10$, and $k = -1$. In Figure 3(bottom-right), we choose $p = 9/10, q = 1/10$, and $k = -5$.

The plot of real zeros of $\widetilde{E}_{n,p,q}^{(k)}(x) = 0$ for $1 \leq n \leq 20$ structure are presented(Figure 4).

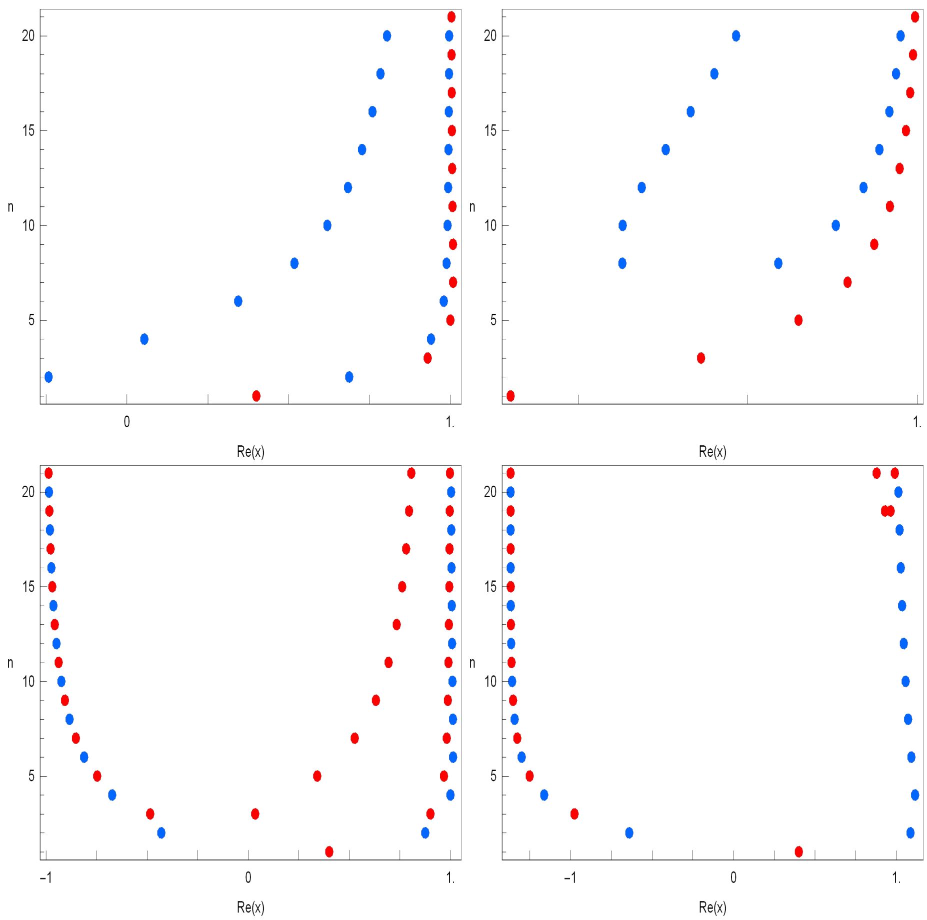

Figure 3: Real zeros of $\widetilde{E}_{n,p,q}^{(k)}(x) = 0$ for $1 \leq n \leq 20$

In Figure 4(top-left), we choose $p = 9/10, q = 1/10$, and $k = 1$. In Figure 4(top-right), we choose $p = 9/10, q = 1/10$, and $k = 5$. In Figure 4(bottom-left), we choose $p = 9/10, q = 1/10$, and $k = -1$. In Figure 4(bottom-right), we choose $p = 9/10, q = 1/10$, and $k = -5$.

Next, we calculated an approximate solution satisfying (p,q)-poly-Eulert polynomials $\widetilde{E}^{(k)}_{n,p,q}(x) = 0$ for $x \in \mathbb{C}$. The results are given in Table 1 and Table 2.

Table 1. Approximate solutions of $\widetilde{E}^{(-5)}_{n,p,q}(x) = 0, p = 9/10, q = 1/10$

degree n	x
1	0.40000
2	$-0.64015, \quad 1.0846$
3	$-0.97595, \quad 0.71267 - 0.32117i, \quad 0.71267 + 0.32117i$
4	$-1.1619, \quad 0.24937 - 0.68250i, \quad 0.24937 + 0.68250i,$ 1.1131
5	$-1.2512, \quad -0.01718 - 0.76730i, \quad -0.01718 + 0.76730i,$ $0.86775 - 0.23513i, \quad 0.86775 + 0.23513i$
6	$-1.2998, \quad -0.22150 - 0.76057i, \quad -0.22150 + 0.76057i,$ $0.55131 - 0.56698i, \quad 0.55131 + 0.56698i, \quad 1.0902$

Table 2. Approximate solutions of $\widetilde{E}^{(5)}_{n,p,q}(x) = 0, p = 9/10, q = 1/10$

degree n	x
1	0.40000
2	$0.22222 - 0.58608i \quad 0.22222 + 0.58608i$
3	$-0.11575 - 0.76525i, \quad -0.11575 + 0.76525i, \quad 0.68089$
4	$-0.28591 - 0.75902i, \quad -0.28591 + 0.75902i,$ $0.51087 - 0.33499i, \quad 0.51087 + 0.33499i$
5	$-0.46789 - 0.68975i, \quad -0.46789 + 0.68975i,$ $0.28053 - 0.71282i, \quad 0.28053 + 0.71282i, \quad 0.82471$
6	$-0.54973 - 0.64739i, \quad -0.54973 + 0.64739i,$ $0.15514 - 0.78522i, \quad 0.15514 + 0.78522i,$ $0.61959 - 0.14392i, \quad 0.61959 + 0.14392i$

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