Neutrosophic Subrings of A Ring

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ABSTRACT
The concept of neutrosophic subring of a ring, is initiated and some examples are discussed. Some of their properties are mentioned. The characterization of neutrosophic subring of a ring is obtained.

Keywords: Ring, subring, neutrosophic subring, (α, β, γ) – cut of neutrosophic subring, level subring of neutrosophic subring.

1. INTRODUCTION

In 1965, Lofti. A. Zadeh [9] introduced the concept of fuzzy sets, where each element of the set had a degree of membership. Zadeh had initiated fuzzy set theory as a modification of the ordinary set theory. In 1983, the notion of intuitionstic fuzzy set was introduced by K. Atanassov [1, 2] as a generalization of fuzzy set, where each element had the degree of membership and non- membership. The notion of neutrosophic set was initiated by Smarandache [7, 8]. The neutrosophic theory has set up the key stone for new mathematical theories inducing both the classical and fuzzy concepts.
In classical theory, subrings associated to any ring, play a vital role. On considering this, in this article, we tried to discuss the algebraic nature of neutrosophic subrings of a ring.

2. Preliminaries

In this segment, the general idea of rings and neutrosophic sets are recalled. Throughout this article, it is assumed that every ring is commutative and has a multiplicative identity element.

Definition 2.1
Let (R, +, .) be a ring. A non – empty subset S of R is called a subring of R if it satisfies the following condition:
$x, y \in S \Rightarrow x - y, xy \in S$

Example 2.2
(Z, +, .) is a ring.

Definition 2.3
Let X be a non – empty set. A set $A = \{<x, \mu_A(x), \sigma_A(x), \upsilon_A(x)>\}$ is called a Neutrosophic set of X, where $x \in X$ and the mappings $\mu_A, \sigma_A, \upsilon_A: X \rightarrow [0, 1]$. Here μ_A is called as the membership function; σ_A is called as the indeterministic membership function and υ_A is called as the non-membership function and there is no restriction on sum of $(\mu_A(x), \sigma_A(x), \upsilon_A(x))$ so $0 \le \mu_A(x) + \sigma_A(x) + \upsilon_A(x) \le 3$.

Definition 2.4
Let A be any Neutrosophic set of a set X and $A = \{<x, \mu_A(x), \sigma_A(x), \upsilon_A(x)>\}$. Then, (α, β, γ) – cut of A is defined as the subset $\{x \in X / \mu_A(x) \ge \alpha; \sigma_A(x) \le \beta \text{ and } \upsilon_A(x) \le \gamma\}$ of X, where $\alpha \in Im\ \mu$, $\beta \in Im\ \sigma$ and $\gamma \in Im\ \upsilon$ and the (α, β, γ) – cut of A is denoted by $A_{\alpha,\beta,\gamma}$.

3. Neutrosophic subring

Here the concept Neutrosophic subring of a ring is initiated and some examples are discussed. Some of their properties are established. The characterization of neutrosophic subring of a ring is obtained

Definition 3.1
A Neutrosophic set A of a ring R, is considered as a Neutrosophic subring of R, if it satisfies the following conditions: For all x, y $\in$ R,
(i) $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$
(ii) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,
(iii) $\sigma_A(x-y) \leq \min\{\sigma_A(x), \sigma_A(y)\}$
(iv) $\sigma_A(xy) \leq \min\{\sigma_A(x), \sigma_A(y)\}$,
(v) $\upsilon_A(x-y) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}$
(vi) $\upsilon_A(xy) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}$.

Example 3.2
Consider the Neutrosophic set A of a ring (Z, +, .).

$$\mu_A(x) = \begin{cases} .9 \text{ if } x \in <5> \\ .2 \quad Z \sim <5> \end{cases} \quad \sigma_A(x) = \begin{cases} .4 \text{ if } x \in <5> \\ .6 \quad Z \sim <5> \end{cases} \quad \upsilon_A(x) = \begin{cases} .3 \text{ if } x \in <5> \\ .7 \quad Z \sim <5> \end{cases}$$

Then A is a Neutrosophic subring of Z.

Example 3.3
Consider the Neutrosophic set B of a ring (Z, +, .).

$$\mu_B(x) = \begin{cases} .4 \text{ if } x \in <4> \\ .9 \quad Z \sim <4> \end{cases} \quad \sigma_B(x) = \begin{cases} .9 \text{ if } x \in <5> \\ .2 \quad Z \sim <5> \end{cases} \quad \upsilon_B(x) = \begin{cases} .6 \text{ if } x \in <5> \\ .3 \quad Z \sim <5> \end{cases}$$

Then B is not a Neutrosophic subring of Z.
Hereafter, A is assumed as any Neutrosophic subring of a ring R.

Proposition 3.4
For any A, $\mu_A(1) \leq \mu_A(x) \leq \mu_A(0)$; $\sigma_A(1) \geq \sigma_A(x) \geq \sigma_A(0)$ and $\upsilon_A(1) \geq \upsilon_A(x) \geq \upsilon_A(0)$, for all x $\in$ R where 0 is the additive identity and 1 is the multiplicative identity in R.
Proof
Given A is any Neutrosophic subring of a ring R with additive identity 0 and multiplicative identity 1.
To prove $\mu_A(1) \leq \mu_A(x) \leq \mu_A(0)$, for all x $\in$ R.
Allow x $\in$ R be arbitrary. Then,
$\mu_A(x) = \mu_A(1.x) \geq \min\{\mu_A(1), \mu_A(x)\} \geq \mu_A(1)$ ------------→ (1)
And $\mu_A(0) = \mu_A(x-x) \geq \min\{\mu_A(x), \mu_A(x)\} \geq \mu_A(x)$ ----------------→ (2)
From (1) and (2), we have $\mu_A(1) \leq \mu_A(x) \leq \mu_A(0)$, for all x $\in$ R.
To prove $\sigma_A(1) \geq \sigma_A(x) \geq \sigma_A(0)$, for all x $\in$ R.
Allow x $\in$ R be arbitrary. Then,
$\sigma_A(x) = \sigma_A(1.x) \leq \min\{\sigma_A(1), \sigma_A(x)\} \leq \sigma_A(1)$ -------------→ (3)
And $\sigma_A(0) = \sigma_A(x-x) \leq \min\{\sigma_A(x), \sigma_A(x)\} \leq \sigma_A(x)$ ----------------→ (4)
From (3) and (4), we have $\sigma_A(1) \geq \sigma_A(x) \geq \sigma_A(0)$, for all x $\in$ R.
To prove $\upsilon_A(1) \geq \upsilon_A(x) \geq \upsilon_A(0)$, for all x $\in$ R.
Allow x $\in$ R be arbitrary. Then,
$\upsilon_A(x) = \upsilon_A(1.x) \leq \min\{\upsilon_A(1), \upsilon_A(x)\} \leq \upsilon_A(1)$ -------------→ (5)
And $\upsilon_A(0) = \upsilon_A(x-x) \leq \min\{\upsilon_A(x), \upsilon_A(x)\} \leq \upsilon_A(x)$ ----------------→ (6)
From (5) and (6), we have $\upsilon_A(1) \geq \upsilon_A(x) \geq \upsilon_A(0)$, for all x $\in$ R.

Proposition 3.5
For any A, $\mu_A(x) = \mu_A(-x)$; $\sigma_A(x) = \sigma_A(-x)$ and $\upsilon_A(x) = \upsilon_A(-x)$, for all x $\in$ R.
Proof
Allow A be any Neutrosophic subring of a ring R, Allow x $\in$ R be arbitrary. Then,
$-x = 0 + (-x) = 0 - x$ -------------------------→ (1)
Now, $\mu_A(-x) = \mu_A(0-x) \geq \min\{\mu_A(0), \mu_A(x)\} \geq \mu_A(x)$ ------→ (2)
Again $\mu_A(x) = \mu_A(-(-x)) \geq \mu_A(-x)$, by (2)
$\geq \mu_A(x)$, by (2)
$\Rightarrow \mu_A(x) = \mu_A(-x)$.
And $\sigma_A(-x) = \sigma_A(0-x) \leq \min\{\sigma_A(0), \sigma_A(x)\} \leq \sigma_A(x)$ -------→ (3)
Again $\sigma_A(x) = \sigma_A(-(-x)) \leq \sigma_A(-x)$, by (3)
$\leq \sigma_A(x)$, by (3)

$\Rightarrow \sigma_A(x) = \sigma_A(-x)$.
And $\upsilon_A(-x) = \upsilon_A(0-x) \leq \min\{\upsilon_A(0), \upsilon_A(x)\} \leq \upsilon_A(x)$ ---------→ (4)
Again $\upsilon_A(x) = \upsilon_A(-(-x)) \leq \upsilon_A(-x)$, by (4)
$\leq \upsilon_A(x)$, by (4)
$\Rightarrow \upsilon_A(x) = \upsilon_A(-x)$.
Hence $\mu_A(x) = \mu_A(-x)$; $\sigma_A(x) = \sigma_A(-x)$ and $\upsilon_A(x) = \upsilon_A(-x)$, for all $x \in R$.

Proposition 3.6
For any A, for all $x, y \in R$,
$\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$;
$\sigma_A(x + y) \leq \min\{\sigma_A(x), \sigma_A(y)\}$ and
$\upsilon_A(x + y) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}$.
Proof
Allow $x, y \in R$ be arbitrary. Then,
$x + y = x + (-(-y)) = x - (-y)$
Now, $\mu_A(x + y) = \mu_A(x - (-y)) \geq \min\{\mu_A(x), \mu_A(-y)\}$
$\geq \min\{\mu_A(x), \mu_A(y)\}$, by Proposition 3.5
$\Rightarrow \mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}, \forall x, y \in R$
And, $\sigma_A(x + y) = \sigma_A(x - (-y)) \leq \min\{\sigma_A(x), \sigma_A(-y)\}$
$\leq \min\{\sigma_A(x), \sigma_A(y)\}$, by Proposition 3.5
$\Rightarrow \sigma_A(x + y) \leq \min\{\sigma_A(x), \sigma_A(y)\}, \forall x, y \in R$
And, $\upsilon_A(x + y) = \upsilon_A(x - (-y)) \leq \min\{\upsilon_A(x), \upsilon_A(-y)\}$
$\leq \min\{\upsilon_A(x), \upsilon_A(y)\}$, by Proposition 3.5
$\Rightarrow \upsilon_A(x + y) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}, \forall x, y \in R$

Proposition 3.7
For any A, if $\mu_A(x - y) = \mu_A(0)$, then $\mu_A(x) = \mu_A(y)$ where $x, y \in R$.
Proof
Allow $x, y \in R$.
Assume that $\mu_A(x-y) = \mu_A(0)$ -------→ (1)
Here, $x = x + 0 = x + ((-y) + y) = (x + (-y)) + y = (x - y) + y$
$\Rightarrow$ $\mu_A(x) = \mu_A((x - y) + y) \geq \min\{\mu_A(x - y), \mu_A(y)\}$, by Proposition 3.6
$= \min\{\mu_A(0), \mu_A(y)\}$, by (1)
$= \mu_A(y)$ -------------→ (2), by Proposition 3.4
Again, $y = 0 + y = (x - x) + (-(-y)) = (x - x) - (-y) = x - (x - y)$
$\Rightarrow$ $\mu_A(y) = \mu_A(x - (x - y)) \geq \min\{\mu_A(x), \mu_A(x - y)\}$
$\geq \min\{\mu_A(x), \mu_A(0)\}$ ---------→ (3), by (1)
But $\min\{\mu_A(x), \mu_A(0)\} = \mu_A(x)$, by Proposition 3.4
(3) $\Rightarrow$ $\mu_A(y) \geq \mu_A(x)$ ------------→ (4)
From (2) and (4), we have $\mu_A(x) = \mu_A(y)$.

Proposition 3.8
For any A, if $\sigma_A(x - y) = \sigma_A(1)$, then $\sigma_A(x) = \sigma_A(y)$ where $x, y \in R$.
Proof
Allow $x, y \in R$.
Assume that $\mu_A(x-y) = \mu_A(1)$ -------→ (1)
Here, $x = x + 0 = x + ((-y) + y) = (x + (-y)) + y = (x - y) + y$
$\Rightarrow$ $\sigma_A(x) = \sigma_A((x - y) + y) \leq \min\{\sigma_A(x - y), \sigma_A(y)\}$, by Proposition 3.6
$= \min\{\sigma_A(1), \sigma_A(y)\}$, by (1)
$= \sigma_A(y)$ -------------→ (2), by Proposition 3.4
Again, $y = 0 + y = (x - x) + (-(-y)) = (x - x) - (-y) = x - (x - y)$
$\Rightarrow$ $\sigma_A(y) = \sigma_A(x - (x - y)) \leq \min\{\sigma_A(x), \sigma_A(x - y)\}$
$\leq \min\{\sigma_A(x), \sigma_A(1)\}$, by (1)
$= \sigma_A(x)$, by Proposition 3.4
(3) $\Rightarrow$ $\sigma_A(y) \geq \sigma_A(x)$ ------------→ (3)
From (2) and (3), we have $\sigma_A(x) = \sigma_A(y)$

Proposition 3.9
For any A, if $\upsilon_A(x - y) = \upsilon_A(1)$, then $\upsilon_A(x) = \upsilon_A(y)$ where $x, y \in R$.

Proposition 3.10
For any A, $\mu_A(x + y) = \mu_A(y)$ for all $x, y \in R$ if and only if $\mu_A(x) = \mu_A(0)$.
Proof
Allow $x, y \in R$ be arbitrary.
Assume that $\mu_A(x + y) = \mu_A(y)$, for all $y \in R$.
As $0 \in R$, $\mu_A(x + 0) = \mu_A(0)$
$\Rightarrow \mu_A(x) = \mu_A(0)$
Conversely, assume that $\mu_A(x) = \mu_A(0)$ -------------→ (1)
Now $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$, by Proposition 3.6
$= \min\{\mu_A(0), \mu_A(y)\}$, by (1)
$= \mu_A(y)$, by Proposition 3.4
$\Rightarrow \mu_A(x + y) \geq \mu_A(y)$, for all $x, y \in R$ ----------→ (2)
Again $y = y + 0 = y + (x - x) = (y + x) - x = (x + y) - x$
Hence $\mu_A(y) = \mu_A((x + y) - x) \geq \min\{\mu_A(x + y), \mu_A(x)\}$
$= \min\{\mu_A(x + y), \mu_A(0)\}$, by (1)
$= \mu_A(x + y)$, by Proposition 3.4
$\Rightarrow \mu_A(y) \geq \mu_A(x + y)$ ----------→ (3)
From (2) and (3), we have $\mu_A(x + y) = \mu_A(y)$.

Proposition 3.11
For any A, $\sigma_A(x + y) = \sigma_A(y)$ for all $x, y \in R$ if and only if $\sigma_A(x) = \sigma_A(1)$.
Proof
Allow $x, y \in R$ be arbitrary. Assume that $\sigma_A(x + y) = \sigma_A(y)$, for all $y \in R$.
As $0 \in R$, $\sigma_A(x + 0) = \sigma_A(0)$
$\Rightarrow \sigma_A(x) = \sigma_A(0)$
Conversely, assume that $\sigma_A(x) = \sigma_A(1)$ -------------→ (1)
Now $\sigma_A(x + y) \leq \min\{\sigma_A(x), \sigma_A(y)\}$, by Proposition 3.6
$= \min\{\sigma_A(1), \sigma_A(y)\}$, by (1)
$= \sigma_A(y)$, by Proposition 3.4
$\Rightarrow \sigma_A(x + y) \leq \sigma_A(y)$, for all $x, y \in R$ ----------→ (2)
Again $y = y + 0 = y + (x - x) = (y + x) - x = (x + y) - x$
Hence $\sigma_A(y) = \sigma_A((x + y) - x) \leq \min\{\sigma_A(x + y), \sigma_A(x)\}$
$= \min\{\sigma_A(x + y), \sigma_A(0)\}$, by (1)
$= \sigma_A(x + y)$, by Proposition 3.4
$\Rightarrow \sigma_A(y) \leq \sigma_A(x + y)$ ----------→ (3)
From (2) and (3), we have $\sigma_A(x + y) = \sigma_A(y)$.

Proposition 3.12
For any A, $\upsilon_A(x + y) = \upsilon_A(y)$ for all $x, y \in R$ if and only if $\upsilon_A(x) = \upsilon_A(1)$.

Proposition 3.13
For any A, if $\mu_A(x) < \mu_A(y)$ for some $x, y \in R$, then $\mu_A(x - y) = \mu_A(x) = \mu_A(y - x)$.
Proof
Given A is a neutrosophic subring of R.
Allow $x, y \in R$.
Here $x - y = -(-(x - y)) = -(-x + y) = -(y - x)$.
Therefore, $\mu_A(x - y) = \mu_A(-(y - x)) = \mu_A(y - x)$, by Proposition 3.5
Assume that $\mu_A(x) < \mu_A(y)$ -------→ (1)
Now $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} = \mu_A(x)$ --------→ (2), by (1)
And $x = x + 0 = x + (-y + y) = (x - y) + y$
Therefore, $\mu_A(x) = \mu_A((x - y) + y) \geq \min\{\mu_A(x - y), \mu_A(y)\}$, by Proposition 3.6
$= \mu_A(x - y)$ as $\mu_A(x) < \mu_A(y)$
$\Rightarrow \mu_A(x) \geq \mu_A(x - y)$ --------→ (3)
From (2) and (3), we have $\mu_A(x) = \mu_A(x - y)$.

Proposition 3.14
For any A, if $\sigma_A(x) > \sigma_A(y)$ for some $x, y \in R$, then $\sigma_A(x - y) = \sigma_A(x) = \sigma_A(y - x)$.
Proof
Given A is a neutrosophic subring of R. Allow $x, y \in R$.
Here $x - y = -(-(x - y)) = -(-x + y) = -(y - x)$.
Therefore, $\sigma_A(x - y) = \sigma_A(-(y - x)) = \sigma_A(y - x)$, by Proposition 3.5
Assume that $\sigma_A(x) < \sigma_A(y)$ -------→ (1)
Now $\sigma_A(x - y) \le \min\{\sigma_A(x), \sigma_A(y)\} = \sigma_A(y) < \sigma_A(x)$ --------→ (2), by (1)
And $x = x + 0 = x + (-y + y) = (x - y) + y$
Therefore, $\sigma_A(x) = \sigma_A((x - y) + y) \le \min\{\sigma_A(x - y), \sigma_A(y)\}$, by Proposition 3.6
$= \sigma_A(x - y)$ as $\sigma_A(x) > \sigma_A(y)$
$\Rightarrow \sigma_A(x) \ge \sigma_A(x - y)$ --------→ (3)
From (2) and (3), we have $\sigma_A(x) = \sigma_A(x - y)$.

Proposition 3.15
For any A, if $\upsilon_A(x) > \upsilon_A(y)$ for some $x, y \in R$, then $\upsilon_A(x - y) = \upsilon_A(x) = \upsilon_A(y - x)$.
Now, the relation between a non – empty subset of a ring and the neutrosophic subring of the ring, defined in terms of that subset, is established. Then it is also proved that the converse relation is also true.

Theorem 3.16
Let H be any non – empty subset of a ring R, $H \neq R$. If A is a neutrosophic subring of R, defined by

$\mu_A(x) = \begin{cases} s_1 \text{ if } x \in H \\ t_1 \text{ if } x \in R \sim H \end{cases}$ where $s_1, t_1 \in [0, 1]$, $s_1 > t_1$,

and $\sigma_A(x) = \begin{cases} s_2 \text{ if } x \in H \\ t_2 \text{ if } x \in R \sim H \end{cases}$ where $s_2, t_2 \in [0, 1]$, $s_2 < t_2$,

and $\upsilon_A(x) = \begin{cases} s_3 \text{ if } x \in H \\ t_3 \text{ if } x \in R \sim H \end{cases}$ where $s_3, t_3 \in [0, 1]$, $s_3 < t_3$,

then H is a subring of R.

Proof
Allow H be any non– empty subset of a ring R, $H \neq R$.
Allow $x, y \in R$ be arbitrary.

Consider the functions, $\mu_A(x) = \begin{cases} s_1 \text{ if } x \in H \\ t_1 \text{ if } x \in R \sim H \end{cases}$ where $s_1, t_1 \in [0, 1]$, $s_1 > t_1$,

and $\sigma_A(x) = \begin{cases} s_2 \text{ if } x \in H \\ t_2 \text{ if } x \in R \sim H \end{cases}$ where $s_2, t_2 \in [0, 1]$, $s_2 < t_2$,

and $\upsilon_A(x) = \begin{cases} s_3 \text{ if } x \in H \\ t_3 \text{ if } x \in R \sim H \end{cases}$ where $s_3, t_3 \in [0, 1]$, $s_3 < t_3$,

Assume that A is a neutrosophic subring of R. Then,
(vii) $\mu_A(x - y) \ge \min\{\mu_A(x), \mu_A(y)\}$
(viii) $\mu_A(xy) \ge \min\{\mu_A(x), \mu_A(y)\}$,
(ix) $\sigma_A(x - y) \le \min\{\sigma_A(x), \sigma_A(y)\}$
(x) $\sigma_A(xy) \le \min\{\sigma_A(x), \sigma_A(y)\}$,
(xi) $\upsilon_A(x - y) \le \min\{\upsilon_A(x), \upsilon_A(y)\}$
(xii) $\upsilon_A(xy) \le \min\{\upsilon_A(x), \upsilon_A(y)\}$,
for all $x, y \in R$.
To prove H is a subring of R.
Allow $a, b \in H$ be arbitrary.
Then, $\mu_A(a) = s$ and $\mu_A(b) = s$.
Here $\min\{\mu_A(a), \mu_A(b)\} = \min\{s, s\} = s$
Hence all the values of $\mu_A(a - b)$ and $\mu_A(ab)$ are greater than or equal to s.

But μ_A has only two values s and t with s > t.
Therefore, all the values of $\mu_A(a – b)$ and $\mu_A(ab)$ are equal to s.
This implies (a – b) and (ab) $\in$ H.
This proves that H is a subring of R.

Theorem 3.17
If H is any subring of a ring R, H $\neq$ R, then the neutrosophic subset μ of R defined by

$\mu_A(x) = \begin{cases} s_1 \text{ if } x \in H \\ t_1 \text{ if } x \in R \sim H \end{cases}$ where $s_1, t_1 \in [0, 1]$, $s_1 > t_1$,

and $\sigma_A(x) = \begin{cases} s_2 \text{ if } x \in H \\ t_2 \text{ if } x \in R \sim H \end{cases}$ where $s_2, t_2 \in [0, 1]$, $s_2 < t_2$,

and $\upsilon_A(x) = \begin{cases} s_3 \text{ if } x \in H \\ t_3 \text{ if } x \in R \sim H \end{cases}$ where $s_3, t_3 \in [0, 1]$, $s_3 < t_3$,

is a neutrosophic subring of R.

Proof
Allow H be any subring of a ring R, H $\neq$ R.
Consider the neutrosophic subset μ of R defined by

$\mu_A(x) = \begin{cases} s_1 \text{ if } x \in H \\ t_1 \text{ if } x \in R \sim H \end{cases}$ where $s_1, t_1 \in [0, 1]$, $s_1 > t_1$,

and $\sigma_A(x) = \begin{cases} s_2 \text{ if } x \in H \\ t_2 \text{ if } x \in R \sim H \end{cases}$ where $s_2, t_2 \in [0, 1]$, $s_2 < t_2$,

and $\upsilon_A(x) = \begin{cases} s_3 \text{ if } x \in H \\ t_3 \text{ if } x \in R \sim H \end{cases}$ where $s_3, t_3 \in [0, 1]$, $s_3 < t_3$,

Allow x, y $\in$ R be arbitrary.
To prove A is a neutrosophic subring of R.
It is enough to prove that
(i) $\mu_A(x – y) \geq \min \{\mu_A(x), \mu_A(y)\}$
(ii) $\mu_A(xy) \geq \min \{\mu_A(x), \mu_A(y)\}$,
(iii) $\sigma_A(x – y) \leq \min \{\sigma_A(x), \sigma_A(y)\}$
(iv) $\sigma_A(xy) \leq \min \{\sigma_A(x), \sigma_A(y)\}$,
(v) $\upsilon_A(x – y) \leq \min \{\upsilon_A(x), \upsilon_A(y)\}$
(vi) $\upsilon_A(xy) \leq \min \{\upsilon_A(x), \upsilon_A(y)\}$,
for all x, y $\in$ R.
We prove this in three cases:
Case (i): x, y $\in$ H
Then, $\mu_A(x) = s_1$, $\mu_A(y) = s_1$
$\Rightarrow \min \{\mu_A(x), \mu_A(y)\} = \min \{s_1, s_1\} = s_1$.
And $\sigma_A(x) = s_2$, $\sigma_A(y) = s_2$
$\Rightarrow \min \{\sigma_A(x), \sigma_A(y)\} = \min \{s_2, s_2\} = s_2$.
And $\upsilon_A(x) = s_3$, $\upsilon_A(y) = s_3$
$\Rightarrow \min \{\upsilon_A(x), \upsilon_A(y)\} = \min \{s_3, s_3\} = s_3$.
Here, x, y $\in$ H $\Rightarrow$ x – y, xy $\in$ H, since H is a subring of R.
Now, $\mu_A(x – y) = s_1 = \min \{\mu_A(x), \mu_A(y)\} \Rightarrow \mu_A(x – y) \geq \min \{\mu_A(x), \mu_A(y)\}$
$\mu_A(xy) = s_1 = \min \{\mu_A(x), \mu_A(y)\} \Rightarrow \mu_A(xy) \geq \min \{\mu_A(x), \mu_A(y)\}$
And $\sigma_A(x – y) = s_2 = \min \{\sigma_A(x), \sigma_A(y)\} \Rightarrow \sigma_A(x – y) \leq \min \{\sigma_A(x), \sigma_A(y)\}$
$\sigma_A(xy) = s_2 = \min \{\sigma_A(x), \sigma_A(y)\} \Rightarrow \sigma_A(xy) \leq \min \{\sigma_A(x), \sigma_A(y)\}$
And $\upsilon_A(x – y) = s_3 = \min \{\upsilon_A(x), \upsilon_A(y)\} \Rightarrow \upsilon_A(x – y) \leq \min \{\upsilon_A(x), \upsilon_A(y)\}$
$\upsilon_A(xy) = s_3 = \min \{\upsilon_A(x), \upsilon_A(y)\} \Rightarrow \upsilon_A(xy) \leq \min \{\upsilon_A(x), \upsilon_A(y)\}$
Then all the inequalities are satisfied in this case.
Case (ii): x, y $\in$ R ~ H
Then, $\mu_A(x) = t_1$, $\mu_A(y) = t_1$

$\Rightarrow$ min $\{\mu_A(x), \mu_A(y)\}$ = min $\{t_1, t_1\} = t_1$.
And $\sigma_A(x) = t_2$, $\sigma_A(y) = t_2$
$\Rightarrow$ min $\{\sigma_A(x), \sigma_A(y)\}$ = min $\{t_2, t_2\} = t_2$.
And $\upsilon_A(x) = t_3$, $\upsilon_A(y) = t_3$
$\Rightarrow$ min $\{\upsilon_A(x), \upsilon_A(y)\}$ = min $\{t_3, t_3\} = t_3$.
Here, $x, y \in H \Rightarrow x - y, xy \in H$, since H is a subring of R.
Now, $\mu_A(x - y) = t_1 = \min\{\mu_A(x), \mu_A(y)\} \Rightarrow \mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$
$\mu_A(xy) = t_1 = \min\{\mu_A(x), \mu_A(y)\} \Rightarrow \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$
And $\sigma_A(x - y) = t_2 = \min\{\sigma_A(x), \sigma_A(y)\} \Rightarrow \sigma_A(x - y) \leq \min\{\sigma_A(x), \sigma_A(y)\}$
$\sigma_A(xy) = t_2 = \min\{\sigma_A(x), \sigma_A(y)\} \Rightarrow \sigma_A(xy) \leq \min\{\sigma_A(x), \sigma_A(y)\}$
And $\upsilon_A(x - y) = t_3 = \min\{\upsilon_A(x), \upsilon_A(y)\} \Rightarrow \upsilon_A(x - y) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}$
$\upsilon_A(xy) = t_3 = \min\{\upsilon_A(x), \upsilon_A(y)\} \Rightarrow \upsilon_A(xy) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}$
Then all the inequalities are satisfied in this case.
Case (iii): $x \in H, y \in R \sim H$
Then, $\mu_A(x) = s_1$, $\mu_A(y) = t_1$
$\Rightarrow$ min $\{\mu_A(x), \mu_A(y)\}$ = min $\{s_1, t_1\} = t_1$.
And $\sigma_A(x) = s_2$, $\sigma_A(y) = t_2$
$\Rightarrow$ min $\{\sigma_A(x), \sigma_A(y)\}$ = min $\{s_2, t_2\} = s_2$.
And $\upsilon_A(x) = s_3$, $\upsilon_A(y) = t_3$
$\Rightarrow$ min $\{\upsilon_A(x), \upsilon_A(y)\}$ = min $\{s_3, t_3\} = s_3$.
Here, $x, y \in H \Rightarrow x - y, xy \in H$, since H is a subring of R.
Now, $\mu_A(x - y) = s_1 > t_1 = \min\{\mu_A(x), \mu_A(y)\} \Rightarrow \mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$
$\mu_A(xy) = s_1 > t_1 = \min\{\mu_A(x), \mu_A(y)\} \Rightarrow \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$
And $\sigma_A(x - y) = s_2 < t_2 = \min\{\sigma_A(x), \sigma_A(y)\} \Rightarrow \sigma_A(x - y) \leq \min\{\sigma_A(x), \sigma_A(y)\}$
$\sigma_A(xy) = s_2 < t_2 = \min\{\sigma_A(x), \sigma_A(y)\} \Rightarrow \sigma_A(xy) \leq \min\{\sigma_A(x), \sigma_A(y)\}$
And $\upsilon_A(x - y) = s_3 < t_3 = \min\{\upsilon_A(x), \upsilon_A(y)\} \Rightarrow \upsilon_A(x - y) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}$
$\upsilon_A(xy) = s_3 < t_3 = \min\{\upsilon_A(x), \upsilon_A(y)\} \Rightarrow \upsilon_A(xy) \leq \min\{\upsilon_A(x), \upsilon_A(y)\}$
Then all the inequalities are satisfied in this case.
Thus μ is a neutrosophic subring of R.

Remark 3.18
$¥_H$ is the characteristic function of the subset H of R.

That is, $¥_H = \begin{cases} 1 \text{ if } x \in H \\ 0 \text{ if } x \in R \sim H \end{cases}$

Corollary 3.19
If a non – empty subset H of a ring R, is a subring of R, then $¥_H$ is a neutrosophic subring of R.
Proof
Take $s_1 = 1, t_1 = 0$; $s_2 = 0, t_2 = 1$; and $s_3 = 0, t_3 = 1$ in the above theorem.
Then $¥_H$ is a neutrosophic subring of R.
From Theorem 3.16 and Theorem 3.17, we have,

Theorem 3.20
Let H be any non – empty subset of a ring R, H ≠ R. Let A be any neutrosophic subset of R defined by

$\mu_A(x) = \begin{cases} s_1 \text{ if } x \in H \\ t_1 \text{ if } x \in R \sim H \end{cases}$ where $s_1, t_1 \in [0, 1]$, $s_1 > t_1$,

and $\sigma_A(x) = \begin{cases} s_2 \text{ if } x \in H \\ t_2 \text{ if } x \in R \sim H \end{cases}$ where $s_2, t_2 \in [0, 1]$, $s_2 < t_2$,

and $\upsilon_A(x) = \begin{cases} s_3 \text{ if } x \in H \\ t_3 \text{ if } x \in R \sim H \end{cases}$ where $s_3, t_3 \in [0, 1]$, $s_3 < t_3$.

Then μ is a neutrosophic subring of R iff H is a subring of R.
Here, the concept of level subring of a ring is introduced. Then the necessary and sufficient condition of a neutrosophic subset of a ring to be a neutrosophic subring of that ring is obtained.

Proposition 3.21

For any A, the (α, β, γ) – cut of A, $A_{\alpha, \beta, \gamma}$, where α∈Im μ, β∈Im σ and γ∈Im υ are subrings of R.

Proof

Given μ is any neutrosophic subring of R and t ∈ Im μ is arbitrary.
Consider the (α, β, γ) – cut of A, $A_{\alpha, \beta, \gamma} = \{x\in R / \mu_A(x) \geq \alpha, \sigma_A(x) \leq \beta, \upsilon_A(x) \leq \gamma\}$.
By Proposition 2.4, $\mu_A(x) \leq \mu_A(0)$, for all x∈R.
$\Rightarrow \mu_A(0) \geq \mu_A(x) \geq \alpha$; $\sigma_A(0) \leq \sigma_A(x) \leq \beta$; $\upsilon_A(0) \leq \upsilon_A(x) \leq \gamma$
Hence $0 \in A_{\alpha, \beta, \gamma}$, for all α, β, γ.
Thus $A_{\alpha, \beta, \gamma} \neq \phi$
Allow x, y ∈ $A_{\alpha, \beta, \gamma}$. Then,
$\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$; $\sigma_A(x) \leq \beta$, $\sigma_A(y) \leq \beta$, $\upsilon_A(x) \leq \gamma$; $\upsilon_A(y) \leq \gamma$-----------→ (1)
Now $\mu_A(x - y) \geq \min \{\mu_A(x), \mu_A(y)\} \geq \alpha$, by (1)
And $\sigma_A(x - y) \leq \min \{\sigma_A(x), \sigma_A(y)\} \leq \beta$, by (1)
And $\upsilon_A(x - y) \leq \min \{\upsilon_A(x), \upsilon_A(y)\} \leq \gamma$, by (1)
$\Rightarrow x - y \in A_{\alpha, \beta, \gamma}$. -----------→ (2)
Again $\mu_A(xy) \geq \min \{\mu_A(x), \mu_A(y)\} \geq \alpha$, by (1)
And $\sigma_A(xy) \leq \min \{\sigma_A(x), \sigma_A(y)\} \leq \beta$, by (1)
And $\upsilon_A(xy) \leq \min \{\upsilon_A(x), \upsilon_A(y)\} \leq \gamma$, by (1)
$\Rightarrow xy \in A_{\alpha, \beta, \gamma}$. -----------→ (3)
From (2) and (3), we get,
$x, y\in A_{\alpha, \beta, \gamma} \Rightarrow x - y, xy \in A_{\alpha, \beta, \gamma}$.
Thus the (α, β, γ) – cut of A, $A_{\alpha, \beta, \gamma}$, where α ∈ Im μ, β ∈ Im σ and γ∈ Im υ are subrings of R.

Theorem 3.22: [Characterization Theorem]

A neutrosophic subset A of a ring R, is a neutrosophic subring of R iff, the (α, β, γ) – cuts of A, $A_{\alpha, \beta, \gamma}$, where α ∈ Im μ, β ∈ Im σ and γ∈ Im υ are subrings of R.

Proof

First part is shown Proposition 2.16.
Conversely, assume that (α, β, γ) – cuts of A, $A_{\alpha, \beta, \gamma}$, where α ∈ Im μ, β ∈ Im σ and γ∈ Im υ are subrings of R.
Then for all x, y ∈ $A_{\alpha, \beta, \gamma}$, iff $\mu_A(x) \geq \alpha$, $\mu_A(y) \geq \alpha$; $\sigma_A(x) \leq \beta$, $\sigma_A(y) \leq \beta$, $\upsilon_A(x) \leq \gamma$; $\upsilon_A(y) \leq \gamma$
To prove A is a neutrosophic subring of R.
Allow x, y ∈ R be arbitrary.
It is enough to prove that,

(i) $\mu_A(x - y) \geq \min \{\mu_A(x), \mu_A(y)\}$
(ii) $\mu_A(xy) \geq \min \{\mu_A(x), \mu_A(y)\}$,
(iii) $\sigma_A(x - y) \leq \min \{\sigma_A(x), \sigma_A(y)\}$
(iv) $\sigma_A(xy) \leq \min \{\sigma_A(x), \sigma_A(y)\}$,
(v) $\upsilon_A(x - y) \leq \min \{\upsilon_A(x), \upsilon_A(y)\}$
(vi) $\upsilon_A(xy) \leq \min \{\upsilon_A(x), \upsilon_A(y)\}$,

Allow $\min \{\mu_A(x), \mu_A(y)\} = r$; $\min \{\sigma_A(x), \sigma_A(y)\} = s$; $\min \{\upsilon_A(x), \upsilon_A(y)\} = t$
Now consider $\min \{\mu_A(x), \mu_A(y)\} = r$
Then Either $\mu_A(x) = r$ and $\mu_A(y) \geq \mu_A(x) = r$ Or $\mu_A(y) = r$ and $\mu_A(x) \geq \mu_A(y) = r$
$\Rightarrow \mu_A(x) \geq r$ and $\mu_A(y) \geq r$
Again, consider $\min \{\sigma_A(x), \sigma_A(y)\} = s$
Then Either $\sigma_A(x) = s$ and $\sigma_A(y) \leq \sigma_A(x) = s$ Or $\sigma_A(y) = s$ and $\sigma_A(x) \leq \sigma_A(y) = s$
$\Rightarrow \sigma_A(x) \leq s$ and $\sigma_A(y) \leq s$
Again, consider $\min \{\upsilon_A(x), \upsilon_A(y)\} = t$
Then Either $\upsilon_A(x) = t$ and $\upsilon_A(y) \leq \upsilon_A(x) = t$ Or $\upsilon_A(y) = t$ and $\upsilon_A(x) \leq \upsilon_A(y) = t$
$\Rightarrow \upsilon_A(x) \leq t$ and $\upsilon_A(y) \leq t$
Hence $\mu_A(x) \geq r$ and $\mu_A(y) \geq r$; $\sigma_A(x) \leq s$ and $\sigma_A(y) \leq s$; $\upsilon_A(x) \leq t$ and $\upsilon_A(y) \leq t$.
$\Rightarrow x, y \in A_{r, s, t}$.
$\Rightarrow x - y, xy \in A_{r, s, t}$., since $A_{r, s, t}$. is a subring of R.
$\Rightarrow \mu_A(x - y) \geq r$ and $\mu_A(xy) \geq r$; $\sigma_A(x - y) \leq s$ and $\sigma_A(xy) \leq s$;
$\upsilon_A(x - y) \leq t$ and $\upsilon_A(xy) \leq t$ ---------------→ (1)

For (i):

Allow $\mu_A(x - y) = s_1$ -------------→ (2)

To prove $\mu_A(x - y) \geq \min \{\mu_A(x), \mu_A(y)\}$
That is to prove $s_1 \geq r$
Suppose $s_1 < r$. -------------→ (3)
From (2) and (3), we have, $\mu_A(x - y) = s_1 < r$.
$\Rightarrow \mu_A(x - y) < r$.
This is a contradiction to (1).
Hence $s_1 \geq r$.
Thus $\mu_A(x - y) \geq \min \{\mu_A(x), \mu_A(y)\}$.

For (iii):
Allow $\sigma_A(x - y) = s_2$ -------------→ (4)
To prove $\sigma_A(x - y) \leq \min \{\sigma_A(x), \sigma_A(y)\}$
That is to prove $s_2 \leq s$
Suppose $s_2 > s$. -------------→ (5)
From (4) and (5), we have, $\sigma_A(x - y) = s_2 > s$.
$\Rightarrow \sigma_A(x - y) > s$.
This is a contradiction to (1).
Hence $s_2 \leq s$.
Thus $\sigma_A(x - y) \leq \min \{\sigma_A(x), \sigma_A(y)\}$.

Similarly, we can prove others.
Thus A is a neutrosophic subring of R.

Definition 3.23
Let A be any neutrosophic subring of R; the (α, β, γ) – cuts of A, $A_{\alpha, \beta, \gamma}$, where $\alpha \in \text{Im } \mu, \beta \in \text{Im } \sigma$ and $\gamma \in \text{Im } \upsilon$ are subrings of R. Then the subring (α, β, γ) – cut of A, $A_{\alpha, \beta, \gamma}$, of R is called a level subring of A.

Remark 3.24
Let A be any neutrosophic subset of R. Then A is neutrosophic subring of R; iff the level subrings of A are subrings of R.

CONCLUSION
The concept of neutrosophic subrings of a ring, are studied with examples. Some of their properties are also studied. The characterization of neutrosophic subring of a ring is obtained as The neutrosophic subset of a ring is a neutrosophic subring of the ring; iff the level subrings of the neutrosophic subrings, are subrings of the ring.

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