Fixed Points And Fractals With Strong Coupling Were Obtained Using Neutrtrophic Metric Spaces

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ABSTRACT

In this paper, the author defined strong coupled fixed point, cyclic, Hutchinson operator, coupling, Neutrosophic Iterated Coupling System(NICS), strong coupled fractal and neutrosophic contractive coupling in Neutrosophic Metric Space (NMS). We used an example to demonstrate that there is a single strong coupled fixed point in a neutrosophic contractive coupling with regard to non-empty closed subsets in NMS. Also, we have proved that Hutchinson operator is a neutrosophic contractive coupling. Next, we have proved that there is a unique strong coupled fractal in NICS.

Keywords: Neutrosophic metric space**,** Strong coupled fixed point, Hutchinson operator, coupling, neutrosophic iterated coupling system, Strong coupled fractal.

1. INTRODUCTION

The main aim of this research is to give a method of generating strong coupled fractals in Neutrosophic Metric Spaces (NMS). Strong coupled fractals is explained by Choudhury et al.[2] using the Hutchinson-Barnsley operator. Strong coupled fractals are formed here by the creation of the Hutchinson-Barnsley operator[1], which corresponds to a specific Iterated Function System termed Iterated Coupling System.
Iterated Function Systems (IFS) are a formalism for generating exactly self-similar fractals based on work of Hutchinson[8] (1981) and Mandelbrot (1982), and popularized by Barnsley (1988). Xiao JZ et. al. [23] defined the concept of IFS. Strong coupled fractals are cyclic generalizations of coupled mappings. The concept of Coupled mappings are defined by the authors Jeyaraman et al [10], [22], Preeti Sengar et. al. [16], Rajesh Shrivastava et. al [17] and the cyclic was defined by the authors Jeyaraman et. al.[11], Pasupathi et. al. [15]. The fixed point problem related with the coupled mapping is known as the coupled fixed point problem. Starting with two arbitrary points drawn from the two subsets that form the coupling, we create two iterations, each of which converges to the coupled fixed point. Furthermore, it is demonstrated that such a point is unique.
NMS concept is the extension of fuzzy and intuitionistic fuzzy metric space. There are different types of NMS which is defined by many authours [9-10,12-13 & 18-20] and has vast application now-a-days. Fuzzy metric space is defined by George and Veeramani [3]. Over the past three decades, fuzzy fixed point theory has seen incredibly broad and varied development that are referred in [4,5,6 and 7]. Shakila and Jeyaraman [14, 21] defined the concept of Hutchinson-Barnsley Operator in Neutrosophic Metric Spaces.
In this paper, the author present a Neutrosophic Contractive Coupling and acquire a strong coupled fixed point result. The couplings are then utilized to introduce a Neutrosophic Iterated Coupling System. Finally, we use the fixed-point result to produce strong coupled fractals using the Hutchinson-Barnsley operator. Examples are provided to demonstrate both the fixed-point theorem and the fractal generating process.

2. Preliminaries

Definition :2.1 [12]

A 6-tuple $(\Lambda, \Gamma, \Theta, \Upsilon, *, \diamond)$ is said to be a Neutrosophic Metric Space (NMS) if Λ is an arbitrary set, $*$ is neutrosophic continuous t-norm, $\diamond$ is neutrosophic continuous t-conorm, Γ, Θ, Υ are neutrosophic on $\Lambda \times \Lambda$ meets the following requirements: For all $\gamma, \theta, \upsilon \in \Lambda$ and $\zeta, \mu > 0$

(1) $0 \leq \Gamma(\gamma,\theta,\zeta) \leq 1, 0 \leq \Theta(\gamma,\theta,\zeta) \leq 1, 0 \leq \Theta(\gamma,\theta,\zeta) \leq 1,$
(2) $\Gamma(\gamma,\theta,\zeta) + \Theta(\gamma,\theta,\zeta) + \Theta(\gamma,\theta,\zeta) \leq 3,$
(3) $\Gamma(\gamma,\theta,\zeta) = 1$ iff $\gamma = \theta,$
(4) $\Gamma(\gamma,\theta,\zeta) = \Gamma(\theta,\gamma,\zeta),$
(5) $\Gamma(\gamma,\theta,\zeta) * \Gamma(\theta,\upsilon,\mu) \leq \Gamma(\gamma,\upsilon,\zeta+\mu)$ for all $\zeta,\mu > 0,$
(6) $\Gamma(\gamma,\theta,\cdot): (0,\infty) \to (0,1]$ is neutrosophic continuous,
(7) $\lim_{\zeta \to \infty} \Gamma(\gamma,\theta,\zeta) = 1$ for all $\zeta > 0,$
(8) $\Theta(\gamma,\theta,\zeta) = 0$ iff $\gamma = \theta,$
(9) $\Theta(\gamma,\theta,\zeta) = \Theta(\theta,\gamma,\zeta),$
(10) $\Theta(\gamma,\theta,\zeta) \diamond \Theta(\theta,\upsilon,\mu) \geq \Theta(\gamma,\upsilon,\zeta+\mu)$ for all $\zeta,\mu > 0,$
(11) $\Theta(\gamma,\theta,\cdot): (0,\infty) \to (0,1]$ is neutrosophic continuous,
(12) $\lim_{\zeta \to \infty} \Theta(\gamma,\theta,\zeta) = 0$ for all $\zeta > 0,$
(13) $\Upsilon(\gamma,\theta,\zeta) = 0$ iff $\gamma = \theta,$
(14) $\Upsilon(\gamma,\theta,\zeta) = \Upsilon(\theta,\gamma,\zeta),$
(15) $\Upsilon(\gamma,\theta,\zeta) \diamond \Upsilon(\theta,\upsilon,\mu) \geq \Upsilon(\gamma,\upsilon,\zeta+\mu)$ for all $\zeta,\mu > 0,$
(16) $\Upsilon(\gamma,\theta,\cdot): (0,\infty) \to (0,1]$ is neutrosophic continuous,
(17) $\lim_{\zeta \to \infty} \Upsilon(\gamma,\theta,\zeta) = 0$ for all $\zeta > 0,$
(18) If $\zeta < 0$ then $\Gamma(\gamma,\theta,\zeta) = 0,\ \Theta(\gamma,\theta,\zeta) = 1$ and $\Upsilon(\gamma,\theta,\zeta) = 1.$

Then (Γ,Θ,Υ) is referred NMS on Λ. The functions Γ,Θ &Υ denote degree of closedness, neutralness and non-closedness between γ and θ with respect to ζ respectively.

Definition :2.2 [19]
Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS. $\mathfrak{B}(\gamma,r,\zeta) = \{\ \theta \in \Lambda;\ \Gamma(\gamma,\theta,\zeta) > 1-r,\ \Theta(\gamma,\theta,\zeta) < r$ and $\Upsilon(\gamma,\theta,\zeta) < r\ \}$ defines the open ball $\mathfrak{B}(\gamma,r,\zeta)$ with centre $\gamma \in \Lambda,\ 0 < r < 1$ and $\zeta > 0$. The family $\{\mathfrak{B}(\gamma,r,\zeta) :\ \gamma \in \Lambda,\ 0 < r < 1$ and $\zeta > 0\}$ is a basis for a Hausdorff topology on Λ.

Definition :2.3 [19]
Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS.
(1) A sequence $\{\gamma_n\}$ in Λ is said to be convergent if there exists some $\gamma \in \Lambda$ such that $\lim_{n \to \infty} \Gamma(\gamma,\theta,\zeta) = 1,\ \lim_{n \to \infty} \Theta(\gamma,\theta,\zeta) = 0,\ \lim_{n \to \infty} \Upsilon(\gamma,\theta,\zeta) = 0$ for all $\zeta > 0$.
(2) A sequence $\{\gamma_n\}$ in Λ is said to be Cauchy sequence if $\lim_{m,n \to \infty} \Gamma(\gamma_n,\gamma_m,\zeta) = 1,$ $\lim_{m,n \to \infty} \Theta(\gamma_n,\gamma_m,\zeta) = 0,\ \lim_{m,n \to \infty} \Theta(\gamma_n,\gamma_m,\zeta) = 0$ for all $\zeta > 0$.

Definition :2.4 [21]
Let an NMS $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ have two non-empty Compact subsets, Ξ and Ω.The Hausdorff NMS $\mathcal{H}_\Gamma,\mathcal{H}_\Theta,\mathcal{H}_\Upsilon$ on $P(\Lambda)$ is defined by
$H_\Gamma(\Xi,\Omega,\zeta) = min\{\varrho(\Xi,\Omega,\zeta),\vartheta(\Xi,\Omega,\zeta)\}$ where
$\varrho(\Xi,\Omega,\zeta) = \inf_{\xi \in \Xi} \sup_{\omega \in \Omega} \Gamma(\xi,\omega,\zeta),\ \vartheta(\Xi,\Omega,\zeta) = \inf_{\omega \in \Omega} \sup_{\xi \in \Xi} \Gamma(\xi,\omega,\zeta),$
$H_\Theta(\Xi,\Omega,\zeta) = max\{\varrho(\Xi,\Omega,\zeta),\vartheta(\Xi,\Omega,\zeta)\}$ where
$\varrho(\Xi,\Omega,\zeta) = \sup_{\xi \in \Xi} \inf_{\omega \in \Omega} \Theta(\xi,\omega,\zeta),\ \vartheta(\Xi,\Omega,\zeta) = \sup_{\omega \in \Omega} \inf_{\xi \in \Xi} \Theta(\xi,\omega,\zeta),$
$H_\Upsilon(\Xi,\Omega,\zeta) = max\{\varrho(\Xi,\Omega,\zeta),\vartheta(\Xi,\Omega,\zeta)\}$ where
$\varrho(\Xi,\Omega,\zeta) = \sup_{\xi \in \Xi} \inf_{\omega \in \Omega} \Upsilon(\xi,\omega,\zeta),\ \vartheta(\Xi,\Omega,\zeta) = \sup_{\omega \in \Omega} \inf_{\xi \in \Xi} \Upsilon(\xi,\omega,\zeta), \zeta > 0.$

Lemma :2.5
Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS. Suppose $\{\Xi_j\}_{j=1}^{p},\ \{\Omega_j\}_{j=1}^{p} \subseteq P(\Lambda),\ \Xi = \cup_{j=1}^{p} \Xi_j$ and $\Omega = \cup_{j=1}^{p} \Omega_j$. Then for all $\zeta > 0,\ H_\Gamma(\Xi,\Omega,\zeta) \geq \min_{1 \leq j \leq \wp} H_\Gamma(\Xi_j,\Omega_j,\zeta),\quad H_\Theta(\Xi,\Omega,\zeta) \leq \max_{1 \leq j \leq \wp} H_\Theta(\Xi_j,\Omega_j,\zeta)$ and $H_\Upsilon(\Xi,\Omega,\zeta) \leq \max_{1 \leq j \leq \wp} H_\Upsilon(\Xi_j,\Omega_j,\zeta).$

Theorem :2.6
Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS. Then $(P(\Lambda),\mathcal{H}_\Gamma,\mathcal{H}_\Theta,\mathcal{H}_\Upsilon,*,\diamond)$ is a NMS. Also, if $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ is a Complete NMS then $(P(\Lambda),H_\Gamma,H_\Theta,H_\Upsilon,*,\diamond)$ is also a Complete NMS.

Definition :2.7
Let $T:\Lambda \times \Lambda \to \Lambda$ be a mapping. An element $(\gamma,\theta)\in\Lambda\times\Lambda$ is considered a Coupled Fixed Point of T if $T(\gamma,\theta)=\gamma$ and $T(\theta,\gamma)=\theta$.
A coupled fixed point is referred to as a strong coupled fixed point if γ = θ, in which case T(γ, γ) = γ. A strong coupled fixed point is defined as (γ, γ) ∈Λ x Λ.

Definition :2.8
Suppose Ξ and Ω are two non-empty subsets of Λ. A mapping $T:\Lambda\times\Lambda\to\Lambda$ is said to be cyclic $(with\ respect\ to\ \Xi\ and\ \Omega)$ if $T(\Xi)\subset\Omega$ and $T(\Omega)\subset\Xi$.

Definition :2.9
Suppose Ξ and Ω are two non-empty subsets of Λ. A mapping $T:\Lambda\times\Lambda\to\Lambda$ is considered Coupling with respect to Ξ and Ω if $T(\gamma,\theta)\in\Omega$ and $T(\theta,\gamma)\in\Xi$ for every $\gamma\in\Xi$ and $\theta\in\Omega$.

Definition :2.10
Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS and $\{T_{j,}\ j\in\mathbb{N}_n\}$ be a finite set of continuous couplings on Λ, each with regard to two non-empty closed subsets Ξ and Ω of Λ. The Hutchinson operator corresponding to $\{T_{j,}\ j\in\mathbb{N}_n\}$, $\hat{Z}:P(\Lambda)\times P(\Lambda)\to P(\Lambda)$, is defined as $\hat{Z}(\Xi,\Omega)=\bigcup_{j=1}^{n}\widehat{T_{j,}}(\Xi,\Omega)$, where $\widehat{T_{j,}}(\Xi,\Omega)=\{T_j(\xi,\omega):\xi\in\Xi,\omega\in\Omega\}$. The aforementioned definition is only applicable to couplings that are continuous.

Definition :2.11
A Neutrosophic Iterated Coupling System (NICS) consists of a Complete NMS $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$. Assume that the NMS is composed of two closed, non-empty subsets Ξ and Ω of Λ and is made up of limited number of couplings $T_j:\Lambda\times\Lambda\to\Lambda$ with respect to Ξ and Ω for all $j\in\mathbb{N}_n$. We denote it by $\langle(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond);\Xi,\Omega,T_j, j\in\mathbb{N}_n\rangle$.

Definition :2.12
Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a Complete NMS and $\hat{E}:P(\Lambda)\times P(\Lambda)\to P(\Lambda)$ be a mapping. A fractal $\Xi\in P(\Lambda)$ is considered strong coupled if $\hat{E}(\Xi,\Xi)=\Xi$.

Definition :2.13
Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS and Ξ and Ω consists of non-empty subsets of Λ. With regard to Ξ and Ω, we refer to a coupling T: Λ x Λ→Λ as a Neutrosophic Contractive Coupling in the event that
ρ ∈ (0,1) exists and such that

$$\Gamma(T(\gamma,\theta),T(\eta,\sigma),\rho\zeta)\geq(\Gamma(\gamma,\eta,\zeta))^{\frac{1}{2}}*(\Gamma(\theta,\rho,\zeta))^{\frac{1}{2}}, \quad (2.13.1)$$

$$\Theta(T(\gamma,\theta),T(\eta,\sigma),\rho\zeta)\leq(\Theta(\gamma,\eta,\zeta))^{\frac{1}{2}}\diamond(\Theta(\theta,\rho,\zeta))^{\frac{1}{2}}, \quad (2.13.2)$$

$$\Upsilon(T(\gamma,\theta),T(\eta,\sigma),\rho\zeta)\leq(\Upsilon(\gamma,\eta,\zeta))^{\frac{1}{2}}\diamond(\Upsilon(\theta,\rho,\zeta))^{\frac{1}{2}}$$
(2.13.3)

where $\gamma.\sigma\in\Xi$ and $\theta,\eta\in\Omega$.
In this case, the coupling's contractivity factor is represented by the constant ρ.

3. Strong Coupled Fixed-Point in NMS

Theorem :3.1
Suppose Ξ and Ω are two non-empty closed subsets of a complete NMS $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ satisfying (Definition 2.1 of (7), (12) and (17)). Assume that T∗Λ x Λ→Λ represents a Neutrosophic contractive coupling concerning Ξ and Ω. T has a single strong linked fixed point if $\Xi\cap\Omega\neq\emptyset$ and $*$ is a t-norm, $\diamond$ is a t-conorm such that $*\geq*_{q,}\ \diamond\leq\diamond_q$. Moreover, for arbitrary choice of $\gamma_0\in\Xi$ and $\theta_0\in\Omega$, the strong coupled fixed point is reached by both sequences $\{\gamma_n\}$, $\{\theta_n\}$, which are formed as $\gamma_{n+1}=T(\theta_n,\gamma_n)$ and $\theta_{n+1}=T(\gamma_n,\theta_n)$
Proof: The construction of $\{\gamma_n\}$ and $\{\theta_n\}$ implies that for all $n\geq0$, $\gamma_n\in\Xi$ and $\theta_n\in\Omega$. Now,
$\Gamma(\gamma_n,\theta_n,\zeta)=\Gamma(T(\theta_{n-1},\gamma_{n-1}),T(\gamma_{n-1},\theta_{n-1}),\zeta)$

$$\geq \left(\Gamma\left(\theta_{n-1},\gamma_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} * \left(\Gamma\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\geq \left(\Gamma\left(\theta_{n-1},\gamma_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} *_{q} \left(\Gamma\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\Gamma\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right) = \Gamma\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho}\right)$$

$$\geq \left(\Gamma\left(\theta_{n-2},\gamma_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} * \left(\Gamma\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}$$

$$\geq \left(\Gamma\left(\theta_{n-2},\gamma_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} *_{q} \left(\Gamma\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} = \Gamma\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right) \dots \geq \Gamma\left(\gamma_0,\theta_{0,}\frac{\zeta}{\rho^n}\right)$$

$$\Theta(\gamma_n,\theta_n,\zeta) = \Theta(T(\theta_{n-1},\gamma_{n-1}),T(\gamma_{n-1},\theta_{n-1}),\zeta)$$

$$\leq \left(\Theta\left(\theta_{n-1},\gamma_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} \diamond \left(\Theta\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\leq \left(\Theta\left(\theta_{n-1},\gamma_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} \diamond_{q} \left(\Theta\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\Theta\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right) = \Theta\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho}\right)$$

$$\leq \left(\Theta\left(\theta_{n-2},\gamma_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond \left(\Theta\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}$$

$$\leq \left(\Theta\left(\theta_{n-2},\gamma_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_{q} \left(\Theta\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} = \Theta\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right) \dots \leq \Theta\left(\gamma_0,\theta_{0,}\frac{\zeta}{\rho^n}\right)$$

$$\Upsilon(\gamma_n,\theta_n,\zeta) = \Upsilon(T(\theta_{n-1},\gamma_{n-1}),T(\gamma_{n-1},\theta_{n-1}),\zeta)$$

$$\leq \left(\Upsilon\left(\theta_{n-1},\gamma_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} \diamond \left(\Upsilon\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\leq \left(\Upsilon\left(\theta_{n-1},\gamma_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} \diamond_{q} \left(\Upsilon\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\Upsilon\left(\gamma_{n-1},\theta_{n-1,}\frac{\zeta}{\rho}\right) = \Upsilon\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho}\right)$$

$$\leq \left(\Upsilon\left(\theta_{n-2},\gamma_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond \left(\Upsilon\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}$$

$$\leq \left(\Upsilon\left(\theta_{n-2},\gamma_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_{q} \left(\Upsilon\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} = \Upsilon\left(\gamma_{n-2},\theta_{n-2,}\frac{\zeta}{\rho^2}\right) \dots \leq \Upsilon\left(\gamma_0,\theta_{0,}\frac{\zeta}{\rho^n}\right)$$

When we apply (Definition (2.1) condition (7), (12) and (17) to the inequality above, when $n \to \infty$, we obtain

$$\lim_{n\to\infty} \Gamma(\gamma_n,\theta_n,\zeta) = 1, \qquad (3.1.1)$$

$$\lim_{n\to\infty} \Theta(\gamma_n,\theta_n,\zeta) = 0 \qquad (3.1.2)$$

$$\lim_{n\to\infty} \Upsilon(\gamma_n,\theta_n,\zeta) = 0 \text{ for all } \zeta > 0 \qquad (3.1.3)$$

Again for all $n \in \mathbb{N}$ and $\zeta > 0$,

$$\Gamma(\gamma_{n+1},\theta_n,\zeta) = \Gamma(T(\theta_n,\gamma_n),T(\gamma_{n-1},\theta_{n-1}),\zeta)$$

$$\geq \left(\Gamma\left(\gamma_{n-1,}\theta_n,\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} * \left(\Gamma\left(\gamma_n,\theta_{n-1,}\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\geq \Gamma\left(\gamma_{n-1,}\theta_n,\frac{\zeta}{\rho}\right) * \Gamma\left(\gamma_n,\theta_{n-1,}\frac{\zeta}{\rho}\right) \text{ (t-norm is monotonic)}$$

$$\Gamma\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-1},\theta_{n-1}),\frac{\zeta}{\rho}\right) * \Gamma\left(T(\theta_{n-1},\gamma_{n-1}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho}\right)$$

$$\geq\left(\left(\Gamma\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} * \left(\Gamma\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right) * \left(\left(\Gamma\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} * \left(\Gamma\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right)$$

$$\geq\left(\Gamma\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} *_q \left(\Gamma\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} *_q \left(\left(\Gamma\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} *_q \left(\Gamma\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right)$$

$$=\Gamma\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right) *_q \Gamma\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)$$

$$=\Gamma\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-3},\theta_{n-3}),\frac{\zeta}{\rho^2}\right) *_q \Gamma\left(T(\theta_{n-3},\gamma_{n-3}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho^2}\right)$$

$$\geq\left(\left(\Gamma\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} * \left(\Gamma\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right) *_q \left(\left(\Gamma\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \left(\Gamma\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right)$$

$$\geq\left(\left(\Gamma\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} *_q \left(\Gamma\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right) *_q \left(\left(\Gamma\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} *_q \left(\Gamma\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right)$$

$$=\Gamma\left(\gamma_{n-3},\theta_{n-2},\tfrac{\zeta}{\rho^3}\right) *_q \Gamma\left(\gamma_{n-2},\theta_{n-3},\tfrac{\zeta}{\rho^3}\right)=\cdots=\Gamma\left(\gamma_0,\theta_1,\tfrac{\zeta}{\rho^n}\right) *_q \Gamma\left(\gamma_1,\theta_0,\tfrac{\zeta}{\rho^n}\right) \qquad (3.1.4)$$

$$\Theta(\gamma_{n+1},\theta_n,\zeta)=\Theta(T(\theta_n,\gamma_n),T(\gamma_{n-1},\theta_{n-1}),\zeta)$$

$$\leq\left(\Theta\left(\gamma_{n-1},\theta_n,\tfrac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} \diamond \left(\Theta\left(\gamma_n,\theta_{n-1},\tfrac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\leq\Theta\left(\gamma_{n-1},\theta_n,\tfrac{\zeta}{\rho}\right) \diamond \Theta\left(\gamma_n,\theta_{n-1},\tfrac{\zeta}{\rho}\right) \text{ (t-conorm is monotonic)}$$

$$\Theta\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-1},\theta_{n-1}),\frac{\zeta}{\rho}\right) \diamond \Theta\left(T(\theta_{n-1},\gamma_{n-1}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho}\right)$$

$$\leq\left(\left(\Theta\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond \left(\Theta\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right) \diamond \left(\left(\Theta\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond \left(\Theta\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right)$$

$$\leq\left(\Theta\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Theta\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_q \left(\left(\Theta\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Theta\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right)$$

$$=\Theta\left(\gamma_{n-1},\theta_{n-2},\tfrac{\zeta}{\rho^2}\right) \diamond_q \Theta\left(\gamma_{n-2},\theta_{n-1},\tfrac{\zeta}{\rho^2}\right)$$

$$=\Theta\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-3},\theta_{n-3}),\frac{\zeta}{\rho^2}\right) \diamond_q \Theta\left(T(\theta_{n-3},\gamma_{n-3}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho^2}\right)$$

$$\leq\left(\left(\Theta\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond \left(\Theta\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right) \diamond_q \left(\left(\Theta\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond \left(\Theta\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right)$$

$$\leq\left(\left(\Theta\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Theta\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right) \diamond_q \left(\left(\Theta\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Theta\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right)$$

$$=\Theta\left(\gamma_{n-3},\theta_{n-2},\tfrac{\zeta}{\rho^3}\right) \diamond_q \Theta\left(\gamma_{n-2},\theta_{n-3},\tfrac{\zeta}{\rho^3}\right)=\cdots=\Theta\left(\gamma_0,\theta_1,\tfrac{\zeta}{\rho^n}\right) \diamond_q \Theta\left(\gamma_1,\theta_0,\tfrac{\zeta}{\rho^n}\right) \qquad (3.1.5)$$

$$\Upsilon(\gamma_{n+1},\theta_n,\zeta)=\Upsilon(T(\theta_n,\gamma_n),T(\gamma_{n-1},\theta_{n-1}),\zeta)\leq\left(\Upsilon\left(\gamma_{n-1},\theta_n,\tfrac{\zeta}{\rho}\right)\right)^{\frac{1}{2}} \diamond \left(\Upsilon\left(\gamma_n,\theta_{n-1},\tfrac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}$$

$$\leq\Upsilon\left(\gamma_{n-1},\theta_n,\tfrac{\zeta}{\rho}\right) \diamond \Upsilon\left(\gamma_n,\theta_{n-1},\tfrac{\zeta}{\rho}\right) \text{ (t-conorm is monotonic)}$$

$$\Upsilon\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-1},\theta_{n-1}),\frac{\zeta}{\rho}\right) \diamond \Upsilon\left(T(\theta_{n-1},\gamma_{n-1}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho}\right)$$

$$\leq \left(\left(\Upsilon\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond \left(\Upsilon\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right) \diamond \left(\left(\Upsilon\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond \left(\Upsilon\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right)$$

$$\leq \left(\Upsilon\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Upsilon\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_q \left(\left(\Upsilon\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Upsilon\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right)\right)^{\frac{1}{2}}\right)$$

$$= \Upsilon\left(\gamma_{n-1},\theta_{n-2},\frac{\zeta}{\rho^2}\right) \diamond_q \Upsilon\left(\gamma_{n-2},\theta_{n-1},\frac{\zeta}{\rho^2}\right)$$

$$= \Upsilon\left(T(\theta_{n-2},\gamma_{n-2}),T(\gamma_{n-3},\theta_{n-3}),\frac{\zeta}{\rho^2}\right) \diamond_q \Upsilon\left(T(\theta_{n-3},\gamma_{n-3}),T(\gamma_{n-2},\theta_{n-2}),\frac{\zeta}{\rho^2}\right)$$

$$\leq \left(\left(\Upsilon\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond \left(\Upsilon\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right) \diamond_q \left(\left(\Upsilon\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond \left(\Upsilon\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right)$$

$$\leq \left(\left(\Upsilon\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Upsilon\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right) \diamond_q \left(\left(\Upsilon\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}} \diamond_q \left(\Upsilon\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right)\right)^{\frac{1}{2}}\right)$$

$$= \Upsilon\left(\gamma_{n-3},\theta_{n-2},\frac{\zeta}{\rho^3}\right) \diamond_q \Upsilon\left(\gamma_{n-2},\theta_{n-3},\frac{\zeta}{\rho^3}\right) = \cdots = \Upsilon\left(\gamma_0,\theta_1,\frac{\zeta}{\rho^n}\right) \diamond_q \Upsilon\left(\gamma_1,\theta_0,\frac{\zeta}{\rho^n}\right). \quad (3.1.6)$$

Following the sequence in the same manner as before $\Gamma(\gamma_n,\theta_{n+1},\zeta)$, $\Theta(\gamma_n,\theta_{n+1},\zeta)$ and $\Upsilon(\gamma_n,\theta_{n+1},\zeta)$ for all $n \in \mathbb{N}$ and $\zeta > 0$, we get

$$\Gamma(\gamma_n,\theta_{n+1},\zeta) \geq \Gamma\left(\gamma_0,\theta_1,\frac{\zeta}{\rho^n}\right) *_q \Gamma\left(\gamma_1,\theta_0,\frac{\zeta}{\rho^n}\right) \quad (3.1.7)$$

$$\Theta(\gamma_n,\theta_{n+1},\zeta) \leq \Theta\left(\gamma_0,\theta_1,\frac{\zeta}{\rho^n}\right) \diamond_q \Theta\left(\gamma_1,\theta_0,\frac{\zeta}{\rho^n}\right) \quad (3.1.8)$$

$$\Upsilon(\gamma_n,\theta_{n+1},\zeta) \leq \Upsilon\left(\gamma_0,\theta_1,\frac{\zeta}{\rho^n}\right) \diamond_q \Upsilon\left(\gamma_1,\theta_0,\frac{\zeta}{\rho^n}\right) \quad (3.1.9)$$

For all $n \in \mathbb{N}$ and $\zeta > 0$, let $\Phi_n(\zeta) = \Gamma\left(\gamma_0,\theta_1,\frac{\zeta}{\rho^n}\right) *_q \Gamma\left(\gamma_1,\theta_0,\frac{\zeta}{\rho^n}\right)$,
$X_n(\zeta) = \Theta\left(\gamma_0,\theta_1,\frac{\zeta}{\rho^n}\right) \diamond_q \Theta\left(\gamma_1,\theta_0,\frac{\zeta}{\rho^n}\right)$ and $\Psi_n(\zeta) = \Upsilon\left(\gamma_0,\theta_1,\frac{\zeta}{\rho^n}\right) \diamond_q \Upsilon\left(\gamma_1,\theta_0,\frac{\zeta}{\rho^n}\right)$.
Using (3.1.4), (3.1.7) and (Definition(2.3) condition (7) we get $\lim_{n\to\infty} \Phi_n(\zeta) = 1$,
using (3.1.5), (3.1.8) and (Definition (2.3) condition (12)) we get $\lim_{n\to\infty} X_n(\zeta) = 0$ and
using (3.1.6), (3.1.9) and (Definition (2.3) condition (17)) we get $\lim_{n\to\infty} \Psi_n(\zeta) = 0$ for all $\zeta > 0$.
Also for $\mathfrak{p} > n$ and $0 < \rho < 1$, $1 > 1 - \rho^{\mathfrak{p}-n} = (1-\rho)(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-1})$.
Hence, for every > 0, $\zeta > \zeta(1-\rho)(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-1})$.
Now, we demonstrate that in Ξ, $\{\gamma_n\}$ is a Cauchy sequence. For $\mathfrak{p} > n$,
We take into account the next two scenarios.
Case I: $\mathfrak{p} - n$ is even.

$$\Gamma(\gamma_n,\gamma_{\mathfrak{p}},\zeta) \geq \Gamma\left(\gamma_n,\gamma_{\mathfrak{p}},\zeta(1-\rho)(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-1})\right)$$
$$\geq \Gamma(\gamma_n,\theta_{n+1},\zeta(1-\rho)) * \Gamma(\theta_{n+1},\gamma_{n+2},\zeta(1-\rho)\rho) * \ldots * \Gamma(\gamma_{p-2},\theta_{\mathfrak{p}-1},\zeta(1-\rho)\rho^{\mathfrak{p}-n-2})$$
$$* \Gamma(\theta_{\mathfrak{p}-1},\gamma_p,\zeta(1-\rho)\rho^{\mathfrak{p}-n-1})$$
$$\geq \left(\Gamma\left(\gamma_0,\theta_1,\frac{\zeta(1-\rho)}{\rho^n}\right) *_q \Gamma\left(\gamma_1,\theta_0,\frac{\zeta(1-\rho)}{\rho^n}\right)\right) * \Gamma\left(\gamma_0,\theta_1,\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right) *_q$$
$$\Gamma\left(\gamma_1,\theta_0,\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right) * \ldots * \Gamma\left(\gamma_0,\theta_1,\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{\rho^{\mathfrak{p}-1}}\right) *_q \Gamma\left(\gamma_1,\theta_0,\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{\rho^{\mathfrak{p}-1}}\right) \text{ (using (3.1.4), (3.1.7))}$$
$$= \underbrace{\Phi_n(\zeta(1-\rho)) * \Phi_n(\zeta(1-\rho)) * \ldots * \Phi_n(\zeta(1-\rho))}_{(\mathfrak{p}-n)\,times}$$

$$\Theta(\gamma_n,\gamma_{\mathfrak{p}},\zeta) \leq \Theta\left(\gamma_n,\gamma_{\mathfrak{p}},\zeta(1-\rho)(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-1})\right)$$
$$\leq \Theta(\gamma_n,\theta_{n+1},\zeta(1-\rho)) \diamond \Theta(\theta_{n+1},\gamma_{n+2},\zeta(1-\rho)\rho) \diamond \ldots \diamond \Theta(\gamma_{p-2},\theta_{\mathfrak{p}-1},\zeta(1-\rho)\rho^{\mathfrak{p}-n-2})$$
$$\diamond \Theta(\theta_{\mathfrak{p}-1},\gamma_p,\zeta(1-\rho)\rho^{\mathfrak{p}-n-1})$$
$$\leq \left(\Theta\left(\gamma_0,\theta_1,\frac{\zeta(1-\rho)}{\rho^n}\right) \diamond_q \Theta\left(\gamma_1,\theta_0,\frac{\zeta(1-\rho)}{\rho^n}\right)\right) \diamond \Theta\left(\gamma_0,\theta_1,\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right) \diamond_q$$

$\Theta\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\diamond\ldots\diamond\Theta\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{\rho^{\mathfrak{p}-1}}\right)\diamond_{\mathfrak{q}}\ \Theta\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{\rho^{\mathfrak{p}-1}}\right)$ (using (3.1.5), (3.1.8))

$=\underbrace{X_n(\zeta(1-\rho))\diamond X_n(\zeta(1-\rho))\diamond\ldots\diamond X_n(\zeta(1-\rho))}_{(\mathfrak{p}-n)times}$

$\Upsilon(\gamma_n,\gamma_{\mathfrak{p}},\zeta)\le\Upsilon\left(\gamma_n,\gamma_{\mathfrak{p}},\zeta(1-\rho)(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-1})\right)$

$\le\Upsilon(\gamma_n,\theta_{n+1},\zeta(1-\rho))\diamond\Upsilon(\theta_{n+1},\gamma_{n+2,}\zeta(1-\rho)\rho)\diamond\ldots\diamond\Upsilon(\gamma_{p-2},\theta_{\mathfrak{p}-1},\zeta(1-\rho)\rho^{\mathfrak{p}-n-2})$

$\diamond\Upsilon(\theta_{\mathfrak{p}-1},\gamma_p,\zeta(1-\rho)\rho^{\mathfrak{p}-n-1})$

$\le\left(\Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)}{\rho^n}\right)\diamond_{\mathfrak{q}}\ \Upsilon\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)}{\rho^n}\right)\right)\diamond\Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\diamond_{\mathfrak{q}}$

$\Upsilon\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\diamond\ldots\diamond\Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{\rho^{\mathfrak{p}-1}}\right)\diamond_{\mathfrak{q}}\ \Upsilon\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{\rho^{\mathfrak{p}-1}}\right)$ (using (3.1.5), (3.1.8))

$=\underbrace{\Psi_n(\zeta(1-\rho))\diamond\Psi_n(\zeta(1-\rho))\diamond\ldots\diamond\Psi_n(\zeta(1-\rho))}_{(\mathfrak{p}-n)\ times}.$

Case II: $\mathfrak{p}-n$ is odd.

$\Gamma(\gamma_n,\gamma_{\mathfrak{p}},\zeta)\ge\Gamma\left(\gamma_n,\gamma_{\mathfrak{p}},\zeta(1-\rho)\left(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-2}+\frac{\rho^{\mathfrak{p}-n-1}}{2}+\frac{\rho^{\mathfrak{p}-n-1}}{2}\right)\right)$

$\ge\Gamma(\gamma_n,\theta_{n+1},\zeta(1-\rho))*\Gamma(\theta_{n+1},\gamma_{n+2,}\zeta(1-\rho)\rho)*\ldots*\Gamma(\gamma_{p-1},\theta_{\mathfrak{p}},\zeta(1-\rho)\rho^{\mathfrak{p}-n-2})$

$*\Gamma\left(\theta_{\mathfrak{p}-1},\gamma_p,\zeta(1-\rho)\frac{\rho^{\mathfrak{p}-n-1}}{2}\right)*\Gamma\left(\theta_{\mathfrak{p}},\gamma_p,\zeta(1-\rho)\frac{\rho^{\mathfrak{p}-n-1}}{2}\right)$

$\ge\left(\Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)}{\rho^n}\right)*_{\mathfrak{q}}\ \Gamma\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)}{\rho^n}\right)\right)$

$*\left(\Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)*_{\mathfrak{q}}\ \Gamma\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\right)*\ldots$

$*\left(\Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-2}}{\rho^{\mathfrak{p}-2}}\right)*_{\mathfrak{q}}\ \Gamma\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-2}}{\rho^{\mathfrak{p}-2}}\right)\right)$

$*\left(\Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{2\rho^{\mathfrak{p}-1}}\right)*_{\mathfrak{q}}\ \Gamma\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{2\rho^{\mathfrak{p}-1}}\right)\right)*\Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{2\rho^{\mathfrak{p}}}\right)$

$=\underbrace{\Phi_n(\zeta(1-\rho))*\Phi_n(\zeta(1-\rho))*\ldots*\Phi_n\left(\frac{\zeta(1-\rho)}{2}\right)}_{(\mathfrak{p}-n)\ times}*\ \Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)}{2\rho^{n+1}}\right)$

$\Theta(\gamma_n,\gamma_{\mathfrak{p}},\zeta)\le\Theta\left(\gamma_n,\gamma_{\mathfrak{p}},\zeta(1-\rho)\left(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-2}+\frac{\rho^{\mathfrak{p}-n-1}}{2}+\frac{\rho^{\mathfrak{p}-n-1}}{2}\right)\right)$

$\le\Theta(\gamma_n,\theta_{n+1},\zeta(1-\rho))\diamond\Theta(\theta_{n+1},\gamma_{n+2,}\zeta(1-\rho)\rho)\diamond\ldots\diamond\Theta(\gamma_{p-1},\theta_{\mathfrak{p}},\zeta(1-\rho)\rho^{\mathfrak{p}-n-2})$

$\diamond\Theta\left(\theta_{\mathfrak{p}-1},\gamma_p,\zeta(1-\rho)\frac{\rho^{\mathfrak{p}-n-1}}{2}\right)\diamond\Theta\left(\theta_{\mathfrak{p}},\gamma_p,\zeta(1-\rho)\frac{\rho^{\mathfrak{p}-n-1}}{2}\right)$

$\le\left(\Theta\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)}{\rho^n}\right)\diamond_{\mathfrak{q}}\ \Theta\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)}{\rho^n}\right)\right)$

$*\left(\Theta\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\diamond_{\mathfrak{q}}\ \Theta\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\right)\diamond\ldots$

$\diamond\left(\Theta\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-2}}{\rho^{\mathfrak{p}-2}}\right)\diamond_{\mathfrak{q}}\ \Theta\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-2}}{\rho^{\mathfrak{p}-2}}\right)\right)$

$\diamond\left(\Theta\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{2\rho^{\mathfrak{p}-1}}\right)\diamond_{\mathfrak{q}}\ \Theta\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{2\rho^{\mathfrak{p}-1}}\right)\right)\diamond\Theta\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{\mathfrak{p}-n-1}}{2\rho^{\mathfrak{p}}}\right)$

$=\underbrace{X_n(\zeta(1-\rho))\diamond X_n(\zeta(1-\rho))\diamond\ldots\diamond X_n\left(\frac{\zeta(1-\rho)}{2}\right)}_{(\mathfrak{p}-n)\ times}\diamond\ \Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)}{2P^{n+1}}\right)$

$\Upsilon(\gamma_n,\gamma_{\mathfrak{p}},\zeta)\le\Upsilon\left(\gamma_n,\gamma_{\mathfrak{p}},\zeta(1-\rho)\left(1+\rho+\rho^2+\cdots+\rho^{\mathfrak{p}-n-2}+\frac{\rho^{\mathfrak{p}-n-1}}{2}+\frac{\rho^{\mathfrak{p}-n-1}}{2}\right)\right)$

$\le\Upsilon(\gamma_n,\theta_{n+1},\zeta(1-\rho))\diamond\Upsilon(\theta_{n+1},\gamma_{n+2,}\zeta(1-\rho)\rho)\diamond\ldots\diamond\Upsilon(\gamma_{p-1},\theta_{\mathfrak{p}},\zeta(1-\rho)\rho^{\mathfrak{p}-n-2})$

$$\diamond \Upsilon\left(\theta_{p-1},\gamma_p,\zeta(1-\rho)\frac{\rho^{p-n-1}}{2}\right)\diamond \Upsilon\left(\theta_{p},\gamma_p,\zeta(1-\rho)\frac{\rho^{p-n-1}}{2}\right)$$
$$\leq\left(\Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)}{\rho^n}\right)\diamond_q \Upsilon\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)}{\rho^n}\right)\right)$$
$$*\left(\Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\diamond_q \Upsilon\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho}{\rho^{n+1}}\right)\right)\diamond \dots$$
$$\diamond\left(\Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{p-n-2}}{\rho^{p-2}}\right)\diamond_q \Upsilon\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{p-n-2}}{\rho^{p-2}}\right)\right)$$
$$\diamond\left(\Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{p-n-1}}{2\rho^{p-1}}\right)\diamond_q \Upsilon\left(\gamma_1,\theta_{0,}\frac{\zeta(1-\rho)\rho^{p-n-1}}{2\rho^{p-1}}\right)\right)\diamond \Upsilon\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)\rho^{p-n-1}}{2\rho^{p}}\right)$$
$$=\underbrace{\Psi_n\big(\zeta(1-\rho)\big)\diamond\Psi_n\big(\zeta(1-\rho)\big)\diamond\dots\diamond\Psi_n\left(\frac{\zeta(1-\rho)}{2}\right)}_{(p-n)\ times}\diamond\,\Gamma\left(\gamma_0,\theta_{1,}\frac{\zeta(1-\rho)}{2\rho^{n+1}}\right)$$

Combining the above two cases and (Definition (2.1) condition (7), (12) and (17)) and $\Phi_n(\zeta)=1$, $X_n(\zeta)=0$ and $\Psi_n(\zeta)=0$ as $n\to\infty$ for all $\zeta>0$, we find that $\{\gamma_n\}$ is a Cauchy sequence in Ξ. In the same way we may demonstrate that $\{\theta_n\}$ is also a Cauchy sequence in Ω. Since Ξ and Ω are closed subsets, there exists $\gamma\in\Xi$ and $\theta\in\Omega$, such that

$lim_{n\to\infty}\Gamma(\gamma_n,\gamma,\zeta)=1, lim_{n\to\infty}\Gamma(\theta_n,\theta,\zeta)=1,$ (3.1.10)

$lim_{n\to\infty}\Theta(\gamma_n,\gamma,\zeta)=0, lim_{n\to\infty}\Theta(\theta_n,\theta,\zeta)=0$ and (3.1.11)

$lim_{n\to\infty}\Upsilon(\gamma_n,\gamma,\zeta)=0, lim_{n\to\infty}\Upsilon(\theta_n,\theta,\zeta)=0.$ (3.1.12)

Also, $\Gamma(\gamma,\theta,\zeta)\geq\Gamma\left(\gamma,\gamma_n,\frac{\zeta-\rho\zeta}{2}\right)*\Gamma(\gamma_n,\theta_n,\rho\zeta)*\Gamma\left(\theta_n,\theta,\frac{\zeta-\rho\zeta}{2}\right),$ (3.1.13)

$\Theta(\gamma,\theta,\zeta)\leq\Theta\left(\gamma,\gamma_n,\frac{\zeta-\rho\zeta}{2}\right)\diamond\Theta(\gamma_n,\theta_n,\rho\zeta)\diamond\Theta\left(\theta_n,\theta,\frac{\zeta-\rho\zeta}{2}\right)$ and (3.1.14)

$\Upsilon(\gamma,\theta,\zeta)\leq\Upsilon\left(\gamma,\gamma_n,\frac{\zeta-\rho\zeta}{2}\right)\diamond\Upsilon(\gamma_n,\theta_n,\rho\zeta)\diamond\Upsilon\left(\theta_n,\theta,\frac{\zeta-\rho\zeta}{2}\right)$ (3.1.15)

Taking limit as $n\to\infty$ in the above inequalities and using ((3.1.1), (3.1.2), (3.1.3), (3.1.10), (3.1.11) and (3.1.12)) , we get $\gamma=\theta$. Hence, $\Xi\cap\Omega\neq\emptyset$ and $\gamma=\theta\in\Xi\cap\Omega$.

Now, $\Gamma(\gamma_n,T(\gamma,\theta),\zeta)\geq\Gamma(\gamma_n,T(\gamma,\theta),\rho\zeta)=\Gamma(T(\theta_{n-1},\gamma_{n-1}),T(\gamma,\theta),\rho\zeta)$
$\geq\Gamma(\theta_{n-1},\gamma,\rho\zeta)^{\frac{1}{2}}*\Gamma(\gamma_{n-1},\theta,\rho\zeta)^{\frac{1}{2}}=\Gamma(\theta_{n-1},\theta,\rho\zeta)^{\frac{1}{2}}*\Gamma(\gamma_{n-1},\gamma,\rho\zeta)^{\frac{1}{2}},$
$\Theta(\gamma_n,T(\gamma,\theta),\zeta)\leq\Theta(\gamma_n,T(\gamma,\theta),\rho\zeta)=\Theta(T(\theta_{n-1},\gamma_{n-1}),T(\gamma,\theta),\rho\zeta)$
$\leq\Theta(\theta_{n-1},\gamma,\rho\zeta)^{\frac{1}{2}}\diamond\Theta(\gamma_{n-1},\theta,\rho\zeta)^{\frac{1}{2}}=\Theta(\theta_{n-1},\theta,\rho\zeta)^{\frac{1}{2}}\diamond\Theta(\gamma_{n-1},\gamma,\rho\zeta)^{\frac{1}{2}}$ and
$\Upsilon(\gamma_n,T(\gamma,\theta),\zeta)\leq\Upsilon(\gamma_n,T(\gamma,\theta),\rho\zeta)=\Upsilon(T(\theta_{n-1},\gamma_{n-1}),T(\gamma,\theta),\rho\zeta)$
$\leq\Upsilon(\theta_{n-1},\gamma,\rho\zeta)^{\frac{1}{2}}\diamond\Upsilon(\gamma_{n-1},\theta,\rho\zeta)^{\frac{1}{2}}=\Upsilon(\theta_{n-1},\theta,\rho\zeta)^{\frac{1}{2}}\diamond\Upsilon\Gamma(\gamma_{n-1},\gamma,\rho\zeta)^{\frac{1}{2}}.$

Taking limit as $n\to\infty$ in the above inequalities and using ((3.1.10), (3.1.11) and 3.1.12)),
we get $\gamma_n\to T(\gamma,\gamma)$. Since the topology of the NMS is Hausdorff, we get $T(\gamma,\gamma)=\gamma$.

As a result, T's strong coupled fixed point is (γ, γ). To demonstrate the uniqueness of the strong coupled fixed point, consider $\nu\neq\gamma\in\Lambda$ where $T(\nu,\nu)=\nu$. Then

$$\Gamma(\gamma,\nu,\zeta)=\Gamma(T(\gamma,\gamma),T(\nu,\nu),\zeta)=\left(\Gamma\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}*\left(\Gamma\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}\geq\Gamma\left(\gamma,\nu,\frac{\zeta}{\rho}\right), \quad (3.1.16)$$

$$\Theta(\gamma,\nu,\zeta)=\Theta(T(\gamma,\gamma),T(\nu,\nu),\zeta)=\left(\Theta\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}\diamond\left(\Theta\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}\leq\Theta\left(\gamma,\nu,\frac{\zeta}{\rho}\right), \quad (3.1.17)$$

$$\Upsilon(\gamma,\nu,\zeta)=\Upsilon(T(\gamma,\gamma),T(\nu,\nu),\zeta)=\left(\Upsilon\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}\diamond\left(\Upsilon\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\right)^{\frac{1}{2}}\leq\Upsilon\left(\gamma,\nu,\frac{\zeta}{\rho}\right) \quad (3.1.18)$$

By applying ((3.1.16), (3.1.17) and (3.1.18))repeatedly we get for all n:
$\Gamma(\gamma,\nu,\zeta)\geq\Gamma\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\geq\Gamma\left(\gamma,\nu,\frac{\zeta}{\rho^2}\right)\geq\dots\geq\Gamma\left(\gamma,\nu,\frac{\zeta}{\rho^n}\right),$
$\Theta(\gamma,\nu,\zeta)\leq\Theta\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\leq\Theta\left(\gamma,\nu,\frac{\zeta}{\rho^2}\right)\leq\dots\leq\Theta\left(\gamma,\nu,\frac{\zeta}{\rho^n}\right)$ and
$\Upsilon(\gamma,\nu,\zeta)\leq\Upsilon\left(\gamma,\nu,\frac{\zeta}{\rho}\right)\leq\Upsilon\left(\gamma,\nu,\frac{\zeta}{\rho^2}\right)\leq\dots\leq\Upsilon\left(\gamma,\nu,\frac{\zeta}{\rho^n}\right).$

Taking limit as $n\to\infty$ in the above inequalities, using (Definition (2.1) condition (7), (12) and (17)), we have $\Gamma(\gamma,\nu,\zeta)=1,\Theta(\gamma,\nu,\zeta)=0$ and $\Upsilon(\gamma,\nu,\zeta)=0$. Therefore, $\gamma=\nu$. Hence, T has a unique strong coupled fixed point in $\Xi\cap\Omega$.

Example :3.2
Let $\Lambda = \mathbb{R}$ and $\Xi = \left[0, \frac{1}{2}\right]$, $\Omega = \left[-\frac{1}{2}, 0\right]$. Consider the NMS $(\Lambda, \Gamma, \Theta, \Upsilon, *, \diamond)$, where $*$ is the product t-norm, $\diamond$ is the algebraic sum, $\Gamma(\gamma, \theta, \zeta) = e^{-\frac{|\theta-\gamma|}{\zeta}}$, $\Theta(\gamma, \theta, \zeta) = 1 - e^{-\frac{|\theta-\gamma|}{\zeta}}$ and $\Upsilon(\gamma, \theta, \zeta) = e^{\frac{|\theta-\gamma|}{\zeta}} - 1$. Let $T: \Lambda \times \Lambda \to \Lambda$ be a mapping given by

$$T(\gamma, \theta) = \begin{cases} \frac{\theta - \gamma}{6} & if\ (\gamma, \theta) \in \left[-\frac{1}{2}, \frac{1}{2}\right] x \left[-\frac{1}{2}, \frac{1}{2}\right], \\ 2\gamma & otherwise. \end{cases}$$

It is evident from the Definition that T is a coupling with regard to $\Xi \cap \Omega$. We demonstrate that, with regard to $\Xi \cap \Omega$ T is a Neutrosophic contractive coupling

Let $\rho = \frac{1}{3}$. For $\gamma, \sigma \in \Xi$ and $\theta, \eta \in \Omega$, we get $\left(\Gamma(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} = e^{-\frac{|\gamma-\eta|}{2\zeta}}$, $\left(\Gamma(\theta, \sigma, \zeta)\right)^{\frac{1}{2}} = e^{-\frac{|\theta-\sigma|}{2\zeta}}$, $\left(\Theta(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} = 1 - e^{-\frac{|\gamma-\eta|}{2\zeta}}$, $\left(\Theta(\theta, \sigma, \zeta)\right)^{\frac{1}{2}} = 1 - e^{-\frac{|\theta-\sigma|}{2\zeta}}$ and $\left(\Upsilon(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} = e^{\frac{|\gamma-\eta|}{2\zeta}} - 1$, $\left(\Upsilon(\theta, \sigma, \zeta)\right)^{\frac{1}{2}} = e^{\frac{|\theta-\sigma|}{2\zeta}} - 1$.

$\Gamma(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) = e^{-\frac{|T(\gamma,\theta)-T(\eta,\sigma)|}{\rho\zeta}} = e^{-\frac{|(\theta-\gamma)-(\sigma-\eta)|}{6\rho\zeta}} = e^{-\frac{|(\eta-\gamma)+(\theta-\sigma)|}{2\zeta}}$

Also, $(\eta - \gamma) + (\theta - \sigma) \leq |\eta - \gamma| + |\theta - \sigma|$

or $\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta} \leq \frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}$ or $e^{-\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta}} \geq e^{-\frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}}$

or $\Gamma(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) \geq \left(\Gamma(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} * \left(\Gamma(\theta, \sigma, \zeta)\right)^{\frac{1}{2}}$

Similarly, $\Theta(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) = 1 - e^{-\frac{|T(\gamma,\theta)-T(\eta,\sigma)|}{\rho\zeta}} = 1 - e^{-\frac{|(\theta-\gamma)-(\sigma-\eta)|}{6\rho\zeta}} = 1 - e^{-\frac{|(\eta-\gamma)+(\theta-\sigma)|}{2\zeta}}$

Then, $(\eta - \gamma) + (\theta - \sigma) \leq |\eta - \gamma| + |\theta - \sigma|$ or $\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta} \leq \frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}$

or $e^{-\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta}} \geq e^{-\frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}}$ or $1 - e^{-\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta}} \leq 1 - e^{-\frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}}$

or $\Theta(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) \leq \left(\Theta(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} \diamond \left(\Theta(\theta, \sigma, \zeta)\right)^{\frac{1}{2}}$ and

$\Upsilon(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) = e^{\frac{|T(\gamma,\theta)-T(\eta,\sigma)|}{P\zeta}} - 1 = e^{\frac{|(\theta-\gamma)-(\sigma-\eta)|}{6P\zeta}} - 1 = e^{\frac{|(\eta-\gamma)+(\theta-\sigma)|}{2\zeta}} - 1$

Then, $(\eta - \gamma) + (\theta - \sigma) \leq |\eta - \gamma| + |\theta - \sigma|$

or $\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta} \leq \frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}$ or $e^{\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta}} \leq e^{\frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}}$

or $e^{\frac{(\eta-\gamma)+(\theta-\sigma)}{2\zeta}} - 1 \geq e^{\frac{|\eta-\gamma|+|\theta-\sigma|}{2\zeta}} - 1$

Thus, with contractivity factor ρ=1/3, we deduce that T is a Neutrosophic contractive coupling. Thus, Theorem 3.1's requirements are all met. There is a strong coupled fixed point (0,0) of T, according to Theorem (3.1); that is, $T(0,0)$ and also $0 \in \Xi \cap \Omega$. Hence, $\Xi \cap \Omega$ is non-empty.

Corollary :3.3
Let $(\Lambda, \Gamma, \Theta, \Upsilon, *, \diamond)$ be a Complete NMS satisfying (Definition 2.1 of (7), (12) and (17). with $*$ and $\diamond$ being stronger than the product t-norm and algebraic sum t-conorm. Let $T : \Lambda \times \Lambda \to \Lambda$ be a mapping that satisfies the following inequality for every $\gamma, \theta, \eta, \sigma \in \Lambda$:

$$\Gamma(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) \geq \left(\Gamma(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} * \left(\Gamma(\theta, \sigma, \zeta)\right)^{\frac{1}{2}}, \quad (3.3.1)$$

$$\Theta(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) \leq \left(\Theta(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} \diamond \left(\Theta(\theta, \sigma, \zeta)\right)^{\frac{1}{2}} \text{ and} \quad (3.3.2)$$

$$\Upsilon(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) \leq \left(\Upsilon(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} \diamond \left(\Upsilon(\theta, \sigma, \zeta)\right)^{\frac{1}{2}}. \quad (3.3.3)$$

T then has a fixed point that is strongly linked.

Proof:
Take $\Xi = \Omega = \Lambda$ in Theorem(3.1). Theorem (3.1) yields the desired outcome.

Remark: 3.4
In Example (3.2) , T is a Neutrosophic contractive coupling, but the inequality (Definition (2.13) equation (2.13.1), (2.13.2) and (2.13.3)) is not satisfied for all $\gamma, \theta, \eta, \sigma \in \Lambda$: For example, take $\gamma = \frac{1}{2}, \theta = -\frac{1}{2}, \eta = \frac{3}{2}, \sigma = -\frac{3}{2}$. Then $\Gamma(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) = \Gamma\left(-\frac{1}{6}, 3, \rho\zeta\right) = e^{-\frac{19}{6\rho\zeta}}$,

$\Theta(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) = \Theta\left(-\frac{1}{6}, 3, \rho\zeta\right) = 1 - e^{-\frac{19}{6\rho\zeta}}$, $\Upsilon(T(\gamma, \theta), T(\eta, \sigma), \rho\zeta) = \Upsilon\left(-\frac{1}{6}, 3, P\zeta\right) = e^{\frac{19}{6\rho\zeta}} - 1$.

Also, $\left(\Gamma(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} * \left(\Gamma(\theta, \sigma, \zeta)\right)^{\frac{1}{2}} = e^{-\frac{1}{\zeta}}$, $\left(\Theta(\gamma, \eta, \zeta)\right)^{\frac{1}{2}} \diamond \left(\Theta(\theta, \sigma, \zeta)\right)^{\frac{1}{2}} = 1 - e^{-\frac{1}{\zeta}}$ and

$\left(\Upsilon(\gamma,\eta,\zeta)\right)^{\frac{1}{2}} \diamond \left(\Upsilon(\theta,\sigma,\zeta)\right)^{\frac{1}{2}} = e^{\frac{1}{\zeta}} - 1$.

If inequalities ((2.13.1), (2.13.2) and (2.13.3)) holds, $e^{-\frac{19}{6\rho\zeta}} \geq e^{-\frac{1}{\zeta}}$, $\frac{e^{\frac{19}{6\rho\zeta}}-1}{e^{\frac{19}{6\rho\zeta}}} \leq \frac{e^{\frac{1}{\zeta}}-1}{e^{\frac{1}{\zeta}}}$ and

$e^{\frac{19}{6\rho\zeta}} - 1 \leq e^{\frac{1}{\zeta}} - 1$ which implies $\rho \geq \frac{19}{6} > 1$

This demonstrates how Corollary (3.3) of Theorem (3.1) is correctly contained in it.

It should be noted that the aforementioned result holds true for a variety of t-norms and t-conorms that are stronger than the product t-norm and algebraic sum t-conorm, such as the minimum t-norm, the H-type t-norm, the maximum t-conorm, etc.

4. Neutrosophic Contractive Coupling in Hausdorff NMS

Theorem :4.1

Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS satisfying (Definition (2.1) conditions (7), (12) and (17)), Ξ,Ω be two non-empty subsets of Λ, $T : \Lambda \times \Lambda \to \Lambda$ be a continuous Neutrosophic contractive coupling with regard to Ξ and Ω with contractivity factor ρ. Then

$\hat{T} : P(\Lambda) \times P(\Lambda) \to P(\Lambda)$ defined as $\hat{T}(\Xi,\Omega) = \{T(\xi,\omega) : \xi \in \Xi, \omega \in \Omega\}$ is a Neutrosophic contractive coupling with regard to $P(\Xi)$ and $P(\Omega)$ in the NMS $(P(\Lambda),\mathcal{H}_\Gamma,\mathcal{H}_\theta,\mathcal{H}_\Upsilon,*,\diamond)$ with the same contractivity factor.

Proof:

By the Definition (2.1) of $\hat{T}(\Xi,\Omega)$ it follows that for all $\mathfrak{C} \in P(\Xi)$ and $\mathfrak{D} \in P(\Omega)$,

$\hat{T}(\mathfrak{C},\mathfrak{D}) \in P(\Omega)$ and $\hat{T}(\mathfrak{D},\mathfrak{C}) \in P(\Xi)$. Let $\mathfrak{C}_1,\mathfrak{C}_2 \in P(\Xi)$ and $\mathfrak{D}_1,\mathfrak{D}_2 \in P(\Omega)$. Then

$\varrho\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = \varrho(\{T(\mathfrak{c}_1,\mathfrak{d}_1) : \mathfrak{c}_1 \in \mathfrak{C}_1, \mathfrak{d}_1 \in \mathfrak{D}_1\},\{T(\mathfrak{c}_2,\mathfrak{d}_2) : \mathfrak{c}_2 \in \mathfrak{C}_2, \mathfrak{d}_2 \in \mathfrak{D}_2\},\rho\zeta)$

$= \inf_{\substack{\mathfrak{c}_1 \in \mathfrak{C}_1 \\ \mathfrak{d}_1 \in \mathfrak{D}_1}} \sup_{\substack{\mathfrak{c}_2 \in \mathfrak{C}_2 \\ \mathfrak{d}_2 \in \mathfrak{D}_2}} \Gamma(T(\mathfrak{c}_1,\mathfrak{d}_1),T(\mathfrak{c}_2,\mathfrak{d}_2),\rho\zeta)$

$\geq \inf_{\substack{\mathfrak{c}_1 \in \mathfrak{C}_1 \\ \mathfrak{d}_1 \in \mathfrak{D}_1}} \sup_{\substack{\mathfrak{c}_2 \in \mathfrak{C}_2 \\ \mathfrak{d}_2 \in \mathfrak{D}_2}} \left(\Gamma(\mathfrak{c}_1,\mathfrak{c}_2,\zeta)\right)^{\frac{1}{2}} * \left(\Gamma(\mathfrak{d}_1,\mathfrak{d}_2,\zeta)\right)^{\frac{1}{2}}$ (by 2.13.1)

$= \left(\inf_{\mathfrak{c}_1 \in \mathfrak{C}_1} \sup_{\mathfrak{c}_2 \in \mathfrak{C}_2} \Gamma(\mathfrak{c}_1,\mathfrak{c}_2,\zeta)\right)^{\frac{1}{2}} * \left(\inf_{\mathfrak{d}_1 \in \mathfrak{D}_1} \sup_{\mathfrak{d}_2 \in \mathfrak{D}_2} \Gamma(\mathfrak{d}_1,\mathfrak{d}_2,\zeta)\right)^{\frac{1}{2}}$

$= \left(\varrho(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} * \left(\varrho(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}} \geq \left(\mathcal{H}_\Gamma(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Gamma(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$

Similarly, $\vartheta\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \geq \left(\mathcal{H}_\Gamma(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Gamma(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$.

Therefore $\mathcal{H}_\Gamma\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = \min\left\{\varrho\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right),\vartheta\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right)\right\}$

$\geq \left(\mathcal{H}_\Gamma(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Gamma(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$

$\varrho\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = \varrho(\{T(\mathfrak{c}_1,\mathfrak{d}_1) : \mathfrak{c}_1 \in \mathfrak{C}_1, \mathfrak{d}_1 \in \mathfrak{D}_1\},\{T(\mathfrak{c}_2,\mathfrak{d}_2) : \mathfrak{c}_2 \in \mathfrak{C}_2, \mathfrak{d}_2 \in \mathfrak{D}_2\},\rho\zeta)$

$= \sup_{\substack{\mathfrak{c}_1 \in \mathfrak{C}_1 \\ \mathfrak{d}_1 \in \mathfrak{D}_1}} \inf_{\substack{\mathfrak{c}_2 \in \mathfrak{C}_2 \\ \mathfrak{d}_2 \in \mathfrak{D}_2}} \Theta(T(\mathfrak{c}_1,\mathfrak{d}_1),T(\mathfrak{c}_2,\mathfrak{d}_2),\rho\zeta)$

$\leq \sup_{\substack{\mathfrak{c}_1 \in \mathfrak{C}_1 \\ \mathfrak{d}_1 \in \mathfrak{D}_1}} \inf_{\substack{\mathfrak{c}_2 \in \mathfrak{C}_2 \\ \mathfrak{d}_2 \in \mathfrak{D}_2}} \left(\Theta(\mathfrak{c}_1,\mathfrak{c}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\Theta(\mathfrak{d}_1,\mathfrak{d}_2,\zeta)\right)^{\frac{1}{2}}$ (by 2.13.2)

$= \left(\sup_{\mathfrak{c}_1 \in \mathfrak{C}_1} \inf_{\mathfrak{c}_2 \in \mathfrak{C}_2} \Theta(\mathfrak{c}_1,\mathfrak{c}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\sup_{\mathfrak{d}_1 \in \mathfrak{D}_1} \inf_{\mathfrak{d}_2 \in \mathfrak{D}_2} \Theta(\mathfrak{d}_1,\mathfrak{d}_2,\zeta)\right)^{\frac{1}{2}}$

$= \left(\varrho(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\varrho(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}} \leq \left(\mathcal{H}_\theta(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\theta(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$

Similarly, $\vartheta\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \leq \left(\mathcal{H}_\theta(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\theta(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$.

Therefore $\mathcal{H}_\theta\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = max\left\{\varrho\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right),\vartheta\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right)\right\}$

$\leq \left(\mathcal{H}_\theta(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\theta(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$.

$\varrho\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = \varrho(\{T(\mathfrak{c}_1,\mathfrak{d}_1) : \mathfrak{c}_1 \in \mathfrak{C}_1, \mathfrak{d}_1 \in \mathfrak{D}_1\},\{T(\mathfrak{c}_2,\mathfrak{d}_2) : \mathfrak{c}_2 \in \mathfrak{C}_2, \mathfrak{d}_2 \in \mathfrak{D}_2\},\rho\zeta)$

$= \sup_{\substack{\mathfrak{c}_1 \in \mathfrak{C}_1 \\ \mathfrak{d}_1 \in \mathfrak{D}_1}} \inf_{\substack{\mathfrak{c}_2 \in \mathfrak{C}_2 \\ \mathfrak{d}_2 \in \mathfrak{D}_2}} \Upsilon(T(\mathfrak{c}_1,\mathfrak{d}_1),T(\mathfrak{c}_2,\mathfrak{d}_2),\rho\zeta)$

$$\leq \sup_{\substack{\mathfrak{c}_1 \in \mathfrak{C}_1 \\ \mathfrak{d}_1 \in \mathfrak{D}_1}} \inf_{\substack{\mathfrak{c}_2 \in \mathfrak{C}_2 \\ \mathfrak{d}_2 \in \mathfrak{D}_2}} \left(\Upsilon(\mathfrak{c}_1,\mathfrak{c}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\Upsilon(\mathfrak{d}_1,\mathfrak{d}_2,\zeta)\right)^{\frac{1}{2}} \qquad \text{(by 2.13.3)}$$

$$= \left(\sup_{\mathfrak{c}_1 \in \mathfrak{C}_1} \inf_{\mathfrak{c}_2 \in \mathfrak{C}_2} \Upsilon(\mathfrak{c}_1,\mathfrak{c}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\sup_{\mathfrak{d}_1 \in \mathfrak{D}_1} \inf_{\mathfrak{d}_2 \in \mathfrak{D}_2} \Upsilon(\mathfrak{d}_1,\mathfrak{d}_2,\zeta)\right)^{\frac{1}{2}}$$

$$= \left(\varrho(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\varrho(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}} \leq \left(\mathcal{H}_\Upsilon(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\Upsilon(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$$

Similarly, $\vartheta\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \leq \left(\mathcal{H}_\Upsilon(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\Upsilon(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$.

Therefore $\mathcal{H}_\Upsilon\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = max\left\{\varrho\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right),\vartheta\left(\hat{T}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right)\right\}$

$\leq \left(\mathcal{H}_\Upsilon(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\Upsilon(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$.

Thus, $\hat{T} : P(\Lambda) \times P(\Lambda) \to P(\Lambda)$ is a Neutrosophic contractive coupling with regard to $P(\Xi)$ and $P(\Omega)$ in the Hausdorff NMS $(P(\Lambda),\mathcal{H}_\Gamma,\mathcal{H}_\Theta,\mathcal{H}_\Upsilon,*,\diamond)$ with the same contractivity factor P.

Lemma : 4.2

Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a NMS satisfying (Definition (2.1) conditions (7), (12) and (17)). Let $T_1, T_2 \dots T_n$ be a finite number of continuous Neutrosophic contractive coupling on $\Lambda \times \Lambda$ with regard to Ξ and Ω, each having **a** contractivity factor $\rho_1,\rho_2,\dots\rho_n$, correspondingly. Then the Hutchinson operator $\hat{Z} : P(\Lambda) \times P(\Lambda) \to P(\Lambda)$ is a Neutrosophic contractive coupling in the NMS $(P(\Lambda),\mathcal{H}_\Gamma,\mathcal{H}_\Theta,\mathcal{H}_\Upsilon,*,\diamond)$ with respect to $P(\Xi)$ and $P(\Omega)$ with contractivity factor $\rho = max\{\rho_n ; n \in \mathbb{N}_n\}$.

Proof:

By the definition of $\hat{Z}$, it follows that for all $\mathfrak{C} \in P(\Xi)$ and $\mathfrak{D} \in P(\Omega)$, $\hat{Z}(\mathfrak{C},\mathfrak{D}) \in P(\Omega)$ and $\hat{Z}(\mathfrak{D},\mathfrak{C}) \in P(\Xi)$. Let $\mathfrak{C}_1,\mathfrak{C}_2 \in P(\Xi)$ and $\mathfrak{D}_1,\mathfrak{D}_2 \in P(\Omega)$;

$$\mathcal{H}_\Gamma\left(\hat{Z}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{Z}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = \mathcal{H}_\Gamma\left(\bigcup_{j=1}^{n}\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\bigcup_{j=1}^{n}\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right)$$

$$\geq \min_{1\leq j\leq n} \mathcal{H}_\Gamma\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \qquad \text{(by Lemma 2.5)}$$

But, since each $\hat{T}_j$ is a Neutrosophic contractive coupling, we have for $j = 1,2,\dots n$,

$$\mathcal{H}_\Gamma\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \geq \mathcal{H}_\Gamma\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho_j\zeta\right)$$

$$\geq \left(\mathcal{H}_\Gamma(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Gamma(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}} \qquad \text{(by 2.13.1)}$$

Hence, $\min_{1\leq j\leq n} \mathcal{H}_\Gamma\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \geq \left(\mathcal{H}_\Gamma(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Gamma(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$

Therefore, $\mathcal{H}_\Gamma\left(\hat{Z}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{Z}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \geq \left(\mathcal{H}_\Gamma(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Gamma(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$.

$$\mathcal{H}_\Theta\left(\hat{Z}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{Z}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = \mathcal{H}_\Theta\left(\bigcap_{j=1}^{n}\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\bigcap_{j=1}^{n}\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right)$$

$$\leq \max_{1\leq j\leq n} \mathcal{H}_\Theta\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \qquad \text{(by Lemma 2.5)}$$

But, since each $\hat{T}_j$ is a Neutrosophic contractive coupling, we have for $j = 1,2,\dots n$,

$$\mathcal{H}_\Theta\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \leq \mathcal{H}_\Theta\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho_j\zeta\right)$$

$$\leq \left(\mathcal{H}_\Theta(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\Gamma(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}} \qquad \text{(by 2.13.2)}$$

Hence, $\max_{1\leq j\leq n} \mathcal{H}_\Theta\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \leq \left(\mathcal{H}_\Theta(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\Theta(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$

Therefore, $\mathcal{H}_\Theta\left(\hat{Z}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{Z}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \leq \left(\mathcal{H}_\Theta(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\Theta(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}}$.

$$\mathcal{H}_\Upsilon\left(\hat{Z}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{Z}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) = \mathcal{H}_\Upsilon\left(\bigcap_{j=1}^{n}\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\bigcap_{j=1}^{n}\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right)$$

$$\leq \max_{1\leq j\leq n} \mathcal{H}_\Upsilon\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \qquad \text{(by Lemma 2.5)}$$

But, since each $\hat{T}_j$ is a Neutrosophic contractive coupling, we have for $j = 1,2,\dots n$,

$$\mathcal{H}_\Upsilon\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta\right) \leq \mathcal{H}_\Upsilon\left(\hat{T}_j(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_j(\mathfrak{C}_2,\mathfrak{D}_2),\rho_j\zeta\right)$$

$$\leq \left(\mathcal{H}_\Upsilon(\mathfrak{C}_1,\mathfrak{C}_2,\zeta)\right)^{\frac{1}{2}} \diamond \left(\mathcal{H}_\Upsilon(\mathfrak{D}_1,\mathfrak{D}_2,\zeta)\right)^{\frac{1}{2}} \qquad \text{(by 2.13.3)}$$

Hence, $\max_{1 \le \mathrm{j} \le n} \mathcal{H}_{\Upsilon}(\hat{T}_{\mathrm{j}}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{T}_{\mathrm{j}}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta) \le (\mathcal{H}_{\Upsilon}(\mathfrak{C}_1,\mathfrak{C}_2,\zeta))^{\frac{1}{2}} \diamond (\mathcal{H}_{\Upsilon}(\mathfrak{D}_1,\mathfrak{D}_2,\zeta))^{\frac{1}{2}}$

Therefore, $\mathcal{H}_{\Upsilon}(\hat{Z}(\mathfrak{C}_1,\mathfrak{D}_1),\hat{Z}(\mathfrak{C}_2,\mathfrak{D}_2),\rho\zeta) \le (\mathcal{H}_{\Upsilon}(\mathfrak{C}_1,\mathfrak{C}_2,\zeta))^{\frac{1}{2}} \diamond (\mathcal{H}_{\Upsilon}(\mathfrak{D}_1,\mathfrak{D}_2,\zeta))^{\frac{1}{2}}$.

Hence proved.

Theorem :4.3

Let $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ be a complete NMS satisfying (Definition (2.1) condition (7), (12) and (17)). Consider a Neutrosophic Iterated Coupling System $\langle(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond);\Xi,\Omega,T_{\mathrm{j}},\mathrm{j}\in\mathbb{N}_n\rangle$ made up of a finite number of continuous Neutrosophic contractive couplings on $\Lambda\ x\ \Lambda$ with regard to two closed subsets Ξ,Ω of Λ, and the associated Hutchinson operator is denoted by $\hat{Z}: P(\Lambda)\ \mathrm{x}\ P(\Lambda) \to P(\Lambda)$; then there is only one strong coupled fractal for $\hat{Z}$, ie. there exists a $M \in P(\Xi) \cap P(\Omega)$, such that $\hat{Z}(M,M) = M$.

Further, both the iterations $\{\Xi_n\}$ and $\{\Omega_n\}$ constructed as $\Omega_{n+1} = \hat{Z}(\Xi_n,\Omega_n)$, $\Xi_{n+1} = \hat{Z}(\Omega_n,\Xi_n), n \ge 0$, with $\Xi_0 \in P(\Xi)$ and $\Omega_0 \in P(\Omega)$ selected at random, they approach the strong coupled fractal.

Proof:

By (Lemma 4.2), $\hat{Z}$ is a Neutrosophic contractive coupling with contractivity factor $\rho = max\{\rho_n : n \in \mathbb{N}_n\}$. But, since $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$ is complete, $(P(\Lambda),\mathcal{H}_{\Gamma},\mathcal{H}_{\Theta},\mathcal{H}_{\Upsilon},*,\diamond)$ is also complete. Since, Ξ,Ω are closed subsets of Λ, $P(\Xi)$ and $P(\Omega)$ are also closed subsets of the NMS $(P(\Lambda),\mathcal{H}_{\Gamma},\mathcal{H}_{\Theta},\mathcal{H}_{\Upsilon},*,\diamond)$. An application of Theorem (3.1) comes next in the theorem.

Example : 4.4

Let $\Lambda = \mathbb{R}$ and $\Xi = [-2,2], \Omega = [-1,2]$. Consider the NMS $(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond)$, where $*$ is the minimum t-norm and $\diamond$ is the maximum t-conorm. Let $\Gamma(\gamma,\theta,\zeta) = e^{-\frac{|\gamma-\theta|}{\zeta}}$,

$\Theta(\gamma,\theta,\zeta) = \frac{e^{\frac{|\gamma-\theta|}{\zeta}}-1}{e^{\frac{|\gamma-\theta|}{\zeta}}}$ and $\Upsilon(\gamma,\theta,\zeta) = e^{\frac{|\gamma-\theta|}{\zeta}} - 1$.

Let $T_1,T_2 : \Lambda\ \mathrm{x}\ \Lambda \to \Lambda$ given by $T_1(\gamma,\theta) = \frac{\theta-\gamma}{11}$, $T_2(\gamma,\theta) = 1 + \frac{\theta-\gamma}{11}$. For $\gamma \in \Xi = [-2,2]$ and $\theta \in \Omega = [-1,2]$, $T_1(\gamma,\theta),T_2(\gamma,\theta) \in \Omega$ and $T_1(\theta,\gamma),T_2(\theta,\gamma) \in \Xi$. Then T_1,T_2 are couplings with regard to Ξ,Ω. Then the ICS $\langle(\Lambda,\Gamma,\Theta,\Upsilon,*,\diamond);\ \Xi,\Omega,T_{\mathrm{j}},\mathrm{j}\in\mathbb{N}_2\rangle$ generates a strong coupled fractal.

Let $\Xi_0 = \Omega_0 = \left[-\frac{1}{2},\frac{3}{2}\right]$. Next, the following are the first four steps of the iteration that lead to the strong coupled fractal:

$\Xi_1 = \Omega_1 = \left[-\frac{1}{2},\frac{3}{2}\right]$,

$\Xi_2 = T(\Xi_1,\Xi_1) = \left[-\frac{2}{11},\frac{2}{11}\right] \cup \left[\frac{9}{11},\frac{13}{11}\right]$,

$\Xi_3 = T(\Xi_2,\Xi_2) = \left[-\frac{15}{121},\frac{-7}{121}\right] \cup \left[-\frac{4}{121},\frac{4}{121}\right] \cup \left[\frac{7}{121},\frac{15}{121}\right] \cup \left[\frac{106}{121},\frac{114}{121}\right] \cup \left[\frac{117}{121},\frac{125}{121}\right] \cup \left[\frac{128}{121},\frac{136}{121}\right]$

$\Xi_4 = T(\Xi_3,\Xi_3) = \left[-\frac{151}{1331},\frac{-91}{1331}\right] \cup \left[-\frac{30}{1331},\frac{30}{1331}\right] \cup \left[\frac{91}{1331},\frac{151}{1331}\right] \cup \left[\frac{1180}{1331},\frac{1240}{1331}\right] \cup \left[\frac{1301}{1331},\frac{1361}{1331}\right] \cup \left[\frac{1422}{1331},\frac{1482}{1331}\right]$.

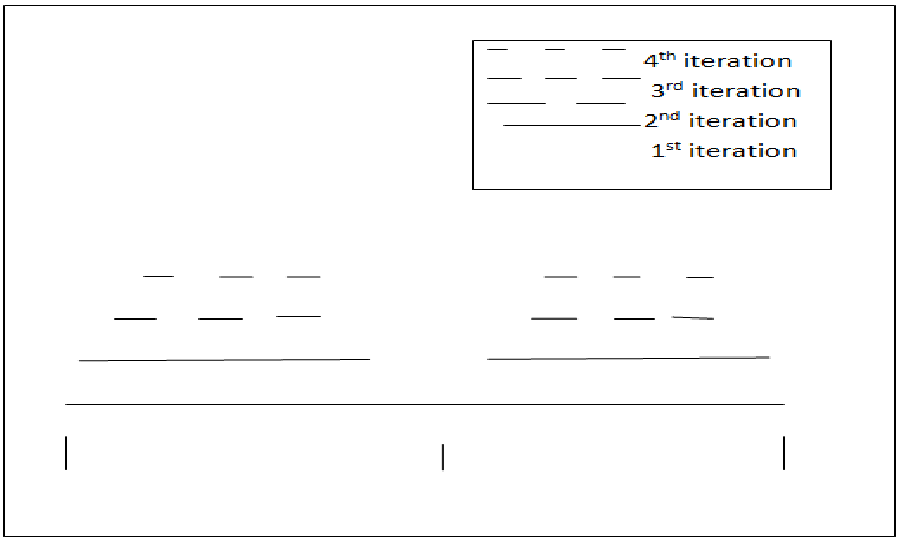

Figure 1. illustrates the first four iterations.

Example : 4.5

Let $\Lambda = \mathbb{R}$ and $\Xi = [-2,2], \Omega = [-1,2]$. Consider the NMS $(\Lambda, \Gamma, \Theta, \Upsilon, *, \diamond)$, where $*$ is the minimum t-norm and $\diamond$ is the maximum t-conorm. Let $\Gamma(\gamma,\theta,\zeta) = e^{-\frac{|\gamma-\theta|}{\zeta}}$, $\Theta(\gamma,\theta,\zeta) = \frac{e^{\frac{|\gamma-\theta|}{\zeta}}-1}{e^{\frac{|\gamma-\theta|}{\zeta}}}$ and

$\Upsilon(\gamma,\theta,\zeta) = e^{\frac{|\gamma-\theta|}{\zeta}} - 1$. Let $T_1, T_2 : \Lambda \times \Lambda \to \Lambda$ given by $T_1(\gamma,\theta) = \frac{\theta-\gamma}{18}$, $T_2(\gamma,\theta) = 1 + \frac{\theta-\gamma}{18}$.

Then the ICS $\langle(\Lambda, \Gamma, \Theta, \Upsilon, *, \diamond);\ \Xi, \Omega, T_j, j \in \mathbb{N}_2\rangle$ has an attractor.

$$\Xi_1 = \Omega_1 = \left[-\frac{1}{2}, \frac{3}{2}\right],$$

$$\Xi_2 = T(\Xi_1, \Xi_1) = \left[-\frac{1}{9}, \frac{1}{9}\right] \cup \left[\frac{8}{9}, \frac{10}{9}\right]$$

$$\Xi_3 = T(\Xi_2, \Xi_2) = \left[-\frac{11}{162}, \frac{-7}{162}\right] \cup \left[-\frac{1}{81}, \frac{1}{81}\right] \cup \left[\frac{7}{162}, \frac{11}{162}\right] \cup \left[\frac{151}{162}, \frac{155}{162}\right] \cup \left[\frac{80}{81}, \frac{82}{81}\right] \cup \left[\frac{169}{162}, \frac{173}{162}\right]$$

$$\Xi_4 = T(\Xi_3, \Xi_3) = \left[-\frac{11}{1458}, \frac{-1}{162}\right] \cup \left[-\frac{13}{2916}, \frac{-46}{729}\right] \cup \left[\frac{-140}{2916}, \frac{-148}{2916}\right] \cup \left[\frac{-149}{2916}, \frac{-170}{2916}\right]$$
$$\cup \left[\frac{-175}{2916}, \frac{13}{2916}\right] \cup \left[\frac{1}{162}, \frac{11}{1458}\right] \cup \left[\frac{140}{2916}, \frac{148}{2916}\right] \cup \left[\frac{149}{2916}, \frac{170}{2916}\right] \cup \left[\frac{175}{2916}, \frac{46}{729}\right]$$
$$\cup \left[\frac{683}{729}, \frac{2741}{2916}\right] \cup \left[\frac{2746}{2916}, \frac{2767}{2916}\right] \cup \left[\frac{2768}{2916}, \frac{2776}{2916}\right] \cup \left[\frac{1447}{1458}, \frac{161}{162}\right] \cup \left[\frac{2903}{2916}, \frac{2929}{2916}\right]$$
$$\cup \left[\frac{163}{162}, \frac{1469}{1458}\right] \cup \left[\frac{3056}{2916}, \frac{3064}{2916}\right] \cup \left[\frac{3065}{2916}, \frac{3086}{2916}\right] \cup \left[\frac{3091}{2916}, \frac{775}{720}\right]$$

CONCLUSION

In this study, the author defined the concept of coupling. In a complete NMS, we have defined a mapping which is a neutrosophic contractive coupling and proved that this mapping has a unique strong coupled fixed point with an application. Also, the author defined Hutchinson operator which is a neutrosophic contractive coupling in the NMS with contractivity factor.

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