Generalized completely monotone functions on some types of white noise spaces

Hossam. A. Ghany[1,*], Ahmed. M. Zabel[2] and Ayat. Nassar[3]

[1] Department of Mathematics, Faculty of Technology, Helwan University,

Sawah Street (11282), Cairo, Egypt.

[*]Corresponding author. h.abdelghany@yahoo.com

[2] Department of Mathematics, Faculty of Science, AzharUniversity,

Naser City (11884), Cairo, Egypt.

[3] Department of Mathematics, Faculty of Science, KafrelSheikh University,

Kafrel Sheikh, Egypt.

Abstract: With this paper, we purpose to introduce and characterized some generalized classes of completely monotone functions on some types of white noise spaces.

Keywords: White Noise; moment function; generalized functions.

2010 Mathematics Subject Classification: 43A62, 60H40, 30G35.

1. Introduction

The basic features of the completely monotone functions constructed on some forms of white noise spaces are provided in this study. if for each $\alpha \in \mathbb{Z}_+^n, (-1)^{|\alpha|} D^\alpha f(x) \geq 0$, then a function f is completely monotone on $\mathbb{R}_+^n$; see [3, 8, 12] for several features of completely monotone functions. According to Bernstein's theorem, f is completely monotone if and only if

$$f(x) = \int_{\mathbb{R}^n} e^{-x.t}\ d\mu(t) \tag{1.1}$$

μ is a positive measure that is based on a subset of $\mathbb{R}_+^n$. Let Q stand for a locally compact basis on the space $\mathbb{R}^d$. $C_b(Q)$ is a linear space of continuous bounded complex-valued functions which is a complete normed space compared to the norm

$$\|f\|_\infty = \sup_{x\in Q} |f(x)| \tag{1.2}$$

f defined on Q, where The space of infinitely differential and bounded functions on Q will be denoted by $C_b^{\infty}(Q)$,Moreover, by $S(Q)$, the linear subspace of $C_b^{\infty}(Q)$ created by the set which contains functions on Qlike that $x^{\alpha} D^{\beta} f(x) \leq C_{\alpha,\beta}$, with$\alpha, \beta \in \mathbb{Z}_+^n$and a constant $C_{\alpha,\beta}$.The space of tempered distributions is represented by $S^{'}(Q)$, Which is linear and continuous functional on $S(Q)$. There are numerous works that explore white noise spaces. Using the Wiener-Itô-Segal isomorphism and other Fock space riggings, some of these works are devoted to the building of test spaces, generalized functions, and operators having to act in these spaces [1,9].The study of PDEs and quantum field theory, where quantum fields are characterized as operator valued distributions, both depend heavily on distributions [5,11]. The works of Berezanskyi and Samoilenko [2] and Hida [9] are where the modern theory of generalized functions of infinitely many variables is derived. As infinite tensor products of one-dimensional spaces, the test and generalized function spaces in [2] were created. The theory of generalized functions was constructed using the classical method in [9], but all functions were functions of a point in the infinite-dimensional space on which the Gaussian measure was defined, which served the same purpose as the Lebesgue measure in the classical theory of generalized functions. The structure of this paper is as follows: section 2, we devoted to introduce and give the main properties of the class of double monotone functions defined on $S(Q)$. In section 3, the main properties of the class of weak monotone functions defined on $S(Q)$ are given.Section 4 introduces a novel method for creating spaces of generalized functions. Section 5 concludes by deriving the principal relationships between the creation of hypercomplex systems and the theory of white noise analysis.

2. Double completely monotone functions on $S(Q)$

Rabidly decreasing functions are the name given to the components of $S(Q)$, which has a family of seminorms for each $\alpha, \beta \in \mathbb{Z}_+^n$

$$\|f\|_{\alpha,\beta} = \sup_{x \in Q} \left| x^{\alpha} D^{\beta} f(x) \right| \tag{2.1}$$

Let $F : Q \to \mathbb{C}$ be a continuous double completely monotone function, i.e., $F = f_1 + i f_2$ and f_1, f_2 are two completely monotone functions. We define

$$\langle \cdot , \cdot \rangle_F : C_0(Q) \times C_0(Q) \to \mathbb{C}$$

by

$$\langle \varphi, \psi \rangle_F := \int_Q \int_Q F(x-y)\, \varphi(x)\, \overline{\psi(y)}\, d\mu(x) d\nu(y) \tag{2.2}$$

where $\mu, \nu \in M(Q)$, the space of Radon measure on Q. The inner product $\langle \cdot , \cdot \rangle_{\mathrm{F}}$ satisfies the following conditions:

I. $\langle \cdot , \cdot \rangle_F$in the first coordinate is complex linear and in the second conjugate complex linear i.e., for any $\varphi, \psi \in C_0(Q)$ and any $c \in \mathbb{C}$

$$\langle c\varphi, \psi \rangle_F = c \langle \varphi, \psi \rangle_F \text{and} \langle \varphi, c\psi \rangle_F = \overline{c} \langle \varphi, \psi \rangle_F$$

II. $\langle \cdot , \cdot \rangle_F$ is conjugate symmetric i.e., for any $\varphi, \psi \in C_0(Q)$

$$\langle \varphi, \psi \rangle_F = \overline{\langle \psi, \varphi \rangle_F}$$

III. $\langle \cdot , \cdot \rangle_F$ is positive definite meaning that for any $\varphi \in C_0(Q)$

$$\langle \varphi, \varphi \rangle_F = L_F(\varphi) \geq 0$$

IV. For all $\varphi \in C_0(Q)$such that $\langle \varphi, \varphi \rangle_F = 0$, then $\varphi = 0$

Theorem 2.1. For any double completely monotone function F on Q, the inner product space $(C_0(Q), \langle \cdot , \cdot \rangle_F)$ is a complex Hilbert space.

Proof. We have that $\langle \cdot , \cdot \rangle_F$ is an inner product space and $C_0(Q)$ is an infinite space so all we need to prove is the completeness for that space , so we assume that we have a Cauchy sequence $\{\varphi_n\}$ and should prove that this Cauchy sequence converges to a limit in $(C_0(Q), \langle \cdot , \cdot \rangle_F)$.

Where

$$\langle \varphi, \psi \rangle_F = \int_Q \int_Q F(x-y)\, \varphi(x)\, \overline{\psi(y)}\, d\mu(x) d\nu(y)$$

$\varphi,\psi \in C_0(Q)$, $\mu,\nu \in M(Q)$ the space of Radon measure on .

$$\begin{aligned} \|\varphi_n - \varphi_m\|^2 &= \langle \varphi_n - \varphi_m, \varphi_n - \varphi_m \rangle \\ &= \int_Q \int_Q F(x-y)(\varphi_n - \varphi_m)(x)\, \overline{(\varphi_n - \varphi_m)(y)}\, d\mu(x) d\nu(y) \\ &= \int_Q \int_Q F(x-y) |\varphi_n(x) - \varphi_m(x)|^2 \, d\mu(x) d\nu(y) \\ &\to 0 \end{aligned}$$

$as\ n,m \to \infty$. This implies

$$|\varphi_n(x) - \varphi_m(x)|^2 \to 0$$

$as\ n,m \to \infty$. So

$$|\varphi_n(x) - \varphi_m(x)| \to 0$$

$as\ n,m \to \infty$.Since $\{\varphi_n\}$ is a Cauchy sequence and we have that $C_0(Q)$ is a complete space which means that $\lim_{n\to\infty} \varphi_n = \varphi\ as\ n \to \infty$ i.e $|\varphi_n(x) - \varphi(x)| \to 0\ as\ n \to \infty$, which tends to that φ belongs to$(C_0(Q), \langle \cdot , \cdot \rangle_F)$, so this space is complete.

Corollary 2.2. For any double completely monotone function F on Q, the space $\mathcal{H}_F \equiv (C_0(Q),\langle\cdot\,,\cdot\rangle_F)$is a subspace of Hilbert space $L^2(\mu)$.

Proof. We want to prove $\mathcal{H}_F \subset L^2(\mu)$ so let $\varphi,\psi \in \mathcal{H}_F$ and we need to reach to these functions in $L^2(\mu)$.Assume that

$$\int_Q |F(x-y)|\, d\mu(x) \le M_1 \quad for\ all\ y \in Q$$

and

$$\int_Q |F(x-y)|\, dv(y) \le M_2 \quad for\ all\ x \in Q$$

and by using (2.2)

$$|\langle\varphi,\psi\rangle_F| = \left|\int_Q \int_Q F(x-y)\, \varphi(x)\, \overline{\psi(y)}\, d\mu(x) dv(y)\right|$$

Where by using (Cauchy – young inequality: If $\frac{1}{p}+\frac{1}{q}=1, then\ ab \le \frac{a^p}{p}+\frac{b^q}{q}\ for\ \ a,b \ge 0$)

i.e.,

$$\begin{aligned}\left|\varphi(x)\overline{\psi(y)}\right| &\le \frac{|\,\varphi(x)|^2}{2}+\frac{|\psi(y)|^2}{2}\\ &\le \int_Q\int_Q \frac{|F(x-y)|}{2} dv(y)|\varphi(x)|^2\, d\mu(x) + \int_Q\int_Q \frac{|F(x-y)|}{2} d\mu(x)|\psi(y)|^2\, dv(y)\\ &\le \frac{M_2}{2}\|\varphi\|^2_{L_2(\mu)} + \frac{M_1}{2}\|\psi\|^2_{L_2(v)}\end{aligned}$$

So $,\psi \in\ L^2(\mu)$.

Let $\mathrm{M_c}$ stand for the set of all continuously real-valued functions ω on $\mathbb{R}^n$ that fulfill the requirements listed below:

1) $0\ =\ \omega(0)\ \le\ \omega(\zeta+\eta)\ \le\ \omega(\zeta)+\omega(\eta);\ \zeta,\eta\ \in \mathbb{R}^n$
2) $\int_{\mathbb{R}^n} \frac{\omega(\zeta)}{(1+|\zeta|)^{n+1}}\ d\zeta <\ \infty$
3) $\omega(\zeta) \ge\ a\ +\ b\ log(1+|\zeta|)$ for some constant a,b
4) $\omega(\zeta)$ is radial.

with the weight function ω in $\mathrm{M_c}$ and open set $\Omega\ \in\ \mathbb{R}^n$ Bj$\ddot{o}$rck extend the Schwartz space by the space $\mathrm{S}_\omega$of all $C^\infty -$ function $\varphi\ \in\ \mathrm{L}^1(\mathbb{R}^n)$:

$$\mathrm{P}_{\alpha,\lambda}(\varphi)\ =\ \sup_{x\in\,\mathbb{R}^n}\ \mathrm{e}^{\lambda\omega(x)}\ |\mathrm{D}^\alpha\varphi(x)| <\ \infty$$

And

$$\Pi_{\alpha,\lambda}(\varphi) = \sup_{\zeta \in \mathbb{R}^n} e^{\lambda\omega(\zeta)} |D^{\alpha}\widehat{\varphi}(\zeta)| < \infty$$

and S'_{ω} the dual space of S_{ω}. Let f be a double completely monotone function and

$$\omega_f(\zeta) = log(1 + |f(\zeta)|) \qquad (2.3)$$

for $s \in \mathbb{R}$ we denote by $\mathcal{H}_f^{\omega,s}$ the set of all generalized distributions $u \in S'_{\omega}$:

$$||u||_f^{\omega,s} = [\int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} |\hat{u}(\zeta)|^2 \, d\zeta \,]^{\frac{1}{2}} \qquad (2.4)$$

Theorem 2.3. The space $\mathcal{H}_f^{\omega,s}$is a Hilbert space with an inner product denoted by

$$\langle u, v \rangle_f^{\omega,s} = \int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} \hat{u}(\zeta)\overline{\hat{v}(\zeta)} d\zeta \qquad (2.5)$$

Proof. We need to prove that the space $\mathcal{H}_f^{\omega,s}$ is complete, so we assume that we have a Cauchy sequence $\{u_m\}$ in $\mathcal{H}_f^{\omega,s}$ and we want to prove that this Cauchy converges to a limit u in $\mathcal{H}_f^{\omega,s}$, where norm defined as:

$$\|u\|_f^{\omega,s} = \left[\int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} |\hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

So

$$\|u_m - u\|_f^{\omega,s} = \left[\int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} |\hat{u}_m(\zeta) - \hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

From (2.3), we have

$$\|u_m - u\|_f^{\omega,s} = \left[\int_{\mathbb{R}^n} (1 + |f(\zeta)|)^{2s} |\hat{u}_m(\zeta) - \hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

$$= \left[\int_{\mathbb{R}^n} (1 + |f(\zeta)|)^{2s+(n+1)} |\hat{u}_m(\zeta) - \hat{u}(\zeta)|^2 (1 + |f(\zeta)|)^{-(n+1)} d\zeta\right]^{\frac{1}{2}}$$

$$\leq \sup(1 + |f(\zeta)|)^{s+(n+1)/2} |\hat{u}_m(\zeta) - \hat{u}(\zeta)| \left[\int_{\mathbb{R}^n} (1 + |f(\zeta)|)^{-(n+1)} d\zeta\right]^{\frac{1}{2}}$$

$$\leq C\|\hat{u}_m - \hat{u}\|_p$$

$$\leq C\|u_m - u\|_{p+n+1}$$

Where $\geq s + (n+1)/2$, we find that $\|u_m - u\|_P \to 0$ as $m \to \infty \ \ \forall p \in N$, which come from that C_0^∞ is dense in S_ω, then $u \in \mathcal{H}_f^{\omega,s}$ which proves the completeness in it and so $\mathcal{H}_f^{\omega,s}$ is a Hilbert space.

Lemma 2.4. Let $u \in \mathcal{H}_f^{\omega,s}$, $< u, \cdot >_f^{\omega,s}$ is the conjugate linear functional on S_ω which uniquely extends to conjugate linear functional on $\mathcal{H}_f^{\omega,s}$ satisfying

1) $\langle\langle u,v \rangle\rangle_f^{\omega,s} = (2\Pi)^{-n} \int_{\mathbb{R}^n} e^{2s\omega_f(\zeta)} \hat{u}(\zeta)\overline{\hat{v}(\zeta)} d\zeta.$
2) $|\langle\langle u,v \rangle\rangle_f^{\omega,s}| \leq ||u||_f^{\omega,s} \, ||v||_f^{\omega,s}$, $u \in \mathcal{H}_f^{\omega,s}$,$v \in \mathcal{H}_f^{\omega,-s}$
3) $\langle\langle u,v \rangle\rangle_f^{\omega,s} = \overline{\langle\langle v,u \rangle\rangle}_f^{\omega,s}$

Theorem 2.5. The space S_ω is dense in $\mathcal{H}_f^{\omega,s}$for all $s \in \mathbb{R}$.

Proof. To prove that S_ω is dense in $\mathcal{H}_f^{\omega,s}$ we need to check two things the first is that $S_\omega \subset \mathcal{H}_f^{\omega,s}$ and the second is that $\overline{S_\omega} = \mathcal{H}_f^{\omega,s}$, for the first let we have a bijective map $g_s: S_\omega \to S_\omega$, $u \mapsto e^{s\omega_f(\zeta)}\hat{u}$**.** With (2.3) and f is a continuous double completely monotone function.

We have from the definition of map that $e^{s\omega_f(\zeta)}\hat{u} \in S_\omega \subset L_2$ which leads to $S_\omega \subset \mathcal{H}_f^{\omega,s}$.

Secondly we want to prove that $\overline{S_\omega} = \mathcal{H}_f^{\omega,s}$, so we must prove that $S_\omega^\perp = \{0\}$(orthogonal complement for S_ω). Where $S_\omega^\perp = \left\{u \in \mathcal{H}_f^{\omega,s} : \langle\langle u,\varphi \rangle\rangle_f^{\omega,s} = 0 \ \ \forall\varphi \in S_\omega\right\}$. We want to get to that $= 0$. i.e. $u \in \mathcal{H}_f^{\omega,s}$ with $u \in S_\omega^\perp$ lead to $\langle\langle u,\varphi \rangle\rangle_f^{\omega,s} = 0 \ \ \forall\varphi \in S_\omega$. We have

$$\langle\langle u,\varphi \rangle\rangle_f^{\omega,s} = \langle e^{s\omega_f(\zeta)}\hat{u}, e^{s\omega_f(\zeta)}\hat{\varphi}\rangle_{L_2} ,$$

Since g_s is bijective , $\forall\phi \in S_\omega$,we find $\langle e^{s\omega_f(\zeta)}\hat{u},\phi\rangle_{L_2} = 0$, since $S_\omega$is dense in L_p, $1 \leq p < \infty$.which mean that $e^{s\omega_f(\zeta)}\hat{u} = 0$, So $u = 0$, i.e. $S_\omega^\perp = \{0\}$, so $\overline{S_\omega} = \mathcal{H}_f^{\omega,s}$. Which complete the proof.

Note that $S_\omega$is dense in L_p comes from that $S_\omega \subset L_p$ .and that C_0^∞ is dense in L_p.

Corollary 2.6 $\mathcal{H}_f^{\omega,t} \subseteq \mathcal{H}_f^{\omega,s}$for $t > s$, the inclusion is continuous and has dense image.

3. Weak completely monotone functions

The purpose of this section is to discuss the idea of Weak completely monotone functions on the Schwartz spaces. Let $\Omega \subseteq \mathbb{R}$ be an open interval, $f \in C^1(\bar{\Omega})$ and $\varphi \in D(\Omega)$, where

$$D(\Omega) := \{\varphi \in C^\infty(\Omega, C) : supp(\varphi) := \overline{\{x;\ \varphi(x) \neq 0\}} \subseteq \Omega \text{ is compact}\}$$

Using integration by parts, we will get

$$\int_\Omega \acute{f}\,(x)\overline{\varphi(x)}dx = -\int_\Omega f\,(x)\overline{\acute{\varphi}(x)}dx$$

Since, $D(\Omega)$is the space of test functions which is dense in $L^P\,(\Omega)$ for $1 \le p \le \infty$, so we can rewrite the above equation using the scalar product of $L_2(\Omega)$ as

$$\langle \acute{f}\,|\varphi\,\rangle = -\langle\, f|\acute{\varphi}\rangle$$

We call a function g that satisfies $\langle\, g|\varphi\,\rangle = -\langle\, f|\acute{\varphi}\rangle$ a weak derivative of f. Let $\Omega \in \mathbb{R}^n$ open, $f \in C^1(\bar{\Omega})$ and $\varphi \in D(\Omega)$, then

$$\langle \frac{\partial}{\partial x_i} f|\varphi\,\rangle = -\langle\, f|\,\frac{\partial}{\partial x_i}\varphi\,\rangle$$

Applying Gauss Theorem, we similarly obtain

$$\langle D^\alpha f\,|\varphi\,\rangle = (-1\,)^{|\alpha|}\langle\, f\,|D^\alpha\varphi\,\rangle$$

Theorem 3.1. For any multi-index $\alpha \in \mathbb{R}$ the differential D^α is a continuous and linear operator from $\mathcal{H}_f^{\omega,s}$ to $\mathcal{H}_f^{\omega,s-|\alpha|}$.

Proof. Where the linearity of the operator is obvious, so all we need to prove is that

$$\|D^\alpha u\|_f^{\omega,s-|\alpha|} \le c\|u\|_f^{\omega,s} \tag{3.1}$$

From (2.4), we have,

$$\|u\|_f^{\omega,s-|\alpha|} = \left[\int_{\mathbb{R}^n} e^{2(s-|\alpha|)\omega_f(\zeta)}\,|\hat{u}(\zeta)|^2 d\zeta\right]^{\frac{1}{2}}$$

So

$$\|D^\alpha u\|_f^{\omega,s-|\alpha|} = \left[\int_{\mathbb{R}^n} e^{2(s-|\alpha|)\omega_f(\zeta)}\,\left|\widehat{D^\alpha u}(\zeta)\right|^2 d\zeta\right]^{\frac{1}{2}}$$

Which equivalent to

$$\|D^{\alpha}u\|_f^{\omega,s-|\alpha|} = \left\|e^{(s-|\alpha|)\omega_f(\zeta)}\widehat{D^{\alpha}u}(\zeta)\right\|_{L_2}$$

For a particular case let $\alpha = 1$

$$\left\|e^{(s-1)\omega_f(\zeta)}\widehat{Du}(\zeta)\right\|_{L_2} = \left\|e^{(s-1)\omega_f(\zeta)}\xi\ \hat{u}(\zeta)\right\|_{L_2}$$
$$\leq C\left\|e^{s\omega_f(\zeta)}\hat{u}(\zeta)\right\|_{L_2}$$
$$= c\|u\|_f^{\omega,s}$$

i.e.,

$$\|D^{\alpha}u\|_f^{\omega,s-1} \leq c\|u\|_f^{\omega,s}$$

where by using induction on $|\alpha|$ we can generalize this for any multi index $\alpha \in \mathbb{R}$ .which follow from this that the linear operator D^{α} is continuous from $\mathcal{H}_f^{\omega,s}$ to $\mathcal{H}_f^{\omega,s-|\alpha|}$.

Theorem 3.2. The pairing $\langle\langle\cdot\ ,\ \cdot\rangle\rangle_f^{\omega,s}$ identifies $\mathcal{H}_f^{\omega,-s}$isometrically with the antidual of $\mathcal{H}_f^{\omega,s}$. If $u \in \acute{D}_f^{\omega}$then $u \in \mathcal{H}_f^{\omega,s}$ if and only if there is a constant c such that $|u(\varphi)| \leq c\|\varphi\|_f^{\omega,-s}$ for $\in D_f^{\omega}$.

proof. Let the anti-dual of $\mathcal{H}_f^{\omega,s}$ be $\left(\mathcal{H}_f^{\omega,s}\right)'$ we will define a map $L: \mathcal{H}_f^{\omega,-s} \to \left(\mathcal{H}_f^{\omega,s}\right)'$ as

$$L_v(u) := \langle v,u\rangle = (2\pi)^{-n}\int \hat{v}(\zeta)\,\overline{\hat{u}(\zeta)}\,d\zeta$$

So we will show firstly that $L: v \to L_v$ is bijective. Let $L_v(u) = 0, so\ \langle v,u\rangle = 0\ and$

$$(2\pi)^{-n}\int \hat{v}(\zeta)\,\overline{\hat{u}(\zeta)}d\zeta = 0$$

This implies

$$(2\pi)^{-n}\int e^{-s\omega_f(\zeta)}\hat{v}(\xi)\,e^{s\omega_f(\zeta)}\overline{\hat{u}(\zeta)}\,d\zeta = 0$$

and

$$(2\pi)^{-n}\int e^{-s\omega_f(\zeta)}\hat{v}(\zeta)\,\psi(\zeta)\,d\zeta = 0\ \ for\ all\ \psi \in S_{\omega}$$

$so, v = 0$ in $\acute{S}_{\omega} and\ v = 0$ in $\mathcal{H}_f^{\omega,-s}$. So, L is one to one. Then we will show that L is surjective. Let $\psi \in \left(\mathcal{H}_f^{\omega,s}\right)'\ and\ \ \psi_1 \in \mathcal{H}_f^{\omega,s}$ we need to reach to $\psi_2 \in \mathcal{H}_f^{\omega,-s}$ such that $L_{\psi_2} = \psi$. So from Resize representation theorem we have, $\psi(u) = \langle\langle\psi_1,u\rangle\rangle_f^{\omega,s}\ for\ all\ u \in \mathcal{H}_f^{\omega,s}$. From the continuous linear function on s_ω then there exists $\psi_2 \in \mathcal{H}_f^{\omega,-s}$such that $\hat{\psi}_2(\zeta) = e^{2s\omega_f(\zeta)}\hat{\psi}_1(\zeta)$ at most, so this leads to

$$\begin{aligned}
\psi(u) &= \langle\langle \psi_1, u\rangle\rangle_f^{\omega,s} \\
&= (2\pi)^{-n}\int e^{2s\omega_f(\zeta)}\hat{\psi}_1(\zeta)\,\overline{\hat{u}(\zeta)}\,d\zeta \\
&= (2\pi)^{-n}\int e^{2s\omega_f(\zeta)}\,e^{-2s\omega_f(\zeta)}\hat{\psi}_2(\zeta)\overline{\hat{u}(\zeta)}\,d\zeta \\
&= \langle \psi_2, u\rangle = L_{\psi_2}(u) \qquad for\ all\ \ u \in \mathcal{H}_f^{\omega,s}
\end{aligned}$$

Hence,

$$\psi = L_{\psi_2}$$

So L is surjective. Next we will show the isometry of . Let $u \in \mathcal{H}_f^{\omega,s}$ and $v \in \mathcal{H}_f^{\omega,-s}$ such that

$$\hat{u}(\zeta) = e^{-2s\omega_f(\zeta)}\hat{v}(\zeta)$$

and

$$\begin{aligned}
L_v(u) &= (2\pi)^{-n}\int \hat{u}(\zeta)\,\overline{\hat{v}(\zeta)}\,d\zeta \\
&= (2\pi)^{-n}\int e^{-2s\omega_f(\zeta)}\hat{v}(\zeta)\,\overline{\hat{v}(\zeta)}\,d\zeta \\
&= (2\pi)^{-n}\int e^{-2s\omega_f(\zeta)}|\hat{v}(\zeta)|^2\,d\zeta \\
&= \left[\|v\|_f^{\omega's}\right]^2 \\
&= \ \|v\|_f^{\omega,-s}\|u\|_f^{\omega,s}
\end{aligned}$$

Which means that $L_v$is isometry from $\mathcal{H}_f^{\omega,-s}$ to $\left(\mathcal{H}_f^{\omega,s}\right)'$.

The second part of the proof is that if $u \in \acute{D}_f^{\omega}$ then $u \in \mathcal{H}_f^{\omega,s}, so\ there\ exists\ a\ constant\ c$ such that

$$|u(\phi)| \leq c\ \|\phi\|_f^{\omega,-s}.$$

So we will assume that $\in \acute{D}_f^{\omega}$, then $u \in \mathcal{H}_f^{\omega,s}$ and we want to prove that

$$|u(\phi)| \leq c\ \|\phi\|_f^{\omega,-s}.$$

We have that

$$|L_u(\phi)| = |\langle u,\phi\rangle|\ \leq \|u\|_f^{\omega,s}\|\phi\|_f^{\omega,-s} \leq c\ \|\phi\|_f^{\omega,-s}$$

This implies

$$|u(\phi)| \leq c\ \|\phi\|_f^{\omega,-s}.$$

Conversely, let $u \in \acute{D}_f^{\omega}$ and $|u(\phi)| \leq c\, \|\phi\|_f^{\omega,-s}$, we want to prove that $u \in \mathcal{H}_f^{\omega,s}$. Where $for\ all\ \ \phi \in D_f^{\omega}$ the map $\phi \mapsto u(\bar{\phi})$ can be extended uniquely to an element of a conjugate linear functional on $\mathcal{H}_f^{\omega,-s}$, with a bounded norm. So $there\ exists\ \psi \in \mathcal{H}_f^{\omega,s}$in sense that $L_{\psi}(\phi) = \langle\langle\psi,\phi\rangle\rangle = u(\bar{\phi}) = \langle u,\phi\rangle$. So $\psi = u\ and\ u \in \mathcal{H}_f^{\omega,s}$ which complete the proof.

4. Reproducing kernel Hilbert space $\mathcal{A}_F$

Let F be a continuous double completely monotone function on $\mathbb{R}^d$, set $F_y(x) := F(x - y)$ for all $x, y \in \mathbb{R}^d$. Define:

$$(\varphi * F)(x) := \int_{\mathbb{R}^d} \varphi(y) F_y(x) dy \ , \varphi \in S^d \tag{4.1}$$

and

$$\langle\, \varphi * F, \psi * F\, \rangle_{\mathcal{A}_F} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x - y)\varphi(x)\psi(y)dxdy \tag{4.2}$$

for all $\varphi, \psi \in S^d$. Then $\varphi * F;\ \varphi \in S^d$ forms a pre-Hilbert space $\mathcal{A}_F$ with inner-product $\langle\, \cdot\, ,\, \cdot\, \rangle_{\mathcal{A}_F}$

Lemma 4.1. A function $\varphi\ *\ F$ is in $\mathcal{A}_F$ if and only if $\widehat{\varphi}\ \in\ L^2(\mu)$ and

$$\|\varphi * F\|_{\mathcal{A}_F}^2 \ = \int_{\mathbb{R}^d} |\hat{\varphi}(\zeta)|^2\, d\mu(\zeta) \tag{4.3}$$

where μ is the tempered measure.

Proof. The first statement is obvious from the previous definitions in section 4., so we will prove (4.3) . Where we have that F is a continuous double completely monotone function, so we can use Bernstein's theorem

$$F(x) = \int_{\mathbb{R}^d} e^{-x\zeta}\, d\mu(\zeta)$$

So we have that, with $\phi \in S^d$ (Schwartz space on $\mathbb{R}^d$)

$$\begin{aligned}\int_{\mathbb{R}^d} F(x)\, \phi(x)\, dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-x\zeta}\, \phi(x)\, d\mu(\zeta)\, dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-x\zeta}\, \phi(x) dx\, d\mu(\zeta) \\ &= \int_{\mathbb{R}^d} \hat{\phi}(\zeta)\, d\mu(\zeta)\end{aligned}$$

And so we can conclude that

$$\|\phi * F\|_{\mathcal{A}_F}^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(x-y)\, \phi(x)\overline{\phi(y)}\; dx\, dy$$

$$= \int_{\mathbb{R}^d} \left|\hat{\phi}(\zeta)\right|^2 d\mu(\zeta)$$

Theorem 4.2. Let $k \in N$, and $\Lambda \in D^k$ and set$F_\Lambda = \sum_{\zeta\in\Lambda} e^{-i\zeta x}$,$x \in \mathbb{R}^k$; as a tempered completely monotone distribution, and let$\mathcal{A}_{F_\Lambda}$be the generalized RKHS of Schwartz. Then a function h on $\mathbb{R}^k$is in $\mathcal{A}_{F_\Lambda}$if and only if it has a convolution factorization

$$h = \varphi * F_\Lambda$$

where ϕ is a measurable function such that $\hat{\phi}(\lambda)$ exists for all $\lambda \in \Lambda$, and $\hat{\phi}(\lambda)$, $\lambda \in \Lambda$ belongs to $l_2(\Lambda)$ and

$$\|h\|_{\mathcal{A}_F}^2 = \sum_{\zeta\in\Lambda} \left|\hat{\phi}(\zeta)\right|^2 \tag{4.4}$$

Proof. We have that $F_\Lambda(x) = \sum_{\zeta\in\Lambda} e^{-ix\zeta}$, $x \in \mathbb{R}^k$ as a tempered completely monotone distributions , we will prove that

$$\|\phi * F_\Lambda\|_{\mathcal{A}_F}^2 = \sum_{\zeta\in\Lambda} \left|\hat{\phi}(\zeta)\right|^2$$

Where $\phi \in S^k$(the Schwartz space on $\mathbb{R}^k$) , $\hat{\phi}$ is the standard Fourier transform , from (4.1),

$$(\phi * F_\Lambda)(x) = \int_{\mathbb{R}^k} \phi(y) F_\Lambda(x-y)\, dy$$

$$= \int_{\mathbb{R}^k} \phi(y) \sum_{\zeta\in\Lambda} e^{-i\zeta(x-y)}\, dy$$

$$= \sum_{\zeta\in\Lambda} \int_{R^k} \phi(y) e^{-i\zeta(x-y)}\, dy$$

$$= \sum_{\zeta\in\Lambda} \int_{\mathbb{R}^k} \phi(y)\, e^{i\zeta y}\, dy\, e^{-i\zeta x}$$

$$= \sum_{\zeta\in\Lambda} \int_{\mathbb{R}^k} \overline{\phi(y) e^{-i\zeta y}}\, dy\, e^{-i\zeta x}$$

Hence

$$(\phi * F_\Lambda)(x) = \sum_{\zeta\in\Lambda} \overline{\hat{\phi}(\zeta)}\, e^{-i\zeta x}$$

And so,

$$\|\phi * F_\Lambda\|^2_{\mathcal{A}_F} = \langle \phi * F_\Lambda , \phi * F_\Lambda \rangle_{\mathcal{A}_{F_\Lambda}}$$

$$= \int\limits_{\mathbb{R}^k} \int\limits_{\mathbb{R}^k} F_\Lambda(x-y)\, \phi(x)\overline{\phi(y)}\ dx\, dy$$

$$= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \sum_{\zeta\in\Lambda} e^{-i\zeta(x-y)}\, \phi(x)\overline{\phi(y)}\ dx\, dy$$

Using Fubini's theorem:

$$\sum_{\zeta\in\Lambda} \int\limits_{\mathbb{R}^k} \int\limits_{\mathbb{R}^k} e^{-i\zeta(x-y)}\, \phi(x)\overline{\phi(y)}\ dx\, dy = \sum_{\zeta\in\Lambda} \left[\int\limits_{\mathbb{R}^k} e^{-i\zeta x}\phi(x)\ dx \int\limits_{\mathbb{R}^k} e^{i\zeta y}\overline{\phi(y)}\ dy \right]$$

$$= \sum_{\zeta\in\Lambda} \left[\int_{\mathbb{R}^k} e^{-i\zeta x}\phi(x)\ dx \int_{\mathbb{R}^k} \overline{e^{-\iota\zeta y}\phi(y)}\ dy \right]$$

$$= \sum_{\zeta\in\Lambda} \hat{\phi}(\zeta)\, \overline{\hat{\phi}(\zeta)}$$

$$= \sum_{\zeta\in\Lambda} \left|\hat{\phi}(\zeta)\right|^2$$

So,

$$\|\phi * F_\Lambda\|^2_{\mathcal{A}_F} = \sum_{\zeta\in\Lambda} \left|\hat{\phi}(\zeta)\right|^2$$

5. Concluding Remarks

In this work, we introduced and gave the main properties of the class of double monotone functions defined on $S(Q)$. Moreover, the main properties of the class of weak monotone functions defined on $S(Q)$ are given.Finally, a novel method for creating spaces of generalized functions are given. Tempered distributions refer to the set of all continuous linear functional on $S(Q)$, and it is represented by the symbol $\acute{S}(Q)$. suppose $l \in \acute{S}(Q)$ and $\alpha \in \mathbb{Z}^d_+$. The weak derivative $D^\alpha l$, often known as the derivative of the sense of distributions, is obtained by

$$(D^\alpha l)(f) = (-1)^\alpha\, l(D^\alpha f)$$

for $f \in (Q)$. This corresponds to $D^\alpha l\{g\} = l\{ D^\alpha g\}$. Noting thatdistributions are always weakly derivative. If assume that $Q = \mathbb{R}^n$. So, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Let x^α be denote the product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\mathbb{Z}^n_+$ represents a set of n-tuples.$(\alpha_1, \cdots, \alpha_n)$where each α_i is an integer that is not negative , $|\alpha| = \sum_{i=1}^n \alpha_i$and D^αdenote the partial differential operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ .The particular case, which follows the space of rapidly decreasing function on $\mathbb{R}^n$ is denoted as $S(Q) = S(\mathbb{R}^n)$ (also known as the Schwartz space), and its dual space of a tempered distribution on $\mathbb{R}^n$ is denoted as $\acute{S}(Q) = \acute{S}(\mathbb{R}^n)$.

Acknowledgements
The authors are thankful to the editor for their valuable comments and suggestions.

Funding
There is no sponsor.

Availability of data and materials
The data used to support the findings of this study are available from the corresponding author upon request.

Competing interests
The authors declare that they have no competing interests concerning the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. Yu. M. Berezansky and Yu. G. Kondratiev, Spectral methods in infinite dimensional analysis, vol. 1, 2, Kluwer, Dordrecht 1995. http://dx.doi.org/10.1007/978-94-011-0509-5
2. Yu. M. Berezansky and Yu. S. Samoilenko, Nuclear spaces of functions of infinitely many variables, Ukr. Mat. Zh., 25, No. 6, (1973), 723-737. http://dx.doi.org/10.1007/BF01090792
3. C. Berg, J. P. R. Christensen, P. Ressel, Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions, Springer-Verlag: Berlin, Heidelberg, NewYork, 1984.http://dx.doi.org/10.1007/978-1-4612-1128-0
4. G. Björck, Linear partial differential operators and generalized distributions, Ark. Mat.6, (1965),351-407. http://dx.doi.org/10.1007/BF02590963
5. I. M. Gelfand and G. E. Shilov, Generalized Function, Academic Press, Inc.,1964.

6. H. A. Ghany, Multidimensional negative definite functions on the product of commutative hypergroups Kuwait Journal of Science 43 (4), (2016) 121-131. https://journalskuwait.org/kjs/index.php/KJS/index
7. H. A. Ghany, Exact solutions for stochastic generalized Hirota-Satsuma coupled KdV equations, Chinese Journal of Physics, vol. 49, no. 4, (2011) 926-940. https://www.researchgate.net/publication/266890714
8. H. A. Ghany, Harmonic analysis in the product of commutative hypercomplex systems. Journal of Computational Analysis and Applications, 25 (5), (2018) 875-888. http://www.ijpam.eu
9. T. Hida, Analysis of Brownian Functional, Carleton Math. Lect. Notes, No. 13 (1975). https://doi.org/10.1007/BFb0120763
10. H. Hölden, B. Øsendal, J. Ubøe and T. Zhang, Stochastic partial differential equations, Springer Science+Business Media, LLC, (2010). https://doi.org/10.1007/978-1-4684-9215-6_4
11. J. Lighthill, Introduction to Fourier Analysis and Generalized Function, Cambridge universitypress,1958. https://doi.org/10.1017/CBO9781139171427
12. A. S. Okb El Bab and H. A. Ghany, Harmonic analysis on hypergroups systems, AIP Conf. Proc., 312 (2010) 1309. https://doi.org/10.1063/1.3525130