Lattice $CS_T(L_1)$ Obtained By The Substitution Sum In Formal Context

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ABSTRACT
Formal Concept Analysis (FCA) is an applied branch of Lattice theory, widely used in computer science, which derives implicit relationships between objects described through a set of attributes on one hand and these attributes on the other. Any finite lattice can be represented by a formal concept, which can be obtained from the formal context of ob- jects and attributes. Let T be a (0, 1) sublattice of a lattice L_1. In this paper, a concept lattice $CS_T(L_1)$ is constructed by the substitution- sum in the formal context, where T is a formal context of a (0, 1) sublattice of the lattice L_1. The structural properties of the concept lattice are studied and a characterization for the meet and join irreducible elements of $CS_T(L_1)$ is given. Further, some congruence relations are defined on the lattice $CST(L_1)$.

T	T
X	*L*

Keywords: Substitution Sum, Irreducible elements, Formal concept Analysis, Convex doubling.

1. INTRODUCTION

Formal concept Analysis (FCA) is a technique mainly adopted for data analysis. It derives latent connections between objects categorized through a set of attributes on one hand and these attributes on the other hand. It is an important mathematical application of computer science that is highly used in knowledge representation, knowledge acquisition, linguistics and data visualization [1], [2], [3]. It helps in processing a wide class of data types by providing a framework in which various techniques of data analysis can be formulated. Central to formal concept analysis is the notion of the formal context. Every formal context K is isomorphic to $(J(L_1), M(L_1), I)$ and every formal context generates a unique Direct Product of Lattices concept lattice [4]. The substitution sum and substitution product were introduced by Luksch and Wille for the concept analytic evaluation of pair comparison tests [4] and further described in detail by Stephan [5], [6]. Wille and Ganter further, compiled all these various types of formal contexts and have characterized the corresponding concept lattices, one of them being the substitution sum in which a context of any lattice is placed in an empty cell of another context where there is no object attribute relation [4]

L	L
L	X

The lattice $CS(L_1)$ of convex sublattices of a lattice L_1, w a s first introduced and studied by Koh.L [7], [8]. Further, a new partial order was defined by S. Lavanya et. al [9] with respect to which $CS(L_1)$ forms a lattice. Using the concept of substitution sums

L	L
L	∅

lattices CS(L) and TS(L) respectively have been constructed and a characterization for their irreducible elements is provided [10]. Accordingly, a concept lattice $CS_T(L_1)$ has been constructed using substitution sum

T	T
X	*L*

And the structural properties of the lattice are studied. Furthermore, we have observed that the construction of the concept lattices $TS(L_1)$ and $CS_T(L_1)$ is similar to the interval doubling construction in lattices introduced by Alan Day in order to prove the Whitman's structure theorem in a simpler way, further characterized the class of CN of all of all lattices obtained by doublings using convex sets [11].

Another characterizations for the class CN was given by Geyer in terms of concept lattices [12]. Also a congruence relation on the convex sublattice of a lattice corresponding to the congruence relation on a lattice is defined and studied by S Parameshwara Bhatta and Ramananda HS [13]. Further, the notion of strong congruence relations has been defined by Ramananda HS on Atomistic lattices [14]. Accordingly, some congruence relations are studied on the concept lattice $CS_T(L_1)$ introduced in this paper.
The paper proceeds as follows: The reader is introduced to the notations and the definitions used in this paper. Section 3 studies the structural properties of the lattice $CS_T(L_1)$ obtained by the substitution sum. In section 4, noting the fact that every formal context K is isomorphic to $(J(L_1), M(L_1), I)$, where the join irreducible elements $J(L_1)$ represent the objects and the meet ir- reducible elements $M(L_1)$ represent the attributes, the join and the meet irreducible elements of the lattice $CS_T(L_1)$ are characterized and thereby the concept lattice $CS_T(L_1)$ is constructed by the substitution sum. Further, some congruence relations on $CS_T(L_1)$ are studied in section 5.

2. Notations and Definitions

We provide some insights into the notions of formal concept analysis. There a dermay refer to Formal Concept Analysis [4] for basic concepts. The formal context $K_0:=(G_0,M_0,I_0)$ is the incidence relation I_0 between the object set G_0 and the attribute set M_0.$g_0I_0m_0$ or $(g_0,m_0) \in I_0$ denotes that the object $g_0 \in G_0$ has an attribute $m_0 \in M_0$.
A Formal Concept is a pair (C,D) with $C \subseteq G_0$,$D\subseteq M_0$ with $C'=D$ and $D'=C$ where C' denotes those attributes in M_o to the objects in Cand D' denotes those objects in $G_o$common to the attributes in D.
Let $K_1:=(G_1,M_1,I_1)$and $K_2:=(G_2,M_2,I_2)$ be the formal
contexts such that $(g_1,m_1)\notin I_1$ in K_1. Suppose that $G_2\neq\emptyset\neq M_2$ and $G_1\backslash g_1\cap G_2=\emptyset=(M_1\backslash m)\cap M_2$.
The Substitution Sum of $K_1$with K_2 on (g_1,m_1) is defined to be the Context $K_1(g_1,m_1)K_2:=(G_0,M_0,I_0)$ with $G_0=(G_1\backslash g_1)\cup G_2)$,
$M_0=(M_1\backslash m_1)\cup M_2$, and
$I_0=(h1, n1) \in I_1, h_1 \neq g_1, n_1 \neq m_1 \cup G_2 \times g_1^{\{I_1\}} \cup m_1^{\{I_1\}} \times M_2 \cup I_2$
(see [14]).
LetTandL$_1$betwolattices.ThenthedirectproductT $\times L_1$
in which the binary operation $\vee$ and $\wedge$on L_1 are such that for any (a_1,b_1) and (a_2,b_2)inL_1,$(a_1,b_1)\vee(a_2,b_2) = (a_1\vee a_2,b_1\vee b_2)$and $(a_1,b_1) \wedge(a_2,b_2) = (a_1\wedge a_2,b_1\wedge b_2)$.
Throughout this paper, T is a (0,1) sublattice of L_1, where L_1 denotes a finite lattice, the maximum element of L_1 is represented as1 and its minimum element is represented as 0. By(0,1) sub lattice T of L_1, we mean that the minimum and maximum elements of T coincide with those of L1.For elements m, j of L_1, m is covered by j denoted by $m\prec j$. An element x of a lattice L_1 is said to be meet-irreducible if there exists a unique element $x^+\in L_1$ such that $x\prec x^+$. These to fall meet- irreducible elements of a latticeL_1 is denoted by $M(L_1)$. An element is said to be join-irreducible in L_1 if there exists a unique element $y^- \in L_1$ such that $y^-\prec y$. These to fall join-irreducible elements is denoted by $J(L_1)$. For notations and more details, there a dermay refer[15].
Let α be a congruence relation on a lattice L_1. Then L_1/α represents the Quotient Lattice of L_1 modulo α and for $x\in L_1$,
x/α represents the congruence class containing x[16].

3. Formal context for $CS_T(L_1)$

In this section, we shall introduce some results that provide the structural properties of the lattice $CS_T(L_1)$ obtained by the substitution sum

T	T
X	L

Lemma 3.1. Let T be a(0,1) sublattice of L_1.Then $CS_T(L_1)$=
$\{(a,b): a \in T, b \in L_1$ with $a\leq b\}$ is a cover preserving sublattice of T × L_1.
Proof. Let $(a,b),(x,y)\in T\times L_1$. Then $(a,b)\leq(x,y)$ if and only if $a\leq x$ and $b\leq y$. Let (a,b), (x,y) $\in CS_T(L_1)$. Then $(a,b)\wedge(x,y)= (a \wedge x,b \wedge y) = (a,b) \in CS_T(L_1)$.
Also$(a,b)\vee(x,y)= (a\vee x,b\vee y)=(x,y)\in CS_T(L_1)$. This establishes that the meet and join operations preserve the cover structure within $CS_T(L_1)$.
To prove $CS_T(L_1)$ is a cover preserving sublattice of $T\times L_1$;it suffices to prove that I f$(a,b)\prec b\prec y$ or $a\prec x$ and b=y in L_1.
(x,y), then a=x in T and Let $(a,b)\prec(x,y)$.Suppose that a=x in T and $b<x<y$ for $CS_T(L_1)$
some $c \in L_1$. This leads to the inequality $(a,b)<(a,c)<(x,y)$in $CS_T(L_1)$, a contradiction to the initial

assumption.
Now, consider a≠x in T. Let a<c<x in T and b=y in L_1, then (a,b)<(c,b)<(x,y) in $CS_T(L_1)$, a contradiction. Hence a≺x. If b≠y in L_1, then b<y. But then,T (a,b)<(a,y)<(x,y)in $CS_T(L_1)$,again a contradiction. This affirms that $CS_T(L_1)$ is a cover-preserving sublattice of $T \times L_1$.
Theorem3.2. Let a,b ∈ T and x ∈ L_1 such that 0≺a≺b≺x≺1 in L_1, then the following relations hold in $CS_T(L_1)$.

(i) (0,1)≺(a,1)≺(b,1)≺(1,1).
(ii) (0,0)≺(0,a)≺(0,b)≺(0,x)≺(0,1).
(iii) (0,a)≺(a,a)≺(a,b)≺(a,x)≺(a,1).
(iv) (0,b)≺(a,b)≺(b,b)≺(b,x)≺(b,1).
(v) (a,a)II(0,b),(a,x)II(b,b),(a,1)II(b,x).

These relations highlight the order structure within $CS_T(L_1)$ corresponding to the given conditions on a, b and x.
Proof. By the definition of $CS_T(L_1)$, we have $CS_T(L_1)$ is a cover preserving sublattice of T×L_1. Morover, the argument presented in Lemma 3.1, establish the validity of the relations (i),(ii),(iii),(iv).
By the definition of $CS_T(L_1)$, the relation (a,x)<(b,b) is not possible, since x≮b in L_1.Similarly, (b,b) <(a,x) is not possible since b ≮a. Consequently, (a,x) II (b,b). Analogously, employing the same argument we have, (a,a)II (0,b)and(a,1)II(b,x),thus proving(v).

4. Formal context for $CS_T(L_1)$

It is emphasized to note that any concept lattice can be constructed by its formal context having join irreducible elements as objects and meet irreducible elements as attributes. Hence, we characterize the irreducible elements of $CS_T(L_1)$ and explore the interrelations between them. These relations characterize the formal context which help to construct the lattice $CS_T(L_1)$.

Theorem 4.1. (i) Let a ∈ M(T), then (a,1) ∈ M($CS_T(L_1)$).
(ii) Let a ∈ T where a=max {a_i ∈ T:a_i<x} and x ∈ M(L_1) then (a,x) ∈ M($CS_T(L_1)$).
(iii) Let a ∈ J(T). Then {a} ∈ J($CS_T(L_1)$).
(iv) If x ∈ J(L_1), then (0,x) ∈ J($CS_T(L_1)$).
Conversely, let A ∈ M($CS_T(L_1)$). Then one of the following holds:
(a) A=(a,1) for some a ∈ M(T).
(b) A=(a,x) for some a=max {a_i ∈ T:a_i<x in L_1} and x ∈ M(L_1).
(c) A=(0,x) for some x ∈ M(L_1) and x ∉ T. Furthermore, If A ∈ J($CS_T(L_1)$), then one of the following holds:
(d) A=(a,a) for some a ∈ J(T).
(e) A=(0,a) for some a ∈ J(L_1).
Proof. Let a ∈ M(T). Then a≺a^+ uniquely .Correspondingly in T
$CS_T(L_1)$,(a,1)≺(a^+,1). The uniqueness of this covering is evident, as there is no element x ∈ L_1 such that (a,1)≺(a,x) in $CS_T(L_1)$. Furthermore, if (a,1)≺(a^{++},1) in $CS_T(L_1)$, then a≺a^{++}, which is a contradiction.
Let a ∈ T and a=max {a_i ∈ T: a_i<x}. In this case, there is no element a $^+$∈ T such that a<a^+. Moreover, when x ∈ M(L) implies a unique covering relationx≺x^*.
1
Consequently, (a,x) is covered uniquely by (a,x^*) in $CS_T(L_1)$.
Let a ∈ J(T).Then a^-≺a uniquely. Then (a^-,a)≺ (a,a).
T Suppose ,there exists (x,a)≺ $CS_T(L_1)$
$CS_T(L_1)$ (a,a). Then we have,(a^-,a)IIx,a). This implies that a^-II x and x≺a in T. This is a contradiction since a ∈ J(T).
Let x ∈ J(L), which implies x^-≺x. Then (0,x^-)≺ L_1 $CS_T(L_1)$ (0,x).
1
This covering is unique since there is no a ∈ T such that (a,x)≺ $CS_T(L_1)$(0,x). Therefore, (0,x) ∈ J(L_1).
Conversely,
(a) Let A=(a,1) ∈ M($CS_T(L_1)$) implies there exists unique (b,1)∈ $CS_T(L_1)$ such that (a,1)≺ (b,1). From lemma 3.2, we have a ≺b in T. Suppose that a ≺c in T with c ≠b.Then, correspondingly we have (a,1)≺(c,1) in $CS_T(L_1)$. We observe that (b,1)II (c,1) in $CS_T(L_1)$, for if (b,1)<(c,1), then we must have b<c, not possible since a≺b and a≺b. Further, if (c,1)<(a,1), then this implies c< b, contradiction since a≺b and a≺b uniquely in T. Thus, a ∈ M(T).
(b) Let A=(a,x) ∈ M($CS_T(L_1)$).

To prove that a=max $\{a_i \in T: a_i < x \text{ in } L_1\}$and $x \in M(L_1)$.
Suppose not, we assume two cases.
Case (1): Assuming $a \neq \max \{a_i \in T: a_i < x \text{ in } L_1\}$ and $x \in M(L_1)$. Then there exists b∈ T such that $a < b < x < y \leq 1$ in L_1 where $y \in L_1 - T$ .By lemma 3.2, correspondingly in $CS_T(L_1)$, we have (a,a) < (a,b) < (a,x) < (a,y) and (a,b) <(b,b) <(b,x) <(b,y).We realize that
(a,x) < (b,x) and (a,x) <(a,y) and therefore (b,x) II (a,y) which is a contradiction to our assumption that $(a,x) \in M(CS_T(L_1))$.
Case(2):We now assume that $a = \max \{a_i \in T : a_i < x \text{ in } L_1\}$and $x \notin M(L_1)$.Then there exists x^+ and x^{++} in L_1 such that $x \prec x^+$ and $x \prec x^{++}$ in L_1.By lemma 3.2, correspondingly for the first case $(a,x) < (a,x^+)$ and in the second
case$(a,x) < (a,x^{++})$ in $CS_T(L_1)$.This is not possible as $(a, x) \in M(CS_T(L_1))$.
(c) Let$(0,x) \in M(CS_T(L_1))$.Then (0,x) has a unique upper bound in $(0,x^+)$ in $CS_T(L_1)$. Correspondingly, $x \prec x^+$ uniquely in L_1. Suppose $x \in T$ and $x \in M(L_1)$, then $(0,x) \prec (0,x^+)$ in $CS_T(L_1)$. Further, $(0,x) \prec (x,x)$ and hence $(0,x^+)$ II(x,x).This is a contradiction since $(0,x) \in M(CS_T(L_1))$. Therefore, $x \notin T$.
(d) Let $(a,a) \in J(CS_T(L))$. Then there exists $(a^-,a) \prec (a,a)$–
uniquely in $CS_T(L)$ and therefore ,$a \prec a$ in T.
Suppose $a \notin J(T)$. There exists $c \in T$ with $c \neq a^-$ such that $c \prec a$ in T. This gives rise to $(c,a) \prec (a,a)$ in $CS_T(L_1)$; a contradiction to our assumption that $(a, a) \in M(CS_T(L_1))$.
(e) Let $(0,a) \in J(CS_T(L))$. Then, there exists a unique$(0,a^-) \in CS_T(L)$ such that $(0,a^-) \prec (0,a)$. Then clearly $a^- \prec a$,this covering must be unique in L_1.
Remark 4.2. $(a,a) \in M(CS_T(L_1))$ If and only if $a \in M(T) \cap M(L_1)$.
Remark 4.3. If $x \in J(L_1)$ and $x \notin J(T)$. Then for any $a \in T$,(a,x) will not remain as a join irreducible in $CS_T(L_1)$, unless x = 0.
The theorem 4.5 characterizes the formal context of the lattice $CS_T(L_1)$.
Proposition 4.4([4]). For any finite lattice L_1 there is, up to iso-morphism, a unique reduced context $K(L_1)$ with $L_1 \cong B(K(L_1))$, that is $K(L_1) := (J((L_1),M(L_1),\leq)$
Theorem 4.5. Let L1 be a lattice. Define:
(i) $G_1=\{A \in CS_T(L_1):A=(a,a) \text{ for some } a \in J(T)\}$.
(ii) $G_2= \{A \in CS_T(L_1): A=(0,a) \text{ for some } a \in J(L_1)\}$.
(iii) $M_1= \{A \in CS_T(L_1): A=(m,1) \text{ for some } m \in M(T)\}$.
(iv) $M_2=\{N_1 \cup N_2 \cup N_3\}$ where
$N_1 =\{A \in CS_T(L_1):A=(a,x)$ for some a=(a,x) for some $a=\max\{a_i \in T:a_i<x;x \in M(L_1)\}\}$.
$N_2 = \{A=(a,a) \text{ for some } a \in M(L_1)\}$
$N_3 = \{A=(0,x) \text{ for some } x \in M(L_1)\}\}$
Define a context K $(CS_T(L_1)) = (G_1 \cup G_2, M_1 \cup M_2, I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5 \cup I_6)$ as follows: for $(g,g) \in G_1$, $(0,g) \in G_2$, $(m,1) \in M_1$, $(m,m) \in M_2$,$(m,v) \in M_2$,$(0,m) \in M_2$,
(1) (g,g) I_1(m,1)if g≤minT.
(2) (g,g) I_2(m,m)if g ≤minL_1.
(3) (g,g) I_3(m,v)if g≤vinL_1andv∈$M(L_1)$.
(4) (0,g) I_4(m,1)for all g and m ∈ L_1.
(5) (0,g) I_5(m,m)if a≤g in L_1.
(6) (0,g) I_6(c,v)if g≤v in L_1.
Then, K ($CS_T(L_1$)is the formal context of $CS_T(L_1)$.
Proof. From the theorem 4.1, it is clear that $J(CS_T(L_1))=G_1 \cup G_2$ and $M(CS_T(L_1))=M_1 \cup M_2$. We note that (i), (ii), (iii), (iv)defines the object-attribute relations in the formal context of $CS_T(L_1)$ are defined by the (1), (2), (3), (4), (5), (6) respectively using the lemma
3.2. From Proposition 4.4 $K(CS_T(L_1))$ is the formal context of $CS_T(L_1)$.
Hence, given any two lattices T and L_1, such that T is a (0,1)

$$\begin{array}{c|c} T & T \\ \hline X & L_1 \end{array}$$

sublattice of L_1, the substitution sum K $(CS_T(L_1))=CS_T(L_1)$.

Table 1. Formal Context of T

	1	2	3
a		X	X
b	X	X	
c	X		

Table 2. Formal Context of L_1

	1	2	3
a		X	X
b	X		X
c	X	X	

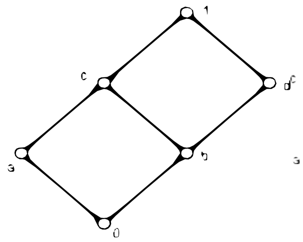

Figure 1. Lattice T

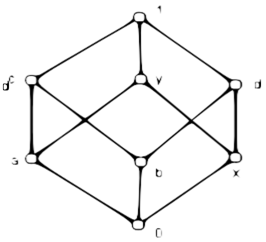

Figure 2. Lattice L1

Table 3. Context of $CS_T(L_1)$

	1	2	3	$1^{\perp}$	$2^{\perp}$	$3^{\perp}$
a		X	X		X	X
b	X	X		X	X	
c	X			X		
$a^{\perp}$	X	X	X		X	X
$b^{\perp}$	X	X	X	X	X	
$c^{\perp}$	X	X	X	X		X

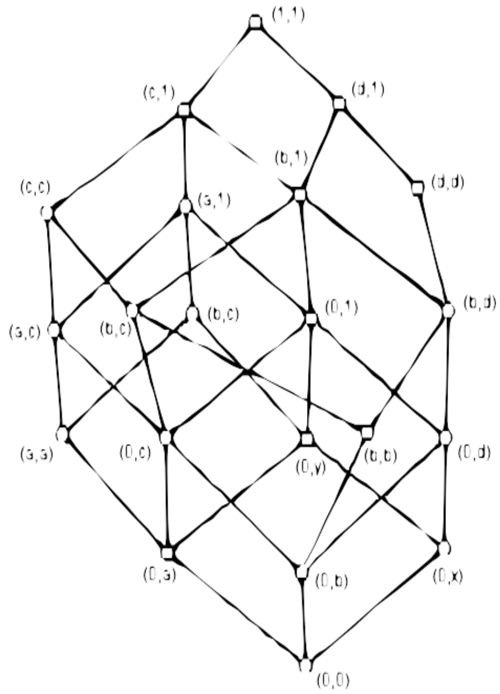

Figure 3. Lattice $CS_T(L_1)$

5. Congruence relations on $CS_T(L_1)$

In this section we study some congruence relations on $CS_T(L_1)$.

Theorem 5.1

Let α be a congruence relation on $CS_T(L_1)$ defined by (0, 0) ≡ (0, 1)α and β be another congruence relation on $CS_T(L_1)$ defined by (0, 1) ≡ (1, 1)β.
Then $CS_T(L_1)/\alpha \cong T$ and $CS_T(L_1)/\beta \cong L_1$.
Proof. Let f : $CS_T(L_1)/\alpha \longrightarrow T$ be defined by [(a, 1)]α → a where a ∈T.
Let g : $CS_T(L_1)/\beta \longrightarrow L_1$ be defined by [(0, x)]β → x where x ∈ L_1. Under the congruence relation (0, 0) ≡ (0, 1)α, (a, b) ∈ [(a, 1)]α and (a, b) ∈ $CS_T(L_1)$ ∀ a ∈ T. Therefore f is well defined.
Similarly under the congruence relation, (0, 1) ≡ (1, 1)β, only (a, b) ∈ [(0, x)]β, ∀x ∈ L1. Therefore, g is well defined. Let f ([[(a, 1)]α) = g([[(b, 1)]β). Then, a = b in T. Note that we have (0, 1) < (a, 1) in $CS_T(L_1)$ ∀ a ∈ T. Under the congruence relation, (0, 0) ≡ (0, 1)α, we have for all (a, 1) ∈ $CS_T(L_1)$, (0, 0) ∨(a, 1) ≡ (0, 1) ∨(a, 1)α,
⇒(a, 1) ≡ (a, 1)α ⇒ (a, 1) ∈ [(a, 1)]α, a ∈ T.
Similarly (b, 1) ∈ [(b, 1)]α, b ∈ T. Since a = b in T, we have (a, 1) = (b, 1) in $CS_T(L_1)$. We have [(a, 1]α = [(b, 1)]α.
Therefore, f is injective. Similarly g is injective.
Let a ∈ T. we have (a, 1) ∈ CST(L_1) and (a, 1) ∈ [(a, 1)]α. Clearly,
f([[(a, 1)]α) = a. Therefore, f is surjective.
Let [(a, 1)]α ≤ [(b, 1)]α. Then, (a, 1) ≤ (b, 1) in CST(L1). Therefore, a ≤ b in T, thus, proving f is order preserving. Similarly, g is order preserving.
Let a ≤ b in T. Then, (a, 1) ≤ (b, 1) in CST(L_1). Under the congruence relation (0, 0) ≡ (0, 1)α, (a, 1) ∈ [(a, 1)]α and (b, 1) ∈ [(b, 1)]α. Therefore,[(a, 1)]α ≤ [(b, 1)]α, thus proving f −1 is order preserving. Similarly, g^{-1} is order preserving.

Theorem 5.2

Let Θ be the congruence relation on T. Define a congruence relation ψΘ on $CS_T(L_1)$ as follows: (a, a) ≡ (b, b)(ψΘ)
if and only if a ≡ b(Θ). Then $CS_T/\Theta(L_1/\Theta) \cong CS_T(L_1)/\psi\Theta$.
Proof. Define a function f : CST/Θ(L_1/Θ) −→ $CS_T(L_1)/\psi\Theta$ given by f([a]Θ, [b]Θ) = (a, b)ψΘ. f is well defined.
Let [(a_1, b_1)]ψΘ = [(a_2, b_2)]ψΘ in $CS_T(L_1)/\psi\Theta$.
We have (a_2, a_2) ≤ (a_2, b_2) ≤(b_2, b_2) in $CS_T(L_1)$.
$a_1 \equiv b_1(\Theta)$ and $a_2 \equiv b_2(\Theta)$ in L_1
implies [a_1]Θ = [b_1]Θ and [a_2]Θ = [b_2]Θ. Further, [(a_1, b_1)]ψΘ = [(a_2, b_2)]ψΘ implies (a_1, b_1) ≡ (a_2, b_2)ψΘ
⇒ $a_1 \equiv a_2(\Theta)$ and b1 ≡ b_2(Θ)
⇒ [a1]Θ = [a2]Θ and [b1]Θ = [b2]Θ.
([a1]Θ, [b1]Θ) = (([a2]Θ, [b2]Θ) in $CS_T(L_1)/\Theta$).
Therefore f is injective.
Let [(a, b)]ψ(Θ). We prove that ([x]Θ, [y]Θ) ∈ $CS_T/\Theta(L_1/\Theta)$
such that f(([x]Θ, [y]Θ)) ∈ [(a, b)]Θ.
Let (x, y) ∈ $CS_T(L_1)$ such that (x, y) ≡ (a, b)ψ(Θ).
Then [(x, y)]ψΘ = [(a, b)ψΘ.
Also, x ≡ a(Θ) and y ≡ b(Θ). ⇒ [x]Θ = [a]Θ and [y]Θ = [b]Θ.
f {([x]Θ), [y]Θ)} = [(a, b)]ψΘ. Therefore f is surjective.
Let ([a_1]Θ, [b_1]Θ) ≤ ([a_2]Θ, [b_2]Θ) in CST/Θ(L_1/Θ).
We have [a1]Θ ≤[a_2]Θ and [b_1]Θ ≤[b_2]Θ.
Let (a_1, x) ∈ $CS_T(L_1)/\psi\Theta$ such that f {([a_1]Θ, [b_1]Θ)} = [(a_1, x)]ψΘ
where x ∈ [b_1]Θ and f{([a_2]Θ, [b_2]Θ)} = [(a_2, y)]ψΘ where y ∈[b_2]Θ.
Since [a_1]Θ ≤[a_2]Θ and [b_1]Θ ≤[b_2]Θ, we have for every a_1 ∈[a_1]Θ,
∃ s ∈ [a_2]Θ such that $a_1 \le s$ and for every x ∈ [b_1]Θ, ∃ t ∈ [b_2]Θ such that x ≤ t. Therefore, (a_1, x)ψΘ ≤ (s, t)ψΘ.
Similarly, (a_2, y)ψΘ ≤ (a_2, t)ψΘ.
Note that [(a_1, x)]ψΘ = [(a_1, b_1)]ψΘ and [(s, t)]ψΘ = [(a_2, b_2)]ψΘ). Therefore, [(a_1, b_1)]ψΘ ≤ [(a_2, b_2)]ψΘ. Hence f is order preserving.
Let [(a_1, b_1)]ψΘ ≤[(a_2, b_2)]ψΘ in $CS_T(L_1)/\psi\Theta$.
Now, (a_1, b_1) ≤ (a_2, b_2) in CST(L_1). Also, (a_1, b_1) ≡ (a, b)ψΘ and (a2, b2) ≡ (a, b)ψΘ.

Then $a_1 \equiv a(\Theta)$ in T and $b1 \equiv b(\Theta)$ in L_1. Also $a_2 \equiv a(\Theta)$ in T and
$b_2 \equiv b(\Theta)$ in L_1. This implies $[a_1]\Theta = [a]\Theta = [a_2]\Theta$ and
$[b_1]\Theta = [b]\Theta = [b_2]\Theta$ implies $([a_1]\Theta, [b1]\Theta) = ([a]\Theta, [b]\Theta) = ([a2]\Theta, [b2]\Theta)$.
Therefore, $([a_1]\Theta, [b_1]\Theta) \leq ([a_2]\Theta, [b_2]\Theta)$ in $CST/\Theta(L_1/\Theta)$, thus proving that, f^{-1} is order preserving.
□

Theorem 5.3.
Let Θ be a congruence relation on L_1, such that
Θ is trivial on T. Define a congruence relation $\psi\Theta$ on $CS_T(L_1)$ as follows: $a \equiv b\ \Theta$ if and only if $(0, a) \equiv (0, b)\ \psi\Theta$. Then $CS_T(L_1)/\Theta) \sim= CS_T(L_1)/\psi\Theta$.
The proof of this theorem runs similarly as theorem 5.2.

CONCLUSION
This paper paves the idea for the study of more concept lattices obtained from Substitution sum.

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