Stable Attractors on a Certain Two-dimensional Piecewise Linear Map

Khanison Youtuam[a], Benjaporn Thipar[a],
Nararat Thakthuang[a] and Wirot Tikjha[a,b,⋆]

[a] Faculty of Science and Technology,Pibulsongkram Rajabhat University, Phitsanulok,Thailand 65000.
[b]Centre of Excellence in Mathematics, PERDO, CHE, Thailand.
⋆Corresponding author. E-mail: wirottik@psru.ac.th

June 17, 2023

Abstract

In this article we study the behaviors of a piecewise linear map with initial condition in the second quadrant. There is a unique equilibrium point and two 4-cycles of the map. We found regions of initial condition that solutions become equilibrium point or 4-cycles. We divided the second quadrant into sub-regions and identify behaviors of solutions in each sub-region by direct calculations, and formulated inductive statements to explain the behaviors of the map without using stability theorems.

Key words: Coexisting attractors, Periodic solution, Equilibrium point, Piecewise linear map.
2010 Mathematics Subject Classification: 39A10 and 65Q10.

1 Introduction

Lozi map (Lozi, 1978) is a well known two dimensional piecewise linear map which is a simplified version of Hénon map and has a strange attractor. There are many applications of piecewise linear maps in models such as power electronic converters and switching circuits (Banerjee & Verghese, 2001; Zhusu baliyev &Mosekilde, 2003). We know that multistability (Simpson, 2010; Zhusubaliyev et al., 2008) can be found in piecewise linear map. Bifurcations sequence in a family of piecewise linear maps were cosidered in articles (Gardini & Tikjha, 2019; Tikjha & Gardini, 2020) and also a transition between invertibility and non-invertibility of piecewise linear map were studied in article (Gardini & Tikjha, 2020). A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a map is called *eventually periodic with prime period-p (or minimal period-p)* if there exists an integer $N > 0$

and a smallest positive integer p such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period p; that is,

$$(x_{n+p}, y_{n+p}) = (x_n, y_n) \text{ for all } n \geq N.$$

As we all known that piecewise linear function is not differentiable. In the case of system that can reduce to equation (one-dimensional map), we are unable to verify stability via stability theorem such as Schwazian derivative (D. Singer, 1978). An open problem about a piecewise linear map was mentioned in (Grove et al.,2012):

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases} \tag{1}$$

with initial condition $(x_0, y_0) \in \mathbb{R}^2$. Many papers studied the open problem for example: Gove et al. (2012) found that every solution is eventually prime period-3 solutions except for the unique equilibrium solution. In article (Tikjha et al., 2010; 2015; 2017) and (Tikjha & Lapiere, 2020), they studied some special cases of system (1), and showed that there are periodic attractors. They showed that every solution is eventually either periodic attractors or equilibrium point by using direct calculation and inductive statement. Recently in article (Aiewcharoen et al. 2021; Laoharenoo et al. 2023), they investigated a family of systems that contain absolute value similar to (1) and they showed that all solutions become the equilibrium point. Moreover, they also showed that there exist a prime period 5 when $b \leq -6$. In article (Lapiere & Tikjha, 2021), they also studied the special case of (1) with $a = b = d = -1$ and $c = 1$. Our goal is to continue investigate the special case of (1) with $a = c = -1$, $b = -3$ and $d = 1$ which Tikjha and Piasu (Tikjha & Piasu, 2020) reported the condition of solutions becoming equilibrium point or periodic with prime period 4. They investigated initial point only in region of the first quadrant. We aim to extend the initial condition in second quadrant and find all possible behaviors of solutions for this map and then characterize the coexisting attractors between equilibrium point and periodic with prime period 4 (4-cycle) and their basin of attractions.

2 Main Results

In this section we will study the following two dimensional map:

$$x_{n+1} = |x_n| - y_n - 3, y_{n+1} = x_n - |y_n| + 1 \tag{2}$$

with initial condition belonging to second quadrant. This map has the unique equilibrium point $(-1, -1)$ that can be computed by solving the system:

$$\begin{cases} \bar{x} = |\bar{x}| - \bar{y} - 3 \\ \bar{y} = \bar{x} - |\bar{y}| + 1 \end{cases}.$$

As in (Tikjha & Piasu, 2020), there are 4-cycles of the system (2) given by $P_{4.1} = \{((-5, -1), (3, -5), (5, -1), (3, 5))\}$ and $P_{4.2} = \{((1, -3), (1, -1), (-1, 1), (-3, -1))\}$.

The 4-cycles are found by numerical calculation. It is easy to verify that $P_{4.1}$ and $P_{4.2}$ are 4-cycles. Let (x_0, y_0) be in the second quadrant of xy plane, $Q_2 := \{(x, y) \in \mathbb{R}^2 | x < 0 \text{ and } y > 0\}$. We have the first iteration as the following:

$$\begin{cases} x_1 = |x_0| - y_0 - 3 & = -x_0 - y_0 - 3 \\ y_1 = x_0 - |y_0| + 1 & = x_0 - y_0 + 1 \end{cases} \tag{3}$$

Before we calculate the next iteration, we have to know the sign (negative or non-negative) of x_1 and y_1 which are the function of x_0 and y_0. The sign of x_1 will change when initial point (x_0, y_0) above or below the line $f(x) = -x - 3$ (resp. $g(x) = x + 1$ for y_1). Now we divide the second quadrant into three sub-regions as Fig. 1 that we will investigate in the next sub-section.

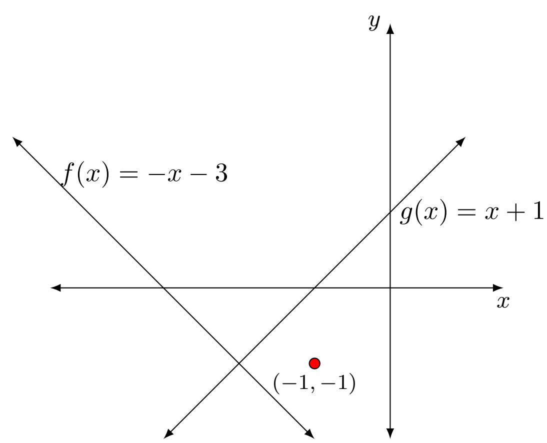

Figure 1: The second quadrant is separated into three sub-regions by the lines $f(x)$ and $g(x)$. The red point is the equilibrium point of system (2).

2.1 Stable equilibrium point

In this section we will investigate rightmost region of second quadrant that is when initial condition belonging to the green region as Fig.2 From (3), we have

$$\begin{cases} x_2 = 2y_0 - 1 \\ y_2 = -2x_0 - 3 < 0. \end{cases} \tag{4}$$

Firstly, we will investigate when $x_2 \geq 0$ that is initial condition $\frac{1}{2} \leq y_0 \leq 1$ as in Fig.3. So the next iteration can be written in the form:

$$\begin{cases} x_3 = 2x_0 + 2y_0 - 1 \\ y_3 = -2x_0 + 2y_0 - 3 < 0. \end{cases} \tag{5}$$

3

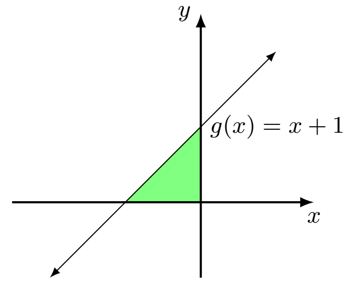

Figure 2: The region of initial points such that x_1 is negative and y_1 is positive.

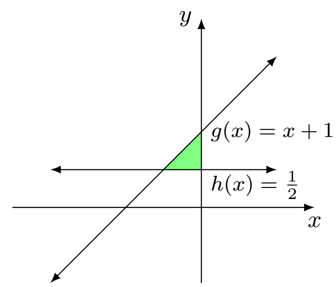

Figure 3: The region of initial points such that x_2 is non-negative.

Again we separate region in Fig.3 into two parts above and below a line $i(x) = -x + \frac{1}{2}$ as in Fig.4. For an initial condition above the line $i(x) = -x + \frac{1}{2}$, the forth iteration is in the form:

$$\begin{cases} x_4 = 4x_0 - 1 < 0 \\ y_4 = 4y_0 - 3. \end{cases} \tag{6}$$

If initial conditions are in green region in Fig.4 with above line $i(x)$ and $y_0 \in \left[\frac{1}{2}, \frac{3}{4}\right]$, we have $y_4 \leq 0$. By direct calculations we have: $\begin{cases} x_5 = -4x_0 - 4y_0 + 1 < 0 \\ y_5 = 4x_0 + 4y_0 - 3 < 0 \end{cases}$, and $\begin{cases} x_6 = -1 \\ y_6 = -1 \end{cases}$. The solution of this region is eventually equilibrium point within sixth iteration. For initial conditions are in green region in Fig. 4 with above line $i(x)$ and $y_0 \in \left(\frac{3}{4}, 1\right]$, we have the following closed form of solution: $\begin{cases} x_5 = -4x_0 - 4y_0 + 1 < 0 \\ y_5 = 4x_0 - 4y_0 + 3 < 0 \end{cases}$, and so

$$\begin{cases} x_6 = 8y_0 - 7 \\ y_6 = -8y_0 + 5 < 0 \end{cases}. \tag{7}$$

Note that the closed form of the sixth iteration with this region is independent from x_0. It is easy to verify that when $y_0 \in \left(\frac{3}{4}, \frac{7}{8}\right]$, $x_6 \leq 0$ and so $x_7 = y_7 = -1$.

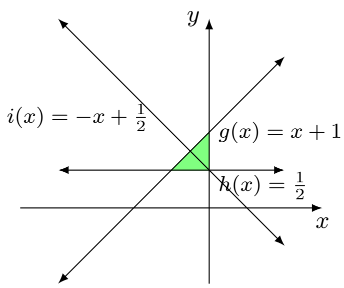

Figure 4: The third iteration of (5) x_3 change sign when initial point (x_0, y_0) crosses the line $i(x)$.

This means that the solution of this region is also eventually equilibrium point within seventh iteration. The remain region is when $y_0 \in \left(\frac{7}{8}, 1\right]$ which we have $x_6 > 0$. The following inductive statement will be used to prove that every solution is eventually equilibrium point for this remain region. Let $a_n = \frac{2^{2n+1}-1}{2^{2n+1}}, b_n = \frac{2^{2n+2}-1}{2^{2n+2}}, \delta_n = 2^{2n+2} - 1$ and $P(n)$ be the following statement : "$y_0 \in (a_n, 1]$, then

$$\begin{cases} x_{4n+3} = 2^{2n+2}y_0 - \delta_n \\ y_{4n+3} = -1 \end{cases}.$$

If $y_0 \in (a_n, b_n]$ then $x_{4n+3} \leq 0$ and so

$$\begin{cases} x_{4n+4} = -2^{2n+2}y_0 + \delta_n - 2 < 0 \\ y_{4n+4} = 2^{2n+2}y_0 - \delta_n \leq 0 \end{cases}, \begin{cases} x_{4n+5} = -1 \\ y_{4n+5} = -1 \end{cases}.$$

If $y_0 \in (b_n, 1]$ then $x_{4n+3} > 0$ and so

$$\begin{cases} x_{4n+4} = 2^{2n+2}y_0 - \delta_n - 2 < 0 \\ y_{4n+4} = 2^{2n+2}y_0 - \delta_n > 0 \end{cases}, \begin{cases} x_{4n+5} = -2^{2n+3}y_0 + 2\delta_n - 1 < 0 \\ y_{4n+5} = -1 \end{cases},$$

$$\begin{cases} x_{4n+6} = 2^{2n+3}y_0 - 2\delta_n - 1 \\ y_{4n+6} = -2^{2n+3}y_0 + 2\delta_n - 1 < 0 \end{cases}.$$

If $y_0 \in (b_n, a_{n+1}]$ then $x_{4n+6} \leq 0$, and so

$$\begin{cases} x_{4n+7} = -1 \\ y_{4n+7} = -1 \end{cases}.$$

If $y_0 \in (a_{n+1}, 1]$ then $x_{4n+6} > 0$. "

We shall show that $P(1)$ is true. For $y_0 \in (a_1, 1] = \left(\frac{7}{8}, 1\right]$ and $\delta_1 = 15$, we have $x_6 = 8y_0 - 7 > 0, y_6 = -8y_0 + 5 < 0$ and so

$$\begin{cases} x_{4(1)+3} = x_7 = 16y_0 - 15 = 2^{2(1)+2}y_0 - \delta_1 \\ y_{4(1)+3} = y_7 = -1 \end{cases}.$$

If $y_0 \in (a_1, b_1] = (\frac{7}{8}, \frac{15}{16}]$ then $x_7 \leq 0$ and so

$$\begin{cases} x_{4(1)+4} = x_8 = -16y_0 + 13 = -2^{2(1)+2}y_0 + \delta_1 - 2 < 0 \\ y_{4(1)+4} = y_8 = 16y_0 - 15 = 2^{2(1)+2}y_0 - \delta_1 \leq 0 \end{cases},$$

$$\begin{cases} x_{4(1)+5} = x_9 = -1 \\ y_{4(1)+5} = y_9 = -1 \end{cases}.$$

If $y_0 \in (b_1, 1] = \left(\frac{15}{16}, 1\right]$ then $x_7 > 0$ and so

5

$$\begin{cases} x_{4(1)+4} = x_8 = 16y_0 - 17 = 2^{2(1)+2}y_0 - \delta_1 - 2 < 0 \\ y_{4(1)+4} = y_8 = 16y_0 - 15 = 2^{2(1)+2}y_0 - \delta_1 > 0 \end{cases},$$
$$\begin{cases} x_{4(1)+5} = x_9 = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\delta_1 - 1 < 0 \\ y_{4(1)+5} = y_9 = -1 \end{cases},$$
$$\begin{cases} x_{4(1)+6} = x_{10} = 32y_0 - 31 = 2^{2(1)+3}y_0 - 2\delta_1 - 1 \\ y_{4(1)+6} = y_{10} = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\delta_1 - 1 < 0 \end{cases}.$$

If $y_0 \in (b_1, a_2] = (\frac{15}{16}, \frac{31}{32}]$ then $x_{10} \le 0$ and so
$$\begin{cases} x_{4(1)+7} = x_{11} = -1 \\ y_{4(1)+7} = y_{11} = -1 \end{cases}.$$

If $y_0 \in (a_2, 1] = (\frac{31}{32}, 1]$ then $x_{4(1)+6} = x_{10} = 32y_0 - 31 > 0$. Therefore $P(1)$ is true. It means that for this region and initial condition $y \in \left(\frac{7}{8}, \frac{31}{32}\right]$, the solution is eventually equilibrium point $(-1, -1)$. Next Suppose $P(k)$ is true. We shall show that $P(k+1)$ is true. For $y_0 \in (a_{k+1}, 1] = \left(\frac{2^{2k+3}-1}{2^{2k+3}}, 1\right]$, then
$$\begin{cases} x_{4k+6} = 2^{2k+3}y_0 - 2\delta_k - 1 > 0 \\ y_{4k+6} = -2^{2k+3}y_0 + 2\delta_k - 1 < 0 \end{cases}. \text{ Then}$$
$$\begin{cases} x_{4(k+1)+3} = 2^{2(k+1)+2}y_0 - (2^{2(k+1)+2} - 1) = 2^{2(k+1)+2}y_0 - \delta_{k+1} \\ y_{4(k+1)+3} = -1 \end{cases}.$$

If $y_0 \in (a_{k+1}, b_{k+1}] = (\frac{2^{2k+3}-1}{2^{2k+3}}, \frac{2^{2k+4}-1}{2^{2k+4}}]$ then $x_{4k+7} = x_{4(k+1)+3} \le 0$ (by substituting boundary of y_0) and so
$$\begin{cases} x_{4(k+1)+4} = -2^{2(k+1)+2}y_0 + \delta_{k+1} - 2 < 0 \\ y_{4(k+1)+4} = 2^{2(k+1)+2}y_0 - \delta_{k+1} \le 0 \end{cases},$$
$$\begin{cases} x_{4(k+1)+5} = -1 \\ y_{4(k+1)+5} = -1 \end{cases}.$$

If $y_0 \in (b_{k+1}, 1] = \left(\frac{2^{2k+4}-1}{2^{2k+4}}, 1\right]$ then $x_{4k+7} = x_{4(k+1)+3} > 0$ and so
$$\begin{cases} x_{4(k+1)+4} = 2^{2(k+1)+2}y_0 - \delta_{k+1} - 2 < 0 \\ y_{4(k+1)+4} = 2^{2(k+1)+2}y_0 - \delta_{k+1} > 0 \end{cases},$$
$$\begin{cases} x_{4(k+1)+5} = -2^{2(k+1)+3}y_0 + 2\delta_{k+1} - 1 < 0 \\ y_{4(k+1)+5} = -1 \end{cases},$$
$$\begin{cases} x_{4(k+1)+6} = 2^{2(k+1)+3}y_0 - 2\delta_{k+1} - 1 \\ y_{4(k+1)+6} = -2^{2(k+1)+3}y_0 + 2\delta_{k+1} - 1 < 0 \end{cases}.$$

If $y_0 \in (b_{k+1}, a_{k+2}] = (\frac{2^{2k+4}-1}{2^{2k+4}}, \frac{2^{2k+5}-1}{2^{2k+5}}]$ then $x_{4(k+1)+6} \le 0$ and so $x_{4(k+1)+7} = -1$ and $y_{4(k+1)+7} = -1$.

If $y_0 \in (a_{k+2}, 1] = \left(\frac{2^{2k+5}-1}{2^{2k+5}}, 1\right]$ then $x_{4(k+1)+6} = x_{4k+10} = 2^{2(k+1)+3}y_0 - 2\delta_{k+1} - 1 > 0$. Hence $P(k+1)$ is also true. By mathematical induction $P(n)$ is true for any positive integer n. From the inductive statement we have that every solution with initial condition y_0 between a_n and b_n is eventually equilibrium point. It is easy to see that the limits of sequences a_n and b_n are equal to 1. Therefore we can confirm that with initial condition, the green region in Fig. 4 with above line $i(x)$, the solution is eventually equilibrium point.

For an initial condition below or in the line $i(x) = -x + \frac{1}{2}$, the initial condition satisfy $x_0 \le -y_0 + \frac{1}{2}$ then $x_3 = 2x_0 + 2y_0 - 1 \le 0$. We have the forth iteration as $x_4 = -4x_0 + 1 < 0$ and $y_4 = 4y_0 - 3$. In this green region below

$i(x)$ of Fig. 4, y_0 is at most $\frac{3}{4}$. Then $y_4 = 4y_0 - 3 < 0$ and so $x_5 = y_5 = -1$. So we proved the following lemma.

Proposition 2.1. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of the map (2) and initial condition $(x_0, y_0) \in \{(x, y) \in Q_2 | y \leq x + 1$ and $\frac{1}{2} \leq y \leq 1\}$. Then every solution is eventually equilibrium point.*

Now we consider the below part of the Fig. 3, which (x_0, y_0) satisfies the following conditions: $x_1 = -x_0 - y_0 - 3 < 0, y_1 = x_0 - y_0 + 1 \geq 0$ and $x_0 < 0, y_0 > 0$. We have $x_2 = 2y_0 - 1$ and $y_2 = -2x_0 - 3$. In this case we consider when $0 < y_0 < \frac{1}{2}$. So $x_2 < 0$ and (x_0, y_0) belong to green portion of Fig. 5. The

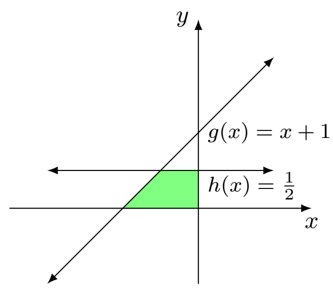

Figure 5: The green region of initial points such that x_2 is negative.

next iteration can be written in the form:

$$\begin{cases} x_3 = 2x_0 - 2y_0 + 1 \\ y_3 = -2x_0 + 2y_0 - 3 < 0. \end{cases} \tag{8}$$

We separate x_3 into two cases, above and below line $k(x) = x + \frac{1}{2}$ as in Fig. 6, when (x_0, y_0) is above $k(x)$ then $x_3 < 0$ while it is positive when (x_0, y_0) below $k(x)$.

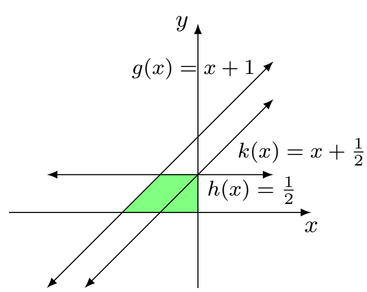

Figure 6: The x_3 of (8) change sign when initial point (x_0, y_0) crosses the line $k(x)$.

7

In the case of $x_3 \leq 0$ (above $k(x)$), we immediately have $x_4 = y_4 = -1$. For the case of $x_3 > 0$, we have

$$\begin{cases} x_4 = 4x_0 - 4y_0 + 1 \\ y_4 = -1 \end{cases}.$$

For $x_4 = 4x_0 - 4y_0 + 1 \leq 0$, we have

$$\begin{cases} x_5 = -4x_0 + 4y_0 - 3 < 0 \\ y_5 = 4x_0 - 4y_0 + 1 \leq 0 \end{cases}$$

and so $x_6 = y_6 = -1$. In the case of $x_4 = 4x_0 - 4y_0 + 1 > 0$, that is in remain region of initial condition in $\Delta = \{(x_0, y_0) \in Q_2 | 4x_0 - 4y_0 + 1 > 0\}$ as Fig.7. We will use an inductive statement to verify that the remain solution is even-

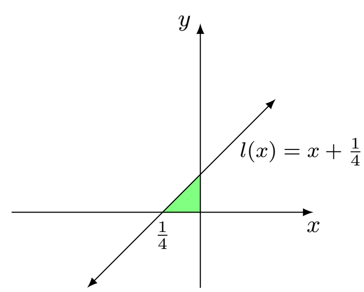

Figure 7: The green region is the initial points belonging to Δ.

tually equilibrium point. Let $\Delta_n = \{(x, y) \in Q_2 \mid 2^{2n}x - 2^{2n}y + 1 > 0\}, D_n = \{(x, y) \in Q_2 \mid 2^{2n+1}x - 2^{2n+1}y + 1 > 0\}$ and $\mathcal{Q}(n)$ be the following statement: "$(x_0, y_0) \in \Delta_n$ then

$$\begin{cases} x_{4n+1} = 2^{2n}x_0 - 2^{2n}y_0 - 1 < 0 \\ y_{4n+1} = 2^{2n}x_0 - 2^{2n}y_0 + 1 > 0 \end{cases}, \quad \begin{cases} x_{4n+2} = -2^{2n+1}x_0 + 2^{2n+1}y_0 - 3 < 0 \\ y_{4n+2} = -1 \end{cases},$$

$$\begin{cases} x_{4n+3} = 2^{2n+1}x_0 - 2^{2n+1}y_0 + 1 \\ y_{4n+3} = -2^{2n+1}x_0 + 2^{2n+1}y_0 - 3 < 0 \end{cases}.$$

If $(x_0, y_0) \in \Delta_n - D_n$ then $x_{4n+3} \leq 0$ and so $x_{4n+4} = y_{4n+4} = -1$.
If $(x_0, y_0) \in D_n$ then $x_{4n+3} > 0$ and so

$$\begin{cases} x_{4n+4} = 2^{2n+2}x_0 - 2^{2n+2}y_0 + 1 \\ y_{4n+4} = -1 \end{cases}.$$

If $(x_0, y_0) \in D_n - \Delta_{n+1}$ then $x_{4n+4} \leq 0$

$$\begin{cases} x_{4n+5} = -2^{2n+2}x_0 + 2^{2n+2}y_0 - 3 < 0 \\ y_{4n+5} = 2^{2n+2}x_0 - 2^{2n+2}y_0 + 1 \leq 0 \end{cases}, \text{ and so } x_{4n+6} = y_{4n+6} = -1.$$

If $(x_0, y_0) \in \Delta_{n+1}$ then $x_{4n+4} > 0$." We shall show that $\mathcal{Q}(1)$ is true. For $(x_0, y_0) \in \Delta_1 = \{(x, y) \in Q_2 \mid 4x - 4y + 1 > 0\}$. We have

$$\begin{cases} x_{4(1)+1} = 4x_0 - 4y_0 - 1 = 2^{2(1)}x_0 - 2^{2(1)}y_0 - 1 < 0 \\ y_{4(1)+1} = 4x_0 - 4y_0 + 1 = 2^{2(1)}x_0 - 2^{2(1)}y_0 + 1 > 0 \end{cases},$$

$$\begin{cases} x_{4(1)+2} = -8x_0 + 8y_0 - 3 = -2^{2(1)+1}x_0 + 2^{2(1)+1}y_0 - 3 < 0 \\ y_{4(1)+2} = -1 \end{cases},$$

$$\begin{cases} x_{4(1)+3} = 8x_0 - 8y_0 + 1 = 2^{2(1)+1}x_0 - 2^{2(1)+1}y_0 + 1 \\ y_{4(1)+3} = -8x_0 + 8y_0 - 3 = -2^{2(1)+1}x_0 + 2^{2(1)+1}y_0 - 3 < 0 \end{cases}.$$

8

If $(x_0, y_0) \in \Delta_1 - D_1 = \{(x, y) \in Q_2 \mid 0 < 4x - 4y + 1 \text{ and } 8x - 8y + 1 \leq 0\}$ then $x_7 \leq 0$ and so $x_{4(1)+4} = y_{4(1)+4} = -1$.
If $(x_0, y_0) \in D_1 = \{(x, y) \in Q_2 \mid 8x - 8y + 1 > 0\}$ then $x_7 > 0$ ans so

$$\begin{cases} x_{4(1)+4} = 16x_0 - 16y_0 + 1 = 2^{2(1)+2}x_0 - 2^{2(1)+2}y_0 + 1 \\ y_{4(1)+4} = -1 \end{cases}.$$

If $(x_0, y_0) \in D_1 - \Delta_2 = \{(x, y) \in Q_2 \mid 0 < 8x - 8y + 1 \text{ and } 16x - 16y + 1 \leq 0\}$ then $x_8 = 16x_0 - 16y_0 + 1 \leq 0$. Then

$$\begin{cases} x_{4(1)+5} = -16x_0 + 16y_0 - 3 = -2^{2(1)+2}x_0 + 2^{2(1)+2}y_0 - 3 < 0 \\ y_{4(1)+5} = 16x_0 - 16y_0 + 1 = 2^{2(1)+2}x_0 - 2^{2(1)+2}y_0 + 1 \leq 0 \end{cases},$$ and so $x_{4(1)+6} = y_{4(1)+6} = -1$.

If $(x_0, y_0) \in \Delta_2 = \{(x_0, y_0) \in Q_2 \mid 16x - 16y + 1 > 0\}$ then $x_{4(1)+4} > 0$. Hence $\mathcal{Q}(1)$ is true. Suppose $\mathcal{Q}(k)$ is true. Next, we show that $\mathcal{Q}(k+1)$ is true. Since $\mathcal{Q}(k)$ is true, we have $x_{4k+4} = 2^{2k+2}x_0 - 2^{2k+2}y_0 + 1 > 0$, and $y_{4k+4} = -1$ when $(x_0, y_0) \in \Delta_{k+1} = \{(x, y) \in Q_2 \mid 2^{2k+2}x - 2^{2k+2}y + 1 > 0\}$ and so

$$\begin{cases} x_{4(k+1)+1} = 2^{2(k+1)}x_0 - 2^{2(k+1)}y_0 - 1 < 0 \\ y_{4(k+1)+1} = 2^{2k+1}x_0 - 2^{2k+1}y_0 + 1 > 0 \end{cases},$$

$$\begin{cases} x_{4(k+1)+2} = -2^{2k+1+1}x_0 + 2^{2k+1+1}y_0 - 3 < 0 \\ x_{4(k+1)+2} = -1 \end{cases},$$

$$\begin{cases} x_{4(k+1)+3} = 2^{2(k+1)+1}x_0 - 2^{2(k+1)+1}y_0 + 1 \\ y_{4(k+1)+3} = -2^{2(k+1)+1}x_0 + 2^{2(k+1)+1}y_0 - 3 < 0 \end{cases}.$$

If $(x_0, y_0) \in \Delta_{k+1} - D_{k+1} = \{(x, y) \in Q_2 \mid 0 < -2^{2k+2}x + 2^{2k+2}y + 1 \text{ and } 2^{2(k+1)+1}x - 2^{2(k+1)+1}y + 1 \leq 0\}$ then $x_{4(k+1)+3} \leq 0$ and so

$$\begin{cases} x_{4(k+1)+4} = -1 \\ y_{4(k+1)+4} = -1 \end{cases}.$$

If $(x_0, y_0) \in D_{k+1} = \{(x, y) \in Q_2 \mid 2^{2(k+1)+1}x - 2^{2(k+1)+1}y + 1 > 0\}$ then $x_{4(k+1)+3} > 0$ and so

$$\begin{cases} x_{4(k+1)+4} = 2^{2(k+1)+2}x_0 - 2^{2(k+1)+2}y_0 + 1 \\ y_{4(k+1)+4} = -1 \end{cases}.$$

If $(x_0, y_0) \in D_{k+1} - \Delta_{k+2} = \{(x, y) \in Q_2 \mid 0 < 2^{2(k+1)+1}x - 2^{2(k+1)+1}y + 1 \text{ and } 2^{2(k+1)+2}x_0 - 2^{2(k+1)+2}y_0 + 1 \leq 0\}$ then $x_{4(k+1)+4} \leq 0$ and so

$$\begin{cases} x_{4(k+1)+5} = -2^{2(k+1)+2}x_0 + 2^{2(k+1)+2}y_0 - 3 < 0 \\ y_{4(k+1)+5} = 2^{2(k+1)+2}x_0 - 2^{2(k+1)+2}y_0 + 1 \leq 0 \end{cases}.$$

$$\begin{cases} x_{4(k+1)+6} = -1 \\ y_{4(k+1)+6} = -1 \end{cases}.$$

If $(x_0, y_0) \in \Delta_{k+2} = \{(x, y) \in Q_2 \mid 2^{2(k+1)+2}x - 2^{2(k+1)+2}y + 1 > 0\}$ then $x_{4(k+1)+4} > 0$. Hence $\mathcal{Q}(k+1)$ is also true. By mathematical induction $\mathcal{Q}(n)$ is true for any positive integer $n \geq 1$. So we proved the following lemma.

Proposition 2.2. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of the map (2) and initial condition $(x_0, y_0) \in \{(x, y) \in Q_2 | y \leq x + 1$ and $0 < y < \frac{1}{2}\}$. Then every solution is eventually equilibrium point.*

Now we complete the proof that every solution is eventually equilibrium point with initial point in the green region of Fig.2.

2.2 Coexisting attractors

This section we will consider the case that $x_1 = -x_0 - y_0 - 3 < 0$ and $y_1 = x_0 - y_0 + 1 < 0$ which means that initial point belong to cyan region of Fig.8. Then we have the next iteration in the form

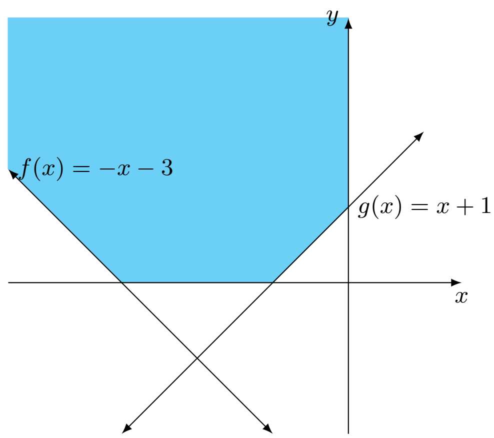

Figure 8: The region that x_1 and y_1 are negative, that (x_0, y_0) is in cyan.

$$\begin{cases} x_2 = 2y_0 - 1 \\ y_2 = -2y_0 - 1 < 0 \end{cases}.$$

That is the second iteration and the remain solutions are independent from x_0. If $y_0 \leq \frac{1}{2}$ then $x_2 \leq 0$ then $x_3 = y_3 = -1$. In the case of $\frac{1}{2} < y_0 \leq \frac{3}{4}$, we have $x_2 > 0$ and so

$$\begin{cases} x_3 = 4y_0 - 3 \leq 0 \\ y_3 = -1 \end{cases}, \quad \begin{cases} x_4 = -4y_0 + 1 < 0 \\ y_4 = 4y_0 - 3 \leq 0 \end{cases}, \quad \begin{cases} x_5 = -1 \\ y_5 = -1 \end{cases}.$$

If $y_0 \geq \frac{5}{4}$ then

$$\begin{cases} x_3 = 4y_0 - 3 > 0 \\ y_3 = -1 \end{cases}, \quad \begin{cases} x_4 = 4y_0 - 5 \geq 0 \\ y_4 = 4y_0 - 3 > 0 \end{cases}, \quad \begin{cases} x_5 = -5 \\ y_5 = -1 \end{cases}.$$

If $\frac{3}{4} < y_0 \leq \frac{7}{8}$ then

$$\begin{cases} x_3 = 4y_0 - 3 > 0 \\ y_3 = -1 \end{cases}, \quad \begin{cases} x_4 = 4y_0 - 5 < 0 \\ y_4 = 4y_0 - 3 > 0 \end{cases}, \quad \begin{cases} x_5 = -8y + 5 < 0 \\ y_5 = -1 \end{cases},$$

$$\begin{cases} x_6 = 8y - 7 < 0 \\ y_6 = -8y + 5 < 0 \end{cases}, \quad \begin{cases} x_7 = -1 \\ y_7 = -1 \end{cases}.$$

Now we can conclude that solutions with initial point in green portion of Fig. 9 become equilibrium point within seventh iteration while solutions with initial point in red portion of Fig. 9 become 4-cycle within fifth iteration. The remain region, cyan region of Fig. 9, is $\frac{7}{8} < y_0 < \frac{5}{4}$ which we have third iteration to fifth iteration are the same as in the case $\frac{3}{4} < y_0 \leq \frac{7}{8}$ but the sixth iteration is $x_6 = 8y_0 - 7 > 0$ and $y_6 = -8y_0 + 5 < 0$. The remain iterations can be proved to become equilibrium point or 4-cycle by using induction. We will use the following inductive statement to verify. Let $A_n = \frac{2^{2n+2}-1}{2^{2n+2}}, l_n = \frac{2^{2n+1}-1}{2^{2n+1}}, u_n = \frac{2^{2n}+1}{2^{2n}}$

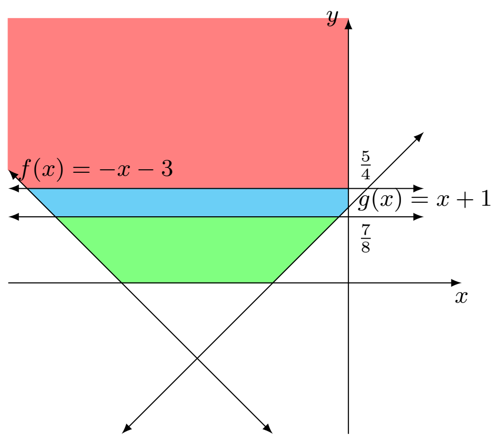

Figure 9: The red (green) region is initial points of solutions that are eventually 4-cycle $P_{4.1}$ (equilibrium point) while the remain region is in cyan.

and $\gamma_n = 2^{2n+2} - 1$ and $R(n)$ be the following statement : "for $y_0 \in (l_n, u_n)$, then $x_{4n+3} = 2^{2n+2}y_0 - \gamma_n, y_{4n+3} = -1$. If $y_0 \in (l_n, A_n]$ then $x_{4n+3} \leq 0$ and so

$$\begin{cases} x_{4n+4} = -2^{2n+2}y_0 + \gamma_n - 2 < 0 \\ y_{4n+4} = 2^{2n+2}y_0 - \gamma_n \leq 0 \end{cases}, \quad \begin{cases} x_{4n+5} = -1 \\ y_{4n+5} = -1 \end{cases}.$$

If $y_0 \in (A_n, u_n)$ then $x_{4n+3} > 0$ and so

$$\begin{cases} x_{4n+4} = 2^{2n+2}y_0 - \gamma_n - 2 \\ y_{4n+4} = 2^{2n+2}y_0 - \gamma_n > 0 \end{cases}.$$

If $y_0 \in [u_{n+1}, u_n)$ then $x_{4n+4} \geq 0$ and so $x_{4n+5} = -5$ and $y_{4n+5} = -1$.
If $y_0 \in (A_n, u_{n+1})$ then $x_{4n+4} < 0$ and so

$$\begin{cases} x_{4n+5} = -2^{2n+3}y_0 + 2\gamma_n - 1 < 0 \\ y_{4n+5} = -1 \end{cases}, \quad \begin{cases} x_{4n+6} = 2^{2n+3}y_0 - 2\gamma_n - 1 \\ y_{4n+6} = -2^{2n+3}y_0 + 2\gamma_n - 1 < 0 \end{cases}$$

If $y_0 \in (A_n, l_{n+1}]$ then $x_{4n+6} \leq 0$ and so $x_{4n+7} = y_{4n+7} = -1$
If $y_0 \in (l_{n+1}, u_{n+1})$ then $x_{4n+6} > 0$."

We shall first show that $P(1)$ is true. For $y_0 \in (l_1, u_1) = (\frac{7}{8}, \frac{5}{4})$ and $\gamma_1 = 15$ we have $x_6 = 8y_0 - 7 > 0, y_6 = -8y_0 + 5 < 0$ and so

$$\begin{cases} x_{4(1)+3} = 16y_0 - 15 = 2^{2(1)+2}y_0 - \gamma_1 \\ y_{4(1)+3} = -1 \end{cases}$$

If $y_0 \in (l_1, A_1] = (\frac{7}{8}, \frac{15}{16}]$ then $x_7 \leq 0$ and so

$$\begin{cases} x_{4(1)+4} = -16y_0 + 13 = -2^{2(1)+2}y_0 + \gamma_1 - 2 < 0 \\ y_{4(1)+4} = 16y_0 - 15 = 2^{2(1)+2}y_0 - \gamma_1 \leq 0 \end{cases}, \quad \begin{cases} x_{4(1)+5} = -1 \\ y_{4(1)+5} = -1 \end{cases}.$$

If $y_0 \in (A_1, u_1) = (\frac{15}{16}, \frac{5}{4})$ then $x_7 > 0$ and so

$$\begin{cases} x_{4(1)+4} = 16y_0 - 17 = 2^{2(1)+2}y_0 - \gamma_1 - 2 \\ y_{4(1)+4} = 16y_0 - 15 = 2^{2(1)+2}y_0 - \gamma_1 > 0 \end{cases}.$$

If $y_0 \in [u_2, u_1) = [\frac{17}{16}, \frac{5}{4})$ then $x_8 \geq 0$ and so $x_{4(1)+5} = -5, y_{4(1)+5} = -1$.
If $y_0 \in (A_1, u_2) = (\frac{15}{16}, \frac{17}{16})$ then $x_8 < 0$ and so

$$\begin{cases} x_{4(1)+5} = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\gamma_1 - 1 < 0 \\ y_{4(1)+5} = -1 \end{cases},$$
$$\begin{cases} x_{4(1)+6} = 32y_0 - 31 = 2^{2(1)+3}y_0 - 2\gamma_1 - 1 \\ y_{4(1)+6} = -32y_0 + 29 = -2^{2(1)+3}y_0 + 2\gamma_1 - 1 < 0 \end{cases}.$$

If $y_0 \in (A_1, l_2] = (\frac{15}{16}, \frac{31}{32}]$ then $x_{10} \leq 0$ and so $x_{4(1)+7} = y_{4(1)+7} = -1$.
If $y_0 \in (l_2, u_2) = (\frac{31}{32}, \frac{17}{18})$ then $x_{4(1)+6} = 32y_0 - 31 = 2^{2(1)+3}y_0 - 2\gamma_1 - 1 > 0$. Thus $R(1)$ is true. So the base case of induction is done. Similar to $P(n)$, one can prove that step case is also true. By mathematical induction $R(n)$ is true for any positive integer $n \geq 1$. From inductive statement one can infer that solution will become 4-cycle ($P_{4.1}$) when $y_0 \in [u_{n+1}, u_n)$ while solution will become equilibrium point when $y_0 \in (l_n, A_n]$ and $y_0 \in (A_n, l_{n+1}]$. One can see that limit of sequence A_n, l_n, and u_n are 1. So the cyan region of Fig. 9 will collapse into a single line $L := \{(x, 1) | x \in [-4, 0]\}$. For $(x_0, y_0) \in L$ one can verify that $(x_2, y_2) = (1, -3) \in P_{4.2}$. It means the solution will become 4-cycle ($P_{4.2}$) when $y_0 \in L$.

Proposition 2.3. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of the map (2) and initial condition $(x_0, y_0) \in \{(x, y) \in Q_2 | y > x + 1$ and $y > -x - 3\}$. Then every solution is eventually equilibrium point.*

We can conclude that there are three attractors: equilibrium point, $P_{4.1}$ and $P_{4.2}$. The basin of attraction of equilibrium point is green portion of Fig.10 while $P_{4.1}$ has red portion of Fig. 10 and $P_{4.2}$ has L being the basin.

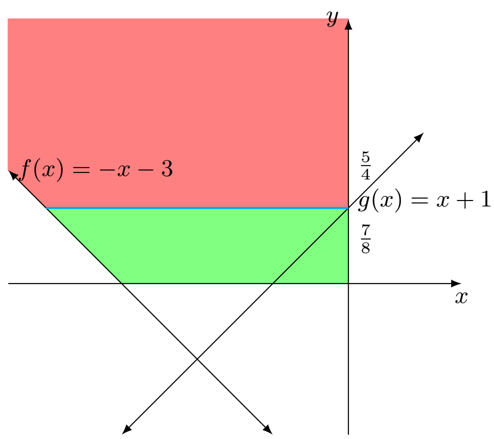

Figure 10: Basin of attraction of $P_{4.1}, P_{4.2}$ is in red and cyan respectively, while the basin of attraction of equilibrium point is in green.

12

3 Conclusion and discussion

We investigated the system of piecewise linear map (2) with initial condition in the second quadrant. By separating the second quadrant into three sub-regions as in Fig.1, we have the following behaviors of solutions. In the rightmost region of second quadrant (initial point below the line $g(x)$), every solution is eventually equilibrium point. For the middle region of second quadrant (initial point above the lines $f(x)$ and $g(x)$), the solution is eventually either equilibrium point or 4-cycle. We proved it by direct calculations and induction. For the last region of second quadrant (below the line $f(x)$) x_1 is positive and y_1 is negative. The behaviors of solution are more complicated than the other two sub-regions and interesting to study that we leave for future work. The behaviors of the map (2) are agree to Tikjha & Piasu (2020) that attractors are only equilibrium point and 4-cycles. It is possible to have equilibrium point and 4-cycles as attractors. But we do still not confirm that until knowing behaviors of solutions with initial condition (x_0, y_0) completely in $\mathbb{R}^2$.

References

[1] Banerjee, S., & Verghese, G.C. (2001). Nonlinear Phenomena in Power Electronics, Attractors, Bifurcations, Chaos, and Nonlinear Control. IEEE Press.

[2] Gardini, L., & Tikjha, W. (2019). The role of virtual fixed points and center bifurcations in a Piecewise Linear Map. Int. J. Bifurcation and Chaos, 29 (14) doi: https://doi.org/10.1142/S0218127419300416

[3] Gardini, L., & Tikjha, W. (2020). Dynamics in the transition case invertible/non-invertible in a 2D Piecewise Linear Map. Chaos, Solitons and Fractals, 137 (2020) https://doi.org/10.1016/j.chaos.2020.109813

[4] Grove, E.A., Lapierre, E.,& Tikjha, W. (2012). On the Global Behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n|$. Cubo Mathematical Journal, 14, 125–166.

[5] Laoharenoo A., Boonklurb R. and Rewlirdsirikul W., Complete analysis of global behavior of certain system of piecewise linear difference equation. Aust. J. Math. Anal. Appl.Vol. 20 (2023), No. 1, Art. 12, 33 pp.

[6] Lapierre, E. and Tikjha, W. (2021) 'On the global behaviour of a system of piecewise linear difference equations', Int. J. Dynamical Systems and Differential Equations, Vol. 11, Nos. 3/4, pp.341–358.

[7] Lorenz, E. , Computational chaos a prelude to computational instability, Physica D. 35, (1989) 299-317.

[8] Lozi, R. (1978). Un attracteur etrange du type attracteur de Henon. J. Phys. (Paris) 39, 9-10.

[9] Mira, C. , Gardini, L., Barugola, A., Cathala, J.-C., Chaotic Dynamics in Two-dimensional Noninvertible Maps, World Scientific, Singapore, 1996.

[10] Simpson, D. J. W. (2010). Bifurcations in piecewise-smooth continuous systems. World Scientific.

[11] Singer, David. "Stable Orbits and Bifurcation of Maps of the Interval." SIAM Journal on Applied Mathematics 35, no. 2 (1978): 260-67. Accessed August 13, 2020. www.jstor.org/stable/2100664

[12] Tikjha, W. and Gardini, L. (2020). Bifurcation sequences and multistability in a two-dimensional Piecewise Linear Map, Int. J. Bifurcation and Chaos, 30(6), doi:S0218127420300141

[13] Tikjha, W., and Lapierre, E. (2020). Periodic solutions of a system of piecewise linear difference equations. Kyungpook Mathematical Journal, 60(2), 401-413

[14] Tikjha, W., Lapierre, E.G., and Sitthiwirattham T. (2017). The stable equilibrium of a system of piecewise linear difference equations. Advances in Difference Equations67 (10 pages); doi:10.1186/s13662-017-1117-2

[15] Tikjha, W., Lenbury, Y. and Lapierre, E.G. (2010). On the Global Character of the System of Piecewise Linear Difference Equations $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n - |y_n|$. Advances in Difference Equations, 573281 (14 pages); doi:10.1155/2010/573281

[16] Tikjha, W., Lapierre, E. G. and Lenbury, Y. (2015). Periodic solutions of a generalized system of piecewise linear difference equations. Advances in Difference Equations 2015:248 doi:10.1186/s13662-015-0585-5.

[17] Tikjha, W., and Piasu, K. (2020). A necessary condition for eventually equilibrium or periodic to a system of difference equations. Journal of Computational Analysis and Applications, 28(2), 254 – 261.

[18] Whitehead R.R., N.MacDonald, A chaotic mapping that displays its own homoclinic structure, Physica D: Nonlinear Phenomena, Volume 13, Issue 3, November 1984, Pages 401-407.

[19] Zhusubaliyev, Zh.T.,and Mosekilde, E. (2003). Bifurcations and Chaos in piecewise-smooth dynamical systems, Nonlinear Science A, Vol. 44, World Scientific.

[20] Zhusubaliyev Zh. T, Mosekilde E. and Banerjee S. (2008). Multiple-attractor bifurcations and quasiperiodicicity in piecewise-smooth maps. Int. J. Bifurcation and Chaos, 18(6), 1775–1789